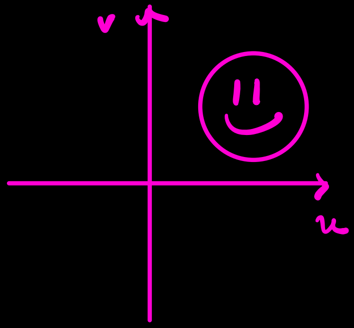
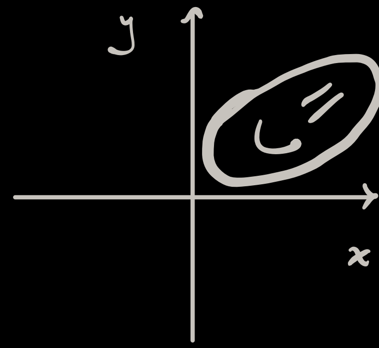


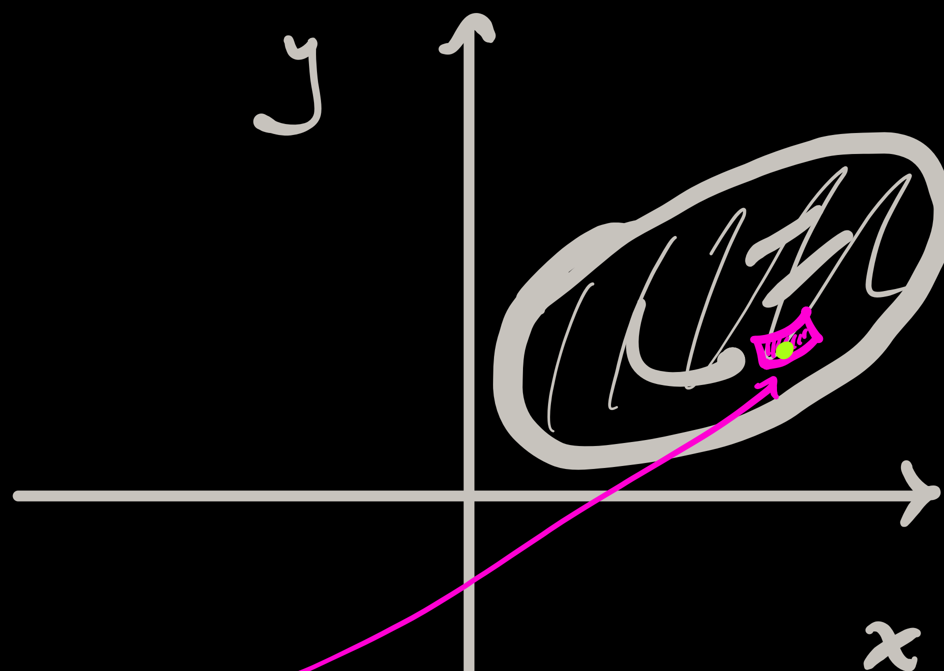
The Jacobian of a change of variables



I. Explanation of where $\mathrm{Jac}(\Phi)$ comes from.

II. Verifying "known" examples.

$$\iint f(x,y)\, dA$$



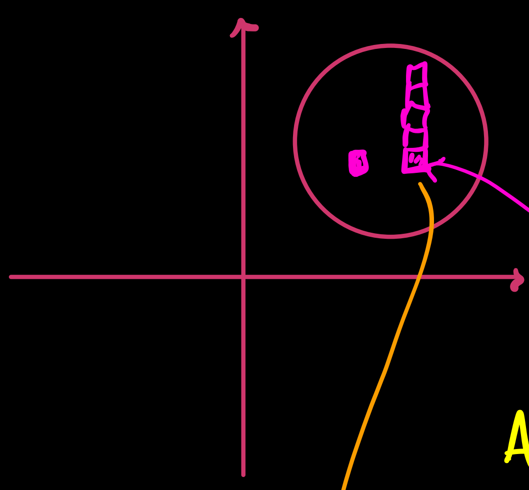


Image:

(Area) × Sampled density.

Add

Sum

$$\underbrace{\text{Area}(\square)}\ \boxed{\frac{\text{Area}(\square)}{\text{Area}(\square)}\ \text{Sampled-density}}\,.$$

$$\text{Jacobian} = \lim_{\substack{\text{sides of small square} \\ \square \to 0}} \frac{\text{Area}(\square)}{\text{Area}(\text{small } \square)}$$

Luckily we've figured out this limit for you:

$$\Phi \quad \begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

Write down:

$$\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \partial y/\partial v \end{vmatrix}$$

Take 2x2 determinant

(Then, whatever comes out, stick abs. value around it!)

Translating to u-v-world:

$$\iint f\left(x(u,v), y(u,v)\right) \cdot \underbrace{\mathrm{Jac}(\Phi)}_{\substack{\text{This} \\ \text{determinant} \\ \text{of} \\ \text{partials}}} \, du\,dv$$

☺

II. What is the "Polar coordinate" Jacobian?

$$\Phi : \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{matrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{matrix}$$

$\shortparallel$

Compute $\text{Jac}(\Phi)$!

$$\det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$\cos\theta\,(r\cos\theta) - \sin\theta\,(-r\sin\theta)$$

$$= r\left[\cos^2\theta + \sin^2\theta\right]$$

$$= r.$$

<u>Test of Mental Fortitude:</u> "Spherical"

Φ:

$$x = \rho \sin\varphi \cos\theta$$

$$y = \rho \sin\varphi \sin\theta$$

$$z = \rho \cos\varphi$$

$$\frac{\partial x}{\partial \rho} \quad \frac{\partial x}{\partial \theta} \quad \frac{\partial x}{\partial \varphi}$$

Compute $\text{Jac}(\Phi)$.

$$\frac{\partial y}{\partial \rho} \quad \frac{\partial y}{\partial \theta} \quad \frac{\partial y}{\partial \varphi}$$

$$\frac{\partial z}{\partial \rho} \qquad \frac{\partial z}{\partial \theta} \qquad \frac{\partial z}{\partial \varphi}$$

$$\sin\varphi\cos\theta \qquad -\rho\sin\varphi\sin\theta \qquad \rho\cos\varphi\cos\theta$$

$$\sin\varphi\sin\theta \qquad \rho\sin\varphi\cos\theta \qquad \rho\cos\varphi\sin\theta$$

$$\rightarrow \quad \cos\varphi \qquad 0 \qquad -\rho\sin\varphi$$

$$\cos\varphi\left[-\rho^2\sin\varphi\cos\varphi\sin^2\theta - \rho^2\sin\varphi\cos\varphi\cos^2\theta\right] \leftarrow \text{opportunity.}$$

$$-\rho^2\sin\varphi\cos^2\varphi\,\underbrace{\cancel{\left[\sin^2\theta+\cos^2\theta\right]}}_{1}$$

$$-0\cdot[\sim]$$

$$+\,-\rho\sin\varphi\left[\rho\sin^2\varphi\cos^2\theta + \rho\sin^2\varphi\sin^2\theta\right]$$

$$-\rho^2\sin\varphi\cdot\sin^2\varphi\,\underbrace{\cancel{\left[\cos^2\theta+\sin^2\theta\right]}}_{1}$$

$$\ldots\ -\rho^2\sin\varphi\,\underbrace{\cancel{\left[\cos^2\varphi+\sin^2\varphi\right]}}_{1}$$

Our determinant.

Switch it positive!

$$\boxed{\rho^2 \sin\varphi}$$