Integration over Strange Regions

$\int$ BOOTCAMP MWF @ 8 pm, Discord.

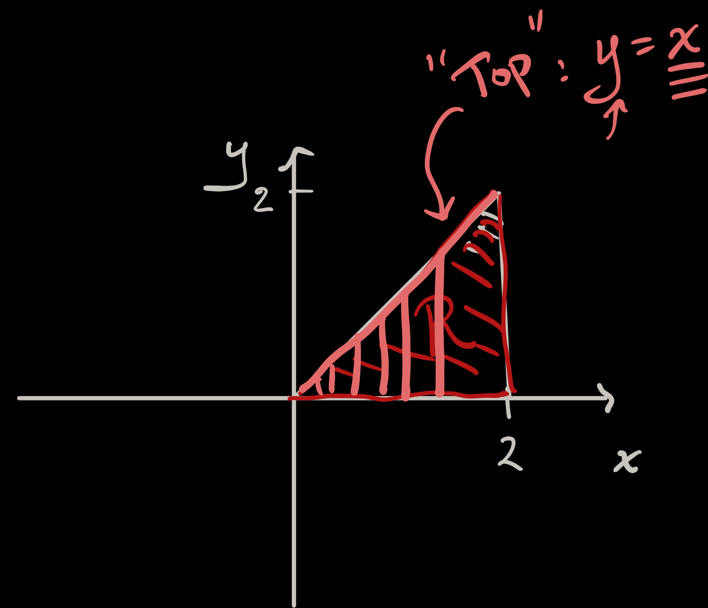
I. Simple example.

II. Medium example.

III. Harder example.

Example: $f(x,y) = xy^2$

$$\iint_R f(x,y)\, dA$$



Fubini still works: Focus 1 variable at a time still works.

$$\int_0^2 \left(\int_0^x \boxed{xy^2}\, dy \right) dx$$

S1

$$\frac{x \cdot y^3}{3}\Bigg|_{y=0}^{y=x}$$

S1:

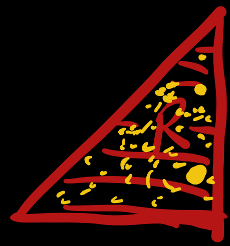
$$\overset{\star}{=} \frac{xy^3}{3}\Bigg|_{y=0}^{y=x}$$

$$= \left[\frac{x^4}{3}\right] - [0] = \boxed{\frac{x^4}{3}}$$

S2:

$$\int_0^2 \frac{x^4}{3}\, dx \overset{\star}{=} \frac{x^5}{15}\Bigg|_{x=0}^{x=2}$$

$$= \frac{2^5}{15} - 0 = \boxed{\frac{32}{15}}.$$



Density Function of Gold

$$xy^2 \cdot \frac{\text{Gold}}{\text{cm}^2}$$

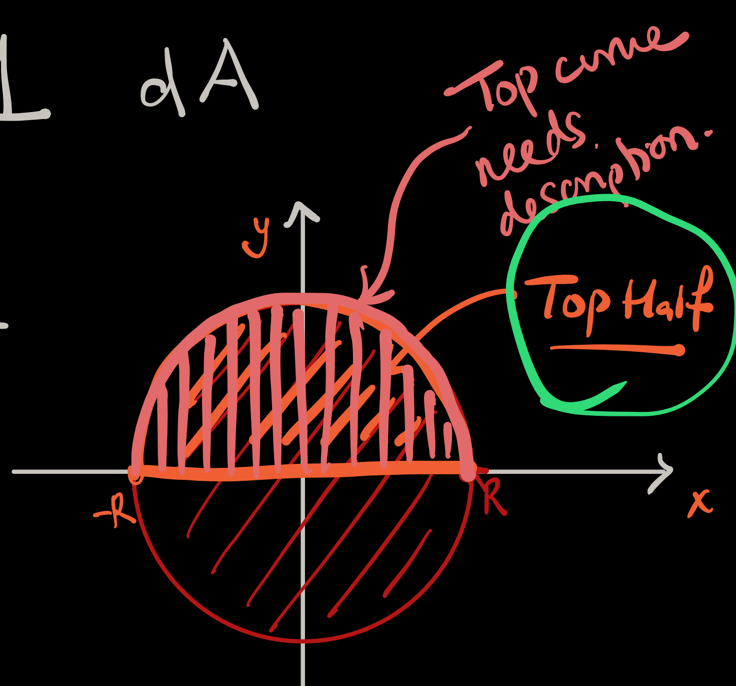
$$\iint = \underline{\text{Total Gold}}.$$

II. Medium Problem:

Compute Area of a circle of radius R.

Trick: $$\iint 1 \, dA$$

 =



$$x^2 + y^2 = R^2$$

$$y^2 = R^2 - x^2$$

R

Fubini says:

$$y = \sqrt{R^2 - x^2}$$

$$\int_{-R} \underbrace{\left(\int_0^{\sqrt{R^2-x^2}} 1 \, dy \right)}_{S1} dx$$

$$S1: \quad \star = y \Big|_{y=0}^{y=\sqrt{R^2-x^2}}$$

$$= \sqrt{R^2-x^2} - 0$$

$$= \boxed{\sqrt{R^2-x^2}}$$

$$S2: \int_{-R}^{R} \sqrt{R^2-x^2} \, dx$$

anti-derivative?

$$\int \sqrt{R^2-x^2}\;\boxed{dx}$$

Substitute:

$$x = R\sin\theta \qquad -\frac{\pi}{2} \text{ to } \frac{\pi}{2}$$

$$dx = R\cos\theta\, d\theta$$

changes into

$$\int \sqrt{R^2 - R^2\sin^2\theta}\cdot R\cos\theta\, d\theta$$

$$= \int R\cdot \underbrace{\sqrt{1-\sin^2\theta}}_{\sqrt{\cos^2\theta}} \cdot R\cdot\cos\theta\, d\theta$$

$$= \int R\cos\theta\cdot R\cdot\cos\theta\, d\theta$$

$$= \int R^2 \cos^2\theta \, d\theta$$

$$= R^2 \int \cos^2\theta \, d\theta$$

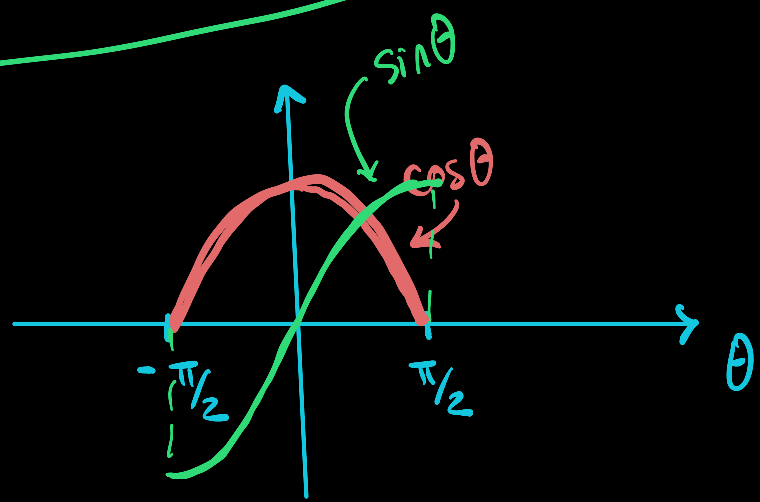
Trig identities which can simplify the situation.

integration by parts can work.

Bootcamp

$$R^2 \boxed{\int_{-\pi/2}^{\pi/2} \cos^2\theta \; d\theta} \neq \frac{1}{2}\pi$$

GENIUS TRICK :



$$\int_{-\pi/2}^{\pi/2} \sin^2\theta \, d\theta = \int_{-\pi/2}^{\pi/2} \cos^2\theta \, d\theta$$

$$A \qquad = \qquad B$$

$$A+B = \int_{-\pi/2}^{\pi/2} 1 \, d\theta = \pi$$

$$B = \frac{1}{2}\pi$$

$$= R^2 \cdot \frac{1}{2}\pi$$

$\Longrightarrow$ Area of circle is $\boxed{\pi R^2.}$

Next Q: What is the Volume of

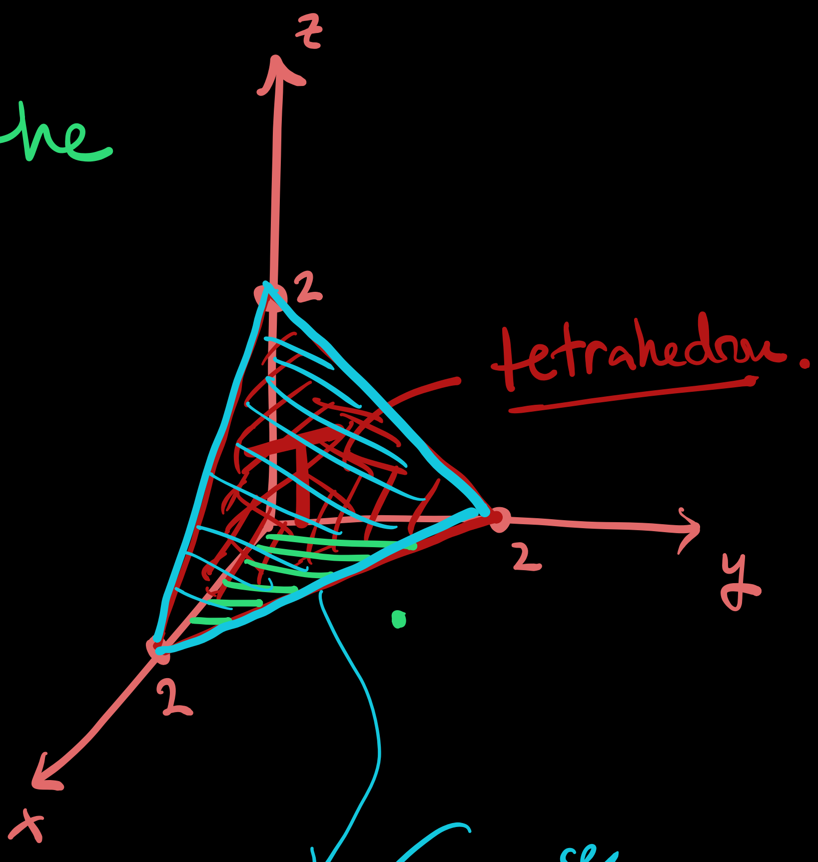
a radius R ball?

$$\int_{?}^{?}\int_{?}^{?}\int_{?}^{?} 1 \, dV$$

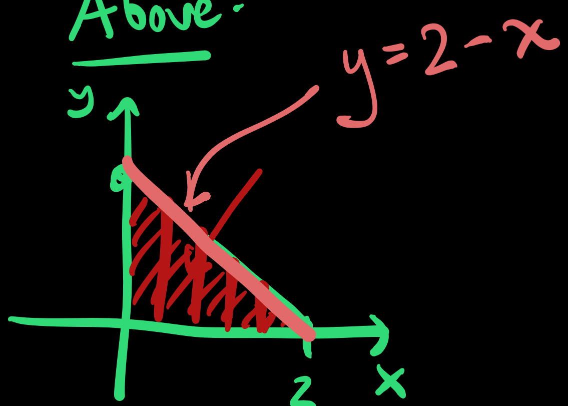
III. Harder example:

Integrate $f(x,y,z) = x+y+z$

Over the



Above:



Fubini:

$$\int_0^2 \left(\int_0^{2-x} \left(\int_0^{???} (x+y+z)\, dz \right) dy \right) dx$$

Suspense