Étale Cohomology 1
The Weil Conjectures

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Introduction

The Weil conjectures are a prime of example of how math is all about asking the right questions. Posed in 1949 by Andre Weil and finally proven in full by Deligne in 1974, these conjectures required many years of development and teamwork, bringing ideas from all over mathematics, and sprouting ideas which are still impactful to this day.

The main reference for this paper is Milne's notes [Mil13], with a little help from [Mil80], [FK88], and Deligne's original article [Del74].

Prerequisites for reading the paper include basic algebraic geometry, sheaf cohomology ([Har77]) and basic algebraic number theory ([Neu99]). We also assume a familiarity with Galois cohomology (see [Ser97] or [Ber10]) as well as basic homological algebra (see [Rot09]). The plan is to provide most of the proofs, and to outline the ideas behind the proofs we don't provide.

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Contents

1 Preliminary Considerations

1.1 The History of the Weil Conjectures

As with many mathematical ideas, the first motivations for study that would lead to the Weil Conjectures was Gauss' work on the laws of reciprocity. He introduced certain sums, called *Gauss sums*, tools for determining properties of number fields and reciprocity laws, and in the study of these sums he was led to evaluate the number of solutions of certain algebraic equations modulo primes, e.g.

$$ax^3 - by^3 \equiv 1 \pmod p, \qquad ax^4 - by^4 \equiv 1 \quad (\text{mod } ()p)$$

where $a, b \in \mathbb{F}_p$ (essentially, the entire calculation takes place "in $\mathbb{F}_p$", the finite field with p elements). So, we want to find ways of calculating the number of solutions to certain algebraic equations modulo p, i.e. we want to find $\#X(\mathbb{F}_p)$, where X is some algebraic variety.

Another classical connection arose from Riemann's study of his well known *Riemann zeta function*

$$\zeta(s) = \sum_{n \geqslant 1} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}$$

Riemann found that $\zeta(s)$ has analytic continuation to all $s \in \mathbb{C}$, satisfies a functional equation, and conjectured that all non-trivial zeros had $\mathrm{Re}(s) = 1/2$ (the famous Riemann Hypothesis). The product formula hints at the possibility of "local factors" over each prime p, so is there a connection to algebraic geometry over $\mathbb{F}_p$? It becomes easier to see when we generalize: Dedekind formulated his own zeta functions, the *Dedekind zeta functions* which are attached to any number field K with ring of integers $\mathcal{O}_K$:

$$\zeta_K(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_K} (N\mathfrak{a})^{-s} = \prod_{\mathfrak{p}} (1 - (N\mathfrak{p})^{-s})^{-1}$$

Hecke proved that these have analytic continuation and functional equations. Here the product formula gives us a connection to the theory of algebraic curves. The prime ideals $\mathfrak{p} \subset \mathcal{O}_K$ are points on the curve $\mathrm{Spec}(\mathcal{O}_K)$ and $N\mathfrak{p}$, the norm of the prime ideals, is exactly the number of elements of the field $\mathcal{O}_K/\mathfrak{p} \simeq \mathbb{F}_q$ for some q.

Emil Artin took this idea from the world of number fields to function fields, $\mathbb{F}_q(T)$. Here we get a direct description of the geometry, where now we have the exact same equation as above, but now we care about the number of points of an algebraic curve over $\mathbb{F}_{q^m}/\mathbb{F}_q$. Artin found that he could express the zeta function as a rational function by writing $Z(q^{-s})$ as $Z(u)$, with analytic continuation and functional equation, and he conjectured that all zeros of the zeta function would be on the circle $|u| = q^{1/2}$.

The case of elliptic curves (which we cover below) was proved by Hasse, and the case for all algebraic curves was proven by Weil, who then went on to conjecture that the same ideas would hold for higher dimensional algebraic varieties.

Specifically, let X be a non-singular d-dimensional projective algebraic variety over the field $\mathbb{F}_q$. The *zeta function* $\zeta(X,t)$ of X is by definition the uniquely determined series such that

$$\zeta(X,0) = 1$$

$$\frac{\zeta'(X,t)}{\zeta(X,t)} = \sum_{n=1}^{\infty} \#X(\mathbb{F}_{q^n})t^{n-1}$$

where $\#X(\mathbb{F}_{q^n})$ is the number of points of X with coordinates in the extension $\mathbb{F}_{q^n}/\mathbb{F}_q$. This can also be expressed as

$$\zeta(X,q^{-s}) = \exp\left(\sum_{n=1}^{\infty} \frac{\#X(\mathbb{F}_{q^n})}{n} q^{-ns}\right)$$

or

$$\zeta(X,t) = \exp\left(\sum_{n=1}^{\infty} \#X(\mathbb{F}_{q^n})\frac{t^n}{n}\right)$$

The Weil Conjectures state that:

1. (Rationality) $\zeta(X,t)$ is a rational function, which can be represented in the following form
$$\zeta(X,t) = \frac{P_1(t)\cdots P_{2d-1}(t)}{P_0(t)\cdots P_{2d}(t)}$$
where each $P_i(t)$ is an integral polynomial. Furthermore, $P_0(t) = 1-t$ and $P_{2d}(t) = 1 - q^d t$, and each $P_i(t) = \prod_j (1-\alpha_{ij}t)$ for some $\alpha_{ij} \in \mathbb{C}$.

2. (Functional equation and Poincaré duality) The zeta function satisfies
$$\zeta\left(X, \frac{1}{q^d t}\right) = \pm q^{\frac{dE}{2}} t^E \zeta(X,t)$$
where E is the Euler characteristic of X (alternating sum of dimension of cohomology).

3. (Betti numbers) If there exists a non-singular variety Y over a number field K with choice of embedding $K \hookrightarrow \mathbb{C}$ such that $X = Y_{\mathbb{F}_q}$ (i.e. X is a good reduction mod p of Y, $q = p^m$ where $m = [K : \mathbb{Q}]$), then
$$\deg P_i = \dim\left(H_i(Y_{\mathbb{C}})\right)$$
In other words, the degree of P_i is the ith Betti number of the space of complex points of Y.

4. ("Riemann Hypothesis") $|\alpha_{ij}| = q^{\frac{i}{2}}$ for all $1 \leqslant i \leqslant 2d-1$ and all j.

Weil based these conjectures on ideas from algebraic topology and showed that given a certain well-behaved theory of cohomology for algebraic varieties, mirroring the theory in algebraic topology, a large portion of the conjectures could be proven. We cover this in the section on "Weil cohomology theories".

The difficulty in finding an appropriate cohomology theory caused much work on the foundations of algebraic geometry, with the outcome being the publishing of *Éléments de géométrie algébrique* (EGA, 1960-1967) and *Séminaire de Géométrie Algébrique du Bois Marie* (SGA, 1960-1969) by Grothendieck, M. Artin (not the same one as above), Verdier, and many others. From this we get the theory of schemes and étale cohomology. They showed the first three parts of the Weil conjectures by proving "basic" properties of étale cohomology. The proof these properties are highly non-trivial, and will be the topic of the bulk of this paper. Unfortunately, the "Riemann Hypothesis" analogue remained open.

Finally in 1974, Deligne filled in the missing pieces required to prove the final part of the conjectures (and, on the way, proved many other interesting bounds, including the Ramanujan-Petersson conjecture and eventually the hard Lefschetz theorem over finite fields). Deligne's proof of the Weil conjectures will be the final topic of this paper.

1.2 An Easy Case: Elliptic Curves

One of the first cases of the Weil Conjectures that we can prove is for elliptic curves. First, we will have a quick reminder of what an elliptic curve is, then we will review some properties to be used in the proof, and then we will write out the proof. Finally, we will quickly speak of more intrinsic reasons as to why this case is easy, which will be put into better perspective as we continue into the theory of étale cohomology. For a better understanding of elliptic curves, see [Sil09].

Definition 1.2.1. Let k be a field. An *elliptic curve* (E, O) (over k) is a smooth projective curve with genus 1 ($E \to \operatorname{Spec} k$) with a chosen *base point* $O \in E$ (thought of as a section $O : \operatorname{Spec} k \hookrightarrow E$). This can be reinterpreted in many different ways:

1. A curve in $\mathbb{P}^2$ defined by a non-singular cubic equation. When $\operatorname{char} k \neq 2, 3$, then we can write this as
$$y^2 = x^3 + ax + b$$
with $a, b \in k$ and $\Delta \neq 0$, where
$$\Delta = -16(4a^3 + 27b^2)$$
2. Over the complex numbers $\mathbb{C}$, we can think of it as a genus 1 Riemann surface, i.e. a torus, defined by the quotient $\mathbb{C}/\Lambda$, where $\Lambda \subset \mathbb{C}$ is a lattice (meaning that $\Lambda \simeq \mathbb{Z}^2$).

Note. We will often drop O from the notation for (E, O), simply calling it E. For our purposes, we care about elliptic curves over $k = \mathbb{F}_q$, since the Weil Conjectures concern varieties over the finite fields $\mathbb{F}_q$. Recall that elliptic curves come equipped with a natural group structure, which we can immediately see from an isomorphism

$$E \xrightarrow{\sim} \mathrm{Pic}^0(E) := \mathrm{Div}^0(E)/\sim$$

$$P \mapsto [P] - [O]$$

where O is the distinguished base point of E. Over the complex numbers, this structure is also naturally inherited from the projection map $\mathbb{C} \to \mathbb{C}/\Lambda$.

Definition 1.2.2. Let $(E_1, O_1), (E_2, O_2)$ be elliptic curves over a field k. An *isogeny* is a morphism $f : E_1 \to E_2$ such that $f(O_1) = O_2$.

Note. Recall that an isogeny is a group homomorphism and that for every isogeny $f : E_1 \to E_2$ there exists a *dual isogeny* $f^\vee : E_2 \to E_1$ such that the following diagram commutes

$$\begin{tikzcd} E_2 \arrow[r, "\simeq"] \arrow[d, "f^\vee"] & \mathrm{Pic}^0(E_2) \arrow[d, "f^*"] \\ E_1 \arrow[r, "\simeq"] & \mathrm{Pic}^0(E_1) \end{tikzcd}$$

(where f^* is the pullback of Picard groups induced by $f : E_1 \to E_2$) with the following properties

$$f^\vee f = [\deg f] : E_1 \to E_1$$

$$(f + g)^\vee = f^\vee + g^\vee$$

Definition 1.2.3. Let E/k be an elliptic curve. Let $E[n]$ denote the n-torsion points of E ($\overline{k}$-points). We define the *Tate module of* E, $T_\ell(E)$, as the inverse limit

$$T_\ell(E) := \varprojlim_n \{E[\ell^n]\}$$

Proposition 1.2.4. *Let E/k be an elliptic curve with $\mathrm{char}\, k \neq \ell$. Then, for $\mathrm{char}\, k \nmid n$, $E[n] \simeq (\mathbb{Z}/n\mathbb{Z})^2$ and $T_\ell(E) \simeq \mathbb{Z}_\ell^2$. In the complex case, if $E = \mathbb{C}/\Lambda$, then we also have that $T_\ell(E) \simeq \Lambda \otimes \mathbb{Z}_\ell$.*

By the above proposition and the natural action of the Galois group $G_k = \mathrm{Gal}(\overline{k}/k)$ on E, the Tate module is equivalent to a Galois representation:

$$\rho : G_k \to \mathrm{GL}_2(\mathbb{Z}_\ell)$$

This allows for many nice calculations.

Proposition 1.2.5 (Weil Pairing). *Let E be an elliptic curve over a field k with fixed algebraic closure $\overline{k}$ and let μ_n denote the group of nth roots of unity in $\overline{k}$. There exists a map*

$$e_n : E[n] \times E[n] \to \mu_n$$

(on $\overline{k}$-points) with the following properties for all $x, y, z \in E[n]$

(a) Bilinear: $e_n(x + y, z) = e_n(x, z)e_n(y, z)$;

(b) Alternating: $e_n(x, x) = 0$ (which implies $e_n(x, y) = -e_n(y, x)$);

(c) Non-Degenerate: if $e_n(x, y) = 0$ for all $y \in E[n]$, then $x = 0$;

(d) Galois-equivariant: $e_n(\sigma x, \sigma y) = \sigma e_n(x, y)$ for all $\sigma \in \mathrm{Gal}(\overline{k}/k)$.

(e) Let $w \in E[nm]$, then $e_{nm}(w, x) = e_n(mw, x)$.

Taking the inverse limit over ℓ^n, we get a pairing for $T_\ell(E)$ which is an isomorphism

$$\bigwedge^2 T_\ell(E) \xrightarrow{\sim} \mathbb{Z}_\ell(1)$$

where $\mathbb{Z}_\ell(1) = \varprojlim_n \{\mu_{\ell^n}\}$ (this module will show up again in our definition of the Tate twist).

Sketch. Let's sketch out how we obtain the Weil pairing. We recall the following set of isomorphisms

$$\widehat{E} := \mathrm{Pic}^0(E) \cong \mathrm{Ext}^1_S(E, \mathbb{G}_m)$$

where E/S is an elliptic curve over a scheme S, $\mathrm{Pic}^0(-)$ is the degree 0 part of the Picard functor, $\mathrm{Ext}^1_S(-,-)$ is the first right-derived functor of $\mathrm{Hom}_S(-,-)$ for (quasi?-)coherent sheaves on S_{et} (or S_{ft}, the flat site?), and $\mathbb{G}_m := (\mathbb{A}^1_S \smallsetminus \{0\})$ is the multiplicative group scheme over S. This can be defined more generally for abelian varieties, but the key part of the case of elliptic curves is that

$$E \xrightarrow{\sim} \widehat{E}$$

$$P \mapsto [P] - [O]$$

is an isomorphism. Now, we use the Ext-functor to construct our pairing, and then the isomorphisms will clarify the situation. We start with the exact sequence

$$0 \to E[m] \to E \xrightarrow{m} E \to 0$$

and then we apply $\mathrm{Hom}_S(-, \mathbb{G}_m)$ and extend to the long exact sequence using Ext:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_S(E[m], \mathbb{G}_m) \arrow[r, "\delta"] \arrow[d, equal] & \mathrm{Ext}^1_S(E, \mathbb{G}_m) \arrow[r, "m"] \arrow[d, equal] & \mathrm{Ext}^1_S(E, \mathbb{G}_m) \arrow[d, equal] \\
0 \arrow[r] & \mathrm{Hom}_S(E[m], \mu_m) \arrow[r, "\delta"] & \widehat{E} \arrow[r, "m"] & \widehat{E}
\end{tikzcd}$$

The bottom line says that δ gives us an isomorphism

$$\mathrm{Hom}_S(E[m], \mu_m) \cong \widehat{E}[m]$$

and this gives us a perfect pairing

$$E[m] \times \widehat{E}[m] \cong E[m]^2 \to \mu_m$$

Taking $m = \ell^n$ and then taking the limit we get the usual Weil pairing

$$T_\ell(E) \times T_\ell(E) \to \mathbb{Z}_\ell(1)$$

□

Proposition 1.2.6. *Let $f : E_1 \to E_2$ be an isogeny and let $x \in E_1[n]$ and $y \in E_2[n]$. Then*

$$e_n(f(x), y) = e_n(x, f^\vee(y))$$

As a corollary of the Weil pairing, we get a useful computational tool:

Corollary 1.2.7. *Let E be an elliptic curve over a field k and let $f : E \to E$ be an endomorphism. Then*

$$\deg(f) = \det(f_\ell)$$

where $f_\ell : T_\ell(E) \to T_\ell(E)$ is the induced map on the Tate module.

Sketch of proof. Let $x, y \in T_\ell(E)$, then

$$e_n(f_\ell(x), f_\ell(y)) = \det(f_\ell) e_n(x, y)$$

and

$$e_n(f_\ell(x), f_\ell(y)) = e_n(f_\ell^\vee(f_\ell(x)), y) = e_n([\deg f]_\ell(x), y) = \deg f e_n(x, y)$$

□

Theorem 1.2.8. *Let $E/\mathbb{F}_q$ be an elliptic curve. Then there is an $a \in \mathbb{Z}$ such that*

$$Z(E/\mathbb{F}_q; T) = \frac{1 - aT + qT^2}{(1-T)(1-qT)}$$

Further, we have a functional equation

$$Z\left(E/\mathbb{F}_q; \frac{1}{qT}\right) = Z(E/\mathbb{F}_q; T)$$

and

$$1 - aT + qT^2 = (1 - \alpha T)(1 - \beta T) \quad \textit{with} \quad |\alpha| = |\beta| = \sqrt{q}$$

Proof. First, remember that we have the Tate module representation for endomorphisms of E:

$$End(E) \to End(T_\ell(E)), \quad \psi \mapsto \psi_\ell$$

Which we can then use to calculate the values

$$\det(\psi_\ell) = \deg(\psi) \quad \text{and} \quad tr(\psi_\ell) = 1 + \deg(\psi) - \deg(1 - \psi)$$

In particular, both of these values are in $\mathbb{Z}$ and are independent of ℓ.

Next, we use the qth-power Frobenius endomorphism $\phi : E \to E$ defined by $(x, y) \mapsto (x^q, y^q)$. Note that, while the Frobenius is totally inseparable, $(1 - \phi)$ is a separable morphism, so

$$\#E(\mathbb{F}_q) = \# \ker(1 - \phi) = \deg(1 - \phi)$$

We then use our representation to compute

$$\det(\phi_\ell) = \deg(\phi) = q$$

$$\text{tr}(\phi_\ell) = 1 + \deg(\phi) - \deg(1 - \phi) = 1 + q - \#E(\mathbb{F}_q)$$

To ease notation, we denote $a := tr(\phi_\ell)$.
Hence, the characteristic polynomial of ϕ_ℓ is

$$\det(T - \phi_\ell) = T^2 - \text{tr}(\phi_\ell)T + \det(\phi_\ell) = T^2 - aT + q$$

Since the characteristic polynomial has coefficients in $\mathbb{Z}$, we can factor it over $\mathbb{C}$ as

$$\det(T - \phi_\ell) = T^2 - aT + q = (T - \alpha)(T - \beta)$$

For every rational number $m/n \in \mathbb{Q}$, we have

$$\det\left(\frac{m}{n} - \phi_\ell\right) = \frac{\det(m - n\phi_\ell)}{n^2} = \frac{\deg(m - n\phi)}{n^2} \geqslant 0$$

Thus, the quadratic polynomial $\det(T - \phi_\ell)$ is nonnegative for all $T \in \mathbb{R}$, so either it has complex conjugate roots or it has a double root. In either case we have $|\alpha| = |\beta|$, and

$$\alpha\beta = \det(\phi_\ell) = \deg(\phi) = q$$

Similarly, for each integer $n \geqslant 1$, the (q^n)th-power Frobenius morphism satisfies $\#E(\mathbb{F}_{q^n} = \deg(1 - \phi^n)$. It follows (from Jordan normal form with α, β in the diagonal) that the characteristic polynomial of ϕ_ℓ^n is given by

$$\det(T - \phi_\ell^n) = (T - \alpha^n)(T - \beta^n)$$

In particular,

$$\begin{aligned} \#E(\mathbb{F}_{q^n} &= \deg(1 - \phi^n) \\ &= \det(1 - \phi_\ell^n) \\ &= 1 - \alpha^n - \beta^n + q^n \end{aligned}$$

Now, we compute

$$\begin{aligned}\log Z(E/\mathbb{F}_q;T) &= \sum_{n\geqslant 1}\frac{\#E(\mathbb{F}_{q^n})T^n}{n}\\ &= \sum_{n\geqslant 1}\frac{(1-\alpha^n-\beta^n+q^n)T^n}{n}\\ &= -\log(1-T)+\log(1-\alpha T)+\log(1-\beta T)-\log(1-qT)\end{aligned}$$

Hence, by usual log calculations,

$$Z(E/\mathbb{F}_q;T)=\frac{(1-\alpha T)(1-\beta T)}{(1-T)(1-qT)}$$

Using this and the fact that

$$a=\alpha+\beta=tr(\phi_\ell)=1+q-\deg(1-\phi)\in\mathbb{Z}$$

We get that

$$1-aT+qT^2=(1-\alpha T)(1-\beta T)$$

and, therefore

$$Z(E/\mathbb{F}_q;T)=\frac{1-aT+qT^2}{(1-T)(1-qT)}$$

which we can see satisfies the functional equation.

□

Note. The Weil Conjecture for elliptic curves are "easy" because we have quick access to the Tate module, which we shall later see is dual to the étale cohomology group $H^1_{ét}(E,\mathbb{Z}_\ell)$. Later, we will have to worry about higher cohomology groups, but elliptic curves are 1-dimensional, so we only have to look at $H^i_{ét}(E,\overline{\mathbb{Q}}_\ell)$ for $i=0,1,2$ (by cohomological dimension), and $H^2_{ét},H^0_{ét}$ are simply isomorphic to $\mathbb{Q}_\ell$, while $H^1_{ét}$ is isomorphic to $T_\ell E\otimes\overline{\mathbb{Q}}_\ell$ (this is a sort of "cohomological dimension" consideration).

Another key point is the connection between the Weil pairing and what we will later call *Poincaré duality*. Just as the Weil pairing gives a perfect pairing of the "cohomology groups"

$$T_\ell(E)\otimes T_\ell(E)\to\bigwedge^2 T_\ell(E)\xrightarrow{\sim}\mathbb{Z}_\ell$$

Poincaré duality gives us a nondegenerate pairing

$$H^r_c(X,\mathcal{F})\times\mathrm{Ext}^{2d-r}_X(\mathcal{F},\mathbb{Z}_\ell(1)^{\otimes d})\to H^{2d}_c(X,\mathbb{Z}_\ell(1)^{\otimes d})\cong\mathbb{Z}_\ell$$

In fact, the ways in which these dualities arise are also similar: we look at the Ext groups as a sort of "dual" to the usual cohomology, and by analyzing the long exact sequence and considering things like cohomological dimension (plus some other reductions via base change and more) we reach the conclusion (easier said than done).

The real miracle of all of this is that both of these are *Galois equivariant*, meaning that we can extract information about Galois representations from this. In our case, it means that the Frobenius can be "safely applied everywhere", which (after some considerations with algebraic cycles, which also follows from Poincaré duality) leads us to the *Lefschetz fixed point formula*, allowing us to easily calculate the number of points of a variety over a finite field, and thereby control the zeta function.

1.3 Weil Cohomology Theories

When Weil proposed the conjectures he had a specific path to the proof in mind. He saw the success of different cohomology theories in algebraic topology, and he knew that if such a thing could be brought to application on algebraic varieties that the basic results of the conjectures would simply follow (besides the "Riemann Hypothesis"). Nowadays, cohomology theories which satisfy these criteria are called "Weil Cohomology Theories" and many more have been created since 1949.

The primary examples which inspired the definition of a Weil cohomology theory at the time were singular cohomology and de Rham cohomology:

Definition 1.3.1. For a ("nice") topological space X, we define the *singular cohomology of X (with coefficients in the group G)*, $H^i_{sing}(X;G)$, as the cohomology of the singular cochain complex

$$\cdots \to \mathrm{Hom}(C_{n-1}(X),G) \xrightarrow{\delta_n} \mathrm{Hom}(C_n,G) \xrightarrow{\delta_{n+1}} \mathrm{Hom}(C_{n+1},G) \to \dots$$

where

$$C_n(X) = \mathbb{Z}[\{\sigma : \Delta^n \to X\}]$$

where Δ^n is the standard n-simplex (think "n-dimensional triangle") and differential δ_n defined as $\mathrm{Hom}(\partial_n)$ with

$$\partial_n : C_n \to C_{n-1}$$

$$\partial_n \sigma = \sum_{k=0}^{n} (-1)^k [p_0, \dots, p_{k-1}, p_{k+1}, \dots, p_n]$$

For more details, see [Hat01].

Definition 1.3.2. For some smooth manifold M, we define the *de Rham cohomology of M*, $H^i_{dR}(M)$, as the cohomology of the de Rham complex

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \to \dots$$

where $\Omega^0(M)$ is the space of smooth functions on M, $\Omega^1(M)$ is the space of 1-forms on M, $\Omega^2(M)$ is the space of 2-forms on M, and so forth. The differential is simply the exterior derivative. For more details, see [BT82].

Both of these cohomology theories satisfy certain properties (on "nice enough" spaces, say compact, connected, oriented smooth manifolds). We will state this as a proposition without proof. For a proof, see the above mentioned references, or any other introductory text on algebraic or differential topology.

Proposition 1.3.3. *Let X, Y be "nice" manifolds. Then, singular cohomology satisfies the following properties*

1. *Functoriality: if $f : X \to Y$ is a continuous map, then there are associated maps in cohomology for all $i \geqslant 0$*
$$f^* : H^i_{sing}(Y, \mathbb{C}) \to H^i_{sing}(X, \mathbb{C})$$
2. *Finiteness: for any $i \geqslant 0$ $H^i_{sing}(X, \mathbb{C})$ are finite dimensional $\mathbb{C}$-vector spaces;*
3. *Cohomological Dimension: if X is of dimension d $H^i_{sing}(X, \mathbb{C})$ vanish for $i > 2d$;*
4. *Poincaré Duality: There is an isomorphism (depending on choice of orientation of X)*
$$H^{2d}_{sing}(X, \mathbb{C}) \simeq \mathbb{C}$$
and for all $i \leqslant d$ natural bilinear forms (called cup products)
$$H^i_{sing}(X, \mathbb{C}) \times H^{2d-i}_{sing}(X, \mathbb{C}) \to H^{2d}_{sing}(X, \mathbb{C}) \simeq \mathbb{C}$$
which are perfect pairings of finite-dimensional vector spaces; in particular H^i_{sing} is dual to H^{2d-i}_{sing}, and they have the same dimension.
5. *Künneth Formula: For any $i \geqslant 0$, there are canonical isomorphisms*
$$\bigoplus_{j+k=i} H^j_{sing}(X, \mathbb{C}) \otimes H^k_{sing}(Y, \mathbb{C}) \xrightarrow{\sim} H^i_{sing}(X \times Y, \mathbb{C})$$
6. *Lefschetz Trace Formula: Let $f : X \to X$ be a differentiable map with isolated fixed points, and $L(f)$ the number of fixed points (counted with multiplicity). Then, we have the equality*
$$L(f) = \sum_{i=0}^{2d} (-1)^i \mathrm{tr}(f^* \mid H^i_{sing}(X, \mathbb{C}))$$

A similar proposition can be made for de Rham cohomology. Keeping this proposition in mind, we make the following definition:

Definition 1.3.4. Let k and K be fields with $\mathrm{char} K = 0$ and let X be a smooth projective variety of dimension d. A *Weil cohomology theory* is a contravariant functor
$$H^* : \{\text{smooth projective varieties over } k\} \to \{\text{graded } K\text{-algebras}\}$$

$$H^*(X) = \bigoplus_i H^i(X)$$

satisfying the following properties:

(i) Finiteness: $H^i(X)$ are finite dimensional K-vector spaces;

(ii) Cohomological Dimension: $H^i(X)$ vanish for $i > 2d$;

(iii) Poincaré Duality: There is an isomorphism (depending on choice of orientation of X)

$$H^{2d}(X) \simeq K$$

and for all $i \leqslant d$ natural bilinear forms (called cup products)

$$H^i(X) \times H^{2d-i}(X) \to H^{2d}(X) \simeq K$$

which are perfect pairings of finite-dimensional vector spaces; in particular H^i is dual to H^{2d-i}, and they have the same dimension.

(iv) Künneth Formula: Let Y be another smooth projective variety. For any $i \geqslant 0$, there are canonical isomorphisms

$$H^*(X) \otimes H^*(Y) \xrightarrow{\sim} H^*(X \times Y)$$

(v) Lefschetz Trace Formula: Let $f : X \to X$ be a morphism with isolated fixed points, satisfying certain separability assumption on $1 - df$ acting on the tangent space at the fixed points, and $L(f)$ the number of fixed points (counted with multiplicity). Then, we have an equality

$$L(f) = \sum_{i=0}^{2d} (-1)^i \mathrm{tr}(f^* \mid H^i(X))$$

Note. For our application, the Lefschetz Trace Formula will be applied to the *Frobenius automorphism*, defined by

$$(\Phi_p, \Phi_p^{\#}) : X \to X$$

$$\Phi_p(x) = x, \qquad x \in X$$

$$\Phi_p^{\#}(g) = g^p \qquad g \in \mathcal{O}_X$$

where p is a prime and $(X, \mathcal{O}_X)$ is a scheme over $\mathbb{F}_p$. This morphism is special in the fact that it fixes all of the points of X, so we can then use it to count the points on a variety over a finite field.

Ideally, we would also want some sort of comparison theorem, which gives an isomorphism

$$H^i(X) \otimes \mathbb{C} \simeq H^i_{sing}(X_{\mathbb{C}}, \mathbb{C})$$

when our variety is well-defined over $\mathbb{C}$. As we shall see, étale cohomology will satisfy this and will be shown to be a Weil cohomology theory.

How does this help us?

If we had a Weil cohomology theory, then we could prove the following parts of the Weil conjectures:

Proposition 1.3.5. *Let $H^*(-)$ be a Weil cohomology theory as defined above and let X be a smooth projective variety over $\mathbb{F}_q$ of dimension d. Then, the local zeta function of X is rational, of the form*

$$\zeta(X,t) = \frac{P_1(t)\cdots P_{2d-1}}{P_0(t)\cdots P_{2d}(t)}$$

where each $P_i(t)$ is an integral polynomial. Furthermore, $P_0(t) = 1-t$, $P_{2d}(t) = 1-q^d t$, and each $P_i(t) = \prod_j (1-\alpha_{ij}t)$ for $\alpha_{ij} \in \mathbb{C}$. In addition it also satisfies a functional equation of the form

$$\zeta\left(X, \frac{1}{q^d t}\right) = \pm q^{\frac{dE}{2}} t^E \zeta(X,t)$$

where E is the Euler characteristic of X.

Proof. Let ϕ denote the qth-power Frobenius map on X, defined as the identity map $X \to X$ topologically, with sheaf map

$$\phi^{\#} : \mathcal{O}_X \to \mathcal{O}_X$$

$$\phi^{\#}(g) = g^q, \qquad g \in \mathcal{O}_X$$

Applying the Lefschetz fixed points formula, we get that

$$\#X(\mathbb{F}_{q^n}) = \sum_{i=0}^{2d} (-1)^d \mathrm{tr}((\phi^n)^* \mid H^i(X))$$

giving us

$$\sum_{n=0}^{\infty} \#X(\mathbb{F}_{q^n})\frac{t^n}{n} = \sum_{i=0}^{2d}(-1)^i \sum_{n=0}^{\infty} \mathrm{tr}((\phi^n)^* \mid H^i(X))\frac{t^n}{n}$$

by basic linear algebra, we compute to check that any endomorphism f of a finite dimensional vector space V satisfies the formal power series identity

$$\sum_{n=1}^{\infty} \mathrm{tr}(f^n \mid V)\frac{t^n}{n} = -\log\det(1-fT)$$

(check the characteristic polynomial), so that we get

$$\begin{aligned}
\zeta(X,t) &= \exp\left(\sum_{n=0}^{\infty} \#X(\mathbb{F}_{q^n})\frac{t^n}{n}\right) \\
&= \exp\left(\sum_{i=0}^{2d}(-1)^i \sum_{n=0}^{\infty} \operatorname{tr}((\phi^n)^* \mid H^i(X))\frac{t^n}{n}\right) \\
&= \exp\left(\sum_{i=0}^{2d}(-1)^{i+1} \log\det(1-\phi^* t)\right) \\
&= \prod_{i=0}^{2d} \det(1-\phi^* t \mid H^i(X))^{(-1)^{i+1}}
\end{aligned}$$

Letting $P_i(t) = \det(1-\phi^* t \mid H^i(X))$ we get that $\zeta(X,t) \in K(t)$, but since it is also an element of $\mathbb{Q}[\![t]\!]$, then $\zeta(X,t) \in \mathbb{Q}(t)$ (see Bourbaki, Algebre, IV.5, Exercise 3; or [Mil13] pg.157).

For integrality of the polynomials P_i, we first let $\deg x = [k(x) : \mathbb{F}_q]$ for $x \in X$ (a closed point). Recall that to give a point of X with coordinates in $\mathbb{F}_{q^m}$ is a map $\operatorname{Spec}(\mathbb{F}_{q^m}) \to X$. To give such a map with image $x \in X$ is the same as giving a morphism $k(x) \to \mathbb{F}_{q^m}$. We let $N_m(x)$ denote the contribution of these morphisms to the total number $\#X(\mathbb{F}_{q^m})$. By the theory of finite fields, we get that

$$N_m(x) = \begin{cases} \deg x & \text{if } \deg x \mid m \\ 0 & \text{otherwise} \end{cases}$$

by basic log calculations and this, we get

$$\log\frac{1}{1-t^{\deg x}} = \sum_{n=1}^{\infty}\frac{t^{n\deg x}}{n} = \sum_{m=1}^{\infty} N_m(x)\frac{t^m}{m}$$

meaning that

$$\zeta(X,t) = \prod_{x\in X_0}\frac{1}{1-t^{\deg x}}$$

so, $\zeta(X,t) \in 1 + t\mathbb{Z}[\![t]\!] \subset 1 + t\mathbb{Z}_\ell[\![t]\!]$.

Next, we simplify our exposition by considering

$$\zeta(X,t) = \frac{P(t)}{Q(t)}$$

where $P(t)$ and $Q(t)$ are relatively prime and are elements of $1 + t\mathbb{Q}[t]$ (and thereby elements of $1 + t\mathbb{Q}_\ell[t]$).

Next, we show that the coefficients of $P(t)$ and $Q(t)$ have ℓ-adic absolute values $\leqslant 1$, meaning that $P(t), Q(t) \in \mathbb{Z}_\ell[t]$ for all ℓ, so, we get that $P(t), Q(t) \in \mathbb{Z}[t]$.

After possibly extending to a finite extension of $\mathbb{Q}_\ell$, we may assume that $Q(t) = \prod(1 - c_i t)$ splits. If $|c_i|_\ell > 1$, then $|c_i^{-1}|_\ell < 1$ and the power series $\zeta(X, c_i^{-1})$ converges. But then

$$\zeta(X, c_i^{-1}) \cdot Q(c_i^{-1}) = P(c_i^{-1})$$

So, $Q(c_i^{-1}) = 0$ implies that $P(c_i^{-1}) = 0$, but this is impossible. Therefore $|c_i|_\ell \leqslant 1$. A similar argument holds for $P(t)$.

For the specific results on the P_i, note that Poincaré duality give us $P_0(t) = 1 - t$ and $P_{2d}(t) = 1 - q^d t$. Finally, the functional equation follows from computation (using Poincaré duality):

Consider the pairing

$$H^{2d-r}(X) \times H^r(X) \to H^{2d}(X) \xrightarrow{\eta_X} K$$

By definition of ϕ_*, for $x \in H^{2d-r}(X)$ and $x' \in H^r(X)$, using the cup product:

$$\eta_X(\phi_*(x) \smile x') = \eta_X(x \smile \phi^*(x'))$$

This means that ϕ_* acting on $H^{2d-r}(X)$ and ϕ^* acting on $H^r(X)$ have the same eigenvalues. Therefore, since $\phi_* \circ \phi^* = q^d$ (by degree arguments), we get that $\phi^* = \frac{q^d}{\phi_*}$, so, if $\alpha_1, \ldots, \alpha_s$ are the eigenvalues of ϕ^* acting on $H^r(X)$, then $\frac{q^d}{\alpha_1}, \ldots, \frac{q^d}{\alpha_s}$ are the eigenvalues of ϕ^* acting on H^{2d-r}. From this, we can complete the calculation:

$$\begin{aligned}
\zeta\left(X, \frac{1}{q^d t}\right) &= \prod_{i=0}^{2d} \det\left(1 - \frac{\phi^*}{q^d t} \mid H^i(X)\right)^{(-1)^{i+1}} \\
&= \exp\left(\sum_{i=0}^{2d} (-1)^{i+1} \log\det\left(1 - \frac{\phi^*}{q^d t}\right)\right) \\
&= \pm q^{\frac{dE}{2}} t^E \left(\prod_{i=0}^{2d} \det(1 - \phi^* t \mid H^i(X))^{(-1)^{i+1}}\right) \\
&= \pm q^{\frac{dE}{2}} t^E \zeta(X, t)
\end{aligned}$$

where $E = \sum_i (-1)^i \dim(H^i(X))$ (the Euler characteristic). □

Note. We implicitly used every property of a Weil cohomology theory. Without finiteness, formal power series identity for vector spaces

$$\sum_{n=1}^{\infty} \operatorname{tr}(f^n \mid V) \frac{t^n}{n} = -\log\det(1 - fT)$$

would not hold. Without cohomological dimension, we could have a possibly infinite product of rational functions. The use of Poincaré duality and the Lefschetz trace formula are more obvious.

One thing which we have not discussed is the choice of "coefficient field" (or sheaf) for our Weil cohomology theory. This choice is crucial, and, in fact, is the main problem which presents itself when constructing étale cohomology. We already see that we need a torsion-free ground field (i.e. a field over $\mathbb{Q}$), but there are certain troubles in directly using a sheaf associated to a field over $\mathbb{Q}$, so we will have to get creative.

In addition, if we have a "comparison theorem" between cohomology on complex analytic spaces and smooth projective varieties, then we also get the following result:

Proposition 1.3.6 (Betti Numbers)**.** *Let $H^*(-)$ be a Weil cohomology theory. Let Y be a smooth projective variety over a number field K with fixed embedding $K \hookrightarrow \mathbb{C}$ and let $X = Y \times_K \mathrm{Spec}(\mathbb{F}_q)$ for $q = p^n$ be a good reduction of Y at p (so, X is a smooth projective variety over $\mathbb{F}_q$). Suppose there is an isomorphism for all i*

$$H^i(Y) \otimes \mathbb{C} \simeq H^i_{sing}(Y_{\mathbb{C}}, \mathbb{C})$$

Then, using the same notation as above, we get that

$$\deg P_i = \dim(H^i(Y_{\mathbb{C}}))$$

i.e. the degree of the ith integral polynomial in the decomposition of $\zeta(X, s)$ equals the ith Betti number of $Y_{\mathbb{C}} = Y \times_K \mathrm{Spec}(\mathbb{C})$ (the complex points of Y).

Proof. By the above identity $P_i(t) = \det(1 - \phi^* t \mid H^i(X))$, we get that

$$\deg P_i = \dim H^i(X) = \dim H^i(Y) = \dim H^i_{sing}(Y_{\mathbb{C}})$$

□

1.4 Deligne's Contribution

Before jumping into our construction of étale cohomology, let's quickly look at how Deligne proved the "Riemann Hypothesis" part of the Weil conjectures. First, we reformulate:

Theorem 1.4.1. *Let X be a smooth projective variety over $\mathbb{F}_q$. Then all of the eigenvalues of the Frobenius automorphisms*

$$\phi^* : H^n(X, \mathbb{Q}_\ell) \to H^n(X, \mathbb{Q}_\ell)$$

are algebraic numbers. These eigenvalues and all their conjugates λ have complex absolute value

$$|\lambda| = q^{n/2}$$

Note. That this is equivalent to our above formulation of the "Riemann Hypothesis" follows from our proof of the rest of the Weil conjectures using a nice enough cohomology theory.

As per usual, we make our lives easier by reducing 1.4.1 to a simpler case. We shall show that it suffices to prove it for the middle cohomology groups of varieties of even dimension and that it suffices to prove an approximate result.

Lemma 1.4.2. *It suffices to prove* 1.4.1 *after* $\mathbb{F}_q$ *has been replaced by an arbitrary extension* $\mathbb{F}_{q^m}$.

Proof. The Frobenius map $\phi_m : X \to X$ defined relative to the field $\mathbb{F}_{q^m}$ is exactly ϕ^m (this ϕ being the Frobenius defined relative to $\mathbb{F}_q$). Therefore, if $\alpha_1, \alpha_2, \ldots$ are the eigenvalues of ϕ on $H^r(X, \mathbb{Q}_\ell)$, then $\alpha_1^m, \alpha_2^m, \ldots$ are the eigenvalues of ϕ^m on $H^r(X, \mathbb{Q}_\ell)$. If α^m satisfies the conditions of 1.4.1 relative to q^m, then α satisfies the condition relative to q. □

Thus, 1.4.1 is really a statement about $X/\mathbb{F}_q$: if it is true for one model of X over a finite field, then it is true for all.

Proposition 1.4.3. *Assume that for all nonsingular projective varieties X of even dimension d over $\mathbb{F}_q$, every eigenvalue α of ϕ on $H^d(X, \mathbb{Q}_\ell)$ is an algebraic number such that*

$$q^{\frac{d}{2}-\frac{1}{2}} < |\alpha'| < q^{\frac{d}{2}+\frac{1}{2}}$$

for all conjugates α' of α. Then, 1.4.1 holds for all nonsingular projective varieties.

Proof. Let X be a smooth projective variety of dimension d (not necessarily even) over $\mathbb{F}_q$, and let α be an eigenvalue of ϕ on $H^d(X, \mathbb{Q}_\ell)$. The Künneth formula shows that α^m occurs among the eigenvalues of ϕ acting on $H^{dm}(X^m, \mathbb{Q}_\ell)$. The statement in the lemma applied to an even power of X shows that

$$q^{\frac{md}{2}-\frac{1}{2}} < |\alpha'|^m < q^{\frac{md}{2}+\frac{1}{2}}$$

On taking the mth root, and letting $m \to \infty$ over even integers, we find that

$$|\alpha'| = q^{\frac{d}{2}}$$

We now prove 1.4.1 by induction on dimension of X. For $\dim X = 0$, it is obvious, and for $d = 1$, only the eigenvalues of $H^1(X, \mathbb{Q}_\ell)$ aren't obvious, but we just showed that for $d = r = 1$

$$|\alpha'| = q^{\frac{d}{2}} = q^{\frac{r}{2}} = q^{\frac{1}{2}}$$

so, we may assume that $d \geqslant 2$.

Recall from the proof of the functional equation for $\zeta(X, t)$, that the Poincaré duality theorem implies that if α is an eigenvalue of ϕ on $H^r(X, \mathbb{Q}_\ell)$, then q^d/α is an eigenvalue of ϕ on $H^{2d-r}(X, \mathbb{Q}_\ell)$. Thus it suffices to prove the theorem for $r > d$. Bertini's Theorem ([Har77], II.8.18) shows that there is a hyperplane $H \subset \mathbb{P}^m$ such that $Z := H \cap X$ is a nonsingular variety. By our reduction, we may assume that H and Z are defined over $\mathbb{F}_q$. Then, the Gysin sequence reads

$$\cdots \to H^{r-2}(Z, \mathbb{Q}_\ell(-1)) \to H^r(X, \mathbb{Q}_\ell) \to H^r(X \smallsetminus Z, \mathbb{Q}_\ell) \to \cdots$$

$X \smallsetminus Z$ is affine, so $H^r(X \smallsetminus Z, \mathbb{Q}_\ell) = 0$ for $r > d$. Thus, the Gysin map

$$i_* : H^{r-2}(Z, \mathbb{Q}_\ell(-1)) \to H^r(X, \mathbb{Q}_\ell)$$

is surjective for $r > d$. By induction that the eigenvalues of ϕ on $H^{r-2}(Z, \mathbb{Q}_\ell)$ are algebraic numbers whose conjugates have absolute value $q^{(r-2)/2}$. Since $\phi \circ i_* = q(i_* \circ \phi)$, the eigenvalues of ϕ acting on $H^r(X, \mathbb{Q}_\ell)$ are algebraic numbers whose conjugates have absolute value $q^{r/2}$. $\square$

Now, we "just" have to prove this fundamental estimate:

$$q^{\frac{d}{2}-\frac{1}{2}} < |\alpha'| < q^{\frac{d}{2}+\frac{1}{2}}$$

for the middle cohomology of even dimensional nonsingular varieties over $\mathbb{F}_q$. We do this by first dealing with the case of affine curves by finding a Lefschetz formula for nonconstant sheaves. Then, we study how to fiber a variety X with a Lefschetz pencil, blow-up the variety to get X^*, and study the higher direct images of $\mathbb{Q}_\ell$ under $\pi : X^* \to \mathbb{P}^1$. This will require a delicate analysis of the monodromy operator, among other things. Deligne's first proof in [Del74] brings this all together quite elegantly, but not in a way which satisfied him, so he wrote another proof in [Del80], which we will review as well (where a lot of other results and machinery will be covered).

1.5 Recalling Sheaf Cohomology

Now, we have reduced the Weil conjectures minus the "Riemann Hypothesis" to a question of finding a Weil cohomology theory. The first thing to mention, is that standard sheaf cohomology on the Zariski topology of a smooth projective variety is not a Weil cohomology theory. We will quickly review sheaf cohomology:

Let X be a topological space. We make the open subsets of X into a category $\mathbf{Top}_X$ with the inclusions $U \subset V$ (where $U, V \subset X$ are open sets) corresponding to $U \to V$ as the only morphisms, and define a presheaf to be a contravariant functor on this category (for our purposes, we focus on contravariant functors to the category $\mathbf{Ab}$ of abelian groups, or the category $\mathbf{Mod}_X$ of $\mathcal{O}_X$ modules, when $(X, \mathcal{O}_X$ is a locally ringed space). Thus, such a presheaf $\mathcal{F} : \mathbf{Top}_X \to \mathbf{Ab}$ attaches to every open subset $U \subset X$ an abelian group $\mathcal{F}(U)$ and to every inclusion $V \subset U$ a restriction map $\rho_V^U : \mathcal{U} \to \mathcal{V}$ in such a way that $\rho_U^U = \mathrm{id}_{\mathcal{F}(U)}$ and whenever $W \subset V \subset U$

$$\rho_W^U = \rho_W^V \circ \rho_V^U$$

A *sheaf* is a presheaf $\mathcal{F}$ which satisfies the following properties

1. (Locality) A section $f \in \mathcal{F}(U)$ is determined by its restriction $\rho_{U_i}^U(f)$ to the set of an open covering $\{U_i\}_{i \in I}$ of U, i.e. given $f, g \in \mathcal{F}(U)$ such that $\rho_{U_i}^U(f) = \rho_{U_i}^U(g)$ for all $i \in I$, then $f = g$.

2. (Gluing) If for an open covering $\{U_i\}_{i \in I}$ of U we have that for some $f_i \in \mathcal{F}(U_i)$ that
$$\rho^{U_i}_{U_i \cap U_j} f_i = \rho^{U_j}_{U_i \cap U_j} f_j$$
for all $i, j \in I$, then there exists $f \in \mathcal{F}(U)$ such that $\rho^{U}_{U_i}(f) = f_i$ for all $i \in I$.

In other words, (and this definition becomes more useful when defining sheaves for étale cohomology), $\mathcal{F}$ is a sheaf if for every open covering $\{U_i\}_{i \in I}$ of an open subset $U \subset X$, the sequence

$$0 \to \mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \xrightarrow{f_{ij} - f_{ji}} \prod_{i \in I} \mathcal{F}(U_i \cap U_j)$$

is exact (where $f_{ij} = \rho^{U_i}_{U_i \cap U_j}$ for $f \in U_i$).

Grothendieck showed that the sheaves on X form an abelian category (when they have values in an abelian category). Thus, we have a notion of an injective sheaf: it is a sheaf $\mathcal{I}$ such that for any subsheaf $\mathcal{F}'$ of $\mathcal{F}$, every homomorphism $\mathcal{F}' \to \mathcal{I}$ extends to a homomorphism $\mathcal{F} \to \mathcal{I}$. Grothendieck showed that every sheaf can be embedded into an injective sheaf. The global sections functor $\mathcal{F} \mapsto \mathcal{F}(X)$ from the category of sheaves on X to the category of abelian groups is left exact but not in general right exact. We define $H^r(X, -)$ to be the rth right derived functor of the global sections functor. Thus, given a sheaf $\mathcal{F}$, we choose an exact sequence

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \cdots$$

with each $\mathcal{I}^r$ an injective sheaf, and we set $H^r(X, \mathcal{F})$ equal to the rth cohomology group of the complex of abelian groups

$$\mathcal{I}^0(X) \to \mathcal{I}^1(X) \to \mathcal{I}^2(X) \to \cdots$$

while injective resolutions are useful for defining the cohomology groups, they are not useful for computing them. Instead, one defines a sheaf $\mathcal{F}$ to be flabby if the restriction maps are surjective for all open subsets and shows that $H^r(X, \mathcal{F}) = 0$ if $\mathcal{F}$ is flabby. Thus, resolutions of flabby sheaves can be used to compute sheaf cohomology. Here is a sketch of the proof:

Sketch of proof. Let $(X, \mathcal{O}_X)$ be a locally ringed space. Recall that for all injective $\mathcal{O}_X$-modules $\mathcal{I}$ that

$$\mathrm{Hom}(\mathcal{O}_U, \mathcal{I}) = \mathcal{I}(U)$$

where $\mathcal{O}_U = i_!(\mathcal{O}_X|_U)$ is the restriction of the structure sheaf to the open $U \subset X$ (this follows from the definition of injective modules).

Let $V \subseteq U \subseteq X$ be open subsets. Then we have an inclusion

$$0 \to \mathcal{O}_V \to \mathcal{O}_U$$

of sheaves of $\mathcal{O}_X$-modules. Since $\mathcal{I}$ is injective, we get a surjection

$$\mathrm{Hom}(\mathcal{O}_U, \mathcal{I}) \to \mathrm{Hom}(\mathcal{O}_V, \mathcal{I}) \to 0$$

but by the above property we get

$$\mathcal{I}(U) \to \mathcal{I}(V) \to 0$$

so, all injective sheaves are flabby.

Let $\mathcal{F}$ be a flabby sheaf. Embed $\mathcal{F}$ into an injective object $\mathcal{I}$ and let $\mathcal{G} = \mathcal{I}/\mathcal{F}$ (the quotient):

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0$$

Then, all three sheaves are flabby by the first part of the proof, which gives us an exact sequence

$$0 \to \mathcal{F}(X) \to \mathcal{I}(X) \to \mathcal{G}(X) \to 0$$

But, since $\mathcal{I}$ is injective, we have $H^i(X, \mathcal{I}) = 0$ for all $i > 0$. In the long exact sequence from cohomology, we get $H^1(X, \mathcal{F}) = 0$ and $H^i(X, \mathcal{F}) \simeq H^{i-1}(X, \mathcal{G})$ for $i \geqslant 2$, but $\mathcal{G}$ is also flasque, so $H^1(X, \mathcal{G}) = 0$, and so we get the result by induction on i. □

An example of how the Zariski topology fails to yield interesting information in sheaf cohomology is:

Example 1. Recall that a topological space is said to be irreducible if any two nonempty open subsets of X have nonempty intersection. Grothendieck showed the following fact

Theorem 1.5.1. *If X is an irreducible topological space, then $H^i(X, \mathcal{F}) = 0$ for all constant sheaves and all $i > 0$.*

Proof. Let $\mathcal{F}$ be the constant sheaf associated to the group Λ. Let $U \subset X$ be an open subset and assume that $U_1 \cup U_2 = U$ but $U_1 \cap U_2 = U$ for open subsets. This contradicts the irreducibility of X, and so every open subset $U \subseteq X$ is connected. This implies that $\mathcal{F}(U) = \Lambda$ for all open $U \subseteq X$, so all restriction maps are obviously surjective, meaning that $\mathcal{F}$ is flabby. Recall that for flabby sheaves $\mathcal{F}$, we have that $H^i(X, \mathcal{F}) = 0$ for all $i > 0$, and so we have proved the theorem. □

This theorem leads to an even larger problem, which immediately shows that for sheaf cohomology on a Noetherian topological space (a large class of interesting algebraic varietiesm including Noetherian schemes) does not satisfy Poincaré duality, $H^{2d}(X, \mathcal{F}) \simeq \Lambda$ for $\mathcal{F}$ the constant sheaf associated with Λ (crucial to showing that $P_{2d} = 1 - q^d t$).

Theorem 1.5.2 (Grothendieck's Vanishing Theorem)**.** *Let X be a Noetherian topogical space. If $\dim(X) = d$, the $H^i(X, \mathcal{F}) = 0$ for all $i > d$ and any abelian sheaf $\mathcal{F}$ on X.*

Sketch of proof. See `https://stacks.math.columbia.edu/tag/02UU` or Grothendieck's Tohoku paper.

Essentially, we use $j : U \subset X$ open and $Z = X \setminus U$ with $i : Z \hookrightarrow X$ closed to get the exact sequence

$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0$$

to reduce to the case of X irreducible and then using another theorem to reduce to $\mathcal{F} = j_! \mathbb{Z}_U$, from which the result follows from the above theorem. $\square$

2 The Étale Site

Before defining étale cohomology, we need to reformulate the foundations:

2.1 What is a topology?

The first step in formulating étale cohomology is to rethink the notion of "topology". Instead of defining sheaves to be contravariant functors on $\mathbf{Top}_X$ (a.k.a $\mathbf{Zar}_X$ for schemes) satisfying locality and gluing, we want to generalize to other categories of schemes over X (with different morphisms than just inclusion of open subsets).

First, we ask ourselves, what sort properties does a category need to be a good "topology"? For this, we need to be able to define sheaves on the category, so we look back at the exact sequence

$$0 \to \mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \xrightarrow{f_{ij} - f_{ji}} \prod_{i \in I} \mathcal{F}(U_i \cap U_j)$$

The important part of this sequence is that we can define *open covers* $\{U_i\}_{i \in I}$ of objects $U \in \mathbf{Zar}_X$, so we axiomatize what makes an open cover a "covering":

Definition 2.1.1. A *covering* is a set of morphisms in a category $\mathscr{C}$ satisfying the following properties

1. Any isomorphism is a covering (inspired by $\{U \subset U\}$ is an open cover of U);

2. If $\{V_i \to U\}$ is a covering and $W \to U$ is a map, then $V_i \times_U W \in \mathscr{C}$ and $\{V_i \times_U W \to W\}$ is a covering (inspired by if $\{V_i \subset U\}$ is an open cover and $W \subset U$, then $\{(V_i \cap W) \subset W\}$ is an open cover);

3. If $\{W_{ij} \to V_i\}$ and $\{V_i \to U\}$ are coverings, then $\{W_{ij} \to U\}$ is a covering (inspired by if $\{W_{ij} \subset V_i\}$ and $\{V_i \subset U\}$ are open covers, then $\{W_{ij} \subset U\}$ is an open cover).

A *Grothendieck topology* on a given category is a defined set of coverings which satisfy these properties. A category together with a Grothendieck topology is called a *site*.

Note. From the definition above (and from a little bit of thinking about the properties of the fiber product) we see that the fiber product provides the perfect generalization of the intersection of open sets, and that this should be (part of) a guiding intuition when we start to define more exotic sites and Grothendieck topologies.

Example 2. The first examples that we can give are:

1. The *small Zariski site*, denoted X_{zar}, has as objects the open immersions of varieties into X, with morphisms the maps which commute with immersion. The coverings are the usual open covers of X, and this gives us what is essentially the standard Zariski topology on X.

2. The *big Zariski site*, denoted X_{Zar}, has as objects *all* varieties over X, with coverings the same as for the small Zariski site: collections of open immersions which are "jointly surjective".

The main distinction here is not the Grothendieck toplogy (since we have essentially chosen the same coverings), but rather the underlying category. The big site has more maps which are not a part of any covering, e.g. there exist $Y \to X$ which may not be a part of any covering, but in the small site all $Y \to X$ can be a part of a covering of X (since they are all open immersions).

Definition 2.1.2. Let $\mathbf{T}_1$ and $\mathbf{T}_2$ be two sites. A functor $\mathrm{Cat}(\mathbf{T}_2) \to \mathrm{Cat}(\mathbf{T}_1)$ preserving fibre products and transforming coverings into coverings is called a continuous map $\mathbf{T}_1 \to \mathbf{T}_2$.

Note. The reason why this is "backwards" is the same reason why continuous functions are defined by the fact that they pullback open sets, i.e. for $f : X \to Y$ a continous map of topological space we have that $f^{-1}(U) \subseteq X$ is open for $U \subseteq Y$ open.

Now we can define sheaves on these categories as contravariant functors which satisfy the usual exact sequence (has to be an equalizer for categories which are not abelian, but for our purpose we only work with sheaves with values in abelian categories)

$$0 \to \mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \xrightarrow{f_{ij}-f_{ji}} \prod_{i\in I} \mathcal{F}(U_i \times_X U_j)$$

for all coverings $\{U_i \to U\}$ on our site of choice.

2.2 Étale Morphisms

So, we have to think: what category of schemes over X we want to define, specifically, what sort of morphisms are useful to us?

The first morphisms which come to mind as being useful are flat morphisms. A flat morphism can be characterized by the fact that the fibers "vary continuously in families"; for example, a surjective morphism $f : X \to Y$ of smooth varieties is flat if and only if the fibers are equidimensional; and, a finite morphism to a reduced scheme is flat if and only if all fibers have the same number of points (counting multiplicity). Many interesting categories can be defined, including the *fppf* and *fpqc* topologies, both which are built from (faithfully) flat morphisms (with some extra requirements). These topologies do not fit the bill, though, as there are too many morphisms, so we need something in between a "flat topology" and the Zariski topology. (The flat sites are discussed in greater detail in the appendix on descent.)

The answer comes to us in the form of *étale morphisms*, which can be conceptually described as the analogue in algebraic geometry of a local isomorphisms of manifolds in differential geometry; a covering of Riemann surfaces with no branch points (i.e. ramification points) in complex analysis; or an unramified extension in algebraic number theory. Already, we see a certain "unification of ideas" coming to fore, drawing from number theory, complex analysis, and geometry. Specifically, we define

Definition 2.2.1. Let $f : Y \to X$ be a morphism of schemes. We say that it is *locally finite-type* for every $y \in Y$ there exists an affine open neighborhoud $\mathrm{Spec}(A) = U \subset Y$ of y and an affine open $\mathrm{Spec}(B) = V \subset X$ with $f(U) \subset V$ such that the induced ring homomorphism $B \to A$ is of finite-type, i.e. A is isomorphic to a quotient of $B[x_1, \ldots, x_n]$ as a B-algebra for some n.

Definition 2.2.2. Let $f : Y \to X$ be a morphism of locally finite-type. We say that f is *unramified at* $x \in X$ if for all $y \in Y$ such that $f(y) = x$, $\mathcal{O}_{Y,y}/\mathfrak{m}_x\mathcal{O}_{Y,y}$ is a separable extension of $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$. We say that f is *unramified* if it is unramified at every $x \in X$.

Equivalently, $f : Y \to X$ is unramified if $\Omega^1_{Y/X} = 0$ (where Ω^1 is the cotangent sheaf). We see this by using Nakayama's lemma to lift the local case of rings $A \to B$ (given by $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ for $\mathrm{Spec}(B) = U \subset Y$ and $\mathrm{Spec}(A) \subset X$) to the case of fields $A \to B$, from which it is a well known fact that if B/A is separable, then $\Omega^1_{B/A} = 0$ (see [Mat87]).

Definition 2.2.3. Let $f : Y \to X$ be morphism of schemes. We say that f is *étale* if it is unramified and flat. One can think of an étale morphism as a flat morphism with discrete, finite fibers with no branching points. If X is a reduced scheme, then the fibers will all have the same number of points (since non-reduced schemes could have nilpotent points, a.k.a "formal neighborhoods").

Note. To see the analogy between étale morphisms and local isomorphisms in the case of smooth manifolds, we recall the definition of locally isomorphic as

$$f : M \to N \quad \text{such that} \quad f_* : T_p(M) \xrightarrow{\sim} T_{f(p)}(N) \quad \text{for all } p \in M$$

in other words, if we think of algebraic varieties $f : Y \to X$ over S, we get

$$f^*\Omega^1_{X/S} \simeq \Omega^1_{Y/S}$$

Then, using the well-known exact sequence

$$f^*\Omega^1_{X/S} \to \Omega^1_{Y/S} \to \Omega^1_{Y/X} \to 0$$

we get that $\Omega^1_{Y/X} = 0$.

To build intuition and to see why we can form a site over a scheme X with étale morphisms, we state some facts about them:

Proposition 2.2.4. *(a) Any open immersion is étale, so the small Zariski site is a subcategory of the (yet to be defined) small étale site.*

(b) Let $f : Y \to Z$ and $g : Z \to X$ be étale morphisms. Then $g \circ f : Y \to X$ is an étale morphism.

(c) Let $f : Y \to X$ be étale and let $f' : Z \to X$ be a morphism of schemes. Then, the base change $Y \times_X Z \to Z$ is an étale morphism.

(d) If $\phi \circ \psi$ and ϕ are étale, then ψ is étale.

Proof. For the case of flat morphisms all four of these facts are known (and are not too hard to prove: see any standard reference on algebraic geometry e.g. [Har77]).

We are left to prove these statements when the morphisms are unramified.

For (a), all open immersions $f : Y \to X$ we have that $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective, so we get an isomorphism of quotients $\mathcal{O}_{Y,y}/\mathfrak{m}_x\mathcal{O}_{Y,y} \simeq \mathcal{O}_{X,x}/\mathfrak{m}_x\mathcal{O}_{X,x}$, which is obviously a separable extension.

For (b), we note that separability is transitive, i.e. if F/H is a separable field extension and H/K is a separable field extension, then F/K is separable field extension, so the result on étale morphisms follows immediately.

For (c), we note that we only need to check that for $p : Y \times_X Z \to Z$ that $\Omega^1_{Y \times_X Z/Z} = p^*(\Omega^1_{Y/X}) = 0$, which is clear since $\Omega^1_{Y/X} = 0$ by the definition of unramified.

For (d), using the exact sequence

$$f^*\Omega^1_{X/S} \to \Omega^1_{Y/S} \to \Omega^1_{Y/X} \to 0$$

we get the result by setting $\phi : X \to S$ and $\psi : Y \to X$. $\square$

Proposition 2.2.5. *Let $f : Y \to X$ be an étale morphism.*

(a) For all $y \in Y$, $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{X,f(y)}$ have the same Krull dimension;

(b) The f is quasi-finite;

(c) The morphism f is open;

(d) If X is reduced, then Y is reduced;

(e) If X is normal, then Y is normal;

(f) If X is regular, then Y is regular.

Proof. This follows from "standard descent arguments" which we review in the appendix. More specifically, certain "nice properties" of schemes are preserved by quasi-compact flat morphisms of schemes (fpqc-morphisms) or finitely presentated flat morphisms (fppf-morphisms) and étale morphisms share these properties (and more). This is all contingent on analysis of tensor products in the affine case, for which flat rings and modules are especially well-behaved. □

Having built understanding of what an étale morphism is and how it behaves, we make the following key definition:

Definition 2.2.6. Let X be a scheme. The (small) *étale site of* X, denoted $X_{ét}$, has as underlying category $\mathbf{Ét}/X$, whose objects are étale morphism $U \to X$ and whose arrows are the X-morphisms $\phi : U \to V$. The coverings are surjective families of étale morphisms (i.e. $\{U_i \to U\}$ étale such that $\bigcup_i U_i \to U$ is surjective).

Note. There are other variants of the étale site, which have different uses.

First, is the "big" étale site, denoted $X_{Ét}$, defined similarly to the big Zariski site; the underlying category is $\mathbf{Sch}/X$ with coverings being surjective families of étale morphisms.

Another is the *finite-étale* topology, denoted $X_{fét}$, useful for understanding the *étale fundamental group*, with the same underlying category as the small étale site, but with coverings being surjective families of finite étale morphisms.

2.3 Étale Sheaves

There are obvious continuous maps

$$X_{Ét} \to X_{ét} \to X_{zar}$$

Also, any sheaf $\mathcal{F}$ on $X_{ét}$ defines by restriction a sheaf on U_{zar} for every $U_{zar} \to X$ étale. First, we want a criterion for checking when a presheaf on the étale site is a sheaf:

Proposition 2.3.1. *Let $\mathcal{F}$ be a presheaf on $X_{ét}$. If $\mathcal{F}$ satisfies the sheaf conditions for Zariski open coverings and for étale open coverings $V \to U$ consisting of a single map with V and U both affine, then $\mathcal{F}$ is a sheaf on $X_{ét}$.*

Proof. If $\mathcal{F}$ satisfies the sheaf condition for Zariski open coverings, the $\mathcal{F}\left(\bigsqcup U_i\right) = \prod \mathcal{F}(U_i)$. From this, it follows that, because

$$\left(\bigsqcup U_i\right) \times_U \left(\bigsqcup U_i\right) = \bigsqcup_{(i,j)\in I\times I} (U_i \times_U U_j)$$

we have that

$$\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \times_U U_j)$$

and

$$\mathcal{F}(U) \to \mathcal{F}\left(\bigsqcup U_i\right) \rightrightarrows \mathcal{F}\left((\bigsqcup U_i) \times_U (\bigsqcup U_i)\right)$$

are equivalent sequences, so checking the sheaf conditions for a covering of the form $\{U_i \to U\}$ is equivalent to checking the sheaf condition for a covering of the form $\{\bigsqcup U_i \to U\}$.

Let $f : V \to U$ be a covering, with $U = \bigcup U_i$ where U_i is affine open and $f^{-1}(U_i) = \bigcup V_{ij}$ a union of affine opens in V. For each i, j, $f(V_{ij}) \subset U_i$ is open and V_{ij} is quasi-compact, so there is a finite set J_i such that $\{V_{ij} \to U_i\}_{j \in J_i}$ is a covering. Consider the diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{F}(V) \arrow[r] \arrow[d] & \mathcal{F}(V \times_U V) \arrow[d] \\
\prod_i \mathcal{F}(U_i) \arrow[r] \arrow[d] & \prod_i \prod_j \mathcal{F}(V_{ij}) \arrow[r] \arrow[d] & \prod_i \prod_{j,k} \mathcal{F}(V_{ij} \times_U V_{ik}) \\
\prod_{i,n} \mathcal{F}(U_i \cap U_n) \arrow[r] & \prod_{i,n} \prod_{j,k} \mathcal{F}(V_{ij} \cap V_{nk}) &
\end{tikzcd}$$

By above, the columns are exact. The middle row is the product of exact sequences, by the assumption that the sheaf condition holds for affine covers, so it is exact. It follows that the bottom row is injective and that $\mathcal{F}$ is a separated presheaf. Then, by diagram chasing, the top row is exact, and it suffices to check on affine $V \to U$. $\square$

We will see later that there are many methods of comparison between sheaves on the Zariski site and sheaves on the étale site, specifically using the Grothendieck spectral sequence for certain comparisons between Zariski sheaf cohomology and étale sheaf cohomology.

Here are some examples of sheaves on the étale site:

Example 3. Let $A \to B$ be a ring homomorphism corresponding to a surjective étale morphism $V \to U$ of affine varieties (or schemes). In checking the second condition in the above proposition, we shall usually make use only of the fact that $A \to B$ is faithfully flat (i.e. $A \to B$ is flat and $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is surjective), and will not be needing the fact that it is unramified.

1. (*The structure sheaf on $X_{ét}$*): For any $U \to X$ étale, define $\mathcal{O}_{X_{ét}}(U) = \Gamma(U, \mathcal{O}_U)$. Certainly, its restriction to U_{zar} is a sheaf for any U étale over X, so we only need to check the case of $V \to U$ affine (by the above proposition). To prove this, we need an algebraic proposition:

   Proposition 2.3.2. *For every faithfully flat homomorphism $A \to B$, the following sequence is exact*

   $$0 \to A \to B \xrightarrow{b \mapsto 1 \otimes b - b \otimes 1} B \otimes_A B$$

This proposition gives us exactly the "sheaf condition" sequence when we move to the structure sheaf for affine schemes, so $\mathcal{O}_{X_{\acute{e}t}}$ is a sheaf on $X_{\acute{e}t}$.

2. (*The sheaf defined by a scheme* Z): An X-scheme Z defines a contravariant functor:
$$\mathrm{Hom}_X(-, Z) : \mathbf{\acute{E}t}/X \to \mathbf{Sets}$$
$$U \mapsto \mathrm{Hom}_X(U, Z)$$
This is clearly a sheaf of sets on the Zariski open coverings, so it suffices to show that
$$Z(A) \to Z(B) \rightrightarrows Z(B \otimes_A B)$$
is exact for any faithfully flat ring homomorphism $A \to B$. If Z is affine, i.e. $Z = \mathrm{Spec}(R)$, then the sequence becomes
$$\mathrm{Hom}_{A-alg}(R, A) \to \mathrm{Hom}_{A-alg}(R, B) \rightrightarrows \mathrm{Hom}_{A-alg}(R, B \otimes_A B)$$
whose exactness follows immediately from the same algebraic proposition used to prove that the structure sheaf is a sheaf. For the case of a non-affine Z, we recall that for an inverse system of sheaves $(\mathcal{F}_i, \phi_{ij})$, the presheaf $U \mapsto \varprojlim\{\mathcal{F}_i(U)\}$ is a sheaf, so, understanding Z as the direct limit of the affine covering $\{Z_i\}$ along with all intersections (in this case fibre products) and appropriate transition (a.k.a. gluing) maps, we have the following inverse limit
$$\mathrm{Hom}_X(-, Z) \simeq \varprojlim \mathrm{Hom}_X(-, Z_i)$$
where the Z_i and their intersections are all affine, so, the non-affine case reduces to the affine case (essentially, we've realized the functor $\mathrm{Hom}_X(-, Z)$ as a projective limit of appropriate subfunctors).

Note that if Z is a group scheme, then $\mathrm{Hom}_X(-, Z)$ will be a sheaf of groups. Examples of important group schemes which we often think of as sheaves of groups include $\mathbb{G}_a$, $\mathbb{G}_m$, and GL_n.

3. (*Constant sheaves*): Let X be a variety or a quasi-compact scheme. For every set Λ, define
$$\underline{\Lambda}(U) = \Lambda^{\pi_0(U)}$$
products of copies of Λ indexed by the set $\pi_0(U)$ of connected components of U. With the obvious restriction maps, this is a sheaf, called the *constant sheaf* on $X_{\acute{e}t}$ defined by Λ. If Λ is finite, then it is the sheaf defined by the scheme $X \times \Lambda$ (disjoint union of copies of X indexed by Λ). When Λ is a group, then $\underline{\Lambda}$ is a sheaf of groups.

4. (*The sheaf defined by a coherent* $\mathcal{O}_X$*-module*): Let $\mathcal{M}$ be a sheaf of coherent $\mathcal{O}_X$-modules on X_{zar} (in the usual algebro-geometric sense). For every étale map $\phi : U \to X$, we obtain a coherent $\mathcal{O}_U$-module $\phi^*\mathcal{M}$ on U_{zar}. There is a presheaf on $X_{\acute{e}t}$ defined by
$$\mathcal{M}^{\acute{e}t}(U) := \Gamma(U, \phi^*\mathcal{M})$$

To verify that $\mathcal{M}^{ét}$ is a sheaf, it suffices to show that the following sequence is exact when $A \to B$ is fully faithful, with M the A-module corresponding to $\mathcal{M}$:

$$0 \to M \to B \otimes_A M \rightrightarrows B \otimes_A B \otimes_A M$$

This is true; all that we've done is tensor the above sequence which we checked by M (i.e. it is proved to be exact in essentially the same way).

5. (*The sheaves on* $\mathrm{Spec}(k)$): Let k be a field and let $\textbf{Ét}/k$ be the category of étale k-algebras. A presheaf $\mathcal{F}$ of abelian groups on $(\mathrm{Spec}(k))_{ét}$ can be regarded as a covariant functor $\mathcal{F} : \textbf{Ét}/k \to \textbf{Ab}$. Such a functor, by our above criterion and the fact that $\textbf{Ét}/k$ consists of all finite separable extensions of k, will be a sheaf if and only if $\mathcal{F}(\prod A_i) = \bigoplus \mathcal{F}(A_i)$ for every finite family $\{A_i\}$ of étale k-algebras and $\mathcal{F}(k') \xrightarrow{\sim} \mathcal{K}^{\mathrm{Gal}(K/k')}$ for every finite Galois extension K/k' of fields with k' of finite degree over k (in the context of Galois cohomology, $\mathcal{F}$ would be called a Galois functor, satisfying the Galois descent condition).

   Choose a separable closure k^s of k, and let $G_k = \mathrm{Gal}(k^s/k)$. For $\mathcal{F}$ a sheaf on $(\mathrm{Spec}(k))_{ét}$, define

   $$M_{\mathcal{F}} = \varinjlim \mathcal{F}(k')$$

   where k' runs through the subfields $k' \subset k^s$ that are finite and Galois over k. Then, $M_{\mathcal{F}}$ is a discrete G_k-module.

   Conversely, if M is a discrete G_k-module, we define for any étale k-algebra A

   $$\mathcal{F}_M(A) = \mathrm{Hom}_{G_k}(F(A), M)$$

   where $F(A)$ is the G_k-set $\mathrm{Hom}_{k-alg}(A, k^s)$ (think "k^s-points of $\mathrm{Spec}(A)$"). Then, $\mathcal{F}_M$ is a sheaf on $\mathrm{Spec}(k)_{ét}$.

   The functors $\mathcal{F} \mapsto M_{\mathcal{F}}$ and $M \mapsto \mathcal{F}_M$ define an equivalence of between the category of sheaves on $(\mathrm{Spec}(k))_{ét}$ and the category of discrete G_k-modules. This case will be explored in detail in a later section.

Now that we have a few examples of étale sheaves on our tool-belts, we can consider what the category of étale sheaves, $\mathbf{Sh}(X_{ét})$, might look like. First, it is easier to discuss the category of presheaves $\mathbf{PreSh}(X_{ét})$, by seeing that is is merely the category of contravariant functors $\mathcal{F} : X_{ét}^{op} \to \mathbf{Ab}$ with morphisms being the natural transformations, i.e. $\Phi : \mathcal{F} \to \mathcal{G}$ is defined as a family of maps $\Phi = \{\{\Phi_U\}_{U \in X_{ét}}$ such that for all $U, V \in X_{ét}$ and $f \in \mathrm{Hom}_X(U, V)$, the following diagram commutes

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\Phi_U"] \arrow[d, "\mathcal{F}(f)"] & \mathcal{G}(U) \arrow[d, "\mathcal{G}(f)"] \\ \mathcal{F}(V) \arrow[r, "\Phi_V"] & \mathcal{G}(V) \end{tikzcd}$$

Next, we just realize that $\mathbf{Sh}(X_{ét})$ is a full subcategory of $\mathbf{PreSh}(X_{ét})$, so we just carry over the same morphisms but with restrictions on objects.

The next question becomes, when we are given a presheaf, how do we "sheafify" it?

Definition 2.3.3. Let $\mathcal{P} \to a\mathcal{P}$ be a homomorphism from a presheaf $\mathcal{P}$ to a sheaf $a\mathcal{P}$. Then, $a\mathcal{P}$ is said to be the *sheaf associated with* $\mathcal{P}$ (or the *sheafification of* $\mathcal{P}$) if all other homomorphisms from $\mathcal{P}$ to a sheaf factors uniquely through $a\mathcal{P}$, in other words

$$\mathrm{Hom}(\mathcal{P}, \mathcal{F}) \simeq \mathrm{Hom}(a\mathcal{P}, \mathcal{F})$$

for all sheaves $\mathcal{F}$. Clearly, the sheafification is unique up to unique isomorphism (if it exists).

Before we discuss how to sheafify or criterion for a sheaf associated to a presheaf, we need a couple more definitions

Definition 2.3.4. Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on $X_{ét}$. We say that α is *locally surjective* if, for every $U \in X_{ét}$ and $s \in \mathcal{G}(U)$, there exists a covering $\{U_i \to U\}$ such that $s|_{U_i}$ is in the image of $\mathcal{F}(U_i) \to \mathcal{G}(U_i)$ for each i.

Lemma 2.3.5. *Let $i : \mathcal{P} \to \mathcal{F}$ be a homomorphism from a presheaf $\mathcal{P}$ to a sheaf $\mathcal{F}$. Assume*

(a) The only sections of $\mathcal{P}$ to have the same image in $\mathcal{F}(U)$ are those which are locally equal;

(b) i is locally surjective.

Then $(\mathcal{F}, i)$ is the sheaf associated with $\mathcal{P}$.

Proof. Let $i' : \mathcal{P} \to \mathcal{F}'$ be a second map from $\mathcal{P}$ into a sheaf, and let $s \in \mathcal{F}(U)$ for some $U \in X_{ét}$). By (b), we know that some covering $\{U_i \to U\}$ there exist $s_i \in \mathcal{P}(U_i)$ such that $i(s_i) = s|_{U_i}$. By (a), $i'(s_i) \in \mathcal{F}'(U_i)$ is independent of the choice of s_i, and moreover that the restrictions of $i'(s_i)$ and $i'(s_j)$ to $\mathcal{F}'(U_i \times_U U_j)$ agree. We define $\alpha(s)$ to be the unique element of $\mathcal{F}'(U)$ that restricts to $i'(s_i)$ for all i. Then $s \mapsto \alpha(s) : \mathcal{F} \to \mathcal{F}'$ is unique such that $i \circ \alpha = i'$. $\square$

Now, let's talk about an important tool for the study of sheaves:

Definition 2.3.6. Let $\mathcal{P}$ be a presheaf. Sections $s_1, s_2 \in \mathcal{P}(U)$ are said to be *locally equal* if $s_1|_{U_i} = s_2|_{U_i}$ for all U_i in some covering $\{U_i \to U\}$. The sheaf criterion implies that locally equal sections of a sheaf are equal.

Definition 2.3.7 (Stalks)**.** An *étale neighborhood* of a point $x \in X$ is an étale map $g : U \to X$ together with a point $u \in U$ such that $g(u) = x$. A morphism of étale neighborhoods is defined in the obvious manner, i.e. a regular map $(U, u) \to (V, v)$ étale such that $u \mapsto v$ and the maps commute.

Let $\overline{x} \to X$ be a geometric point of X such that $\overline{x} \mapsto x$ and let $\mathcal{F} \in \mathbf{Sh}(X_{ét})$. We define the *stalk of* $\mathcal{F}$ *at* $x \in X$ as

$$\mathcal{F}_{X,\overline{x}} := \varinjlim_{(U,u)} \mathcal{F}(U)$$

where the limit is taken over the étale neigborhoods of x

Example 4. A couple examples of stalks include:

(a) The stalk of $\mathcal{O}_{X_{ét}}$ at $\overline{x}$ is the strictly local ring at $\overline{x}$, $\mathcal{O}_{X,\overline{x}}$.

(b) For a scheme Z of finite type over X, the stalk of $\mathrm{Hom}_X(-,Z)$ at $\overline{x}$, is $Z(\mathcal{O}_{X,\overline{x}})$. For example,

   (i) μ_n gives $\mu_n(\mathcal{O}_{X,\overline{x}})$, the nth roots of unity of $\mathcal{O}_{X,\overline{x}}$.

   (ii) $\mathbb{G}_a$ gives $\mathcal{O}_{X,\overline{x}}$, regarded as an additive abelian group.

   (iii) $\mathbb{G}_m$ gives $\mathcal{O}^{\times}_{X,\overline{x}}$, regarded as a multiplicative abelian group.

   (iv) GL_n give $\mathrm{GL}_n(\mathcal{O}_{X,\overline{x}})$, the group of $n \times n$ matrices with coefficients in $\mathcal{O}_{X,\overline{x}}$ (and nonzero determinant).

(c) Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module. The stalk of $\mathcal{M}^{et}$ at $\overline{x}$ is $\mathcal{M}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,\overline{x}}$, where $\mathcal{M}_x$ is the stalk of $\mathcal{M}$ at x as a sheaf for the Zariski topology.

(d) Let k be a field. For a sheaf $\mathcal{F}$ on $\mathrm{Spec}(k)$, that stalk at $\overline{x} = \mathrm{Spec}(k^s) \to \mathrm{Spec}(k)$ is the G_k-module $M_{\mathcal{F}}$ regarded as an abelian group.

As a corollary of the above lemma, we get

Corollary 2.3.8. *If* $i : \mathcal{P} \to \mathcal{F}$ *satisfies the conditions (a) and (b) of the lemma above, then*

$$i_{\overline{x}} : \mathcal{P}_{\overline{x}} \to \mathcal{F}_{\overline{x}}$$

is an isomorphism for all geometric point $\overline{x}$.

So, intuitively, what differs a sheaf from a presheaf is exactly how it "glues" from stalks up to open sets. This makes a big difference, as it allows for us to analyze if a sequence of sheaves is exact by checking the stalks (this follows from general facts on sheaves). It also allows the following construction:

Definition 2.3.9. For each $x \in X$, choose a geometric point $\overline{x} \to X$ with image x. For a presheaf $\mathcal{P}$ on X, define

$$\mathcal{P}^* := \prod_{x \in X} (\mathcal{P}_{\overline{x}})^{\overline{x}}$$

where $(\mathcal{P}_{\overline{x}})^{\overline{x}}$ is the skyscraper sheaf at x associated to the abelian group $\mathcal{P}_{\overline{x}}$. Then, $\mathcal{P}^*$ is a sheaf (since is it a product of sheaves) and the natural morphism $\mathcal{P} \to \mathcal{P}^*$ it satisfies condition (a), i.e. the only sections of $\mathcal{P}$ to have the same image in $\mathcal{P}^*(U)$ are those that are locally equal.

From this, we deduce a theorem

Theorem 2.3.10. *For every presheaf $\mathcal{P} \in \mathbf{PreSh}(X_{ét})$, there exists an associated sheaf $i : \mathcal{P} \to a\mathcal{P}$. The map i induces isomorphisms on the stalks, and the functor $a : \mathbf{PreSh}(X_{ét}) \to \mathbf{Sh}(X_{ét})$ is exact.*

Proof. Take $a\mathcal{P}$ to be the subsheaf of $\mathcal{P}^*$ generated by $i(\mathcal{P})$ (i.e. the subsheaf of $\mathcal{P}^*$ such that, for all i, $s \in a\mathcal{P}(U)$ restrict to $s|_{U_i} \in (\mathcal{P})(U_i)$ for some covering $\{U_i \to U\}$). Then $i : \mathcal{P} \to a\mathcal{P}$ satisfies the conditions (a) and (b), from which the first two statements follow.

For an exact sequence $\mathcal{P}' \to \mathcal{P} \to \mathcal{P}''$, then $\mathcal{P}'_{\overline{x}} \to \mathcal{P}_{\overline{x}} \to \mathcal{P}''_{\overline{x}}$ is exact for all $x \in X$, but this can be identifies with $(a\mathcal{P}')_{\overline{x}} \to (a\mathcal{P})_{\overline{x}} \to (a\mathcal{P}'')_{\overline{x}}$, which, by sheaf conditions, implies that $a\mathcal{P}' \to a\mathcal{P} \to a\mathcal{P}''$ is exact. □

As we know, the category of sheaves (with values in an abelian category) over the Zariski site form an abelian category. By similar argument (which can be generalized to include sheaves on any site), the sheaves on the étale site also form an abelian category. It is encouraged that one thinks this through, as an exercise, to get used to the formal constructions surrounding the étale site. The basic idea is that morphisms of sheaves are determined by behavior at the stalks, and then that the co-image of a morphisms to its image is an isomorphism at the stalks (this is also why $\mathbf{PreSh}(X_{ét})$ does not form an abelian category; we can't "glue" up from the stalks. Essentially, the sheaf conditions guarantee exactness and compatibility of exactness at every "level"). Since we defined stalks in the same "categorical" sense as with usual sheaves (i.e. as a limit), we immediately get the same formal results. This is how "abstract nonsense" is useful, but we also have to be careful not to brush results like this under the rug.

Example 5. Here are some important exact sequences of sheaves, which will be used later

1. (Kummer Sequence) Let n be an integer that is not divisible by the characteristic of any residue field of X. Consider the sequence

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 0$$

To prove that this is exact, we have to check at the stalks, i.e. if

$$0 \to \mu_n(A) \to A^\times \xrightarrow{n} A^\times \to 0$$

is exact for every strictly local ring $A = \mathcal{O}_{X,\overline{x}}$ of X. This is obvious except at the second $A^\times$, and here we have to show that every element of $A^\times$ is an nth power. However, for all $a \in A$

$$\frac{d(T^n - a)}{dT} = nT^{n-1} \neq 0$$

in the residue field of A, and so $T^n - a$ splits in $A[T]$ (since we are dealing with étale morphisms), and so the nth root of a is in A.

2. (Artin-Schreier Sequence) Let X be a variety over a field k of characteristc $p \neq 0$ and consider the sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_a \xrightarrow{t \mapsto t^p - t} \mathbb{G}_a \to 0$$

Again, in order to check that this sequence is exact, we have to check at the stalks; is the sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to A \xrightarrow{t \mapsto t^p - t} A \to 0$$

exact for every strictly local ring $A = \mathcal{O}_{X,\overline{x}}$ of X. This is obvious except at the second $A^\times$, but

$$\frac{d(T^p - T - a)}{dt} = -1 \neq 0$$

in the residue field of A, and so $T^p - T - a$ splits in $A[T]$.

Note. If p divides the characteristic of some residue field of X, then

$$0 \to \mu_p \to \mathbb{G}_m \xrightarrow{p} \mathbb{G}_m \to 0$$

will not be exact for the étale topology on X. However, it will be exact for the flat topology, because the equation $T^p - a$ defines a flat covering of U for any $a \in \Gamma(U, \mathcal{O}_U)$.

2.4 Direct and Inverse Images of Sheaves

Given a morphism of schemes, we want to understand how étale sheaves behave under pushforward and pullback, a.k.a direct and inverse images of sheaves.

Definition 2.4.1 (Direct images of sheaves)**.** Let $\pi : X \to Y$ be a morphism of schemes, and let $\mathcal{P}$ be a presheaf on $Y_{\acute{e}t}$. For $U \to X$ étale, define

$$\pi_* \mathcal{P}(U) := \mathcal{P}(U \times_X Y$$

Since $U \times_X Y \to Y$ is étale, this definition makes sense. With the obvious restriction maps, $\pi_* \mathcal{P}$ becomes a presheaf on $X_{\acute{e}t}$.

Lemma 2.4.2. *If $\mathcal{F}$ is a sheaf, then $\pi_* \mathcal{F}$ is a sheaf.*

Proof. For a scheme V over X, let V_Y denote the base change $V \times_X Y$ over Y. Then $V \mapsto V_Y$ is a functor taking étale maps to étale maps, surjective families of maps to surjective families, and fibre products over X to fibre products over Y.

Let $\{U_i \to U\}$ be a covering in $X_{\acute{e}t}$. Then $\{U_{iY} \to U_Y\}$ is a covering in $Y_{\acute{e}t}$, and so

$$\mathcal{F}(U_Y) \to \prod \mathcal{F}(U_{iY}) \rightrightarrows \prod \mathcal{F}\left(U_{iY} \times_X U_{jY}\right)$$

is exact. But this equals the sequence

$$(\pi_* \mathcal{F})(U) \to \prod (\pi_* \mathcal{F})(U_i) \rightrightarrows \prod (\pi_* \mathcal{F})\left(U_i \times_X U_j\right)$$

which is therefore also exact. ∎

Obviously, the functor $\pi_* : \mathbb{P}\smallsetminus\mathbb{S}\eqsim(Y_{ét}) \to \mathbf{PreSh}(X_{ét})$ is exact. Therefore, its restriction to sheaves, $\pi_* : \mathbf{Sh}(Y_{ét}) \to \mathbf{Sh}(X_{ét})$ is left exact. Local surjectivity does not necessarily carry over, so it is not generally right exact.

A geometric point $\overline{y} \to Y$ defines a geometric point $\overline{y} \to Y \to X$, which we denote $\overline{x}$. Clearly, $(\pi_*\mathcal{F})_{\overline{x}} = \varinjlim \mathcal{F}(V)$, where the limit is over all étale neighborhoods of $\overline{y}$ of the form U_Y for some étale neighborhood of $\overline{x}$. Thus, there is a canonical map $(\pi_*\mathcal{F})_{\overline{x}} \to \mathcal{F}_{\overline{y}}$.

Proposition 2.4.3. *Let X be a scheme.*

(a) Let $\pi : V \hookrightarrow X$ be an open immersion. Then, for $x \in V$ and $\mathcal{F} \in \mathbf{Sh}(V_{ét})$, we have that

$$(\pi_*\mathcal{F})_{\overline{x}} = \mathcal{F}_{\overline{x}}$$

For points outside of V, we don't have a general conclusion.

(b) Let $\pi : Z \to X$ be a closed immersion. Then, for $\mathcal{F} \in \mathbf{Sh}(X_{ét})$,

$$(\pi_*\mathcal{F})_{\overline{x}} = \begin{cases} \mathcal{F}_{\overline{x}} & \text{for } x \in Z \\ 0 & \text{for } x \notin Z \end{cases}$$

(c) Let $\pi : Y \to X$ be a finite map. Then

$$(\pi_*\mathcal{F})_{\overline{x}} = \bigoplus_{y \mapsto x} \mathcal{F}_{\overline{y}}^{d(y)}$$

where $d(y)$ is the separable degree of $k(y)$ over $k(x)$. For example, if π is a finite étale map of degree d of varieties over an algebraically closed field, then

$$(\pi_*\mathcal{F})_{\overline{x}} = \mathcal{F}_{\overline{x}}^d$$

Proof. When $V \hookrightarrow X$ is an immersion (either open or closed), the fibre product $U \times_X V = \phi^{-1}(V)$ for any morphism $\phi : U \to X$.

(a) If $x \in V$, then for any "sufficiently small" étale neighborhood $\phi : U \to X$ of $\overline{x}$, $\phi(U) \subset V$, and so $U = \phi^{-1}(V) = U \times_X V$. Thus, the étale neighborhoods of $\overline{x}$ of the form $U_V = U \times_X V$ form a cofinal set, so the limits defining the stalks are equal, which proves the equality. Concerning points outside of V, we point out that $(\pi_*\mathcal{F})_{\overline{x}}$ need not be zero.

(b) If $x \notin Z$, then for any "sufficiently small" étale neighborhood $\phi : U \to X$ of $\overline{x}$, $\phi(U) \cap Z = \varnothing$, and so $U_Z = U \times_X Z = \phi^{-1}(Z) = \varnothing$; thus $\mathcal{F}(U_Z) = \mathcal{F}(\varnothing) = 0$.

When $x \in Z$ we see that every étale map $\overline{\phi} : \overline{U} \to Z$ with $x \in \phi(\overline{U})$ "extends" to an étale map $\phi : U \to X$. In terms of rings, this amounts to showing that an étale homomorphism $\overline{A} \to \overline{B}$, $\overline{A} = A/\mathfrak{a}$, lifts to an étale homomorphism $A \to B$. We may assume that $\overline{B} = (\overline{A}[T]/(\overline{f}(T)))_b$ by standard properties of étale morphisms. Choose $f(T) \in A[t]$ lifting $\overline{f}(T)$, and set $B = (A[T]/(f(T)))_b$ for appropriate b.

(c) (sketch of proof) Over each geometric point $\overline{x} \to X$, the fiber Y_x is a finite discrete set of points, and since we care about the étale neighborhoods, we only need to worry about the separable degree of $k(y)/k(x)$ for any $y \in \pi^{-1}(x)$, i.e. the intermediate fields $k(y)/F/k(x)$ which are separable over $k(x)$.

$\square$

Corollary 2.4.4. *The functor π_* is exact if π is finite or a close immersion.*

Definition 2.4.5 (Inverse images of sheaves)**.** Let $\pi : Y \to X$ be a morphism of schemes. We shall define a left adjoint for the functor π_*. Let $\mathcal{P} \in \mathbf{PreSh}(X_{\acute{e}t})$. For $V \to Y$ étale, define

$$\mathcal{P}'(V) = \varinjlim \mathcal{P}(U)$$

where the direct limit is over the commutative diagrams

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

for $U \to X$ étale. For any presheaf $\mathcal{Q} \in \mathbf{PreSh}(Y_{\acute{e}t})$, there are naturaly one-to-one correspondences between

- morphisms $\mathcal{P}' \to \mathcal{Q}$;
- Families of maps $\mathcal{P}(U) \to \mathcal{Q}(V)$, indexed by commutative diagrams as above, compatible with restriction maps;
- morphisms $\mathcal{P} \to \pi_*\mathcal{Q}$

Thus,

$$\mathrm{Hom}_Y(\mathcal{P}', \mathcal{Q}) \simeq \mathrm{Hom}_X(\mathcal{P}, \pi_*\mathcal{Q})$$

functorially in $\mathcal{P}$ and $\mathcal{Q}$. Unfortunately, $\mathcal{P}'$ need not be a sheaf even when $\mathcal{P}$ is. Therefore, for a sheaf $\mathcal{F} \in \mathbb{S}\eqsim(X_{\acute{e}t})$, we define $\pi^*\mathcal{F} = a(\mathcal{F}')$. Then, for any sheaf $\mathcal{G} \in \mathbf{Sh}(Y_{\acute{e}t})$

$$\mathrm{Hom}_Y(\pi^*\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_Y(\mathcal{F}', \mathcal{G}) \simeq \mathrm{Hom}_X(\mathcal{F}, \pi_*\mathcal{G})$$

and so π^* is left-adjoint to $\pi_* : \mathbf{Sh}(Y_{\acute{e}t}) \to \mathbf{Sh}(X_{\acute{e}t})$.

Note. Let $i : \overline{x} \to X$ be a geometric point. For any sheaf $\mathcal{F} \in \mathbf{Sh}(X_{\acute{e}t})$, $(i^*\mathcal{F})(\overline{x}) = \mathcal{F}_{\overline{x}}$ (clear by the definitions of $\mathcal{F}_{\overline{x}}$ and i^*). Therefore, for any morphism $\pi : Y \to X$ and geometric point $i : \overline{y} \to Y$

$$(\pi^*\mathcal{F})_{\overline{y}} = i^*(\pi^*\mathcal{F})(\overline{y}) = \mathcal{F}_{\overline{x}}$$

where $\overline{x}$ is the geometric point $\overline{y} \xrightarrow{i} Y \xrightarrow{\pi} X$.

Since this is true for all geometric points of Y, we see that π^* is exact, and therefore π_* preserves injectives (by homological algebra).

Let X be a scheme, and let $j : U \hookrightarrow X$ be an open immersion. As we noted above, for a sheaf $\mathcal{F}$ on $U_{\acute{e}t}$, the stalks of $j_*\mathcal{F}$ need not be zero at points outside of U. We now define a functor $j_!$, the "extension-by-zero", such that $j_!\mathcal{F}$ does have this property.

Definition 2.4.6. Let $\mathcal{P}$ be a presheaf on $U_{\acute{e}t}$. For any $\phi : V \to X$ étale, define

$$\mathcal{P}_!(V) = \begin{cases} \mathcal{P}(V) & \text{if } \phi(V) \subset U \\ 0 & \text{otherwise} \end{cases}$$

Then $\mathcal{P}_!$ is a presheaf on $X_{\acute{e}t}$, and for any presheaf $\mathcal{Q}$ on $X_{\acute{e}t}$, a morphism $\mathcal{P} \to \mathcal{Q}|_U$ extends uniquely to a morphism $\mathcal{P}_! \to \mathcal{Q}$. Thus

$$\mathrm{Hom}_X(\mathcal{P}_!, \mathcal{Q}) \simeq \mathrm{Hom}_U(\mathcal{P}, \mathcal{Q}|_U)$$

Unfortunately,$\mathcal{P}_!$ need not be a sheaf even if $\mathcal{P}$ is. Thus, for $\mathcal{F}$ a sheaf on $U_{\acute{e}t}$, we define $j_!\mathcal{F} = a(\mathcal{F}_!)$. Then, for any sheaf $\mathcal{G}$ on $X_{\acute{e}t}$

$$\mathrm{Hom}_X(j_!\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_X(\mathcal{F}_!, \mathcal{G}) \simeq \mathrm{Hom}_U(\mathcal{F}, \mathcal{G}|_U)$$

and so $j_!$ is left adjoint to $j^* : \mathbf{Sh}(X_{\acute{e}t}) \to \mathbf{Sh}(U_{\acute{e}t})$.

Proposition 2.4.7. *Let $j : U \hookrightarrow X$ be an open immersion. For any sheaf $\mathcal{F}$ on $U_{\acute{e}t}$ and geometric point $\overline{x} \to X$*

$$(j_!\mathcal{F})_{\overline{x}} = \begin{cases} \mathcal{F}_{\overline{x}} & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

Proof. By the fact that the sheaf associated to a presheaf has isomorphic stalks, it suffices to prove this for $\mathcal{F}_!$, from which the results follows by definition. $\square$

Then, by similar reasoning as with π^* and π_*, we get that:

Corollary 2.4.8. *The functor $j_! : \mathbf{Sh}(U_{\acute{e}t}) \to \mathbf{Sh}(U_{\acute{e}t})$ is exact, and j^* preserves injectives.*

From these functors, $j_!, j^*, i_*, i^*$, we get the following result:

Proposition 2.4.9. *Let $j : U \hookrightarrow X$ be an open immersion, and let $Z = X \setminus U$ with closed immersion $i : Z \hookrightarrow X$. For any $\mathcal{F} \in \mathbf{Sh}(X_{\acute{e}t})$, there are canonical morphisms $j_!j^*\mathcal{F} \to \mathcal{F}$ and $\mathcal{F} \to i_*i^*\mathcal{F}$ corresponding to adjointess to the identity map on $j^*\mathcal{F}$ and $i^*\{$, respectively. Then, the sequence*

$$0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0$$

is exact.

Proof. We check on stalks: For $x \in U$, the sequence of stalks is

$$0 \to \mathcal{F}_{\overline{x}} \xrightarrow{id} \mathcal{F}_{\overline{x}} \to 0 \to 0$$

and for $x \notin U$, the sequence of stalks is

$$0 \to 0 \to \mathcal{F}_{\overline{x}} \xrightarrow{id} \mathcal{F}_{\overline{x}} \to 0$$

Both are visibly exact. ☐

Beyond this, the functor $j_!$ will be used in defining cohomology with compact support, which will then be important for our proof of the fundamental theorems.

Topological Invariance of the Étale Site

There is one last important property of the étale site which we must review:

Proposition 2.4.10. *Let $f : X \to Y$ be a morphism of schemes. Assume f is integral, universally injective and surjective, in other words, f is a universal homeomorphism. The functor*

$$V \mapsto V_X = X \times_Y V$$

defines an equivalence of categories

$$\{schemes\ V\ \text{\'etale}\ over\ Y\} \Longleftrightarrow \{schemes\ U\ \text{\'etale}\ over\ X\}$$

A proof can be found at [Sta20, Section 04DY].

Note. From this equivalence, we also get an equivalence between $\mathbf{Sh}(X_{\acute{e}t})$ and $\mathbf{Sh}(Y_{\acute{e}t})$, which is where we really use this equivalence, in the context of sheaf cohomology.

Though we will not provide a general proof, we will give a specific case which we will repeatedly use: if $f : X \to Y$ is a (purely) inseparable morphism, then we have the equivalence $\mathbf{Sh}(Y_{\acute{e}t}) \Leftrightarrow \mathbf{Sh}(X_{\acute{e}t})$. To see why this might be true, we note that if we have K/k a purely inseparable extension of fields, and if we let K^s/K be a separable closure with Galois group $G_K = \mathrm{Gal}(K^s/K)$. Then, if we denote the separable closure of $k \subset K^s$ as k^s/k, then we have that $G_K = G_k = \mathrm{Gal}(k^s/k)$. In this way, the category of G_K-modules and G_k-modules are equivalent, so their Galois cohomology is equivalent. Generalizing this thought to schemes, we see that an inseparable morphism should give an equivalence between the corresponding categories of sheaves. In particular, étale sheaf theory on a scheme X over a field k is equivalent to étale sheaf theory on $X \times_k k_p$, where k_p is the perfect closure of k.

2.5 Étale Fundamental Group

3 Étale Cohomology

We now want to analyze the cohomology of sheaves on the étale site.

3.1 Basic Definitions

Étale cohomology is essentially defined in exactly the same way as sheaf cohomology for the Zariski site. After defining étale cohomology and proving that it is well-defined, we will compute and study some concrete examples in increasing order of dimension. We will find a connection with Galois cohomology in the case of $\mathrm{Spec}(k)$, as we have hinted at before, a connection with the Tate module in the case of algebraic curves, and start to understand what is "new" about étale cohomology by studying the simplest-complicated case: that of a surface.

Finally, we discuss the difficulty of torsion vs torsion-free cohomology, and construct the correct torsion-free cohomology theory which will then be shown to be a Weil cohomology theory in a later section (after much work).

Let X be a scheme. The global sections functor

$$\Gamma : \mathbf{Sh}(X_{\acute{e}t}) \to \mathbf{Ab}$$

defined by $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$, is left exact. We define the *rth étale cohomology of X* $H^r_{et}(X, -)$ to be its rth right derived functor, i.e.

$$H^r_{et}(X, -) = R^r\Gamma(X, -)$$

Note: we will often drop the H^r_{et} in favor of H^r for simplicities sake.

Explicitly, we do exactly as with sheaves on the Zariski site: for a given sheaf $\mathcal{F}$, choose an injective resolution

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \cdots$$

and apply the functor $\Gamma(X, -)$ to obtain a complex

$$\Gamma(X, \mathcal{I}^0) \to \Gamma(X, \mathcal{I}^1) \to \Gamma(X, \mathcal{I}^2) \to \cdots$$

This is no longer exact (in general), and $H^r_{et}(X, \mathcal{F})$ is defined to be the rth cohomology group (catchphrase: kernel-over-image).

Problem: What if we can't find an injective resolution? Solution: Exploit the stalk!

Proposition 3.1.1 (Existence of enough injectives)**.** *Let X be a scheme. Every sheaf $\mathcal{F} \in \mathbf{Sh}(X_{\acute{e}t})$ can be embedded into an injective sheaf.*

Proof. For each $x \in X$, choose a geometric point $i_x : \overline{x} \to X$ with image x and an embedding $\mathcal{F}_{\overline{x}} \hookrightarrow I(x)$ of the abelian group $\mathcal{F}_{\overline{x}}$ into an injective abelian group.

We have shown above that the pullback is exact, so the pushforward must preserve injectives (because they are adjoint). Then, $\mathcal{I}^x := i_{x*}(I(x))$ is injective. Since a product of injective objects is injective, $\mathcal{I} := \prod_{x \in X} \mathcal{I}^x$ will be an injective sheaf. $\square$

Having solved this problem, by universal properties of derived functors, we have created a bona-fided cohomology theory for schemes. There are other variations which we can define:

1. Let Z be a closed subscheme of X, and let $U = X \smallsetminus Z$. For any sheaf $\mathcal{F} \in \mathbf{Sh}(X_{ét})$, define the *cohomology of $\mathcal{F}$ with support on Z* $H^r_Z(X, \mathcal{F})$ to be the rth right derived functor of

$$\Gamma_Z(X, \mathcal{F}) := \ker(\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F}))$$

2. For a fixed sheaf $\mathcal{F}_0$

$$\mathcal{F} \mapsto \mathrm{Hom}_X(\mathcal{F}_0, \mathcal{F})$$

is left exact, and we denote its rth right derived functor by $\mathrm{Ext}^r(\mathcal{F}_0, -)$. As an examples, we have that $\mathrm{Ext}^r(\underline{\mathbb{Z}}, -) \simeq H^r(X, -)$.

3. Let X be a scheme, with open immersion $j : X \to \overline{X}$, where $\overline{X}$ is a complete scheme and let $\mathcal{F} \in \mathbf{Sh}(X_{ét})$. We define the *cohomology with compact support* to be

$$H^i_c(X, \mathcal{F}) := H^i(\overline{X}, j_! \mathcal{F})$$

We will later prove that the cohomology groups are independent of the choice of open immersion $j : X \hookrightarrow \overline{X}$ if $\mathcal{F}$ is a torsion sheaf (this will require certain base change theorems).

Obviously, if we are given an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

we get a long exact sequence by passing to cohomology

$$\cdots \to \mathrm{Ext}^r(\mathcal{F}_0, \mathcal{F}') \to \mathrm{Ext}^r(\mathcal{F}_0, \mathcal{F}) \to \mathrm{Ext}^r(\mathcal{F}_0, \mathcal{F}'') \to \cdots$$

but, what isn't so obvious, is that since $\mathrm{Hom}_X(\mathcal{F}_0, -)$ is functorial in $\mathcal{F}_0$, so also is $\mathrm{Ext}^r(\mathcal{F}_0, -)$

Proposition 3.1.2. *A short exact sequence of sheaves on $X_{ét}$*

$$0 \to \mathcal{F}_0' \to \mathcal{F}_0 \to \mathcal{F}_0'' \to 0$$

gives rise to a long exact sequnce

$$\cdots \to \mathrm{Ext}^r(\mathcal{F}_0'', \mathcal{F}) \to \mathrm{Ext}^r(\mathcal{F}_0, \mathcal{F}) \to \mathrm{Ext}^r(\mathcal{F}_0', \mathcal{F}) \to \cdots$$

for any sheaf $\mathcal{F}$.

Proof. If $\mathcal{I}$ is injective, then

$$0 \to \mathrm{Hom}_X(\mathcal{F}_0'', \mathcal{I}) \to \mathrm{Hom}_X(\mathcal{F}_0, \mathcal{I}) \to \mathrm{Hom}_X(\mathcal{F}_0', \mathcal{I}) \to 0$$

is exact. For any injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$, we get an exact sequence of complexes

$$0 \to \mathrm{Hom}_X(\mathcal{F}_0'', \mathcal{I}^\bullet) \to \mathrm{Hom}_X(\mathcal{F}_0, \mathcal{I}^\bullet) \to \mathrm{Hom}_X(\mathcal{F}_0', \mathcal{I}^\bullet) \to 0$$

which, according to a standard result in homological algebra (see [Rot09]), gives rise to a long exact sequence of cohomology groups

$$\cdots \to \mathrm{Ext}^r(\mathcal{F}_0'', \mathcal{F}) \to \mathrm{Ext}^r(\mathcal{F}_0, \mathcal{F}) \to \mathrm{Ext}^r(\mathcal{F}_0', \mathcal{F}) \to \cdots$$

□

From this result, we can prove something interesting about cohomology with compact support:

Theorem 3.1.3. *Let X be a schemes. Let Z be a closed subscheme of X with $U = X \smallsetminus Z$. For any sheaf $\mathcal{F} \in \mathbf{Sh}(X_{\acute{e}t})$, there is a long exact sequence*

$$\cdots \to H^r_Z(X, \mathcal{F}) \to H^r(X, \mathcal{F}) \to H^r(U, \mathcal{F}) \to H^{r+1}_Z(X, \mathcal{F}) \to \cdots$$

The sequence is functorial in the pairs $(X, X \smallsetminus Z)$ and $\mathcal{F}$.

Proof. Let $U \xrightarrow{j} X \xleftarrow{i} Z$ be as in the statement of the theorem. Let $\underline{\mathbb{Z}}$ denote the constant sheaf on X defined by $\mathbb{Z}$, and consider the exact sequence

$$0 \to j_! j^* \underline{\mathbb{Z}} \to \underline{\mathbb{Z}} \to i_* i^* \underline{\mathbb{Z}} \to 0$$

For any sheaf $\mathcal{F} \in \mathbf{Sh}(X_{\acute{e}t})$,

$$\mathrm{Hom}_X(j_! j^* \underline{\mathbb{Z}}, \mathcal{F} = \mathrm{Hom}_U(j^*, j^* \mathcal{F}) = \mathcal{F}(U)$$

and so $\mathrm{Ext}^r(j_! j^* \underline{\mathbb{Z}}, \mathcal{F}) = H^r(U, \mathcal{F})$. From the exact sequence

$$0 \to \mathrm{Hom}(i_* i^* \underline{\mathbb{Z}}, \mathcal{F}) \to \mathrm{Hom}(\underline{\mathbb{Z}}, \mathcal{F}) \to \mathrm{Hom}(j_! j^* \underline{\mathbb{Z}}, \mathcal{F})$$

we find that

$$\mathrm{Hom}_X(i_* i^* \underline{\mathbb{Z}}, \mathcal{F}) = \ker(\mathcal{F}(X) \to \mathcal{F}(U)) = \Gamma_Z(X, \mathcal{F})$$

and so $\mathrm{Ext}^r(i_* i^* \underline{\mathbb{Z}}, \mathcal{F}) = H^r_Z(X, \mathcal{F})$. Therefore, the long exact sequences of Ext's

$$\cdots \to \mathrm{Ext}^r(i_* i^* \underline{\mathbb{Z}}, \mathcal{F}) \to \mathrm{Ext}^r(\underline{\mathbb{Z}}, \mathcal{F}) \to \mathrm{Ext}^r(j_! j^* \underline{\mathbb{Z}}, \mathcal{F}) \to \cdots$$

corresponding to the short exact sequence above, is exactly the sequence required for the conclusion of the theorem. □

3.2 The Leray Spectral Sequence

Definition 3.2.1. Let $\pi : Y \to X$ be a morphism of schemes. Recall that for a sheaf $\mathcal{F} \in \mathbf{Sh}(Y_{\acute{e}t})$, we defined $\pi_*\mathcal{F}$ to be the sheaf on $X_{\acute{e}t}$ such that for $U \to X$ étale

$$\Gamma(U, \pi_*\mathcal{F}) = \Gamma(U_Y, \mathcal{F})$$

The functor π_* is left exact, and hence we consider its right derived functors $R^i\pi_*$. We call the sheaves $R^i\pi_*\mathcal{F}$ the *higher direct images* of $\mathcal{F}$.

Proposition 3.2.2. *For any $\pi : Y \to X$ and sheaf $\mathcal{F} \in \mathbf{Sh}(Y_{\acute{e}t})$, $R^i\pi_*\mathcal{F}$ is the sheaf on $X_{\acute{e}t}$ associated with the presheaf $U \mapsto H^i(U_Y, \mathcal{F})$.*

Proof. Let $\pi_p : \mathbf{PreSh}(Y_{\acute{e}t} \to \mathbf{PreSh}(X_{\acute{e}t}$ be the functor sending a presheaf $\mathcal{P}$ on $Y_{\acute{e}t}$ to the presheaf $U \mapsto \Gamma(U_Y, \mathcal{P})$ on $X_{\acute{e}t}$. From the definition of π_*, we get a commutative diagram

$$\begin{tikzcd} \mathbf{PreSh}(Y_{\acute{e}t} \arrow[r, "\pi_p"] & \mathbf{PreSh}(X_{\acute{e}t} \arrow[d, "a"] \\ \mathbf{Sh}(Y_{\acute{e}t} \arrow[u, "i"] \arrow[r, "\pi_*"] & \mathbf{Sh}(X_{\acute{e}t} \end{tikzcd}$$

where i is the forgetful functor and a is the "sheafification" functor. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$. Then, because a and π_p are exact,

$$R^r\pi_*\mathcal{F} := H^r(\pi_*\mathcal{I}^\bullet) = H^r((a \circ \pi_p \circ i)\mathcal{I}^\bullet) = (a \circ \pi_p)H^r(i\mathcal{I}^\bullet)$$

Note that $H^r(i\mathcal{I}^\bullet)$ is the presheaf $U \mapsto H^r(U, \mathcal{F})$, and so $\pi_p(H^r(i\mathcal{I}^\bullet))$ is the presheaf $U \mapsto H^r(U_Y, \mathcal{F})$. $\square$

Corollary 3.2.3. *The stalk of $R^i\pi_*\mathcal{F}$ at $\overline{x} \to X$ is $\varinjlim H^i(U_Y, \mathcal{F})$ where the limit is over all étale neighborhoods (U, u) of $\overline{x}$.*

Example 6. Let X be a connected, normal variety, and let $g : \eta \to X$ be the inclusion of the generic point of X. Then

$$(R^r g_*\mathcal{F})_{\overline{x}} = H^r(\mathrm{Spec}(K_{\overline{x}}), \mathcal{F})$$

where $K_{\overline{x}}$ is the field of fractions of $\mathcal{O}_{X,\overline{x}}$. Moreover, in this case g_* takes constant sheaves to constant sheaves.

Let K^s/K be a separable closure with $K = K_\eta$ and let $G_K = \mathrm{Gal}(K^s/K)$. Let

$$M_{\mathcal{F}} = \varinjlim_{K^s/K'/K \text{ finite}} \mathcal{F}(K')$$

Then, $\mathcal{F}$ is constant if G acts trivially on $M_{\mathcal{F}}$ and locally constant if the action of G on $M_{\mathcal{F}}$ factors through a finite quotient.

The map $\mathrm{Spec}(K^s) \to \mathrm{Spec}(K) \to X$ is a geometric point of X, which we denote $\overline{\eta}$. We have that the strictly local ring

$$\mathcal{O}_{X,\overline{eta}} = \varinjlim_{(U,u)} \mathcal{U} = \varinjlim_{K^s/K'/K \text{ finite}} K' = K^s$$

Thus, $(R^r g_* \mathcal{F})_{\overline{eta}} = H^r(G_{K^s}, M_{\mathcal{F}}) = M_{\mathcal{F}}$ if $r = 0$ and is 0 otherwise.

In general, we have that

$$(R^r g_* \mathcal{F})_{\overline{x}} = H^r(G_{K_{\overline{x}}}, M_{\mathcal{F}})$$

For examples, let A be a Dedekind domain and let $X = \mathrm{Spec}(A)$. Let $\tilde{A}$ be the integral closure of A in K^s. A closed point $x \in X$ is a nonzero prime ideal of $\mathfrak{p} \subset A$, and the choice of a prime ideal $\tilde{\mathfrak{p}} \subset \tilde{A}$ lying over $\mathfrak{p}$ determines a geometric point $\overline{x} \to x \to X$. In this case $K_{\overline{x}} = (K^s)^{I(\tilde{\mathfrak{p}})}$ where $I(\tilde{\mathfrak{p}}) \subset G_K$ is the inertia group of $\tilde{\mathfrak{p}}$. Thus, $(R^r g_* \mathcal{F})_{\overline{x}} = H^r(I(\tilde{\mathfrak{p}}), M_{\mathcal{F}})$.

Now, we introduce a tool for computation with higher direct images (for proof, see the appendix on spectral sequences):

Theorem 3.2.4 (Leray spectral sequence). *Let $\pi : Y \to X$ be a morphism of schemes. For any sheaf $\mathcal{F}$ on $Y_{\acute{e}t}$, there is a spectral sequence*

$$H^r(X, R^s \pi_* \mathcal{F}) \Rightarrow H^{r+s}(Y, \mathcal{F})$$

Example 7. Let $X \to P := \mathrm{Spec}(k)$ be a variety over a field k. Let k^s be a separable closure of k, and let $\overline{X} := X \times_k k^s$ be the base change with morphism $\psi : X \to \overline{X}$. Let $G_k = \mathrm{Gal}(k^s/k)$. As we will describe in the next section, we can identify $\mathbf{Sh}(P_{\acute{e}t})$ with the category $\mathbb{M} \rtimes_{G_k}$ of discrete G_k-modules, π_* becomes identified with the functor

$$\mathcal{F} \mapsto \mathcal{F}(\overline{X}) = \varinjlim_{k'} \mathcal{F}(X_{k'})$$

where the limit is over finite intermediate field extensions $k^s/k'/k$. Thus, in this case the Leray spectral sequence becomes

$$H^r(\Gamma, H^s(\overline{X}, \psi^* \mathcal{F}) \Rightarrow H^{r+s}(X, \mathcal{F})$$

so, we can use Galois cohomology to compute étale cohomology. This, combined with the above example involving the higher direct image, and the facts about Galois cohomology which we will recall in the next section will allow us to compute étale cohomology groups of curves with relative ease.

3.3 Étale Cohomology for Points

This section will describe étale cohomology over a $\mathrm{Spec}(k)$ for a field k; in other words, étale cohomology for a point. We will see that there is a direct comparison with Galois cohomology, which we should keep in mind to help with intuition later on. Proofs for this section will not be fully written out, as we assume a familiarity with Galois cohomology. For proofs, see [Ber10] and [Ser97].

Let k^s/k be a choice of separable closure of k, and let $G_k = \mathrm{Gal}(k^s/k)$ be the corresponding Galois group. Let A be an étale k-algebra, and let M be a discrete G_k-module. Recall our previous construction of two functors

$$M_{\mathcal{F}} = \varinjlim \mathcal{F}(k')$$

$$\mathcal{F}_M(A) = \mathrm{Hom}_{G_k}(F(A), M)$$

which establish an equivalence between the category of sheaves on $(\mathrm{Spec}(k))_{\acute{e}t}$ and the category of discrete G_k-modules.

What is left to understand is how the global sections functor $\Gamma(X, -)$ is equivalent to the Galois invariance functor $(-)^{G_k}$, thereby showing that their rth right-derived functors will be equal. To convince yourself that this is true, and to get some intuition, we think of an example:

Example 8. Recall the scheme $\mathbb{G}_a$, and let

$$\mathcal{F} := \mathrm{Hom}_{\mathrm{Spec}(k)}(-, \mathbb{G}_m)$$

be the corresponding representable sheaf on $(\mathrm{Spec}(k))_{\acute{e}t}$. The G_k-module associated to $\mathcal{F}$ is the additive group of k^s, the chosen separable closure of k. Applying the global sections functor to $\mathcal{F}$, we get

$$\Gamma(\mathrm{Spec}(k), \mathcal{F}) = \mathbb{G}_a(k)$$

and the Galois invariance functor applied to the additive group of k^s gives us

$$(k^s)^{G_k} = \mathbb{G}_a(k)$$

where we view $\mathbb{G}_a(k)$ as an additive group. Similar arguments apply to all representable sheaves, which we can then generalize to all constructible sheaves, which then generalizes to all sheaves which are important to us.

Now that we see the equivalence, we state some important theorems and sketch the proofs. Let k be a field with chosen separable closure k^s and Galois group $G_k := \mathrm{Gal}(k^s/k)$. Let $X := \mathbb{S}\mathrm{l}(k)$.

Theorem 3.3.1 (Hilbert Theorem 90)**.** *For the sheaf $\mathcal{O}_X^*$ a.k.a the representable sheaf corresponding to $\mathbb{G}_m$, we have that*

$$H^1(X, \mathcal{O}_X^*) = 0$$

Proof. By the classical Hilbert Theorem 90,

$$H^1(X, \mathcal{O}_X^*) \simeq H^1(G_k, (k^s)^\times) = 0$$

A more geometric interpretation is

$$H^1(X, \mathcal{O}_X^*) \simeq Pic\ (X) = 0$$

□

Theorem 3.3.2 ("Cohomological Dimension")**.** $H^i(X, \mathcal{O}_X) = 0$ *for all* $i > 0$.

Proof. $H^i(G_k, k^s) = 0$ for all $i > 0$ (viewing k^s as the underlying additive group). □

Theorem 3.3.3 (Tsen's Theorem)**.** *Let K be the function field of an algebraic curve over an algebraically closed field. The Brauer group*

$$\mathrm{Br}(K) = H^2(G_K, K^\times) = 0$$

Geometrically, this means that the second cohomology group of the generic point $\eta \to Y$, for an algebraic curve Y, vanishes:

$$H^2(\eta, \mathcal{O}_\eta^*) = 0$$

See [Sta20, Section 0A2M] for details.

Corollary 3.3.4. *Tsen's theorem, along with the Hilbert Theorem 90, imply that $H^i(G_K, K^\times) = 0$ for all $i > 0$, where K is as above.*

In the next subsection, using Tsen's theorem and these other results, along with the higher direct images along generic points, we will obtain results concerning the cohomology of algebraic curves, which we will then combine with information about Lefschetz pencils (certain fiberings $X \to \mathbb{P}^1$) to obtain information about the cohomology of surfaces. This is then generalized later as well, to all higher dimensional varieties.

Theorem 3.3.5 (Cohomology of roots of unity)**.** *Assume that* $\mathrm{char}(k)$ *does not divide n. Then, we have a group isomorphism*

$$H^1(X, \mu_n) \simeq k^\times / n(k^\times)$$

Proof. Since $\mathrm{char}(k)$ does not divide n, we have the Kummer sequence

$$0 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 0$$

which induces an exact sequence in cohomology

$$k^\times \xrightarrow{n} k^\times \to H^1(G_k, \mu_n) \to H^1(G_k, \mathbb{G}_m)$$

Using Hilbert 90 and exactness of this sequence, we obtain the desired isomorphism. □

Theorem 3.3.6 (Cup products)**.** *There exists a $\mathbb{Z}$-bilinear map*

$$\smile : H^p(X, \mathcal{F}) \times H^q(X, \mathcal{G}) \to H^{p+q}(X, \mathcal{F} \otimes_{\underline{\mathbb{Z}}} \mathcal{G})$$

Moreover, after identifying $\mathcal{F} \otimes_{\underline{\mathbb{Z}}} \mathcal{G}$ and $\mathcal{G} \otimes_{\underline{\mathbb{Z}}} \mathcal{F}$, we have for $\alpha \in H^p(X, \mathcal{F})$ and $\beta \in H^q(X, \mathcal{G})$ that

$$\alpha \smile \beta = (-1)^{pq}(\beta \smile \alpha)$$

Proof. This follows from the cup-product

$$\smile : H^p(G_k, B) \times H^q(G_k, A) \to H^{p+q}(G_k, A \otimes_{\mathbb{Z}} B)$$

which can be seen by cocycle computations by constructing $(p+q)$-cocycles $\alpha \smile \beta : G_k^{p+q} \to A \otimes_{\mathbb{Z}} B$ from a p-cocycle α and a q-cocycle β:

$$(\sigma_1, \ldots, \sigma_{p+q}) \mapsto (\alpha_{\sigma_1, \ldots, \sigma_p}) \otimes (\sigma_1 \cdots \sigma_p \beta_{\sigma_{p+1}, \ldots, \sigma_{p+q}})$$

□

This will be useful when we want to understand Poincaré duality, as we will need a general cup product for all varieties.

Going forward, it is useful to remember Galois cohomology and to think of étale cohomology as a direct generalization. This will help with intuition and calculation. As before, we will exploit the stalks of sheaves, which can be interpreted as the pushforward along a geometric point, therefore allowing us to use the higher direct images of the Galois cohomology of that said point to then understand the general cohomology on higher dimensional varieties (by the Leray spectral sequence).

3.4 Étale Cohomology for Curves

For this section, we will work on proving the following proposition:

Proposition 3.4.1. *Let X be a complete connected smooth curve over an algebraically closed field k whose characteristic is relatively prime to the natural number n, with μ_n denoting the constant sheaf of nth roots of unity (isomorphic to ($\mathbb{Z}/n\mathbb{Z}$ after choice of nth roots of unity in k). Then:*

(a) $H^0(X, \mu_n) \simeq \mu_n(k)$*;*

(b) $H^1(X, \mu_n) \simeq \mathrm{Pic}(X)[n]$*;*

(c) $H^2(X, \mu_n) \simeq \mathbb{Z}/n\mathbb{Z}$

(d) $H^r(X, \mu_n) = 0$ *for* $r > 2$*.*

where $\mu_n(k)$ are a the nth roots of unity in k (isomorphic to $\mathbb{Z}/n\mathbb{Z}$ after choice of roots of unity) and $\mathrm{Pic}(X)[n] \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$ is the n-torsion subgroup of $\mathrm{Pic}(X)$.

The proof uses the Kummer sequence

$$0 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 0$$

and rests on the following lemma (a generalization of Hilbert's Theorem 90):

Lemma 3.4.2. *Let X be a smooth curve over an algebraically closed field k. Then, for $i \geqslant 2$,*

$$H^i(X, \mathbb{G}_m) = 0$$

Proof. Let K_X denote the sheaf of rational functions on X, i.e. $K_X(U)$ is the set of fractions of rational functions on $U \to X$ étale. Let $\mathrm{Div}_X(U)$ denote the set of divisors on U, i.e. the free abelian group of closed points on $U \to X$ étale. There is a natural exact sequence of sheaves

$$0 \to \mathbb{G}_m \to K_X^\times \to \mathrm{Div}_X \to 0$$

By the long exact sequence of cohomology, the lemma will follow if

(i) $H^i(X, K_X^\times) = 0$ for $i > 0$;

(ii) $H^i(X, \mathrm{Div}_X) = 0$ for $i > 0$.

We may assume that X is irreducible. Let K be the function field of X, and let

$$j : Spec\ (K) \to X$$

be the natural embedding. We have that

$$K_X^\times \simeq j_* \mathbb{G}_m$$

and, so, we want to show that

$$R^i j_* \mathbb{G}_m = 0$$

for $i > 0$. The ith direct image is the sheaf associated to the functor

$$U \mapsto H^i(\mathrm{Spec}(K) \times_X U, \mathbb{G}_m)$$

Base changes preserve étale morphisms, so $\mathrm{Spec}(K) \times_X U$ is étale, and thus made up of finitely many finite separable extensions of K, and we know that their cohomology vanishes (by our discussion in the previous section; consequence of Galois cohomology). The Leray spectral sequence then gives us

$$H^i(X, j_* \mathbb{G}_m) \simeq H^i(\mathrm{Spec}(K), \mathbb{G}_m)$$

which vanishes for $i > 0$, again thanks to the fact that K is the function field of an algebraic curve over an algebraically closed field.

Every closed point $x \in X$ can be interpreted as a geometric point

$$x : \mathrm{Spec}(k) \to X$$

with

$$\mathrm{Div}_X = \bigoplus_{x : \mathrm{Spec}(k) \to X} x_*(\underline{\mathbb{Z}})$$

(where $\underline{\mathbb{Z}}$ is the locally constant sheaf associated to $\mathbb{Z}$). All sheaves on $\mathrm{Spec}(k)$ are acyclic (cohomology vanishes), since k is algebraically closed (see previous section). Therefore, Div_X is also acyclic, because cohomology is compatible with direct sums. □

We can now calculate the cohomology $H^*(X, \mathbb{Z}/n\mathbb{Z})$ by means of the Kummer sequence

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 0$$

As a corollary of our generalized "Hilbert's Theorem 90" we immediately get that $H^i(X, \mu_n) = 0$ for $i > 2$ and

$$H^1(X, \mu_n) = \ker\left(\mathrm{Pic}(X) \xrightarrow{n} \mathrm{Pic}(X)\right)$$

$$H^2(X, \mu_n) = \mathrm{coker}\left(\mathrm{Pic}(X) \xrightarrow{n} \mathrm{Pic}(X)\right)$$

by the fact that the Picard group can be defined as $\mathrm{Pic}(X) := H^1(X, \mathcal{O}_X^\times)$. Recall that the group of degree zero line bundles $Pic^0\ (X)$ carries the structure of an abelian variety of dimension g, where g is the genus of X, also known as the Jacobian variety, $Jac\ (X)$. By the theory of abelian varieties (multiplication-by-n is an isogeny), we get that

$$\mathrm{Pic}^0(X) \xrightarrow{n} \mathrm{Pic}^0(X)$$

is surjective. Then, by using the exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z} \to 0$$

we get

$$H^1(X, \mu_n) = ker\left(\mathrm{Pic}^0(X) \xrightarrow{n} \mathrm{Pic}^0(X)\right) \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$$

since the n-torsion points of a g-dimensional abelian variety over an algebraically closed field of characteristic relatively prime to n is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{2g}$ ([Mum85] or [Mil08]) and

$$H^2(X, \mu_n) = \mathrm{coker}\left(Z \xrightarrow{n} \mathbb{Z}\right) \simeq \mathbb{Z}/n\mathbb{Z}$$

Then, since X is complete of dimension 1, it is also projective, and, therefore, the global sections

$$H^0(X, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X) = k$$

since k is algebraically closed, and, so,

$$H^0(X, \mu_n) = \mu_n(k)$$

completing our proof of the above proposition. □

To generalize the proposition 3.4.1, we recall that for an smooth connected affine curve Y, we can find an embedding

$$\iota : Y \hookrightarrow X$$

for some complete smooth connected curve X, and that any divisor on Y can be extended to a degree 0 divisor on X by using the set of finite points $S = X \smallsetminus \iota(Y)$, so, using the proof above, we get that

$$\mathrm{Pic}(Y) \xrightarrow{n} \mathrm{Pic}(Y)$$

is surjective, so

$$H^2(Y, \mu_n) = \mathrm{coker}\left(\mathrm{Pic}(X) \xrightarrow{n} \mathrm{Pic}(X)\right) = 0$$

Since every finite abelian group is a direct product of cyclic groups, the corresponding statement holds for arbitrary finite constant sheaves of order relatively prime to the characteristic of k.

Now, we turn to cohomology with compact support:

Proposition 3.4.3. *For any connected regular curve U over an algebraically closed field k and an integer n not divisible by the characteristic of k, there is a canonical isomorphism*

$$H^2_c(U, \mu_n) \simeq \mathbb{Z}/n\mathbb{Z}$$

Proof. Let $j : U \hookrightarrow X$ be the canonical inclusion of U into a complete regular curve, and let $i : Z \hookrightarrow X$ be the complement of U in X. Regard μ_n as a sheaf on X. from the sequence

$$0 \to j_!j^*\mu_n \to \mu_n \to i_*i^*\mu_n \to 0$$

we obtain an exact sequence

$$\cdots \to H^r_c(U, \mu_n) \to H^r(X, \mu_n) \to H^r(X, i_*i^*\mu_n) \to \cdots$$

but (by the Leray spectral sequence)

$$H^r(X, i_*i^*\mu_n) \simeq H^r(Z, i^*\mu_n) = 0 \quad \text{for } r > 0$$

and so

$$H^2_c(U, \mu_n) \simeq H^2(X, \mu_n) \simeq \mathbb{Z}/n\mathbb{Z}$$

□

Let's say something about torsion which is *not* prime to the characteristic:

Theorem 3.4.4. *Let X be a scheme of finite-type over a separably closed field k of characteristic p. If X is affine, then $H^q(X, \mathbb{Z}/p\mathbb{Z}) = 0$ for any $q \geqslant 2$. If X is proper, then $H^q(X, \mathbb{Z}/p\mathbb{Z}) = 0$ for any $q > \dim X$, and we have an exact sequence*

$$0 \to H^q(X, \mathbb{Z}/p\mathbb{Z}) \to H^q_{\text{Zar}}(X, \mathcal{O}_X) \xrightarrow{\wp} H^q_{\text{Zar}}(X, \mathcal{O}_X) \to 0$$

for every q, where $\wp : H^q_{\text{Zar}}(X, \mathcal{O}_X) \to H^q_{\text{Zar}}(X, \mathcal{O}_X)$ is the homomorphism induced by

$$\wp : \mathcal{O}_X \to \mathcal{O}_X, \qquad s \mapsto s^p - s$$

Proof. We may assume that k is algebraically closed by **??**. By the Artin-Schreier sequence, we get the following long exact sequence in cohomology

$$0 \to H^0(X, \mathbb{Z}/p\mathbb{Z}) \to H^0(X, \mathcal{O}_{X_{et}}) \xrightarrow{\wp} H^0(X, \mathcal{O}_{X_{et}}) \to \cdots$$

$$\cdots \to H^q(X, \mathbb{Z}/p\mathbb{Z}) \to H^q(X, \mathcal{O}_{X_{et}}) \xrightarrow{\wp} H^q(X, \mathcal{O}_{X_{et}}) \to \cdots$$

We also know that $H^q(X, \mathcal{O}_{X_{et}}) \cong H^q_{\text{Zar}}(X, \mathcal{O}_X)$ for all q. If X is affine, we have that $H^q_{\text{Zar}}(X, \mathcal{O}_X) = 0$ for all $q \geqslant 1$, so our exact sequence tells us that $H^q(X, \mathbb{Z}/p\mathbb{Z}) = 0$ for all $q \geqslant 2$.

If X is proper, then each $H^q_{\text{Zar}}(X, \mathcal{O}_X)$ is a finite dimensional k-vector space. By the general theory of the Frobenius, we have that $\wp$ is surjective on any finite dimensional k-vector space, and so we extract short exact sequences from our long exact sequence:

$$0 \to H^q(X, \mathbb{Z}/p\mathbb{Z}) \to H^q_{\mathrm{Zar}}(X, \mathcal{O}_X) \xrightarrow{\wp} H^q_{\mathrm{Zar}}(X, \mathcal{O}_X) \to 0$$

and, by algebraic geometry, we have that if $q > \dim X$, then $H^q_{\mathrm{Zar}}(X, \mathcal{O}_X) = 0$, and so we get that $H^q(X, \mathbb{Z}/p\mathbb{Z}) = 0$. $\square$

Finally, we close the section with a case of Poincaré duality along with a sketch of the proof:

Theorem 3.4.5. *Let U be a regular connected curve over an algebraically closed field with $\mathrm{char}(k) \nmid n$ an integer. For any finite locally constant n-torsion sheaf $\mathcal{F}$ and integer $r \geqslant 0$, there is a canonical perfect pairing of finite groups*

$$H^r_c(U, \mathcal{F}) \times H^{2-r}(U, \check{\mathcal{F}}(1)) \to H^2_c(U, \mu_n) \simeq \mathbb{Z}/n\mathbb{Z})$$

where the sheaf $\check{\mathcal{F}}(1)$ is

$$V \mapsto \mathrm{Hom}_V(\mathcal{F}|_V, \mu_n|_V)$$

Recall that a pairing $M \times N \to C$ is perfect if the induced maps

$$M \to \mathrm{Hom}(N, C), \qquad N \to \mathrm{Hom}(M, C)$$

are isomorphisms.

Sketch of proof. This is a multi-staged proof. First, we recall a useful lemma from homological algebra: the Five Lemma. Specifically, the two "sublemmas" which constitute a usual proof, which I like to call the "Four Lemmas" (the diligent reader is suggested to reprove them by diagram chasing). Given a commutative diagram of the form

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "a"] & \bullet \arrow[r] \arrow[d, "b"] & \bullet \arrow[r] \arrow[d, "c"] & \bullet \arrow[d, "d"] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

with exact rows, we can conclude

$$b, d \text{ injective and } a \text{ surjective } \Rightarrow c \text{ is injective;}$$

$$a, c \text{ surjective and } d \text{ injective } \Rightarrow b \text{ is surjective.}$$

Then, we note that $H^r_c(U, \mathcal{F}) := H^r(X, j_!\mathcal{F}) = 0$ for all $r > 2$ (see our proposition above). This reduces us to the case of $r = 0, 1, 2$. For convenience sake, we write

$$T^r(U, \mathcal{F}) = H^{2-r}_c(U, \check{\mathcal{F}}(1))^\vee = \mathrm{Hom}(H^{2-r}_c(U, \check{\mathcal{F}}(1))^\vee, \mathbb{Z}/n\mathbb{Z})$$

We have to show that the map

$$\phi^r(U, \mathcal{F}) : H^r_c(U, \mathcal{F}) \to T^r(U, \mathcal{F})$$

defined by the pairing is an isomorphism of finite groups for all finite locally constant sheaves $\mathcal{F}$ on U.

Because $\mathrm{Hom}(-, \mathbb{Z}/\mathbb{Z})$ preserves the exactness of sequence of $\mathbb{Z}/n\mathbb{Z}$-modules, so also does $\mathcal{F} \mapsto \check{\mathcal{F}}(1)$ (by checking stalks). It follows that an exact sequence of finite locally constant sheaves

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

gives rise to an exact sequence

$$\cdots \to T^s(U, \mathcal{F}') \to T^s(U, \mathcal{F}) \to T^s(U, \mathcal{F}'') \to T^{s+1}(U, \mathcal{F}') \to \cdots$$

From here, we start our 7 step approach:

1. Let $\pi : U' \to U$ be a finite map. The theorem is true for $\mathcal{F}$ on U' if and only if it is true for $\pi_*\mathcal{F}$ on U. This follows from the fact that π_* is exact and preserves injectives (when it is finite!).

2. Let $V = U \setminus x$ for some point $x \in U$. Recall the exact sequence associated to a pair (U, V), which we derived above; there is an exact commutative diagram with isomorphisms indicated:

$$\begin{tikzcd}
H^r_x(U, \mu_n) \arrow[r] \arrow[d, "\sim"] & H^r(U, \mu_n) \arrow[r] \arrow[d, "\phi^r(U,\mu_n)"] & H^r(V, \mu_n) \arrow[r] \arrow[d, "\phi^r(V,\mu_n)"] & H^{r+1}_x(U, \mu_n) \arrow[d, "\sim"] \\
H^{2-r}(x, \mathbb{Z}/n\mathbb{Z})^\vee \arrow[r] & H^{2-r}_c(U, \mathbb{Z}/n\mathbb{Z})^\vee \arrow[r] & H^{2-r}_c(V, \mathbb{Z}/n\mathbb{Z})^\vee \arrow[r] & H^{3-r}(x, \mathbb{Z}/n\mathbb{Z})^\vee
\end{tikzcd}$$

When $r = 0, 1$ the left and right hand columns are 0, but when $r = 2$, we have

$$H^0(x, \mathbb{Z}/n\mathbb{Z}) \times H^0(x, \mathbb{Z}/n\mathbb{Z}) \to H^0(x, \mathbb{Z}/nF) \simeq \mathbb{Z}/n\mathbb{Z}$$

The lower sequence is obtained by the compact cohomology of

$$0 \to j_! j^* \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to i_* i^* \mathbb{Z}/n\mathbb{Z} \to 0$$

3. <u>Claim:</u> The map $\phi^0(U, \mathbb{Z}/n\mathbb{Z})$ is an isomorphism of finite groups.

<u>Proof:</u> In this case, the pairing is

$$H^0_c(U, \mathbb{Z}/n\mathbb{Z}) \times H^2(U, \mu_n) \to H^2_c(U, \mu_n) \simeq \mathbb{Z}/n\mathbb{Z}$$

Here $H^0_c(U, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$ and its action on $H^2(U, \mu_n)$ is defined by the natural $\mathbb{Z}/n\mathbb{Z}$-module structure on $H^2(U, \mu_n)$.

4. <u>Claim:</u> The theorem is true for $r = 0$ and $\mathcal{F}$ locally constant.

<u>Proof:</u> Let $U' \to U$ be a finite étale covering such that $\mathcal{F}|_{U'}$ is constant. We can embed $\mathcal{F}|_{U'}$ into a sheaf $\mathcal{F}' = (\mathbb{Z}/n\mathbb{Z})^s$ on U' for some s. On applying π_* to $\mathcal{F}|_{U'} \hookrightarrow \mathcal{F}'$ and composing the result with the natural

inclusion $\mathcal{F} \hookrightarrow \pi_* \pi^* \mathcal{F}$, we define $\mathcal{F}''$ to the be the cokernel, and obtain the sequence

$$0 \to \mathcal{F} \to \pi_* \mathcal{F}' \to \mathcal{F}'' \to 0$$

Consider the diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\sim"] & \bullet \arrow[r] \arrow[d, "\sim"] & H^0(U, \mathcal{F}) \arrow[r] \arrow[d] & H^0(U, \pi_* \mathcal{F}') \arrow[r] \arrow[d] & H^0(U, \mathcal{F}'') \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & T^0(U, \mathcal{F}) \arrow[r] & T^0(U, \pi_* \mathcal{F}') \arrow[r] & T^0(U, \mathcal{F}'') \end{tikzcd}$$

The map $\phi^0(U, \pi_* \mathcal{F}')$ is an isomorphism by steps 1 and 3. The five lemma show that $\phi^0(U, \mathcal{F})$ is injective. Since this is true for all locally constant sheaves $\mathcal{F}$, it means that $\phi^0(U, \mathcal{F}'')$ is injective, from which the five lemma shows that $\phi^0(U, \mathcal{F})$ is surjective. Finally, $H^0(U, \mathcal{F})$ is obviously finite.

5. Claim: The map $\phi^1(U, \mathbb{Z}/n\mathbb{Z})$ is injective.

   Proof: Recall [Mil13][p78, §11.3]: $H^1(U, \mathbb{Z}/n\mathbb{Z}) = \mathrm{Hom}(\pi_1(U, \overline{u}), \mathbb{Z}/n\mathbb{Z})$. Let $s \in H^1(U, \mathbb{Z}/n\mathbb{Z})$ and let $\pi : U' \to U$ be the Galois covering corresponding to the kernel of s. Then, s maps to zero in $H^1(U', \mathbb{Z}/n\mathbb{Z}) \simeq H^1(U, \pi_* \mathbb{Z}/n\mathbb{Z})$. Let $\mathcal{F}''$ be the cokernel of $\mathbb{Z}/n\mathbb{Z} \to \pi_*(\mathbb{Z}/n\mathbb{Z})$. Then, in the long exact sequence of cohomology, we get

$$\begin{tikzcd} H^0(U, \pi_* \mathbb{Z}/n\mathbb{Z}) \arrow[r] \arrow[d, "\sim"] & H^0(U, \mathcal{F}'') \arrow[r] \arrow[d, "\sim"] & H^1(U, \mathbb{Z}/n\mathbb{Z}) \arrow[r] \arrow[d] & H^1(U, \pi_* \mathbb{Z}/n\mathbb{Z}) \arrow[d] \\ T^0(U, \pi_* \mathbb{Z}/n\mathbb{Z}) \arrow[r] & T^0(U, \mathcal{F}'') \arrow[r] & T^1(U, \mathbb{Z}/n\mathbb{Z}) \arrow[r] & T^1(U, \pi_* \mathbb{Z}/n\mathbb{Z}) \end{tikzcd}$$

   From our choice of map π, $s \mapsto 0 \in H^1(U, \pi_* \mathbb{Z}/n\mathbb{Z})$, and a diagram chase shows that if s also maps to zero in $T^1(U, \mathbb{Z}/n\mathbb{Z})$, then it is zero, and therefore only zero maps to zero, and so $\phi^1(U, \mu_n)$ is injective.

6. Claim: The maps $\phi^r(U, \mathbb{Z}/n\mathbb{Z})$ are isomorphisms of finite groups for $r = 1, 2$.

   Proof: First, we take the case $U = X$, but it follows immediately, since we know exactly what the cohomology groups are by our previous lemma 3.4.1. For the case of $U \neq X$, apply the five lemma to the diagram in step 2 while removing points of $X \smallsetminus U$ one at a time, and recall that a "compactification" (i.e. mapping $U \hookrightarrow X$) only requires a finite number of points for algebraic curves.

7. Finally, to complete the proof for all locally constant $\mathcal{F}$, we apply the argument in step 4 twice (take the long exact sequence, but this time the isomorphisms are $H^0 \simeq T^0$ to show that $r = 1$ is true, and then $H^1 \simeq T^1$ to show $r = 2$).

$\square$

Note. Using similar techniques as our proof of the Weil conjectures for ellptic curves, we can use this case of Poincaré duality to prove the Weil conjectures for curves. The key part is using the well-developed theory of divisors on curves to prove the Lefschetz fixed point formula in this case.

Weil proved his conjectures for the case of curves, but he didn't use this result. Instead, he studied the divisors and used them as a "proxy cohomology theory" to get something similar to étale cohomology. Unfortunately, the general theory of divisors on higher dimensional schemes is not as well-behaved, and neither is the theory of algebraic cycles understood well-enough for application either. The connection between étale cohomology and algebraic cycles is a deep topic which is still far from being fully realized. See Tate's Conjecture.

3.5 Étale Cohomology for Surfaces

We will now cover the étale cohomology of surfaces, where will will develop the theory of Lefschetz pencils which we will generalize later to prove the final pieces of the Weil conjectures (specifically the "Riemann Hypothesis"). This subsection should be seen as a prelude to Deligne's proof, as many of these ideas will be seen again later. As such, we will use some of the results of the section title "Fundamental Theorems" without proof. This subsection should help us contextualize the importance of these fundamental theorems, just as the section on Weil cohomology theories has already done.

We will not prove the existence of the Lefschetz pencil here, but we will get a good grasp on its properties and uses. Proof of this theorem can be found in [Igu56] or SGA 7, XVII. For a sketch of the ideas, see [Mil80][p.197-199]. For the rest of this section X will denote a smooth, complete (hence projective) surface (i.e. $\dim X = 2$) over an algebraically closed field k.

Theorem 3.5.1 (Existence of Lefschetz pencils)**.** *Let X and k be as above. There exists a surface X^* that is obtained from X by blowing up a finite number of points and a map $\pi : X^* \to \mathbb{P}^1$ satisfying the following conditions:*

(i) π is proper, flat, and has a section;

(ii) the generic fiber of π is a smooth curve;

(iii) the closed fibers of π are connected with at most a single node as a singularity.

"Proof". We will not even attempt to sketch the proof in much detail. Suffice it to say that we use an embedding $X \hookrightarrow \mathbb{P}^m$ and the dual projective space $\check{\mathbb{P}}^m$ (the set of hyperplanes in $\mathbb{P}^m$ or, equivalently, the set of linear forms on $\mathbb{P}^m$ "modulo homothety"). By studying the dual $\check{X} \subset \check{\mathbb{P}}^m$ of X, we can see that there is a nice line $i : \mathbb{P}^1 \hookrightarrow \mathbb{P}^m$ which "captures" all of the hyperplanes which $X \hookrightarrow \mathbb{P}^m$ intersects (the axis of intersection), with certain singularities (studied via the Jacobian criteria). We then blow-up X according to the behavior of the singularities, and then construct a map $X^* \to \mathbb{P}^1$. $\square$

So, now that we know Lefschetz pencils exist, what makes then useful?

Proposition 3.5.2. *Let S be a smooth curve over an algebraically closed field k and let $\pi : X \to S$ be a proper flat map. Assume that $\mathcal{O}_S \xrightarrow{\sim} \pi_*\mathcal{O}_X$ (as sheaves on S_{Zar}) and that for the closed point $s \in S$, X_s is reduced. Then:*

(a) $H^0(X_s, \mathcal{O}_{X_s}) \simeq k$*;*

(b) $p_a(X_s) = p_a(X_\eta)$*, where X_η is the generic fiber of π and $p_a := \dim_k H^1(\mathcal{O}) = h^1(\mathcal{O})$ is the arithmetic genus.*

Proof. The condition $\mathcal{O}_S \simeq \pi_*\mathcal{O}_X$ implies that all fibers are geometrically connected (Zariski connectedness theorem, Hartshorne [Har77][2,III,.11.3]).

Thus, as X_s is reduced, $H^0(X_s, \mathcal{O}_{X_s}) = k$, which proves (a). Since the function $s \mapsto \chi(X_s, \mathcal{O}_{X_s}) = \sum(-1)^i h^i(X_s, \mathcal{O}_{X_s})$ is constant on S, (b) follows from (a). ☐

Theorem 3.5.3. *Let S be a smooth curve over an algebraically closed field k, let $\pi : X \to S$ have irreducible fibers and satisfy the Lefschetz pencil conditions:*

(i) π is proper, flat, and has a section;

(ii) the generic fiber of π is a smooth curve;

(iii) the closed fibers of π are connected with at most a single node as a singularity.

Let T be the finite set of points of $s \in S$ such that $\pi^{-1}(s) = X_s$ is singular and let n be relatively prime to $\mathrm{char}(k)$*. Then:*

(a) For any geometric point $\overline{s}$ of S, the canonical maps

$$(R^i\pi_*\mu_n)_{\overline{s}} \to H^i(X_{\overline{s}}, \mu_n)$$

are isomorphisms.

(b) $R^0\pi_*\mu_n = \mu_n$*;*

$R^1\pi_*\mu_n$ *is constructible and* $R^1\pi_*\mu_n|_{S\smallsetminus T}$ *is locally constant;*

$R^2\pi_*\mu_n = \mathbb{Z}/n\mathbb{Z}$*;*

$R^i\pi_*\mu_n = 0$ *for* $i > 2$*.*

(c) $R^i\pi_*\mu_n \xrightarrow{\sim} g_*g^*R^i\pi_*\mu_n$*, where $g : \eta \hookrightarrow S$ is the generic point of S.*

Proof. Recall that $(R^i\pi_*\mu_n)_{\overline{s}} = H^r(\tilde{X}, \mu_n)$, where $\tilde{X} = X \times_S \mathrm{Spec}(\mathcal{O}_{S,\overline{s}})$. If $s = \eta$, then $\tilde{X} = X \otimes_K K^s = X_{\overline{\eta}}$, and so the canonical maps are the identity maps. Otherwise, they are the maps

$$H^i(\tilde{X}, \mu_n) \to H^i(X_{\overline{s}}, \mu_n)$$

that restrict a class on $\tilde{X}$ to the closed fiber of $\tilde{X}/\mathrm{Spec}(\mathcal{O}_{S,\bar{s}})$. That these maps are isomorphisms are a special case of the proper base change theorem (which we will cover later, 5.2.1).

It follows from the hypothesis that $\pi_*\mathcal{O}_X = \mathcal{O}_S$ as sheaves on $S_{ét}$ and hence $\pi_*\mathbb{G}_m = \mathbb{G}_m$ and $\pi_*\mu_n = \mu_n$.

Write $F = R^1\pi_*\mu_n$; the stalk $F_{\bar{s}}$ is $\mathrm{Pic}(X_{\bar{s}})[n]$, which is finite. If we choose, for s a closed point of S, an embedding $\mathcal{O}_{S,s} \hookrightarrow k(\overline{\eta})$, then the induced map $\phi_s : F_s \to F_{\overline{\eta}}^{I_s}$ (where $I_s \subset \mathrm{Gal}(k(\overline{\eta})/k(\eta))$ is the inertia group corresponding to the embedding) may be identified with

$$\mathrm{Pic}(X_s)[n] \xleftarrow{\sim} \mathrm{Pic}(\tilde{X})[n] \xrightarrow{\sim} \mathrm{Pic}(\tilde{X}_{\tilde{\eta}})[n] = \mathrm{Pic}(X_{\overline{\eta}})[n]^{I_s}$$

(where $\tilde{\eta}$ is the generic point of $\tilde{X}$). Thus, ϕ_s is an isomorphism whose inverse $p_s : F_{\overline{\eta}}^{I_s} \to F_s$ sends a divisor class on $\tilde{X}_{\tilde{\eta}}$ to the intersection of $\tilde{X}_s = X_s$ with the Zariski closure of the divisor class.

If $s \notin T$, then X_s and $X_{\overline{\eta}}$ are smooth curves of the same genus g according to the above proposition. Thus $[F_s] = n^{2g} = [F_{\overline{\eta}}]$, and we find that $F_s \simeq F_{\overline{\eta}}^{I_s} = F_{\overline{\eta}}$. It follows that F is constructible and $F|_{S \smallsetminus T}$ is locally constant.

For any complete integral curve Y over an algebraically closed field, $H^2(Y, \mu_n) \simeq \mathbb{Z}/n\mathbb{Z}$; this was proved for the smooth case in the earlier section on curves, and the general case follows by comparing the cohomology of Y with its normalization.

The map $R^1\pi_*\mathbb{G}_m \to R^2\pi_*\mu_n$ factors through the degree map $R^1\pi_*\mathbb{G}_m = \underline{\mathrm{Pic}}_{X/S} \to \mathbb{Z}$ (since it does on each stalk) and the results map $\mathbb{Z}/n\mathbb{Z} \to R^2\pi_*\mu_n$ is an isomorphism on each stalk and is therefore an isomorphims.

Finally, $R^r\pi_*\mu_n = 0$ for $r > 2$ because $(R^r\pi_*\mu_n)_{\bar{s}} = H^r(X_{\bar{s}}, \mu_n) = 0$ for each $r > 2$ (shown in the previous section). Part (c) is obvious for $r \neq 1$, and $F \xrightarrow{\sim} g_*g^*F$ because $\phi_s : F_s \xrightarrow{\sim} F_{\overline{\eta}}^{I_s}$ for all s. □

We next consider the structure of $F = R^1\pi_*\mu_n$ at $s \in T$ (at the points with singularities).

Definition 3.5.4. The group of *vanishing cycles* at a closed point $s \in S$ is

$$V_s := \ker(F_s \to \mathrm{Pic}(X'_s))$$

where X'_s is the normalization of X_s. Since $F_s = \mathrm{Pic}(X_s)[n]$, we have that $F_s = 0$ for $s \notin T$. We usually identify V_s and F_s as subgroups of the stalk $F_{\overline{\eta}}$ at the generic point $\eta \in S$.

Note. We see that $F_{\overline{\eta}} = \mathrm{Pic}(X_\eta)[n] = \mathrm{Jac}(X_\eta)(k(\overline{\eta}))[n]$, so there is a canonical, nondegenerate, skew-symmetric pairing (generalization of the Weil pairing)

$$e_n : F_{\overline{\eta}} \times F_{\overline{\eta}} \to \mu_n(k)$$

This pairing may be identified with the cup product pairing on $H^1(X_{\overline{\eta}}, \mu_n)$. For any $\sigma \in \mathrm{Gal}(k(\overline{\eta})/k(\eta))$

$$e_n(\sigma\gamma_1, \sigma\gamma_2) = \sigma e_n(\gamma_1, \gamma_2) = e_n(\gamma_1, \gamma_2)$$

Before our next theorem, we will need a lemma:

Lemma 3.5.5. *Let Y be a complete curve over k and assume that Y has a node at $Q \in Y$ and no other singularities. Let $\psi : Y' \to Y$ be the normalization of Y, and let Q_1, Q_2 be the points of Y' mapping to Q. Then there is an exact sequence*

$$0 \to \mathbb{G}_m \to \mathrm{Jac}(Y') \to \mathrm{Jac}(Y) \to 0$$

in which the first map is described as follows: let $a \in \mathbb{G}_m(k) = k^\times$; there exists a function $f \in k(Y') = k(y)$ such that $f(Q_1) = a$ and $f(Q_2) = 1$; the image of a in $\mathrm{Jac}(Y)(k)$ is the class of the divisor (f).

Proof. Consider the exact sequence

$$0 \to \mathbb{G}_{m,Y} \to \psi_* \mathbb{G}_{m,Y'} \to \mathbb{G}_{m,Q} \to 0$$

and recall that $\mathrm{Jac}(Y)(k) = \ker(H^1(Y, \mathbb{G}_m) \xrightarrow{\deg} \mathbb{Z})$. Then, in cohomology, we get

$$\cdots \to H^0(Q, \mathbb{G}_m) \to H^1(Y, \mathbb{G}_m) \to H^1(Y', \mathbb{G}_m) \to H^1(Q, \mathbb{G}_m) = 0$$

which we split into the short exact sequence

$$0 \to \mathbb{G}_m \to \mathrm{Jac}(Y') \to \mathrm{Jac}(Y) \to 0$$

□

Theorem 3.5.6. *Let $\pi : X \to S$ be as above, and let $s \in T$. Then $F_s \simeq (\mathbb{Z}/n\mathbb{Z})^{2g-1}$ whre $g = p_a(X_\eta)$, and $V_s \simeq \mathbb{Z}/n\mathbb{Z}$. Moreover, V_s is the exact annihilator of F_s under the pairing $e_n : F_{\overline{\eta}} \times F_{\overline{\eta}} \to \mu_n(k)$.*

Proof. From the exact sequence in the lemma, with $Y = X_s$, we get an exact sequence

$$0 \to \mu_n(k) \to F_s \to \mathrm{Jac}(X_s')(k)[n] \to 0$$

Since X_s' has genus $p_a(X_\eta) - 1$, then $\mathrm{Jac}(X_s')(k)[n] \simeq (\mathbb{Z}/n\mathbb{Z})^{2g-2}$, so $F_s \simeq (\mathbb{Z}/n\mathbb{Z})^{2g-1}$ and V_s is the kernel $\mu_n(k) \hookrightarrow F_s$.

The fact that V_s is an exact annihilator of F_s follows from the fact that the pairing e_n is nondegenerate and $V_s, F_s, F_{\overline{\eta}}$ have order n, n^{2g-1}, n^{2g} respectively.

□

Corollary 3.5.7. *F is tamely ramified at any $s \in T$.*

Proof. We must show that there is a subgroup $H \subset I_s \subset \mathrm{Gal}(k(\overline{\eta})/k(\eta))$ (where I_s is the inertia subgroup of s) of finite index prime to $\mathrm{char}(k)$, which acts trivially on $F_{\overline{\eta}}$.

For $\gamma_0, \gamma \in F_{\overline{\eta}}$ and $\tau \in I_s$

$$e_n(\tau(\gamma_0), \tau(\gamma)) = e_n(\gamma_0, \gamma)$$

Thus, for any $\gamma \in F_s = F_{\overline{\eta}}^{I_s}$

$$e_n(\tau(\gamma_0 - \gamma_0, \gamma) = e_n(\tau(\gamma_0), \gamma) - e_n(\gamma_0, \gamma) = 0$$

that is $\tau(\gamma_0) - \gamma$ is orthogonal to F_s. The above theorem shows that $\tau(\gamma_0) - \gamma_0 \in V_s$, and it follows that if we choose γ_0 to generate $F_{\overline{\eta}}/F_s$, then $\tau \mapsto \tau(\gamma_0) - \gamma_0$ is a homomorphism $I_s \to V_s \simeq \mathbb{Z}/n\mathbb{Z}$ whose kernel is a subgroup of finite index. $\square$

In order to apply these results to an arbitrary surface, we must know how the cohomology behaves when a point is blown up, that is, when we pass from X to X^* as in the beginning of the subsection.

Proposition 3.5.8. *Let $q : X' \to X$ be the map obtained by blowing up X at a closed point $i : P \hookrightarrow X$, and let n be prime to* $\mathrm{char}(k)$*. Then*

$$R^r q_* \mu_n = \begin{cases} \mu_n & r = 0 \\ 0 & r = 1 \\ i_* \mathbb{Z}/n\mathbb{Z} & r = 2 \\ 0 & r \geqslant 3 \end{cases}$$

Proof. Obviously $q_*\mu_n \xrightarrow{\sim} \mu_n$ and $R^r q_* \mu_n$ is the sheaf associated with the presheaf $U \mapsto H^r(U', \mu_n)$ where U' is obtained from U by blowing up all the points of U lying over P. As $U' = U$ if the image of U does not contain P, it follows that $(R^r q_* \mu_n)_{\overline{x}} = 0$ if $r > 0$ and $x \neq P$. One may then use the proper base change theorem (proven in a later section, 5.2.1) to show that $(R^r q_* \mu_n)_P = H^r(\mathbb{P}^1_k, \mu_n)$ as $X'_P = \mathbb{P}^1_k$, from which the result follows. $\square$

Note. We can also prove this more directly by applying the Kummer sequence

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 0$$

Since for $U \to X$ étale, $\mathrm{Pic}(U') = \mathrm{Pic}(U) \oplus (\bigoplus_{P'} \mathbb{Z})$ for $P' \mapsto P$.

Let $\pi : X \to S = \mathbb{P}^1$ be a Lefschetz pencil with irreducible fibers, and let

$$F(-1) = F \otimes \mu_n^\vee = R^1 \pi_* \mathbb{Z}/n\mathbb{Z}$$

so that $F(-1)_{\overline{\eta}} = H^1(X_{\overline{\eta}}, \mathbb{Z}/n\mathbb{Z})$ and $F(-1)_s = H^1(X_s, \mathbb{Z}/n\mathbb{Z})$ for any $s \in T$, with $V_s \simeq \mu_n(k)$, and so

$$V_s(-1) \simeq \mu_n(k) \otimes \mu_n * k)^\vee = \mathbb{Z}/n\mathbb{Z}$$

has a canonical generator. The image of this generator in $F(-1)_{\overline{\eta}}$ will be written δ_s and is called the *canonical vanishing cycle at s*. Note that δ_s depends on choice of embedding $\mathcal{O}_{S,s} \hookrightarrow k(\overline{\eta})$. The e_n-pairing induces a pairing

$$F(-1)_{\overline{\eta}} \times F(-1)_{\overline{\eta}} \to (\mathbb{Z}/n\mathbb{Z})(-1)$$

which may be identified with the canonical pairing

$$\smile\colon H^1(X_{\overline{\eta}}, \mathbb{Z}/n\mathbb{Z}) \times H^1(X_{\overline{\eta}}, \mathbb{Z}/n\mathbb{Z}) \to H^2(X_{\overline{\eta}}, \mathbb{Z}/n\mathbb{Z})$$

As before, it is nondegenerate and skew-symmetric with the exact annihilator of $F_s(-1)$ being $V_s(-1) = \langle \delta_s \rangle$.

We write ϵ_s for the character $I_s \to \mu_n(k)$ of the inertia group at s such that $\sigma(t^{1/n}) = \epsilon_s(\sigma)t^{1/n}$ for t a uniformizing parameter at s.

We next give an explicit description of the cohomology groups $H^r(X, \mathbb{Z}/n\mathbb{Z})$ when $\pi : X \to S = \mathbb{P}^1$ is a Lefschetz pencil with irreducible fibers. To simplify notation, we write $\Lambda = \mathbb{Z}/n\mathbb{Z}$, so that $\Lambda(1) = \mu_n$ and $R^1\pi_*\Lambda(1) = F$. The Leray spectral sequence for π is

$$H^r(S, R^s\pi_*\Lambda(1)) \Rightarrow H^{r+s}(X, \Lambda(1))$$

We have already shown that

$$R^r\pi_*\Lambda(1) = \begin{cases} \Lambda(1) & r = 0 \\ F & r = 1 \\ \Lambda & r = 2 \\ 0 & r \geqslant 3 \end{cases}$$

and as any constant sheaf on $\mathbb{P}^1$ has zero first cohomology group, this gives

$$H^0(X, \Lambda(1)) = \Lambda(1)$$

$$H^1(X, \Lambda(1)) = H^0(S, F)$$

$$H^2(X, \Lambda(1)) = H^1(S, F) \oplus \langle \gamma_E \rangle \oplus \langle \gamma_F \rangle$$

$$H^3(X, \Lambda(1)) = H^2(S, F)$$

$$H^4(X, \Lambda(1)) = \Lambda(-1)$$

$$H^r(X, \Lambda(1)) = 0, \quad r > 4$$

where $\Gamma_E \in H^2(X, \Lambda(1))$ corresponds to the canonical generator of $H^2(S, \Lambda(1))$ and $\langle \gamma_E \rangle \simeq \Lambda \simeq H^2(S, \Lambda(1))$ is the free Λ-module that γ_E generates. It may also be described as the cohomology class of the divisor $E = \alpha(S)$, where α is a section of $\pi : X \to \mathbb{P}^1$ (from the map $\mathrm{Pic}(X) = H^1(X, \mathbb{G}_m) \to H^2(X, \Lambda(1))$). Similarly, γ_F is the cohomology class of any smooth fiber of π.

All that remains is to compute the cohomology of $F = R^1\pi_*\Lambda(1)$ on $S = \mathbb{P}^1$. By excision,

$$H^r_T(S, F) \simeq \bigoplus_{s \in T} H^r_s(\tilde{S}_{(s)}, F)$$

where $\tilde{S}_{(s)} = \mathrm{Spec}(\mathcal{O}_{S,s})$. Moreover, by the exact sequence in cohomology

$$0 \to H^0_s(\tilde{S}_{(s)}, F) \to H^0(\tilde{S}_{(s)}, F) \to H^0(\tilde{K}_{(s)}, F) \to H^1_s(\tilde{S}_{(s)}, F) \to \cdots$$

(where $\tilde{K}_{(s)}$ is the field of fractions of $\tilde{S}_{(s)}$) and the fact that $\tilde{S}_{(s)}$ is strictly local, so $H^r(\tilde{S}_{(s)}, F) = 0$ for $r > 0$ while $H^0(\tilde{S}_{(s)}, F) = F_x$, we get that

$$H^r_s(\tilde{S}_{(s)}, F) = 0 \qquad \text{for } r \neq 2$$

$$H^2(\tilde{S}_{(s)}, F) = H^1(\tilde{K}_{(s)}, F) = H^1(I_s, F) = H^1(I_s/I^1_s, F)$$

where I^1_s is the first ramification group (which we can quotient out by since the order F is prime to the characteristic).

Let $\pi_1^t = \mathrm{Gal}(k_t/k(\eta))$, where $k_t = \bigcup L$, for $k(\eta) \subset L \subset k(\overline{\eta})$ finite over $k(\eta)$ such that the normalization S_L of S in L is unramified over $S \smallsetminus T$ and tamely ramified over T, and writing $(S \smallsetminus T)_t = \varprojlim (S \smallsetminus T)_L$. Then, by the Hochschild-Serre spectral sequence (see appendix),

$$H^r(\pi_1^t, H^s((S \smallsetminus T)_t, F)) \Rightarrow H^{r+s}(S \smallsetminus T, F)$$

and the fact that U_t is an affine curve, with $H^1((S \smallsetminus T)_t, F) = \mathrm{Hom}(\pi_1, F_{\overline{\eta}}) = 0$, we get an isomorphism

$$H^r(S \smallsetminus T, F) = H^r(\pi_1^t, F_{\overline{\eta}})$$

which is zero for $r > 1$. The cohomology sequence for the pair $(S, S \smallsetminus T)$ now shows that

$$H^0(S, F) = H^0(S \smallsetminus T, F) = H^0(\pi_1^t, F_{\overline{\eta}})$$

$$H^1(S, F) = \ker\left(H^1(\pi_1^t, F_{\overline{\eta}}) \to \bigoplus_{s \in T} H^1(I_s^t, F_{\overline{\eta}}) \right)$$

$$H^2(S, F) = \mathrm{coker}\left(H^1(\pi_1^t, F_{\overline{\eta}}) \to \bigoplus_{s \in T} H^1(I_s^t, F_{\overline{\eta}}) \right)$$

From this we state a theorem

Theorem 3.5.9. *If $\pi : X \to \mathbb{P}^1$ is a Lefschetz pencil with irreducible fibers and n is prime to* $\mathrm{char}(k)$*, then:*

$$H^0(X, \Lambda(1)) = \Lambda(1)$$

$$H^1(X, \Lambda(1)) = F_{\overline{\eta}}^{\pi_1^t}$$

$$H^2(X, \Lambda(1)) = \langle \gamma_E \rangle \oplus \langle \gamma_F \rangle \oplus \ker\left(H^1(\pi_1^t, F_{\overline{\eta}}) \to \bigoplus_{s \in T} H^1(I_s^t, F_{\overline{\eta}}) \right)$$

$$H^3(X, \Lambda(1)) = \mathrm{coker}\left(H^1(\pi_1^t, F_{\overline{\eta}}) \to \bigoplus_{s \in T} H^1(I_s^t, F_{\overline{\eta}}) \right)$$

$$H^4(X, \Lambda(1)) = \Lambda(-1)$$

and

$$H^r(X, \Lambda(1)) = 0 \qquad r > 4$$

where

$$F_{\overline{\eta}} = H^1(X_{\overline{\eta}}, \Lambda(1)) = \ker\left(\mathrm{Jac}(X_{\overline{\eta}})(k) \xrightarrow{n} \mathrm{Jac}(X_{\overline{\eta}})(k)\right)$$

$T \subset \mathbb{P}^1$ is the set over which the fibers are singular, $\pi_1^t = \mathrm{Gal}(k_t/k(\eta)$ (as before, $k_t = \bigcup L$, where L is finite intermediate field extension unramified at $\mathbb{P} \smallsetminus T$ and tamely ramified at T), and γ_E, γ_F are the cohomology classes of a section and smooth fiber respectively.

Proof. This is a restatement of the above. □

We then combine this with the pushforward along the blow-up to get the general case:

Theorem 3.5.10. *Let X be a smooth complete surface over an algebraically closed field k. Let $q : X^* \to X$ be a map obtained by blowing up X at a finite set of point $P = \{p_1, \ldots, p_m\}$ and $\pi : X^* \to \mathbb{P}^1$ be a Lefschetz pencil with set of singularities $T = \{s_1, \ldots, s_t\} \subset \mathbb{P}^1$. Then, we have the following results*

$$H^0(X, \Lambda(1)) = \Lambda(1)$$

$$H^1(X, \Lambda(1)) = F_{\overline{\eta}}^{\pi_1^t}$$

$$H^2(X, \Lambda(1)) \oplus \Lambda^m = \langle \gamma_E \rangle \oplus \langle \gamma_F \rangle \oplus \ker\left(H^1(\pi_1^t, F_{\overline{\eta}}) \to \bigoplus_{s \in T} H^1(I_s^t, F_{\overline{\eta}})\right)$$

$$H^3(X, \Lambda(1)) = \operatorname{coker}\left(H^1(\pi_1^t, F_{\overline{\eta}}) \to \bigoplus_{s \in T} H^1(I_s^t, F_{\overline{\eta}})\right)$$

$$H^4(X, \Lambda(1)) = \Lambda(-1)$$

and

$$H^r(X, \Lambda(1)) = 0 \qquad r > 4$$

(with notation as above)

Proof. By the result on the higher direct images of along blow-ups, we have that

$$R^r q_* \Lambda(1) = \begin{cases} \Lambda(1) & r = 0 \\ 0 & r = 1 \\ \Lambda_P & r = 2 \\ 0 & r \geqslant 3 \end{cases}$$

where Λ_P is the skyscraper sheaf $(\Lambda_P)_{p_i} = \Lambda$ for all $p_i \in P$. Combining this with the Leray spectral sequence, we get that

$$H^0(X^*, \Lambda(1)) = H^0(X, \Lambda(1))$$

$$
\begin{aligned}
H^1(X^*, \Lambda(1)) &= H^1(X, \Lambda(1)) \\
H^2(X^*, \Lambda(1)) &= H^2(X, \Lambda(1)) \oplus H^0(X, \Lambda_P) \\
H^3(X^*, \Lambda(1)) &= H^3(X, \Lambda(1)) \oplus H^1(X, \Lambda_P) \\
H^4(X^*, \Lambda(1)) &= H^4(X, \Lambda(1)) \oplus H^2(X, \Lambda_P) \\
H^r(X^*, \Lambda(1)) &= H^r(X, \Lambda(1)) \oplus H^{r-2}(X, \Lambda_P) \qquad r > 4
\end{aligned}
$$

Since the skyscraper sheaves are flasque, their higher cohomology vanishes, and we have that $H^0(X, \Lambda_P) = \Lambda^m$. The result follows. $\square$

Note. The above theorems remain valid if we consider $\Lambda = \mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$, $\ell \neq \mathrm{char}(k)$.

As we shall later see, we will become very concerned about the *monodromy* of the Lefschetz pencils, i.e. the behavior at the set of singular fibers T, and how the Galois group π_1^t behaves on and around the fibers of T. We will see that the singularities are (fortunately) of a simple nature, and that we will be able to conclude many similar things in the general case by using the inertia groups, Picard groups, etc.

3.6 Étale Cohomology for Abelian Varieties

We will quickly review what the étale cohomology groups of abelian varieties are, just to see how simple this case is, clarifying why the Weil conjectures were immediately proven for this case, as well as giving a new perspective on the proof for elliptic curves. Recall that an abelian variety is a projective group variety, a generalization of elliptic curves to higher dimensions. For a review of the theory, see [Mil08] or [Mum85].

Theorem 3.6.1. *Let A be an abelian variety of dimension g over a separably closed field k, and let ℓ be a prime different from* $\mathrm{char}(k)$.

(a) There is a canonical isomorphism

$$H^1(A_{\acute{e}t}, \mathbb{Z}_\ell) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell A, \mathbb{Z}_\ell)$$

where $T_\ell A$ is the ℓ-adic Tate module of A, defined similarly to the Tate module of an elliptic curve.

(b) The cup-product pairings define isomorphisms for all r

$$\bigwedge^r H^1(A_{\acute{e}t}, \mathbb{Z}_\ell) \to H^r(A_{\acute{e}t}, \mathbb{Z}_\ell)$$

In particular, $H^r(A_{\acute{e}t}, \mathbb{Z}_\ell)$ is a free $\mathbb{Z}_\ell$-module of rank $\binom{2g}{r}$.

Note. This all "follows from the fact" that the Tate module can be thought of as satisfying the following isomorphism

$$T_\ell A \simeq \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$$

where we think of the abelian variety as a complex torus $A \simeq V/\Lambda$, where V is some g dimensional complex vector space and $\Lambda \simeq \mathbb{Z}^{2g}$ is a lattice in V (this is not quite true in all cases, e.g. when the characteristic is positive, but it is a fine heuristic)(In general, we also have to worry about polarization or the existence of Riemann forms, but this is not a paper on abelian varieties or complex tori).

From this, we deduce the fact that

$$H_1(A, \mathbb{Z}_\ell) \simeq H_1(V/\Lambda, \mathbb{Z}) \otimes \mathbb{Z}_\ell \simeq \Lambda \otimes \mathbb{Z}_\ell \simeq T_\ell A$$

and remember the fact that cohomology is dual to homology, along with the fact that the Künneth formula on a torus T gives us an isomorphism

$$\bigwedge^i (H^1(T, \mathbb{Z})) \simeq H^i(T, \mathbb{Z})$$

From this theorem, it is quite elementary to generalize the proof of the Weil conjectures for elliptic curves to the case of abelian varieties, with the one caveat being that the "Riemann Hypothesis" requires some extra work, but quickly follows from directly calculating the eigenvalues of the Frobenius automorphism on the Tate modules (and a little bit of extra thought). The diligent reader is encouraged to work this all out.

4 More on Sheaves

For this section we refer to [KW01] and [Fu15].

4.1 Constructible Sheaves

Intuitively, we should think of constructible sheaves as a link between torsion sheaves (which connect us to ℓ-adic sheaves, to be defined below) and local systems. We can think of our reductions like this

$$\ell\text{-adic} \Rightarrow \text{torsion} \Rightarrow \text{constructible} \Rightarrow \text{local systems} \Leftrightarrow \pi_1(X)\text{-respresentations}$$

This is in fact how we show that lisse ℓ-adic sheaves correspond to smooth Galois representations

Beyond that, constructible sheaves are the correct sheaves for most finiteness theorems, limit constructions, and other useful things.

An important class of sheaves for the étale site is the class of *constructible sheaves*. Recall that for any sheaf of sets $\mathcal{F}$ on X, there exists a family of X-schemes $\{X_n\}$ and a surjective sheaf mapping

$$\bigsqcup \tilde{X}_n \to \mathcal{F}$$

(where $\tilde{X}_n$ is the sheaf represented by X_n, i.e. $\tilde{X}_n := \mathrm{Hom}_X(-, X_n)$). The sheaf $\mathcal{F}$ is called constructible if one can find a finite family $\{X_i\}$ to satisfy this condition. In the case of the étale site, it turns out that $\mathcal{F}$ is the quotient of a representable sheaf by a representable equivalence relation, so one can interpret $\mathcal{F}$ as an étale algebraic space (see [Knu71]). We also have the following four equivalent definitions (see [FK88][p39-53] for proofs throughout this section):

Definition 4.1.1. A sheaf $\mathcal{F}$ on a scheme X is called constructible if it satisfies one of the following (equivalent) properties

(i) If X can be written as the union of finitely many locally closed subschemes $Y \subset X$ for which $\mathcal{F}|_Y$ is finite locally constant;

(ii) For every closed subscheme $Y \subset X$, there is an open nonempty subset $U \subset Y$ for which the inverse image $\mathcal{F}|_U$ is locally constant;

(iii) Fpr every irreducible closed subscheme $Y \subset X$ there is an étale neighborhood $U \to Y$ of the generic point of Y for which the inverse image of $\mathcal{F}$ on U is finite locally constant;

(iv) For every closed integral subscheme $Y \subset X$, there is a nonempty étale scheme $V \to Y$ for which the inverse image of $\mathcal{F}$ on V is constructible.

Lemma 4.1.2. *Every sheaf representable as an étale scheme $U \to X$*

$$\mathrm{Hom}_X(-, U)$$

is a constructible sheaf of sets.

Constructible sheaves are nice because they have an important compactness property:

Lemma 4.1.3. *Let $\{\mathcal{F}_i \to \mathcal{F}\}_{i \in I}$ be a surjective family of mappings of arbitrary sheaves to the constructible sheaf $\mathcal{F}$. Then, there is a finite surjective subfamily*

$$\mathcal{F}_{i_1}, \ldots, \mathcal{F}_{i_r} \to \mathcal{F}$$

that is, one for which

$$\bigsqcup_v \mathcal{F}_{i_v} \to \mathcal{F}$$

is surjective.

and, there is an important characterization of constructible sheaves:

Proposition 4.1.4. *For a sheaf $\mathcal{F}$ of sets on X, the following three conditions are equivalent:*

(a) $\mathcal{F}$ is constructible;

(b) There is a sheaf $\mathcal{G} = \mathrm{Hom}_X(-, Y)$ representable by some étale scheme $Y \to X$ and a surjective mapping $\mathcal{G} \to \mathcal{F}$;

(c) *There is an equivalence relation $R \rightrightarrows Y$ in* $\mathbf{Et}/X$ *such that* $\mathcal{F}$ *is the cokernel of the equivalence relation*

$$\mathrm{Hom}_X(-, R) \rightrightarrows \mathrm{Hom}_X(-, Y)$$

of representable sheaves.

Corollary 4.1.5. *Every subsheaf of a constructible sheaf is constructible.*

Corollary 4.1.6. *Every sheaf of sets is the filtered direct limit of its constructible subsheaves.*

Proposition 4.1.7. *Let* $\mathcal{F}$ *be a constructible sheaf of* $\mathbb{Z}/n\mathbb{Z}$*-modules on a scheme* X*. There is an injective homomorphism*

$$\mathcal{F} \hookrightarrow \bigoplus_{v=1}^{m} (f_v)_*(\mathbb{Z}/n\mathbb{Z})$$

for some finite (not necessarily surjective) morphisms

$$f_v : X_v \to X$$

Note. This can be generalized beyond $\mathbb{Z}/n\mathbb{Z}$-modules to abelian groups by the idea that abelian groups are just $\mathbb{Z}$-modules (but there is a *bit* more finesse involved).

Constructible sheaves of abelian groups are particular (well-behaved) examples of torsion sheaves (i.e. sheaves for which all stalks are torsion groups). Specifically,

Proposition 4.1.8. *Ther constructible sheaves of abelian groups are precisely the Noetherian objects in the category of all torsion sheaves, and every torsion sheaf is the filtered direct limit of its constructible subsheaves.*

The Noetherian property guarantees that we can use certain reductions and inductive principles when working with constructible sheaves. In addition, constructible sheaves behave well under the direct image of finite morphisms:

Proposition 4.1.9. *Let* $f : Y \to X$ *be a finite morphism of schemes and let* $\mathcal{F}$ *be a constructible sheaf on* Y*. Then,* $f_*\mathcal{F}$ *is also constructible.*

Constructible sheaves are useful because cohomology (which we are about to discuss in detail) commutes with filtered direct limits of sheaves, i.e. for $\mathcal{F} = \varinjlim_i \mathcal{F}_i$, we have that

$$H^r(X, \mathcal{F}) = \varinjlim_i H^r(X, \mathcal{F}_i)$$

so, when we study torsion sheaves, it is enough to study constructible sheaves. This, along with the construction of a limit of étale schemes allows for the following theorem which will be useful (and which we may be guilty of implicitly citing without explanation):

Proposition 4.1.10. *Let $X = \varprojlim X_\alpha$ be a projective limit of a projective system of schemes.. Let $\mathcal{F}$ be a constructible sheaf of sets on X. Then, there exists an index α_0 and a constructible sheaf $\mathcal{F}_{\alpha_0}$ on X_{α_0} whose inverse image on X coincides with $\mathcal{F}$.*

Let $\mathcal{F}_{\alpha_0}$ and $\mathcal{G}_{\alpha_0}$ be sheaves of sets on some X_{α_0} If $\mathcal{F}_{\alpha_0}$ is constructible, then

In particular, if $\mathcal{F}$ is representable by an étale X_{α_0}-scheme U_{α_0}, this means

Let $\mathcal{F}_{\alpha_0}$ be a torsion sheaf on an X_{α_0}. Then, for $\alpha \geqslant \alpha_0$, the following étale cohomology groups are equal

$$H^*(X, \mathcal{F}) = \varinjlim H^*(X_\alpha, \mathcal{F}_\alpha)$$

In particular

$$\mathcal{F}(X) = \varinjlim \mathcal{F}_\alpha(X_\alpha)$$

Any sheaf $\mathcal{F}$ on $X_{\acute{e}t}$ is represented by a unique locally separated algebraic space $\tilde{\mathcal{F}}$. Unfortunately, if $\mathcal{F}$ is a sheaf on the big étale site $X_{\acute{E}t}$ and $\mathcal{F}' = \mathcal{F}|_{X_{\acute{e}t}}$ it is not true that $\mathcal{F} = \tilde{\mathcal{F}}'$ on $X_{\acute{e}t}$. This would be true if and only if for the continuous morphism of site $f : X_{\acute{E}t} \to X_{\acute{e}t}$ defined by the identity morphism, we have that

$$f^* f_* \mathcal{F} \simeq \mathcal{F}$$

we call such a sheaf *locally constructible*. We can see from this that the sheaf $\mathcal{F}$ is constructible if it is locally constructible and $\tilde{\mathcal{F}}$ is of finite-type over X (this "follows" from the fact that the sheaf represented by a group scheme is constructible if and only if it is of finite-type over X). This is another way to characterize constructible sheaves.

4.2 ℓ-adic Sheaves

So far, we have only considered étale cohomology with respect to torsion sheaves, but one condition for a Weil cohomology theory is that the cohomology groups are vector spaces over a certain torsion-free coefficient field. One can show the following:

Let X be a finitely generated scheme of dimension n over an algebraically closed base field, and let $\mathcal{F}$ be an arbitrary étale sheaf on X. Then, the cohomology groups $H^i(X, \mathcal{F})$ for $i > n$ are torsion groups. In addition, if X is normal and $\mathcal{F}$ is locally constant, then, in fact, all cohomology groups $H^i(X, \mathcal{F})$, i>0, are torsion gropus. The reason for this is that all higher Galois cohomology groups for an arbitrary Galois module are torsion groups.

This can be better illustrated by drawing conclusions from the calculations which we made in the last subsection:

Example 9. Let X be a smooth projective connected curve X of genus g over an algebraically closed base field k. Let ℓ be a prime number different from the characteristic of k. Then, by using the higher direct image along generic fibers and properties of Galois cohomology, we find that

$$H^i(X, \underline{\mathbb{Q}_\ell}) = \begin{cases} \mathbb{Q}_\ell & \text{for } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$H^i(X, \underline{\mathbb{Z}_\ell}) = \begin{cases} \mathbb{Z}_\ell & \text{for } i = 0 \\ 0 & \text{for } i = 1 \\ (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{2g} & \text{for } i = 2 \\ \mathbb{Q}_\ell/\mathbb{Z}_\ell & \text{for } i = 3 \\ 0 & \text{otherwise} \end{cases}$$

For the case of $\mathbb{Q}_\ell$, we take the structure sheaf on the generic fiber $X \times_k \eta$, getting the function field K_X as its pushforward. This reduces the problem to one of Galois cohomology (like in the previous section), where we take the Galois group $G = \mathrm{Gal}(K_X^{sep}/K_X)$ and have it act trivially on $\mathbb{Z}_\ell$ and $\mathbb{Q}_\ell$. Using the exact sequence

$$0 \to \mathbb{Z}_\ell \to \mathbb{Q}_\ell \to \mathbb{Q}_\ell/\mathbb{Z}_\ell \to 0$$

and the fact that $H^i(G, \mathbb{Q}_\ell) = 0$ for $i > 0$, we get by the long exact sequence in cohomology (taking the Galois-invariance functor)

$$H^1(G, \mathbb{Z}_\ell) = 0$$

$$H^{i+1}(G, \mathbb{Z}_\ell) \simeq H^i(G, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \qquad \text{for i>0}$$

so, since $\mathbb{Q}_\ell/\mathbb{Z}_\ell \simeq \varinjlim_n \mathbb{Z}/\ell^n\mathbb{Z}$, we get $\varinjlim H^i(G, \mathbb{Z}/\ell^n\mathbb{Z}) = H^i(G, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$, so, using the calculations on torsion sheaves in the last subsection,

$$H^2(G, \mathbb{Z}_\ell) \simeq H^1(G, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \simeq (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{2g}$$

$$H^3(G, \mathbb{Z}_\ell) \simeq H^2(G, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \simeq \mathbb{Q}_\ell/\mathbb{Z}_\ell$$

and, for $i > 3$

$$H^i(G, \mathbb{Z}_\ell) \simeq H^{i-1}(G, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$$

But, if we instead take the limits of the cohomology groups themselves, we get exactly what we "should get":

Example 10.

$$\varprojlim_n H^i(X, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}) = \begin{cases} \mathbb{Z}_\ell & \text{for } i = 0, 2 \\ (\mathbb{Z}_\ell)^{2g} & \text{for } i = 1 \\ 0 & \text{for } i > 2 \end{cases}$$

$$\varprojlim_n H^i(X, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}) \otimes \mathbb{Q}_\ell = \begin{cases} \mathbb{Q}_\ell & \text{for } i = 0, 2 \\ (\mathbb{Q}_\ell)^{2g} & \text{for } i = 1 \\ 0 & \text{for } i > 2 \end{cases}$$

More generally, we shall consider certain projective systems of torsion sheaves, and instead of considering the cohomology of the projective limit, we instead consider the projective limit of the cohomology groups, i.e.

$$H^i\left(X, \varprojlim_n \mathcal{F}_n\right) \qquad \text{vs} \qquad \varprojlim_n H^i(X, \mathcal{F}_n)$$

Definition 4.2.1. Accordingly, we we have the following notation for the rest of the paper

$$H^i(X, \mathbb{Z}_\ell) = \varprojlim_n H^i(X, \underline{\mathbb{Z}/\ell^n\mathbb{Z}})$$

and

$$H^i(X, \mathbb{Q}_\ell) = H^i(X, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell$$

More generally, we define an ℓ-adic sheaf to be the "sheaf" corresponding to a projective system $\{F_n, u_n\}$ satisfying the following properties

(i) $F_n = 0$ for $n < 0$;

(ii) $\ell^{n+1}F_n = 0$ for all n;

(iii) For all $n > 0$, an isomorphism

$$F_n \otimes_{\mathbb{Z}/\ell^{n+1}\mathbb{Z}} \mathbb{Z}/\ell^n\mathbb{Z} = \simeq F_n/\ell^n F_n \simeq F_{n-1}$$

(iv) For all n, F_n is a module of finite length.

where we take the cohomology in the same manner as above for $H^i(X, \mathbb{Z}_\ell)$.

Note that ℓ-adic sheaves aren't actually sheaves on the étale site, but that we can treat the entire projective system as a sheaf of sorts. The following propositions verify that this is a valid approach.

We call an ℓ-adic sheaf $\mathcal{F} = \{\mathcal{F}_n\}$ locally constant if all of the sheaves $\mathcal{F}_n$ are locally constant.

A morphism between ℓ-adic sheaves is a set of morphisms between the corresponding projective systems compatible with the transition maps, denote $\mathrm{Hom}_\ell(\mathcal{F}, \mathcal{G})$

Proposition 4.2.2. *Let $\mathcal{F} = \{\mathcal{F}_n\}$ be an ℓ-adic sheaf on a scheme X. Then there is a dense open subscheme $U \subset X$ for which the restricted system $\mathcal{F}|_U = \{\mathcal{F}|_U\}$ is locally constant.*

Proof. We may assume that X is irreducible; then, we must simply find a non-empty open subscheme $U \subset X$ for which $\mathcal{F}|_U$ is locally constant.

It suffices to find U such that $\mathcal{F}_0$ and the kernels $\ell^n\mathcal{F}_n$ of the surjective mappings $\mathcal{F}_n \to \mathcal{F}_{n-1}$ are locally constant on U. But, the map factors

$$\mathcal{F}_n \to \mathcal{F}_0 \simeq \mathcal{F}_n/\ell\mathcal{F}_n \to \ell^n\mathcal{F}_n$$

We know that $\mathcal{F}_0$ is Noetherian, so the chain of kernels is stationary. $\square$

Proposition 4.2.3. *Let $\pi : X \to S$ be a compactifiable morphism, and $\mathcal{F} = \{\mathcal{F}_n\}$ an ℓ-adic sheaf on X. Then the system $R^i_c\pi_*\mathcal{F} = \{R^i_c\pi_*\mathcal{F}_n\}$ are ℓ-adic sheaves.*

Note. We will not present a proof here. We recall that a morphism $\pi : X \to S$ is compactifiable if there is a commutative diagram

$$\begin{tikzcd} X \arrow[rr, hook, "j"] \arrow[dr, "\pi"'] & & \overline{X} \arrow[dl, "\overline{\pi}"] \\ & S & \end{tikzcd}$$

with $j : X \hookrightarrow \overline{X}$ an open immersion, $\overline{X}$ a complete variety and $\overline{\pi}$ a proper morphism. Then, we set

$$R^i_c\pi_*\mathcal{F} = R^i\overline{\pi}_*(j_!\mathcal{F})$$

to be the *higher direct images with compact support.* These cohomology groups are independent of embedding, essentially by the proper base change theorem 5.2.1 and the Grothendieck spectral sequence.

Definition 4.2.4. Let $\mathcal{F} = \{\mathcal{F}_n\}$ be an ℓ-adic sheaf on a scheme X. Then, $\mathcal{F} \otimes \mathbb{Q}_\ell = \{\mathcal{F}_n \otimes \mathbb{Q}_\ell\}$ is called a *sheaf of $\mathbb{Q}_\ell$-vector spaces.* The morphisms are defined as

$$\operatorname{Hom}(\mathcal{F} \otimes \mathbb{Q}_\ell, \mathcal{G} \otimes \mathbb{Q}_\ell) := \operatorname{Hom}_\ell(\mathcal{F}, \mathcal{G}) \otimes \mathbb{Q}_\ell$$

Similarly, all other sheaf theoretic constructions (stalks, cohomology, global sections, etc.) are all defined by taking the corresponding object for ℓ-adic sheaves (as a projective system) and then tensoring with $\mathbb{Q}_\ell$.

Note. For most of the results which we will derive, we will work with torsion sheaves, for simplicities sake. Note that many of these results can be carried over to the case of ℓ-adic sheaves by taking projective limits. This makes ℓ-adic sheaves and sheaves of $\mathbb{Q}_\ell$-vector spaces the correct way for us to incorporate torsion-free coefficients into our cohomology theory.

4.3 Derived Categories

Instead of hashing out the entire theory of derived categories, we devote a section in the appendix to discussing them alongside spectral sequences. For more info on derived categories, please see my other notes on the subject (there are many of them on my website). Here, we will define and analyze the derived categories of sheaves which are of interest to us and then state some lemmas which we will require later, but we might skimp on the details of the proofs a little bit, instead referring to some standard text.

Though it would be nice for the reader to have a good understanding/intuition for derived categories, it is infeasible for me to write anything which wouldn't be far too long for my purposes here, plus the fact that it is my opinion that the best way to get acquainted with the theory and to become convinced of its merits is to see it in action.

Let me comment on *why* we even want to work with derived categories, besides as a conceptual simplification of certain proofs (since much of it can be done "manually" with injective resolutions and spectral sequences). Our first goal after this section is to prove the fundamental theorems of étale cohomology and to do so in the language of derived categories. For the purpose of the base change theorems, there will be little different from the case of good-old sheaves, but we will see that the Künneth formula quite easily falls from the formalism of derived categories. The real power will start to shine through once we get into Poincaré duality. To generalize duality to potentially singular morphisms of schemes, derived categories are required. This is best seen in the construction of the "upper-shriek functor" which we denote $f^!$, culminating the theory of biduality for derived categories of ℓ-adic construcible sheaves (giving us tight control on certain cycle classes).

5 Base Change Theorems

Now we turn to proving that étale cohomology has the properties of a Weil cohomology theory, thereby proving every part of the Weil conjectures, except for the "Riemann Hypothesis". Cohomological dimension is simple, but the other properties require more work, with preliminary theorems on base changes, finiteness, and other reduction techniques required to prove them.

5.1 Cohomological Dimension 1

Definition 5.1.1. Let X be a scheme and let ℓ be a prime number. The *ℓ-cohomological dimension $cd_\ell(\mathscr{C}/X)_E$* of a site $(\mathscr{C}/X)_E$ is the smallest integer n (or ∞) such that $H^i(X_E, \mathcal{F}) = 0$ for all $i > n$ and all ℓ-torsion sheaves $\mathcal{F}$. We write $cd_\ell(A)$ for $cd_\ell(\mathrm{Spec}A)_{\acute{e}t}$.

To prove the property of cohomological dimension, we will need a generalization of Tsen's Theorem:

Lemma 5.1.2. *Let K be a field of transcendence degree d over its subfield k. Then*

$$cd_\ell(K) \leqslant cd_\ell(k) + d$$

See [Sha72][p119] for proof. We will also need the following lemma:

Lemma 5.1.3. *Let X be an irreducible and reduced scheme, with $\mathcal{F}$ a sheaf on the generic point $g : \eta \hookrightarrow X$. Then, the sheaf $R^j g_* \mathcal{F}$ has support in dimension $\leqslant n - j$.*

Proof. Let $x \in X$, let $A = \mathcal{O}_{X,x}$, and let $A_1, \ldots, A_s$ be the quotients of A by its minimal ideals, and let K_r be the field of fractions of A_r. Then, we have the following isomorphism

$$(R^j g_* \mathcal{F})_{\overline{x}} \simeq \bigoplus H^j(K_r, \mathcal{F}_{K_r})$$

But, for any $x \in X$, there exists a separably closed field k^s, a scheme X' of finite-type over k', and a closed point $x' \in X'$ such that $\mathcal{O}_{X,x} \simeq \mathcal{O}_{X',x'}$ (this follows from Noether normalization). Then, if $d = \dim\overline{\{x\}}$, K_r is of transcendence degree $\leqslant n-d$ over k' and so $H^j(K_r, F|_{K_r}) = 0$ for $j > n-d$, i.e $d > n-j$, by Tate's theorem. □

Theorem 5.1.4. *If X is a scheme of finite-type over a separably closed field k, then $cd_\ell(X_{ét}) \leqslant 2\dim(X)$.*

Proof. We precede by induction. As shown in our section "Étale cohomology on a point" the theorem is true for $n = 0$, so we assume that it holds for $n-1$.

Let $\mathcal{F}$ be an ℓ-torsion sheaf on $X_{ét}$. If $\mathcal{F}$ has support in dimension $\leqslant n-1$, i.e. if $\mathcal{F} = \bigcup i_{n*}\mathcal{F}_n$, where $i_n : Z_n \hookrightarrow X$ are closed subschemes of dimension $\leqslant n-1$, then, since cohomology commutes with (pseudo)-direct limits, we have that

$$H^i(X, \mathcal{F}) = \bigcup H^i(Z_n, \mathcal{F}_n)$$

which vanishes for $i > 2n-2$ by the induction hypothesis.

Let $\{x_1, \ldots, x_r\}$ be the generic points of irreducible components of X and $g_s : x_s \hookrightarrow X$ the inclusion maps. The kernel and cokernel of the canonical map

$$\mathcal{F} \to \bigoplus g_{s*}g_s^*\mathcal{F}$$

have support in dimension $\leqslant n-1$, and so the above remark reduces us to considering a sheaf of the form $g_{s*}\mathcal{F}$. Thus, we may assume that X is irreducible and reduced.

From the induction hypothesis and the lemma above, we find that

$$H^i(X, R^j g_*\mathcal{F}) = 0$$

for $i > (2n-j)$, $j \neq 0$. Thus, the Leray spectral sequence

$$H^i(X, R^j g_*\mathcal{F}) \Rightarrow H^{i+j}(k(X), \mathcal{F})$$

gives an isomorphism

$$H^i(X, g_*\mathcal{F}) \simeq H^i(k(X), \mathcal{F})$$

for $i > 2n$, and the last group is zero □

Now, let's prove something slightly more general in terms of base rings, but less general in terms of dimension. This will follow from our analysis of the cohomology of curves:

Theorem 5.1.5. *Let X be a scheme of finite-type over a separably closed field k with $\dim X \leqslant 1$. For any torsion sheaf $\mathcal{F}$ on X, we have $H^q(X, \mathcal{F}) = 0$ for all $q \geqslant 3$. If X is affine, we have that $H^q(X, \mathcal{F}) = 0$ for all $q \geqslant 2$.*

Proof. First, we recall that fact that all torsion sheaves $\mathcal{F}$ are the direct limit of their constructible subsheaves and so the fact that limits of constructible sheaves commute with cohomology, we may assume that $\mathcal{F}$ is constructible.

Then, by similar reasoning, we may assume that k is algebraically closed (since algebraically closures are limits of finite extensions). In addition, by **??** we may assume that X is reduced.

Let $\eta_1, \ldots, \eta_m$ be all the points in X such that $\dim\overline{\{\eta_i\}} = 1$ for all $i = 1, \ldots, m$. Let

$$S = \prod_{i=1}^{m} \mathcal{O}_{X,\eta_i}$$

and

$$j : \mathrm{Spec}R \to X$$

the canonical morphism. In the derived category the canonical morphism $\mathcal{F} \to \mathbf{R}j_*j^*\mathcal{F}$ can be extended to a distinguished triangle via the mapping cone:

$$\mathcal{F} \to \mathbf{R}j_*j^*\mathcal{F} \to \Delta \to$$

Since $\mathcal{F}$ is a complex with only one elements, we can take homology of this distinguished triangle to get an exact sequence

$$0 \to \mathscr{H}^{-1}(\Delta) \to \mathcal{F} \to j_*j^*\mathcal{F} \to \mathscr{H}^0(\Delta) \to 0$$

and the fact

$$\mathbf{R}^q j_*j^*\mathcal{F} \cong \mathscr{H}^q(\Delta)$$

for any $q \geqslant 1$. Recall the fact that $H^q(\mathrm{Spec}\tilde{\mathcal{O}}_{X,\eta_i}, j^*\mathcal{F}) = 0$ for all $q \geqslant 1$ (since the cohomology of Henselian rings corresponds to the cohomology of their closed point, and clearly $\Gamma(X, \mathcal{F}) = \mathcal{F}_s$ is exact; see **??** for details), and so, since the η_i are all of the non-closed points, we conclude that $\mathscr{H}^q(\Delta)$ is a skyscraper sheaf for all q. Then, since the cohomology of skyscraper sheaves is trivial (see **??**), we conclude that the Grothendieck spectral sequence

$$E_2^{p,q} := H^p(X, \mathscr{H}^q(\Delta)) \Rightarrow H^{p+q}(X, \Delta)$$

degenerates, and

$$H^q(X, \Delta) \cong H^0(X, \mathscr{H}^q(\Delta))$$

Moreover

$$H^q(X, \mathbf{R}j_*j^*\mathcal{F}) \cong H^q(\mathrm{Spec}R, j^*\mathcal{F})$$

Next, we apply the cohomology functor $H^*(X, -)$ to the distinguished triangle above to get

$$\cdots \to H^q(X, \mathcal{F}) \to H^q(\mathrm{Spec}R, j^*\mathcal{F}) \to H^0(X, \mathscr{H}^q(\Delta)) \to H^{q+1}(X, \mathcal{F}) \to \cdots$$

Now, we want to analyze this sequence to get our results.

Recall that since X is reduced, each $\mathcal{O}_{X,\eta_i}$ is a field of transcendence degree one over k. By recalling the connection to Galois cohomology and Tsen's Theorem **??**, we get that $H^q(\mathrm{Spec}R, j^*\mathcal{F}) = 0$ for all $q \geqslant 2$, and thereby we get

$$H^q(X, \mathcal{F}) \cong H^0(X, \mathscr{H}^{q-1}(\Delta))$$

for any $q \geqslant 3$. To see that this cohomology is trivial, we point to the fact that $\mathbf{R}^q j_* j^* \mathcal{F}$ is, by definition, the sheaf associated to the presheaf

$$V \mapsto H^q((\mathrm{Spec}R) \times_X V, j^*\mathcal{F})$$

for any étale X-scheme V. Then, we can see that $(\mathrm{Spec}R) \times_X V$ must be a disjoint union of the spectra of some finite separable field extensions of $\mathcal{O}_{X,\eta_i}$, and so Tsen's theorem gives us our desired results once again.

Now we turn to the case of X affine. By our preliminary work on constructible sheaves, we start by assuming that $\mathcal{F}$ is a constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules for some n, and so there exists an epimorphism

$$\phi : f_! \mathbb{Z}/n\mathbb{Z} \to \mathcal{F}$$

for some separated étale morphism of finite-type $f : U \to X$. From our above discussion, we have that $H^3(X, \ker \phi) = 0$, and so the exact sequence of cohomology tells us that

$$H^2(X, f_! \mathbb{Z}/n\mathbb{Z}) \to H^2(X, \mathcal{F})$$

is surjective, and so we are reduced to proving that $H^2(X, f_! \mathbb{Z}/n\mathbb{Z}) = 0$. Reusing the notation η_i from above, we note that since f is étale we have that

$$f_{\eta_i} : U \times_{\mathcal{O}_X} \mathcal{O}_{X,\eta_i} \to \mathrm{Spec}\mathcal{O}_{X,\eta_i}$$

are finite morphisms, so by [Fu15] 1.10.10, there exists an open neighborhood V_i of η_i such that $f_{V_i} : f^{-1}(V_i) \to V_i$ is finite. Let $V = \cup_i V_i$ and let $j : V \to X$ be the open immersion. Then, the canonical morphism

$$f_! \mathbb{Z}/n\mathbb{Z} \to j_* j^* f_! \mathbb{Z}/n\mathbb{Z}$$

is an isomorphism when restricted to V. Since $X \smallsetminus V$ is finite over k, the qth cohomology groups of the kernel and cokernel of the canonical morphism above vanish for all $q \geqslant 1$. So

$$H^2(X.f_! \mathbb{Z}/n\mathbb{Z}) \cong H^2(X, j_* j^* f_! \mathbb{Z}/n\mathbb{Z}) \cong H^2(X, j_*(f_V)_! \mathbb{Z}/n\mathbb{Z})$$

but since f_V is finite, we have that $(f_V)_! \mathbb{Z}/n\mathbb{Z} \cong (f_V)_* \mathbb{Z}/n\mathbb{Z}$, and it follows that

$$j_* j^* f_! \mathbb{Z}/n\mathbb{Z} \cong (j \circ f_V)_* \mathbb{Z}/n\mathbb{Z}$$

Since $j \circ f_V$ is étale we are reduced to proving that $H^2(X, f_* \mathbb{Z}/n\mathbb{Z}) = 0$ for any separated étale morphism $f : U \to X$ of finite type. By the Zariski Main Theorem (see section in [Liu02]), we have a commutative diagram

$$\begin{tikzcd} U \arrow[r, hook, "i"] \arrow[d, "f"] & \overline{U} \arrow[dl, "\overline{f}"] \\ X & \end{tikzcd}$$

such that i is a dominant open immersion and $\overline{f}$ is finite. we have

$$H^2(X, f_*\mathbb{Z}/n\mathbb{Z}) \cong H^2(X, \overline{f}_* i_* \mathbb{Z}/n\mathbb{Z}) \cong H^2(\overline{U}, i_*\mathbb{Z}/n\mathbb{Z})$$

Note that $\overline{U}$ is affine and $\dim \overline{U} \leqslant 1$. The kernel and cokernel of the morphism $\mathbb{Z}/n\mathbb{Z} \to i_*\mathbb{Z}/n\mathbb{Z}$ are supported in $\overline{U} \setminus i(U)$, and, since i is dominant, this is finite over k. So, by similar reasoning as above, we have

$$H^2(\overline{U}, i_*\mathbb{Z}/n\mathbb{Z}) \cong H^2(\overline{U}, \mathbb{Z}/n\mathbb{Z})$$

We may assume that U is reduced by **??**. Let $\pi : \tilde{U} \to \overline{U}$ be the normalization. Recall that π is finite, induces isomorphisms above generic points, and hence induces an isomorphism above a dense open subset of $\overline{U}$. So, once again, the kernel and cokernl of the canonical morphism $\mathbb{Z}/n\mathbb{Z} \to \pi_*\mathbb{Z}/n\mathbb{Z}$ are supported on closed subscheme of $\overline{U}$ finite over k. It follows that

$$H^2(\overline{U}, \mathbb{Z}/n\mathbb{Z}) \cong H^2(\overline{U}, \pi_*\mathbb{Z}/n\mathbb{Z}) \cong H^2(\tilde{U}, \mathbb{Z}/n\mathbb{Z})$$

but $\tilde{U}$ is a smooth affine curve, so the results of 3.4 give us our desired result (even for when $\mathrm{char}(k) \mid n$). □

Note. We will use this version of cohomological dimension to prove our proper base change theorem, and then we can use our proper base change theorem to prove a theorem concerning cohomology with proper support, which will then prove a very general theorem of cohomological dimension. For applications to the Weil conjectures, the first theorem on cohomological dimension suffices, but we also want to point to the more general case. We will state the theorem now, even though not all of the notation will be made clear until later.

Theorem 5.1.6. *Let $f : X \to Y$ be an S-compactifiable morphism and let n be the supremum of dimensions of fibers of f. Then, for any torsion sheaf $\mathcal{F}$ on X, we have $\mathbf{R}^q f_! \mathcal{F} = 0$ for all $q > 2n$.*

Proof. By 5.3.4, for any $y \in Y$ we have

$$(\mathbf{R}^q f_! \mathcal{F})_y \cong H^q_c(X \times_Y \overline{k(y)}, \mathcal{F}|_{X \times_Y \overline{k(y)}})$$

where $\overline{k(y)}$ is a separable closure of the residue field $k(y)$. Thus, we are reduced to showing that $H^q_c(X, \mathcal{F}) = 0$ for any $q > 2\dim X$ for $X \to \mathrm{Spec} k$ compactifiable and k a separably closed field. This is exactly the case of 5.1.4. □

5.2 Proper Base Change

Base change theorems arise from the following situation: consider a Cartesian square

$$\begin{array}{ccc} X_T & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{g} & S \end{array}$$

where $X_T := X \times_S T$. For any sheaf $\mathcal{F}$ on X, applying f_* to the canonical $\mathcal{F} \to g'_* g'^* \mathcal{F}$ gives us

$$f_* \mathcal{F} \to f_* g'_* g'^* \mathcal{F} \cong g_* f'_* g'^* \mathcal{F}$$

Then, we use the adjointness of the pair (g^*, g_*) to get

$$g^* f_* \mathcal{F} \to f'_* g'^* \mathcal{F}$$

Another way to construct this morphism is as follows: apply g'^* to the canonical morphism $f^* f_* \mathcal{F} \to \mathcal{F}$ to get

$$f'^* g^* f_* \mathcal{F} \cong g'^* f^* f_* \mathcal{F} \to g'^* \mathcal{F}$$

and then we use the adjointness of (f'^*, f'_*) to get

$$g^* f_* \mathcal{F} \to f'_* g'^* \mathcal{F}$$

Similarly, for any $K^\bullet \in D^+(X)$, let $K^\bullet \to I^\bullet$ and $g'^* I^\bullet \to J^\bullet$ be quasi-isomorphisms such that $I^\bullet$ and $J^\bullet$ are bounded below complexes of injective sheaves on X and X_T respectively. We have morphisms

$$g^* \mathbf{R} f_* K^\bullet \cong g^* g_* I^\bullet \to f'_* g'^* I^\bullet \to f'_* J^\bullet \cong \mathbf{R} f'_* g'^* K^\bullet$$

i.e. a morphism

$$g^* \mathbf{R} f_* K^\bullet \to \mathbf{R} f'_* g'^* K^\bullet$$

in particular, for every q we have

$$g^* \mathbf{R}^q f_* K^\bullet \to \mathbf{R}^q f'_* g'^* K^\bullet$$

Theorem 5.2.1. *Let $f : X \to S$ be proper in the Cartesian square above. Then, for any torsion sheaf $\mathcal{F}$ on X, the canonical morphisms*

$$g^* \mathbf{R}^q f_* \mathcal{F} \to \mathbf{R}^q f'_* g'^* \mathcal{F}$$

are isomorphism. For any object $K^\bullet \in D^+_{\mathrm{tor}}(X)$, the canonical morphism

$$g^* \mathbf{R} f_* K^\bullet \to \mathbf{R} f'_* g'^* K^\bullet$$

is an isomorphism. In addition, if $\mathbf{R} f_*$ and $\mathbf{R} f'_*$ are of finite cohomological dimension, then the same assertion holds for all $K^\bullet \in D_{\mathrm{tor}}(X)$

We will state two corollaries first:

Corollary 5.2.2. *Let everything be as above, except $T = s$, where $s \to S$ is a geometric point. Then*

$$(\mathbf{R}^q f_* \mathcal{F})_s \cong H^q(X_s, \mathcal{F}|_{X_s})$$

Proof. Just put $T = s$ as instructed. □

Corollary 5.2.3. *Let A be a strictly Henselian ring, set $S = \mathrm{Spec}A$, let X_0 denote the special fiber of X, with T being the closed point. Then*

$$H^q(X, \mathcal{F}) \cong H^q(X_0, \mathcal{F}|_{X_0})$$

Proof. Let $s = T$ denote the closed point of $S = \mathrm{Spec}A$. After passing to appropriate limits (since everything is constructible or a limit of constructible sheaves), we can note that

$$(\mathbf{R}^q f_* \mathcal{F})_s \cong H^q(X_0, \mathcal{F}|_{X_0})$$

and

$$(\mathbf{R}^q f_* \mathcal{F})_s \cong H^q(X, \mathcal{F})$$

□

Proof of 5.2.1. First, we note that these theorems are true in the non-Noetherian case (see EGA for proofs) but we will only cover the proof of the Noetherian case. In addition, this proof carries over to ℓ-adic sheaves by passing to limits. We apply the following list of reductions

1. By properties of constructible sheaves, and since any torsion sheaf is a direct limit of its constructible subsheaves, we can reduce to the case of $\mathcal{F}$ being a $\mathbb{Z}/n\mathbb{Z}$-module;

2. Chow's lemma allows us to reduce to the case of $f : X \to S$ projective;

3. We will prove later in 5.2.5 that the case a sheaf $\mathcal{F}$ on X implies that case of $K^\bullet \in D^+_{\mathrm{tor}}(X)$ with the cohomological dimension condition being clear from technicalites on the derived category;

4. We note that the affine case, i.e. $S = \mathrm{Spec}A$ and $T = \mathrm{Spec}B$ clearly implies the above case (see [Sta20, Tag 0EZR] to understand why base change questions are always étale local – essentially, it follows from the compatibility of localization with morphisms of sites);

5. Since B is an étale A-algebra, we can write it as a direct limit of finitely generated A-algebras, and the fact that direct limits commute with $\mathbf{R}f_*$ allows us to assume that $T \to S$ is of finite-type;

6. If $\pi : U \to T$ is an étale morphism and let s be a section over U of the kernel or cokernel of the morphism $g^*\mathbf{R}f_*\mathcal{F} \to \mathbf{R}f'_*g'^*\mathcal{F}$. Suppose for any point $u \in U$ which is closed in the fiber $(g\pi)^{-1}(g\pi(u))$ that we have an isomorphism of stalks. Then, there exists an étale neighborhood W_u of $\overline{u}$ over U such that $s|_{W_u} = 0$. Then, if we take the union over all such u, we have that $\cap_u W_u$ is an étale cover of U. It follows that $s = 0$, so we are reduced to the case of

$$(g^*\mathbf{R}^q f_*\mathcal{F})_t \to (\mathbf{R}^q f'_* g'^*\mathcal{F})_t$$

where $t \to T$ is a point which is closed in the fiber $g^{-1}(g(t))$.

Then, we want to prove the following:

Claim: 5.2.3 implies that the following is an isomorphism

$$(g^*\mathbf{R}^q f_*\mathcal{F})_t \to (\mathbf{R}^q f'_* g'^*\mathcal{F})_t$$

Let $\tilde{A}$ (resp. $\tilde{B}$) be the strict henselization of $\mathcal{O}_{S,g(t)}$ (resp. $\mathcal{O}_{T,t}$).By **??** (essentially, since the strict henselization is a limit and we can commute cohomology and limits, we get)

$$(g^*\mathbf{R}^q f_*\mathcal{F})_t \cong (\mathbf{R}^q f_*\mathcal{F})_{g(t)} \cong H^q(X \times_S \tilde{A}, \mathcal{F})$$

$$(\mathbf{R}^q f'_* g'^*\mathcal{F})_t \cong H^q(X \times_S \tilde{B}, \mathcal{F})$$

(where the sheaves $\mathcal{F}$ are restricted appropriately), and then 5.2.3 gives us

$$H^q(X \times_S \tilde{A}, \mathcal{F}) \cong H^q(X \times_S \overline{k(g(t))}, \mathcal{F})$$

$$H^q(X \times_S \tilde{B}, \mathcal{F}) \cong H^q(X \times_S \overline{k(t)}, \mathcal{F})$$

where, once again, the sheaves $\mathcal{F}$ are restricted appropriately, and $\overline{k(g(t))}$ (resp. $\overline{k(t)}$) is a separable closure of $k(g(t))$ (resp. $k(t)$). Since t is closed in the fiber $g^{-1}(g(t))$, $k(t)$ is algebraic over $k(g(t))$, and hence they have the same algebraic closure. So, $\overline{k(g(t))}$ and $\overline{k(t)}$ only differ by some inseparable extension, but inseparable extensions don't affect cohomology (see **??**). Then, once again being careful about some passages to limits, we conclude that

$$H^q(X \times_S \overline{k(g(t))}, \mathcal{F}) \cong H^q(X \times_S \overline{k(t)}, \mathcal{F})$$

and so

$$(g^*\mathbf{R}^q f_*\mathcal{F})_t \cong (\mathbf{R}^q f'_* g'^*\mathcal{F})_t$$

All that is left is to prove 5.2.3. If our Henselian ring A were also complete, then this would amount to relating the cohomology of a formal scheme with the cohomology of its special fiber, or, thinking analytically, given a proper flat map $f : X \to C$ to a smooth curve $C/\mathbb{C}$ with f smooth away from $c_0 \in C(\mathbb{C})$ and constant $\mathcal{F}$, we can think of this as saying that the inclusion $X^{an}_{c_0} \hookrightarrow X^{an}$ is a homotopy equivalence over a small neighborhood of c_0 in C^{an}.

First, we prove a proposition:

Proposition 5.2.4. *Let* $\dim X_s \leqslant 1$. *For* n *invertible in* $\mathcal{O}_X$, *the natural mapping*

$$H^i(X, \mathbb{Z}/n\mathbb{Z}) \to H^i(X_s, \mathbb{Z}/n\mathbb{Z})$$

is bijective when $i = 0$ *and surjective for* $i \geqslant 1$.

Since $k(s)$ is separably closed, we know that $H^i(X_s, \mathbb{Z}/n\mathbb{Z})$ vanishes for $i > 2$ and also for $n = 2$ if n is a power of $\mathrm{char}(k)$.

For the case $i = 0$, the proposition follows from the Zariski connectedness theorem for a proper scheme over a Henselian ring (the number of connected components of X is the same as the number of connected components of X_s). This can also be thought of as the fact that we can lift idempotent elements of the special fiber $\Gamma(X_s, \mathcal{O}_{X_s})$ to $\Gamma(X, \mathcal{O}_X)$. This follows by argument that we can uniquely lift to the infinitesimal fiber $X_n \to \mathrm{Spec}(A/\mathfrak{m}^{n+1})$, which equates to a lifting to the formal neighborhood $\Gamma(X, \mathcal{O}_X) \otimes_A \widehat{A}$ in the limit, but by properties of Henselian rings, the idempotents in $\Gamma(X, \mathcal{O}_X)$ match the idempotents in $\Gamma(X, \mathcal{O}_X) \otimes_A \widehat{A}$ (see EGA, IV, 18.5.11).

For the case $i = 1$, we recall that the group $H^1(X_s, \mathbb{Z}/n\mathbb{Z})$ parametrizes the set of isomorphism classes of finite étale $\mathbb{Z}/n\mathbb{Z}$-torsors over X_s, and similarly for X. Therefore, the surjectivity of

$$H^1(X, \mathbb{Z}/n\mathbb{Z}) \twoheadrightarrow H^1(X_s, \mathbb{Z}/n\mathbb{Z})$$

comes down to the assertion that any finite étale $\mathbb{Z}/n\mathbb{Z}$-torsor

$$Y_0 \xrightarrow{\mathbb{Z}/n\mathbb{Z}} X_s$$

can be lifted to a finite étale $\mathbb{Z}/n\mathbb{Z}$-torsor over X.

Let $X_n = X \times_A \mathrm{Spec}(A/\mathfrak{m}^{n+1})$ be the nth infinitesimal neighborhood. The natural maps $X_{n+1} \to X_n$ are inseparable, so the induce equivalences of categories of sheaves and of étale schemes over X_n and X_{n+1}, meaning that a finite étale covering space of the special fiber $X_s = X_1$ can be repeatedly lifting to higher and higher X_n. This means that we can extend a finite étale cover of the special fiber to a finite étale cover of the formal completion of X along X_s.

From the fundamental existence theorem of Grothendieck for proper mappings, it follows that every finite étale covering of the special fiber can be extended to a finite étale covering of $X \times_A \mathrm{Spec}(\varprojlim_v A/\mathfrak{m}^v)$.

We now want to apply the Artin approximation theorem, but the theorem only applies to rings A which are strict Henselizations of finitely generated algebras over $\mathbb{Z}$ (see appendix for details). Fortunately, every strict Henselian ring is a direct limit of such rings, and we have already shown (section on constructible sheaves), that we take projective limits of projective systems of schemes along with a direct limit of constructible sheaves while still maintaining the cohomology groups, so we can focus our attention on the case where Artin approximation works (along with the case of $\mathcal{F}$ constructible).

We now consider the following functor F on the category of (Noetherian) A-algebras B:

$F(B) = \{$isomorphism classes of finite étale $\mathbb{Z}/n\mathbb{Z}$-covering spaces of $X \times_S \operatorname{Spec}(B)\}$

It is locally of finite presentation and

$$F(\varinjlim B_v) = \varinjlim F(B_v)$$

From the Artin approximation theorem, it follows: For the given $v \geqslant 1$ there is a finite étale covering space of X that agrees over $X \times_A \operatorname{Spec}(A/\mathfrak{m}^v)$ with the covering space constructed for $X \times_A \operatorname{Spec}(\varprojlim_n A/\mathfrak{m}^n)$.

For the case $i = 2$, we may assume that n is invertible on S and we may work with μ_n-coefficients. We have the identification $H^2(X_s, \mu_n) \simeq H^1(X_s, \mathbb{G}_m)/(n)$ due to the Kummer sequence, provided that $H^2(X_s, \mathbb{G}_m)[n] = \operatorname{Br}(X_s)[n] = 0$. By scalar extension to the algebraic closure and general techniques with Brauer groups of curves, we reduce this vanishing to the well-known vanishing in the smooth case over an algebraically closed field, so we have to prove the surjectivity of $\operatorname{Pic}(X) \to \operatorname{Pic}(X_s)$. This can be rephrased as *every line bundle on the special fiber X_s can be extended to all of X.*

By similar argument as above, it is enough to prove that every line bundle on the infinitesimal fiber $X_n = X \times_A \operatorname{Spec}(A/\mathfrak{m}^{n+1})$ can be extended to X_{n+1}. This follows from the fact that the same topological space underlies every X_n, so we consider sheaves on the Zariski topology, and the sequence

$$0 \to \mathscr{J} \to \mathcal{O}^*_{X_{n+1}} \to \mathcal{O}^*_{X_n} \to 0$$

where

$$\mathscr{J} = \mathfrak{m}^n(\mathcal{O}_X)/\mathfrak{m}^{n+1}(\mathcal{O}_X) = \ker\left(\mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n}\right)$$

is exact. Since $\mathscr{J}$ is coherent and X_s is one-dimensional, $H^2(X, \mathscr{J}) = 0$, so the long exact sequence of cohomology gives us the surjectivity of

$$H^1(X_s, \mathcal{O}^*_{X_{n+1}}) \to H^1(X_s, \mathcal{O}^*_{X_n})$$

Thus, concludes our proof of the proposition. Now, we just have to imply the rest of the theorem:

First, we recall that any constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules $\mathcal{F}$ can be realized as a subsheaf $\mathcal{F} \hookrightarrow \mathcal{G}$, where $\mathcal{G}$ is a direct sum of sheaves of the form $\pi_*(\mathbb{Z}/n\mathbb{Z})$. From the diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & H^0(X, \mathcal{F}) & \longrightarrow & H^0(X, \mathcal{G}) & \longrightarrow & H^0(X, \mathcal{G}/\mathcal{F}) \\
& \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & H^(X_s, \mathcal{F}_s) & \longrightarrow & H^0(X_s, \mathcal{G}_s) & \longrightarrow & H^0(X_s, (\mathcal{G}/\mathcal{F})_s
\end{array}$$

on sees that $H^0(X,\mathcal{F}) \to H^0(X_s,\mathcal{F}_s)$ is injective. This holds for all constructible sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules, and hence for all sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules.

Using the injectivity of $H^0(X,\mathcal{G}/\mathcal{F}) \to H^0(X_s,(\mathcal{G}/\mathcal{F})_s)$, we can conclude that that mapping

$$H^0(X,\mathcal{F}) \to H^0(X_s,\mathcal{F}_s)$$

is bijective for all sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules.

We now prove by induction on i that the mapping

$$H^i(X,\mathcal{F}) \to H^i(X_s,\mathcal{F}_s)$$

is an isomorphism for all sheaves $\mathcal{F}$ of $\mathbb{Z}/n\mathbb{Z}$-modules. We assume that it is true for all $i < p$ and then prove the isomorphism for $i = p$. We can restrict ourselves to constructible sheaves $\mathcal{F}$, and we use the fact that $\mathcal{F}$ can be embedded in a torsion sheaf $\mathcal{G}$ for which the mapping

$$H^i(X,\mathcal{G}) \to H^i(X_s,\mathcal{G}_s)$$

is surjective for all i.

First, we show that $H^p(X,\mathcal{G}) \to H^p(X_s,\mathcal{G}_s)$ is injective, and therefore an isomorphism. We consider a short exact sequence

$$0 \to \mathcal{G} \to \mathscr{I} \to \mathscr{K} \to 0$$

where we let $\mathscr{I}$ be injective in the category of $\mathbb{Z}/n\mathbb{Z}$-modules. In the diagram

$$\begin{tikzcd}
H^{p-1}(\mathscr{I}) \arrow[r] \arrow[d, "s"] & H^{p-1}(\mathscr{K}) \arrow[r] \arrow[d, "s"] & H^p(\mathcal{G}) \arrow[r] \arrow[d] & 0 \arrow[d] \\
H^{p-1}(\mathscr{I}_s) \arrow[r] & H^{p-1}(\mathscr{K}) \arrow[r] & H^p(\mathcal{G}) \arrow[r] & H^p(\mathscr{I}_s)
\end{tikzcd}$$

the first two vertical arrows are isomorphisms (induction hypothesis). The Five lemma then gives us injectivity of $H^p(\mathcal{G}) \to H^p(\mathcal{G}_s$. We now consider the diagram of long exact cohomology sequences

$$\begin{tikzcd}
\cdots H^{p-1}(\mathcal{G}) \arrow[r] \arrow[d] & H^{p-1}(\mathcal{G}/\mathcal{F}) \arrow[r] \arrow[d] & H^p(\mathcal{F}) \arrow[r] \arrow[d] & H^p(\mathcal{G}) \arrow[r] \arrow[d] & H^p(\mathcal{G}/\mathcal{F}) \cdots \arrow[d] \\
\cdots H^{p-1}(\mathcal{G}_s) \arrow[r] & H^{p-1}((\mathcal{G}/\mathcal{F})_s) \arrow[r] & H^p(\mathcal{F}_s) \arrow[r] & H^p(\mathcal{G}_s) \arrow[r] & H^p((\mathcal{G}/\mathcal{F})_s) \cdots
\end{tikzcd}$$

It follows that $H^p(\mathcal{F}) \to H^p(\mathcal{F}_s)$ is injective. Since $\mathcal{F}$ is arbitrary, this injectivity holds for $\mathcal{G}/\mathcal{F}$. Then surjectivity follows from the diagram, giving us the desired bijection

$$H^p(X,\mathcal{F}) \xrightarrow{\sim} H^p(X_s,\mathcal{F}_s)$$

Now, we completely understand the case of $X \to S$ with a one-dimensional fiber. By composing

$$(\mathbb{P}^1_A)^N \to (\mathbb{P}^1_A)^{N-1} \to \cdots \mathbb{P}^1_A \to S$$

Then using the finite covering map

$$p : (\mathbb{P}^1_A)^N \to \mathbb{P}^N_A \qquad (= (\mathbb{P}^1_A)^N/S_n)$$

$$([a_1, b_1], \dots, [a_N, b_N]) \mapsto [h_0, \dots, h_N]$$

where

$$\prod_{i=1}^{N}(a_iT + b_i) = h_0 + h_1T + h_2T^2 + \dots + h_NT^N$$

as polynomial in a variable T. It is easy to check that p is quasi-finite and surjective, so it is a finite surjection by properness. The base change theorem holds for sheaves of the form $p_*\mathcal{F}$, because base change holds for finite morphisms, because the higher direct images vanish. The mapping $\mathcal{G} \to p_*p^*\mathcal{G}$ is injective for every sheaf $\mathcal{G}$ on $\mathbb{P}^N_A$. Hence $\mathcal{G}$ is quasi-isomorphic to a bounded-below complex of sheaves $\mathcal{F}^\bullet$ where the base change theorem holds for every $\mathcal{F}^n$. The base change theorem follows for $\mathcal{F}^\bullet$ and then for $\mathcal{G}$.

Since we are already assuming that $X \to S$ is projective, we can assume that there exists a closed immersion $X \hookrightarrow \mathbb{P}^N_A$, so after pushing $\mathcal{F}$ forward, we can assume $X = \mathbb{P}^N_A$, and then by the above covering, and composition of higher direct images (Leray spectral sequence argument) we can reduce to the case $X = \mathbb{P}^1_A$ for A a strictly Henselian local ring. Therefore, we have proved the theorem.

□

Note. It is clear that this theorem generalizes to the case of morphism $X \to Y$ of S-schemes, with base change to $X \times_Y Y'$ for some $Y' \to Y$ proper.

Now, as promised, we prove the following:

Lemma 5.2.5. *Let $X_0 \to X$ be a morphism. For every torsion sheaf on X, assume that, for some n, the canonical homomorphism*

$$H^i(X, \mathcal{F}) \to H^i(X_0, \mathcal{F}|_{X_0})$$

is an isomorphism for each $i < n$ and a monomorphism for $i = n$. Then, for every complex $K^\bullet$ satisfying $\mathscr{H}^i(K^\bullet) = 0$ for all $i < 0$, the canonical homomorphism

$$H^i(X, K^\bullet) \to H^i(X_0, K^\bullet|_{X_0})$$

is an isomorphism for each $i < n$ and a monomorphism for each $i = n$.

Proof. Let $m \geqslant 0$ be an integer such that $\mathscr{H}^i(K^\bullet) = 0$ for all $i < m$. We prove the lemma by descending induction on m. We have the following biregular spectral sequence

$$E_2^{p,q} = H^p(X, \mathscr{H}^q(K^\bullet)) \Rightarrow H^{p+q}(X, K^\bullet)$$

If $p+q < m$, we have either $p < m$ or $q < m$, and hence we have $H^p(X, \mathscr{H}^q(K^\bullet)) = 0$. It follows that $H^i(X, K^\bullet) = m$ for any $i < m$. Hence, the lemma holds for all those $K^\bullet$ with $\mathscr{H}^i(K^\bullet) = 0$ for all $i < n+1$.

Suppose that the lemma holds for those $K^\bullet$ with $\mathscr{H}^i(K^\bullet) = 0$ for all $i < m+1$. Let K be a complex with $\mathscr{H}^i(K^\bullet) = 0$ for all $i < m$. We have a distinguished triangle

$$\mathscr{H}^m(K^\bullet)[-m] \to K^\bullet \to \tau_{\geqslant m+1}K^\bullet \to$$

(first morphism is "inserting mth homology at mth position") where $\tau_{\geqslant m+1}K^\bullet$ is the truncated complex

$$\cdots \to 0 \to K^{m+1}/\operatorname{im} d_m \to K^{m+2} \to \cdots$$

Consider the commutative diagram

$$\begin{tikzcd}
H^{i-1}(X, \tau_{\geqslant m+1}K^\bullet) \arrow[r] \arrow[d] & H^{i-m}(X, \mathscr{H}^m(K^\bullet)) \arrow[r] \arrow[d] & H^i(X, K^\bullet) \arrow[r] \arrow[d] & {} \\
H^{i-1}(X_0, \tau_{\geqslant m+1}K^\bullet|_{X_0}) \arrow[r] & H^{i-m}(X_0, \mathscr{H}^m(K^\bullet)|_{X_0}) \arrow[r] & H^i(X_0, K^\bullet|_{X_0}) \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & H^i(X, \tau_{\geqslant m+1}K^\bullet) \arrow[r] \arrow[d] & H^{i+1-m}(X, \mathscr{H}^m(K^\bullet)) \arrow[d] \\
{} \arrow[r] & H^i(X_0, \tau_{\geqslant m+1}K^\bullet|_{X_0}) \arrow[r] & H^{i+1-m}(X_0, \mathscr{H}^m(K^\bullet)|_{X_0})
\end{tikzcd}$$

The five lemma then gives us the results. $\square$

5.3 Cohomology with Compact Support

Let's first recall the definition of cohomology with compact support. If $X \to Y$ is a morphisms of S-schemes, we say that it is *S-compactifiable* if Y is quasi-compact, quasi-separated, and there exists $P \to S$ proper such that there is $\overline{f} : X \to Y \times_S P$ making the following diagram of S-schemes commutes

$$\begin{tikzcd}
X \arrow[r, "\overline{f}"] \arrow[dr, "f"'] & Y \times_S P \arrow[d, "p_1"] \\
& Y
\end{tikzcd}$$

A theorem of Nagata says that any separated morphism of finite-type between Noetherian schemes is compactifiable.

Proposition 5.3.1. *Let's review some properties of compactifiable morphisms.*

(i) S-compactifiable morphisms are separated and of finite-type;

(ii) *Let $f : X \to Y$ be a separated quasi-finite S-morphism such that Y is quasi-compact and quasi-separated. Then f is S-compactifiable;*

(iii) *Let $f : X \to Y$ and $g : Y \to Z$ be two S-compactifiable morphisms. Then gf is S-compactifiable;*

(iv) *Let $f : X \to Y$ be an S-compactifiable morphism. For any morphism $Y' \to Y$, such that Y' is quasi-compact and quasi-separated, the base change $f' : X \times_Y Y' \to Y'$ is S-compactifiable;*

(v) *Let $f : X \to Y$ and $g : Y \to Z$ be two S-morphisms. Suppose g is quasi-compact and quasi-separated and gf is S-compactifiable. Then, f is S-compactifiable.*

Proof. (i), (ii), and (iv) are clear, with (v) following from (iii) and (iv). All that remains is to prove (iii), but that is just a matter of simple diagram-chasing, best done on a blackboard or paper. $\square$

Thus, an S-scheme is compactifiable if it has a morphism into some proper S-scheme, and then, Zariski's Main Theorem, EGA IV 18.12.13, tells us that we can find some finite morphism and open immersion such that the "compactification" is a composite of the two. Then, by the proposition we are about to state, we get that any S-compactifiable morphism is the composite of an open immersion and a proper S-compactifiable morphism. So, intuitively, we can think of S-compactifiable meaning "can be immersed into a compact scheme" or "is an open subscheme of a compact scheme".
To better formalize this intuition, we make the following definition:

Definition 5.3.2. Let $f : X \to Y$ be a compactifiable S-morphism. A *compactification* of f is a factorization of f as a composite

$$X \overset{j}{\hookrightarrow} \overline{X} \xrightarrow{\overline{f}} Y$$

such that j is an open immersion and $\overline{f}$ is a proper S-compactifiable morphism. A "morphism of compactifications" is the obvious thing making all compactifications commute. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be S-compactifiable morphisms. A compactification of (f, g) is a commutative diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] & \overline{X} \arrow[r, hook] \arrow[d] & \overline{X}' \arrow[d] \\ & Y \arrow[r, hook] \arrow[dr] & \overline{Y} \arrow[d] \\ & & Z \end{tikzcd}$$

such that the horizontal arrows are open immersions and vertical arrows are proper S-compactifiable morphisms. We can similarly define morphisms between these compactifications and we can even define compactifications of more and more morphisms.

With notation as above, we recall the definition of cohomology with compact support: for $X \to Y$ S-compactifiable, we take the compactification $X \to \overline{X} \to Y$ and then we define for any $K^\bullet \in D^+_{\text{tor}}(X)$

$$\mathbf{R}f_!K^\bullet := \mathbf{R}\overline{f}_*j_!K^\bullet$$

in the more familiar land of cohomology over a separably closed field k, this looks like

$$\mathbf{R}\Gamma_c(X, K^\bullet) := \Gamma(\mathrm{Spec}k, \mathbf{R}f_!K^\bullet) \cong \mathbf{R}\Gamma(\overline{X}, j_!K^\bullet)$$
$$H^q_c(X, K^\bullet) = \mathscr{H}^q(\mathbf{R}\Gamma_c(X, K^\bullet) \cong H^q(\overline{X}, j_!K^\bullet)$$

Recall that $j_!\mathcal{F}$ is the "extension by zero", defined as the sheaf associated to the following presheaf: for any $\phi : V \to \overline{X}$ étale, define

$$\mathcal{F}_!(V) = \begin{cases} \mathcal{F}(V) & \text{if } \phi(V) \subset j(X) \\ 0 & \text{otherwise} \end{cases}$$

(i.e. $j_!\mathcal{F} = a(\mathcal{F}_!)$, where $a(\mathcal{F}_!)$ is the subsheaf generated by the presheaf in the corresponding product of skyscraper sheaves at each stalk; see the section on direct images of sheaves for details.)

First, we point out that the general definition of $f_!$ lower shriek coincides with the zeroth cohomology group here. We need to cover some technicalities: why is $\mathbf{R}f_!$ independent of choice of compactification? This follows from a technical lemma, which follows from the proper base change theorem 5.2.1.

Lemma 5.3.3. *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[r, hook, "k"] \arrow[d, "f"] & \overline{X} \arrow[d, "\overline{f}"] \\ Y \arrow[r, hook, "j"] & \overline{Y} \end{tikzcd}$$

where j and k are open immersions, and f and $\overline{f}$ are proper morphisms. Then, for any $K^\bullet \in D^+_{\text{tor}}(X)$, we have a canonical isomorphism

$$j_!\mathbf{R}f_*K^\bullet \cong \mathbf{R}\overline{f}_*k_!K^\bullet$$

Example 11. One more subtly to point out: in general, $\mathbf{R}f_!$ is not the right derived functor of $f_!$. For example, in the case where $S = \mathrm{Spec}k$ for a separably closed field, and X a smooth affine curve over k, we have

$$\Gamma_c(X, \mathcal{F}) \cong \bigoplus_{x \in |X|} \Gamma_x(X, \mathcal{F})$$

It follows that the right derived functor of $\Gamma_c(X, -)$ is isomorphic to $\bigoplus_{x \in |X|} \mathbf{R}\Gamma_x(X, -)$. It can be shown that $H^1_x(X, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$, and so the first cohomology of the right derived functor of $\Gamma_c(X, -)$ is infinite. On the other hand, we will later see that $H^1_c(X, \mathbb{Z}/n\mathbb{Z})$ is finite.

Theorem 5.3.4. *This will involve three results:*

(i) Consider a Cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

where $X' = X \times_Y Y'$. Suppose that f is S-compactifiable, and Y' is quasi-compact and quasi-separated. Then, for any $K^\bullet \in D^+_{\mathrm{tor}}(X)$, we have a canonical isomorphism

$$g^* \mathbf{R} f_! K^\bullet \xrightarrow{\sim} \mathbf{R} f'_! g'^* K^\bullet$$

(ii) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two S-compactifiable morphisms. For any $K^\bullet \in D^+_{\mathrm{tor}}(X)$, we have a canonical isomorphism

$$\mathbf{R} g_! \mathbf{R} f_! K^\bullet \xrightarrow{\sim} \mathbf{R}(gf)_! K^\bullet$$

and we have a biregular spectral sequence

$$E_2^{p,q} = \mathbf{R}^p g_! \mathbf{R}^q f_! K^\bullet \Rightarrow \mathbf{R}^{p+q}(gf)_! K^\bullet$$

(iii) Let $f : X \to Y$ be an S-compactifiable morphism, $j : U \hookrightarrow X$ an S-compactifiable open immersion, and $i : A \to X$ a closed immersion with $A = X \smallsetminus U$. For any $K^\bullet \in D^+_{\mathrm{tor}}(X)$, we have a distinguished triangle

$$\mathbf{R}(fj)_! j^* K^\bullet \to \mathbf{R} f_! K^\bullet \to \mathbf{R}(fi)_! i^* K^\bullet \to$$

and a long exact sequence

$$\cdots \to \mathbf{R}^q(fj)_! j^* K^\bullet \to \mathbf{R}^q f_! K^\bullet \to \mathbf{R}(fi)_! i^* K^\bullet \to \mathbf{R}^{q+1}(fj)_! j^* K^\bullet \to \cdots$$

Proof. We start by proving (i): let $X \overset{j}{\hookrightarrow} \overline{X} \xrightarrow{\overline{f}} Y$ be a compactification of f. Fix notation by the following commutative diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "j'"] & X \arrow[d, "j"] \\ \overline{X}' \arrow[r, "\overline{g}"] \arrow[d, "\overline{f}'"] & \overline{X} \arrow[d, "\overline{f}"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

where X' and $\overline{X}'$ are the corresponding fiber products making the squares Cartesian. By commutativity and the fact that j, j' are open immersions with $j_!, j'_!$ being extension by zero, we have

$$j'_! g'^* K \cong \overline{g}^* j_! K$$

and by proper base change 5.2.1, we have

$$g^* \mathbf{R}\overline{f}_* j_! K \cong \mathbf{R}\overline{f}'_* \overline{g}^* j_! K$$

putting those together, we get

$$g^* \mathbf{R}\overline{f}_* j_! K \cong \mathbf{R}\overline{f}'_* j'_! g'^* K$$

recalling the definition $\mathbf{R}f_! := \mathbf{R}\overline{f}_* j_!$, this means

$$g^* \mathbf{R}f_! K \cong \mathbf{R}f'_! g'^* K$$

and one can check that this is independent of compactification.

Next, we prove (ii): let

$$\begin{tikzcd}
X \arrow[r, hook, "j_1"] \arrow[dr, "f"] & \overline{X} \arrow[r, hook, "j_2"] \arrow[d, "\overline{f}"] & \overline{X}' \arrow[d, "\overline{f}'"] \\
& Y \arrow[r, hook, "k"] \arrow[dr, "g"] & \overline{Y} \arrow[d, "\overline{g}"] \\
& & Z
\end{tikzcd}$$

be a compactification of (f, g). By 5.3.3, we have

$$k_! \mathbf{R}\overline{f}_* j_{1!} K \cong \mathbf{R}\overline{f}'_* j_{2!} j_{1!K}$$

so

$$\begin{aligned}
\mathbf{R}g_! \mathbf{R}f_! K &\cong \mathbf{R}\overline{g}_* k_! \mathbf{R}\overline{f}_* j_{1!} K \\
&\cong \mathbf{R}\overline{g}_* \mathbf{R}\overline{f}'_* j_{2!} j_{1!} K \\
&\cong \mathbf{R}(\overline{g}\overline{f}')_* (j_2 j_1)_! K \\
&\cong \mathbf{R}(gf)_! K
\end{aligned}$$

The spectral sequence is clear, since we have the Grothendieck spectral sequence, for $L = \mathbf{R}f_! K$,

$$E_2^{p,q} = \mathbf{R}^p \overline{g}_* (\mathscr{H}^q(k_! L)) \Rightarrow \mathbf{R}^{p+q} \overline{g}_* k_! L$$

Finally, we prove (iii): by the recollement of sheaves, we have a distinguished triangle

$$j_! j^* K \to K \to i_* i^* K \to$$

and hence we can apply the exact functor $\mathbf{R}f_!$ to get a distinguished triangle

$$\mathbf{R}f_! j_! j^* K \to \mathbf{R}f_! K \to \mathbf{R}f_! i_* i^* K \to$$

which, by our above results, is the same as

$$\mathbf{R}(fj)_! j^* K \to \mathbf{R}f_! K \to \mathbf{R}(fi)_! i^* K$$

□

5.4 Künneth Formula

We want to prove the Künneth formula for cohomology with compact support, with the proof of the more general Künneth formula only being possible after we have established a couple more facts (generic base change and generic local acyclicity, after Poincaré duality). The case we are focused on now follows from the more general projection formula:

Theorem 5.4.1 (Projection formula)**.** *Let $f : X \to Y$ be an S-compactifiable morphism, and let A be a torsion ring.*

1. *For any $K \in D^-(X, A)$ and $L \in D^-(X, A)$, we have a canonical isomorphism*
$$L \otimes^{\mathbf{L}}_A \mathbf{R}f_! K \xrightarrow{\sim} \mathbf{R}f_!(f^* L \otimes^{\mathbf{L}}_A K)$$
2. *For any $K \in D^b_{\mathrm{tf}}(X, A)$, we have $\mathbf{R}f_! K \in D^b_{\mathrm{tf}}(Y, A)$, and we have a canonical isomorphism*
$$L \otimes^{\mathbf{L}}_A \mathbf{R}f_! K \xrightarrow{\sim} \mathbf{R}f_!(f^* L \otimes^{\mathbf{L}}_A K)$$
for any $L \in D(Y, A)$.
3. *Let $A \to B$ be a homomorphism of torsion rings. For any $K \in D^-(X, A)$ and $M \in D^-(Y, B)$, we have a canonical isomorphism*
$$M \otimes^{\mathbf{L}}_A \mathbf{R}f_! K \xrightarrow{\sim} \mathbf{R}f_!(f^* M \otimes^{\mathbf{L}}_A K)$$
in $D^-(Y, B)$

Proof. We prove (i) and note that the others are proved similarly with some care for technicalities regarding the derived categories involved (see specifically **??** and **??**). Let $X \overset{j}{\hookrightarrow} \overline{X} \xrightarrow{\overline{f}} Y$ be a compactification of f. For any complex of sheaves of A-modules $M^\bullet$ on X and $N^\bullet$ on $\overline{X}$, we have a canonical isomorphism

$$j_!(j^* N^\bullet \otimes_A M^\bullet) \cong N^\bullet \otimes_A j_! M^\bullet$$

Taking $M^\bullet$ to be a complex of flat sheaves of A-modules quasi-isomorphic to K and $N^\bullet$ a complex quasi-isomorphic to $\overline{f}^* L$, we get an isomorphism in the derived category

$$j_!(j^* \overline{f}^* L \otimes^{\mathbf{L}}_A K) \cong \overline{f}^* L \otimes^{\mathbf{L}}_A j_! K$$

and so, taking this isomorphism along with the definition of $\mathbf{R}f_!$, we get

$$\mathbf{R}f_!(f^*L \otimes^{\mathbf{L}}_A K) \cong \mathbf{R}\overline{f}_* j_!(j^*\overline{f}^* L \otimes^{\mathbf{L}}_A K) \cong \mathbf{R}\overline{f}_*(\overline{f}^* L \otimes^{\mathbf{L}}_A j_! K$$

and by taking the definition, we have

$$L \otimes^{\mathbf{L}}_A \mathbf{R}f_! K \cong L \otimes^{\mathbf{L}}_A \mathbf{R}\overline{f}_* j_! K$$

and so, we need to prove that

$$L \otimes^{\mathbf{L}}_A \mathbf{R}\overline{f}_* j_! K \to \mathbf{R}\overline{f}_*(\overline{f}^* L$$

is an isomorphism. As per usual, at this point we reduce to finding that this is an isomorphism of stalks for all geometric points $s \to Y$, where s is the spectrum of a separably closed field:

$$(L \otimes^{\mathbf{L}}_A \mathbf{R}\overline{f}_* j_! K)_s \to (\mathbf{R}\overline{f}_*(\overline{f}^* L \otimes^{\mathbf{L}}_A j_! K))_s$$

We denote the base change of $\overline{f}$:

$$\overline{f}_s : \overline{X}_s = \overline{X} \times_Y s \to s$$

By the proper base change theorem 5.2.1, we have

$$(L \otimes^{\mathbf{L}}_A \mathbf{R}\overline{f}_* j_! K)_s \cong L_s \otimes^{\mathbf{L}}_A \mathbf{R}\overline{f}_{s*}((j_! K)|_{\overline{X}_s})$$

and

$$(\mathbf{R}\overline{f}_*(\overline{f}^* L \otimes^{\mathbf{L}}_A j_! K))_s \cong \mathbf{R}\overline{f}_{s*}(\overline{f}^*_s L_s \otimes^{\mathbf{L}}_A (j_! K)|_{\overline{X}_s})$$

So, it suffices to prove that the following is an isomorphism

$$L_s \otimes^{\mathbf{L}}_A \mathbf{R}\overline{f}_{s*}((j_! K)|_{\overline{X}_s}) \to \mathbf{R}\overline{f}_{s*}(\overline{f}^*_s L_s \otimes^{\mathbf{L}}_A (j_! K)|_{\overline{X}_s})$$

First, we note that in this case $\mathbf{R}f_*$ has finite cohomological dimemsion, since we are working over a separably closed field. In addition, we can see that $\mathscr{H}^i(L_s)$ are constant sheaves on s for all i. Thus, our result follows from this proposition $\square$

Proposition 5.4.2. *Let A be a ring and let $f : X \to Y$ be a morphism of schemes. Suppose $\mathbf{R}f_*$ has finite cohomological dimension. For any $\mathcal{F}^\bullet \in D^-(X, A)$ and any $\mathcal{G}^\bullet \in D^-(Y, A)$ such that $\mathscr{H}^i(\mathcal{G}^\bullet)$ are locally constant sheaves, we have a canonical isomorphism*

$$\mathcal{G}^\bullet \otimes^{\mathbf{L}}_A \mathbf{R}f_* \mathcal{F}^\bullet \cong \mathbf{R}f_*(f^*\mathcal{G}^\bullet \otimes^{\mathbf{L}}_A \mathcal{F}^\bullet)$$

Proof. The finite cohomological dimension of $\mathbf{R}f_*$ helps us verify that both $- \otimes^{\mathbf{L}}_A \mathbf{R}f_* \mathcal{F}^\bullet$ and $\mathbf{R}f_*(f^* - \otimes^{\mathbf{L}}_A \mathcal{F}^\bullet)$ are way-out left functors. So, it suffices to prove

$$\mathcal{G} \otimes^{\mathbf{L}}_A \mathbf{R}f_* \mathcal{F}^\bullet \cong \mathbf{R}f_*(f^*\mathcal{G} \otimes^{\mathbf{L}}_A \mathcal{F}^\bullet)$$

for any locally constant sheaf $\mathcal{G}$ of A-modules. The problem is local on the étale topology of Y, so we may assume that $\mathcal{G}$ is a constant sheaf, which we associate to an A-module M. Let

$$\cdots \to L^{-1} \to L^0 \to 0$$

be a resolution of M by free A-modules, and let $\mathcal{F}^\bullet \to \mathcal{F}'^\bullet$ be a quasi-isomorphism such that $\mathcal{F}'^\bullet$ is a bounded above complex of $\mathbf{R}f_*$-acyclic sheaves. We put these quasi-isomorphism in to get

$$\mathcal{G} \otimes_A^{\mathbf{L}} \mathbf{R}f_*\mathcal{F}^\bullet \cong L^\bullet \otimes_A f_*\mathcal{F}'^\bullet$$

and

$$f^*\mathcal{G} \otimes_A^{\mathbf{L}} \mathcal{F}^\bullet \cong f^*L^\bullet \otimes_A \mathcal{F}'^\bullet$$

Since each L^i is free and each $\mathcal{F}'^j$ is $\mathbf{R}f_*$-acyclic, each $f^*L^i \otimes_A \mathcal{F}'^j$ is $\mathbf{R}f_*$-acyclic, so we have

$$\mathbf{R}f_*(f^*\mathcal{G} \otimes_A^{\mathbf{L}} \mathcal{F}^\bullet) \cong f_*(f^*L^\bullet \otimes_A \mathcal{F}'^\bullet)$$

Since the L^i are free, the following morphism is an isomorphism

$$L^\bullet \otimes_A f_*\mathcal{F}'^\bullet \to f_*(f^*L^\bullet \otimes_A \mathcal{F}'^\bullet)$$

Hence

$$L^\bullet \otimes_A^{\mathbf{L}} \mathbf{R}f_*\mathcal{F}^\bullet \cong \mathbf{R}f_*(f^*L^\bullet \otimes_A^{\mathbf{L}} \mathcal{F}^\bullet)$$

□

Corollary 5.4.3. *Consider a Cartesian diagram of S-schemes*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

where $X' = X \times_Y Y'$. Suppose that g is S-compactifiable and X is quasi-compact and quasi-separated. Let A be a torsion ring. For any $K \in D^-(X, A)$ and $L \in D^-(Y, A)$, we have a canonical isomorphism

$$K \otimes_A^{\mathbf{L}} f^*\mathbf{R}g_!L \xrightarrow{\sim} \mathbf{R}g'_!(g'^*K \otimes_A^{\mathbf{L}} f'^*L)$$

Proof. By 5.3.4 and 5.4.1, we have

$$K \otimes_A^{\mathbf{L}} f^*\mathbf{R}g_!L \cong K \otimes_A^{\mathbf{L}} \mathbf{R}g'_!f'^*L \cong \mathbf{R}g'_!(g'^*K \otimes_A^{\mathbf{L}} f'^*L)$$

□

Corollary 5.4.4 (Künneth formula)**.** *Let X_i, Y_i ($i = 1, 2$) and Z be S-schemes, let $f_i : X_i \to Y_i$ and $Y_i \to Z$ be S-compactifiable morphisms, and let $p_i : X_1 \times_Z X_2 \to X_i$ and $q_i : Y_1 \times_Z Y_2 \to Y_i$ be the projections. For any $K_i \in D^-(X_i, A)$, we have*

$$q_1^*\mathbf{R}f_{1!}K_1 \otimes_A^{\mathbf{L}} q_2^*\mathbf{R}f_{2!}K_2 \cong \mathbf{R}(f_1 \times f_2)_!(p_1^*K_1 \otimes_A^{\mathbf{L}} p_2^*K_2)$$

Proof. Warning: the notation in this proof is quite horrible (because we want to state it in full generality, but we will have more specific and easier to read cases down below), so please check that each equation makes sense and keep a copy of this following diagram of Cartesian squares handy:

$$\begin{tikzcd}
X_1 \times_Z X_2 \arrow[r, "f_1''"] \arrow[d, "f_2''"] & Y_1 \times_Z X_2 \arrow[r, "b_1''"] \arrow[d, "f_2'"] & X_2 \arrow[d, "f_2"] \\
X_1 \times_Z Y_2 \arrow[r, "f_1'"] \arrow[d, "b_2''"] & Y_1 \times_Z Y_2 \arrow[r, "q_2"] \arrow[d, "q_1"] & Y_2 \arrow[d, "b_2"] \\
X_1 \arrow[r, "f_1"] & Y_1 \arrow[r, "b_1"] & Z
\end{tikzcd}$$

By the corollary above 5.4.3, we have isomorphisms

$$\begin{aligned} q_1^*\mathbf{R}f_{1!}K_1 \otimes_A^{\mathbf{L}} q_2^*\mathbf{R}f_{2!}K_2 &\cong \mathbf{R}f'_{2!}(f_2'^* q_1^*\mathbf{R}f_{1!}K_1 \otimes_A^{\mathbf{L}} b_1''^* K_2) \\ &\cong \mathbf{R}f'_{2!}\mathbf{R}f''_{1!}(f_2''^* b_2''^* K_1 \otimes_A^{\mathbf{L}} f_1''^* b_1''^* K_2) \end{aligned}$$

and then we look at our handy-dandy commutative diagram above to see that $(f_i'' b_i'')^* = p_i^*$ for $i = 1, 2$ and $\mathbf{R}f'_{2!}\mathbf{R}f''_{1!} = \mathbf{R}(f_2' f_1'')_! = \mathbf{R}(f_1 \times f_2)_!$, and so our desired result follows. $\square$

For those of less familiar with the formalism of derived categories, let's put the Künneth formula into a more digestible form:

Corollary 5.4.5. *Let X and Y be proper schemes over a separably closed field k, and let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X and Y respectively. Assume that $\mathcal{F}$ and the groups $H^r(X, \mathcal{F})$ are flat. The maps*

$$H^r(X, \mathcal{F}) \to H^r(X \times Y, \mathcal{F}|_{X \times Y})$$

$$H^s(Y, \mathcal{G}) \to H^s(X \times Y, \mathcal{G}|_{X \times Y})$$

and the cup-product pairings

$$H^r(X \times Y, \mathcal{F}|_{X \times Y}) \times H^s(X \times Y, \mathcal{G}|_{X \times Y}) \to H^{r+s}(X \times Y, \mathcal{F} \boxtimes \mathcal{G})$$

induce isomorphisms

$$\bigoplus_{r+s=m} H^r(X, \mathcal{F}) \otimes H^s(Y, \mathcal{G}) \xrightarrow{\sim} H^m(X \times Y, \mathcal{F} \boxtimes \mathcal{G})$$

for all m, where $\mathcal{F} \boxtimes \mathcal{G} = p_1^*\mathcal{F} \otimes p_2^*\mathcal{G}$.

5.5 Smooth Base Change

Most proofs of smooth base change rely on the theory of local acyclicity, but we'll use a slightly different approach. Note that difference in the statement of smooth base change and proper base change – intuitively, smooth base change depends on a property of our chosen scheme $X \to S$, but proper base change depends on a property of our chosen base change $T \to S$. In some sense, these are exactly the same, but they are "morally different" – we will see better once we start applying these base change theorems what I mean by that.

Theorem 5.5.1 (Smooth Base Change). *Consider a Cartesian diagram of schemes*

$$\begin{array}{ccc} X_t & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{g} & S \end{array}$$

where f is smooth and g is quasi-compact and quasi-separated. Then, for any $K^\bullet \in D^+_{\mathrm{tor}}(T)$ such that the torsion at every stalk is of order invertible on S, we have that

$$f^*\mathbf{R}g_*K^\bullet \cong \mathbf{R}g'_*f'^*K^\bullet$$

Proof. As with proper base change, the case of general chain complexes follows from the case of $\mathcal{F}$ a torsion sheaf (whose torsion at every stalk is of order invertible on S). Similar to before as well, we instead consider the individual isomorphisms

$$f^*\mathbf{R}^q g_*K^\bullet \cong \mathbf{R}^q g'_*f'^*K^\bullet$$

and note that for $q = 0$, this is just the fact that pushforwards and pullbacks commute when f is fppf with geometrically reduced fibers, which follows from the fact that f is smooth.

Letting $\mathcal{F}[n]$ denote the n-torsion subsheaf of $\mathcal{F}$, we note that $\mathcal{F} = \varinjlim \mathcal{F}[n]$. Then, since f^*, f'^* are left-adjoints, we know that they commute with direct limits, but we also know (by our previous study of limits of sheaves) that $\mathbf{R}^q g_*$ and $\mathbf{R}^q g'_*$ commute with direct limits as well, so we are reduced to $\mathcal{F}[n]$, a $\mathbb{Z}/n\mathbb{Z}$-modules (for n invertible on S).

Similar to the proper base change case, we also reduce to the étale local case (see [Sta20, Tag 0EZR] for details), but this time we are working locally on X and S, instead of on T and S. By the local structure of smooth morphisms ("smooth schemes are étale-locally like affine spaces"), we may assume X and S are affine and $X \to S$ factors through an étale morphism $X \to \mathbb{A}^d_S$. Thus, we can decompose $X \to S$ as

$$X \to \mathbb{A}^d_S \to \mathbb{A}^{d-1}_S \to \cdots \to \mathbb{A}^1_S \to S$$

and then it is clear that the theorem follows from the case where X and S are affine and $X \to S$ is of relative dimension 1 (just study the tower of Cartesian squares corresponding to our decomposition of $X \to S$).

The next reduction follows from [Sta20, Tag 0F08]: we now just need to prove that $\mathbf{R}^q g'_*\mathbb{Z}/d\mathbb{Z} = 0$ for all $d \mid n$ whenever we have a Cartesian square where $X \to S$ is affine and smooth of relative dimension 1 and $S = \mathrm{Spec}A$ with $\mathrm{Frac}(A) = L$ algebraically closed with K/L an extension of algebraically closed fields.

Recall that $\mathbf{R}^q h_* \mathbb{Z}/d\mathbb{Z}$ is the sheaf associated to the presheaf

$$U \mapsto H^q(U \times_X Y, \mathbb{Z}/d\mathbb{Z}) = H^q(U \times_S \mathrm{Spec}(K), \mathbb{Z}/d\mathbb{Z})$$

Thus, it suffices to show that given some element $\xi \in H^q(U \times_S \mathrm{Spec}(K), \mathbb{Z}/d\mathbb{Z})$, there exists some étale covering $\{U_i \to U\}$ such that $\xi|_{U_i} = 0$ in (i.e. is zero in $H^q(U_i \times_S \mathrm{Spec}(K), \mathbb{Z}/d\mathbb{Z})$. Of course, we may take U to be affine, and thus $U \times_S \mathrm{Spec}(K)$ is a smooth affine curve over K and hence we have $H^q(U \times_S \mathrm{Spec}(K), \mathbb{Z}/d\mathbb{Z}) = 0$ for all $q > 1$ (by our previous study of dimensionality).

Now, we are just left with the task of showing that given any $\xi \in H^1(U \times_S \mathrm{Spec}(K), \mathbb{Z}/d\mathbb{Z})$ that there is some étale cover $\{U_i \to U\}$ such that ξ vanishes on the pullbacks. This follows from analysis of Galois groups of prime-to-d-torsion, giving us certain covers for which it is clear that the Galois group annihilates our ξ, and therefore (at each stalk and therefore on the entire cover) we have that ξ is annihilated in the first cohomology group. □

5.6 Comparison Theorem

The next theorem shows that étale cohomology and classical complex cohomology agree as well as it is reasonable to hope; it is unreasonable to hope for agreement for non-torsion coefficients. For example, if X is a smooth complete curve of genus g over $\mathbb{C}$, then $H^1(X(\mathbb{C}), \mathbb{Z}) \simeq \mathbb{Z}^{2g}$, but

$$H^1(X_{ét}, \mathbb{Z}) = \mathrm{Hom}_{Conts}(\pi_1(X), \mathbb{Z}) = 0$$

as $\pi_1(X)$, being profinite, has only finitely many discrete quotients. However,

$$H^1(X(\mathbb{C}), \mathbb{Z}/n\mathbb{Z}) \simeq (\mathbb{Z}/n\mathbb{Z})^{2g} \simeq H^1(X_{ét}, \mathbb{Z}/n\mathbb{Z})$$

Theorem 5.6.1. *Let X be a smooth schem over $\mathbb{C}$. For any finite abelian group M, for all $i \geqslant 0$ we have an isomorphism*

$$H^i(X(\mathbb{C}), M) \simeq H^i(X_{ét}, \underline{M})$$

where $\underline{M}$ is the constant sheaf on X associated with M.

Note. This theorem together with the smooth base change theorem implies that for a smooth variety X over $\mathbb{Z}$ with $X(\mathbb{C}) = X \times_{\mathbb{Z}} \mathrm{Spec}(\mathbb{C})$ and $X(\mathbb{F}_q) = X \times_{\mathbb{Z}} \mathrm{Spec}(\mathbb{F}_q)$ (with both being smooth as well), we get that

$$\dim H^i_{sing}(X(\mathbb{C}), \mathbb{Q}_\ell) = \dim H^i_{ét}(X(\mathbb{F}_q), \mathbb{Q}_\ell)$$

This is exactly what we need to prove the part of the Weil conjectures concerning the Betti numbers and the degree of the polynomials in the product formula for the zeta function.

Sketch of Proof. Details may be found in SGA 4, XI.

For $i = 0$, the comparison theorem simply asserts that $X(\mathbb{C})$ and X have the same number of connected components (relative to the complex and Zariski topologies respectively), which is well-known.

For $i = 1$, the theorem states that there is a one-to-one correspondence between the Galois coverings of $X(\mathbb{C})$ with automorphism group M and the similar coverings of X. This is a consequence of the following result of Grauert and Remmert:

Lemma 5.6.2 (Riemann existence theorem)**.** *Let X be a scheme of locally-finite type over $\mathbb{C}$, and let X^{an} be the associated complex analytic space. The functor $Y \mapsto Y^{an}$ defines an equivalence between the categor of finite étale coverings Y/X and the category of similar coverings of X^{an}.*

For $i > 1$, we proceed as follows. Let X_{cx} denote the small site $(X^{an})_E$, where E is the class of local isomorphisms. Since an inclusion $U \hookrightarrow X(\mathbb{C})$ of an open subset is a local isomorphism, we have a morphism of sites $X_{cx} \to X(\mathbb{C})$. This morphism induces an isomorphism on cohomology

$$H^i(X(\mathbb{C}), M) \simeq H^i(X_{cx}, M)$$

There is a morphism of sites $f : X_{cx} \to X_{ét}$, where the inverse image of U étale over X is U^{an}. There is a Leray spectral sequence

$$H^i(X_{ét}, R^j f_* \mathcal{F}) \Rightarrow H^{i+j}(X_{cx}, \mathcal{F})$$

Thus, it remains to show that $R^j f_* \mathcal{F} = 0$ for $j > 0$. This follows from the next lemma (which we will not prove):

Lemma 5.6.3. *Let $\gamma \in H^i(X_{cx}, \mathcal{F})$, $i > 0$, where $\mathcal{F}$ is a locally constant torsion sheaf on X_{cx} with finite fibers. Then, for any $x \in X(\mathbb{C})$, there exists an étale morphism $U \to X$ whose image contains x and which is such that $\gamma|_{U_{cx}} = 0$.*

□

6 (Poincaré) Duality

While the only case of duality which we will need for our application to the Weil conjectures is that of Poincaré duality for smooth projective schemes, we will take the time to develop a general theory of duality for the derived categories of sheaves which we are working with, introducing the upper-shriek operator $f^!$ and culiminating in a statement of general biduality.

6.1 Trace morphisms

For now, we will be working on some fixed scheme S with an integer n which is invertible on S. We are interested in smooth S-compactifiable morphisms $f : X \to Y$ of relative dimension d, and we want to consider sheaves $\mathcal{F}$ of

$\Lambda = \mathbb{Z}/n\mathbb{Z}$-modules on X (and then "take the limit" to get results on ℓ-adic sheaves).

We will need to introduce certain sheaves which will essentially act as ways to "maintain Galois equivariance" in all of our maps.

Definition 6.1.1 (Tate Twists)**.** Recall that μ_n denotes the sheaf obtained from the kernel of the morphism

$$n : \mathcal{O}_{X,et}^{\times} \to \mathcal{O}_{X,et}^{\times}, \qquad s \mapsto s^n$$

For any integer d and any sheaf $\mathcal{F}$ of $\Lambda = \mathbb{Z}/n\mathbb{Z}$-modules on X, we define

$$\Lambda(d) = \begin{cases} \mu_n^{\otimes d} & \text{if } d > 0 \\ \Lambda & \text{if } d = 0 \\ \mathscr{H}\text{om}(\mu_n^{\otimes(-d)}, \Lambda) & \text{if } d < 0 \end{cases}$$

and

$$\mathcal{F}(d) = \mathcal{F} \otimes_{\Lambda} \Lambda(d)$$

The first key step in developing a theory of duality is to find *trace morphisms* from the top cohomology groups (since we know by our previous results that $\mathbf{R}^q f_! f^* \mathcal{F}$ vanishes for $q > 2d$), i.e.

$$\mathrm{Tr}_f : \mathbf{R}^{2d} f_! f^* \mathcal{F}(d) \to \mathcal{F}$$

we then use this together with the canonical homomorphisms

$$\begin{aligned} Ext^i\ (\mathcal{F}, \mathcal{G}) &\cong \mathrm{Hom}_{D(X,\Lambda)}(\mathcal{F}, \mathcal{G}[i]) \\ &\to \mathrm{Hom}_{D(\Lambda)}(\mathbf{R}\Gamma_c(X, \mathcal{F}), \mathbf{R}\Gamma_c(X, \mathcal{G}[i])) \\ &\to \mathrm{Hom}(H_c^j(X, \mathcal{F}), H_c^{i+j}(X, \mathcal{G})) \end{aligned}$$

to get the cup product

$$H_c^j(X, \mathcal{F}) \times \mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) \to H_c^{i+j}(X, \mathcal{G})$$

and then for $\mathcal{G} = \Lambda$ and $j = r$, $i = 2d - r$ we have

$$H_c^r(X, \mathcal{F}) \times \mathrm{Ext}^{2d-r}(\mathcal{F}, \Lambda(d)) \to H_c^{2d}(X, \Lambda(d))$$

and then compose with the trace morphism to get the Poincaré duality pairing, which we then have to prove is perfect

$$H_c^r(X, \mathcal{F}) \times \mathrm{Ext}^{2d-r}(\mathcal{F}, \Lambda(d)) \to \Lambda$$

This mirrors how the pairing on $V \otimes V^* \cong \mathrm{End}(V) \xrightarrow{tr} K$ is used to formulate duality for vector spaces V over a field K (or how duality for modules over local rings is formulated and then generalized to the context of duality for coherent sheaf cohomology, a.k.a Grothendieck duality).

As with many of our earlier sections, we start with the "zero-dimensional case", use some explicit techniques to generalize to the "one-dimensional case", and then apply some induction (with some help of derived categories) to get the general "d-dimensional case". Of course, this is easier said than done.

If $f : X \to Y$ is étale (read: "smooth of relative dimension 0"), then $f_!$ is left-adjoint to f^*, and so the adjunction isomorphism gives us

$$\mathrm{Tr}_{X/Y} : f_! f^* \mathcal{F} \to \mathcal{F}$$

Next, we move to the "one-dimensional case", but we start with $X \to \mathrm{Spec}(k)$ a smooth projective curve over an algebraically closed field k. From our study of the cohomology of curves (Kummer sequence, Picard groups, pushforwards, etc.) we have the canonical isomorphism

$$H^2(X, \mu_n) \cong \mathrm{Pic}(X)/n\mathrm{Pic}(X)$$

then, the degree map $\deg : \mathrm{Pic}(X) \to \mathbb{Z}$ gives the isomorphism $\mathrm{Pic}(X)/n\mathrm{Pic}(X) \cong \Lambda$ and so we compose this with the isomorphism above to get the trace map

$$\mathrm{Tr}_{X/k} : H^2(X, \Lambda(1)) \to \Lambda$$

Now, let's start removing conditions: suppose that $X \to \mathrm{Spec}(k)$ is a smooth irreducible curve, with $\overline{X}$ the compactification. The excision exact sequence gives us

$$\cdots \to H^1_c(\overline{X} \smallsetminus X, \Lambda(1)) \to H^2_c(X, \Lambda(1)) \to H^2(\overline{X}, \Lambda(1)) \to H^2_c(\overline{X} \smallsetminus X, \Lambda(1)) \to \cdots$$

but $\overline{X} \smallsetminus X$ is finite, so its cohomology groups vanish for $q > 0$, so we have the isomorphism

$$H^2_c(X, \Lambda(1)) \cong H^2(\overline{X}, \Lambda(1))$$

which we then compose with our trace morphism above to get

$$\mathrm{Tr}_{X/k} : H^2_c(X, \Lambda(1)) \to \Lambda$$

What if X is not irreducible? Then we let $X_1, \ldots, X_n$ denote the irreducible components and recall that

$$H^2_c(X, \Lambda(1)) = \bigoplus_i H^2_c(X_i, \Lambda(1))$$

and so, we define $\mathrm{Tr}_{X/k} = \bigoplus_i \mathrm{Tr}_{X_i/k}$. Finally, we get the following proposition:

Proposition 6.1.2. *Let $Y \to \mathrm{Spec}(k)$ be a smooth curve over an algebraically closed field k and let $f : X \to Y$ be an étale morphism. Since $f^*\Lambda(1)_Y \cong \Lambda(1)_X$, we have that the adjunction $f_!\Lambda(1) \to \Lambda(1)$ induces a map $H^2_c(Y, f_!\Lambda(1)) \to H^2_c(Y, \Lambda(1))$. By base change, we have the following composition*

$$S_{X/Y} : H^2_c(X, \Lambda(1)) \cong H^2_c(Y, f_!\Lambda(1)) \to H^2_c(Y, \Lambda(1))$$

Then

$$\mathrm{Tr}_{Y/k} \circ S_{X/Y} \cong \mathrm{Tr}_{X/k}$$

Proof. ☐

Now, for the induction step, a.k.a. "towards a higher dimensional trace morphism". Here is the idea:

1. Assuming that we have two morphisms $X \to Y \to Z$ with trace morphisms $\mathrm{Tr}_{X/Y}$ and $\mathrm{Tr}_{Y/Z}$, we construct the trace morphism $\mathrm{Tr}_{X/Z}$ using the formalism of the derived category;

2. We assume that we have a factorization of any $X \to Y$ as $X \to \mathbb{A}^d_Y \to Y$ with definable trace morphisms everywhere. We then factor this as

$$X \to \mathbb{A}^d_Y \to \mathbb{A}^{d-1}_Y \to \cdots \to \mathbb{A}^1_Y \to Y$$

   and so each step besides $X \to \mathbb{A}^d_Y$ is the 1-dimensensional trace morphism;

3. We show that the trace morphism we get is independent of the factorization $X \to \mathbb{A}^d_Y \to Y$;

4. (work on more technical details).

6.2 Smooth Projective Schemes

For the case of a smooth projective scheme X of constant dimension d over a separably closed base field k, the key to proving Poincaré duality is the construction of a canonical *trace homomorphism*

$$S = S_{X/k} : H^{2d}(X, \Lambda(d)) \to \Lambda$$

(where $\Lambda = \mathbb{Z}/n\mathbb{Z}$ as in the section on purity and the Gysin sequence). We use this homomorphism together with the cup product

$$\mathrm{Ext}^p(\mathcal{G}, \Lambda(d)) \times H^q(X, \mathcal{G}) \to H^{p+q}(X, \Lambda(d))$$

to get a pairing

$$\mathrm{Ext}^p_\Lambda(\mathcal{G}, \Lambda(d)) \times H^{2d-p}(X, \mathcal{G}) \to \Lambda$$

where one statement of Poincaré duality is that this pairing is nondegenerate, i.e. there is an isomorphism

$$\mathrm{Ext}^p_\Lambda(\mathcal{G}, \Lambda(d)) \xrightarrow{\sim} \mathrm{Hom}_\Lambda(H^{2d-p}(X, \mathcal{G}), \Lambda)$$

to obtain the case of non-complete k-schemes, we must use cohomology with compact support.

The classical Poincaré duality theorem says that for an *oriented* connected m-dimensional manifold U, there is a canonical isomorphism

$$H^r_c(U, \Lambda) \xrightarrow{\sim} H_{m-r}(U, \Lambda)$$

Using the duality of cohomology and homology, we can rewrite this as a perfect pairing of finite groups

$$H_c^r(U,\Lambda) \times H^{m-r}(U,\Lambda) \to H_c^m(U,\Lambda) \simeq \Lambda$$

The key point here is that we made a choice of *orientation.* One way to package this choice without having to use an oriented manifold is to equip our manifold with an "orientation sheaf" $\mathcal{O}$, for which there is a isomorphism $H_c^m(U,\mathcal{O}) \simeq \Lambda$. We can then restate Poincaré duality as a perfect pairing

$$H_c^r(U,\mathcal{O}) \times H^{m-r}(U,\Lambda) \to H_c^m(U,\mathcal{O}) \simeq \Lambda$$

The choice of isomorphism $H_c^m(U,\mathcal{O}) \simeq \Lambda$ is essentially the same as choosing an orientation. For the case of a complex manifold, the only choice that we must make is the choice of $\sqrt{-1} \in \mathbb{C}$; this then gives us an automatic choice of $e^{2\pi\sqrt{-1}/n}$ of primitive nth roots of unity.

More generally, we will need to make a choice for our nth roots of unity. More precisely, our orientation sheaf will be $\Lambda(d) = \mu_n^{\otimes d}$. Let X be a nonsingular variety of dimension d over an algebraically closed field k. For an closed point $P \in X$, the Gysin sequence gives an isomorphism

$$H^0(P,\Lambda) \xrightarrow{\sim} H_P^{2d}(X,\Lambda(d))$$

There is a canonical map $H_P^{2d}(X,\Lambda(d)) \to H_c^{2d}(X,\Lambda(d))$, and we let $\mathrm{cl}_X(P)$ denote the image of 1 under the composition of these maps. More generally, we define:

Definition 6.2.1. Let X be a smooth algebraic variety of constant dimension d over an algebraically closed field, and let $j : Y \hookrightarrow X$ be an irreducible reduced closed subvariety of codimension s. We compose the restriction mapping

with the trace mapping

and get an element

$$\alpha := S_Y \circ j^* \in \mathrm{Hom}_\Lambda(H_c^{2(d-s)}(X,\Lambda(s)),\Lambda)$$

by Poincaré duality (which we are about to prove) this group is canonically isomorphic to $H^{2s}(X,\Lambda(s))$. We denote by $\mathrm{cl}_X(Y)$ the image of α in $H^{2s}(X,\Lambda(s))$ of this canonical isomorphism. We call this the *cohomology calss associated with* $Y \subset X$.

Note. By extending linearly, we get a homomorphism

$$\mathrm{cl}_X : Z^s(X) \to H^{2s}(X,\Lambda(s))$$

from the group $Z^s(X)$ of algebraic cycles of codimension s on X and the cohomology group.

In addition, it can be shown that the cup product takes the place of the intersection product, giving us a homomorphism from the Chow ring $CH^*(X)$ to the cohomology ring $H^*(X)$. There are other technicalities involved, and we won't need the full generality of algebraic cycles, but suffice it to say that this is an important viewpoint (e.g. the Tate conjecture and Hodge conjecture are concerned with this homomorphism).

Before proving Poincaré duality, we will need two lemmas:

Lemma 6.2.2. *For any separated variety X of dimension d over a separably closed field, $H_c^{2d}(X, \Lambda(d)) \simeq \Lambda$.*

Proof. We proceed via induction on d, so we assume that the theorem holds for all separated varieties X of dimension $< d$.

If X_0 is a dense open subvariety of X, then the exact sequence

$$\cdots \to H_c^r(X_0, \Lambda(d)) \to H_c^r(X, \Lambda(d)) \to H_x^r(X \smallsetminus X_0, \Lambda(d)) \to \cdots$$

and cohomological dimension show that $H_c^{2d}(X_0, \Lambda(d)) \simeq H_c^{2d}(X, \Lambda(d))$. Thus, we may replace X by such an X_0 (or X_0 by such an X).

The following paragraph is informal, but it is to summarize why we can now assume that there is a smooth variety S and a projective smooth map $\pi : X \to S$ whose fibers are connected of dimension 1:

It is possible to embed $X_0 \hookrightarrow \mathbb{P}^r$ in such a way that the closure $\overline{X}_0 = X$ of X_0 is normal. Thus, the singularities of $\overline{X}_0$ have codimension $\geqslant 2$. After replacing the embedding $X \hookrightarrow \mathbb{P}^r$ by a multiple $X \hookrightarrow \mathbb{P}^R$, one finds that there is a particularly good projection map $\mathbb{P}^R \to \mathbb{P}^{d-1}$ for X. After blowing up X at the center of the projection map, one obtains the desired projective smooth morphism $\pi : X \to S$ (this will actually be what is called an *elementary fibration*).

There is a commutative diagram corresponding to the Kumer sequence in cohomology:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \underline{\mathrm{Pic}}_{X/S} & \xrightarrow{n} & \underline{\mathrm{Pic}}_{X/S} & \longrightarrow & R^2\pi_*\Lambda(1) & \longrightarrow & \cdots \\
 & & \big\downarrow{\scriptstyle \deg} & & \big\downarrow{\scriptstyle \deg} & & & & \\
\cdots & \longrightarrow & \mathbb{Z} & \xrightarrow{n} & \mathbb{Z} & \longrightarrow & \Lambda & \longrightarrow & 0
\end{array}$$

The map $\underline{\mathrm{Pic}}_{X/S} \to \Lambda$ factors through $R^2\pi_*\Lambda(1)$, since it does on each fiber, and the resulting map $R^2\pi_*\Lambda(1) \to \Lambda$ is an isomorphism, since it is on each fiber (consequence of our study of cohomology on curves). The Leray spectral sequence gives an isomorphism

$$H_c^{2d}(X, \Lambda(d)) \xrightarrow{\sim} H_c^{2d-2}(S, R^2\pi_*\Lambda(d)) \simeq H_c^{2d-2}(S, \Lambda(d-1)) \simeq \Lambda$$

(since $\dim S < \dim X$) $\square$

Lemma 6.2.3. *Let $\pi : Y \to X$ be a separated étale morphism where X is a smooth separated variety of dimension d over a separably closed field. Let $P \in Y$ be a closed point, and let $Q = \pi(P)$. The trace map*

$$\pi_* : H_c^{2d}(Y, \Lambda(d)) \to H_c^{2d}(X, \Lambda(d))$$

induced by the map $R_c^0\pi_*\Lambda(d) = \pi_!\pi^*\Lambda(d) \xrightarrow{tr} \Lambda(d)$ sends $\mathrm{cl}_Y(P)$ to $\mathrm{cl}_x(Q)$

Proof. Consider a compactification $Y \overset{j}{\hookrightarrow} \overline{Y} \xrightarrow{\overline{\pi}} X$ of π with $\overline{\pi}$ finite. There is a commutative diagram

$$\begin{array}{ccccc} H^{2d}_{\pi^{-1}(Q)}(\overline{Y}, j_!\Lambda(d)) & \longrightarrow & H^{2d}_P(\overline{Y}, j_!\Lambda(d)) & \longrightarrow & H^{2d}_c(Y, \Lambda(d)) \\ \downarrow & & & & \downarrow \\ H^{2d}_Q(X, \Lambda(d)) & & \longrightarrow & & H^{2d}_c(X, \Lambda(d)) \end{array}$$

in which the vertical morphisms are induced by $tr : \overline{\pi}_* j_!\Lambda(d) \to \Lambda(d)$. Thus is suffices to show that some inverse image of $\mathrm{cl}_{\overline{Y}}(P)$ in

$$H^{2d}_{\pi^{-1}(Q)}(\overline{Y}, j_!\Lambda(d))$$

maps to $\mathrm{cl}_X(Q)$. By excision, we may assume that X is strictly local, but then Y is a disjoint union of copies of some open subsets of X, and the assertion is obvious. □

For any sheaf $\mathcal{F}$ we let $\check{\mathcal{F}}(d) := \underline{\mathrm{Hom}}(\mathcal{F}, \Lambda(d))$.

Theorem 6.2.4 (Poincaré duality)**.** *Let X be a smooth separated variety of dimension d over a separably closed field k.*

(a) There is a unique isomorphism

$$\eta(X) : H_c^{2d}(X, \Lambda(d)) \xrightarrow{\sim} \Lambda$$

such that $\mathrm{cl}_X(P) \mapsto 1$ for any closed point $P \in X$;

(b) For any constructible sheaf of Λ-modules $\mathcal{F}$ on X, the canonical pairings

$$H_c^r(X, \mathcal{F}) \times \mathrm{Ext}_X^{2d-r}(\mathcal{F}, \Lambda(d)) \to H_c^{2d}(X, \Lambda(d)) \simeq \Lambda$$

are nondegenerate;

(c) Equivalently, for any locally constant, constructible sheaf $\mathcal{F}$ of Λ-modules on X, the cup product pairings

$$H_c^r(X, \mathcal{F}) \times H^{2d-r}(X, \check{\mathcal{F}}(d)) \to H_c^{2d}(X, \Lambda(d)) \simeq \Lambda$$

are nondegenerate.

Proof of (a). By the Kummer sequence, we know that $H^1(\mathbb{P}^m, \Lambda(r)) = 0$, and then by induction on the Gysin sequence we can see that

$$H^r(\mathbb{P}^m_k, \Lambda) \simeq \begin{cases} \Lambda(-r/2) & \text{if } r \text{ is even, } 0 \leqslant r \leqslant 2m \\ 0 & \text{otherwise} \end{cases}$$

This then gives us that the Gysin map for any linear subspace L^r of codimension r in $\mathbb{P}^m$

$$\Lambda = H^0(L^r, \Lambda) \to H^{2r}(\mathbb{P}^m, \Lambda(r))$$

is an isomorphism. Thus, $H^{2r}(\mathbb{P}^m, \Lambda(r))$ is generated by $\mathrm{cl}_{\mathbb{P}^m}(L^r)$ (image of 1 under the Gysin map).

Since $\mathrm{Pic}(\mathbb{P}^m) \simeq \mathbb{Z}$ is generated by the class of any hyperplane, $\mathrm{cl}_{\mathbb{P}^m}(L^1)$ is independent of L^1, and since any L^r is the transverse intersection of r hyperplanes, we get that

$$\mathrm{cl}(L^r) = \mathrm{cl}(L^1) \smile \cdots \smile \mathrm{cl}(L^1)$$

is also independent of L^1.

So, $H^{2d}(\mathbb{P}^d, \Lambda(d))$ is generated by $\mathrm{cl}(P)$ for any closed point $P \in \mathbb{P}^d$, and this class is independent of P; thus, we have a unique choice for $\eta(\mathbb{P}^d)$.

Let X be a smooth separated variety over a separably closed field and fix a closed point $P_0 \in X$ □

Proof of (b). We now present a seven part proof by induction on dimension d of part (b). Note that we have already proved the case of $d = 1$ in our section on curves and that the case of $d = 0$ is trivial. We write

$$\phi^r(X, \mathcal{F}) : \mathrm{Ext}^{2d-r}_X(\mathcal{F}, \Lambda(d)) \to H^r_c(X, \mathcal{F})^\vee$$

for the map induced by the pairing in the theorem.

Step 1: Let $\pi : X' \to X$ be a finite étale morphism, where X and X are smooth separated varieties of dimension d over an algebraically closed field. For a sheaf $\mathcal{F}$ on X', $\phi^r(X', \mathcal{F})$ is an isomorphism if and only if $\phi^r(X, \pi_*\mathcal{F})$ is an isomorphism.

Proof. The degree of π is constant, so we only have to show that

$$\begin{tikzcd} H^{2d}_c(X', \Lambda(d)) \arrow[rr, "tr"] \arrow[dr, "\eta(X')"'] & & H^{2d}_c(X, \Lambda(d)) \arrow[dl, "\eta(X)"] \\ & \Lambda & \end{tikzcd}$$

commutes, but this follows from 6.2.3 □

Step 2: $\phi^r(X, \mathcal{F})$ is an isomorphism if $\mathcal{F}$ has support on a smooth closed subvariety $Z \neq X$ of X.

Proof. Regard $\mathcal{F}$ as a sheaf on Z. We have to show that there exists a commutative diagram where $a = \dim Z$.

Let $j : X \hookrightarrow \overline{X}$ be a compactification of X and let $\overline{Z}$ be the closure of Z in $\overline{X}$. Thus, we have a commutative diagram where everything injects:

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "j"] & X \arrow[d, "j"] \\ \overline{Z} \arrow[r] & \overline{X} \end{tikzcd}$$

In addition, $j_! i_* = i_* j_!$ and

$$H_c^{2a}(Z, \Lambda(a)) \xrightarrow{\sim} H^{2a}(\overline{Z}, \Lambda(a))$$

and

$$H_c^{2d}(X, \Lambda(d)) \xrightarrow{\sim} H^{2d}(\overline{X}, \Lambda(d))$$

□

Step 3:

Step 4:

Step 5:

Step 6:

Step 7:

Step 3 now completes the proof of the theorem, because any constructible sheaf is locally constant over an open subvariety of X. □

6.3 Upper Shriek

7 Grothendieck-Lefschetz Trace Formula

7.1 A Few Frobenii

7.2 Weil Sheaves

We will prove the Lefschetz trace formula by using Poincaré duality to construct the étale cohomology class of an algebraic cycle Z on X of codimension d,

$$\mathrm{cl}_X(Z) \in H^{2d}(X, \mathbb{Z}/n\mathbb{Z})$$

If we consider a morphism $f : X \to Y$ of smooth schemes, then the inverse image $f^*(Z)$ satisfies a compatibility formula:

$$\mathrm{cl}_Y(f^*(Z)) = H^{2d}(f)(\mathrm{cl}_X(Z))$$

(where $H^{2d}(f)$ is the induced map $H^{2d}(Y, \mathbb{Z}/n\mathbb{Z}) \to H^{2d}(X, \mathbb{Z}/n\mathbb{Z})$).

This formula will then be used on the graph mapping

$$\Gamma_f : X \to X \times X$$

corresponding to the morphism $f : X \to X$. We use the Künneth formula to compute the cohomology class of the diagonal $\Delta \subset X \times X$ and then apply the compatibility formula to determine the cohomology class of the fixed point cycle $\Gamma_f^*(\Delta)$. As we already know, this is then used in conjunction with the Frobenius automorphism to count the points of a smooth projective variety over a finite field, allowing us to prove rationality, the functional equation, and the connection with Betti numbers.

In the following two lemmas and theorems, X is a smooth projective variety over an algebraically closed field, and p, q denote the first and second projection maps $X \times X \rightrightarrows X$. We identify $\mathbb{Q}_\ell$ with $\mathbb{Q}_\ell(1)$ and write $H^*(X)$ for the cohomology ring $\bigoplus H^r(X, \mathbb{Q}_\ell)$, where ℓ is prime to the characteristic. For a morphism $\phi : X \to Y$ we let $\Gamma_\phi \subset X \times Y$ denote the graph of ϕ.

First, we note that one can compute $H^*(X \times X, \Lambda)$ using the Künneth formula, since the following map is an isomorphism

$$H^*(X) \otimes H^*(X) \to H^*(X \times X)$$

$$\alpha \otimes \beta \mapsto p^*(\alpha) \smile q^*(\beta)$$

Lemma 7.2.1. *Let $\phi : X \to X$ be an endomorphism of X. For any $b \in H^*(X)$,*

$$p_* \left(\mathrm{cl}_{X \times X}(\Gamma_\phi) \smile q^*(b) \right) = \phi^*(b)$$

Proof. The map $(1, \phi) : X \to X \times X$ has image Γ_ϕ, and $p \circ (1, \phi) = \mathrm{id}$

$$\begin{aligned} p_*(\mathrm{cl}_{X \times X}(\Gamma_\phi) \smile q^*(b)) &= p_*((1, \phi)_*(1) \smile q^*(b)) \\ &= p_*(1, \phi)_*(1 \smile (1, \phi)^* q^*(b)) \\ &= (\mathrm{id})_*(1 \smile \phi^*(b)) \\ &= \phi^*(b) \end{aligned}$$

□

Lemma 7.2.2. *Let $\phi : X \to X$ be an endomorphism of X. Let $\{e_i\}$ be a basis for $H^*(X)$, and let $\{e'_i\}$ be the dual basis (so, if $e_i \in H^r$, then $e'_i \in H^{2d-r}$). Then*

$$\mathrm{cl}_{X \times X}(\Gamma_\phi) = \sum_i \phi^*(e_i) \otimes e'_i$$

Proof. Let $\mathrm{cl}_{X \times X}(\Gamma_\phi) = \sum a_i \otimes e'_i$. Then, according to 7.2.1,

$$\begin{aligned} \phi^*(e_j) &= p_* \left(\left(\sum a_i \otimes e'_i \right) \smile (1 \otimes e_j) \right) \\ &= p_*(a_j \otimes e_{2d}) \\ &= a_j \end{aligned}$$

where e_{2d} is the canonical generator of $H^{2d}(X)$. □

Theorem 7.2.3 (Lefschetz trace formula)**.** *Let $\phi : X \to X$ be an endomorphism such that $(\Gamma_\phi \cdot \Delta)$ is defined. Then*

$$(\Gamma_\phi \cdot \Delta) = \sum_{r=0}^{2d} (-1)^r \mathrm{tr}\,(\phi \mid H^r(X, \mathbb{Q}_\ell))$$

Proof. Let $\{e_i^r\}$ be a basis for $H^r(X, \mathbb{Q}_\ell)$ and let $\{f_i^{2d-r}\}$ be the corresponding dual basis for $H^{2d-r}(X, \mathbb{Q}_\ell)$. Then

$$\mathrm{cl}_{x \times X}(\Gamma_\phi) = \sum_{r,i} \phi^*(e_i^r) \otimes f_i^{2d-r}$$

and

$$\begin{aligned} cl_{X \times X}\ (\Delta) &= \sum_{r,i} e_i^r \otimes f_i^{2d-r} \\ &= \sum_{r,i} (-1)^{r(2d-r)} f_i^{2d-r} \otimes e_i^r \\ &= \sum_{r,i} (-1)^r f_i^{2d-r} \otimes e_i^r \end{aligned}$$

Thus

$$\begin{aligned} \mathrm{cl}_{X \times X}(\Gamma_\phi \cdot \Delta) &= \sum_{r,i} (-1)^r \phi^*(e_i^r) f_i^{2d-r} \otimes e^{2d} \\ &= \sum_{r=0}^{2d} (-1)^r \mathrm{tr}(\phi^*)(e^{2d} \otimes e^{2d}) \end{aligned}$$

as $\phi^*(e_i^r) f_j^{2d-r}$ is the coefficient of e_j^r when $\phi^*(e_i^r)$ is expressed in terms of the basis $\{e_j^r\}$. On applying $\eta(X \times X)$ to both sides, we obtain the required formula, since $e^{2d} \mapsto 1$ under $\eta(X)$. $\square$

7.3 Grothendieck's Formula for L-functions

8 Deligne's Weil I

Now that we have a very general theory of L-functions and Lefschetz trace formulas (along with all the fundamental theorems of étale cohomology), let's go over Deligne's proof the the "Riemann Hypothesis":

Theorem 8.0.1. *Let X be a d-dimensional nonsingular variety over $\mathbb{F}_q$ with d an even integer. Every eigenvalue α of the Frobenius ϕ on $H^d(X, \mathbb{Q}_\ell)$ is an algebraic number such that*

$$q^{\frac{d-1}{2}} < |\alpha| < q^{\frac{d+1}{2}}$$

9 Deligne's Weil II

While Deligne's original proof of the "Riemann Hypothesis" is quite elegant, it seems like Deligne himself wasn't fully satisfied with the results, because six years after [Del74] he published [Del80], a paper going into detail on the theory weights (the "yoga of weights"), Fourier transforms for sheaves, and even a proof of the Hard Lefschetz theorem. Nowadays, if people use results from the Weil conjectures it is often in the form of something from [Del80], as it is more naturally generalized and applied. Let's talk about it.

Philosophically, what Deligne does quite beautifully is finding, unifying, and using many different interpretations of sheaves:

1. The most obvious one is geometric: certain sheaves (locally free of finite rank) can be seen as vector bundles, which also turns us to the "espace étale" interpretation, i.e. spaces as sheaves and sheaves as spaces.

2. For étale cohomology, we get Galois representations *and* representations of the étale fundamental group. In addition, Weil sheaves give us representations of the Weil group.

3. Finally, the most "classical" interpretation of sheaves as functions, which in our case looks sort of like

$$\begin{aligned} D(X_0) &\xrightarrow{\phi} \mathrm{Fun}(X_0(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell) \\ K &\mapsto (x \mapsto \mathrm{tr}\,(\mathrm{Frob}_x \mid x^*K)) \end{aligned}$$

   This view is quite unique, and is the inspiration for the Fourier transform of (complexes of) sheaves.

9.1 Overview

9.2 Weights

9.3 Monodromy

9.4 Real Sheaves

9.5 Fourier Transforms

9.6 The Proofs

9.7 Hard Lefschetz

9.8 Applications?

9.9 Weight-Monodromy

10 Appendix

10.1 Artin Approximation

One key part of intuition regarding the étale site is the fact that local rings $\mathcal{O}_{X,x}$ are all Henselian, and so Artin's approximation theorem gives us an even stronger connection between étale morphisms and the analytic theory. What Artin's theory says is that, basically, any formal power series can be well-approximated by an algebraic function *in the case of Henselian rings over an excellent discrete valuation ring*. So, locally the étale site looks pretty darn close to what an "analytic site" might look like. More practically, this allows for better applications of deformation theory (as we used it for our proof of proper base change), particularly in the construction of moduli spaces via the theory of algebraic spaces (where we glue schemes w.r.t their étale sites rather than their Zariski topology). Here we state and prove Artin's approximation theorem. The best reference still remains Artin's original article [Art69].

Theorem 10.1.1. *Let R be a field or an excellent discrete valuation ring, and let A be the Henselization of an R-algebra of finite type at a prime ideal. Let $\mathfrak{m}$ be a proper ideal of A. Given an arbitrary system of polynomial equations*

$$f(Y) = 0 \qquad Y = (Y_1, \dots, Y_N)$$

with coefficients in A, a solution $\overline{y} = (\overline{y}_1, \dots, \overline{y}_N)$ in the $\mathfrak{m}$-adic completion $\widehat{A}$ of A, then for some integer c there exists a solution $y = (y_1, \dots, y_N) \in A$ with

$$y_i \equiv \overline{y}_i \pmod{\mathfrak{m}^c}$$

Equivalently, let $F : \mathbf{Alg}_A \to \mathbf{Sets}$ be a functor which is locally of finite presentation (e.g. a scheme $X/\mathrm{Spec}(A)$ locally of finite presentation). Given any $\overline{\xi} \in F(\widehat{A})$, there is a $\xi \in F(A)$ such that

$$\xi \equiv \overline{\xi} \pmod{\mathfrak{m}^c}$$

Proof. Some reductions are in order: First, we may assume that $\mathfrak{m}$ is the maximal ideal of A. Next, we may assume that R is a discrete valuation ring, since if it is a field, then we may replace it with the power series ring $R[\![t]\!]$, where t acts trivially on A (this is allowed since R is excellent). Next, we may assume that the ideal $\mathfrak{m}$ lies over the closed point of R, for if it lies over the generic point, then we replace R by its field of fractions, and then replace that with the power series ring.

Moreover, since A is the Henselization of an R-algebra of finite-type at a prime ideal $\mathfrak{p}$, we may assume that $K = A/\mathfrak{m}$ is finite over the residue field k of R, because if it isn't we may replace R by the localization $R[Z_i]_{(\mathfrak{p})}$ where the Z_i map onto the transcendence basis of A over R.

Finally, letting A_0 be the R-algebra of finite type over the maximal ideal $\mathfrak{m}$ such that its Henselization is A, we can make A_0 into a finite algebra over a polynomial ring $R[x_1, \ldots, x_n)$ in such a way that $\mathfrak{m}$ lies over the "origin" $(\mathfrak{m}_R, x_1, \ldots, x_n)$ of $\mathbb{A}^n_R$. This can be seen by constructing the obvious map $R[x_1, \ldots, x_n) \to A_0$.

Now, let $\tilde{R}[x_1, \ldots, x_n]$ denote the Henselization of $R[x_1, \ldots, x_n]$ at the origin $(p, X) := (\mathfrak{m}_R, x_1, \ldots, x_n)$. The $\tilde{R}[X]$-algebra $\tilde{A}_0$ obtained by extending scalars from A_0 is a product of local rings which are the Henselizations of A_0 at the various points lying over the origin. The ring A is among them, and is therefore a finite $\tilde{R}[X]$-algebra.

We claim that it is enough to prove the theorem for the ring $\tilde{R}[X]$ itself. This follows from the fact that we may take the fiber product at the level of schemes

$$\widehat{A} \simeq A \otimes R[X]^\wedge$$

We are therefore reduced to the case of $A = \tilde{R}[X]$ and $\mathfrak{m} = (p, X)$. We proceed by induction on the number of variable $n \geqslant 0$, so we assume that the theorem is true for $\leqslant n-1$ variables. □

10.2 Generic Smoothness

Étale morphisms can be thought of as smooth morphisms of relative dimension 0. Generic smoothness gives a criterion for smoothness (or, for the "important part" of smoothness) which is similar to the valuative criterion for proper morphisms.

10.3 Galois Cohomology; Proofs

Here we outline the proofs from the section "Étale Cohomology of Point" (which was really just a sectoin on Galois cohomology).

10.4 Spectral Sequences & Derived Categories

In the subsection "Higher Direct Images and the Leray Spectral Sequence" we state a result about the Leray spectral sequence without giving proof, sweeping it under the rug. Here we give a (rudimentary) introduction to spectral sequences, specifically to the spectral sequence most important for sheaf cohomology and homological algebra: the Grothendieck spectral sequence. This spectral sequence is the most general, and most other spectral sequences (which we use) are direct consequences. There are other spectral sequences which are not quite direct consequences (mostly from algebraic topology, e.g. the Adams spectral sequence). The standard reference for this is Grothendieck's *Tohoku paper* ([Gro57] and english translation). Most proofs will be sketched rather than done fully, as they are better suited for "diagram chasing" a.k.a. standing in front of a blackboard and pointing at different points on a commutative diagram while hoping that everyone in the audience agrees with you on what's "obvious".

The basic idea behind a spectral sequence is that we have a large graded or filter object, H^*, which we want to study. We create a double complex, $E_r^{p,q}$ and repeatedly take the cohomology of the individual complexes, either horizontal-then-vertical or vertical-then-horizontal. This then helps us approximate the cohomology of the total complex, which we have constructed so as to give us a filtration of H^*. This is summed up with notation as

$$E_r^{p,q} \Rightarrow H^*$$

More specifically, this says that H^* has a canonical decreasing filtration

$$H^* = F^0H^* \supseteq F^1H^* \supseteq \cdots \supseteq F^nH^* \supseteq F^{n+1}H^* = 0$$

such that $\mathrm{gr}_p H^* := F^pH^*/F^{p+1}H^*$ is isomorphic to some subquotient of $E_r^{p,q}$. If H^* is a graded vector space with filtration F^*, it is possible to construct another graded vectors space, the *associated graded vector space*

$$\bigoplus_{p=0}^{\infty} \mathrm{gr}_p(H^*)$$

In the case of a vector space, we have that the associated vector space is directly isomorphic to H^*. If H^* is an arbitrary graded module, there may be extension problems that prevent one from reconstructing H^* from the associated graded vector space.

Since H^* is not easily computed, we can take as a first approximation to H^* the associated graded vector space to some filtration of H^*. *This is the target of the spectral sequence!* We then hope to be able to reconstruct H^* from $\mathrm{gr}_p(H^*)$.

From this, we make a definition:

Definition 10.4.1. Let $m \in \mathbb{N}$. An *E_m-spectral sequence* in an abelian category $\mathcal{A}$ is a system

$$E = (E_r^{p,q}, E^n)$$

satisfying the following properties:

(a) Objects $E_r^{p,q} \in \mathcal{A}$ for all $(p,q) \in \mathbb{Z} \times \mathbb{Z}$ and any integer $r \geqslant m$;

(b) Morphisms $d = d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ with $d \circ d = 0$;

(c) Isomorphisms $\alpha_r^{p,q} : \ker(d_r^{p,q})/\mathrm{im}(d^{p-r,q+r-1}) \xrightarrow{\sim} E_{r+1}^{p,q}$;

(d) Finitely filtered objects $E^n \in \mathcal{A}$ for all $n \in \mathbb{Z}$;

(e) Isomorphisms $\beta^{p,q} : E_\infty^{p,q} \xrightarrow{\sim} \mathrm{gr}_p(E^{p+q})$.

In addition, we require that for large enough r, the morphisms $d_r^{p,q}$ and $d^{p-r,q+r-1}$ vanish, meaning that the objects $E_r^{p,q}$ are independent of r for r sufficiently large, and we denote them by $E_\infty^{p,q}$.

In other words, for $r \geqslant m$, $E_r^{p,q}$ is a system of complexes whose cohomology groups are the objects $E_{r+1}^{p,q}$ of the next system. A spectral sequence is like a book with infinitely many pages $E_m^{p,q}, E_{m+1}^{p,q}, E_{m+2}^{p,q}, \ldots$ and a limit page E^n at the end.

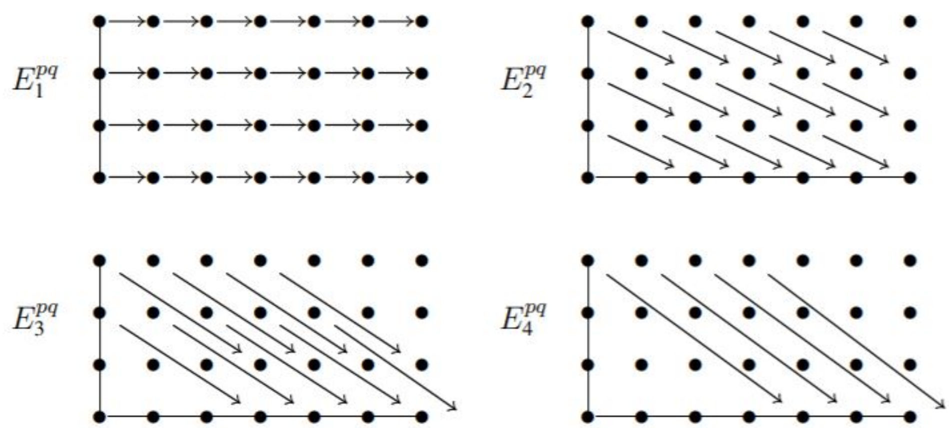

Figure 1: The first 4 pages

For an E_m-spectral sequence $E = (E_r^{p,q}, E^n)$, one usually writes

$$E_m^{p,q} \Rightarrow E^{p+q}$$

The $E_m^{p,q}$ are the *initial terms*, the E^n the *limit terms*, and the $d_r^{p,q}$ *differentials*. For all $m' \geqslant m$, we can forget the first $m' - m$ pages of an E_m-spectral sequence to obtain an $E_{m'}$-spectral sequence in the natural way.

A *morphism of spectral sequences* is the natural thing: a system of morphisms

$$\phi_r^{p,q} : E_r^{p,q} \to E_r'^{p,q}$$

compatible with filtration and commuting with the differentials (including the maps between pages, α and β).

In general, most theorems concerning spectral sequences will look something like this:

Theorem 10.4.2 ("Theorem 1")**.** *There is a spectral sequence with*

$$E_2^{*,*} \simeq \text{“something computable”} \Rightarrow H^*$$

where H^* is something we want to compute.

Note. The important observation to make about such theorems is that we do not usually specify the differentials involved in the spectral sequence, instead it only gives the E_2-terms. Though $E_r^{*,*}$ may be known, without the differentials or some further information, it may be impossible to proceed.

Like with other areas of homological algebra, we instead exploit certain algebraic structures of the $E_2^{p,q}$-terms that we do know, so as to gleen some information, and this is usually quite succesful. For example, in most cases of importance to algebraic geometry we are dealing with spectral sequences involving sheaf cohomology, which we have plenty of information about without having to explicitly understand the differentials. In addition, most useful spectral sequences are of a particular, nice form, which allows even more exploitation:

We shall restrict ourselves to the most important case of an E_2-spectral sequence. If $E_r^{p,q} = 0$ for $p < 0$ or $q < 0$, then we are dealing with a *first quadrant* spectral sequence. In this case, we have $E_r^{p,q} = E_\infty^{p,q}$ for $r > \max\{p, q+1\}$, $r \geqslant 2$. Of basic importance are the two *edge morphisms*

$$E_2^{n,0} \to E^n \to E_2^{0,n}$$

The first is the composite of the morphisms

$$E_2^{n,0} \to E_3^{n,0} \to \cdots \to E_\infty^{n,0} \to E^n$$

which are well defined since $F^{n+1}E^n = 0$, so $E_\infty^{n,0} \simeq \mathrm{gr}_n(E^n) = F^nE^n \subseteq E^n$ and $E^{n+r,1-r} = 0$, so then $\ker d_r^{n,0} = E_2^{n,0}$, meaning that $E_{r+1}^{n,0} \simeq E^{n,0}/\mathrm{im}(d_r^{p-r,q+r-1})$. The second is the composite of the morphisms

$$E^n \to E_\infty^{0,n} \to \cdots \to E_3^{0,n} \to E_2^{0,n}$$

which are well-defined since $F^0E^n = E^n$, so $E_\infty^{0,n} \simeq E^n/F^1E^n$ is a quotient and $E^{-r,n+r-1} = 0$, so $E_{r+1}^{0,n} \simeq \ker(d_r^{0,n}) \subset E_r^{0,n}$ is an inclusion. We now state a proposition which is very useful for computations:

Proposition 10.4.3 (Five term exact sequence)**.** *For any first quadrant E_2-spectral sequence, the sequence*

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \xrightarrow{d} E_2^{2,0} \to E^2$$

is exact.

Generally, this proposition is what we use when computing with a spectral sequence. For example, we implicitly used this when calculating the cohomology of surfaces.

Another useful lemma which can be used in a proof of the Grothendieck spectral sequence is the following:

Lemma 10.4.4. *Let a $(E_r^{p,q}, E^n)$ be a first quadrant E_2-spectral sequence.*

(i) If $E_2^{p,q} = 0$ for all $q > 1$ and all p, then we have a long exact sequence

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \to \cdots$$
$$\cdots \to E_2^{m,0} \to E^m \to E_2^{m-1,1} \to \cdots$$

(ii) If $E_2^{p,q} = 0$ for all $p > 1$ and all q, then the sequences

$$0 \to E_2^{1,n-1} \to E^n \to E_2^{0,n} \to 0$$

are exact for all $m \geqslant 1$.

We now look at the Grothendieck spectral sequence, which is encapsulates most spectral sequences in algebraic geometry and algebraic number theory. Keep in mind a catchphrase: "The Grothendieck spectral sequence is to (well-behaved) derived functors as the chain rule is to derivatives".

Theorem 10.4.5 (Grothendieck spectral sequence)**.** *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be two additive, left-exact functors between abelian categories. If F takes injective objects to G-acyclic objects and if $\mathcal{B}$ has enough injectives, then there is a spectral sequence for each object $A \in \mathcal{A}$ that admits an F-acyclic resolution:*

$$E_2^{pq} = (R^pG \circ R^qF)(A) \Rightarrow R^{p+q}(G \circ F)(A)$$

Note. If we introduce the formalism of derived categories, this simply becomes the existence of a natural transformation $\mathbf{R}(G \circ F) \Rightarrow (\mathbf{R}G) \circ (\mathbf{R}F)$.

For this case, the chain complexes will be resolutions, but we also need resolutions of the chain complexes so that we can fill up the first quadrant properly. For this, we make a definition:

Definition 10.4.6. Let $\mathcal{A}$ be an abelian category with enough injectives. A (right) *Cartan-Eilenberg resolution* of a cochain complex $A^\bullet$ in $\mathcal{A}$ is an upper-half plane complex $I^{\bullet,\bullet}$ of injective objects in $\mathcal{A}$, with augmentation map $A^\bullet \to I^{\bullet,0}$

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & I^{0,1} & \longrightarrow & I^{1,1} & \longrightarrow & I^{2,1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & I^{0,0} & \longrightarrow & I^{1,0} & \longrightarrow & I^{2,0} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & A^{0} & \longrightarrow & A^{1} & \longrightarrow & A^{2} & \longrightarrow & \cdots
\end{array}$$

We require that maps on the horizontal coboundaries and cohomologies are injective resolutions of $B^p(A^\bullet$ and $H^p(A^\bullet)$ respectively. We think of Cartan-Eilenberg resolutions as "resolutions of chain complexes".

Note. Recall that we let

$$\begin{aligned} B^i(A^\bullet) &:= \operatorname{im}(d_{i-1}) \\ Z^i(A^\bullet) &:= \ker(d_i) \\ H^i(A^\bullet) &:= \frac{Z^i(A^\bullet)}{B^i(A^\bullet)} \end{aligned}$$

where d_i is the ith differential of the cochain complex $A^\bullet$.

Lemma 10.4.7. *Every cochain complex admits a Cartan-Eilenberg resolution.*

Sketch of proof. For each p, select injective resolutions $I_B^{p,\bullet}$ of $B^p(A^\bullet)$ and $I_H^{p,\bullet}$ of $H^p(A^\bullet)$. By the horseshoe lemma applied to the short exact sequence

$$0 \to B^p(A^\bullet) \to Z^p(A^\bullet) \to H^p(A^\bullet) \to 0$$

we obtain an injective resolution $I_Z^{p,\bullet}$ of $Z^p(A^\bullet)$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d, dotted] & 0 \arrow[d, dotted] & \\
0 \arrow[r] & B^p(A^\bullet) \arrow[r] \arrow[d] & I_B^{p,1} \arrow[r] \arrow[d, dotted] & I_B^{p,2} \arrow[r] \arrow[d, dotted] & \cdots \\
0 \arrow[r, dotted] & Z^p(A^\bullet) \arrow[r, dotted] \arrow[d] & I_Z^{p,1} \arrow[r, dotted] \arrow[d, dotted] & I_Z^{p,2} \arrow[r, dotted] \arrow[d, dotted] & \cdots \\
0 \arrow[r] & H^p(A^\bullet) \arrow[r] \arrow[d] & I_H^{p,1} \arrow[r] \arrow[d, dotted] & I_H^{p,2} \arrow[r] \arrow[d, dotted] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Use the horseshoe lemma again, but this time with

$$0 \to Z^p(A^\bullet) \to A^p \to B^{p+1}(A^\bullet) \to 0$$

to construct an injective resolution $I_A^{p,\bullet}$ of A^p for every p. We then define $I^{\bullet,\bullet}$ by $I^{p,\bullet} = I_A^{p,\bullet}$. Vertical differentials are obtained from the individual resolutions $I_A^{p,\bullet}$ while horizontal resolutions are obtained by "passage-through-cohomology"

$$d^p : I_A^{p,\bullet} \twoheadrightarrow I_B^{p+1,\bullet} \hookrightarrow I_Z^{p+1,\bullet} \hookrightarrow I_A^{p+1,\bullet}$$

$\square$

Now we sketch a proof of the Grothendieck spectral sequence (enough so that details may be filled in):

"Proof" of Grothendieck spectral sequence. The spectral sequence is obtained as follows: We take the Cartan-Eilenberg resolution of the complex $F(I^\bullet)$, where $A \to I^\bullet$ is an injective resolution, calling it $I^{\bullet,\bullet}$. Applying the functor G to this resolution, we get a differential graded double complex. This is our 0th page. After taking cohomology twice, we get our desired 2nd page, $E_2^{p,q} \simeq (R^pG \circ R^qF)(A)$.

Proving that this spectral sequence converges is made easier by considering the *hypercohomology* of the injective resolution $A \to I^\bullet$, which is essentially just taking the cohomology of the corresponding Eilenberg-Maclane resolution. We consider the two different directional choices: vertical-then-horizontal vs horizontal-then-vertical. In both cases, since F carries injective objects to G-acyclic objects, we get zeros in appropriate place (for all $q > 0$) and from there apply 10.4.4 to get the desired result. □

As a corollary to the Grothendieck spectral sequence, we obtain the Leray spectral sequence:

Corollary 10.4.8 (Leray spectral sequence)**.** *Let $\pi : Y \to X$ be a morphism of schemes. For any sheaf $\mathcal{F}$ on $Y_{\acute{e}t}$, there is a spectral sequence*

$$H^r(X_{\acute{e}t}, R^s\pi_*\mathcal{F}) \Rightarrow H^{r+s}(Y_{\acute{e}t}, \mathcal{F})$$

Proof. Let $F = \pi_*$, $G = \Gamma(X, -)$, and $A \in \mathbf{Sh}(Y_{\acute{e}t})$. Recall that the pushforward preserves injectives and that $\mathbf{Sh}(X_{\acute{e}t})$ has enough injectives. Seeing that $\Gamma(-, X) \circ \pi_* = \Gamma(-, Y)$, we get from the Grothendieck spectral sequence

$$E_2^{pq} = (R^pG \circ R^qF)(A) \Rightarrow R^{p+q}(G \circ F)(A)$$

that

$$H^r(X_{\acute{e}t}, R^s\pi_*\mathcal{F}) \Rightarrow H^{r+s}(Y_{\acute{e}t}, \mathcal{F})$$

which is exactly the desired result. □

A couple other useful corollaries include

Corollary 10.4.9. *Let X be a scheme (or ringed space), and let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules. Then, there exists a spectral sequence*

$$H^p(X, \mathscr{E}\mathrm{xt}^q_X(\mathcal{F}, \mathcal{G})) \Rightarrow \mathrm{Ext}^{p+q}_X(\mathcal{F}, \mathcal{G})$$

This implies that if $\mathcal{F}$ is locally free of finite-type, $\mathrm{Ext}^p_X(\mathcal{F}, \mathcal{G}) \simeq H^p(X, \mathcal{G} \otimes \mathcal{F}^\vee)$.

Proof. Apply the Grothendieck spectral sequence to $F = \Gamma(X, -)$ and $G = \mathscr{H}\mathrm{om}_X(\mathcal{F}, -)$. Then

$$F \circ G = \Gamma(X, \mathscr{H}\mathrm{om}_X(\mathcal{F}, -)) = \mathrm{Hom}_X(\mathcal{F}, -)$$

□

Corollary 10.4.10. *Let $i : X \hookrightarrow Y$ be a closed immersion of schemes (or ringed spaces). Let $\mathcal{F}, \mathcal{G}$ be two $\mathcal{O}_X$-modules, and let $\mathcal{E}$ be an $\mathcal{O}_Y$-module. Suppose $\mathcal{G}$ is locally free of finite type. Then there exists a spectral sequence*

$$\mathrm{Ext}^p_X(\mathcal{F}, \mathscr{E}xt^q_Y(i^*\mathcal{G}, \mathcal{E})) \Rightarrow \mathrm{Ext}^{p+q}_Y(i^*(\mathcal{G} \otimes \mathcal{F}), \mathcal{E})$$

Proof. Apply the Grothendieck spectral sequence to $F = \mathrm{Hom}_X(\mathcal{F}, -)$ and $G = \mathscr{H}\mathrm{om}_Y(i^*\mathcal{G}, -)$ gives the spectral sequence

$$\begin{aligned} E_2^{p,q} &= \mathrm{Ext}^p_X(\mathcal{F}, \mathscr{E}\mathrm{xt}^q_Y(i^*\mathcal{G}, \mathcal{E})) \\ &\Rightarrow R^{p+q}\mathrm{Hom}_X(\mathcal{F}, \mathscr{H}\mathrm{om}_Y(i^*\mathcal{G}, \mathcal{E})) \\ &= R^{p+q}\mathrm{Hom}_Y(i^*(\mathcal{G} \otimes \mathcal{F}), \mathcal{E}) \\ &= \mathrm{Ext}^{p+q}_Y(i^*(\mathcal{G} \otimes \mathcal{F}, \mathcal{E}) \end{aligned}$$

□

Corollary 10.4.11 (Hochschild-Serre Spectral Sequence)**.** *Let G be a group, $K \subset G$ a normal subgroup and A a left G-module. The group cohomology groups $H^n(G, A)$ form the right-derived functors of the invariant functor $A \mapsto A^G = \{a \in A \mid ga = a, \forall g \in G\}$.*

The invariant can be computed in two stages,

$$A^G = (A^K)^{G/K}$$

The Hochschild-Serre spectral sequence is the Grothendieck spectral sequence for the composition of these invariance functors, with

$$E_2^{p,q} = H^p(G/K, H^q(K, A)) \Rightarrow H^n(G, A)$$

We will repeatedly use this spectral sequence for Galois cohomology and for the action of $\pi_1(X)$ on $\mathcal{O}_X$-modules.

Note. One cool corollary of the Hochschild-Serre spectral sequence is the *inflation-restriction sequence*, which follows immediately from the five term exact sequence

$$0 \to H^1(G/N, A^N) \to H^1(G, A) \to H^1(N, A)^{G/N} \to H^2(G/N, A^N) \to H^2(G, A)$$

This is the mathematical equivalent of using a flamethrower to light a candle, as this follows quite easily from the "continuous cochain" definition for the cohomology of profinite groups. There is one useful thing that we do gain: apply 10.4.4 to find out.

10.4.1 Chain Complexes

We quickly review some facts about chain complexes of sheaves. The formalism of triangulated categories and derived categories will not be necessary, but we will use chain complexes to simplify exposition. We use these facts during the especially during section on the Künneth formula and in other sections sprinkled throughout.

If $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are complexes of some sheaves on some scheme X, we write $\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet$ for the total complex of the double complex $\mathcal{F}^r \otimes \mathcal{G}^s$; more precisely

$$(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet)^m = \sum_{r+s=m} \mathcal{F}^r \otimes \mathcal{G}^s$$

and

$$d^m = \sum (d^r_{\mathcal{F}} \otimes 1) + ((-1)^r \otimes d^s_G)$$

Morphisms between chain complexes must commute in the appropriate ways and degrees must match. A *chain homotopy* between two morphisms $f, g : A \to B$ is a sequence of homomorphisms $h_n : A_n \to B_{n+1}$ such that $hd_A + d_B h = f - g$. Two morphisms which have a chain homotopy relating them are said to be *homotopic*, and they induce the same homomorphisms on (co)homology groups. A chain homomorphism is said to be a *quasi-isomorphism* if it induces isomorphisms in (co)homology.

Lemma 10.4.12. *If the maps $g_1, g_2 : \mathcal{G}^\bullet_1 \to \mathcal{G}^\bullet_2$ are homotopic, then so also are*

$$1 \otimes g_1, 1 \otimes g_2 : \mathcal{F}^\bullet \otimes \mathcal{G}^\bullet_1 \to \mathcal{F}^\bullet \otimes \mathcal{G}^\bullet_2$$

for any complex $\mathcal{F}^\bullet$

Lemma 10.4.13. *Let $\mathcal{G}^\bullet_1 \to \mathcal{G}^\bullet_2$ be a quasi-isomorphism and let $\mathcal{F}^\bullet$ be a bounded complex of flat sheaves. Then $1 \otimes g : \mathcal{F}^\bullet \otimes \mathcal{G}^\bullet_1 \to \mathcal{F}^\bullet \otimes \mathcal{G}^\bullet_2$ is a quasi-isomorphism if either:*

(a) $\mathcal{G}^\bullet_1$ and $\mathcal{G}^\bullet_2$ are bounded above, or

(b) $\mathcal{F}^\bullet$ is bounded below.

Lemma 10.4.14. *If $\mathcal{F}^\bullet$ is a complex of sheaves such that $H^r(\mathcal{F}^\bullet) = 0$ for $r >> 0$, then there exists a quasi-isomorphism $\mathcal{P}^\bullet \to \mathcal{F}^\bullet$ with $\mathcal{P}^\bullet$ a bounded above complex of flat sheaves.*

Lemma 10.4.15. *Let $f : X \to S$ be a proper morphism with S quasi-compact. For any complex of sheaves $\mathcal{F}^\bullet$ on X there is a quasi-isomorphism*

$$\mathcal{F}^\bullet \xrightarrow{\sim} A^\bullet(\mathcal{F}^\bullet)$$

with $A^\bullet(\mathcal{F}^\bullet)$ a complex of f_*-acyclic sheaves.

If $H^r(\mathcal{F}^\bullet) = 0$ for $r << 0$, then $A^\bullet(\mathcal{F}^\bullet)$ may be taken to be a bounded below complex of injectives.

If $\mathcal{F}^\bullet \xrightarrow{\sim} A^\bullet_1$ and $\mathcal{F}^\bullet \xrightarrow{\sim} A^\bullet_1$ with $A^\bullet_1$ and $A^\bullet_2$ complexes of f_*-acyclics, then $f_* A^\bullet_1 \simeq f_* A^\bullet_2$.

IF $\alpha : \mathcal{F}^\bullet_1 \to \mathcal{F}^\bullet_2$ is a map of complexes, it is possible to choose $A^\bullet(\mathcal{F}^\bullet_1)$ and $A^\bullet(\mathcal{F}^\bullet_2)$ such that there is a commutative diagram

$$\begin{array}{ccc} \mathcal{F}_1^\bullet & \xrightarrow{\alpha} & \mathcal{F}_2^\bullet \\ \downarrow \simeq & & \downarrow \simeq \\ A^\bullet(\mathcal{F}_1^\bullet) & \xrightarrow{\beta} & A^\bullet(\mathcal{F}_2^\bullet) \end{array}$$

If α is a quasi-isomorphism, so also is $f_*(\beta)$.

Lemma 10.4.16. *Let $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ be complexes of sheaves and assume that $\mathcal{F}^\bullet$ is flat. Then there is a spectral sequence*

$$E_2^{r,s} = \sum_{i+j=s} \mathrm{Tor}_{-r}^{\Lambda}(H^i(\mathcal{F}^\bullet, H^j(\mathcal{G}^\bullet)) \Rightarrow H^{r+s}(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet)$$

(where Λ is the constant sheaf defined by some finite ring).

Lemma 10.4.17. *Let A be a (Noetherian) ring, and let*

$$M^\bullet \xrightarrow{\phi} L^\bullet \xleftarrow{\pi} N^\bullet$$

be maps of complexes of A-modules with π a quasi-isomorphism. If $M^\bullet$ is perfect, there exists a quasi-isomorphism $\psi : M^\bullet \to N^\bullet$ such that $\pi \circ \psi = \phi$.

*Proof can be found in [**milne**], p.264.*

Lemma 10.4.18. *Let A be a local Artin ring, and let A_0 be some quotient ring of A. We write $M \mapsto M_0$, $\phi \mapsto \phi_0$ for the functor $A_0 \otimes_A -$.*

Let $M^\bullet$ and $N^\bullet$ be perfect complexes of A and A_0-modules respectively, and let $\psi : M_0^\bullet \xrightarrow{\sim} N^\bullet$ be a quasi-isomorphism. Then, there exists a perfect complex $L^\bullet$ of A-modules, a quasi-isomorphism $\phi : M^\bullet \to L^\bullet$, and an isomorphism $L_0^\bullet \cong N^\bullet$ such that the following diagram commutes

$$\begin{tikzcd} M_0^\bullet \arrow[r, "\phi_0"] \arrow[d, "\psi"] & L_0^\bullet \arrow[dl, "\cong"] \\ N^\bullet & \end{tikzcd}$$

*Proof can be found in [**milne**], p.264.*

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