Variance linearity for Gaussian excursions

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Hyperuniform structures, rigid point processes and related topics
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Gaussian fields

- **Centered Gaussian fields** : $\mathbf{X} : \mathbb{R}^d \to \mathbb{R}$ such that :
  - $(X(x_1), \dots, X(x_q))$ is a Gaussian vector for $x_1, \dots, x_q \in E$
  - $\mathbb{E}(\mathbf{X}(x)) = 0, x \in E$.
- Stationarity : $\mathbf{X}(x + \cdot) \overset{(d)}{=} \mathbf{X}$ for $x \in \mathbb{R}^d$

Excursion and level sets are privileged observables : for $\ell \in \mathbb{R}$

$$\mathbf{E}_\ell = \{x : \mathbf{X}(x) \geqslant \ell\}$$
$$\mathbf{L}_\ell = \{x : \mathbf{X}(x) = \ell\}.$$

Variance linearity :

- in $\mathbb{R}$:

$$\mathbf{Var}(\mathbf{L}_0 \cap [0, R]) \sim R?$$

- in $\mathbb{R}^d$:

$$\mathbf{Var}(\mathbf{E}_\ell \cap \mathbf{B}(0, R)) \sim R^d?$$

A non-stationary field with stationary HU zeros

Let $\mathbf{X} : \mathbb{C} \to \mathbb{C}$ be a GAF, i.e. a Gaussian field such that

- $\mathbf{X}$ is a.s. holomorphic
- For all $x_1, \dots, x_q, (\mathbf{X}(x_1), \dots, \mathbf{X}(x_q)) \in \mathbb{C}^q$ has a centered standard Complex distribution ($\neq$ standard distribution that is complex)

Then $\mathbf{X}$ is not stationary (maximum principle).

Example : The "planar GAF" (Nazarov, Sodin, Tsirelson, ...)

$$\mathbf{X}(z) = \sum_n \underbrace{a_n}_{iid\mathcal{N}_{\mathbb{C}}(0,1)} \frac{1}{\sqrt{n!}} z^n \in \mathbb{C}$$

characterised by

$$\mathbb{E}(\mathbf{X}(z)\overline{\mathbf{X}(w)}) = \exp(z\bar{w}), z, w \in \mathbb{C}$$

Zeros of the planar GAF (S. Ghosh, Y. Peres 2017)

- The zero set

$$\mathbf{Z} = \mathbf{L}_0 = \mathbf{X}^{-1}(\{0_{\mathbb{C}}\})$$

  is stationary !
- And it is the only one : for $\mathbf{X}'$ another GAF on $\mathbb{C}$, if $\mathbf{Z}' = \mathbf{X}'^{-1}(\{0\})$ is a stationary point process,

$$\mathbf{X}'(z) = \mathbf{X}(\alpha z),$$
$$\mathbf{Z}' = \alpha^{-1}\mathbf{Z}.$$

- This set furthermore is
  - Hyperuniform
  - Rigid (at order 1)
  - Mixing (asymptotically independent)

Stationary real zeros

- A problem going back to the 50's is the study of zeros of a smooth Stationary Gaussian Process (SGP) $\mathbf{X} : \mathbb{R} \to \mathbb{R}$

$$\mathbf{Z} := \mathbf{L}_0 = \{x \in \mathbb{R} : \mathbf{X}(x) = 0\}$$

- **"Nodal"** : the properties of $\mathbf{Z}$ might differ from those of the $\mathbf{L}_\ell, \ell \neq 0$.
- **First order** : $\mathbb{E}(\mathbf{Leb}^1([0, T] \cap \mathbf{Z}))$ is proportionnal to T ("linear")

Tools

A SGP $\mathbf{X} : \mathbb{R}^d \to \mathbb{R}$ is characterised by :

- Its reduced covariance function $\mathbf{C}_\mathbf{X} : \mathbb{R}^d \to \mathbb{R}$ satisfying

$$\mathbb{E}(\mathbf{X}(x)\mathbf{X}(y)) = \mathbf{C}_\mathbf{X}(x-y), x, y \in E$$

- Its spectral measure $\mu_\mathbf{X}$, defined by

$$\mathbf{C}_\mathbf{X}(x) = \int_{\mathbb{R}^d} e^{i\langle t,x\rangle}\mu_\mathbf{X}(dt)$$

- Example : symmetrized atom at $a > 0$

$$\mathbf{C}_\mathbf{X}(x) = \cos(ax),\ \mu_\mathbf{X} = \frac{\delta_a + \delta_{-a}}{2},\ \mathbf{X}(x) = A\cos(ax) + B\sin(ax),$$
$$A, B\ i.i.d. \sim \mathcal{N}(0,1)$$

Zeros number variance

- Define

$$\mathbf{V}_{\mathbf{X}}(T) = \mathbf{Var}(\mathbf{Z} \cap [0, T])$$

- If $\mathbf{X}$ is τ-periodical, $\mathbf{V}_{\mathbf{X}}(T) \sim T^2\mathbf{Var}(\mathbf{Z} \cap [0, \tau])$, hence **quadratic** ($\sim T^2$), except if

$$\mathbf{X}(t) = A\cos(\frac{2\pi x}{\tau}) + B\sin(\frac{2\pi x}{\tau}) \Leftrightarrow \mathbf{C}_{\mathbf{X}}(x) = \mathbf{C}_{\mathbf{X}}(0)\cos(\frac{2\pi x}{\tau}), x \in \mathbb{R}$$

for A, B i.i.d. Gaussian variables.

- **Kac-Rice (1950')** : Expression of $\mathbf{V}_{\mathbf{X}}$ in fonction of $\mathbf{C}_{\mathbf{X}}$.
- **Cramer & Leadbetter (1967)** : $\mathbf{V}_{\mathbf{X}}(T) < \infty$ if $\mathbf{C}_{\mathbf{X}}$ is twice differentiable and a little bit mode : for some $\delta > 0$

$$\int_0^\delta \frac{1}{t^2}(\mathbf{C}'_{\mathbf{X}}(t) - \mathbf{C}''_{\mathbf{X}}(0)t)dt < \infty. \quad (1)$$

Bibliography

- **Geman (1972)** : Sufficient condition ("Geman's condition")
- **Cuzick (1976)** : If **furthermore** $\mathbf{C_X} \in \mathbf{L^2}, \mathbf{C''_X} \in \mathbf{L^2}$, the variance is at most linear

$$\limsup_{T\to\infty} T^{-1}\mathbf{V_X}(T) < \infty$$

Central Limit Theorem under the additional assumption that the variance is at least linear :

$$\lim_{T\to\infty} T^{-1}\mathbf{V_X}(T) = \sigma > 0$$

He proves the sufficient condition :

$$\int \frac{(\mathbf{C'_X}(t))^2}{1-\mathbf{C_X}(t)^2} dt < \frac{\pi}{2}\sqrt{\int x^2 \mu_{\mathbf{X}}(dx)}$$

Chaotic decomposition

- **Idea** : $N_T := \#\mathbf{Z} \cap T$ is L^2 hence obeys the orthogonal Wiener-Ito decomposition

$$N_T = \mathbb{E}(N_T) + \sum_{q=0} C_{2q}(T)$$

where $C_q(T)$ lives in the q-th chaos ; i.e. it is a multiple integral of order q with respect to $d\mathbf{X}$ on $[0, T]$.

- **Slud (1991) :** gets rid of the "at least linear" assumption
- **Kratz & Léon (2001)** : Chaotic decomposition in $(\mathbf{X}, \mathbf{X}')$ (multiple integrals wrt $(\mathbf{X}, \mathbf{X}')$) : generalisations, levels $\ell \neq 0$, ...

$$C_{2q}(T) = \sum_{k=0}^{q} c_{k,q} \int_0^T H_k(\mathrm{X}(t)) H_{q-k}(\mathrm{X}'(t)) dt$$

Variance linearity

Can we have hyperuniform zeros ?

Theorem (Lr 20)

- *The variance is sub-linear only if* $\mathbf{C_X}(x) = \cos(2\pi x/\tau), \tau \geqslant 0$
- *"Special frequency" at* $\sigma = \sqrt{-\mathbf{C}''_{\mathbf{X}}(0)}$ *(ABF '20 terminology)*
- *If the spectral measure is* L^2 *around* $\pm\sigma$*, then*

$$\textit{Linear variance} \;\Leftrightarrow\; \mathbf{C_X} + \mathbf{C_{X''}} \in L^2 \;\Leftrightarrow\; \mathbf{C_X}, \mathbf{C}''_{\mathbf{X}} \in L^2$$

- *Extension to* **linear statistics** *of zeros*

Proof : Based on a study of the second chaos + spectral theory

$$C_2(T) = \frac{-1}{2\sqrt{2\pi}}\left[\int_{-T}^{T} H_0(X(t))H_2(X'(t))dt - \int_{-T}^{T} H_2(X(t))H_0(X'(t))dt\right]$$

with $H_0(x) = 1, H_2(x) = x^2 - 1$ and $\mathrm{Var}(N_T) \geqslant \mathrm{Var}(C_2(T))$

- **Letendre, Ancona '20** : Linear statistics in the linear regime
- **Assaf, Buckley, Feldheim '20** : Similar results + upper bounds, and if $M := \limsup_{t\to\infty} |\mathbf{C_X}| + |\mathbf{C_X'}|/\sigma < \mathbf{C_X}(0)$, (mild mixing condition)

$$\text{Linear variance} \Leftrightarrow \mathbf{C_X} + \mathbf{C_{X''}} \in L^2$$

Proof :

$$\mathrm{Var}(C_{2q}(T)) \leqslant cM^q \mathrm{Var}(C_2(T)).$$

- **Conjecture (ABF '20)** : In all generality

$$\text{Linear variance} \Leftrightarrow \mathbf{C_X} + \mathbf{C_X''} \in L^2$$

An exemple rigid and hyper-fluctuating

Exemple

Let $\mathbf{X}$ *with covariance*

$$\mathbf{C}(x) = \prod_{k=1}^{\infty} \cos(x/k!)$$

The zeros $\mathbf{Z}$ *of* $\mathbf{X}$ *are hyper-fluctuating and maximally rigid* *(*$\mathbf{X}$ *is not too much dependent : it is weakly mixing, as is the PP* $\mathbf{Z}$*, and* $\mathbf{X}$ *is a.s. unbounded.)*

- **Klatt & Last '20** : Other (hyperfluctuating rigid) example in dimension $d \geqslant 2$ with "random grids"

Higher dimensions

- $\mathbf{X} : \mathbb{R}^d \to \mathbb{R}^d$
- $\mathbf{Z} = \mathbf{X}^{-1}(\{0\})$. $\mathrm{Var}(\mathbf{Z} \cap B_0(R)) \geqslant cR^d$?
- For instance $\mathbf{X} = \nabla \mathbf{X}_0$ where $\mathbf{X}_0 : \mathbb{R}^d \to \mathbb{R}$, $\mathbf{Z} =$ critical points
- Critical points "drive" the excursions topology
- Conditions for lower bounds for Number of critical points, Euler characteristic, number of components, see **Estrade & Léon '16** ; **Nicolaescu '17** ;... **Belliaev, McAulay, Muirhead '22**,
- As in many results about Gaussian fields, assumptions imply $\mathbf{C}(x) \to 0$ at infinity (more or less fast)
- Linear variance in all generality ?

Gaussian excursions volume variance (bi-phased media)

- $\mathbf{X} : \mathbb{R}^d \to \mathbb{R}$ stationary
- $\mathbf{V}_{\mathbf{X}}^{\ell}(R) := \mathbf{Var}(\mathrm{Leb}(\{\mathbf{X} > \ell\} \cap \mathbf{B}(0, R)))$

Theorem (Lr 21)

- $\mu_{\mathbf{X}}$: *Spectral measure*
- $\mathbf{U}_n$: *Random walk with i.i.d. increments with law* $\mu_{\mathbf{X}}$
- $\mathbf{K}(\varepsilon) := \sum_n n^{-3/2} \mathbb{P}(\|\mathbf{U}_{2n+1}\| < \varepsilon)$.

Then if $\mathbf{K}(\varepsilon) \sim \varepsilon^{\alpha}$, *then* $\alpha \leqslant d + 1$ *and*

$$V_{\mathbf{X}}^{0}(R) \sim R^{2d} \mathbf{K}(R^{-1}) \sim R^{2d-\alpha}$$

More generally,

$$c_- R^{2d} \mathbf{K}(R^{-1}) \leqslant \mathbf{V}_{\mathbf{X}}^{0}(R) \leqslant c_+ R^{2d} \mathbf{K}(R^{-1}) + I(R)$$

$$c R^{2d} \mathbb{P}(\|\mathbf{U}_2\| < R^{-1}) \leqslant \mathbf{V}_{\mathbf{X}}^{\ell}(R), \ell \neq 0$$

Example : Gaussian planar waves

- Consider fields of the form

$$\mu_{\mathbf{X}}(dx) = \frac{1}{2}(\delta_u + \delta_{-u}) \text{ for some } u \in \mathbb{R}^d$$
$$\mathbf{X}_u(x) = A\cos(\langle u, x\rangle) + B\sin(\langle u, x\rangle)$$

- More generally,

$$\mu_{\mathbf{X}} = \sum_k (\delta_{u_k} + \delta_{-u_k}), u_k \in \mathbb{R}^d$$
$$\mathbf{X}(x) = \sum_k A_k \cos(\langle x, u_k\rangle) + B_k \sin(\langle x, u_k\rangle)$$
$$\mathbf{C}_{\mathbf{X}}(x) = \int_{\mathbb{R}^d} e^{i\langle u,x\rangle}\mu_{\mathbf{X}}(du)$$

- **Remark** : if all $u \in \mathrm{Supp}(\mu_{\mathbf{X}})$ are in $\mathbb{S}^{d-1}$,

$$\Delta\mathbf{X} = -\mathbf{X}$$

Isotropic Gaussian planar wave

If $\mu_{\mathbf{X}}$ is uniform on the unit sphere

$$\mu_{\mathbf{X}}(dx) = \mathbf{1}_{\{\mathbb{S}^{d-1}\}}(x)\mathcal{H}^{d-1}(dx) \Leftrightarrow \Delta\mathbf{X} = -\mathbf{X}\textit{a.s.} \text{ and } \mathbf{X} \text{ isotropic}$$

We can prove for ε small

$$\begin{aligned}
&\mathbb{P}(\|\mathbf{U}_1\| < \varepsilon) = 0 \\
&\mathbb{P}(\|\mathbf{U}_2\| < \varepsilon) \sim \varepsilon^{d-1} \\
&\mathbb{P}(\|\mathbf{U}_n\| < \varepsilon) \sim \varepsilon^{d}, n \geqslant 3
\end{aligned}$$

hence

$$\mathbf{V}_{\mathbf{X}}^{\ell}(R) \geqslant c' R^{d+1} > 0 \quad \text{and} \quad \mathbf{K}(\varepsilon) \sim \varepsilon^{d} \text{ and } \mathbf{V}_{\mathbf{X}}^{0}(R) \sim R^{d}$$

- Variance cancellation phenomenon (cf. Marinucci-Wigman '11, Rossi '19, ...)

Irrational support

We consider spectral measures with finite support, for instance

$$\mathbf{C}(x) = \cos(x) + \cos(\omega x) \text{ where } \omega \in \mathbb{R} \setminus \mathbb{Q}$$
$$\mathbf{X}(x) = A_1 \cos(x) + A_2 \sin(x) + A_3 \cos(\omega x) + A_4 \sin(\omega x)$$

where the A_i are i.i.d. centered standard Gaussian. Let

$$\mathbf{V}(T) = \mathbf{Var}(\mathbf{Leb}^1(\mathbf{E}_0 \cap [0, T])).$$

Theorem

Let $\beta \in [0, 2)$, L a slowly varying function in some sense. Then there are uncountably many $\omega \in \mathbb{R}$ such that

$$0 < c_- T^\beta L(T) \overset{inf.often}{\leqslant} \mathbf{V}(T) \leqslant c_+ T^\beta L(T) < \infty$$

Irrational random walk

- Back on $\mathbb{R}^d$
- Spectral measure

$$\mu = \sum_{k,i} (\delta_{\omega_{k,i}} + \delta_{-\omega_{k,i}}) \underbrace{\mathbf{e}_k}_{\text{basis of } \mathbb{R}^d} \text{ where } \omega_{k,i} \in \mathbb{R} \setminus Q$$

- $\mathbf{X}_j$ i.i.d. with law μ and

$$\mathbf{U}_n = \sum_{j=1}^n \mathbf{X}_j$$
$$\bar{\mathbf{U}}_n = \mathbf{U}_n - [\mathbf{U}_n] \in \mathbb{T}^d$$

- What are

$$\mathbb{P}(0 < \|\mathbf{U}_n\| < \varepsilon)?$$
$$\mathbb{P}(0 < \| \bar{\mathbf{U}}_n\| < \varepsilon)?$$

Random walk (Cont'd)

- Random walks on (continuous and discrete) torus : **Diaconis, Saloff-Coste, Rosenthal, Porod, ...** in the 80s, 90s
- Known results (**Su 1998**)

$$\sup_{I \text{ interval of } [0,1]} |\mathbb{P}(\bar{\mathbf{U}}_n \in I) - \mathbf{Leb}^1(I)| \xrightarrow[n\to\infty]{} 0$$

$$\text{Hence } \sup_{0<\varepsilon<1} |\mathbb{P}(|\bar{\mathbf{U}}_n| < \varepsilon) - 2\varepsilon| \xrightarrow[n\to\infty]{} 0$$

- Similar results on the torus **Prescott & Su '04**
- Need : Uniform bound over n and ε of the form

$$\mathbb{P}(\|\bar{\mathbf{U}}_n\| < \varepsilon) < cn^{-\frac{1}{2}}\varepsilon^{\gamma}.$$

Random walk bounds

- ω is η-approximable :

$$c_- q^{-1-\eta} \leqslant \min_{p\in\mathbb{Z}} |p - \omega q| \overset{inf.often}{\leqslant} c_+ q^{-1-\eta}, q \in \mathbb{N}^*,$$

Theorem (Lr 21)

If the $\omega_{k,i}$ are $\mathbb{Z}-$free and η-approximable, there are finite $c, c', c'' > 0$ such that

$$\mathbb{P}(\|\bar{\mathbf{U}}_n\| < \varepsilon) \leqslant cn^{-d/2}\varepsilon^{\frac{md}{m+\eta}}$$

$$c''\varepsilon^{\frac{1+dm}{m/d+\eta}} \overset{\textbf{inf.often}}{\leqslant} \mathbf{K}(\varepsilon) \leqslant c'\varepsilon^{\frac{1+dm}{m+\eta}}$$

- **Case m=d=1** : If $\eta = 0$ (badly approximable numbers, e.g. $\sqrt{2}$), we retrieve the linear order ε^1, otherwise the optimal bound is larger.
- Lebesgue-(Almost every) ω is η-approx. for any $\eta > 0$

Variance exponent and approximability of ω

- **Several frequencies** :

$$\mathbf{C}(x) = \sum_{i=0}^{m} \cos(\omega_i x) \text{ (with } \omega_0 = 1)$$

the variance depends on the diophantine properties of the vector $(\omega_1, \dots, \omega_m)$, i.e. on the number $\eta \geqslant 0$ such that

$$c_+ \|q\|^{-m-\eta} \overset{\textbf{inf.often}}{\geqslant} \mathrm{dist}(q_1\omega_1 + \cdots + q_m\omega_m, \mathbb{Z}) \geqslant c_- \|q\|^{-m-\eta}$$

- For $\mathbf{Leb}^m-$a.a. $(\omega_1, \dots, \omega_m)$, the variance is in $R^{1-\frac{2}{m+\varepsilon}}, \varepsilon$ arb. small

- In dimension d, if

$$\mathbf{C}(x_1, \dots, x_d) = \cos(x_1) + \cos(x_1\omega) + \cdots + \cos(x_d) + \cos(x_d\omega)$$

the variance on $\mathbf{B}(0, R)$ is in

$$R^{\max(d-1, 2d - \frac{1+2d}{1+\eta})},$$

- Several vectors $\omega_k = (\omega_k, i)_{1 \leqslant i \leqslant m}$, for $1 \leqslant k \leqslant d$,

$$\mathbf{C}(x_1, \dots, x_d) = \sum_{k=1}^{d} \sum_{i=1}^{m} \cos(\omega_{k,i} x_k)$$

The lower bound depends on the properties of **simultaneous** diophantine approximations of the ω_k

Nonatomic measures on $\mathbb{R}$

- **Continuous measures :** If $\mu_{\mathbf{X}}$ has a non zero continuous component, for n large enough
$$\mathbb{P}(|\mathbf{U}_{2n+1}| \leqslant \varepsilon) > c_n \varepsilon$$
for ε small enough, hence
$$\mathbf{V}_{\mathbf{X}}(T) \geqslant cT$$
- **Maruyama's theorem :** $\mathbf{X}$ (and $\mathbf{Z}$) are ergodic iff $\mu_{\mathbf{X}}$ has no atoms.
- **Question** : Hyperuniform ergodic behaviour for some $\mu_{\mathbf{X}}$ with "Cantor-like"/fractal support ?

Thank you for your attention