Solutions to Problem-set 1

JLS

Questions on A* search

1. Figure 1 shows a map of the second floor of the Kilburn Building. Suppose you wanted to find the shortest route between room 2.100 and the nearest women's toilet, using A* search. Suppose you did not have measurements of the corridors and rooms. What could you use as a heuristic? How would you use the algorithm to find the path to the closest women's toilet?

   Answer: One way to create a graph representation of this space is to use doors as nodes, where neighboring doors and doors across the corridor would be connected. Then the distance measure is the number of doors passed along a path on one side of a corridor (i.e. the number of rooms). This would work better as a distance measure were the rooms all of the same size, which they are clearly not. But it is a reasonable approximation. Since there are two women's toilets, when computing the value of the heuristic for the current location, one could count the number of doors between the current location and each of the women's toilets, and use the closest one. Or, run the algorithm twice, once with each toilet as the goal, and then use the closest one. Comment: If I ask a question about A* on an exam, it would be about what it does, when is it used, and how the heuristic is used, but not about algorithm details.

2. What would be a good heuristic for the Towers of Hanoi problem? If you do not know the Towers of Hanoi problem, look it up, perhaps here.

   Answer: Some people found this hard, and it is trickier than I thought. Don't worry if you found it confusing. Let us assume that there are k disks to start with, and there are three pegs. The leftmost peg is where the tower starts, and the rightmost peg is where it must finish. Here are some possible heuristics (all of which would be fine answers):

   - H_1: **The number of disks on the starting (leftmost) peg. This can be 0 without the puzzle being solved, so not always informative. All the disks on the leftmost peg must be moved, so it is admissible. In solving the game, it is necessary to put disks back on the leftmost peg, i.e. moving backward. You might be led to believe that this means it it not monotonic, but, in fact, *it is monotonic.* Suppose state X has n disks on the starting peg, and state Y has n'**

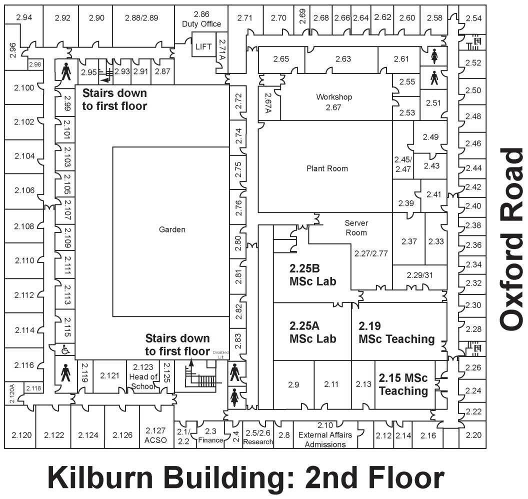

Figure 1: The second floor of the Kilburn Building. The figure refers to question 1. You want to find the shortest route from Kilburn 2.100 to the nearest women's toilet.

disks on the starting peg. If $n' > n$, then $H_1(Y) > H_1(X)$ which implies that $H_1(X) \leq C(X,Y) + H_1(Y)$ trivially, since the number of moves between X and Y, i.e. the true cost $C(X,Y)$, is positive. If n' is less than n, (so you are moving towards the goal), then at least $n - n'$ moves are required to get from X to Y. So,

$$
\begin{aligned}
C(X,Y) &\geq n - n'; \text{ so,} && (1)\\
C(X,Y) &\geq H_1(X) - H_1(Y); \text{ so,} && (2)\\
H_1(X) &\leq C(X,Y) + H_1(Y). && (3)
\end{aligned}
$$

But finally, what if $n = n'$? Assuming that X and Y are different states, then it must take at least one move to get from X to Y. Since $H_1(X) = H_1(Y)$ in this case, it follows that,

$$
\begin{aligned}
H_1(X) &\leq H_1(X) + 1 && (4)\\
&= 1 + H_1(Y) && (5)\\
&\leq C(X,Y) + H_1(Y). && (6)
\end{aligned}
$$

- **H_2: The number of disks not on the finishing peg. When this is 0, the puzzle is completed. Thus, admissible, and more informative than the above. While solving the puzzle, it is necessary to move disks on and off the finishing peg. This is monotonic, using similar arguments as above.**

Comment:

- **I will not ask you to produce a proof that the heuristic is (or isn't) monotonic on an exam.**

- **One idea which was discussed in a Zoom breakout room was to use number of pegs away as a distance. E.g. a disk on the starting peg is two away; a disk on the middle peg is one step away. This is not a good idea (although, I did not realize it at the time we discussed it). The rules allow moves directly from the starting peg to the goal peg, so as described it would not be admissible. It also does not capture the rules of the problem.**
- **It appears that most heuristics which are admissible are monotonic. This is because the true distance, $C(X < Y)$, is greater than the "heuristic distance". It is possible to cook up admissible heuristics which are not monotonic, but it is not that easy to do.**

1 Game trees

1. Consider NimN with $N = 4$. I.e. starting with 4 matches, each player in turns takes 1, 2, or 3 matches. Player taking the last match *loses*.

   (a) Draw the game tree for NimN with $N = 4$. Can either side guarantee a win? Which side?

   Answer: The game tree is shown in Figure 2. You can see that Player 1 can force a win by taking three matches.

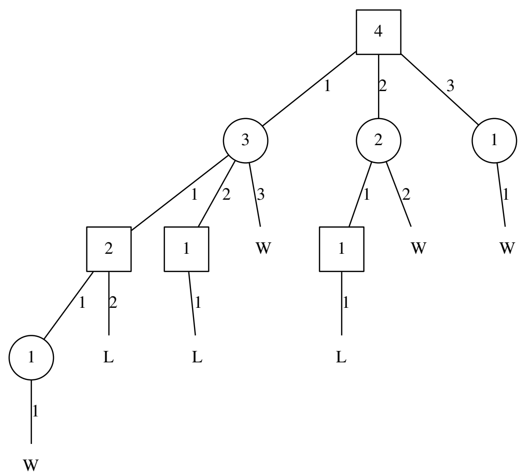

Figure 2: The game tree for NimN with $N = 4$. Square nodes are decision nodes for Player 1; round nodes are for Player 2. The number along the edges denote the number of matches taken. The number within the nodes are the number of nodes remaining. W (L) denotes a win (loss) for Player 1.

Comment: I might ask you to draw a game tree in an exam.

(b) There is a winning strategy for Player 1 or for Player 2 (but not both) for any positive integer value of N? See if you can find it. Try a few other values of N For example, $N = 5, 6, 8, 9$. (Don't draw out the game tree for these. Just think which player could force a win from each starting number of matches.)

Answer:

- **The first observation is that if n is the number of matches at the start of a round, the second player to go can ensure that the number of matches remaining at the end of the round is $n - 4$ by taking 4 minus whatever the opponent took. So if the opponent takes 3 then take 1; if the opponent takes 2 then take 2; if the opponent takes 1 then take 3. This strategy ensures that at each round, the number of matches remaining shifts down by 4. I will call this the default strategy.**
- **The second observation is that *if it is possible to end your turn with the number of matches remaining one greater than an exact multiple of* 4 (such as 5, or 9, or 13), then employing the default strategy will result in you ending a turn with 1 match remaining and winning the game.**
- **Another way to express this is as follows: if a player ends its move with n matches remaining, and $n \mod 4 = 1$, then that player will win the game by employing the default strategy. In conclusion, if $N \mod 4 = 1$, then Player 2 has a winning strategy, by employing the default strategy.**
- **Otherwise, Player 1 has a winning strategy by taking enough matches so as to end its turn with 1 greater than a exact multiple of 4.**
- **More precisely, if $N \mod 4 = 0$, Player 1 takes 3 is its first move, and then wins by employing the default strategy; if $N \mod 4 = 2$, Player 1 takes 1 is its first move, and then wins by employing the default strategy; and if $N \mod 4 = 3$, Player 1 takes 2 is its first move, and then wins by employing the default strategy.**

Comment: I would never ask this question in an exam. It involves a trick which would require too much cognitive load under exam conditions. Also, some people may have seen this before.

2. Consider an auction for a work of art. This is an *English Auction* where the bids increase in value. The item goes to the highest bidder. The rules are: the starting bid must be £10, and all subsequent bids must be raised by exactly £10.

   There are only two bidders at the auction: Alice and Bob. Alice values the artwork at £25, but she is willing to bid £30 for it, but no higher. Bob values the artwork at £20, and is not willing to bid higher than that. Draw the game tree for this auction, taking Alice as player 1.

Each player has two possible actions: raise the bid by £10, and pass. If a player passes after its opponent has made a bid, the opponent gets the item at the cost of its bid. The exception is at the beginning of the game. If no bid has yet been made, then if Player 1 passes followed by Player 2 passing, Player 1 can make the opening bid.

At the terminal positions, write down the value of the result. A player who did not win the auction gets a value of 0 they did not get the artwork, but they did not spend any money. The winner of the auction gets the artwork, but pays money for it, so the value they get is

$$(\text{The value of the artwork to them}) - (\text{the amount they paid for it}) .$$

Answer: Figure 3 shows the game tree of the auction. Squares are Alice; circles are Bob. The numbers in the node shapes are the current bid (if it is not 0). If Alice passes and Bob passes, start again. If Alice passes and Bob bids, Bob must bid £10. If Alice follows by passing, Bob wins the auction and gets the item for £10. Since the item is worth £20 to Bob, he gains £10. Alice gets nothing, but loses nothing. The player who does not win the auction, gets nothing, but loses nothing. However, if Alice raises the bid to £20 after Bob bid £10, Bob would have to raise the bid to £30 to stay in the auction, but Bob is not willing to bid more than £20 (it says in the question), so Bob passes, and Alice gets the item for £20. It is worth £25 to her, so she gains £5. If Alice started the auction with a bid of £10, then if Bob did not bid, Alice gets it for £10 and gains £15. But if Bob raised the bid to £20, Alice could pass. Then Bob gets the item for £20 which is worth £20 to him. So, in a sense, Bob neither gains nor loses. He just exchanged two equally valuable things — cash for art. Alice is willing to bid up to £30 (it says in the question), so if she raises the bid up to £30 after Bob's bid, Bob will have to bid 40 to stay in the auction. But he is not willing to bid more than 20, so he passes and Alice gets the item for £30. It is only worth £25 to her, so she loses £25.

Comment: I might ask you to draw a game tree in an exam.

Games in Normal Form

1. Bill is falling in love with Amy, and Amy is likewise falling for Bill, but neither has told the other. They both know that Amy loves basketball, and is happiest at the basketball center; Bill loves jazz, and is happiest at the jazz club. They both decide simultaneously to go to one of the venues in the hopes of meeting up, but they have not discussed with each other where they will be. Express this as a normal form game with the two players, Bill and Amy, and the two actions being the two venues they could go to. You will need to think of numbers for the payoffs to express the situation: they are both happy if they might up, both unhappy if they go to different venues, and there is an extra degree of happiness if they go to their preferred venues.

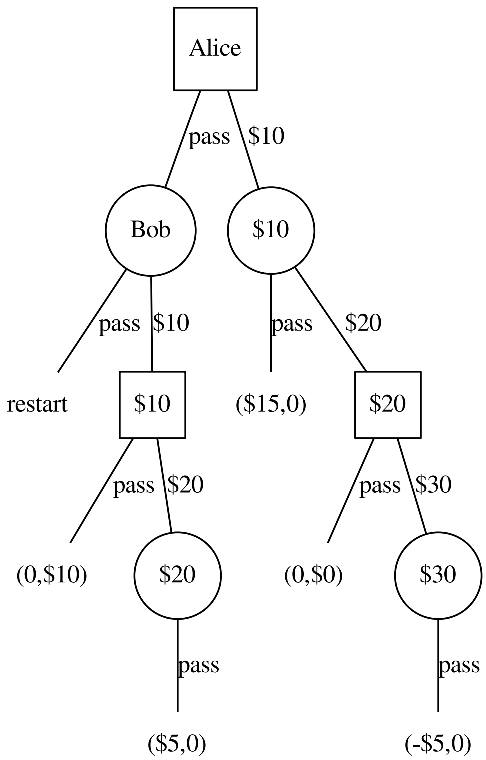

Figure 3: The game tree of the auction. Squares are Alice's decisions; circles are Bob's decisions. The number in the node shape is the current bid, which must be raised by £10 or the item goes to the current highest bidder. (My graph-making software is a bit pathetic and is unable to produce a UK £symbols.)

> **Answer: Qualitatively, if they go to the same venue and meet up, they are both happy. The person at their favorite venue is happiest. If Bob goes to the Jazz and Amy goes to the BBall, then they are both less happy because they did not meet up, but still a bit happy, since at least they are at their favorite venues. If Bob goes to the BBall, and Amy goes to the Jazz, then they are both miserable. Bob does not like basketball that much, Amy hates jazz and they are both alone ☹. Figure 4 shows this qualitatively. However, to express it as a game, we need to assign numbers as payoffs to the decision choices. This is one of the difficulties of game theory, how to assign numbers to benefits or losses which are not win/lose or financial costs or rewards. One attempt to assign numbers to Bob and Amy's feelings is shown in Figure 5. Other numbers could also be used.**

		Bill	
		BBall	Jazz
Amy	BBall	(☺,☺)	(☺,☺)
	Jazz	(☹,☹)	(☺,☺)

Figure 4: The amount of smiles indicates the positivity-negativity of the reward.

		Bill	
		BBall	Jazz
Amy	BBall	$(3, 2)$	$(1, 1)$
	Jazz	$(-1, -1)$	$(2, 3)$

Figure 5: A somewhat arbitrary assignment of numbers to feelings, from miserable (-1) to neutral (0) to very happy (3). Any numbers could be assigned as long is greater positive numbers represent more happy, and more negative represents more unhappy.

Comment: I might ask a question like this in an exam.

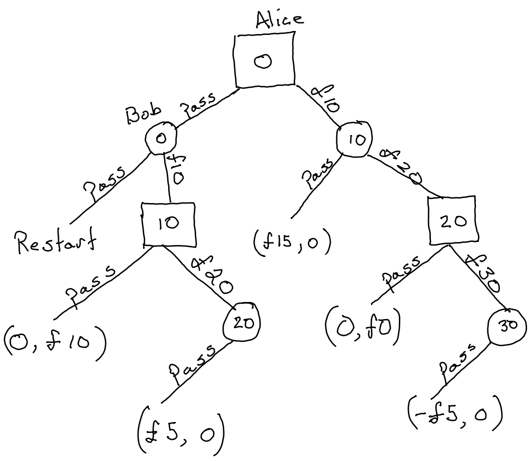

Figure 6: Hand drawn version of Figure 3. I asked you to draw it by hand. Not fair if I don't have to draw it by hand too.