Hyperuniformity(?) and hyperfluctuation in "curved " spaces

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Based on joint works with Tobias Hartnick (Karlsruhe) and Mattias Byléhn (Chalmers/GU)

General framework - Random measures in metric spaces I

Some notation:

Let (X, d) be a metric space, fix a point $x_o \in X$, and assume that the closed ball

$$B_R := \{x \in X \ : \ d(x, x_o) \le R\} \quad \text{is compact for all } R \ge 0.$$

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Homogenity: Suppose that $G < \mathrm{Isom}(X, d)$ is a closed transitive subgroup, and thus

$$X = G.x_o \cong G/K, \quad \text{where } K := \mathrm{Stab}_G(x_o) < G \text{ is compact.}$$

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Let $M^+(X)$ denote the space of positive Radon measures on X, and let μ be a locally square-integrable G-invariant probability measure on $M^+(X)$, i.e.

$$\int_{M^+(X)} p(B_R)^2 \, d\mu(p) < \infty, \quad \text{for all } R \geq 0.$$

General framework - Random measures in metric spaces II

Volume on X: Fix a left Haar measure m_G on G and define

$$\int_X f \, dm_X = \int_G f(g.x_o) \, dm_G(g), \quad \text{for all } f \in \mathcal{L}_c^\infty(X),$$

so that m_X is a G-invariant Radon measure on X ($=$ Volume). Let m_K denote the Haar probability measure on K.

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Note: $Sf \in L^2(\mu)$ for all $f \in \mathcal{L}_c^\infty(X)$ and there exists $\iota_\mu \geq 0$ (intensity of μ) such that

$$\mu(Sf) = \iota_\mu m_X(f), \quad \text{for all } f \in \mathcal{L}_c^\infty(X).$$

Geometric hyperuniformity?

Naive definition: μ is *(geometrically) hyperuniform* if

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Questions:

- How should spectral hyperuniformity be defined?
- Do examples exist?
- What should spectral hyperuniformity correspond to "in the real world"?

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Some potential answers below...

Hyperfluctuations

Fact: If (B_R) form a nice ergodic averaging sequence and μ is G-ergodic, then

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When $X = \mathbb{R}^d$ or a (real/complex) hyperbolic space, then for every $\delta > 0$ there exists a random measure μ_δ (in fact, a point process) such that

$$\mathrm{Var}_{\mu_\delta}(S\chi_{B_R}) \gg m_X(B_R)^{2-\delta}, \quad \text{as } R \to \infty.$$

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Theorem (B-Bylehn '23) If $X =$ quaterionic hyperbolic space, or a higher rank symmetric space (for instance, the space of positive definite matrices in $\mathbb{R}^n$ for $n \geq 3$), then there exists $\delta_X > 0$ such that for <u>every</u> random measure μ on X, we have

$$\mathrm{Var}_\mu(S\chi_{B_R}) \ll m_X(B_R)^{2-\delta_X}, \quad \text{as } R \to \infty.$$

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Let $\mathcal{L}_c^\infty(G, K)$ denote the space of bi-K-invariant bounded Borel functions with bounded supports on G. Note that $\mathcal{L}_c^\infty(G, K) * \mathcal{L}_c^\infty(G, K) \subset \mathcal{L}_c^\infty(G, K)$.

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We say that (G, K) is a **Gelfand pair** if the algebra $(\mathcal{L}_c^\infty(G, K), *)$ is commutative, i.e. if for every $\varphi \in \mathcal{L}^\infty(G, K)$, we have

$$\int_K \varphi(g_1 k g_2)\, dm_K(k) = \int_K \varphi(g_2 k g_1)\, dm_K(k), \quad \text{for all } g_1, g_2 \in G.$$

Fact: If (G, K) is a Gelfand pair, then m_G must be bi-G-invariant (unimodular).

Zonal spherical functions

A bi-K-invariant function $\omega : G \to \mathbb{C}$ is a *zonal spherical function* if it is positive definite and satisfies

$$\int_K \omega(g_1 k g_2)\, dm_K(k) = \omega(g_1)\omega(g_2), \quad \text{for all } g_1, g_2 \in G.$$

Let Ω denote the space of all zonal functions for (G, K), equipped with the topology of uniform convergence on compact sets. We can (sloppily) think of these as K-invariant functions on X.

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Given $f \in \mathcal{L}_c^\infty(X)^K$, define the **spherical Fourier transform** $\widehat{f} : \Omega \to \mathbb{C}$ by

$$\widehat{f}(\omega) = \int_X f(x)\, \overline{\omega(x)}\, dm_X(x), \quad \text{for } \omega \in \Omega.$$

Plancherel and Godement

Fact: There exists a unique positive measure σ (=**Plancherel measure**) on Ω such that

$$\int_X |f(x)|^2 \, dm_X(x) = \int_\Omega |\widehat{f}(\omega)|^2 \, d\sigma(\omega), \quad \text{for all } f \in \mathcal{L}_c^\infty(X)^K.$$

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Theorem (Godement) There exists a unique positive Radon measure $\widehat{\eta}_\mu$ on Ω such that

$$\mathrm{Var}_\mu(Sf) = \int_\Omega |\widehat{f}(\omega)|^2 \, d\widehat{\eta}_\mu(\omega), \quad \text{for all } f \in \mathcal{L}_c^\infty(X)^K.$$

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We will see that $\widehat{\eta}_\mu$ generalizes the notion of a "spectral density" of a locally square-integrable (translation-invariant) random measure on $\mathbb{R}^d$.

The Poisson process on X

Let $\mu = \text{Poisson}(m_X)$ (a.k.a. the "Poisson process on X").

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Then, for every $f \in \mathcal{L}_c^\infty(X)$, we have

$$\text{Var}_\mu(Sf) = \int_X |f(x)|^2 \, dm_X(x),$$

and thus $\widehat{\eta}_\mu = \sigma$ (the Plancherel measure).

DPP's on X associated with projections

Fix a Borel set $B \subset \Omega$ with $\sigma(B) = 1$, and define $\Phi(g) = \int_B \omega(g)\, d\sigma(\omega)$ for $g \in G$, as well as the kernel $\mathcal{K}_\Phi : X \times X \to \mathbb{C}$ by

$$\mathcal{K}_\Phi(g_1 K, g_2 K) = \Phi(g_1^{-1} g_2), \quad g_1 K, g_2 K \in G/K.$$

Since

$$\Phi * \Phi = \Phi * \Phi^* = \Phi \quad \text{and} \quad \Phi(e) = 1,$$

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Let μ denote the corresponding determinantal point process $(\iota_\mu = 1)$. Then,

$$\widehat{\eta}_\mu = (1 - \kappa_\Phi)\, \sigma,$$

where $\kappa_\Phi(\omega) = \int_G |\Phi(g)|^2 \omega(g)\, dm_G(g)$.

Example I: $X = \mathbb{R}^n$, $x_o = 0$, $G = \mathbb{R}^n$, $K = \{0\}$

Classical Fourier analysis: For $\lambda \in \mathbb{R}^n$, let $\omega_\lambda(x) = e^{2\pi i \langle \lambda, x \rangle}$, for $x \in \mathbb{R}^d$. Then,

$$\Omega = \{\omega_\lambda\}_{\lambda \in \mathbb{R}^n} \cong \mathbb{R}^n \quad \text{and} \quad \sigma = m_{\mathbb{R}^n} (= \text{Lebesgue measure}).$$

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Basic lemma of hyperuniformity:

$$\mathrm{Var}_\mu(S\chi_{B_R}) = o(R^n), \;\; R \to \infty \iff \widehat{\eta}_\mu([-\epsilon, \epsilon]^n) = o(\epsilon^n), \;\; \epsilon \to 0.$$

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DPP (n=1): Take $B = [-1/2, 1/2]$, then $\Phi(x) = \frac{\sin(\pi x)}{\pi x}$ for $x \in \mathbb{R}$, and

$$d\widehat{\eta}_\mu(\lambda) = \begin{cases} |\lambda| \, d\lambda & \text{for } |\lambda| \leq 1 \\ d\lambda & \text{for } \lambda| \geq 1 \end{cases} \qquad \text{(hence hyperuniform).}$$

Example II: $X = \mathbb{R}^n$, $x_o = 0$, $G = \mathbb{R}^n \rtimes O(n)$, $K = \{0\} \rtimes O(n)$

Radial Fourier analysis: For $\lambda \in [0, \infty)$, let $\omega_\lambda(x) = a_n J_{(n-2)/2}(2\pi\lambda\|x\|)$, for $x \in \mathbb{R}^d$. Then,

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Example III: $X = \mathbb{H}_{\mathbb{C}}$, $x_o = j$, $G = \mathrm{SL}_2(\mathbb{C})$, $K = \mathrm{SU}(2)$

Let

$$\mathbb{H}_{\mathbb{C}} = \{z + jw \,:\, z \in \mathbb{C}, \ \ w > 0\},$$

viewed as a subspace of Hamiltonian quaternions.

Fact: Every bi-K-invariant function on G is completely determined by its restriction to the sub-semigroup $a_t = \mathrm{Diag}(e^{t/2}, e^{-t/2})$, for $t \geq 0$.

For $\lambda \in \mathbb{R} \cup i[0,1]$, let $\omega_\lambda(a_t) = \frac{\sin(\lambda t)}{\lambda \sinh(t)}$, for $t \geq 0$. Then,

$$\Omega = \{\omega_\lambda\}_{\lambda \in \mathbb{R} \cup i[0,1]} \cong \mathbb{R} \cup i[0,1] \quad \text{and} \quad d\sigma(\lambda) = \lambda^2 \, d\lambda, \quad \lambda \in \mathbb{R}.$$

In particular, the support of σ is a <u>proper</u> subset of Ω and does not contain the "identity" spherical function ($\lambda = i$).

Corollary: If $\mu = \mathrm{Poisson}(m_X)$, then $\sigma([-\epsilon, \epsilon]) = 2\epsilon^3/3$.

Is there a "basic lemma" for this Gelfand pair?

Theorem (B-Bylehn, '23) For every G-invariant locally square-integrable random measure μ on X, we have

$$\varlimsup_{R \to \infty} \frac{\mathrm{Var}_\mu(S\chi_{B_R})}{m_X(B_R)} > 0.$$

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Recall that $\omega_o(a_t) = \frac{t}{\sinh(t)}$.

Theorem (B-Bylehn, '22) Suppose that μ is a G-invariant random measure on X such that $\widehat{\eta}_\mu(i[0,1]) = 0$. Then,

$$\mathrm{Var}_\mu(S(\omega_o\chi_{B_R})) = o(R^3), \ \ R \to \infty \iff \widehat{\eta}_\mu([-\epsilon,\epsilon]) = o(\epsilon^3), \ \ \epsilon \to 0.$$

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The condition $\widehat{\eta}_\mu(i[0,1]) = 0$ is satisfied in "most " situations.

Examples?

Tentative definition: We say that μ is *spectrally hyperuniform* if

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How about DPP's? No - for every DPP μ on X, there exists $c > 0$ such that

$$d\widehat{\eta}_\mu = \underbrace{(1-\kappa_\phi)}_{\geq c \text{ on } \mathrm{supp}(\sigma)} \, d\sigma$$

Very different from $\mathbb{R}^n$!

The Heisenberg nilmanifold

Heisenberg group: $H = \mathbb{C} \oplus \mathbb{R}$, $(z,s)(w,t) = (z+w, s+t+\frac{1}{2}\Im(z\overline{w}))$.

Rotations: For $k \in U(1)$, let $k.(z,s) = (kz, s)$.

Fact: $G = H \rtimes U(1)$ and $K = \{(0,0)\} \rtimes U(1)$ form a Gelfand pair (and $G/K \cong H$).

The space of zonal spherical functions for (G,K) can be identified with the *Heisenberg fan*:

$$\Omega = \{(|\lambda|(2j+1), \lambda) \, : \, \lambda \neq 0, \;\; j \in \mathbb{N}_o\} \sqcup \{(\xi, 0) \, : \, \xi \geq 0\} \subset \mathbb{R}^2,$$

and the Plancherel measure is given by

$$\sigma(\psi) = \frac{1}{4\pi^2} \sum_{j=0}^{\infty} \int_{-\infty}^{\infty} \psi(|\lambda|(2j+1), \lambda)\, |\lambda| \, d\lambda, \quad \text{for all } \psi \in C_c^\infty(\mathbb{R}^2)$$

More precisely, $\omega_{(j,\lambda)}(z,t) = e^{i\lambda t} e^{-|\lambda||z|^2/4} L_j(|\lambda||z|^2/2)$, where L_j denotes the Laguerre polynomial of degree j.

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Involves delicate estimates of Laguerre polynomials and (yet unproven) Diophantine properties of symplectic lattices. Formulas for the diffraction measures were obtained by B.-Hartnick ('18). A key difficulty is to relate the interactions between the contributions from the line $\{(\xi, 0)\}_{\xi \geq 0}$ and the support of σ.