Gibbs point processes and number-rigidity

Thibaut Vasseur
Université Paris Cité

Hyperuniform structures, rigid point processes and related topics

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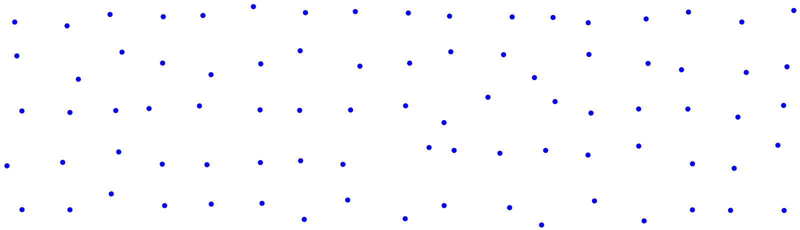

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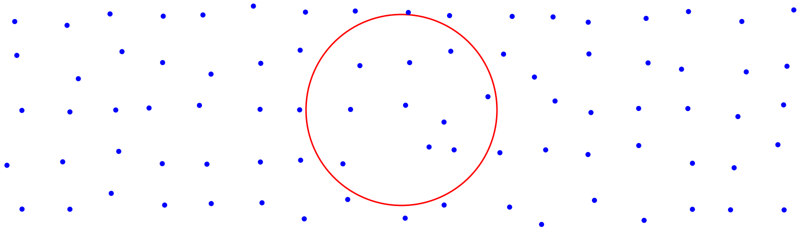

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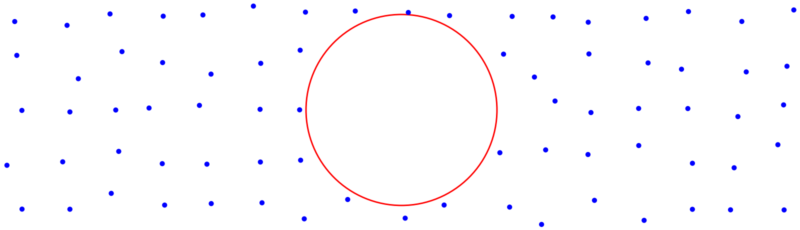

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For Gibbs point processes, dependency is described through an energy function H.

Pairwise interaction

$$H(\gamma) = \sum_{\{x,y\}\subset\gamma} g(y-x)$$

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The Riesz interaction with parameter $s \in (-2, +\infty)$

$$g(x) = \begin{cases} |x|^{-s} & \text{for } s > 0 \\ -\log|x| & \text{for } s = 0 \\ -|x|^{-s} & \text{for } s \in (-2, 0) \end{cases}$$

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$s = d - 2$ is the Coulomb interaction

$\Lambda_n = [-n^{1/d}/2, n^{1/d}/2]$: square of volume n

The finite volume canonical Gibbs measure is

$$P_n^{\beta}(d\gamma) = \frac{1}{Z_n} e^{-\beta H(\gamma)} \, \mathrm{Bin}_{\Lambda_n, n}(d\gamma).$$

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Infinite volume Gibbs point process : an accumulation points P^β of the sequence $(P_n^\beta)_{n\geq 1}$

The periodic setting can be more convenient in the long range case

Modèle périodique

$$H_n(\gamma) = \sum_{\{x,y\}\subset\gamma} g_n(x-y)$$

where $g_n(x) = \sum_{u\in\mathbb{Z}^d} \left[g(x+n^{1/d}u) - n^{-1}\int_{\Lambda_n} g(y+n^{1/d}u)dy \right].$

and $g(x) = |x|^{-s}$ with $s\in(d-1,d)$.

Again, the finite volume canonical Gibbs measure is

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$d=1$, $s=0$ $(g(x) = -\log|x|)$ one acc. point : Sine process

Law of Γ_Δ is inaccessible

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Grand canonical DLR equations

$$\Gamma_\Delta \mid \Gamma_{\Delta^c} = \gamma_{\Delta^c} \sim \frac{1}{Z(\gamma_{\Delta^c})} \exp\left(-\beta(H(\eta) + W(\eta \mid \gamma_{\Delta^c}))\right) \mathrm{Pois}_\Delta(d\eta),$$

where $W(\eta \mid \gamma_{\Delta^c}) = \sum_{x\in\eta} \sum_{y\in\gamma_{\Delta^c}} g(x-y)$.

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Adapted to short range ($s > d$) for which W is well defined.

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Canonical DLR equations

$$\Gamma_\Delta \mid \Gamma_{\Delta^c} = \gamma_{\Delta^c}, \#\Gamma_\Delta = n$$
$$\sim \frac{1}{Z(\gamma_{\Delta^c})} \exp\left(-\beta(H(\eta) + \mathrm{Move}(\eta \mid \gamma_{\Delta^c}))\right) \mathrm{Bin}_{\Delta,n}(d\eta),$$

where $\mathrm{Move}(\eta \mid \gamma_{\Delta^c}) = W(\eta \mid \gamma_{\Delta^c}) - W(\mathrm{ref} \mid \gamma_{\Delta^c})$

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Adding information on the inside helps to define interaction with the outside

The n-points reduced Campbell measure is defined as

$$\mathrm{Campbell}_P^{(n)}(f) = E\left[\sum_{x_1,\dots,x_n\in\Gamma}^{\neq} f(x_1,\dots,x_n,\Gamma\setminus\{x_1,\dots,x_n\})\right]$$

where $f : (\mathbb{R}^d)^n \times \mathrm{Config} \to \mathbb{R}$ positive and measurable.

- If $\Gamma \sim \mathrm{Pois}$ then

$$\mathrm{Campbell}^{(n)}(dX_n d\gamma) = dX_n \, \mathrm{Pois}(d\gamma)$$

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- If $\Gamma \sim \mathrm{Bin}_{\Lambda,N}$ and $n \leq N$ then

$$\mathrm{Campbell}^{(n)}(dX_n d\gamma) = \frac{1}{|\Lambda|^n}\frac{N!}{(N-n)!} d_\Lambda X_n \,\mathrm{Bin}_{\Lambda,N-n}(d\gamma)$$

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- If Γ satisfies the previous Canonical DLR equations then

$$\mathrm{Campbell}^{(\mathrm{n})}(dX_n d\gamma) = \exp(-\beta(H(X_n) + \mathrm{Move}(X_n,\gamma)))dX_n Q_n^\beta(d\gamma)$$

Dereudre, Hardy, Leblé, Maïda

The Sine process satisfies canonical DLR equations and is number-rigid.

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- Consequence of the non-number rigidity $\exists n \geq 1 : Q_n^\beta \ll P^\beta$

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- Consequence of the non-number rigidity $\exists n \geq 1 : Q_n^\beta \ll P^\beta$
- Use the n-points Campbell measure to obtain a contradiction.

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Ideas of the proof :

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- Use the n-points Campbell measure to obtain a contradiction.

We assume $n = 1$, so there exist a function Add such that

$$Q_1^\beta(d\gamma) = e^{-\beta \,\mathrm{Add}(\gamma)} P^\beta(d\gamma)$$

1-point Campbell measure

$$\mathrm{Campbell}(dxd\gamma) = \exp(-\beta(\mathrm{Move}(x,\gamma) + \mathrm{Add}(\gamma)))dxP^{\beta}(d\gamma)$$

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2-points Campbell measure

$$\mathrm{Campbell}^{(2)}(dxdyd\gamma) = \exp(-\beta(\mathrm{Move}(x,\gamma) + \mathrm{Add}(\gamma)))$$
$$\exp(-\beta(\mathrm{Move}(y,\gamma \cup \{x\}) + \mathrm{Add}(\gamma \cup \{x\})))dxdyP^{\beta}(d\gamma)$$

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Manipulating Campbell measures shows that

$$\text{Add}(\gamma \cup \{x\}) + \log|x| = \text{Add}(\gamma \cup \{z\}) + \log|z|$$

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Manipulating Campbell measures shows that

$$\text{Add}(\gamma \cup \{x\}) \to -\infty$$

What we have used

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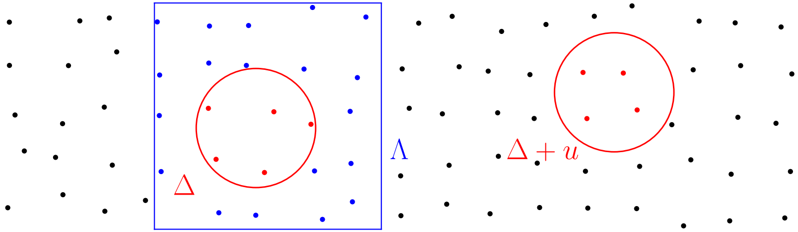
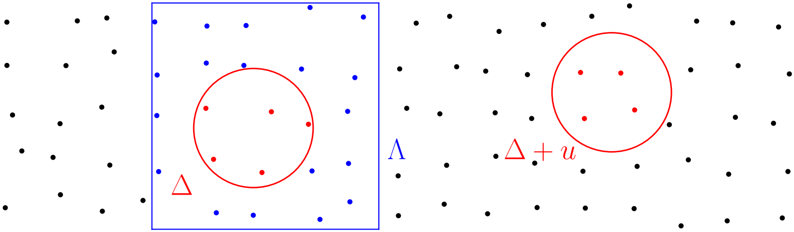
Leblé, V. (ongoing work)

2D coulomb gases satisfy canonical DLR equations and is number-rigid.

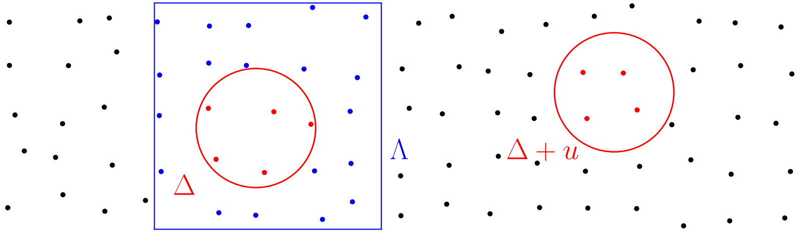


Λ
Δ
$\Delta + u$



DLR equations and a control of the cost of exchanging a k point configuration in Δ with a l points configuration in $\Delta + u$ (independently of u) then for any $\Lambda \setminus \Delta$-local event

$$P(\#\Gamma_\Delta = k, \#\Gamma_{\Delta+u} = l, A) \leq CP(\#\Gamma_\Delta = l, \#\Gamma_{\Delta+u} = k, A) + \varepsilon$$



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$$P(\#\Gamma_\Delta = k \mid \Gamma_{\Delta^c}) > 0 \implies P(\#\Gamma_\Delta = l \mid \Gamma_{\Delta^c}) > 0$$

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Control the cost of exchange configuration in Δ and $\Delta + u$ can be achieved if we show that

$$\lim_{\Lambda \to \mathbb{R}^d} \sum_{y \in \Gamma_\Lambda} g(x-y) - \int_\Lambda g(x-y)dy$$

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For instance, when $s \in (d-1, d)$ a sufficient condition is

$$Var(\#\Gamma_{\Lambda_k}) \leq Ck^{2s-\varepsilon}.$$

Dereudre, V.

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Which implies grand canonical DLR equations.

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Existence of a compensator

$$h(x,\Gamma) = \lim_{n\to+\infty} \sum_{y\in\Gamma_{\Lambda_n}} g(x-y) - \text{Comp}(\Lambda_n, \Gamma_{\Lambda_n^c}, \#\Gamma_{\Lambda_n})$$

Thank you for your attention !