Hyperuniformity on the Sphere

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Hyperuniform structures, rigid point processes
and related topics,
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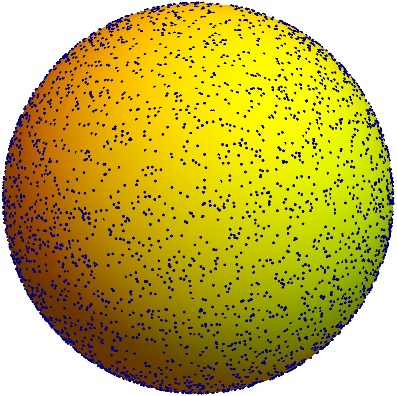
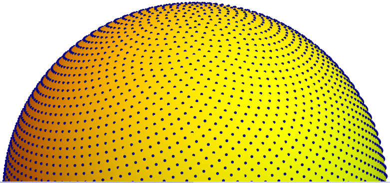
Two point distributions





Uniform distribution

Definition

A sequence of point sets $(X_N)_{N\in\mathbb{N}}$ ($X_N \subset \mathbb{S}^d$) is called uniformly distributed, if

$$\lim_{N\to\infty} \frac{\#\left(X_N \cap C\right)}{N} = \sigma_d(C),$$

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This is equivalent to

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\mathbf{x}\in X_N} f(\mathbf{x}) = \int_{\mathbb{S}^d} f(\mathbf{x})\, d\sigma_d(\mathbf{x})$$

for all continuous (or even Riemann-integrable) functions f.

Uniform distribution

By the density of spherical harmonics in the continuous functions

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{\mathbf{x},\mathbf{y}\in X_N} P_n^{(d)}(\langle \mathbf{x}, \mathbf{y} \rangle) = 0$$

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We denote by $P_n^{(d)}$ the Legendre-polynomials for $\mathbb{S}^d$ normalised by $P_n^{(d)}(1) = 1$. These are Gegenbauer-polynomials for parameter $\lambda = \frac{d-1}{2}$ up to a scaling factor.

Hyperuniformity on the sphere

Definition (Hyperuniformity)

Let $(X_N)_{N\in\mathbb{N}}$ be a sequence of point sets on the sphere $\mathbb{S}^d$. The *number variance* of the sequence for caps of opening angle ϕ is given by

$$V(X_N, \phi) = \mathbb{V}_{\mathbf{x}} \# \left(X_N \cap C(\mathbf{x}, \phi) \right). \tag{1}$$

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- **hyperuniform for large caps**, if

$$V(X_N,\phi) = o\left(N\right) \quad \text{as } N \to \infty \tag{2}$$

for all $\phi \in (0, \frac{\pi}{2})$;

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and all sequences $(\phi_N)_{N\in\mathbb{N}}$ such that

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$$\limsup_{N\to\infty} V(X_N, tN^{-\frac{1}{d}}) = \mathcal{O}(t^{d-1}) \quad \text{as } t\to\infty. \tag{4}$$

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In all three regimes we require the point distribution to behave **better** than i.i.d. random points.

Large caps

If $(X_N)_{N\in\mathbb{N}}$ is hyperuniform for large caps, then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{\mathbf{x},\mathbf{y}\in X_N} P_n^{(d)}(\langle\mathbf{x},\mathbf{y}\rangle)=0$$

for all $n\geq 1$. This implies uniform distribution of $(X_N)_{N\in\mathbb{N}}$.

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for all $n \geq 1$. This implies uniform distribution of $(X_N)_{N\in\mathbb{N}}$. Furthermore, it is not enough to require the defining relation for hyperuniformity for only one value of ϕ.

Small caps

If $(X_N)_{N\in\mathbb{N}}$ is hyperuniform for small caps, then

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{\mathbf{x},\mathbf{y}\in X_N} P_n^{(d)}(\langle \mathbf{x}, \mathbf{y}\rangle) < \infty$$

for all $n \geq 1$. This again implies uniform distribution of $(X_N)_{N\in\mathbb{N}}$.

Threshold order

If $(X_N)_{N\in\mathbb{N}}$ is hyperuniform for caps at threshold order, then

$$\lim_{N\to\infty}\frac{1}{N^2}\sum_{\mathbf{x},\mathbf{y}\in X_N} P_n^{(d)}(\langle \mathbf{x},\mathbf{y}\rangle)=0$$

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In the cases of small caps and caps of threshold order the conclusion of uniform distribution is not immediately obvious, since the range of caps for testing the distribution is quite restricted.

Deterministic hyperuniform point sets

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- sequences of QMC-designs
- many candidates like Fibonacci-points or spiral points, but no proofs…

Spherical designs

A spherical t-design is a point set $X_N \subset \mathbb{S}^d$, such that

$$\sum_{i,j=1}^{N} P_n^{(d)}(\langle x_i, x_j \rangle) = 0$$

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for $1 \leq n \leq t$. This gives exact integration for all polynomials of degree $\leq t$ on $\mathbb{S}^d$.
It was a major breakthrough by Bondarenko, Radchenko, and Viazovska to show that there exist such point sets with $N = \mathcal{O}(t^d)$ points; this is the minimal possible order.

Probabilistic aspects

The original setting of hyperuniformity comes from statistical physics. The points are assumed to be sampled from a point process. The number variance is then the variance with respect to the process.

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In this context the i.i.d. random case is referred to as the "Poissonian point process". This process is – of course – not hyperuniform.

Determinantal point processes

A point process is determinantal on M with kernel $K : M \times M \to \mathbb{R}$, if its joint densities are given by

$$\rho_N(x_1, \ldots, x_N) = \frac{1}{N!} \det \left(K(x_i, x_j)_{i,j=1}^N \right).$$

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The fact that the determinant vanishes, if $x_i = x_j$ for some $i \neq j$, implies a mutual repulsion of the sample points ("particles").

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One special case is especially important and easy to understand:
Let $H \subset L^2(M)$ be a finite dimensional space and K_H be the orthogonal projection to this space. If $N = \dim H$, then the DPP given by the kernel K_H samples exactly N points.

The spherical ensemble on $\mathbb{S}^2$

The kernel

$$\tilde{K}^{(N)}(x,y) = \frac{N(1+x\bar{y})^{N-1}}{4\pi(1+|x|^2)^{N+1}(1+|y|^2)^{N+1}}$$

on $\mathbb{C}^2$ describes the distribution of the eigenvalues of AB^{-1} for two $N \times N$ matrices A, B with i.i.d. complex Gaussian entries.

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$$C_N \prod_{1 \leq i < j \leq N} \|\mathbf{x}_i - \mathbf{x}_j\|^2$$

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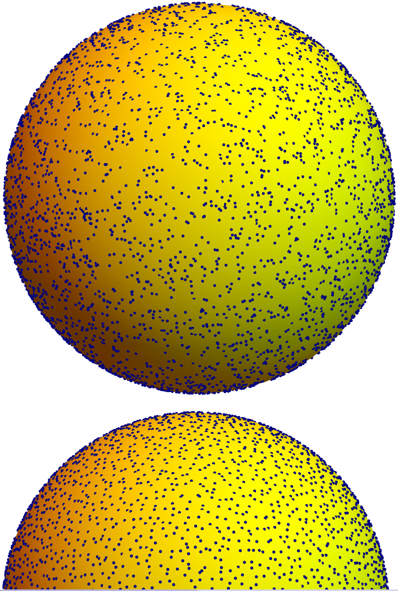
It has been shown by Alishahi and Zamani that samples of the spherical ensemble are hyperuniform for small caps.

Sample of spherical ensemble



The harmonic ensemble on $\mathbb{S}^d$

Let H_L be the span of all spherical harmonics of degree $\leq L$ on $\mathbb{S}^d$. Then the corresponding projection kernel defines a determinantal point process sampling $\dim H \sim L^d$ points. This process was introduced and studied by Beltrán, Marzo, and Ortega-Cerdá.

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They proved *inter alia* that samples of this process are hyperuniform for small caps.
In joint work with J. Brauchart, W. Kusner, and J. Ziefle we could prove that samples of this process are also hyperuniform at threshold order (in a slightly weaker sense).

Jittered sampling

Let A_i $(i = 1, \ldots, N)$ be an area regular partition of $\mathbb{S}^d$ with $\operatorname{diam}(A_i) \leq CN^{-\frac{1}{d}}$ and $\sigma(A_i) = \frac{1}{N}$. Such partitions exist by work of Kuijlaars and Saff and Gigante and Leopardi.

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Then define point process by taking N points idependently uniformly from the sets A_i (one point per set). This process is the determinantal process given by the projection to the space of functions measurable with respect to the finite σ-algebra generated by $(A_i)_{i=1}^N$.

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Jittered sampling points are hyperuniform in all three regimes.

Poissonian pair correlation

A second local statistic of point distributions has been studied in the context of eigenvalue distributions of Schrödinger operators in the physics context. This has then been extended to zeros of the Riemann zeta function and sequences of the form $(\alpha p(n))$, where p is a polynomial of degree ≥ 2.

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A second local statistic of point distributions has been studied in the context of eigenvalue distributions of Schrödinger operators in the physics context. This has then been extended to zeros of the Riemann zeta function and sequences of the form $(\alpha p(n))$, where p is a polynomial of degree ≥ 2.

Definition

A sequence of points $(x_n)_n$ on the unit circle has **Poissonian pair correlation**, if

$$\lim_{N\to\infty}\frac{1}{N}\#\{1\leq i\neq j\leq N \mid N|x_i-x_j|<s\}=2s$$

for all $s>0$.

This is a property shared by i.i.d. random point sets.

Poissonian pair correlation continued

It has been proved by Grepstad and Larcher and independently Aistleitner, Lachmann, and Pausinger that PPC implies uniform distribution. The proofs of this expected fact are surprisingly complicated and intricate.

Higher dimensions

In higher dimensions the obvious generalisation is.

Definition

A sequence of point sets $(X_N)_N$ on a compact manifold M of dimension n has Poissonian pair correlation, if

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ 1 \leq i \neq j \leq N \mid N^{1/n} d(x_i^{(N)}, x_j^{(N)}) < s \right\} = s^n B_n$$

for all $s > 0$; B_n denotes the volume of the n-dimensional unit ball. Here, d denotes the geodesic distance on M.

PPC implies uniform distribution

Steinerberger studied the case of the torus and gave a proof that PPC implies uniform distribution in this case. Marklof gave a very general proof of this fact for general compact manifolds.

Open questions

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- Find more explicit deterministic constructions for hyperuniform point sets for any N.
- Find explicit deterministic constructions for point sets achieving the best possible discrepancy bound (or even a bound better than $N^{-\frac{1}{2}}$)

Thank you for your attention!