On the hyperuniformity of short range Gibbs point processes

HSRPP 2023, 20.-22.2.2023

Daniela Flimmel
Université de Lille

This is a joint work with David Dereudre.

Outline of the talk

1. Hyperuniformity of a point process
   - Definition + Examples
2. Hyperuniformity and Gibbs
   - Conditions a for non-hyperuniformity result
3. Examples of short range interactions
   - Pair potential
   - Widom-Rowlinson
   - Voronoi interaction
   - k-nearest neighbour interaction

Notation

M	...	the space of all locally finite measures on $(\mathbb{R}^d, \mathcal{B})$,
$\mathcal{M}$	...	the smallest σ-field on **M** which makes all the projections $\mu \mapsto \mu(B)$ measurable for all $B \in \mathcal{B}$ and $\mu \in$ **M**,
N	...	the space of all locally finite integer valued measures on $(\mathbb{R}^d, \mathcal{B})$,
$\mathcal{N}$	...	the trace σ-field of $\mathcal{M}$ on $\mathbf{N}$.

Definition (Point process)

A point process on $\mathbb{R}^d$ is a measurable mapping

$$\Gamma : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbf{N}, \mathcal{N}).$$

Γ is stationary if $\Gamma \overset{D}{=} \Gamma + x$ for any $x \in \mathbb{R}^d$.

Marked point process: A point process on $\mathbb{R}^d \times \mathbb{M}$ that is locally finite in the first component.

Agreement: Γ is always assumed to be simple and we look at $\gamma \in \mathbf{N}$ as a locally finite point set of $\mathbb{R}^d$ and write $x \in \gamma$ for $\gamma(x) = 1$.

1. Hyperuniformity of a point process

Definition of hyperuniformity

For a configuration $\gamma \in \mathbf{N}$, denote

$$N_\Lambda := N_\Lambda(\gamma) = \sum_{x \in \gamma} \mathbf{1}_\Lambda(x) \quad \Lambda \subset \mathbb{R}^d \text{ bounded.}$$

Definition

A point process Γ is hyperuniform if

$$\lim_{\Lambda \nearrow \mathbb{R}^d} \frac{\mathsf{Var}(N_\Lambda)}{|\Lambda|} = 0.$$

Definition of hyperuniformity

For a configuration $\gamma \in \mathbf{N}$, denote

$$N_\Lambda := N_\Lambda(\gamma) = \sum_{x \in \gamma} \mathbf{1}_\Lambda(x) \quad \Lambda \subset \mathbb{R}^d \text{ bounded.}$$

Definition

A point process Γ is hyperuniform if

$$\lim_{\Lambda \nearrow \mathbb{R}^d} \frac{\mathsf{Var}(N_\Lambda)}{|\Lambda|} = 0.$$

- What sequence $|\Lambda| \nearrow \mathbb{R}^d$ shall we take?
  - $B(0, R), R \to \infty$?
  - $[-R, R]^d$?
  - RW, W compact convex?
  - all?

Examples of (non)hyperuniform point processes

1. $\Gamma = \mathbb{Z}^d + U$, U uniform, is hyperunifom.

Examples of (non)hyperuniform point processes

1. $\Gamma = \mathbb{Z}^d + U$, U uniform, is hyperunifom.
2. Linear transformation, rotation or i.i.d perturbation of a hyperuniform point process is hyperuniform.

Examples of (non)hyperuniform point processes

1. $\Gamma = \mathbb{Z}^d + U$, U uniform, is hyperunifom.
2. Linear transformation, rotation or i.i.d perturbation of a hyperuniform point process is hyperuniform.
3. Poisson point process Π with intensity λ is not hyperuniform.
    - $\mathrm{Var}(N_\Lambda) = \lambda|\Lambda|$.

Examples of (non)hyperuniform point processes

1. $\Gamma = \mathbb{Z}^d + U$, U uniform, is hyperunifom.
2. Linear transformation, rotation or i.i.d perturbation of a hyperuniform point process is hyperuniform.
3. Poisson point process Π with intensity λ is not hyperuniform.
   - $\mathsf{Var}(N_\Lambda) = \lambda|\Lambda|$.
4. Thomas point process

$$\Gamma = \bigcup_{x \in \Pi} \{x + \mathsf{cluster}(x)\}$$

is not hyperuniform.

Examples of (non)hyperuniform point processes

1. $\Gamma = \mathbb{Z}^d + U$, U uniform, is hyperunifom.
2. Linear transformation, rotation or i.i.d perturbation of a hyperuniform point process is hyperuniform.
3. Poisson point process Π with intensity λ is not hyperuniform.
   - Var$(N_\Lambda) = \lambda|\Lambda|$.
4. Thomas point process
$$\Gamma = \bigcup_{x\in\Pi} \{x + \text{cluster}(x)\}$$
   is not hyperuniform.
5. Stationary determinantal processes, i.e. those with k-th correlation function
$$\rho_k(x_1, \ldots, x_k) = \det[\mathbb{K}(x_i, x_j)_{i,j\in\{1,\ldots,k\}}],$$
   are hyperuniform if
$$\lambda = \mathbb{K}(0,0) = \int_{\mathbb{R}^d} \mathbb{K}(x,0)^2 \mathrm{d}x.$$
   E.g. $2-$dim. Ginibre process, $1-$dim. Sine process,..

2. Hyperuniformity and Gibbs

Small historic window

Theorem (Ginibre, 67)

Let X *be a random variable with values in* $\mathbb{N}$ *and denote* $p_n := \mathbb{P}(X = n)$ *and* $P_n = n!p_n$. *If* $\mathbb{E}\, X^2 < \infty$ *and*

$$\frac{P_{n+2}}{P_{n+1}} \geq \frac{P_{n+1}}{P_n} - F \quad \textit{for some } F > -1,$$

then

$$\frac{\mathit{Var} X}{\mathbb{E}\, X} \geq \frac{1}{1+F} > 0.$$

Small historic window

Theorem (Ginibre, 67)

Let X be a random variable with values in $\mathbb{N}$ and denote $p_n := \mathbb{P}(X = n)$ and $P_n = n!p_n$. If $\mathbb{E}\, X^2 < \infty$ and

$$\frac{P_{n+2}}{P_{n+1}} \geq \frac{P_{n+1}}{P_n} - F \quad \text{for some } F > -1,$$

then

$$\frac{\mathit{Var} X}{\mathbb{E}\, X} \geq \frac{1}{1+F} > 0.$$

Example Let $\Phi : \mathbb{R}^d \to \mathbb{R}, \beta \geq 0$ be

- integrable: $\int (1 - e^{-\beta\Phi(x)}) \mathrm{d}x < \infty$
- locally stable: $\exists B \in \mathbb{R}$ such that $\sum_{i=1}^m \Phi(x_0 - x_i) \geq -B \; \forall m \in \mathbb{N}$.

The Gibbs distribution with pair potential Φ in a bounded set Λ is given by

$$P_n := Z^{-1} z^n \int_{\Lambda^n} e^{-\beta \sum_{i<j} \Phi(x_i - x_j)} \mathrm{d}x_1 \ldots \mathrm{d}x_n$$

Small historic window

Theorem (Ginibre, 67)

Let X be a random variable with values in $\mathbb{N}$ and denote $p_n := \mathbb{P}(X = n)$ and $P_n = n! p_n$. If $\mathbb{E}\, X^2 < \infty$ and

$$\frac{P_{n+2}}{P_{n+1}} \geq \frac{P_{n+1}}{P_n} - F \quad \text{for some } F > -1,$$

then

$$\frac{\mathit{Var} X}{\mathbb{E}\, X} \geq \frac{1}{1+F} > 0.$$

Example Let $\Phi : \mathbb{R}^d \to \mathbb{R}, \beta \geq 0$ be

- integrable: $\int (1 - e^{-\beta \Phi(x)}) \mathrm{d}x < \infty$
- locally stable: $\exists B \in \mathbb{R}$ such that $\sum_{i=1}^{m} \Phi(x_0 - x_i) \geq -B \; \forall m \in \mathbb{N}$.

The Gibbs distribution with pair potential Φ in a bounded set Λ is given by

$$P_n := Z^{-1} z^n \int_{\Lambda^n} e^{-\beta \sum_{i<j} \Phi(x_i - x_j)} \mathrm{d}x_1 \dots \mathrm{d}x_n$$

- (Ruelle, 70): Φ superstable pair potential

Infinite-volume Gibbs point processes

Definition

A measurable function $\lambda^* : \mathbb{R}^d \times \mathbf{N} \to [0, \infty]$ is called Papangelou intensity of a point process Γ if

$$\int \sum_{x \in \gamma} f(x, \gamma \setminus x\})\Gamma(\mathrm{d}\gamma) = \int \int_{\mathbb{R}^d} f(x, \gamma)\lambda^*(x, \gamma)\mathrm{d}x\Gamma(\mathrm{d}\gamma). \qquad \text{(GNZ)}$$

for all positive measurable $f : \mathbb{R}^d \times \mathbf{N} \to \mathbb{R}$.

The point process Γ is called Gibbs if its Papangelou intensity is of the form

$$\lambda^*(x, \gamma) = z \exp\{-\beta h(x, \gamma)\}$$

for some measurable function $h : \mathbb{R}^d \times \mathbf{N} \to \mathbb{R}$ called local energy and parameters $z > 0, \beta \geq 0$.

Definition

A measurable function $\lambda^* : \mathbb{R}^d \times \mathbf{N} \to [0, \infty]$ is called Papangelou intensity of a point process Γ if

$$\int \sum_{x \in \gamma} f(x, \gamma \setminus x\}) \Gamma(\mathrm{d}\gamma) = \int \int_{\mathbb{R}^d} f(x, \gamma) \lambda^*(x, \gamma) \mathrm{d}x \Gamma(\mathrm{d}\gamma). \qquad \text{(GNZ)}$$

for all positive measurable $f : \mathbb{R}^d \times \mathbf{N} \to \mathbb{R}$.

The point process Γ is called Gibbs if its Papangelou intensity is of the form

$$\lambda^*(x, \gamma) = z \exp\{-\beta h(x, \gamma)\}$$

for some measurable function $h : \mathbb{R}^d \times \mathbf{N} \to \mathbb{R}$ called local energy and parameters $z > 0, \beta \geq 0$.

Remark. Usually, we have given the *energy* H defined on finite configurations. Then we take

$$h(x, \gamma) := \lim_{r \to \infty} [H(\gamma \cap \mathcal{B}(0, r) \cup \{x\}) - H(\gamma \cap B(0, r))]$$

Main result

Theorem (Dereudre + F., 2023+)

Let Γ be a stationary point process on $\mathbb{R}^d$ with Papangelou intensity λ^*. Assume that there are some $\delta > 0, \alpha_1, \alpha_2 > 1, \frac{2}{\alpha_1} + \frac{1}{\alpha_2} = 1$ such that

(A1) $\mathbb{E}\,\lambda^*(0,\Gamma)^{\alpha_1} < +\infty$,

(A2) $\int_{\mathbb{R}^d \setminus B(0,\delta)} \left(\mathbb{E} \left| 1 - \frac{\lambda^*(0,\Gamma \cup \{y\})}{\lambda^*(0,\Gamma)} \right|^{\alpha_2} \right)^{1/\alpha_2} \mathrm{d}y < \infty.$

Then Γ is not hyperuniform.

Available also in the marked setting.

- (A1) requires some stability assumption
- (A2) requires control over the range of interaction
  - > short-range interactions

2. Examples of point processes satisfying (A1) and (A2)

Pairwise interactions - Ruelle's regime

The local energy takes form

$$h(x,\gamma) = \sum_{y\in\gamma} \Phi(x-y), \quad \Phi : \mathbb{R}^d \to \mathbb{R}.$$

- Stability: h is stable, i.e. $\sum_{i=1}^m \Phi(x_0 - x_i) \geq -mB$ (or superstable + lower regular)
- Range of interaction There is $R > 0$ such that

$$h(x,\gamma) = h(x, \gamma \cap B(0,R)).$$

$\Rightarrow$ Γ is not hyperuniform.

Ex. Strauss process

$$\Phi(x-y) = \begin{cases} 1, & \text{if } |x-y| \leq R, \\ 0, & \text{otherwise} \end{cases}$$

Pairwise interactions - infinite range

The local energy takes form

$$h(x,\gamma) = \sum_{y\in\gamma} \Phi(x-y), \quad \Phi : \mathbb{R}^d \to \mathbb{R}.$$

- Stability: h is locally stable $h \geq -B$. $\Rightarrow$ (A1) ✓
- Range of interaction Φ is integrable at infinity, i.e. $\int_{\mathbb{R}^d\setminus B(0,\delta)} |1-e^{-\Phi(x)}| \mathrm{d}x < \infty$ for some $\delta > 0$. $\Rightarrow$ (A2) ✓

$$\int_{\mathbb{R}^d\setminus B(0,\delta)} \left(\mathbb{E} \left| 1 - \frac{\lambda^*(0,\Gamma\cup\{y\})}{\lambda^*(0,\Gamma)} \right|^{\alpha_2} \right)^{1/\alpha_2} \mathrm{d}y = \int_{\mathbb{R}^d\setminus B(0,\delta)} \left| 1 - e^{-\Phi(y)} \right| \mathrm{d}y.$$

$\Rightarrow$ Γ is not hyperuniform.

Ex. Riesz gas with $s > d$

$$\Phi(x-y) = \frac{1}{\|x-y\|^s}$$

Widom-Rowlinson interaction

For $R > 0$ and a finite configuration $\gamma \in \mathbf{N}$, the energy of γ is given by

$$H(\gamma) = \left| \bigcup_{x \in \gamma} B(x, R) \right| .$$

Widom-Rowlinson interaction

For $R > 0$ and a finite configuration $\gamma \in \mathbf{N}$, the energy of γ is given by

$$H(\gamma) = \left| \bigcup_{x \in \gamma} B(x, R) \right| .$$

For general $\gamma \in \mathbf{N}$, the local energy is given by

$$h(x, \gamma) = \left| \bigcup_{y \in \gamma} B(y, R) \cup B(x, R) \right| - \left| \bigcup_{y \in \gamma} B(y, R) \right| .$$

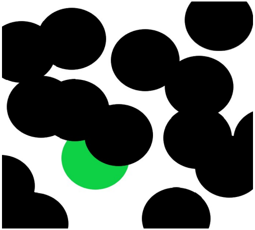
Widom-Rowlinson interaction

For $R > 0$ and a finite configuration $\gamma \in \mathbf{N}$, the energy of γ is given by

$$H(\gamma) = \left| \bigcup_{x \in \gamma} B(x, R) \right| .$$

For general $\gamma \in \mathbf{N}$, the local energy is given by

$$h(x, \gamma) = \left| \bigcup_{y \in \gamma} B(y, R) \cup B(x, R) \right| - \left| \bigcup_{y \in \gamma} B(y, R) \right| .$$



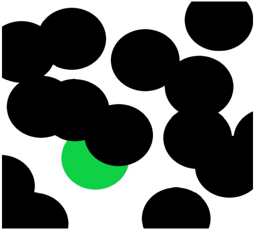
Widom-Rowlinson interaction

For $R > 0$ and a finite configuration $\gamma \in \mathbf{N}$, the energy of γ is given by

$$H(\gamma) = \left| \bigcup_{x \in \gamma} B(x, R) \right| .$$

For general $\gamma \in \mathbf{N}$, the local energy is given by

$$h(x, \gamma) = \left| \bigcup_{y \in \gamma} B(y, R) \cup B(x, R) \right| - \left| \bigcup_{y \in \gamma} B(y, R) \right| .$$



- Stability: The local energy is uniformly bounded

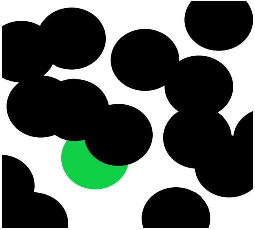
$$0 \leq h \leq |B(0, R)|$$

For $R > 0$ and a finite configuration $\gamma \in \mathbf{N}$, the energy of γ is given by

$$H(\gamma) = \left| \bigcup_{x \in \gamma} B(x, R) \right| .$$

For general $\gamma \in \mathbf{N}$, the local energy is given by

$$h(x, \gamma) = \left| \bigcup_{y \in \gamma} B(y, R) \cup B(x, R) \right| - \left| \bigcup_{y \in \gamma} B(y, R) \right| .$$



- Stability: The local energy is uniformly bounded
$$0 \leq h \leq |B(0, R)|$$
- Range of interaction: The range of interaction is $2R$, i.e.
$$h(x, \gamma) = h(x, \gamma \cap B(x, 2R)).$$

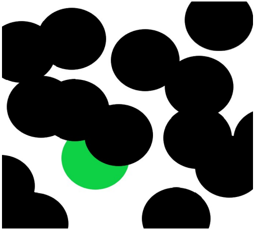
Widom-Rowlinson interaction

For $R > 0$ and a finite configuration $\gamma \in \mathbf{N}$, the energy of γ is given by

$$H(\gamma) = \left| \bigcup_{x \in \gamma} B(x, R) \right| .$$

For general $\gamma \in \mathbf{N}$, the local energy is given by

$$h(x, \gamma) = \left| \bigcup_{y \in \gamma} B(y, R) \cup B(x, R) \right| - \left| \bigcup_{y \in \gamma} B(y, R) \right| .$$



- Stability: The local energy is uniformly bounded
$$0 \leq h \leq |B(0, R)|$$
- Range of interaction: The range of interaction is $2R$, i.e.
$$h(x, \gamma) = h(x, \gamma \cap B(x, 2R)).$$

$\Rightarrow$ Γ is not hyperuniform.

Widom-Rowlinson interaction with random radii

Assume each $x \in \Gamma$ is equipped with a mark R_x from $\mathbb{R}_+$, independently and with the same distribution. For finite marked configuration γ, the energy is given by

$$H(\gamma) = \left| \bigcup_{(x,R_x)\in\gamma} B(x, R_x) \right|.$$

Widom-Rowlinson interaction with random radii

Assume each $x \in \Gamma$ is equipped with a mark R_x from $\mathbb{R}_+$, independently and with the same distribution. For finite marked configuration γ, the energy is given by

$$H(\gamma) = \left| \bigcup_{(x,R_x)\in\gamma} B(x, R_x) \right| .$$

For a locally finite marked configuration γ, the local energy is given by

$$h((x, R_x), \gamma) = \left| \bigcup_{(y,R_y)\in\gamma} B(y, R_y) \cup B(x, R_x) \right| - \left| \bigcup_{(y,R_y)\in\gamma} B(y, R_y) \right| .$$

Widom-Rowlinson interaction with random radii

Assume each $x \in \Gamma$ is equipped with a mark R_x from $\mathbb{R}_+$, independently and with the same distribution. For finite marked configuration γ, the energy is given by

$$H(\gamma) = \left| \bigcup_{(x,R_x)\in\gamma} B(x, R_x) \right| .$$

For a locally finite marked configuration γ, the local energy is given by

$$h((x, R_x), \gamma) = \left| \bigcup_{(y,R_y)\in\gamma} B(y, R_y) \cup B(x, R_x) \right| - \left| \bigcup_{(y,R_y)\in\gamma} B(y, R_y) \right| .$$

- We still have $h \geq 0$. The range of interaction is not bounded, but random and

$$h((x, R_x), \gamma) = h(x, \gamma \cap B(x, 2R_x) \times \mathbb{R}_+).$$

Widom-Rowlinson interaction with random radii

Assume each $x \in \Gamma$ is equipped with a mark R_x from $\mathbb{R}_+$, independently and with the same distribution. For finite marked configuration γ, the energy is given by

$$H(\gamma) = \left| \bigcup_{(x,R_x)\in\gamma} B(x,R_x) \right|.$$

For a locally finite marked configuration γ, the local energy is given by

$$h((x,R_x),\gamma) = \left| \bigcup_{(y,R_y)\in\gamma} B(y,R_y) \cup B(x,R_x) \right| - \left| \bigcup_{(y,R_y)\in\gamma} B(y,R_y) \right|.$$

- We still have $h \geq 0$. The range of interaction is not bounded, but random and

$$h((x,R_x),\gamma) = h(x, \gamma \cap B(x,2R_x) \times \mathbb{R}_+).$$

Lemma (Dereudre+F.)

Let R_0 be the typical mark. If $\mathbb{E}\, e^{\alpha\beta|B(0,1)|R_0^d} < \infty$ for some $\alpha > 0$ then Γ is not hyperuniform.

Voronoi interaction

Let $A \neq \emptyset$ be a locally finite set in $\mathbb{R}^d$. For each $x \in A$, we define a Voronoi cell around x as

$$C(x, A) := \{z \in \mathbb{R}^d : \|z - x\| \leq \|z - y\| \text{ for all } y \in A\}.$$

Let $\mathcal{C}^d$ denote the set of all polytopes in $\mathbb{R}^d$ and $\Phi : \mathcal{C}^d \to \mathbb{R}$.

Voronoi interaction

Let $A \neq \emptyset$ be a locally finite set in $\mathbb{R}^d$. For each $x \in A$, we define a Voronoi cell around x as

$$C(x, A) := \{z \in \mathbb{R}^d : \|z - x\| \leq \|z - y\| \text{ for all } y \in A\}.$$

Let $\mathcal{C}^d$ denote the set of all polytopes in $\mathbb{R}^d$ and $\Phi : \mathcal{C}^d \to \mathbb{R}$.

- For a finite configuration $\gamma \in \mathbf{N}$, we consider energy

$$H(\gamma) = \sum_{x \in \gamma} \Phi(C(x, \gamma)) \mathbf{1}\{|C(x, \gamma)| < \infty\}.$$

Voronoi interaction

Let $A \neq \emptyset$ be a locally finite set in $\mathbb{R}^d$. For each $x \in A$, we define a Voronoi cell around x as

$$C(x, A) := \{z \in \mathbb{R}^d : \|z - x\| \leq \|z - y\| \text{ for all } y \in A\}.$$

Let $\mathcal{C}^d$ denote the set of all polytopes in $\mathbb{R}^d$ and $\Phi : \mathcal{C}^d \to \mathbb{R}$.

- For a finite configuration $\gamma \in \mathbf{N}$, we consider energy

$$H(\gamma) = \sum_{x \in \gamma} \Phi(C(x, \gamma)) \mathbf{1}\{|C(x, \gamma)| < \infty\}.$$

- The local energy is given by

$$h(x, \gamma) := \Phi(C(x, \gamma)) + \sum_{y \sim x} \left[\Phi(C(y, \gamma \cup \{x\})) - \Phi(C(y, \gamma))\right],$$

where $x \sim y$ if $C(x, A \cup \{x, y\}) \cap C(y, A \cup \{x, y\}) \neq \emptyset$.

Voronoi interaction

Assumptions. The function Φ is

(A) additive if $\Phi(C) \leq \Phi(C_1) + \Phi(C_2)$ whenever $C = C_1 \cup C_2$, where $C, C_1, C_2 \in \mathcal{C}^d$,

(I) increasing if $\Phi(C) \leq \Phi(C')$ whenever $C, C' \in \mathcal{C}^d$ and $C \subseteq C'$,

(CV) controlled by the volume if there exists a constant $K > 0$ such that $\Phi(C) \leq \min\{|C|, K\}$ for all $C \in \mathcal{C}^d$.

Voronoi interaction

Assumptions. The function Φ is

(A) additive if $\Phi(C) \leq \Phi(C_1) + \Phi(C_2)$ whenever $C = C_1 \cup C_2$, where $C, C_1, C_2 \in \mathcal{C}^d$,

(I) increasing if $\Phi(C) \leq \Phi(C')$ whenever $C, C' \in \mathcal{C}^d$ and $C \subseteq C'$,

(CV) controlled by the volume if there exists a constant $K > 0$ such that $\Phi(C) \leq \min\{|C|, K\}$ for all $C \in \mathcal{C}^d$.

Stability:

$$\text{(A)+(I)+(CV)} \Rightarrow -K - |C(x,\gamma)| \leq h(x,\gamma) \leq K.$$

Assumptions. The function Φ is

(A) additive if $\Phi(C) \leq \Phi(C_1) + \Phi(C_2)$ whenever $C = C_1 \cup C_2$, where $C, C_1, C_2 \in \mathcal{C}^d$,

(I) increasing if $\Phi(C) \leq \Phi(C')$ whenever $C, C' \in \mathcal{C}^d$ and $C \subseteq C'$,

(CV) controlled by the volume if there exists a constant $K > 0$ such that $\Phi(C) \leq \min\{|C|, K\}$ for all $C \in \mathcal{C}^d$.

Stability:

$$\text{(A)+(I)+(CV)} \Rightarrow -K - |C(x,\gamma)| \leq h(x,\gamma) \leq K.$$

Range of interaction: There is a random variable R with $\mathbb{E}\, e^{c_d R^d} < \infty$ such that

$$h(x,\gamma) = h(x, \gamma \cap B(0,R)).$$

Voronoi interaction

Assumptions. The function Φ is

(A) additive if $\Phi(C) \leq \Phi(C_1) + \Phi(C_2)$ whenever $C = C_1 \cup C_2$, where $C, C_1, C_2 \in \mathcal{C}^d$,

(I) increasing if $\Phi(C) \leq \Phi(C')$ whenever $C, C' \in \mathcal{C}^d$ and $C \subseteq C'$,

(CV) controlled by the volume if there exists a constant $K > 0$ such that $\Phi(C) \leq \min\{|C|, K\}$ for all $C \in \mathcal{C}^d$.

Stability:

$$\text{(A)+(I)+(CV)} \Rightarrow -K - |C(x,\gamma)| \leq h(x,\gamma) \leq K.$$

Range of interaction: There is a random variable R with $\mathbb{E}\, e^{c_d R^d} < \infty$ such that

$$h(x,\gamma) = h(x, \gamma \cap B(0,R)).$$

Lemma (Dereudre + F.)

There is a critical value β_c (explicitely given) depending on z, K and the dimension, such that for all $\beta < \beta_c$, the Gibbs point process Γ is not hyperuniform.

k-nearest neighbour interaction

Let $V^k(x,\gamma)$ denote the set of k nearest neighbours of x in γ and $\Phi : \mathbb{R}^d \to \mathbb{R}$.

k-nearest neighbour interaction

Let $V^k(x,\gamma)$ denote the set of k nearest neighbours of x in γ and $\Phi : \mathbb{R}^d \to \mathbb{R}$.

- The energy for finite configuration γ is

$$H(\gamma) = \sum_{x\in\gamma} \sum_{y\in V^k(x,\gamma)} \Phi(x-y),$$

k-nearest neighbour interaction

Let $V^k(x,\gamma)$ denote the set of k nearest neighbours of x in γ and $\Phi : \mathbb{R}^d \to \mathbb{R}$.

- The energy for finite configuration γ is

$$H(\gamma) = \sum_{x\in\gamma} \sum_{y\in V^k(x,\gamma)} \Phi(x-y),$$

- The local energy is then

$$h(x,\gamma) = \sum_{y\in V^k(x,\gamma)} \Phi(x-y) - \sum_{y\in\gamma} \mathbf{1}_{V^k(y,\gamma\cup\{x\})}(x) \left[\Phi(y-x) - \Phi(y - v^k(y,\gamma))\right].$$

k-nearest neighbour interaction

Let $V^k(x,\gamma)$ denote the set of k nearest neighbours of x in γ and $\Phi : \mathbb{R}^d \to \mathbb{R}$.

- The energy for finite configuration γ is

$$H(\gamma) = \sum_{x\in\gamma} \sum_{y\in V^k(x,\gamma)} \Phi(x-y),$$

- The local energy is then

$$h(x,\gamma) = \sum_{y\in V^k(x,\gamma)} \Phi(x-y) - \sum_{y\in\gamma} \mathbf{1}_{V^k(y,\gamma\cup\{x\})}(x) \left[\Phi(y-x) - \Phi(y - v^k(y,\gamma))\right].$$

Stability: There exist a constant C_d depending only on the dimension d such that $|h(x,\gamma)| \le 3kC_d\|\Phi\|_\infty$.

k-nearest neighbour interaction

Let $V^k(x,\gamma)$ denote the set of k nearest neighbours of x in γ and $\Phi : \mathbb{R}^d \to \mathbb{R}$.

- The energy for finite configuration γ is

$$H(\gamma) = \sum_{x\in\gamma} \sum_{y\in V^k(x,\gamma)} \Phi(x-y),$$

- The local energy is then

$$h(x,\gamma) = \sum_{y\in V^k(x,\gamma)} \Phi(x-y) - \sum_{y\in\gamma} \mathbf{1}_{V^k(y,\gamma\cup\{x\})}(x) \left[\Phi(y-x) - \Phi(y - v^k(y,\gamma))\right].$$

Stability: There exist a constant C_d depending only on the dimension d such that $|h(x,\gamma)| \le 3kC_d\|\Phi\|_\infty$.

Range of interaction: There is a random variable R with $\mathbb{E}\, R^{d+\epsilon} < \infty$ such that $h(x,\gamma) = h(x, \gamma \cap B(x,R))$.

k-nearest neighbour interaction

Let $V^k(x,\gamma)$ denote the set of k nearest neighbours of x in γ and $\Phi : \mathbb{R}^d \to \mathbb{R}$.

- The energy for finite configuration γ is

$$H(\gamma) = \sum_{x\in\gamma} \sum_{y\in V^k(x,\gamma)} \Phi(x-y),$$

- The local energy is then

$$h(x,\gamma) = \sum_{y\in V^k(x,\gamma)} \Phi(x-y) - \sum_{y\in\gamma} \mathbf{1}_{V^k(y,\gamma\cup\{x\})}(x) \left[\Phi(y-x) - \Phi(y - v^k(y,\gamma))\right].$$

Stability: There exist a constant C_d depending only on the dimension d such that $|h(x,\gamma)| \leq 3kC_d \|\Phi\|_\infty$.

Range of interaction: There is a random variable R with $\mathbb{E}\, R^{d+\epsilon} < \infty$ such that $h(x,\gamma) = h(x, \gamma \cap B(x,R))$.

Lemma (Dereudre + F.)

If $|\Phi| \leq K$ or decreasing$+\Phi \geq K$, then Γ is not hyperuniform.

Ex. Coulomb interaction $\Phi(x) = \frac{1}{\|x\|^{d-2}}$.

Thank you for your attention.