Spooky Scary Exam 2

October 22, 2021

Professor: Anand Patel

Submit your hand-written answers by scanning them and uploading preferably one pdf file to Canvas. You don't need to print out the exam. *You can use your class notes, but not webassign materials.*

If you want me to believe that you actually did the problem, and give you full credit, you should clearly show your work.

There are *5* problems with several parts. Each problem is worth *10* points total. There is one extra credit problem worth *2* points -- yay!

Try your best, and make me proud!

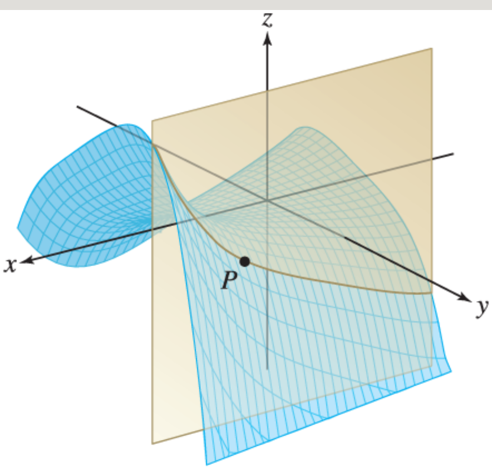
Problem 1

a) Compute $\frac{\partial}{\partial x}\left(\frac{x+y}{y+2}\right)$.

b) Compute $\frac{\partial}{\partial x}\ln(x^2+xyz)$.

c) Compute $f_z(1,2,3)$, where $f(x,y,z)=xy+yz+xz^2$.

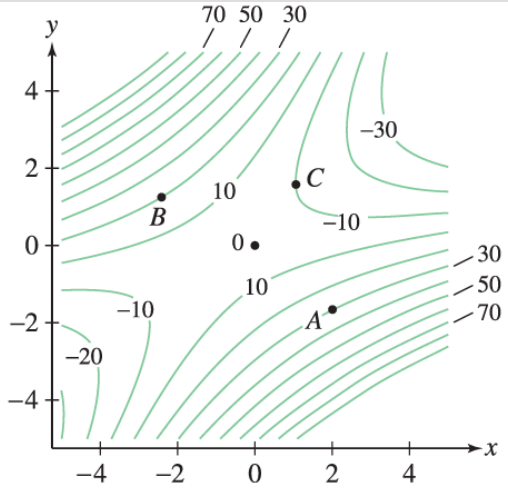
d) Determine whether the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are positive or negative at the point on the graph in this figure:



Rogawski et al., *Calculus: Early Transcendentals*, 4e, © 2019 W. H. Freeman and Company

Graph of f

e) The following is the contour plot of some function $g(x, y)$:



Rogawski et al., *Calculus: Early Transcendentals*, 4e, © 2019 W. H. Freeman and Company

Contour plot of g(x,y)

At which point, A, B or C is g_y the smallest?

Problem 2

a) Find the equation of the tangent plane of $f(x, y) = x^2 + xy$ at the point $(2, 3)$. (Express your answer in the form $z = ax + by + c$.)

b) Find the points on the graph of $f(x, y) = (x + 1)y^2$ where the tangent plane is horizontal.

c) Let $f(x, y) = \sin(x^2 + y)$. Calculate the gradient ∇f.

d) Fill in the question marks: The gradient vector $\nabla f(x, y, z)$ at a point P is always ???????? to the level set $f(x, y, z) = c$ containing P.

Problem 3

a) Compute the directional derivative $D_{\mathbf{u}} f$ at the point $P = (1, 2)$, where $f(x, y) = xy^3 - x^2$, and $\mathbf{u} = \langle 3/5, 4/5 \rangle$.

b) Compute the directional derivative of $g(x, y, z) = z^2 - xy + 2y^2$ along the direction of $\mathbf{v} = \langle 1, -2, 2 \rangle$ at the point $P = (2, 1, -3)$.

c) Does the function $f(x, y) = xy - y^3$ increase if the input $(2, 3)$ is budged in the direction of the vector $\langle 1, 1 \rangle$? Explain.

d) $T(x, y, z)$ is the temperature at location (x, y, z). You know that, at your location $P = (1, 1, 7)$, the gradient vector is $\nabla T = \langle 2, 0, 2 \rangle$. It's very hot at your location P. Which way should you move in order to cool off as rapidly as possible?

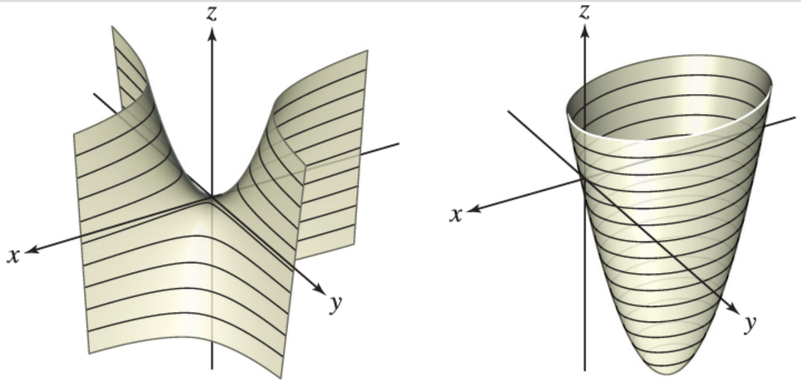
e) (Reminder: Problem 2 part d.) Compute a "$ax + by + cz = d$" equation for the tangent plane of the surface $x^2 + 3y^2 + 4z^2 = 20$ at the point $P = (2, 2, 1)$.

Problem 4

a) Find the critical points of the functions

$f(x, y) = x^2 + 2y^2 - 4y + 6x$ and $g(x, y) = x^2 - 12xy + y$.

Use the Second Derivative Test to determine whether they are local minimum, local maximum, or saddle points. Then, using this, match f and g with the two graphs:



(A) (B)

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which is f, which is g?

b) Find the critical points of $f(x, y) = x^3y + 12x^2 - 8y$. Then use the 2nd Derivative Test to determine whether they are local max/min/ saddle, or state that the test fails.

c) Find the greatest possible product of three positive numbers that add up to 150.

Problem 5

a) Find the maximum and minimum *outputs* of the function $f(x, y) = 2x + 3y$ if (x, y) is forced to live on the circle $x^2 + y^2 = 4$.

b) Find the maximum and minimum *outputs* of the function $T(x, y) = 4x^2 + 9y^2$ if (x, y) is forced to live on the hyperbola $xy = 4$.

c) Find the rectangular box of maximum volume if the sum of the lengths of *all* edges is 300 cm. (At least do some set up, for partial credit.)

For 2 extra points:

The volume of a cylindrical orange Koolaid can is $V = \pi r^2 h$ where r is the radius and h is the height. Suppose the can has a closed top and bottom. Then the total surface area of the can is $S = 2\pi rh + 2 \times \pi r^2$. (Agree?)

If the surface area S is fixed to be $4 + 2\pi$, what is the maximum possible volume of orange Koolaid achievable?

Problem 1.)

a.) $\frac{d}{dx}\left(\frac{x+y}{y+z}\right) = \boxed{\frac{1}{y+z}}$

c.) $\boxed{A}$

b.) $\frac{d}{dx}\ln(x^2+xyz) = \boxed{\frac{2x+yz}{x(x+yz)}}$

C.) $f_z(1,2,3)$, $f(x,y,z) = xy + yz + xz^2$

$f_z(x,y,z) = y + 2xz$

$f_z(1,2,3) = 2 + 2(1)(3) = 2 + 6 = \boxed{8}$

d.) f_x = negative 

f_y = negative 

Problem 2.)

a.) $(2,3)$

$f(x,y) = x^2 + xy \Rightarrow f(2,3) = 2^2 + (2)(3) = 4 + 6 = 10$

$f_x(x,y) = 2x + y \Rightarrow f_x(2,3) = 2(2) + 3 = 4 + 3 = 7$

$f_y(x,y) = x \Rightarrow f_y(2,3) = 2$

$z = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$

$= 10 + 7(x-2) + 2(y-3) = 10 + 7x - 14 + 2y - 6$

$\boxed{z = 7x + 2y - 10}$

b.) $f(x,y) = (x+1)y^2$

$f_x(x,y) = y^2 \Rightarrow y^2 = 0 \Rightarrow y = 0$

$f_y(x,y) = 2xy + 2y \Rightarrow 2x(0) + 2(0) = 0$

$$\boxed{\begin{array}{c} y = 0 \quad x = 0 \\ (0,0) \end{array}}$$

c.) $f(x,y) = \sin(x^2+y)$

$\nabla f = \langle f_x(x,y), f_y(x,y) \rangle$

$f_x(x,y) = 2x\cos(x^2+y)$

$f_y(x,y) = \cos(x^2+y)$

$$\boxed{\nabla f = \langle 2x\cos(x^2+y), \cos(x^2+y) \rangle}$$

d.) Parrallel

Problem 3.)

$$\sqrt{\left(\tfrac{3}{5}\right)^2+\left(\tfrac{4}{5}\right)^2} = \sqrt{\tfrac{9}{25}+\tfrac{16}{25}} = \sqrt{\tfrac{25}{25}}$$
$$= \sqrt{1} = 1$$

a.) $D_u f$ @ $P = (1,2)$

$f(x,y) = xy^3 - x^2$

$U = \langle \frac{3}{5}, \frac{4}{5} \rangle$

$f_x = y^3 - 2x \qquad \nabla f = \langle y^3 - 2x, 3xy^2 \rangle$

$f_y = 3xy^2$

@ $(1,2)$

$\nabla f = \langle 8 - 2, 3(1)(4) \rangle$

$D_{\langle \frac{3}{5}, \frac{4}{5} \rangle} f = \langle 6, 12 \rangle \cdot \langle 3/5, 4/5 \rangle$

$$\frac{18}{5} + \frac{48}{5} = \boxed{\frac{66}{5}}$$

b.) $g(x,y,z) = z^2 - xy + 2y^2$

$v = \langle 1, -2, 2 \rangle \quad P = (2, 1, -3)$

$u = \frac{v}{\|v\|} = \langle \frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \rangle$

$\sqrt{1^2 + 2^2 + 2^2} = \sqrt{1+4+4} = \sqrt{9} = 3$

$g_x = -y$

$g_y = -x + 4y \quad \nabla g = \langle -y, -x+4y, 2z \rangle$

$g_z = 2z$

@ $(2, 1, -3)$

$\nabla g = \langle -1, 2, -6 \rangle$

$D_{\langle 1,-2,2 \rangle} g = \langle -1, 2, -6 \rangle \cdot \langle \frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \rangle = -\frac{1}{3} - \frac{4}{3} - \frac{12}{3} = \boxed{-\frac{17}{3}}$

C.)

$f_x = y$

$f_y = x - 3y^2$

$\nabla f = \langle y, x-3y^2 \rangle$ @ $(2,3)$

$\nabla f = \langle 3, -25 \rangle$

$\nabla f = \langle y, x-3y^2 \rangle$ @ $(3,4)$

$\nabla f = \langle 4, -44 \rangle$

$D_u f = \langle 3, -25 \rangle \cdot \langle 1, 1 \rangle$

$3 - 25 = -22$

$D_u f = \langle 4, -44 \rangle \cdot \langle 1, 1 \rangle$

$4 - 44 = -40$

When the Point (2,3) is budged in the Direction of $\vec{v} = \langle 1, 1 \rangle$ it decreases -22 -> -40 d.) $T(x,y,z)$ @ Location (x,y,z)

me $= P = (1,1,7)$ $\nabla T = \langle 2,0,2 \rangle$

Problem 4.)

a.) $f(x,y) = x^2 + 2y^2 - 4y + 6x \Rightarrow$ $(-3, 1)$ - Local minimum (B)

$g(x,y) = x^2 - 12xy + y \Rightarrow$ $\left(\frac{1}{12}, 2\right)$ - Saddle Point (A)

$f_x = 2x + 6 \Rightarrow 2x = -6 \Rightarrow x = -3$

$f_y = 4y - 4 \Rightarrow 4y = 4 \Rightarrow y = 1$

$g_x = 2x - 12y \Rightarrow 2\left(\frac{1}{12}\right) - 12y \Rightarrow \frac{1}{6} - 12y \Rightarrow -12y = -\frac{1}{6} \Rightarrow y = 2$

$g_y = -12x + 1 \Rightarrow -12x = -1 \Rightarrow x = \frac{1}{12}$

$f_x = 2x + 6 \Rightarrow f_{xx} = 2$ $\qquad D = 2 \cdot 4 - 0^2 = 8 > 0$

$f_y = 4y - 4 \Rightarrow f_{yy} = 4$

$f_{xy} = 0$

$g_x = 2x - 12y \Rightarrow g_{xx} = 2$

$g_y = -12x + 1 \Rightarrow g_{yy} = 0$ $\qquad D = 2 \cdot 0 - (-12)^2 = -144 < 0$

$g_{xy} = -12$

b.)

$$f(x,y) = x^3y + 12x^2 - 8y$$

$$f_x = 3x^2y + 24x \Rightarrow f_{xx} = 6xy + 24 \ @(2,-4) = 6(2)(-4) + 24 = -24$$

$$f_y = x^3 - 8 \Rightarrow f_{yy} = 0$$

$$f_{xy} = 3x^2 \ @(2,-4) = 3(2)^2 = 12$$

$$3x^2y + 24x \Rightarrow 3(2)^2y + 24(2) = 12y + 48 \Rightarrow 12y = -48 \Rightarrow y = -4$$

$$x^3 - 8 \Rightarrow x^3 = 8 \Rightarrow x = 2$$

$(2, -4)$

$$D(2,-4) = -24 \cdot 0 - (12)^2 = -144 < 0$$

- Saddle Point

c.)

$$\frac{150}{3} = 50$$

$$50 \cdot 50 \cdot 50 = 125{,}000$$ (Agree?)

Problem 5.)

a.)

$$f(x,y) = 2x + 3y$$

$$g(x,y) = x^2 + y^2 = 4$$

$$f_x = 2$$
$$f_y = 3$$

$$\nabla f = \langle 2, 3 \rangle$$

$$g_x = 2x$$
$$g_y = 2y$$

$$\nabla g = \langle 2x, 2y \rangle$$

① $2 = \lambda 2x$

② $3 = \lambda 2y$

③ $x^2 + y^2 = 4$

$$\frac{②}{①} = \frac{3}{2} = \frac{2y}{2x} \Rightarrow \frac{3}{2} = \frac{y}{x}$$

$$y = \frac{3x}{2}$$

$$x^2 + \left(\frac{3x}{2}\right)^2 = 4$$

$$x^2 + \frac{9x^2}{4} = 4$$

$$x^2\left(1 + \frac{9}{4}\right) = 4$$

$$\frac{4}{\frac{13}{4}}$$

$$\frac{13}{4}x^2 = 4$$

$$x^2 = \frac{16}{13} \Rightarrow x = \pm\frac{4}{\sqrt{13}}$$

$$y = \frac{3\left(\frac{4}{\sqrt{13}}\right)}{2}$$

$$y = \frac{6\sqrt{15}}{13}$$

$$x = \pm \frac{4}{\sqrt{13}} \quad y = \pm \frac{6\sqrt{13}}{13}$$

$$\left(\frac{4}{\sqrt{13}}, \frac{6\sqrt{13}}{13}\right) \quad \left(-\frac{4}{\sqrt{13}}, -\frac{6\sqrt{13}}{13}\right)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$f(x,y) \qquad\qquad f(x,y)$$

$$= 2\sqrt{13} \qquad\qquad = -2\sqrt{13}$$

$$\uparrow \qquad\qquad \uparrow$$

max   min

b.) $T(x,y) = 4x^2 + 9y^2$

$T_x = 8x$

$T_y = 18y$

$g(x,y) = xy = 4$

$g_x = y$

$g_y = x$

① $8x = \lambda y \Rightarrow \lambda = \frac{8x}{y}$

② $18y = \lambda x \Rightarrow \lambda = \frac{18y}{x}$

③ $xy = 4$

$$\frac{②}{①} = \frac{18y}{8x} = \frac{x}{y} \Rightarrow 18y^2 = 8x^2$$

$$x^2 = \frac{9y^2}{4} \Rightarrow x = \pm\frac{3y}{2}$$

$$\frac{3y^2}{2} = 4 \Rightarrow 3y^2 = 8 \Rightarrow y^2 = \frac{8}{3} \Rightarrow y = \pm\frac{2\sqrt{6}}{3}$$

$$x = \frac{3\left(\frac{2\sqrt{6}}{3}\right)}{2} = \sqrt{6}$$

$$\left(\sqrt{6}, \frac{2\sqrt{6}}{3}\right) \quad \left(-\sqrt{6}, -\frac{2\sqrt{6}}{3}\right)$$

$4x^2 + 9y^2$

$4(\sqrt{6})^2 + 9\left(\frac{2\sqrt{6}}{3}\right)^2$

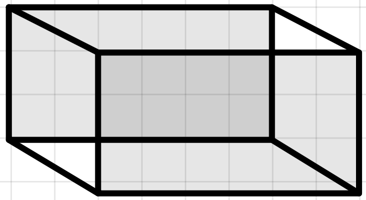
$= 48$

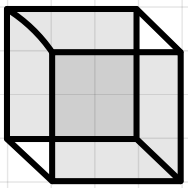
$= 48$

max

No min

c.)





$\frac{300}{12}$ = 25cm each edge

$25 \cdot 25 \cdot 25 = 15{,}625 \text{cm}^3$