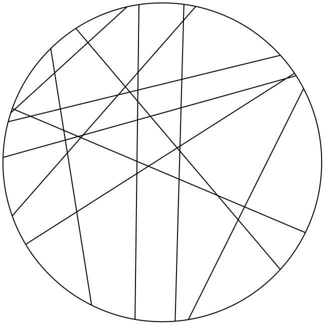
Poisson flats in constant curvature spaces

based on joint work with Carina Betken, Felix Herold, Christoph Thäle

Daniel Hug | Lille, February 20–22, 2023

Hyperplane tessellations



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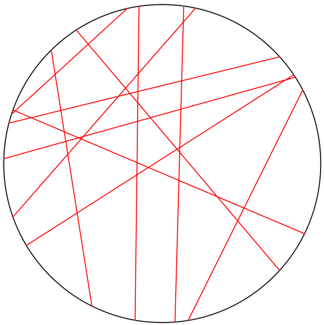
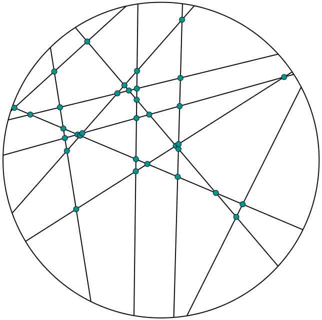


total edge length

Hyperplane tessellations



number of intersection points

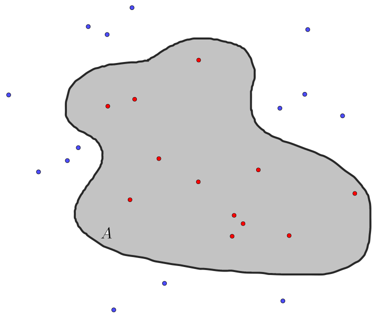
Poisson process

Measurable space $(\mathbb{X}, \mathcal{X})$, σ-finite measure μ on $\mathbb{X}$. **Point process** on $\mathbb{X}$ is a random collection of points in $\mathbb{X}$.

η is a **Poisson process** on $\mathbb{X}$ with intensity measure μ if

- $\eta(A) \sim \mathrm{Po}(\mu(A))$ for $A \in \mathcal{X}$,
- $\eta(B_1), \ldots, \eta(B_n)$ are independent for $n \in \mathbb{N}$, p.d. $B_1, \ldots, B_n \in \mathcal{X}$

Then $\mu = \mathbb{E}\eta(\cdot)$ is the **intensity measure** of η.



Poisson U-statistic

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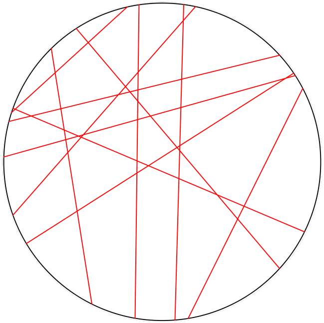
Then $\mu = \mathbb{E}\eta(\cdot)$ is the **intensity measure** of η.

A **Poisson U-statistic** w.r.t. η has the form

$$F = \frac{1}{k!} \sum_{(x_1,\ldots,x_k)\in\eta^k_{\neq}} f(x_1, \ldots, x_k),$$

where $f : \mathbb{X}^k \to [0, \infty)$ is symmetric and integrable w.r.t. μ^k; we call f a **kernel** of order k for F.

Intersection processes in $\mathbb{R}^2$



- $F^{(1)}_{r,t} \triangleq$ "total edge length in B_r"

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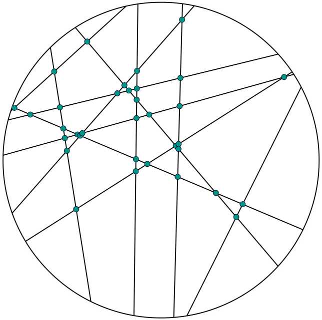

- $F^{(1)}_{r,t} \mathrel{\hat{=}}$ “total edge length in B_r”
- $F^{(0)}_{r,t} \mathrel{\hat{=}}$ “number of intersection points in B_r”

Intersection processes in $\mathbb{R}^d$

- $\mathbb{X} = A(d, d-1)$ carries an isometry invariant measure

$$\mu_{d-1}(B) := \int_{G(d,1)} \int_L \mathbf{1}\{H(L,x) \in B\}\, \mathcal{H}^1(dx)\, \nu_1(dL), \quad B \subset A(d, d-1).$$

- Let η_t, $t > 0$, be a Poisson process on $A(d, d-1)$ with **intensity measure** $t\,\mu_{d-1}$ and **intensity** t.
- For fixed $W \in \mathcal{K}^d$ and $i \in \{0, \dots, d-1\}$ define

$$F^{(i)}_{W,t} := \frac{1}{(d-i)!} \sum_{(H_1,\dots,H_{d-i}) \in \eta^{d-i}_{t,\neq}} \mathcal{H}^i(H_1 \cap \ldots \cap H_{d-i} \cap W).$$

- $F^{(i)}_{r,t} := F^{(i)}_{B_r,t}$

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Results in $\mathbb{R}^d$

Theorem (K. Paroux '98, L. Heinrich, H. Schmidt & V. Schmidt '06)

Let $t > 0$, $d \geq 2$ and $i \in \{0, \dots, d-1\}$ be fixed. Let $N \sim N(0,1)$. Then

$$\frac{F_{r,t}^{(i)} - \mathbb{E} F_{r,t}^{(i)}}{\sqrt{\mathbb{V}\mathrm{ar} F_{r,t}^{(i)}}} \xrightarrow[r \to \infty]{d} N.$$

- K. Paroux: $d = 2$, PhD thesis '97, method of moments.
- L. Heinrich, Hendrik & Volker Schmidt '06: Hoeffding's decomposition of U-statistics & CLT.
- L. Heinrich '09: general scaled W, multivariate CLT.
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A general result in $\mathbb{R}^d$

Theorem (G. Last, M. Penrose, M. Schulte, C. Thäle '14)

Let F be a Poisson U-statistic w.r.t. η_t. Under integrability assumptions on f there is a $c > 0$ such that

$$d\left(\frac{F - \mathbb{E}F}{\sqrt{\mathbb{V}\mathrm{ar}F}}, N\right) \leq c\, t^{-1/2}, \qquad t \geq 1,$$

where $d(\cdot,\cdot)$ is a metric (Wasserstein distance) that quantifies convergence in distribution.

- Based on previous work of Reitzner & Schulte '13, Schulte '13, Schulte '16, which in turn is based on Peccati, Solé, Taqqu, Utzet '10 (Stein's method for normal approximation of Poisson functionals in combination with the Malliavin calculus of variations for Poisson functionals).
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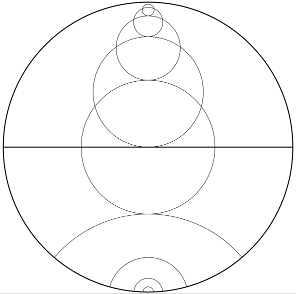
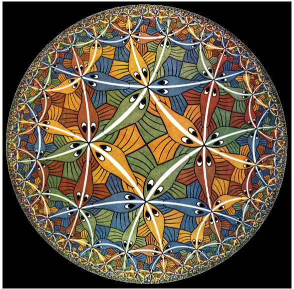
Hyperbolic geometry

- A (complete, simply connected) standard space of constant negative curvature -1.
- Various models (hyperboloid model, half-space model, conformal ball (Poincaré) model, projective disc (Beltrami–Klein) model.
- Unimodular isometry group acts transitively on $\mathbb{H}^d$, in $\mathbb{H}^d$ we have “free mobility”.





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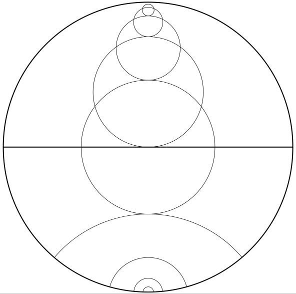

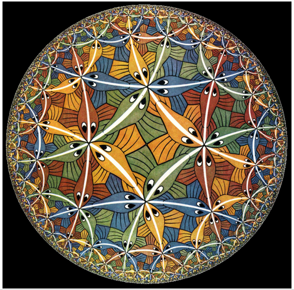

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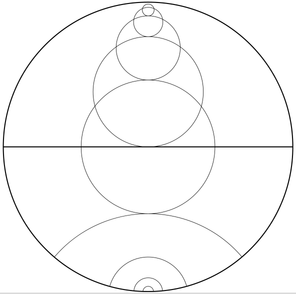
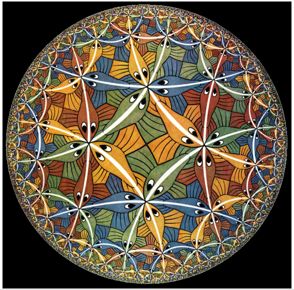


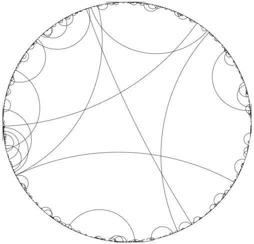


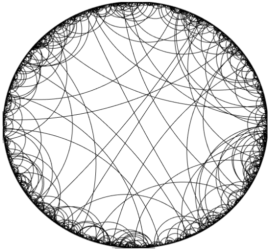
Poisson hyperplane process in hyperbolic space

A **Poisson process** of hyperplanes

$$\eta_t \sim \sum_{i\geq 1} \delta_{H(u_i,r_i)}.$$

in $\mathbb{H}^d$ with intensity measure $t \cdot \mu_{d-1}$ can be obtained via an inhom. Poisson process $X_0 = \sum_{i\geq 1} \delta_{r_i}$ on $\mathbb{R}$ with $\mathbb{E}[X_0(dr)] = t \cosh^{d-1}(r)\, dr$ and i.i.d. $u_i \in \mathbb{S}_p^{d-1}$, $i \geq 1$, also independent of X_0.





Integral-geometric tools

Lemma (Hyperbolic Crofton formula, J.E. Brothers '66)

Let $d \geq 2$, $0 \leq i \leq k \leq d-1$, and let $W \subset \mathbb{H}^d$ be a $(d+i-k)$-rectifiable Borel set. Then

$$\int_{A_h(d,k)} \mathcal{H}^i(W \cap E)\, \mu_k(dE) = \frac{\omega_{d+1}\, \omega_{i+1}}{\omega_{k+1}\, \omega_{d-k+i+1}} \mathcal{H}^{d+i-k}(W).$$

- The case $i = k$ can be checked by a direct calculation and invariance arguments.
- Very general class of admissible sets.
- Example: $k = d-1$, $i = 0$.

Mean values

$F^{(i)}_{W,t}$ is defined as in $\mathbb{R}^d$, but plenty of hyperplanes of η_t do not intersect in $\mathbb{H}^d$.

Multivariate Mecke formula and integral geometry:

Theorem (F. Herold, D. H., C. Thäle ’21)

Let $d \geq 2$, $W \subset \mathbb{H}^d$ mb, $t > 0$ and $i \in \{0, \dots, d-1\}$. Then

$$\mathbb{E}\, F^{(i)}_{W,t} = c_{i,d}\, t^{d-i}\, \mathcal{H}^d(W).$$

Variances

Theorem (F. Herold, D. H., C. Thäle '21)

Let $d \geq 2$, $W \subset \mathbb{H}^d$ *mb,* $t > 0$ *and* $i \in \{0, \ldots, d-1\}$*. Then*

$$\mathbb{V}\mathrm{ar}\, F_{W,t}^{(i)} = \sum_{n=1}^{d-i} c_{i,n,d}\, t^{2(d-i)-n} \int_{A_h(d,d-n)} \mathcal{H}^{d-n}(E \cap W)^2\, \mu_{d-n}(dE).$$

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- What can be said about the cross-sectional measures?

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- What can be said about the cross-sectional measures?
- Extremal problem (imlicitly conjectured by Heinrich '09 in $\mathbb{R}^d$, related discussion in Heinrich '16)?

Variance bounds

Theorem (F. Herold, D. H., C. Thäle '21)

Let $d \geq 2$ and $i \in \{0, \dots, d-1\}$. Then for $r, t \geq 1$:

$$\mathbb{V}\mathrm{ar}\, F_{r,t}^{(i)} \sim t^{2(d-i)-1}\, V_d(B_r) \cdot \begin{cases} 1, & d = 2, \\ \log V_3(B_r), & d = 3, \\ V_d(B_r)^{\frac{d-3}{d-1}}, & d \geq 4. \end{cases}$$

In particular,

$$\mathbb{V}\mathrm{ar}\, F_{r,t}^{(d-1)} \sim t \cdot V_d(B_r)^{\frac{2(d-2)}{d-1}} \sim t \cdot e^{2r(d-2)} \quad \text{if } d \geq 4.$$

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Euclidean case: $\mathbb{V}\mathrm{ar}\, F_{r,t}^{(d-1)} \sim t \cdot V_d(B_r) \cdot V_d(B_r)^{\frac{d-1}{d}}$ for $d \geq 2$.

Normal approximation of Poisson U-statistics

Theorem (M. Reitzner, M. Schulte '13, M. Schulte '16)

Let F be a Poisson U-statistic w.r.t. η_t and kernel f of order k. Then there is a constant c_k such that

$$d\left(\frac{F-\mathbb{E}F}{\sqrt{\mathbb{V}\mathrm{ar}\, F}}, N\right) \le c_k \sum_{u,v=1}^{k} \frac{\sqrt{M_{u,v}(f)}}{\mathbb{V}\mathrm{ar}\, F}.$$

- Here $d(\cdot\,,\,\cdot)$ means $d_W(\cdot\,,\,\cdot)$ or $d_K(\cdot\,,\,\cdot)$.
- Uses Malliavin calculus on Poisson space, chaos expansion of U statistics, based on Peccati, Solé, Taqqu, Utzet '10 (Stein's method for normal approximation of Poisson functionals).

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Specific application

Theorem (M. Reitzner, M. Schulte '13, M. Schulte '16)

Let F be a Poisson U-statistic w.r.t. η_t and kernel f of order k. Then there is a constant c_k such that

$$d\left(\frac{F-\mathbb{E}F}{\sqrt{\mathbb{V}\mathrm{ar}(F)}},N\right)\le c_k\sum_{u,v=1}^{k}\frac{\sqrt{M_{u,v}(f)}}{\mathbb{V}\mathrm{ar}(F)}.$$

Application with $i:=d-k$, $u,v\in\{1,\dots,k\}$, $f=f^{(i)}$ and via integral geometry:

$$f_u^{(i)}(H_1,\dots,H_n)=c_{i,u,d}\,\mathcal{H}^{d-u}(H_1\cap\dots\cap H_u\cap B_r),$$

$$M_{u,v}(f^{(i)}):=\sum_{\sigma\in\Pi_{\ge2}^{\mathrm{con}}(u,u,v,v)}\int_{A_h(d,d-1)^{|\sigma|}}(f_u^{(i)}\otimes f_u^{(i)}\otimes f_v^{(i)}\otimes f_v^{(i)})_\sigma\,d(t\cdot\mu_{d-1})^{|\sigma|},$$

$\Pi_{\ge2}^{\mathrm{con}}(u,u,v,v)$ is a certain collection of partitions σ of $\{1,\dots,2u+2v\}$ into $|\sigma|$ blocks.

CLT for increasing radius

Theorem (F. Herold, D. H., C. Thäle '21)

(a) *If* $d = 2$, $t \geq 1$ *is fixed and* $i \in \{0, 1\}$, *then there is a constant* $c_2 = c_2(t) \in (0, \infty)$ *such that*

$$d\left(\frac{F_{r,t}^{(i)} - \mathbb{E}\, F_{r,t}^{(i)}}{\sqrt{\mathbb{V}\mathrm{ar}\, F_{r,t}^{(i)}}}, N\right) \leq c_2\, r^{1-i}\, e^{-r/2}$$

for $r \geq 1$.

CLT for increasing radius

Theorem (F. Herold, D. H., C. Thäle ’21)

(b) *If* $d = 3$, $t \geq 1$ *is fixed and* $i \in \{0, 1, 2\}$, *then there is a constant* $c_3 = c_3(t) \in (0, \infty)$ *such that*

$$d\left(\frac{F_{r,t}^{(i)} - \mathbb{E}F_{r,t}^{(i)}}{\sqrt{\mathbb{V}\mathrm{ar}F_{r,t}^{(i)}}}, N\right) \leq \begin{cases} c_3\, r^{-1} & : i = 2, \\ c_3\, r^{-1/2} & : i \in \{0, 1\}, \end{cases}$$

for $r \geq 1$.

No CLT for increasing radius

Theorem (F. Herold, D. H., C. Thäle '21)

Let $t \geq 1$ be fixed and $i \in \{0, 1, \dots, d-1\}$.

(c) *If $d \geq 7$ or $d \geq 4$ and $i = d-1$, then*

$$\frac{F_{r,t}^{(i)} - \mathbb{E}\, F_{r,t}^{(i)})}{\sqrt{\mathbb{V}\mathrm{ar}\, F_{r,t}^{(i)}}}$$

does not satisfy a CLT for $r \to \infty$.

Theorem (Z. Kabluchko, D. Rosen, C. Thäle '22+)

Suppose that $d \geq 4$. *Then,*

$$\frac{F_{r,t}^{(d-1)} - \mathbb{E}F_{r,t}^{(d-1)}}{e^{r(d-2)}} \xrightarrow{d} \frac{\omega_{d-1}}{(d-2)2^{d-2}} Z_d, \qquad \text{as } r \to \infty,$$

where Z_d *is an infinitely divisible, zero-mean random variable defined by*

$$Z_d := \lim_{T \to +\infty} \left(\sum_{\substack{s \in \zeta_d \\ s \leq T}} \cosh^{-(d-2)}(s) - 2 \sinh T \right) \quad \text{in } L^2 \text{ and a.s.},$$

and ζ_d *is an inhomogeneous Poisson process on* $[0, \infty)$ *with density function* $s \mapsto 2\cosh^{d-1}(s)$ *with respect to* 1*-dimensional Lebesgue measure.*

Remarks

- For $T > 0$, let

$$Y_T := \sum_{s \in \zeta_d, s \le T} \cosh^{-(d-2)}(s).$$

  Then

$$\mathbb{E} Y_T = 2 \sinh T, \quad \mathbb{V}\mathrm{ar}\, Y_T = \int_0^T \cosh^{3-d}(s)\, \mathrm{d}s \le c < \infty.$$

Remarks

- For $T > 0$, let
$$Y_T := \sum_{s\in\zeta_d, s\le T} \cosh^{-(d-2)}(s).$$
Then
$$\mathbb{E} Y_T = 2\sinh T, \quad \mathbb{V}\mathrm{ar}\, Y_T = \int_0^T \cosh^{3-d}(s)\, ds \le c < \infty.$$
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- The cumulants of Z_d are $\mathrm{cum}_1(Z_d) = \mathbb{E} Z_d = 0$ and

$$\mathrm{cum}_\ell(Z_d) = \lim_{T\to\infty} \mathrm{cum}_\ell(Y_T - \mathbb{E} Y_T) = \int_0^\infty \cosh^{-((d-2)\ell-(d-1))}(s)\, \mathrm{d}s = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{(d-2)\ell-(d-1)}{2}\right)}{\Gamma\left(\frac{(d-2)(\ell-1)}{2}\right)}, \quad \ell \ge 2.$$

- The characteristic function of Z_d is

$$\mathbb{E}e^{\mathrm{i}t\,Z_d} = \exp\left(2\int_0^\infty (e^{\mathrm{i}th(s)} - 1 - \mathrm{i}th(s))\cosh^{d-1}(s)\,\mathrm{d}s\right), \qquad h(s) := \cosh^{-(d-2)}(s),$$

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- Z_d has no Gaussian component, Z_d has finite exponential moments, Z_d has an infinitely differentiable density and all of its derivatives vanish at ∞.

What is so special about $d = 4$ in $\mathbb{H}^d$?

- For $k \in \{0, 1, \dots, d-1\}$, let $\mathbf{A}(d, k)$ be the space of k-flats of $\mathbb{H}^d$.
- Let μ_k be the suitably normalized isometry invariant measure on $\mathbf{A}(d, k)$.
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- Let $m \in \mathbb{N}$ be such that $d - m(d-k) \geq 0$. For a Borel set $W \subset \mathbb{H}^d$ define

$$F_{W,t}^{(m)} := \frac{1}{m!} \sum_{(E_1,\ldots,E_m) \in \eta_{t,\neq}^m} \mathcal{H}^{d-m(d-k)}(E_1 \cap \ldots \cap E_m \cap W).$$

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- If $W = B_r^d$, we replace the index W by r.
- Let $\widetilde{F}_{W,t}^{(m)}$ and $\widetilde{F}_{r,t}^{(m)}$ denote the standardized versions.

Theorem (Central limit theorem for large intensities, C. Betken, D.H., Ch. Thäle '23+)

Let $\kappa \in \{-1, 0, 1\}$. *Consider a PP of k-flats in* $\mathbf{M}^d_\kappa$ *with* $d \geq 2$ *and* $k \in \{0, 1, \dots, d-1\}$.

Let $m \in \mathbb{N}$ *be such that* $d - m(d-k) \geq 0$. *Let N be a standard Gaussian random variable,* $\diamondsuit \in \{K, W\}$ *and let* $W \subset \mathbf{M}^d_\kappa$ *be a Borel set with* $\mathcal{H}^d_\kappa(W) \in (0, \infty)$.

Then there is a constant $C \in (0, \infty)$ *such that*

$$\mathrm{d}_\diamondsuit(\widetilde{F}^{(m)}_{W,t,\kappa}, N) \leq C\, t^{-1/2}$$

for all $t \geq 1$. *In particular,* $\widetilde{F}^{(m)}_{W,t,\kappa}$ *satisfies a central limit theorem, as* $t \to \infty$.

Theorem (Central limit theorem for large radii and $\kappa = -1$, C. Betken, D.H., Ch. Thäle '23+)

Consider a PP of k-flats in $\mathbb{H}^d$ with $d \geq 2$ and $k \in \{0, 1, \ldots, d-1\}$. Let N be a standard Gaussian random variable and $\diamondsuit \in \{K, W\}$. Then there are $C_1, C_2, C_3 \in (0, \infty)$ s.t. for $r \geq 1$:

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(i) *If $2k < d$, then $m = 1$ and*

$$\mathrm{d}_\diamondsuit(\widetilde{F}_r^{(m)}, N) \leq C_1 \begin{cases} e^{-\frac{r}{2}(d-2k+1)} & : \text{ for } k \geq 1, \\ e^{-\frac{r}{2}(d-1)} & : \text{ for } k = 0. \end{cases} \tag{1}$$

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(ii) *If $2k = d$, then $m \in \{1, 2\}$ and*

$$\mathrm{d}_\diamondsuit(\widetilde{F}_r^{(m)}, N) \leq C_2 \begin{cases} e^{-\frac{r}{2}} & : \text{ for } d \geq 4, \\ r^{m-1} e^{-\frac{r}{2}} & : \text{ for } d = 2. \end{cases} \tag{2}$$

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(iii) *If $2k = d + 1$, then $m \in \{1, 2, 3\}$ and*

$$\mathrm{d}_\diamondsuit(\widetilde{F}_r^{(m)}, N) \leq C_3 \begin{cases} r^{-1} & : \text{ for } m = 1, \\ r^{-\frac{1}{2}} & : \text{ for } m \in \{2, 3\}. \end{cases} \tag{3}$$

Let ζ be an inhomogeneous Poisson process on $[0, \infty)$ with intensity function $s \mapsto \omega_{d-k} \cosh^k s \sinh^{d-k-1} s$.

Theorem (Non-Gaussian fluctuations for $m = 1$, $\kappa = -1$, C. Betken, D.H., Ch. Thäle '23)

Consider a PP of k-flats in $\mathbb{H}^d$, where $d \geq 4$ and $k \in \{3, \ldots, d-1\}$. If $2k > d + 1$, then

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and ζ is an inhomogeneous Poisson process on $[0, \infty)$ with intensity function given above.

An extremal problem

Theorem (Maximal variances, C. Betken, D.H., Ch. Thäle '23)

Let $\kappa \in \{-1, 0, 1\}$. Consider a PP of k-flats in $\mathbf{M}_\kappa^d$, $d \geq 2$, with $t > 0$, $k \in \{1, \dots, d-1\}$, $d - m(d-k) \geq 0$. Let $W \subset \mathbf{M}_\kappa^d$ be a Borel set with $\mathcal{H}_\kappa^d(W) \in (0, \infty)$, let $B_W \subset \mathbf{M}_\kappa^d$ be a geodesic ball with $\mathcal{H}_\kappa^d(W) = \mathcal{H}_\kappa^d(B_W)$. If $\kappa = 1$, we assume in addition that $d_\kappa(x, y) \leq \pi/2$ for all $x, y \in W$.

Then

$$\operatorname{Var} F_{W,t,\kappa}^{(m)} \leq \operatorname{Var} F_{B_W,t,\kappa}^{(m)}.$$

Equality holds iff W is isometric to B_W, up to set of $\mathcal{H}_\kappa^d$-measure zero.

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- New Blaschke–Petkantschin type formula in hyperbolic space
- Riesz rearrangement inequality (Pfiefer '90, Burchard, Schmuckenschläger '01, Lieb '77, ...)

Theorem (Blaschke–Petkantschin type formula, C. Betken, D.H., Ch. Thäle)

Let $\kappa \in \{-1, 0, 1\}$, $d \geq 2$, *and* $k \in \{1, \ldots, d-1\}$. *If* $f : \mathbf{M}_\kappa^d \times \mathbf{M}_\kappa^d \to [0, \infty]$ *is a measurable function, then*

$$\int_{\mathbf{A}_\kappa(d,k)} \int_E \int_E f(x, y)\, \mathbf{sn}_\kappa^{d-k}\, d_\kappa(x, y)\, \mathcal{H}_\kappa^k(\mathrm{d}x)\, \mathcal{H}_\kappa^k(\mathrm{d}y)\, \mu_{k,\kappa}(\mathrm{d}E)$$
$$= \frac{\omega_k}{\omega_d} \int_{\mathbf{M}_\kappa^d} \int_{\mathbf{M}_\kappa^d} f(x, y)\, \mathcal{H}_\kappa^d(\mathrm{d}x)\, \mathcal{H}_\kappa^d(\mathrm{d}y).$$

From general variance formulas for Poisson U-statistics and by integral-geometric formulas, we derive that

$$\operatorname{Var} F^{(m)}_{W,t,\kappa} = \sum_{i=1}^{m} c_i t^{2m-i} \int_{\mathbf{A}_\kappa(d,d-i(d-k))} \mathcal{H}^{d-i(d-k)}_\kappa(E \cap W)^2 \, \mu_{d-i(d-k),\kappa}(\mathrm{d}E)$$

with explicitly determined constants c_i, $i \in \{1, \ldots, m\}$ A consequence of the Blaschke–Petkantschin formula is

$$\begin{aligned}\int_{\mathbf{A}_\kappa(d,d-i(d-k))} & \mathcal{H}^{d-i(d-k)}_\kappa(E \cap W)^2 \, \mu_{d-i(d-k),\kappa}(\mathrm{d}E) \\ &= \frac{\omega_{d-i(d-k)}}{\omega_d} \int_W \int_W \frac{1}{\mathbf{sn}^{i(d-k)}\, d_\kappa(x,y)} \, \mathcal{H}^d_\kappa(\mathrm{d}x)\, \mathcal{H}^d_\kappa(\mathrm{d}y).\end{aligned}$$

To these integrals we can apply Riesz-rearrangement inequalities (with some care ...).

Let ζ be an inhomogeneous PP on $[0,\infty)$ with intensity function $s \mapsto \omega_{d-k} \cosh^k s \sinh^{d-k-1} s$.

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Consider a PP of k-flats in $\mathbb{H}^d$, where $d \geq 4$ and $k \in \{3, \dots, d-1\}$. If $2k > d+1$, then

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and ζ is an inhomogeneous Poisson process on $[0,\infty)$ with intensity function given above.

Consider

$$Y_T := \int \mathbf{1}\{s \in [0,T]\} \cosh^{-(k-1)} s \, \zeta(\mathrm{d}s).$$

Then

$$\mathbb{E} Y_T = \frac{\omega_{d-k}}{d-k} \sinh^{d-k} T \qquad \text{and} \qquad \operatorname{Var} Y_T = \omega_{d-k} \int_0^T \cosh^{2-k} s \, \sinh^{d-k-1} s \, \mathrm{d}s.$$

If $2k > d+1$, then $\lim_{T\to\infty} \operatorname{Var} Y_T < \infty$. If follows that $Y_T - \mathbb{E} Y_T \xrightarrow{D} c_k Z$ (also a.s. and in L_2).

The Levy measure of Z lives on $(0,1)$ and has the Lebesgue density

$$y \mapsto \frac{1}{k-1} y^{-\frac{d+k-2}{k-1}} \left(1 - y^{\frac{2}{k-1}}\right)^{\frac{d-k-2}{2}},$$

which satisfies the required integrability condition for $d+1 < 2k$.

Idea of proof

Let $d_h(E) = d_h(E, p)$. There is a function $f_r : [0, \infty) \to [0, \infty)$ such that

$$\mathcal{H}^k(E \cap B_r^d) = f_r \circ d_h(E), \qquad E \in \mathbf{A}_h(d, k).$$

Let η denote a hyperbolic k-flat process in $\mathbb{H}^d$ with intensity measure μ_k. Then

$$d_{h\sharp}\mu_k = \omega_{d-k} \int_0^\infty \mathbf{1}\{s \in \cdot\} \cosh^k s \sinh^{d-k-1} s \, \mathrm{d}s.$$

Since $F_r = \int f_r \circ d_h(E)\, \eta(\mathrm{d}E)$, the characteristic function of F_r is (for $\xi \in \mathbb{R}$)

$$\begin{aligned}\mathbb{E}[e^{\mathrm{i}\xi F_r}] &= \exp\left(\int_{\mathbf{A}_h(d,k)} (e^{\mathrm{i}\xi (f_r \circ d_h)(E)} - 1)\, \mu_k(\mathrm{d}E)\right) = \exp\left(\int_0^r (e^{\mathrm{i}\xi f_r(s)} - 1)\, (d_{h\sharp}\mu_k)(\mathrm{d}s)\right) \\ &= \exp\left(\omega_{d-k} \int_0^r (e^{\mathrm{i}\xi f_r(s)} - 1) \cosh^k s \sinh^{d-k-1} s \, \mathrm{d}s\right).\end{aligned}$$

We conclude that

$$\mathbb{E}\left[\exp\left(\mathrm{i}\xi\frac{F_r-\mathbb{E}[F_r]}{e^{r(k-1)}}\right)\right] = \exp\left(\omega_{d-k}\int_0^r \left(e^{\mathrm{i}\xi g_r(s)} - 1 - \mathrm{i}\xi g_r(s)\right)\cosh^k s\,\sinh^{d-k-1} s\,\mathrm{d}s\right)$$

with $g_r(s) := f_r(s)/e^{r(k-1)}$.

Lemma

Let $s \in [0,\infty)$. Then $g_r(s) \sim g(s) := \frac{\omega_k}{(k-1)2^{k-1}}\cosh^{-(k-1)} s$ as $r \to \infty$.