ECE 269-[redacted] Linear Algebra and Applications, Final (A)

[redacted]

1. (20 points) Prove or disprove each statement:

   (a) Every orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ is diagonalizable over $\mathbb{R}$.

   Solution: False. Counterexample: Consider $Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (or almost any other rotation matrix). Here, the eigenvalues are $\pm i$, which is not diagonalizable over $\mathbb{R}$.
   Interesting fact: the above result holds if the exam asked for diagonalizability over $\mathbb{C}$. But that would be a much harder problem!

   (b) If A is invertible, the singular values of A^{-1} are reciprocals of those of A.

   Solution: True. Let $A = U\Sigma V^*$ be the SVD. Since, A is invertible, U, V, Σ would be invertible square matrix of the same size and $A^{-1} = V\Sigma^{-1}U^*$, which is the SVD of A^{-1}. Hence, singular values are reciprocals of σ_i.

   (c) For a real symmetric matrix, algebraic and geometric multiplicities are equal.

   Solution: True. As shown in the class, real symmetric matrices are diagonalizable (this is aka (the Spectral Theorem), meaning geometric multiplicity equals algebraic multiplicity for all eigenvalues.

   (d) P is positive semi-definite iff all leading principal minors are non-negative.

   Solution: False. Non-negative leading principal minors are necessary but not sufficient. The correct characterization requires all principal minors (not just leading ones) to be non-negative. Counterexample: $P = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ has leading minors 0, 0 but is not PSD.

2. (20 points) Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 0 \end{pmatrix}$.

(a) Compute QR decomposition of A.

Solution: Using Gram-Schmidt on columns a_1, a_2:

$$q_1 = \frac{1}{\sqrt{5}}(1,0,2)^T, \quad \tilde{q}_2 = a_2 - (a_2^T q_1)q_1 = (2,1,0)^T - \frac{2}{\sqrt{5}} q_1.$$

Normalizing $\tilde{q}_2$ gives q_2. Resulting:

$$Q = \begin{pmatrix} 1/\sqrt{5} & 8/\sqrt{105} \\ 0 & 5/\sqrt{105} \\ 2/\sqrt{5} & -4/\sqrt{105} \end{pmatrix}, \quad R = \begin{pmatrix} \sqrt{5} & 2/\sqrt{5} \\ 0 & 21/\sqrt{105} \end{pmatrix}.$$

(b) Find compact SVD of A.

Solution: From $A^T A = \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}$ with eigenvalues 7 and 3. Singular values $\sigma_1 = \sqrt{7}$, $\sigma_2 = \sqrt{3}$. After computations:

$$U = \begin{pmatrix} \frac{3}{\sqrt{14}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{14}} & -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{14}} & \frac{2}{\sqrt{6}} \end{pmatrix} \quad \Sigma = \begin{pmatrix} \sqrt{7} & 0 \\ 0 & \sqrt{3} \end{pmatrix}, \quad V^T = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$$

(c) Find Moore-Penrose inverse $A^\dagger$.

Solution: Using SVD: $A^\dagger = V\Sigma^{-1}U^T = \frac{1}{21}\begin{pmatrix} 1 & -2 & 10 \\ 8 & 5 & -4 \end{pmatrix}$.

(d) Solve $\min_{x \in \mathbb{R}^2} \|Ax - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\|_2$ and find the minimum mean square solution.

Solution: We know that the least square solution would be

$$x^* = A^\dagger \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{21}\begin{pmatrix} 9 \\ 9 \end{pmatrix} = \frac{3}{7}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This results in the mean square error of $\|Ax^* - b\|_2 = 3$.

3. (20 points) Show the following statements:

(a) For an invertible $A \in \mathbb{R}^{n\times n}$ and vectors $u, v \in \mathbb{R}^n$, show that $(A + uv^T)$ is invertible if and only if $1 + v^T A^{-1} u \neq 0$.

Solution: Note that $A + uv^T$ is not invertible, if and only if there exists a non-zero vector x such that $(A + uv^T)x = Ax + uv^T x = 0$. If A is invertible, for such a vector $v^T x \neq 0$ as otherwise, this mean that $Ax = 0$ which implies that A is not invertible.

Due to invertibility of A, $Ax + uv^T x = 0$ iff $x + A^{-1}uv^T x = 0$. If this happens then multiplying both sides (from left) by v^T, we get $v^T x + v^T A^{-1} uv^T x = v^T x(1 + v^T A^{-1} u) = 0$. But $v^T x \neq 0$ implies that $(1 + v^T A^{-1} u) = 0$. Therefore, non-invertibility of $(A + uv^T)$ implies $(1 + v^T A^{-1} u) = 0$. Conversely, if $(1 + v^T A^{-1} u) = 0$. Then, $v \neq 0$ and for any x, we have

$$0 = v^T x(1 + v^T A^{-1} u) = v^T (x + A^{-1} uv^T x) = v^T A^{-1}(A + uv^T)x. \quad (1)$$

Note that if $(A + uv^T)$ is invertible, then so would be $A^{-1}(A + uv^T)$ which renders $v^T A^{-1}(A + uv^T) \neq 0$. But this is in contrast to as $x = (v^T A^{-1}(A + uv^T))^T$ would make the equation non-zero.

(b) If $A, B \in \mathbb{R}^{n\times n}$ are positive semidefinite, show that $A + B = \mathbf{0}$ iff $A = B = \mathbf{0}$, where $\mathbf{0}$ is the zero matrix.

Solution: Suppose that $A \neq 0$. Then, there exists some vector x such that $x^T Ax > 0$. But since $A + B = 0$, this means that

$$0 = x^T (A + B)x = x^T Ax + x^T Bx$$

implying $x^T Bx = -x^T Ax < 0$ which is in contradiction with B being PSD.

(c) For any matrix $A \in \mathbb{R}^{n\times n}$, show $\|A\|_F^2 = \sum_{i=1}^k \sigma_i^2$, where $\sigma_1, \ldots, \sigma_k$ are the singular values of A and $\|\cdot\|_F$ is the Frobenius norm.

Solution: Suppose that the SVD of A be $A = U\Sigma V^*$. Note that

$$\|A\|_F^2 = tr(A^* A) = tr(V\Sigma^2 V^*) = tr(V^* V\Sigma^2) = tr(\Sigma^2) = \sum_{i=1}^k \sigma_i^2. \quad (2)$$

(d) Show that the algebraic multiplicity of an eigenvalue λ for a matrix $A \in \mathbb{R}^{n\times n}$ is invariant (i.e., would be preserved) under similarity transformation.

Solution: Let $B = TAT^{-1}$ for an invertible T. Then

$$c_B(\lambda) = det(\lambda I - B) = det(\lambda I - TAT^{-1}) = det(T(\lambda I - A)T^{-1}) = det(\lambda I - A) = c_A(\lambda). \quad (3)$$

Therefore, A and B would have the same characteristic polynomial and hence, same set of eigenvalues with similar algebraic multiplicities.

4. (20 points) For $\langle x, y\rangle = 2x_1y_1 + 2x_2y_2 + x_3y_3 + x_1y_3 + x_3y_1$ in $\mathbb{R}^3$:

(a) Verify $\langle \cdot, \cdot \rangle$ is an inner product.

Solution: Note that

$$\langle x, y\rangle = x^T \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} y = x^T P x.$$

Therefore,

- **Symmetry**: Due to the above bilinear form driven by a symmetric matrix, it is a symmetric form.
- **Linearity**: The form is linear in both arguments.
- **Positive Definiteness**: $\langle x, x\rangle = x^P x$. But P is positive definite (by invoking the Sylvester criterion) as the three leading principal minors are 2, 4, 2, respectively.

(b) Transform the standard basis into an orthonormal basis using this inner product.

Solution: Following the Gram-Schmidt procedure, you should arrive at:

$$\mathbf{u_1} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{u_2} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \quad \mathbf{u_3} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \sqrt{2} \end{pmatrix}.$$

(c) Project $v = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ onto span $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ using this inner product.

Solution: Let $u = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. The normalized vector along u would be

$$\tilde{u} = \frac{1}{\sqrt{u^T P u}} u = \frac{1}{2} u.$$

As shown in the class, the projection of V on the span of u would

$$v_\pi = \langle v, \tilde{u}\rangle \tilde{u} = \frac{1}{4} u = \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

5. (20 points) Consider the matrix

$$A = \begin{pmatrix} 4 & 6 & 2 \\ 2 & 4 & 2 \\ 4 & 8 & 7 \end{pmatrix}.$$

(a) Find a lower triangular matrix $L \in \mathbb{R}^{3\times 3}$ and an upper triangular matrix $U \in \mathbb{R}^{3\times 3}$ such that $LA = U$.

Solution: We have

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A = \begin{pmatrix} 4 & 6 & 2 \\ 0 & 1 & 1 \\ 4 & 8 & 7 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A = \begin{pmatrix} 4 & 6 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 5 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A = \begin{pmatrix} 4 & 6 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$

Therefore, $L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$ and $U = \begin{pmatrix} 4 & 6 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}$.

(b) Using this find the vector x such that $Ax = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Solution: We need to recursively solve $Ux = L \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \\ -1 \end{pmatrix}$. For this, we recursively have

$$\begin{aligned} x_3 &= -\frac{1}{3} \\ x_2 &= \frac{1}{2} - x_3 = \frac{5}{6} \\ x_1 &= \frac{1}{4}(1 - 6x_2 - 2x_3) = -\frac{5}{6}. \end{aligned}$$

(c) Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ be eigenvalues of A. Find the product $\lambda_1\lambda_2\lambda_3$.

Solution: Note that $det(A) = \lambda_1\lambda_2\lambda_3$. Using $\det(L)\det(A) = \det(U)$ we get $det(A) = \lambda_1\lambda_2\lambda_3 = 12$.

6. (15 points) Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix with the largest eigenvalue λ_1 and $B = \begin{pmatrix} A & v \\ v^T & \alpha \end{pmatrix}$, where $v \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ are arbitrary. Let $\beta_1 \geq \beta_2$ be the two largest eigenvalues of B. Show that

$$\beta_1 \geq \lambda_1 \geq \beta_2.$$

Solution: This is essentially a part of the interlacing property of the eigenvalues for symmetric/Hermitian matrices, that was proved in one of the discussion sessions. For the first inequality, we have

$$\beta_1 = \max_{x\in\mathbb{R}^{n+1}\setminus\{0\}} \frac{x^T Bx}{\|x\|^2} \geq \max_{y\in\mathbb{R}^n\setminus\{0\}} \frac{\begin{bmatrix} y \\ 0 \end{bmatrix}^T \begin{pmatrix} A & v \\ v^T & \alpha \end{pmatrix} \begin{bmatrix} y \\ 0 \end{bmatrix}}{\| \begin{bmatrix} y \\ 0 \end{bmatrix} \|^2} = \max_{y\in\mathbb{R}^n\setminus\{0\}} \frac{y^T Ay}{\|y\|^2} = \lambda_1. \tag{4}$$

For the second inequality, by the Courant-Fischer Theorem, we know that

$$\beta_2 = \min_{\substack{\text{subspace} V \\ \dim(V)=n}} \max_{x\in V\setminus\{0\}} \frac{x^T Bx}{\|x\|^2} \leq \max_{y\in\mathbb{R}^n\setminus\{0\}} \frac{\begin{bmatrix} y \\ 0 \end{bmatrix}^T \begin{pmatrix} A & v \\ v^T & \alpha \end{pmatrix} \begin{bmatrix} y \\ 0 \end{bmatrix}}{\| \begin{bmatrix} y \\ 0 \end{bmatrix} \|^2} = \max_{y\in\mathbb{R}^n\setminus\{0\}} \frac{y^T Ay}{\|y\|^2} = \lambda_1,$$

where the inequality follows by choosing V to be the specific choice $V = \{\begin{bmatrix} y \\ 0 \end{bmatrix} \mid y \in \mathbb{R}^n\}$.