Applications of Integration

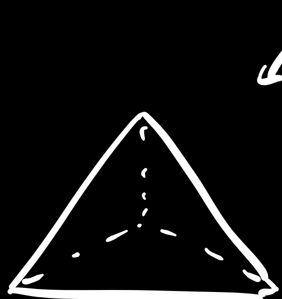
I. Gold.

II. Average Value of $f(x,y)$ over R?

III. What's the Middle of a shape?

I. Gold.

$$\iiint_T f(x,y,z)\, dV$$

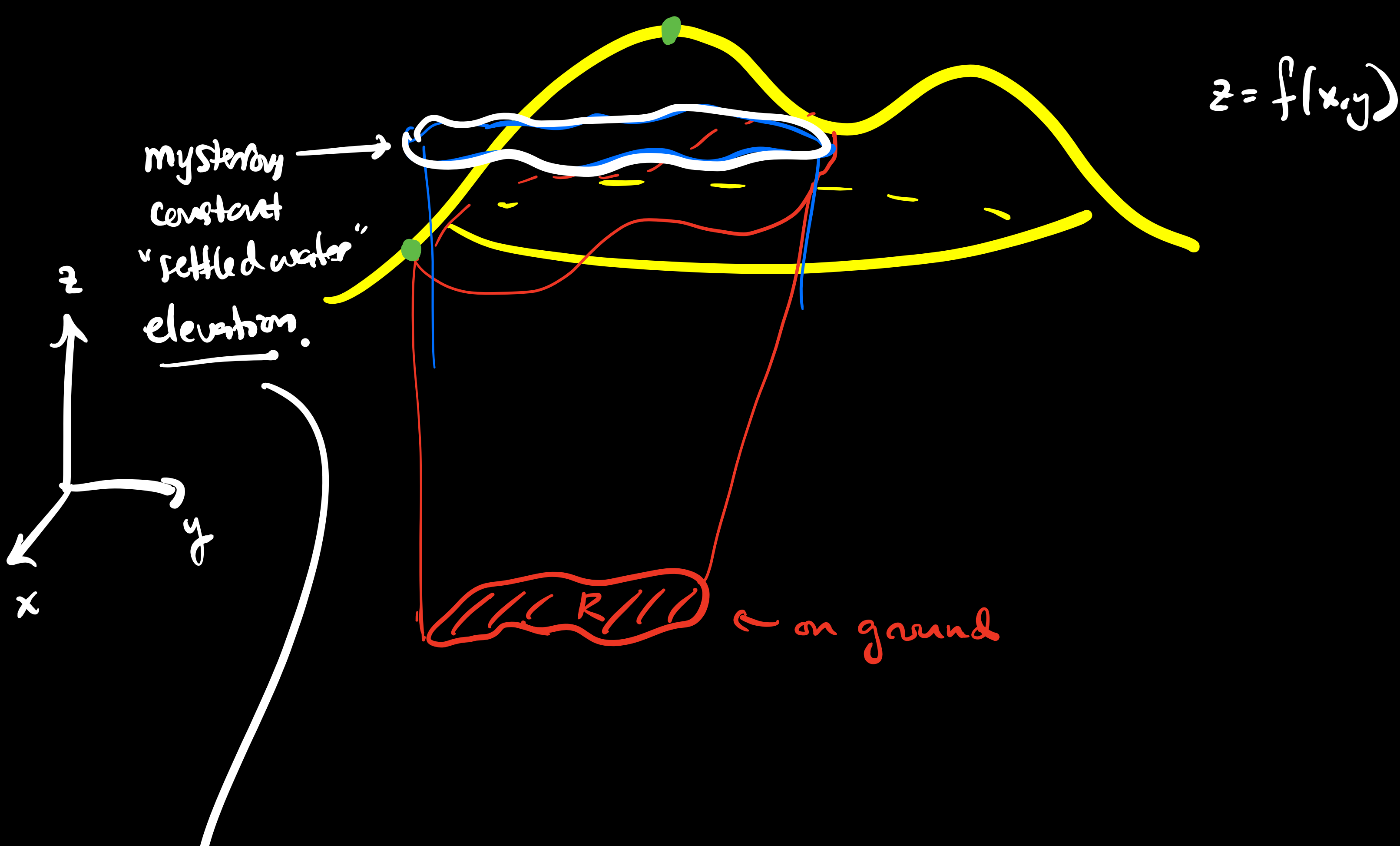


"mass density function: mass/cm^3.

Then, this integral is the Total Mass.

II. Suppose $f(x,y)$ is a function, and R is a region in $\mathbb{R}^2$.

Q: If you had to pick ONE representative output for the function on R, which output should you pick?



$$\text{Average} = \frac{\iint_R f(x,y)\,dA}{\text{Area of } (R)}$$

IF $g(x,y,z)$ is a function, + T is some solid in $\mathbb{R}^3$,

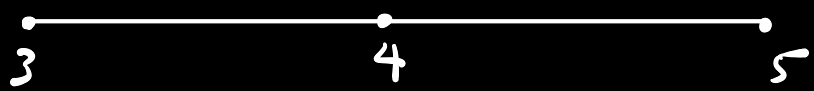
then

$$\frac{\iiint_T g(x,y,z)\,dV}{\text{Volume } (T)}$$

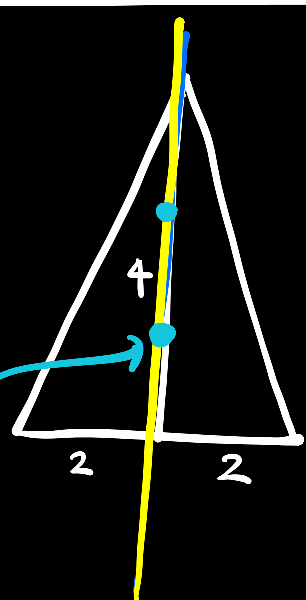
Is the Average value of $g(x,y,z)$ over T.

III. Using this Average-value concept, we can answer:

Q: What is the middle of a shape?

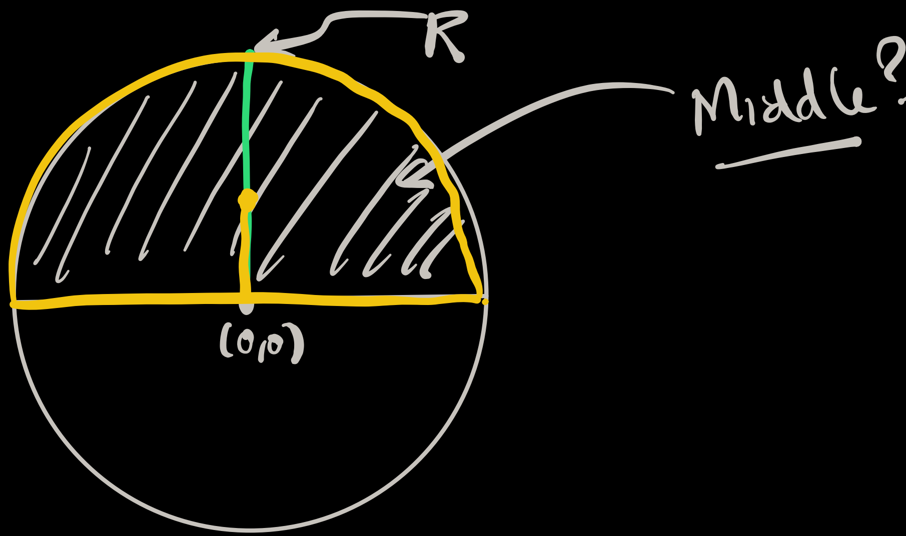


2D.



This is the middle, it turns out.

Isosceles Triangle.



$\frac{4}{3\pi}$ th the way up exactly.

Compute: What is the Average y-coord?

$$\frac{\iint y \, dA}{\iint 1 \, dA} \qquad \iint 1 \, dA = \left(\tfrac{1}{2}\pi R^2\right)$$

$$\iint y \, dA$$

$$\int^{R} \int^{\sqrt{R^2-x^2}} y \quad dy \quad dx$$

$-R \quad 0$

S1: ☆

$$\frac{y^2}{2}\Bigg|_{y=0}^{y=\sqrt{R^2-x^2}}$$

$$= \boxed{\frac{R^2-x^2}{2}} - 0$$

S2:

$$\int_{-R}^{R} \frac{R^2-x^2}{2}\,dx$$

$$= \frac{R^2}{2} - \frac{x^2}{2}$$

$$\overset{☆}{=} \frac{R^2 x}{2} - \frac{x^3}{6}\Bigg|_{x=-R}^{x=R}$$

$= 2 \; R^3 \left(\frac{1}{3}\right)$

$= \frac{2}{3} R^3 =$ Numerator of Average value.

$$\frac{\frac{2}{3} R^3}{\frac{1}{2} \pi R^2} = \boxed{\frac{4}{3\pi} \cdot R}$$

"Centroid" of a 2D shape.

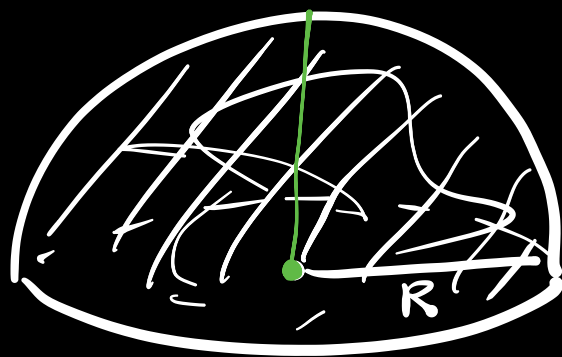
x-coord: $\dfrac{\iint_{\text{Shape}} x \, dA}{\text{Area of Shape.}}$

y-coord: $\iint_{\text{Shape}} y \, dA$

Area of Shape

3D shapes have <u>centers</u> too!

<u>Ex:</u>



Where is the middle?

Don't waste time on magic x or y.

Average (z)?

$$\frac{\iiint z \, dV}{\iiint 1 \, dV}$$

$$\iiint 1 \, dV = \frac{1}{2}\left(\frac{4}{3}\pi R^3\right)$$

$$\iiint z \, dV = \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \int_{0}^{\sqrt{R^2-x^2-y^2}} z \; dz \; dy \; dx$$

$\underline{S1}$: $\overset{\star}{=}$ $\left.\dfrac{z^2}{2}\right|_{z=0}^{z=\sqrt{R^2-x^2-y^2}}$

$$= \frac{R^2-x^2-y^2}{2} \quad (-0)$$

$$\int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{R^2-x^2-y^2}{2} \, dy \, dx$$

$-(x^2+y^2) = -r^2$

Switch to Polar!

$$\int_0^{2\pi} \left[\int_0^{R} \frac{(R^2-r^2)}{2} r \, dr \right] d\theta$$

$$\frac{R^2 r}{2} - \frac{r^3}{2}$$

$$\overset{\star}{=} \left. \frac{R^2 \cdot r^2}{4} - \frac{r^4}{8} \right|^{r=R}$$

$\Big|_{r=0}$

$$\frac{R^4}{4} - \frac{R^4}{8} - 0$$

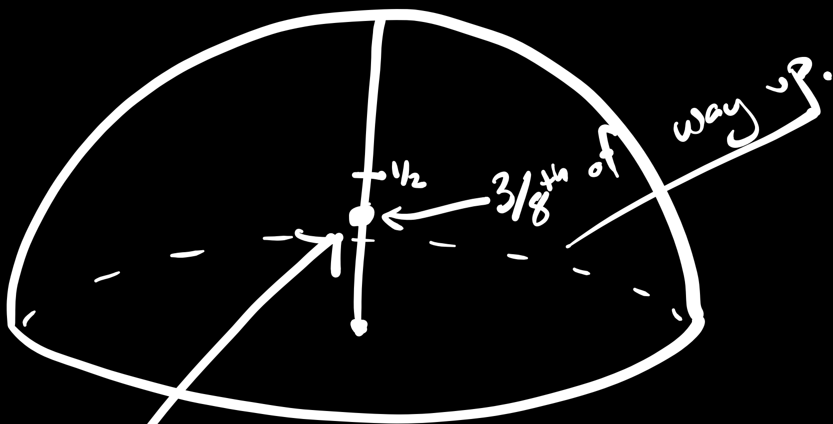
$$= \frac{1}{8} R^4$$

S2: $\displaystyle\int_0^{2\pi} \frac{1}{8} R^4 \, d\theta$

$\overset{\star}{=}$ $\displaystyle \frac{1}{8} R^4 \theta \Big|_{\theta=0}^{\theta=2\pi}$

$$= \frac{1}{8} R^4 \cdot (2\pi) = \frac{\pi}{4} R^4 . = \underline{\text{Num.}}$$

$$\frac{\pi/4 \, R^4}{1/2 \; 4/3 \, \pi R^3} = \frac{\frac{1}{4}}{\cdot 2/3} = 3/8 \, R$$



Was first deduced by Archimedes ~ <u>200 BCE</u>.