MATH 2163
Fall 2021
Exam 1
9/17/21
Time Limit: 1 hour

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This exam contains 7 pages (including this cover page) and 5 problems. Check to see if any pages are missing.

No notes, no calculators, only use your brain. You may direct-message me on Discord for clarifications, though I may refuse to answer some questions. **You should show your work whenever possible, so that I believe that you did not cheat.**
You may **not** consult the internet or textbooks or other people.

You are required to show your work on each problem on this exam. The following rules apply:

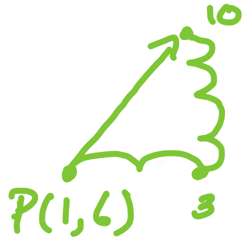
- **Organize your work**, in a reasonably neat and coherent way, in the space provided.
- **Unsupported answers will not receive full credit**.
- **If you are quarantining, hope you get better and all that. But submit your solutions all together as a single file on Canvas.**

Do not write in the table to the right.

Good luck, and make your professor PROUD!

Problem	Points	Score
1	12	
2	12	
3	8	
4	10	
5	8	
Total:	50	

1. (12 points) (a) What is the terminal point Q of the vector $\mathbf{v} = \langle 2, 4\rangle$ based at $P = (1, 6)$?



$\boxed{T = (3, 10)}$

$(1+2, 6+4) = (3, 10)$

(b) The vector $\langle 1, 7\rangle$ is parallel to the vector $\langle 5, ?\rangle$. Find the question mark.

$C \cdot \vec{v} = \vec{w}$ $C = 5$ $1 \cdot 5 = 5$

$5 \langle 1, 7 \rangle = \boxed{\langle 5, 35 \rangle}$

(c) Do all three points $(2, 4), (3, 6)$ and $(5, 9)$ lie on the same line? $\boxed{\text{NO}}$

$A = (2,4)$ $B = (3,6)$ $C = (5,9)$

$\vec{AB} = \langle 1, 2 \rangle$ $\vec{BC} = \langle 2, 3 \rangle$ $\vec{CA} = \langle 3, 5 \rangle$

$\|\vec{AB}\| = \sqrt{5}$ $\|\vec{BC}\| = \sqrt{13}$ $\|\vec{CA}\| = \sqrt{34}$ $CA = AB + BC$

(d) Find the unit vector in the direction of $\langle 2, 3, 5\rangle$.

$\vec{v} = \langle 2, 3, 5 \rangle$ $\frac{1}{\|\vec{v}\|} \cdot \vec{v} = \frac{1}{\sqrt{38}} \cdot \langle 2, 3, 5 \rangle$

$\|\vec{v}\| = \sqrt{4+9+25} = \sqrt{38}$ $\boxed{= \left\langle \frac{2}{\sqrt{38}}, \frac{3}{\sqrt{38}}, \frac{5}{\sqrt{38}} \right\rangle}$

(e) Find the vector $\vec{PQ}$. $P = (-1, -3, 5), Q = (1, -5, 5)$.

$\vec{PQ} = \langle (1+1), (-5+3), (5-5) \rangle = \boxed{\langle 2, -2, 0 \rangle}$

(f) Find b and c which make the vector $\langle 2, 3, 5\rangle$ parallel to $\langle 3, b, c\rangle$.

$\langle 2, 3, 5 \rangle \cdot C = \langle 3, b, c \rangle = \boxed{\left\langle 3, \frac{9}{2}, \frac{15}{2} \right\rangle}$

$2 \cdot C = 3 \Rightarrow C = \frac{3}{2}$ $\boxed{\begin{array}{l} b = 3 \cdot \frac{3}{2} = \frac{9}{2} \\ c = 5 \cdot \frac{3}{2} = \frac{15}{2} \end{array}}$

2. (12 points) (a) Find a parametrization of the line joining the points $(-3, 0, 1)$ and $(3, 5, 5)$.

$P = (-3, 0, 1)$; $Q = (3, 5, 5)$ $\boxed{r(t) = \langle -3+6t, 5t, 1+4t \rangle}$

$\overrightarrow{PQ} = \langle 6, 5, 4 \rangle$ $r(t) = (-3, 0, 1) + t\langle 6, 5, 4 \rangle$

(b) Do the lines $r(t) = (1 + t, 4 - 2t, 3t)$ and $s(t) = (2 - t, 2 + 15t, 3 - 2t)$ intersect? If yes, where. If no, why.

$r(t) = r(s)$ $s(t) = r(s) = (2-s, 2+15s, 3-2s)$

$(1+t, 4-2t, 3t) = (2-s, 2+15s, 3-2s)$

$1+t = 2-s$ ① ✓ $s=2$ $1+t=0$ $1-1 = 2-s$ $4-2(-1) = 2+15(2)$

$4-2t = 2+15s$ ② ✗ $t=-1$ $s=2$ $6 = 2+30 = 6 = 32$

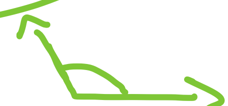
$3t = 3-2s$ ③

$\boxed{\text{t + s do Not Work for All Equations So they don't intersect}}$

(c) Compute the dot product $\langle 0, 1, 3\rangle \cdot \langle -2, -2, 0\rangle$ Are these two vectors perpendicular?

$\langle 0, 1, 3 \rangle \cdot \langle -2, -2, 0 \rangle = 0(-2) + 1(-2) + 3(0) = -2$

$\boxed{\text{No, they are not } \perp.}$



(d) Do the vectors $\underset{\vec{v}}{\langle 2, 3, 4\rangle}$ and $\underset{\vec{w}}{\langle -4, 2, 1\rangle}$ form an acute angle? Why.

$\langle 2, 3, 4 \rangle \cdot \langle -4, 2, 1 \rangle = 2(-4) + 3(2) + 4(1) = -8 + 6 + 4 = 2$

$\boxed{\text{yes, because } \vec{v} \cdot \vec{w} > 0 \text{ forming an acute angle.}}$

(e) $\mathbf{u}, \mathbf{v}$ are two vectors in space, with magnitudes 3 and 7 respectively. The angle between them is $\pi/4$ radians. What is their dot product?

$\frac{\pi}{4}$ is an acute angle $\boxed{\vec{u} \cdot \vec{v} = 10}$

So $\vec{u} \cdot \vec{v} > 0$

(f) Calculate the 2×2 determinant of $\begin{bmatrix} 1 & 7 \\ 8 & 2 \end{bmatrix}$.

$(1 \times 2) - (8 \times 7) = 2 - 56 = \boxed{-54}$

3. (8 points) (a) Find the volume of the parallelopiped created by the three vectors $\langle 1,0,0\rangle, \langle 1,2,3\rangle, \langle 2,0,3\rangle$. (Remember what determinants measure?)

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 3 \\ 2 & 0 & 3 \end{vmatrix} = 1(6) - 0(3-6) + 0(-4) = 6$$

(b) Calculate $\mathbf{v} \times \mathbf{w}$ if $\mathbf{v} = \langle 4,3,2\rangle, \mathbf{w} = \langle 1,-1,2\rangle$.

$$\begin{vmatrix} i & j & k \\ 4 & 3 & 2 \\ 1 & -1 & 2 \end{vmatrix} = i(6+2) - j(8-2) + k(-4-3)$$

$$\boxed{\begin{array}{ccc} i & j & k \\ = \langle 8, & -6, & -7\rangle \end{array}}$$

(c) Calculate $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})$ if $\mathbf{u} = \langle 2020, 2021, 2022\rangle$ and $\mathbf{v} = \langle 69, 420, 80085\rangle$.

$$= (2020((2021 \cdot 80085) - (2022 \cdot 420))$$

$$+ (2021((2020 \cdot 80085) - (69 \cdot 2022))$$

$$+ (2022((2020 \cdot 420) - (69 \cdot 2021)) = \text{Some Large Number}$$

8 See Scratwork for Work. 8/

(d) Find an equation ($ax + by + cz = d$ style) of the plane passing through the three points given. $P = (3,-1,2), Q = (1,1,1), R = (4,1,-4)$.

$$a = \overrightarrow{PQ} = \langle -2, 2, -1\rangle$$

$$b = \overrightarrow{PR} = \langle 1, 2, -6\rangle$$

$$a \times b = \begin{vmatrix} i & j & k \\ -2 & 2 & -1 \\ 1 & 2 & -6 \end{vmatrix}$$

$$= i(-10) - j(13) + k(-6)$$

$$= \langle -10, -13, -6\rangle$$

$$\langle -10, -13, -6\rangle \cdot \langle 3, -1, 2\rangle$$

$$= -30 + 13 - 12 = -29 = 29$$

$$\boxed{-10x - 13y - 6z = 29}$$

4. (10 points) (a) Find parametric equations for the line through $P_0 = (2, -1, 1)$ perpendicular to the plane $2x+5y-3z = 34$. (Enter your answers as a comma-separated list of equations.)

direction Vector $\vec{n} = \langle 2, 5, -3 \rangle$

$r(t) = (2, -1, 1) + t\langle 2, 5, -3 \rangle$ $\boxed{\langle 2+2t, -1+5t, 1-3t \rangle}$

(b) Find a parametrization of the horizontal circle (meaning, parallel to (x, y)-plane) of radius 7 with center $(1, -9, 6)$.

$x = r\cos\theta \quad = 1 + 7\cos\theta$

$y = r\sin\theta \quad = -9 + 7\cos\theta$

$z = z \quad = 6$

$\boxed{\langle 1+7\cos(t), -9+\sin(t), 6 \rangle}$

(c) Convert from rectangular to cylindrical coordinates: $(4, 4, 2021)$.

$r^2 = x^2+y^2 \qquad r = \sqrt{16+16} = \sqrt{32} = 4\sqrt{2}$

$\tan\theta = \frac{y}{x} \qquad \theta = \tan^{-1}\left(\frac{4}{4}\right) = \frac{\pi}{4}$

$z = z \qquad z = 2021$

$\boxed{\left(4\sqrt{2}, \frac{\pi}{4}, 2021\right)}$

(d) Convert from rectangular to spherical coordinates: $(\sqrt{3}/2, 3/2, -1)$.

$\rho^2 = x^2+y^2+z^2 \quad = \sqrt{x^2+y^2+z^2} = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2+\left(\frac{3}{2}\right)^2+1} = 2$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) \quad = \tan^{-1}\left(\frac{3/2}{\sqrt{3}/2}\right) = \frac{\pi}{3}$

$\varphi = \cos^{-1}\left(\frac{z}{\rho}\right) \quad = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

$\boxed{\left(2, \frac{\pi}{3}, \frac{2\pi}{3}\right)}$

(e) Write the equation of the sphere of radius 10 in spherical coordinates.

5. (8 points) (a) Compute the derivative of: $r(t) = (t, t^6, t^4)$.

$r'(t) = \langle 1, 6t^5, 4t^3 \rangle$

(b) Compute the tangent vector for $r(t) = (t^4, 2t^3)$ at $t = 2$. Then, parametrize the tangent line of the path $r(t)$ at that point.

$r'(t) = \langle 4t^3, 6t^2 \rangle$

$r'(2) = \langle 4(2)^3, 6(2)^2 \rangle = \boxed{\langle 32, 24 \rangle}$

(c) Compute the exact arc-length of the path $r(t) = (4t^{1/2}, \ln(t), 2t)$ between times $t = 1$ and $t = 2$. (Make sure you clearly set up the calculation, so I can give some points, even if you can't execute the whole thing.)

$r'(t) = \langle \frac{2}{x^{1/2}}, \frac{1}{t}, 2 \rangle$

$\int_1^2 ||r'(t)|| dt$

$||r'(t)|| = \sqrt{\left(\frac{2}{t^{1/2}}\right)^2 + \left(\frac{1}{t}\right)^2 + (2)^2}$

$= \left(\frac{4}{t}\right) + \frac{1}{t^2} + 4$

$\int_1^2 \sqrt{\frac{4}{t} + \frac{1}{t^2} + 4}\, dt$

Scratchwork, if necessary: 3c.

$u \times v$

$$\begin{vmatrix} i & j & k \\ 2020 & 2021 & 2022 \\ 69 & 420 & 80085 \end{vmatrix} = i((2021\cdot 80085)-(2022\cdot 420)) - j((2020\cdot 80085)-(69\cdot 2022)) + k((2020\cdot 420)-(69\cdot 2021))$$

$$\langle 2020, 2021, 2022\rangle \cdot \langle (2021\cdot 80085)-(2022\cdot 420), (2020\cdot 80085)-(69\cdot 2022), (2020\cdot 420)-(69\cdot 2021)\rangle$$

$$= (2020((2021\cdot 80085)-(2022\cdot 420))$$

$$+ (2021((2020\cdot 80085)-(69\cdot 2022))$$

$$+ (2022((2020\cdot 420)-(69\cdot 2021)) = \text{Some Large Number}$$