HDU-ITMO Joint Institute
杭州电子科技大学 圣光机联合学院

Algorithms and Data Structures
Laboratory work #3

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Classes plan

1. Previous homework problem #2025 “Line Fighting”
2. Problem #1207 “Median on the Plane”
3. Problem #1604 “Country of Fools”
4. Task for homework

Problem #2025 "Line Fighting"

- Link to the problem's description https://acm.timus.ru/problem.aspx?space=1&num=2025&locale=en

- There can be from 2 to k teams taking part in a competition, and there are n fighters altogether in all the teams. Before the competition starts, the fighters are divided into teams: each fighter becomes a member of exactly one team.
- Two fighters fight each other if they are members of different teams.
- Help the organizers to distribute fighters between teams so as to maximize the number of fights and output this number.

Problem #2025 "Line Fighting"

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- Optimal number of teams
  - Do we always should split fighters on exactly k teams?
- Optimal way to distribute fighters between teams
  - Equal teams, all fighters in one team…
- Case of 2 teams:
  - Fights for one team is $m(n-m)$, where m – number of fighters in the team

Problem #2025 "Line Fighting"

- Many teams
  - $total\ fights = \frac{1}{2}\sum_{i=1}^{k} m_i(n - m_i)$
- Simplify formula

$$total\ fights = \frac{1}{2}\sum_{i=1}^{k}(m_i n - m_i^2) =$$

$$\frac{1}{2}\left(n\sum_{i=1}^{k} m_i - \sum_{i=1}^{k} m_i^2\right) = \frac{1}{2}(n^2 - \sum_{i=1}^{k} m_i^2)$$

- Minimize sum of squares

Problem #2025 "Line Fighting"

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- Let's prove mathematically
- $f(m_1, m_2, m_3, \dots, m_k) = m_1^2 + m_2^2 + m_3^2 + \cdots + m_k^2$
  - Main function, we need to find minimum of this function
- $m_1 + m_2 + m_3 + \cdots + m_k = n$
  - Constraint
- Method of Lagrange multipliers
  - $L(m_1, m_2, m_3, \dots, m_k, \lambda) = f(m_1, m_2, m_3, \dots, m_k) - \lambda g(m_1, m_2, m_3, \dots, m_k)$
  - L – Lagrange function or Lagrangian
  - λ – Lagrange multiplier
  - g(x) – constraint function
- $g(m_1, m_2, m_3, \dots, m_k) = m_1 + m_2 + m_3 + \cdots + m_k - n$
- $L = m_1^2 + m_2^2 + m_3^2 + \cdots + m_k^2 - \lambda(m_1 + m_2 + m_3 + \cdots + m_k - n)$

Problem #2025
"Line Fighting"

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- $L = m_1^2 + m_2^2 + m_3^2 + \cdots + m_k^2 - \lambda(m_1 + m_2 + m_3 + \cdots + m_k - n)$
- Let's take partial derivatives, equate them to zero and solve system relative to λ

$$\frac{\partial L}{\partial m_1} = 2m_1 - \lambda$$
$$\frac{\partial L}{\partial m_2} = 2m_2 - \lambda$$
$$\dots$$
$$\frac{\partial L}{\partial m_k} = 2m_k - \lambda$$

$$\begin{cases} \dfrac{\partial L}{\partial m_1} = 0 \\ \dfrac{\partial L}{\partial m_2} = 0 \\ \dots \\ \dfrac{\partial L}{\partial m_k} = 0 \end{cases}$$

$$\begin{cases} 2m_1 - \lambda = 0 \\ 2m_2 - \lambda = 0 \\ \dots \\ 2m_k - \lambda = 0 \end{cases}$$

$$\begin{cases} m_1 = \dfrac{\lambda}{2} \\ m_2 = \dfrac{\lambda}{2} \\ \dots \\ m_k = \dfrac{\lambda}{2} \end{cases}$$

Problem #2025 "Line Fighting"

- Apply result to constraint equation

$$m_1 + m_2 + m_3 + \cdots + m_k = n$$

$$\frac{\lambda}{2} + \frac{\lambda}{2} + \frac{\lambda}{2} + \cdots + \frac{\lambda}{2} = n$$

$$\left(\sum_{i=1}^{k} \frac{\lambda}{2}\right) = k\frac{\lambda}{2} = n$$

$$\lambda = \frac{2n}{k}$$

- And find stationary points

$$\begin{cases} m_1 = \dfrac{\lambda}{2} = \dfrac{2n}{2k} = \dfrac{n}{k} \\ m_2 = \dfrac{\lambda}{2} = \dfrac{2n}{2k} = \dfrac{n}{k} \\ \quad \cdots \\ m_k = \dfrac{\lambda}{2} = \dfrac{2n}{2k} = \dfrac{n}{k} \end{cases}$$

Problem #2025 "Line Fighting"

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- It is easy to check, that in this point function f() has its minimum
- But what k is optimal?
- Let's apply found optimal values for m_i to the function

$$f(m_1, m_2, m_3, \dots, m_k) = m_1^2 + m_2^2 + m_3^2 + \cdots + m_k^2$$

$$f(k) = \left(\frac{n}{k}\right)^2 + \left(\frac{n}{k}\right)^2 + \left(\frac{n}{k}\right)^2 + \cdots + \left(\frac{n}{k}\right)^2 = \sum_{i=0}^{k} \left(\frac{n}{k}\right)^2 = k\left(\frac{n}{k}\right)^2 = \frac{n^2}{k}$$

Problem #2025 "Line Fighting"

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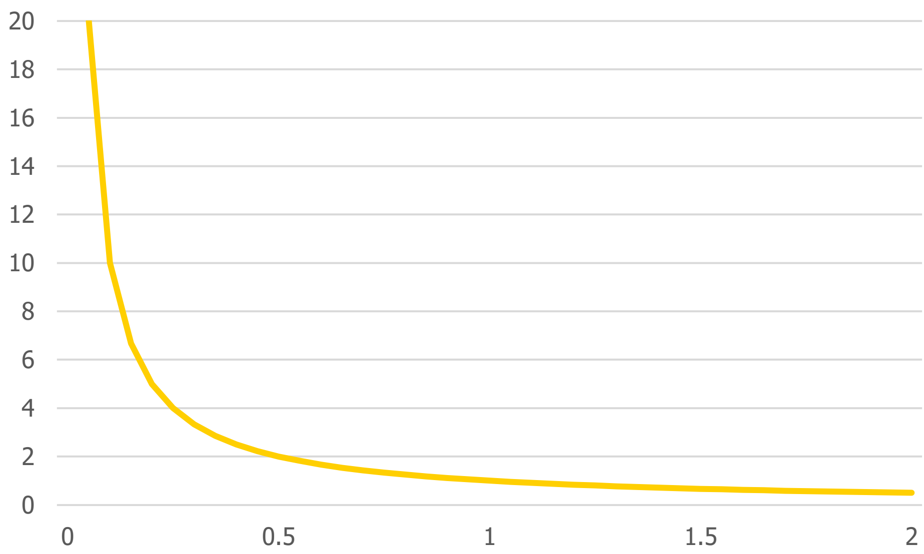
$$f(k) = \frac{n^2}{k}$$

- For k > 0 the plot of this function is decreasing hyperbola



- The value of function is less, when k is greater
- It is better to take maximum available k and split fighters on teams of the same size

Problem #1207 "Median on the Plane"

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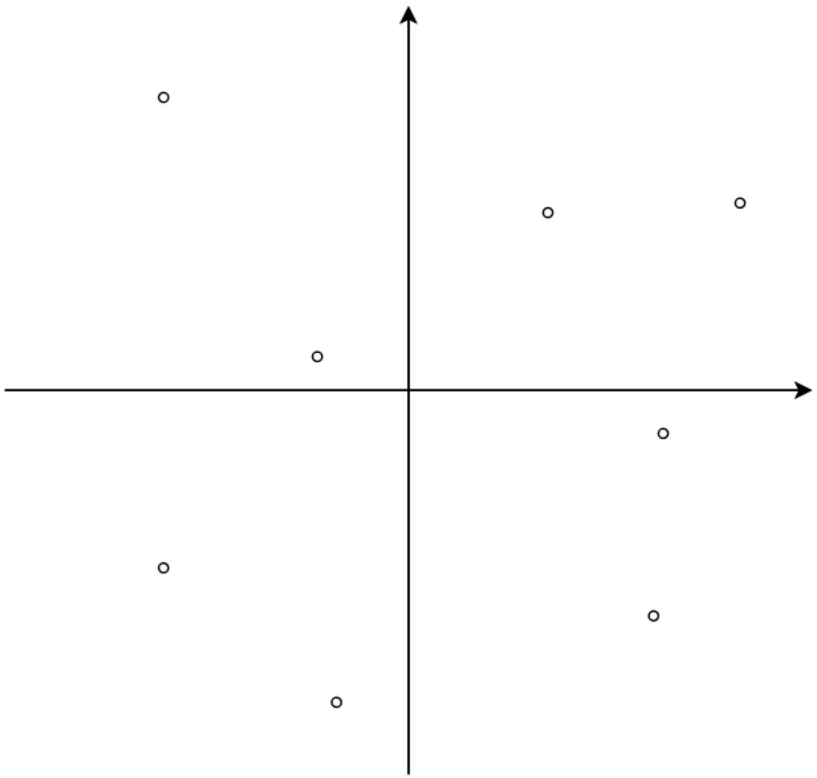
- Link to the problem's description
  https://acm.timus.ru/problem.aspx?space=1&num=1207&locale=en

- There are N points on the plane (N is even). No three points lie on the same straight line. Your task is to select two points in such a way, that straight line they belong to divides the set of points into two equal-sized parts.

Problem #1207 "Median on the Plane"

- Geometric problem
- How to use sorting in this problem?
- Can points be sorted?



Problem #1207 "Median on the Plane"

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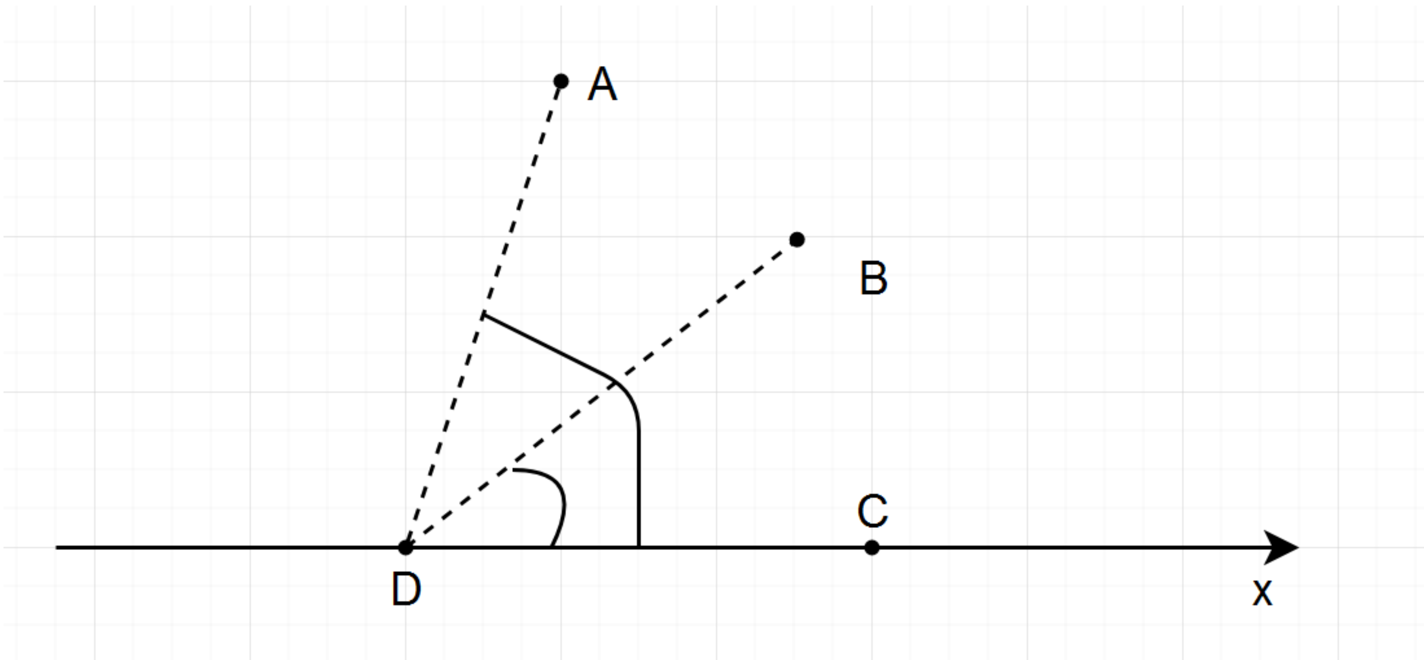
- Angles and trigonometric functions



- Which function to choose?

Problem #1207 "Median on the Plane"

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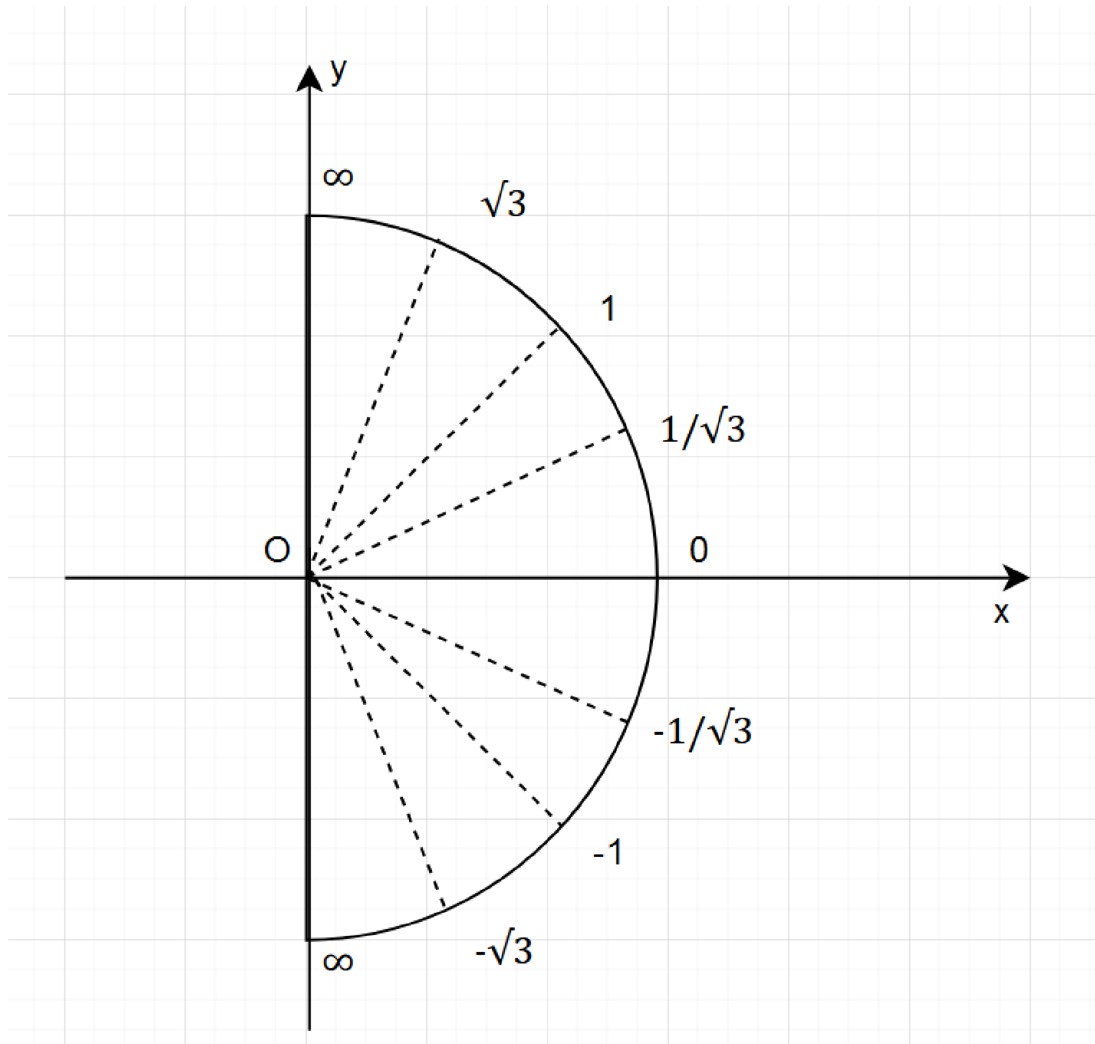
- Continuous function
- Monotonic function
- Tangent (tan) in the right half of coordinate plane!



Problem #1207
"Median on the Plane"

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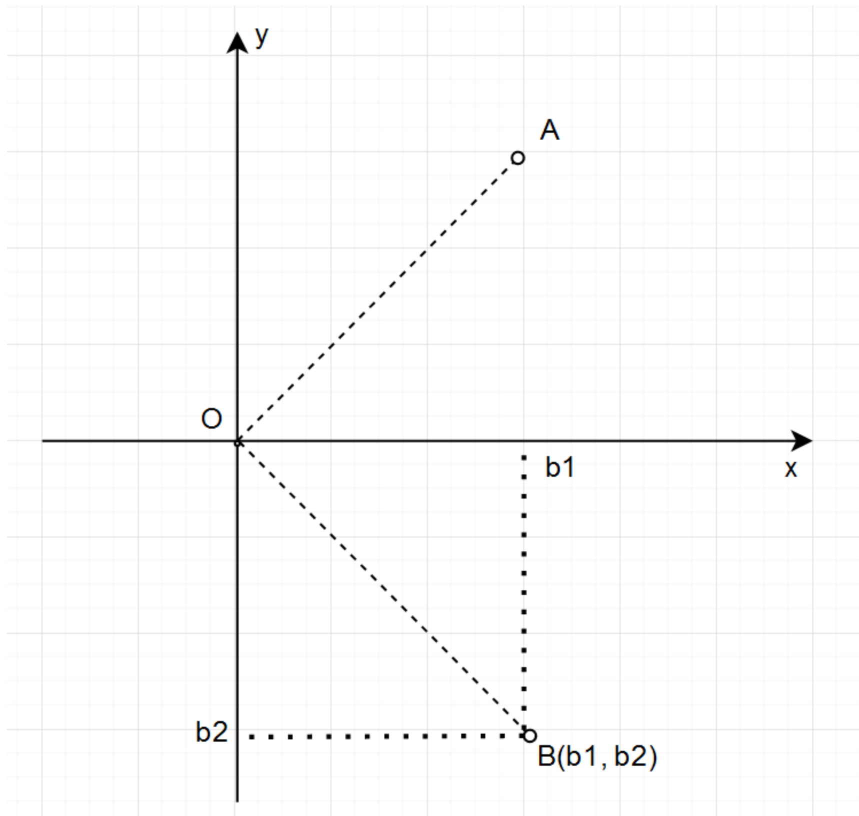
- For point B with coordinates (b_1, b_2) tan of angle between OB and Ox axis can be calculated:
  - $\tan(BOx) = \frac{b_2}{b_1}$ and will be negative
  - For point A it will be positive



Problem #1207 "Median on the Plane"

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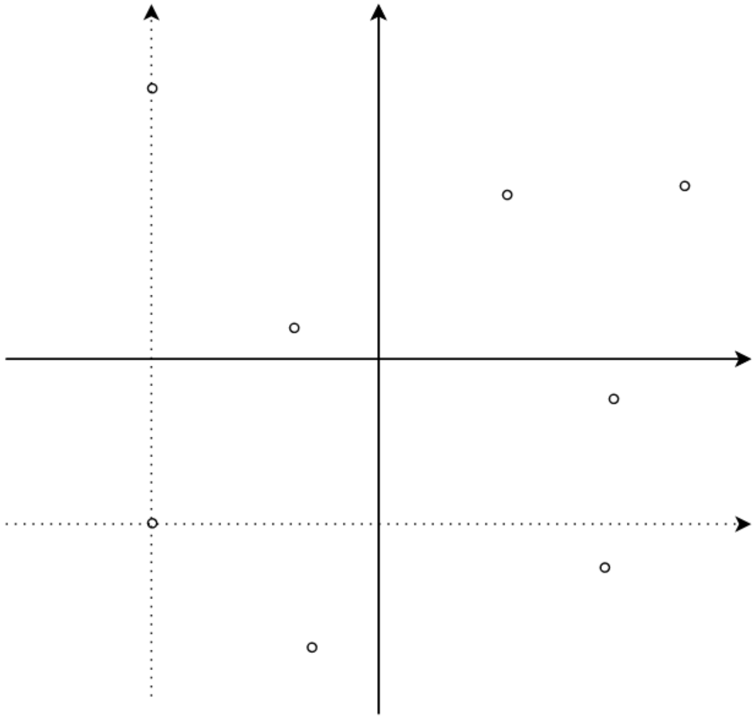
- Issues for using tan
  - No guarantee that all points in the right part of the coordinate square, how to calculate tan?

- Let's "move" coordinate plane center to point with the least abscissa (most "left" point on coordinate plane) by "recalculating" their coordinates



Problem #1207 "Median on the Plane"

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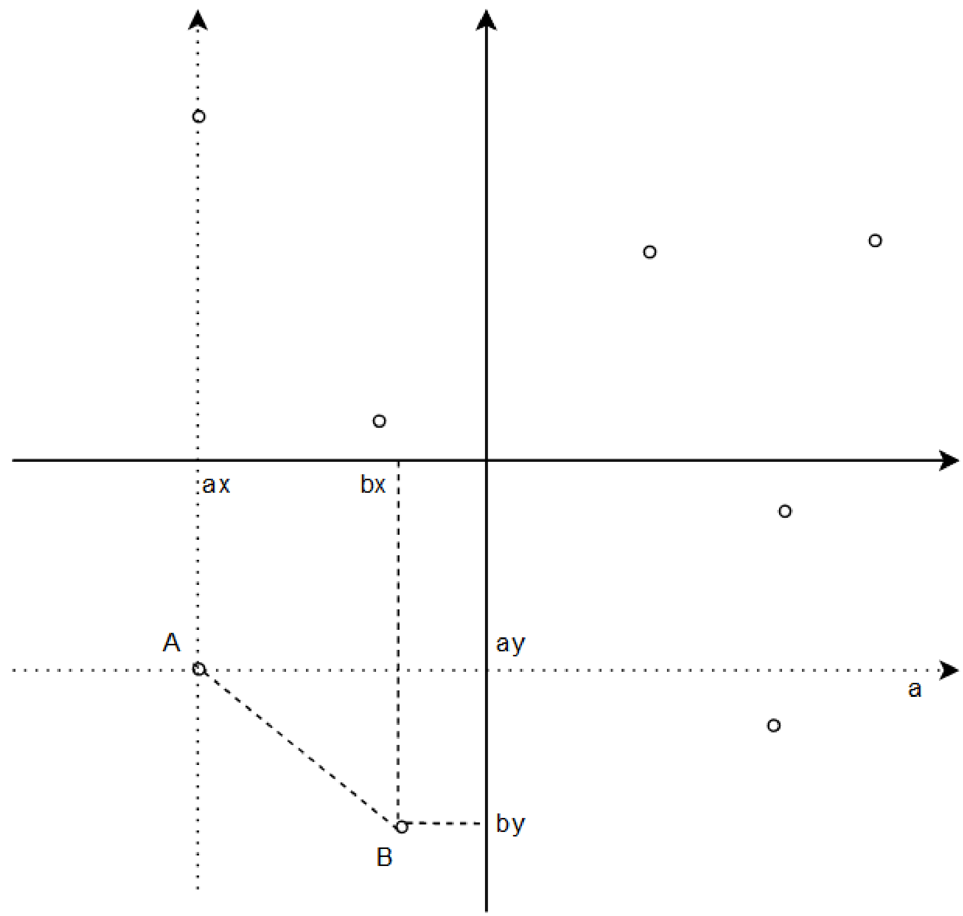
- Another way, is just use relative coordinates to chosen point
- On the image tan for point B (angle between line BA and axe Aa) should be calculated with this formula:

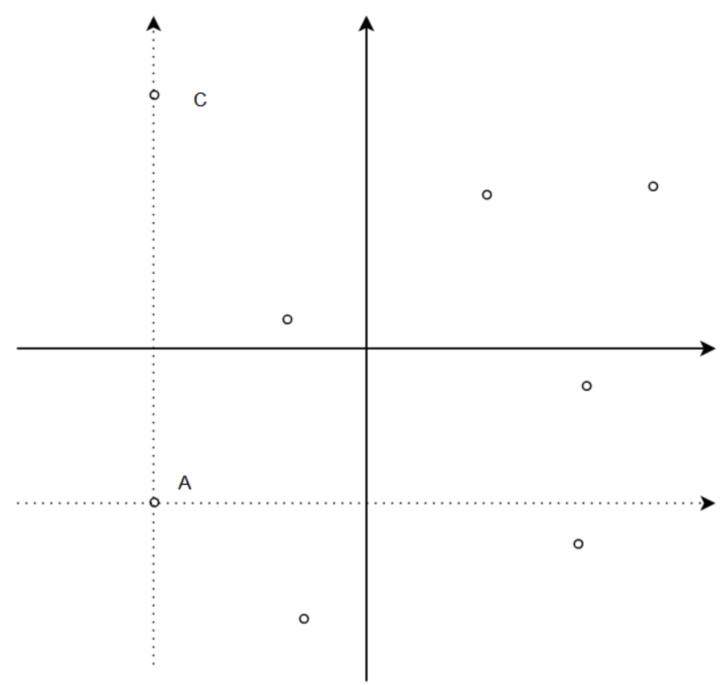
$$\tan(BAa) = \frac{b_y - a_y}{b_x - a_x}$$



Problem #1207 "Median on the Plane"

- Issues for using tan
  - Function tan doesn't exist for several arguments, e.g., for 90-degree angle

- Can be handled manually
- If $b_x - a_x = 0$, you can use specific values (e.g., Infinity or INT_MAX)
- This value should be positive, if $b_y > a_y$; and negative otherwise



Problem #1207 "Median on the Plane"

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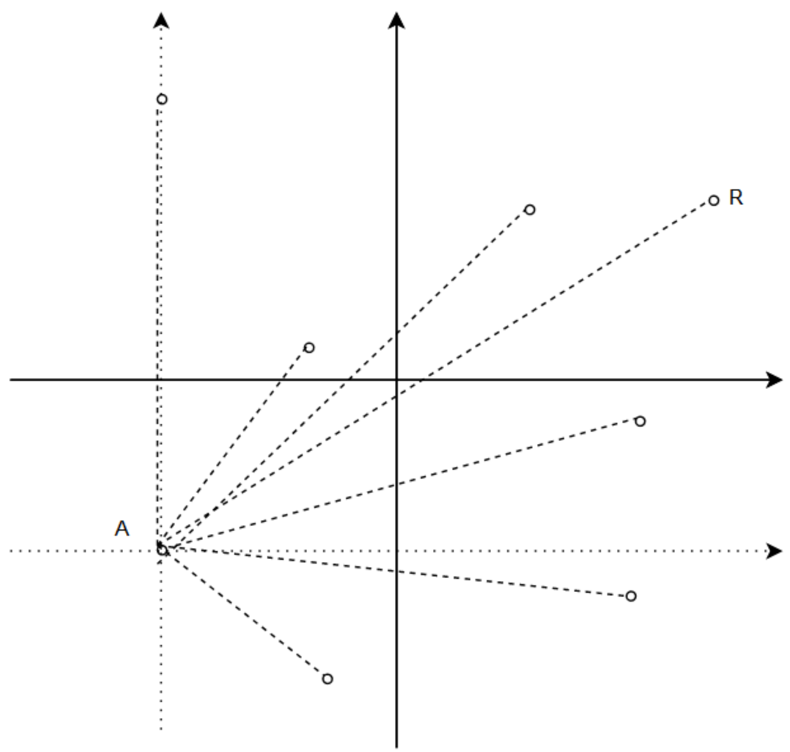
- Next step is sorting all points by value of tan
- The middle point in result of sorting is the answer
- In case of example, this point is R
- This problem also can be solved with other trigonometric and inverse functions (e.g., atan2 for top half of coordinate plane)
- Sorting can be replaced with searching of k^{th} order statistic



Problem #1604 "Country of Fools"

- Link to the problem's description https://acm.timus.ru/problem.aspx?space=1&num=1604&locale=en

- The chief traffic policeman ordered n speed limit signs. When the order arrived, it turned out that the signs had different numbers on them, which showed limits in kilometers per hour. There were k different limits: n_1 signs with the first limit, n_2 signs with the second limit, etc.; $n_1 + \ldots + n_k = n$.
- The chief policeman decided to place the signs on the motorway in such a way that a driver would have to change speed as many times as possible. A speed limitation is valid until the following speed limit sign. For example, if there is the number 60 on the sign, then a car must go until the following sign with the speed of exactly 60 kilometers per hour.

Problem #1604 "Country of Fools"

- Idea 1. Just use all types of signs in a loop

- Perfectly works, when we have equal number of each types
  - For example, 3 signs of 3 types
  - Output is 1-2-3-1-2-3-1-2-3

- Number of third type is greater, than others
  - For example, 2 signs of 2 types and 4 signs of 3rd type
  - Output is 1-2-3-1-2-3-3-3

Problem #1604 "Country of Fools"

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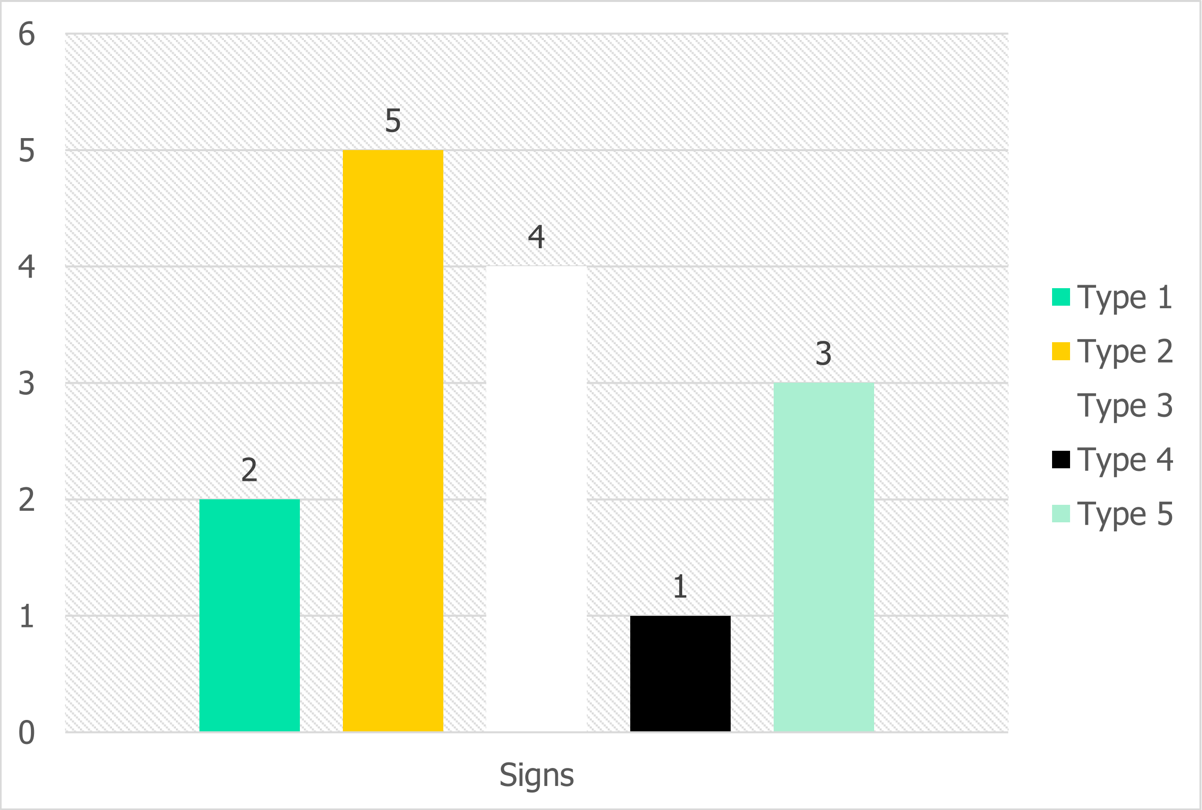
- Idea 2. Let's sort them



Problem #1604 "Country of Fools"

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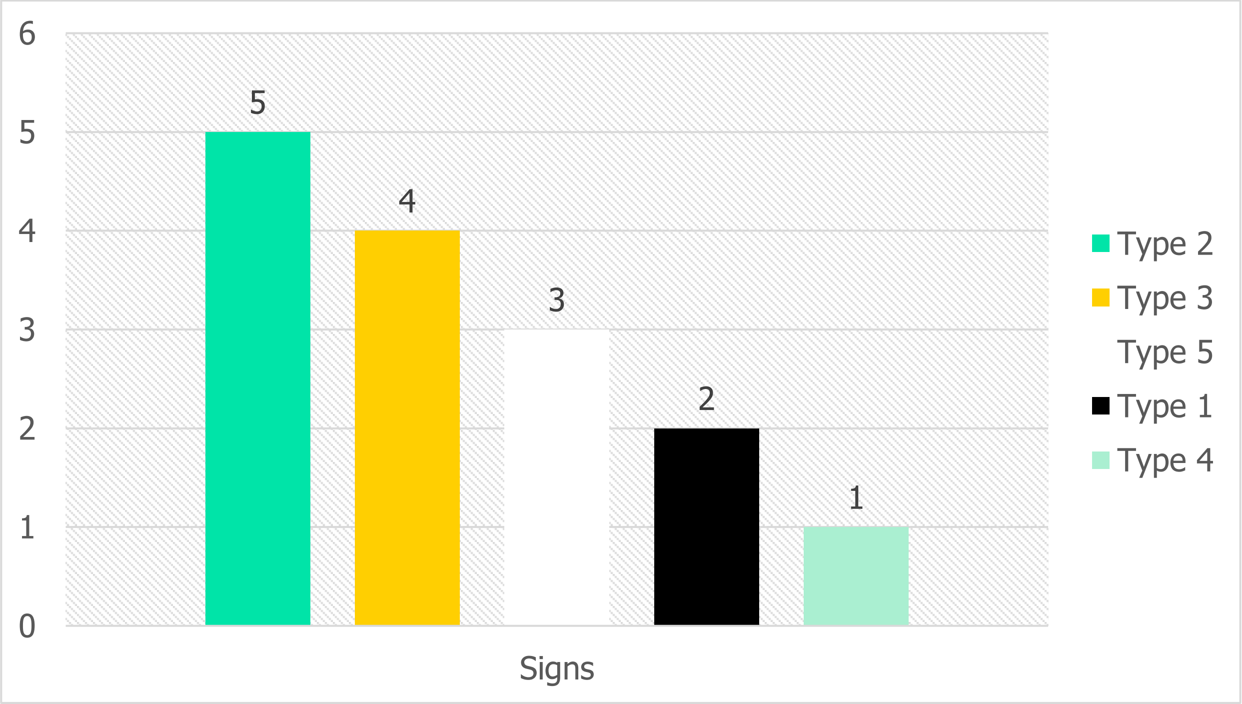
- What to do next?
- What is the main problem of distribution?



Problem #1604 "Country of Fools"

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- Maybe we can rotate type with greatest number of signs and… any other type?
- Let's prove it!
- How good is the input data?
- Metric of current state - number of signs of one type in a row

$$row\ length = max((max\, n_i - (\sum_{i=0}^{k} n_i\ (except\ \ max(n_i) + 1)); 0)$$

Problem #1604 "Country of Fools"

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- Can "row length" become better?
- Can "row length" become worse?
- What if "row length" doesn't change?

- What is behavior of "row length", if we choose on each iteration type with maximum number of signs and any other type?

- Easier to implement special cases of this strategy: on each iteration you can output 2 types with the greater number of signs than other, or rotate types with maximum and minimum number of signs

Problem #1604 "Country of Fools"

- Another solution
- After sorting split all signs of type with maximum amount into buckets
- Example from previous slides
  - Add type 2 as type with maximum amount of numbers

Bucket 1	Bucket 2	Bucket 3	Bucket 4	Bucket 5
2	2	2	2	2

Problem #1604 "Country of Fools"

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- Start from first bucket let split all other types of signs in order of decreasing of their amount
  - Add type 3 as next type with maximum amount

Bucket 1	Bucket 2	Bucket 3	Bucket 4	Bucket 5
2	2	2	2	2
3	3	3	3	

Problem #1604 "Country of Fools"

- Note, that for next type of sign you shouldn't start again from first bucket, but continue from bucket with less signs
  - Add type 5

Bucket 1	Bucket 2	Bucket 3	Bucket 4	Bucket 5
2	2	2	2	2
3	3	3	3	5
5	5			

Problem #1604
"Country of Fools"

- Finally add type 1 and 4
- Each bucket always contains only unique signs – can be output one by one

Bucket 1	Bucket 2	Bucket 3	Bucket 4	Bucket 5
2	2	2	2	2
3	3	3	3	5
5	5	1	1	4

- Please, note, that only one case, when we can't avoid using 2 signs of the same type in a row – when number of this type is greater than sum of all other types

Mandatory task

1. Prepare source code to solve problem #1207 “Median on the Plane”
   https://acm.timus.ru/problem.aspx?space=1&num=1207&locale=en
2. Pass tests on Timus system for this problem
   https://acm.timus.ru/submit.aspx?space=1&num=1207
3. Prepare a report with algorithm complexity and explanation
   Use template.docx to prepare report and send it to hduitmo.ads@yandex.ru with correct subject

Task for homework

You can solve following problems to get extra 2 points for each problem:

1. Problem #1604 "Country of Fools"
   https://acm.timus.ru/problem.aspx?space=1&num=1604&locale=en
   Solution of this problem was already explained
2. Problem #1444 "Elephpotamus"
   https://acm.timus.ru/problem.aspx?space=1&num=1444&locale=en
   N.B. Report for this problem should contain explanation of your solution. What is the difference with problem #1207 "Median on the Plane"?

Thank you!