Integration

Integration BootCamp
Next Week.
MWF.

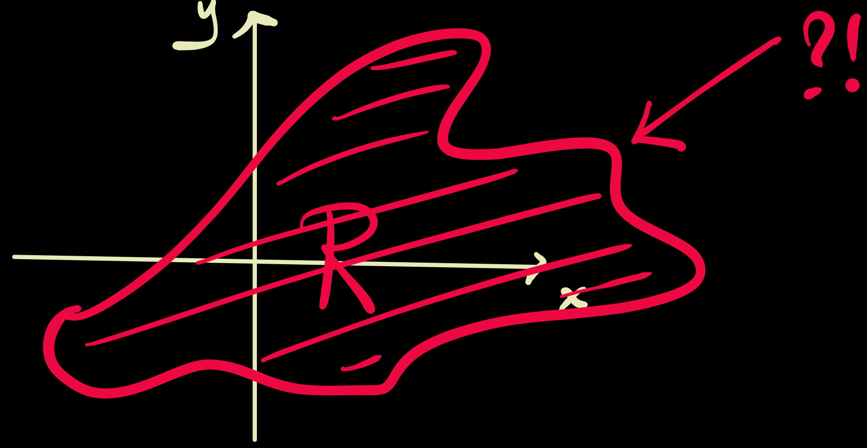
I. Reminder.

II. Easy integrals.
(Fubini's Theorem.)

III. What makes multi-variable integral Tricky?

I. Situation:

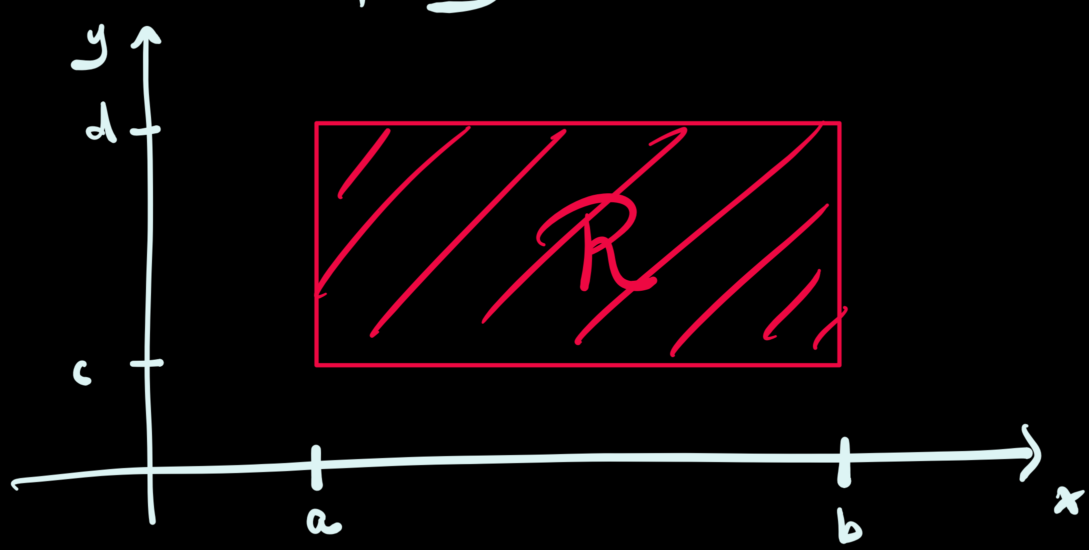
- $f(x,y)$ 2 variable fn.
- Region $R \subset \mathbb{R}^2$



- $$\boxed{\iint_R f(x,y)\, dA}$$

How?

II. Simplify situation:



$$\text{"}\underbrace{[a,b]}_{x} \times \underbrace{[c,d]}_{y}\text{"} \leftarrow \text{how we describe a rectangle.}$$

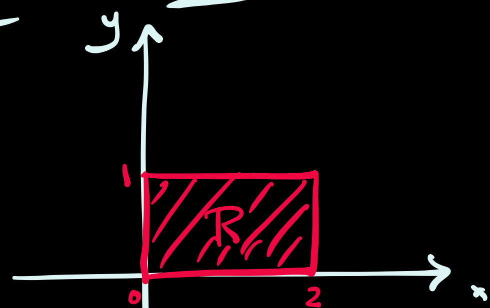
How:

$$\iint_{\text{Rectangle}} f(x,y)\, dA$$

Fubini: You break it up as an "iterated" integral. / You integrate "1-variable at a time".

Ex:

$$\boxed{\iint_{\text{Rectangle}} x^2+y^2 \, dA} \leftarrow \boxed{10/3}.$$



"$[0,2] \times [0,1]$"

$$\int_0^1 \left(\int_0^2 x^2 + y^2 \, dx \right) dy$$

Stage 1:

$$\int_0^2 (x^2+y^2)\, dx$$

$$\left. \frac{x^3}{3} + y^2 x \right|_{x=0}^{x=2}$$

$$\left[\frac{2^3}{3} + y^2(2) \right] - \left[\underbrace{\frac{0^3}{3} + y^2 \cdot (0)}_{0} \right]$$

$$= \underbrace{\frac{8}{3} + 2y^2}_{\text{Stage 1.}}$$

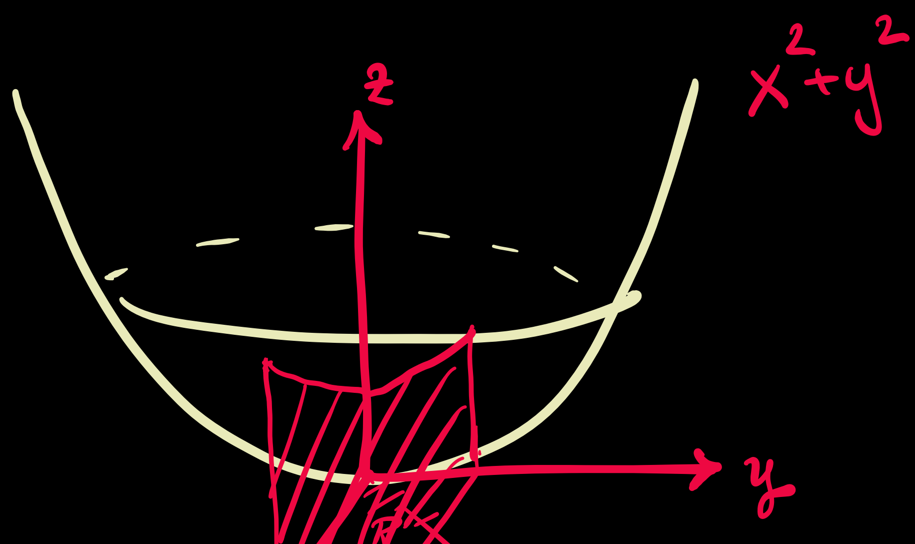
Stage 2:

$$\int_0^1 \frac{8}{3} + 2y^2 \, dy$$

$$\stackrel{\star}{=} \frac{8}{3}y + \frac{2y^3}{3} \Bigg|_{y=0}^{y=1}$$

$$= \left[\frac{8}{3} + \frac{2}{3}\right] - \left[0 + 0\right]$$

$$= \boxed{\frac{10}{3}}$$



Volume of that thing = $\boxed{10/3}$

Ex: Compute:

$$\iint\limits_{[-1,1]\times[-2,3]} \overset{\text{"Hill"}}{\boxed{e^{x}\sin(y)}}\, dA$$

Fubini:

$$\int_{-1}^{1} \underbrace{\left(\int_{-2}^{3} e^{x}\sin(y)\, dy \right)}_{\text{Stage 1:}} dx$$

$$\int_{-2}^{3} e^{x}\sin(y)\, dy$$

$$\overset{\star}{=} \quad e^{x} \cdot (-\cos(y)) \Big|_{y=-2}^{y=3}$$

$$e^{x}(-\cos(3)) - e^{x}(-\cos(-2))$$

$$= \underbrace{e^{x}\left[-\cos(3) + \cos(-2)\right]}_{\text{End stage 1.}}$$

<u>Stage II</u>:

$$\int_{-1}^{1} e^{x} \cdot \left[-\cos(3) + \cos(-2)\right] \, dx$$

$$\overset{\star}{=} \quad e^{x} \cdot \left[-\cos(3) + \cos(-2)\right] \Big|_{x=-1}^{x=1}$$

$$= \boxed{e\left[-\cos(3)+\cos(-2)\right] - e^{-1}\left[-\cos(3)+\cos(-2)\right]}$$

By some miracle: Switching the order yields same answer.

BTW:

$$\iiint_B \boxed{x^2+y^2+z^3}\, dV = \boxed{\frac{16}{3}}.$$

$$B = [0,1]\times[1,2]\times[-1,1]$$

$$\int_{-1}^{1}\left(\int_{1}^{2}\left(\int_{0}^{1} x^2+y^2+z^3 \, dx\right) dy\right) dz$$

3 Stages:

$$S1: \quad \frac{x^3}{3} + y^2x + z^3x \Big|_{x=0}^{x=1}$$

$$= \left(\frac{1}{3} + y^2 + z^3\right)$$

$$S2: \quad \int_1^2 \frac{1}{3} + y^2 + z^3 \quad dy$$

$$\stackrel{\star}{=} \quad \frac{1}{3}y + \frac{y^3}{3} + z^3y \Big|_{y=1}^{y=2}$$

$$= \left(\frac{2}{3} + \frac{8}{3} + 2z^3\right) - \left(\frac{1}{3} + \frac{1}{3} + z^3\right)$$

$$= 8/3 + z^3$$

S2.

S3:

$$\int_{-1}^{1} 8/3 + z^3 \, dz$$

$$\overset{\star}{=} \; 8/3\, z + \frac{z^4}{4} \Bigg|_{z=1}^{z=1}$$

$$\left(8/3 + \underline{1/4}\right) - \left(-8/3 + \underline{1/4}\right)$$

$$= \boxed{16/3}.$$

III. What makes this any more difficult than just Calc II?

New Spooky Tricky Part:

How to integrate over Non-Boxy Regions ???