Estimates for Solutions to Equations of p-Laplace type in Ahlfors regular NTA-domains

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Outline of my Talk

1. History of the problem
2. Introduction
3. Statement of results

History of the problem: Harmonic functions

- Consider the Laplace equation and let Ω be a smooth bounded domain. Let $G(\cdot, y)$ be the Green function w.r.t. Ω, extend G outside Ω as 0. Then we obtain the following representation formula via Riesz representation theorem

$$\int \nabla G(x,y) \cdot \nabla \varphi(x) dx = - \int \varphi(x) d\omega^y(x) \quad \varphi \in C_0^\infty(\mathbb{R}^n \setminus \{y\})$$

  where the measure ω^y is referred to as the harmonic measure at y associated to the Laplace operator.
- For short, if $x_0 \in \Omega$ is a fix point, we let $\omega^{x_0} = \omega$.

History of the problem: Non-tangential

Definition

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, then we define for $y \in \partial\Omega$, and $0 < r$

$$\Gamma_b(x) = \{x \in \Omega : b|x - y| < d(x, \partial\Omega)\} \cap B(y, r) \qquad (1)$$

where $0 < b < 1$. We call $\Gamma_b(x)$ the non-tangential approach region with respect to Ω, y, r.

History of the problem: Non-tangential

Definition

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, let $u : \Omega \to \mathbb{R}$. Then we say that u has a non-tangential limit at $y \in \partial\Omega$ if

$$\lim_{x \in \Gamma_b(y)} u(x) \quad \text{exists.} \tag{2}$$

Definition

Let Ω, u, Γ_b be as above, then the non-tangential maximal function $N[u] : \partial\Omega \to \mathbb{R}$ is defined as

$$N[u](y) = \sup_{x \in \Gamma_b(y)} u(x)$$

History of the problem I

Lemma ('Fatou's theorem')

If u is harmonic in the unit ball B, and u is non-tangentially bounded from below on $E \subset \partial B$ i.e.

$$u(x) \leq c \quad \forall x, y : \; y \in E, \; x \in \Gamma_b(y). \tag{3}$$

Then u has finite non-tangential limit a.e. in E with respect to the harmonic measure ω.

Fatou $n = 2$ (1906), Calderón $n > 2$ (Halfplane) (1950), Hunt & Wheeden (1968) (Starlike Lipschitz)

Question

- *Is ω absolutely continuous with respect to σ?*
- *Is σ absolutely continuous with respect to ω?*

History of the problem II

Theorem (Priwalow $n = 2$ (1956), Dahlberg $n \geq 3$ (1977))

Let $D \subset \mathbb{R}^n$ be a Lipschitz domain. Then the harmonic measure ω is mutually absolutely continuous with respect to the surface measure σ.

Question

Can we express the harmonic measure explicitly in terms of the Green function?

Question

Can we define the boundary values of the normal derivative of the Green function?

History of the problem III

Main result

Theorem (Dahlberg (1977))

Let $n \geq 3$, $\Omega \subset \mathbb{R}^n$, Lipschitz domain. Then, the harmonic measure is an A^∞ measure w.r.t. the surface measure

1. *The gradient of the Green function has a finite normal limit a.e. with respect to the surface measure, i.e.*

$$\lim_{t \to 0^+} \frac{\partial G(Q + t n_Q, y)}{\partial n_Q} \quad \textit{exists,}$$

   where n_Q is the normal at $Q \in \partial\Omega$.

2. *The harmonic measure can be expressed in terms of this normal derivative.*
3. *The normal derivative of the Green function, satisfies a reverse Hölder inequality.*

Question

What about solutions u to equations of the type

$$Lu := \nabla \cdot (A(x)\nabla u(x)) = 0$$

where A is a bounded symmetric measurable matrix?

Theorem

The following are equivalent

1. $\omega \in A^\infty(d\sigma)$, ω *is the corresponding L-harmonic measure, to the above equation.*
2. *There exists* $1 < p < \infty$ *s.t. if u is a solution to the classical L-Dirichlet problem with* $f \in C(\partial B)$ *as data, then*
$$\int_{\partial B} [N[u](y)]^p d\sigma(y) \leq \int_{\partial B} f(y)^p d\sigma(y)$$
3. ω *is absolutely continuous w.r.t.* σ *and* $k = \frac{d\omega}{d\sigma}$ *satisfies a reverse Hölder inequality.* $(⨍_\Delta k^q)^{1/q} \leq c ⨍_\Delta k$, *where q is the Hölder conjugate of p.*

Introduction: NTA-domains

Definition

A bounded domain Ω is called non-tangentially accessible (NTA) if there exist $M \geq 1$ and $r_0 > 0$ such that the following are fulfilled:

(i) corkscrew condition: for any $w \in \partial\Omega, 0 < r < r_0$, there exists $a_r(w) \in \Omega$ satisfying $M^{-1}r < |a_r(w) - w| < r$, $d(a_r(w), \partial\Omega) > M^{-1}r$,

(ii) $\mathbb{R}^n \setminus \bar{\Omega}$ satisfies the corkscrew condition,

(iii) uniform condition: if $w \in \partial\Omega, 0 < r < r_0$, and $w_1, w_2 \in B(w, r) \cap \Omega$, then there exists a rectifiable curve $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = w_1, \gamma(1) = w_2$, and such that

  (a) $H^1(\gamma) \leq M\,|w_1 - w_2|$,

  (b) $\min\{H^1(\gamma([0, t])), H^1(\gamma([t, 1]))\} \leq M\,d(\gamma(t), \partial\Omega)$.

Introduction: Ahlfors domains

Given Ω, σ denotes the restriction of $(n-1)$-dimensional Hausdorff measure to $\partial\Omega$.

Definition

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We say that Ω and $\partial\Omega$ are Ahlfors regular provided that there exist $r_0 > 0$, $C \geq 1$, such that

$$C^{-1} \leq \frac{\sigma(\Delta(w,r))}{r^{n-1}} \leq C$$

whenever $w \in \partial\Omega$, $0 < r < r_0$.

Intro: (A, p)-harmonic functions

We require A to be a positively definite, bounded and symmetric matrix, with Hölder continuous entries.

Definition

Let α, $\beta \in (1, \infty)$ and $\gamma \in (0, 1]$ be given. Let $A = A(x) = \{a_{ij}(x)\}$ where $a_{ij}(x) : \mathbb{R}^n \to \mathbb{R}$ for $i, j \in \{1, ..., n\}$. We say that the function A belongs to the class $M^{0,\gamma}(\alpha, \beta)$ if

$$
\begin{aligned}
&(i) &\qquad& \alpha^{-1}|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j,\\
&(ii) && |a_{ij}(x)| \leq \alpha,\\
&(iii) && a_{ij}(x) = a_{ji}(x),\\
&(iv) && |a_{ij}(x) - a_{ij}(y)| \leq \beta|x - y|^{\gamma},
\end{aligned}
$$

hold whenever $x, y \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, and $i, j \in \{1, 2, ..., n\}$.

Intro: (A, p)-harmonic functions

For $u \in W^{1,p}(\Omega)$ to be (A, p)-harmonic we require
$\nabla \cdot ((A(x)\nabla u \cdot \nabla u)^{p/2-1} A(x)\nabla u) = 0$ in Ω, to hold in a weak sense.
Notice $A = I$ gives us the p-Laplace equation.

Definition

Let $p \in (1, \infty)$ and let $A \in M^{0,\gamma}(\alpha, \beta)$ for some (α, β, γ). Given a bounded domain Ω we say that u is (A, p)-harmonic in Ω provided $u \in W^{1,p}(\Omega)$ and

$$\int (A(x)\nabla u(x) \cdot \nabla u(x))^{p/2-1}(A(x)\nabla u(x) \cdot \nabla\theta(x))dx = 0,$$

whenever $\theta \in W_0^{1,p}(\Omega)$. As a short notation for the above equation we write
$\nabla \cdot ((A(x)\nabla u \cdot \nabla u)^{p/2-1} A(x)\nabla u) = 0$ in Ω.

Intro: (A, p)-harmonic functions: further regularity assumptions

We require further regularity, essentially that entries in A are $C^{1,\gamma}(\Omega)$ with bounded gradients.

Definition

Let α, $\hat{\alpha}$, $\beta \in (1, \infty)$ and $\gamma \in (0, 1]$ be given. Let $A = A(x) = \{a_{ij}(x)\}$ where $a_{ij}(x) : \mathbb{R}^n \to \mathbb{R}$ for $i, j \in \{1, ..., n\}$. We say that the function A belongs to the class $M^{1,\gamma}(\alpha, \hat{\alpha}, \beta)$ if $A \in M^{0,\gamma}(\alpha, \beta)$ and

$$
\begin{aligned}
&(i') \qquad && |\nabla a_{ij}(x)| \leq \hat{\alpha}, && (4)\\
&(ii') \qquad && |\nabla a_{ij}(x) - \nabla a_{ij}(y)| \leq \beta |x - y|^{\gamma}, && (5)
\end{aligned}
$$

hold whenever $x, y \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, and $i, j \in \{1, 2, ..., n\}$.

Intro: (A, p)-harmonic measure

Definition

Let u be a positive (A, p)-harmonic function in a bounded domain Ω, vanishing on $B(w, r) \cap \partial\Omega$. If we extend $u \equiv 0$ in $B(w, r) \setminus \Omega$. Then there exists, see [HKM], a unique finite positive Borel measure μ on $\mathbb{R}^n$, with support in $\partial\Omega$, such that

$$\int (A\nabla u \cdot \nabla u)^{p/2-1} A\nabla u \cdot \nabla\varphi \, dx = -\int \varphi \, d\mu$$

whenever $\varphi \in C_0^\infty(B(w, r))$. We call μ the (A, p)-harmonic measure.

Intro: (A, p)-harmonic measure on level surfaces

Using integration by parts we obtain the following explicit representation on a level surface of u i.e. $D_t = \{u = t > 0\}$.

$$\int_{D_t} (A\nabla u \cdot \nabla u)^{p/2-1} A\nabla u \cdot \nabla \varphi dx = -\int_{\partial D_t} \varphi (A\nabla u \cdot \nabla u)^{p/2} |\nabla u|^{-1} d\sigma$$

The conclusion, instead of studying the existance of the normal derivative of the Green's function, we study the existence of the gradient on the boundary (in terms of non-tangential convergence).

Theorem (1, Avelin, Nyström, 2011)

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain with constant N. Let $w \in \partial\Omega$, $0 < r < r_0$, suppose that u is a positive (A, p)-harmonic function in $\Omega \cap B(w, 4r)$, u is continuous in $\bar{\Omega} \cap B(w, 4r)$ and $u = 0$ on $\Delta(w, 4r)$. Then $d\mu \in A^\infty(\Delta(w, 2r), d\sigma)$. Moreover,

$$\lim_{x \in \Gamma(y) \cap B(w,4r), x \to y} \nabla u(x) =: \nabla u(y) \tag{6}$$

exists for σ-almost every $y \in \Delta(w, 4r)$ and for b, $0 < b < 1$, fixed in the definition of $\Gamma(y)$. Also there exist $q > p$ and a constant c, $1 \le c < \infty$, such that

(i) $N(|\nabla u|) \in L^q(\Delta(w, 2r))$,

(ii) $\displaystyle\int_{\Delta(w,2r)} |\nabla u|^q d\sigma \le c r^{(n-1)(\frac{p-1-q}{p-1})} \Big(\int_{\Delta(w,2r)} |\nabla u|^{p-1} d\sigma \Big)^{q/(p-1)}$,

(iii) $\log |\nabla u| \in BMO(\Delta(w, r))$, $\quad \| \log |\nabla u| \|_{BMO(\Delta(w,r))} \le c$,

(iv) $d\mu = (A\nabla u \cdot \nabla u)^{p/2} |\nabla u|^{-1} d\sigma$ *σ-almost everywhere on $\Delta(w, 2r)$.*

Theorem (2, Avelin, Nyström, 2011)

Let $\Omega \subset \mathbb{R}^n$ be an Ahlfors regular NTA-domain with constants C, M, r_0. Then the same statements and conclusions as given in Theorem 1 hold with one change: in this case there exist $q > p - 1$ and a constant c, $1 \leq c < \infty$, such that the conclusions in (i) and (ii) hold. Furthermore, $\Delta(w, 4r)$ has a tangent plane at σ almost every $y \in \Delta(w, 4r)$ and if $n(y)$ denotes the unit normal to this tangent plane pointing into $\Omega \cap B(w, 4r)$, then $\nabla u(y) = |\nabla u(y)| n(y)$.

T1. Proved for *p*-Laplace by J. Lewis and K. Nyström in *Regularity and free boundary regularity for the p Laplacian in Lipschitz and C^1 domains*. Ann. Acad. Sci. Fenn. Math. (2008) vol. 33 (2) pp. 523-548

T2. Proved for *p*-Laplace by J. Lewis and K. Nyström, *Regularity and Free Boundary Regularity for the p-Laplace Operator in Reifenberg Flat and Ahlfors Regular Domains*, in preparation.

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