Dipole transition for the two-component plasma

J. Boursier
ENS Lyon

In dimension 2, $\Lambda = [0, \sqrt{N}]^2$.

- N particles $x_1, \dots, x_N$ of charge $+1$.
- N particles $y_1, \dots, y_N$ of charge -1.

Energy
$$H_N(X_N, Y_N) = \frac{1}{2} \sum_{i \neq j} -\log |x_i - x_j| + \frac{1}{2} \sum_{i \neq j} -\log |y_i - y_j| + \frac{1}{2} \sum_{i \neq j} \log |x_i - y_j|.$$

Motivations

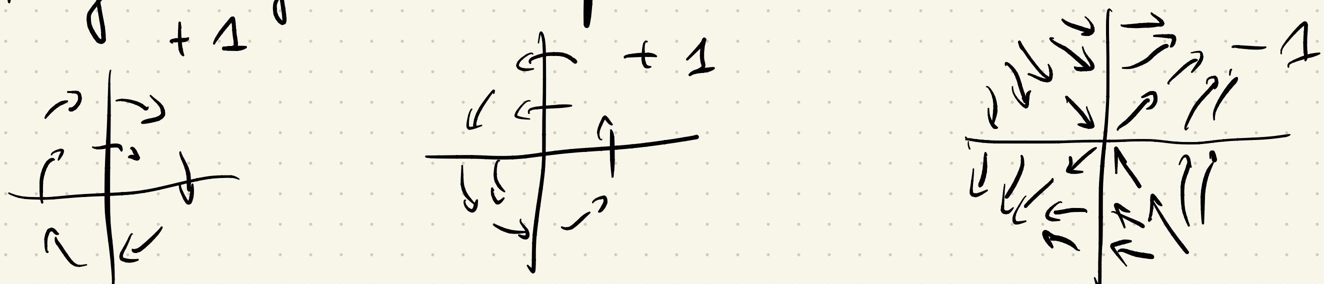
XY model (superfluid hellium, hexatic liquid cristals).

$u_i \in S^1$ spins on a lattice with angle $\theta_i \in [0, 2\pi]$

$$\propto e^{\beta \sum_{i \sim j} \cos(\theta_i - \theta_j)} \, d\theta_1 \dots d\theta_N$$

Coulomb charges!

"Angle field = spin wave + vortex"



Kosterlitz-Thouless '73, Berezinskii '71 : phase transition of infinite order!

Renormalization of the model

$$Z_{N,\beta} = \int_{\Lambda^{2N}} \prod_{i<j} \frac{1}{|x_i - y_j|^\beta} \dots dX_N \, dY_N .$$

$$Z_{N,\beta} < +\infty \quad \text{iff} \quad \int_{[0,1]^4} \frac{1}{|x-y|^\beta} \, dx \, dy < +\infty$$

$$\text{iff} \quad \beta < 2 .$$

$\rightsquigarrow$ introduce lengthscale $\lambda > 0$ and truncate $\log|x|$ at $\log(|x| \vee \lambda)$?

We consider smeared charges $\delta_x^{(\lambda)}$ uniform measure of mass 1 on $B(x,\lambda)$.

Let z_i, $i = 1, \dots, 2N$ be the points x_i or y_i with $d_i = \pm 1$ their charge.

$$H_{N,\lambda} = \frac{1}{2} \sum_{i \neq j} \iint -d_i d_j \log|x-y| \, \delta_{z_i}^{(\lambda)}(x) \, \delta_{z_j}^{(\lambda)}(y).$$

Now

$$Z_{N,\beta}^{\lambda} = \iint e^{-\beta H_{N,\lambda}(Z_{2N})} dZ_{2N}$$

is defined for every λ.

Phase transitions

- $\beta \in (0,2)$: "free" vortices.
- $\beta \geq 2$: particles of opposite signs bond into pairs of size λ (dipoles).
- $\beta = 3$: quadripoles start diverging.
- $\beta = 4 - \frac{2}{p}$: $2p$-poles start diverging.
- $\beta = 4$: KT transition.

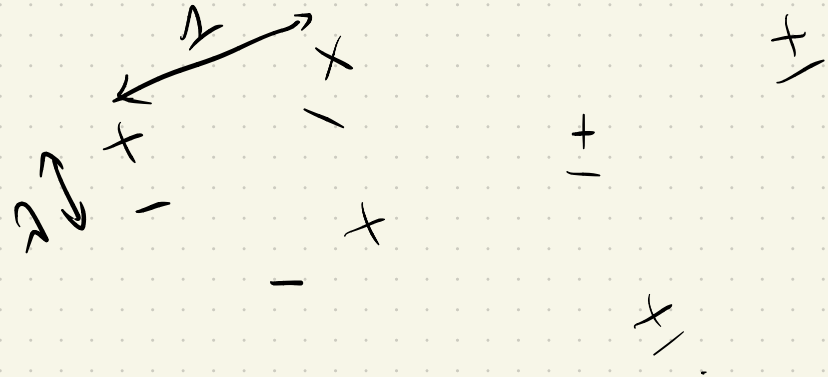
Literature

- $\beta \in (0,2)$. Thermodynamic limit.
  Gunson-Panta, Leblé-Serfaty-Zeitouni '17
- Through Sine-Gordon transformation:
  existence of two phases Fröhlich-Spencer '81
  Log-Sobolev inequality for Sine-Gordon
  Bodineau-Bauerschmidt. 1D Lacoin, Rhodes, Vargas
- Using XY or Villain XY model.
  Garban-Sepulvéda

questions

- Expand $\log Z_{N,\beta}^{\lambda}$ for $\beta \geq 2$ and prove that dipoles of size λ are formed.
- Show the transitions at $\beta_p = 4 - \frac{2}{p}$.
- Study fluctuations $\sum \delta_{x_i} - \sum \delta_{y_i}$ (against test-functions). Typical size ?

What do we expect?



Free energy of a dipole of size λ : $O(\lambda^{2-\beta})$.

$\rightarrow$ should be dominant.

Dipole-dipole interaction : $O\left(\left(\frac{\lambda}{d}\right)^2\right)$.

$\sum$ interactions $= O(N \log N \lambda^2)$?

$\boxed{\begin{array}{l}\text{Thm (B. Serfaty '23}^{+}\text{). For } \beta > 2, \\ \log Z^{\lambda}_{N,\beta} = 2N\log N + N(2-\beta)\log\lambda \\ + N\log C_\beta + O_\varepsilon\left(N\left(\lambda^{2-\varepsilon} + \lambda^{\frac{2(\beta-2)}{4-\beta}-\varepsilon}\right)\right).\end{array}}$

"Cor": typically N dipoles of size $o(\lambda)$.

$\rightarrow$ transition at $\beta = 3$.

$\rightarrow$ could be improved to show $2p$-pole transitions.

Electric reformulation of the energy

$$h_N^{(\lambda)}(x) = -\int \log|x-y|\, d\Big(\sum_i d_i\, \delta_{z_i}^{(\lambda)}\Big)(y).$$

$$H_{N,\lambda} = \frac{1}{2}\sum_{i\neq j} d_i d_j \iint \log|x-y|\, \delta_{z_i}^{(\lambda)}(x)\, \delta_{z_j}^{(\lambda)}(y)$$

$$= \frac{1}{2}\sum_{i,j} \quad '' \quad + N\int \log * \delta_0^{(\lambda)}\, \delta_0^{(\lambda)}$$

$$= -\frac{1}{2}\int h_N^{(\lambda)} \frac{1}{2\pi}\Delta h_N^{(\lambda)} + C_N$$

$$= \frac{1}{4\pi}\int |\nabla h_N^{(\lambda)}|^2 + C_N.$$

Hint on the proof: upper bound

Idea: enlarge balls $B(z_i, \lambda)$ into $B(z_i, \alpha_i)$ for some well-chosen α_i.

→ Leblé-Serfaty-Zeitouni; Serfaty'18.

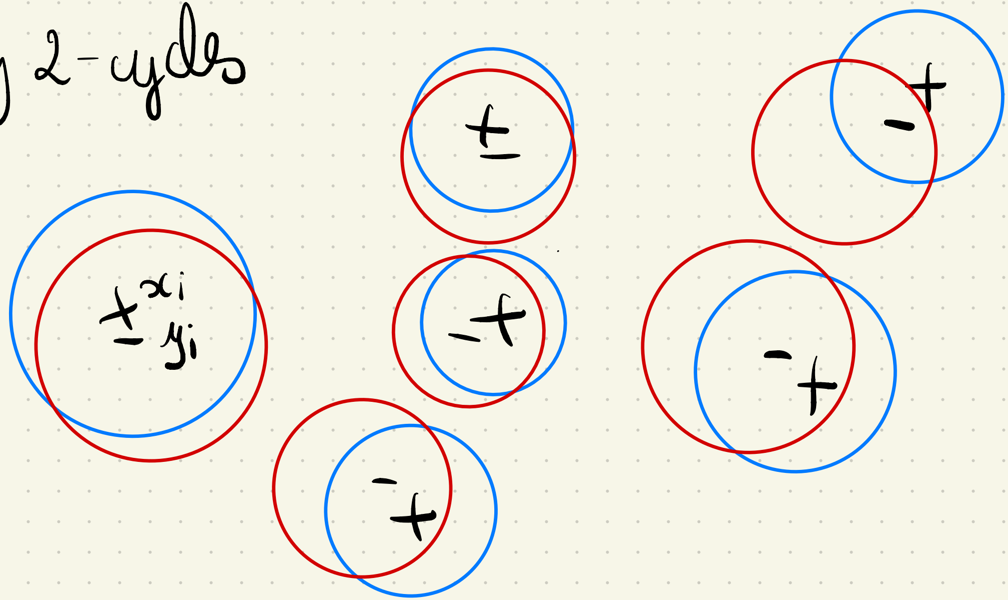
Which α_i??

Try $\alpha_i = \frac{1}{4}$ distance to the 2^{nd} nearest-neighbor

$:= \min(r_2(y_i), r_2(x_i))$.

$r_1(z_i) = 1/4$ distance to the 1^{st} n-n.

$\{x_i, y_i\}$ 2-cycles



Newton's theorem $\rightarrow$ only terms in

$$\underbrace{\iint -\log|x-y|\,\delta_{x_i}^{(\alpha_i)}(x)\,\delta_{y_i}^{(\alpha_i)}(y)}_{=\,O\left(\left(\frac{r_1(z_i)}{r_2(z_i)}\right)^2\right).} - \iint -\log|x-y|\,\delta_{x_i}^{(1)}(x)\,\delta_{y_i}^{(1)}(y).$$

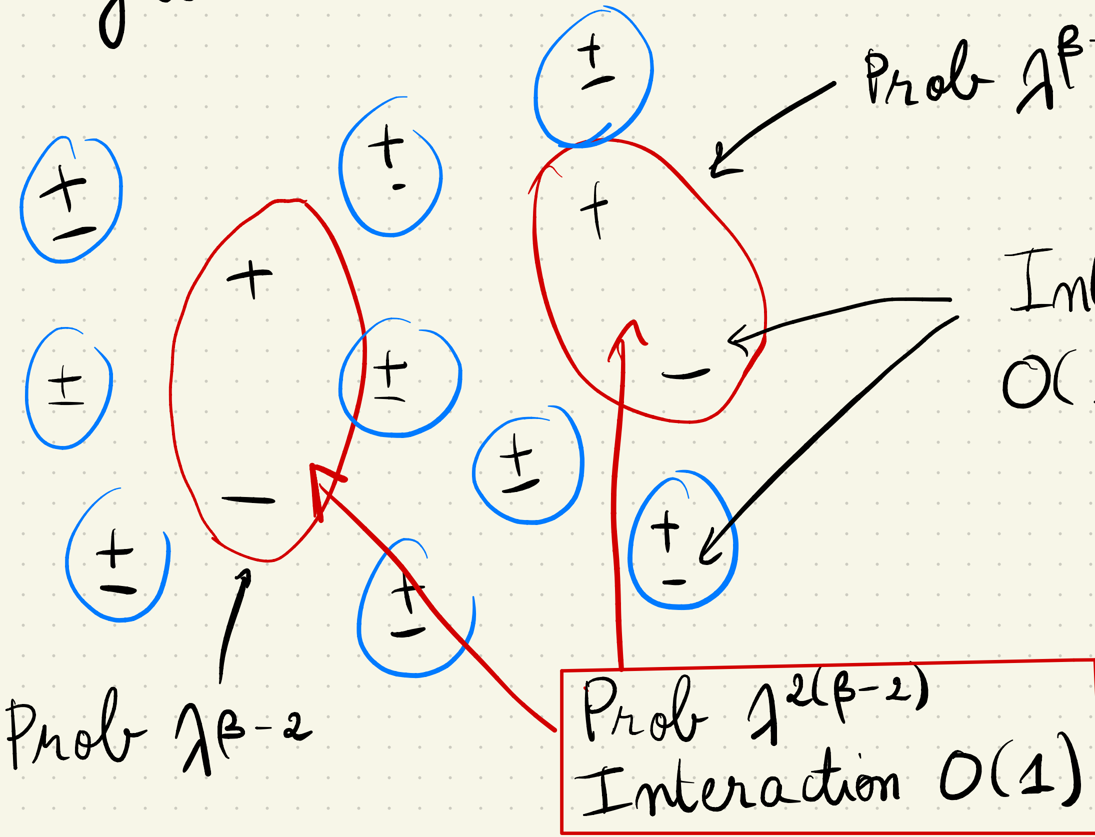
gives (here)

$$\int |\nabla h^{(\alpha)}|^2 - \int |\nabla h^{(\lambda)}|^2 \geq \frac{1}{2}\sum_i \log r_1(z_i) - C \sum_i \left(\frac{r_1(z_i)}{r_2(z_i)}\right)^2 .$$

Error term is quadratic.

Dipole of size 1 $\to$ error of size 1.
$O(N\lambda^{\beta-2})$ such dipoles $\to$ error in $O(N\lambda^{\beta-2})$.

In fact:


Prob $\lambda^{\beta-2}$
Interaction $O(1)$
Prob $\lambda^{\beta-2}$
Prob $\lambda^{2(\beta-2)}$
Interaction $O(1)$

Multilayer decomposition

Pairing of + and − with a greedy algorithm. $x_i - y_{\sigma(i)}$

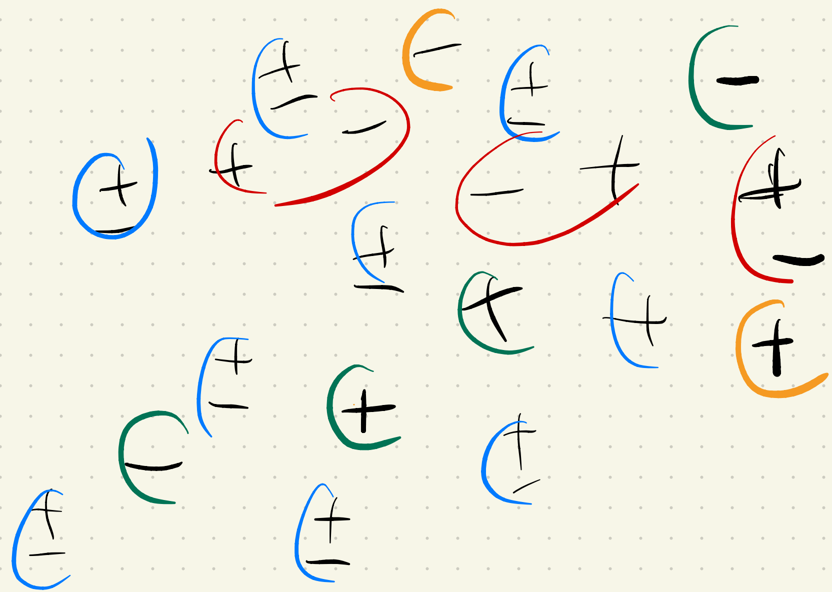
I_1 : size $\leq \lambda^{1-\varepsilon}$

I_2 : size $\in \,]\lambda^{1-\varepsilon}, \lambda^{1-2\varepsilon}]$

I_3 : size $\in \,]\lambda^{1-2\varepsilon}, \lambda^{1-3\varepsilon}]$

$\vdots$

I_K



We propose

$$\alpha_i = \max\left(\lambda, \frac{1}{4}\min_{\substack{j\in I_k \\ j\neq i,\sigma(i)}} |z_i - z_j|\right), \quad i\in I_k.$$

$$H_{N,\lambda} \geq \frac{1}{2}\sum_{i\in\cup I_k} g(z_i - z_{\sigma(i)}) - C\sum_k \sum_i \left(\frac{|z_i - z_{\sigma(i)}|^2}{\alpha_i^2}\right) + \text{Layers interactions}$$
$$- C\left(N - \left|\bigcup_k I_k\right|\right).$$

$\rightarrow$ gives the desired upper bound on $\log Z_{N,\beta}^{\lambda}$.

Fluctuations

$\boxed{\text{Setup}}$ $\beta \in (0,2)$. Prove that for f smooth

$$\operatorname{Var}\left[\sum_{i=1}^{N} f(x_i) - \sum_{i=1}^{N} f(y_i)\right] = o(N) \ ?$$

<u>1CP</u> $\sum f(x_i) - N\int f \, d\mu_V \longrightarrow \mathcal{N}\left(0, \frac{1}{4\pi\beta}\int |\nabla f|^2\right).$

Leblé - Serfaty, Bauerschmidt - Bourgade - Yin - Yau.

<u>Discrete 2CP</u> $\Delta^{-1} q \longrightarrow$ GFF with an effective T_{eff}.

$T_{eff} = T$ for $T > T_c$.

1CP transportation argument

$F = \sum_{i=1}^{N} f(x_i) - N \int f \, d\mu_V$.

Laplace transform $\mathbb{E}[e^{tF}] \to$ new force $-\frac{\beta}{t} \nabla F$.

$\leadsto$ modifies the distribution.

Fluctuation-dissipation : transport particles back to equilibrium.

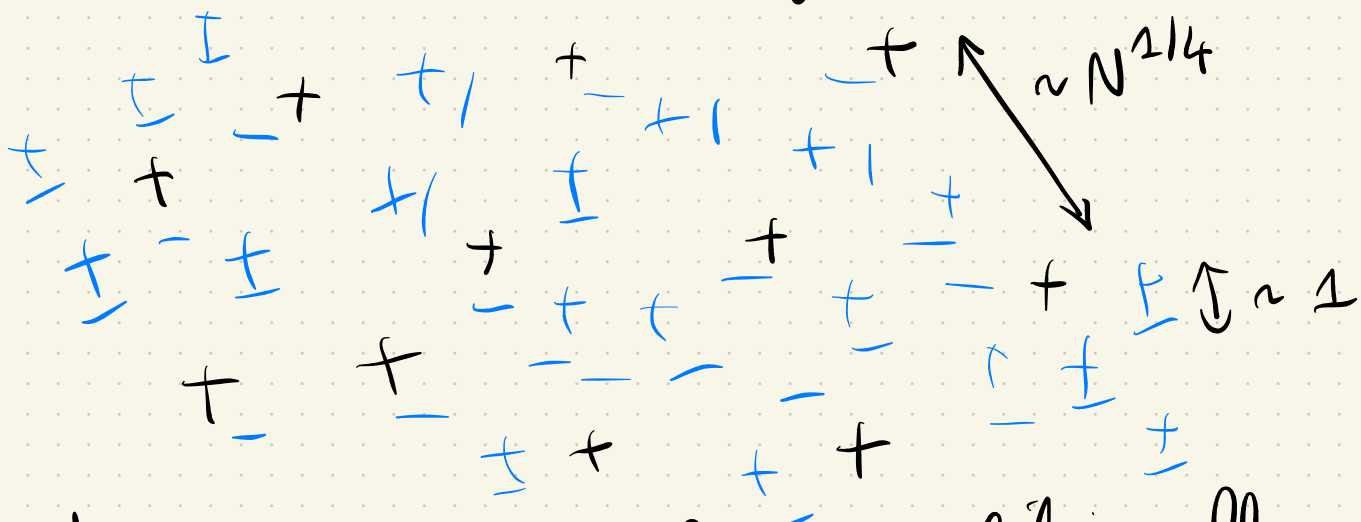
For the 1CP enough to transport macroscopic densities.

$\Rightarrow$ Find ψ such that $(Id + t\psi) \# \mu_V = \mu_V + O(t^2)$.

$\leadsto \psi = \frac{\nabla \xi}{\mu_V}$.

A "cavity" method

Fix $\sqrt{N}$ points of charge $+1$ on Λ. $(x_i)_{i\in I}$



Study of $\mu_N := \sum_{i\in I^c} \delta_{x_i} - \sum \delta_{y_i}$ conditionally on $(x_i)_{i\in I}$.

$$\mu = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}^{(R)} \quad , R \gg 1.$$

I acts on I^c through an approximate 1-body potential $V = -\log|\cdot| * N\mu$.

$\mu_V = -\frac{\mu}{\sqrt{N}}$.

$\to$ Can take $\psi = \frac{\nabla \xi}{\mu_V}$.

Prove $\mathbb{E}\left[e^{\sum_{i\in I^c} f(x_i) - \sum f(y_j) - N\int f \, d\mu_V} \,\middle|\, (x_i)_{i\in I^c}\right] = e^{o(\sqrt{N})}$.

Thank you
for your attention!