Volumes of n-balls.

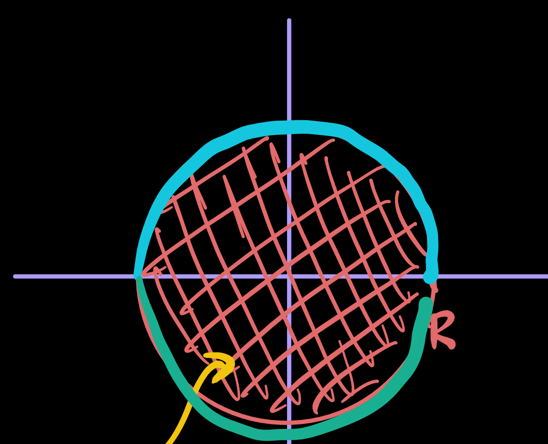
The n-dimensional generalization of a circle, sphere, ...

Treat.

I. Filled in "Radius R" **Circle.** = The collection of points (x,y) in $\mathbb{R}^2$ whose distance (to origin) is $\leq R$

$$\sqrt{x^2+y^2} = R$$

$$x^2+y^2 = R^2$$



"2-ball".

Last time: Calculated "how much" is in this 2-ball: πR^2 (Unit circle $R=1$ gives π.)

$$\boxed{\int_{-\text{🎄}}^{\text{🎄}}\int_{-\sqrt{\text{🎄}^2-x^2}}^{\sqrt{\text{🎄}^2-x^2}} 1 \, dy \, dx = \pi\, \text{🎄}^2}$$

Christmas Tree identity.

II. Volume of the 3-ball

"The ball".

Edge:

$$\boxed{x^2+y^2+z^2=R^2}$$

Do: $\iiint 1 \, dV$

$$\int_{-\star}^{\star}\int_{-\sqrt{\star^2-x^2}}^{\sqrt{\star^2-x^2}}\int_{-\sqrt{\star^2-x^2-y^2}}^{\sqrt{\star^2-x^2-y^2}} 1 \; dz \; dy \; dx = \frac{4}{3}\pi\boxed{\star}^3$$

STAR Identity.

Christmas-Tree-Identity:

Shortcuts to:

$$\int_{-R}^{R} \pi\,\text{🎄}^2 \; dx$$

$$= \int_{-R}^{R} \pi\left(\sqrt{R^2-x^2}\right)^2 dx$$

$$= \int_{-R}^{R} \pi \left(R^2 - x^2\right) dx$$

$$\stackrel{\star}{=} \pi R^2 x - \frac{\pi x^3}{3} \Bigg|_{x=-R}^{x=R}$$

$$= \left[\pi R^3 - \frac{\pi}{3} R^3\right] - \left[-\pi R^3 + \frac{\pi R^3}{3}\right]$$

$$= 2\pi R^3 - \frac{2\pi}{3} R^3 = \boxed{\frac{4}{3}\pi R^3}$$

III. How much space does a $\underbrace{\text{4-ball}}$ take up?

(x, y, z, w)'s

$$\sqrt{x^2 + y^2 + z^2 + w^2} \leq R$$

$$x^2+y^2+z^2+w^2 = R^2 \leftarrow \text{equation of the "Shell"}$$

$$\iiiint 1 \, dV$$

over the 4-ball.

$$\int_{-R}^{R} \boxed{\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} \int_{-\sqrt{R^2-x^2-y^2-z^2}}^{\sqrt{R^2-x^2-y^2-z^2}} 1 \, dw \, dz \, dy} \, dx$$

$$\star = \sqrt{R^2-x^2}$$

Don't reinvent wheel

$$= \int_{-R}^{R} \frac{4}{3}\pi\left(\sqrt{R^2-x^2}\right)^3 dx$$

$$x = R\sin\theta$$

$$dx = \underbrace{R\cos\theta\, d\theta}$$

$$= \int_{-\pi/2}^{\pi/2} \frac{4}{3}\pi R^3 \cos^3\theta \cdot R\cos\theta\, d\theta$$

$$= \frac{4}{3}\pi R^4 \underbrace{\int_{-\pi/2}^{\pi/2} \cos^4\theta \; d\theta}_{\boxed{3\pi/8}}$$

$$= \cancel{4}/\cancel{3}\,\pi \cdot \frac{\cancel{3}\pi}{\cancel{8}} \cdot R^4 = \boxed{\frac{\pi^2}{2} R^4}$$

Next...

$$\iiiint\!\!\int_{\text{radius } R \text{ 5-ball}} 1 \; d\text{"}V\text{"}$$

Iterated thing...

$$\int_{-R}^{R} \frac{\pi^2}{2}\left(\sqrt{R^2-x^2}\right)^4 dx$$

$$= \frac{\pi^2}{2}\int_{-R}^{R} \left(R^4-2R^2x^2+x^4\right) dx$$

$$\star = \frac{\pi^2}{2}\left[R^4x - \frac{2R^2x^3}{3} + \frac{x^5}{5} \Bigg|_{x=-R}^{\underline{\underline{x=R}}} \right]$$

$$= \frac{\pi^2}{\cancel{2}}\cdot \cancel{2}\left[R^5 - \frac{2R^5}{3} + \frac{R^5}{5}\right]$$

$$= \pi^2\cdot\left[\frac{8}{15}\right]R^5$$

$$\frac{8\pi^2}{15} R^5$$

Yes, we know the formulas.

List unit ball sizes (volumes):

Largest!!!

$$2, \quad \pi, \quad \frac{4}{3}\pi, \quad \pi^2/2, \quad \frac{8\pi^2}{15}, \quad \frac{\pi^3}{6}, \quad \frac{16\pi^3}{105}$$

"1-ball" 2-ball 3-ball 4-ball 5-ball 6-ball 7-ball