# Solutions to Problem-set 2

JLS

## Winning strategies in two-player games

**problem 0.1.** *In Lecture 3 and Theorem 1.10 of the notes, the following thereom is stated.*

**1.10.** *For any* ***two-player, zero-sum*** *game with* ***perfect information*** *and* ***no chance****, that ends after a finite number of moves, one of the following is true.*

1. *Player 1 has a winning strategy;*
2. *Player 2 has a winning strategy;*
3. *Player 1 and Player 2 both have strategies which ensure a draw.*

*Are all the assumptions needed? For each assumption, give a counter-example where the assumption is dropped and the theorem is false, or else an argument that the assumption is not needed.*

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***Answer:*** *All assumptions are needed:*

1. *Two-player: If there are more than two players, it is possible that two players win (e.g. $(1/3, 1/3, -2/3)$), or two players can ensure a draw, but one of the other two players wins and the other loses.*
2. *zero-sum: If it is not zero-sum, the payoffs could be positive to both or negative to both.*
3. *perfect information: (This is the most tricky one.) Consider the following game. Player 1 writes the word "apple" or "peach" on a piece of paper but does not show it to Player 2. Player 2 must then guess which it is, apple or peach. If Player 2's guess is correct, Player 2 wins and Player 1 loses. Otherwise, Player 1 wins and Player 2 loses. Neither player can guarantee a win nor a draw.*
4. *no chance: Consider the following game. A coin is flipped. Heads means Player 1 win; tails means Player 2 wins. Neither player can ensure a win nor a draw.*
5. *ends in a finite number of moves:*
   (a) *Player 1 demands a pay rise of £5 per hour.*
   (b) *Player 2 can agree to the pay rise and loses £5 per hour. In thise case, Player 1 wins; Player 2 loses, and the game ends. Or Player 2 can reject the demand and the game continues.*

(c) *From this point on, Player 1 can end the game and lose, giving the win to Player 2, or repeat the demand for a pay rise.*

(d) *From this point on, Player 2 can agree to the pay rise lose, giving the win to Player 1, or reject the pay rise and the game continues.*

*Note: the game need never end. Player 1 can guarantee a loss but not a win (by giving up). Player 2 can guarantee a loss but not a win (by giving in).*

***End of answer***

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## The pie rule

**problem 0.2.** *In games with a strong first-player advantage, such as Hex, it is common to use an addition rule called "the pie rule" (also called "the swap rule") to make the game more fair. Let Alice be Player 1 and Bob be Player 2. After Player 1 (Alice) makes her first move, Player 2 (Bob) has a choice: either Bob accepts the move and continue to play as Player 2 and makes a move in response, or Bob can "swap". If Bob swaps, he becomes Player 1 and Alice becomes Player 2, and needs to move in response to her original move. If this is confusing, perhaps* `https://www.academickids.com/encyclopedia/Pie_rule` *or* `https://sites.google.com/site/boardandpieces/terminology/pie-rule` *Will help. We use the pie rule in the project. The pie rule can only be used by Player 2 and only after the very first move of the game.*

*Suppose Alice and Bob are playing a version of Hex which is weakly solved, and assume both Alice and Bob know the winning strategy.*

(a) *Should Alice make the winning move as her first move? What happens if she does?*

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***Answer:*** *No! Bob will swap.*

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(b) *Suppose Alice makes a non-winning move as her opening move. Does this mean that Bob has a guaranteed win? Speculate why Bob might have a winning move.*

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***Answer:*** *The speculation would be that Bob would not swap but would steal the winning strategy. I think this is not true, but I have yet to prove it. Bob has one less move than Alice, not one more.*

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## Equilibria in normal form games

**problem 0.3.** *Below is a pay-off matrix for a zero-sum, two-player game with perfect information and no chance. The two players are Alice and Bob. Alice has three actions: A–C; Bob has four actions: a–d. The table shows the pay-off to Alice.*

| | | Bob | | | |
|---|---|---|---|---|---|
| | | *a* | *b* | *c* | *d* |
| Alice | *A* | -2 | 0 | -3 | 1 |
| | *B* | -1 | 1 | 1 | -1 |
| | *C* | -2 | -1 | 2 | 3 |

(a) *Find a Nash equilibrium for the game using* dominance. *If it takes more than one step to get to the equilibrium, be sure to write down every step in reducing the matrix until the equilibrium is found. (Remember, a dominant strategy for Player 2 has lower values than other strategies, for all opponent actions.)*

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***Answer:*** *Alice has no dominant strategies. For Bob, a dominates b and d, leaving*

| | | Bob | |
|---|---|---|---|
| | | *a* | *c* |
| Alice | *A* | -2 | -3 |
| | *B* | -1 | 1 |
| | *C* | -2 | 2 |

*Now for Alice, both C and B dominate over A. So remove A, leaving*

| | | Bob | |
|---|---|---|---|
| | | *a* | *c* |
| Alice | *B* | -1 | 1 |
| | *C* | -2 | 2 |

*Now for Bob a is the best, so Bob removes C. Then for Alice B is the best. The result is (B,a) with a payoff of -1 to Alice.*

***End of answer!***

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(b) *Now find the equilibrium using the minimax approach, again giving the steps in your reasoning.*

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***Answer:*** *For Alice, the minima are* $-3$ *for action A,* $-1$ *for action B, and* $-2$ *for action C. Maximising these minima, she choses B. For Bob, the maxima are* $-1$ *for action a,* 1 *for action b, and* 3 *for action c. Minimising these maxima, Bob choses a. The pair (B,a) maximises the minimum for rows and minimises the maxima for columns, and is therefore a Nash equilibria. It is the same found using dominance.* ***End of answer!***

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**problem 0.4.** *This question involves the following game. The two players move simultaneously.*

| | | Brian | |
|---|---|---|---|
| | | *a* | *b* |
| Amy | *A* | $(1,-1)$ | $(-1,1)$ |
| | *B* | $(-1,1)$ | $(2,-2)$ |

*Now answer the following questions.*

(a) *What type of game is this? Name all types of games it is.*

***Answer:*** *A two-player, zero-sum game in normal form, hence with imperfect information.* ***End of answer!***

(b) *Find all of the pure strategy equilbria. Justify that they are equilibria.*

***Answer:*** *There are none.*

***End of answer!***

(c) *Is there a fully mixed strategy equilibrium? If so, find it. Otherwise, give a convincing argument that one does not exist.*

***Answer:***

*Let Amy play action A with probability $x$ and B with probability $(1-x)$. Against Brian playing a, Amy gets a payoff of $x-(1-x)=2x-1$. Against Brian playing b, Amy gets $-x+2(1-x)=2-3x$. The only place where Brian does not have a superior pure strategy is when the two are equal. This is also where the Amy maximises the minima payoff she can get.* $\boxed{x=\frac{3}{5}}$. *Following a similar argument for Brian, where $y$ is the probability he plays action a and $(1-y)$ is the probability he plays action b. We find,* $\boxed{y=\frac{3}{5}}$.

***End of Answer!***

**problem 0.5.** *This question involves the following game. The two players move simultaneously.*

| | | Brian | |
|---|---|---|---|
| | | $a$ | $b$ |
| Amy | $A$ | $(1,1)$ | $(-1,-1)$ |
| | $B$ | $(-1,-1)$ | $(3,2)$ |

*Now answer the following questions.*

(a) *What type of game is this? Name all types of games it is.*

***Answer:*** *A two-player, general-sum (not zero-sum) game in normal form, hence with imperfect information.* ***End of answer!***

(b) *Find all of the pure strategy equilbria. Justify that they are equilibria.*

***Answer:*** *(A,a) and (B,b) are two equilibria. Unilateral deviation from either of these results in a poorer payoff for the player deviating.*

***End of answer!***

(c) *Is there a fully mixed strategy equilibrium? If so, find it. Otherwise, give a convincing argument that one does not exist.*

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***Answer:***

*Let Amy play action A with probability $x$ and B with probability $(1-x)$. Because this is* not *a zero-sum game, Amy can give Brian no incentive to deviate if Brian gets the same payoff to Brian whichever action Brian takes. In other words, Amy needs to set the two strategies equal using Brian's payoffs.*

*Against Brian playing a, Brian gets a payoff of $x-(1-x) = 2x-1$. Against Brian playing b, Brian gets $-x+2(1-x) = 2-3x$. The only place where Brian does not have a superior pure strategy is where the two are equal.* $\boxed{x = \frac{3}{5}}$. *Following a similar argument for Brian, where $y$ is the probability he action a and $(1-y)$ is the probability he plays action b. We find,* $\boxed{y = \frac{2}{3}}$. ***End of Answer!***

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## Game trees

**problem 0.6.** *Figure 1 shows a game tree. In what order are the nodes values determined? Nodes can be evaluated left to right or right to left, but say which you are using.*

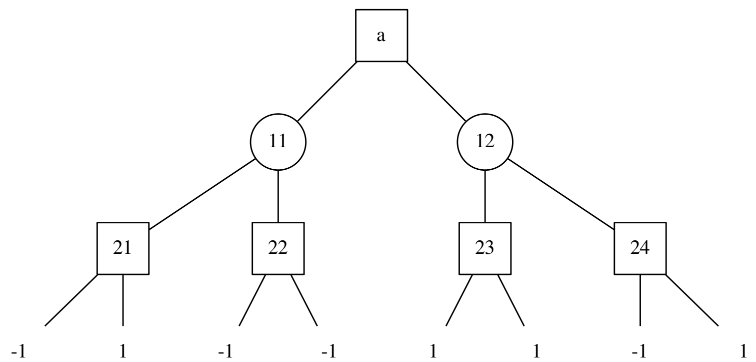


Figure 1: A game tree used for Problem 0.6. Squares are `MAX` nodes and circles are `MIN` nodes.

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***Answer:*** *Evaluating left to right the order is: 21, 22, 11, 23, 24, 12, a.* ***End of answer!***

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**problem 0.7.** *Figure 2 shows a win-loss tree. Apply win-loss pruning (either left to right or right to left). What nodes are* not *evaluated.*

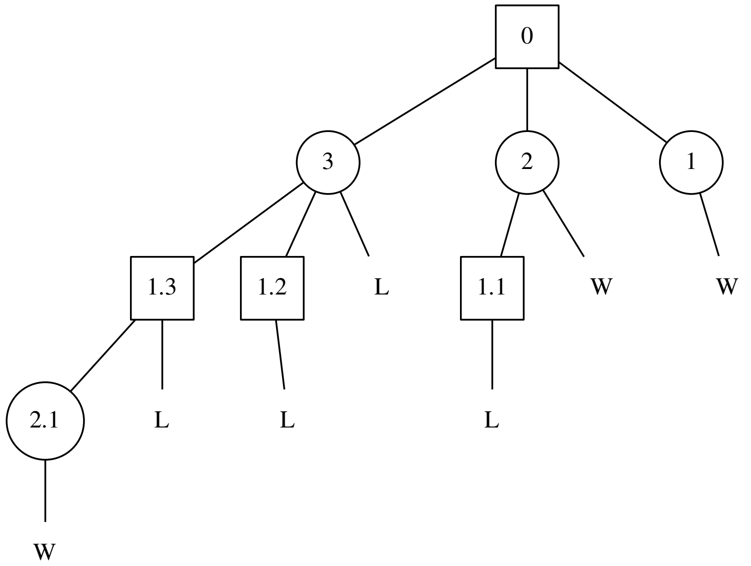


Figure 2: A win-loss tree associated with Problem 0.7. You might recognize this tree.

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***Answer:*** ***Evaluating from left to right:*** *the right side terminal node of node 1.3 is not evaluated because 1.3 is already a winning node. The right-most terminal node of node of node 3 is not evaluated because node 3 is already a losing node from node 1.2. The right-most terminal node of node 2 is not evaluated, because node is already a losing node from node 1.1.*

***Evaluating right to left:*** *all nodes are pruned except 1 and 0.* ***End of answer!***

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**problem 0.8.** *Figure 3 shows a game tree. Apply alpha-beta pruning (either left to right or right to left). What nodes are* not *evaluated. What are the final alpha-beta values of nodes C1, C2, C3 and the root?*

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***Answer:***

**C1:** *Final range is* $[-\infty, 15]$ *and the value is* 15.

**C2:** *Final range is* $[15, -3]$ *and value is* $-3$.

**C3:** *Final range is* $[15, 12]$ *and value is* 12.

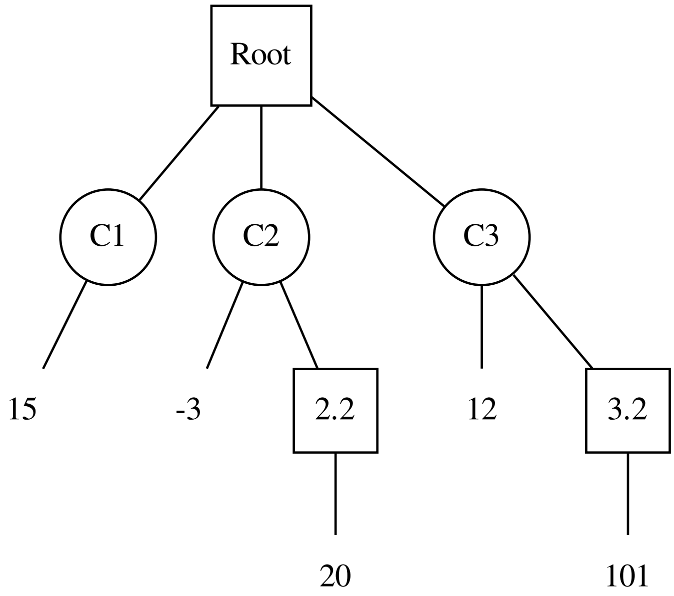


Figure 3: A game tree associated with Problem 0.8. To practice alpha-beta pruning. You might recognize this tree.

**Root:** *Final range is* $[15, \infty]$ *and the value is* $15$.

**Pruned:** *2.2 and 3.2.*

***End of answer!***