NONLINEAR PARABOLIC POTENTIAL THEORY:
PARABOLIC CAPACITY

BENNY AVELIN

1. Part I

1.1. **Heat equation.**

$$Hu = u_t - \Delta u = 0.$$

1.1.1. *Thermal capacity.* The following definitions are due to Neil Watson PLMS. -78, [W].

Definition 1.1. A superparabolic function u in $E \subset \mathbb{R}^{n+1}$ is a l.s.c. function satisfying the comparison principle on cylinders and it is finite on a dense subset of E.

Definition 1.2. Consider the set $E \subset \mathbb{R}^{n+1}$ and let $A \subset E$

$$R_A^v = \inf\{u : u \geq v 1_A, u \text{ is superparabolic in } E\},$$

the Réduite or the Reduction of v over A. Usually the function v is a superparabolic function in E, we will mostly be concerned with R_A^1 which can be called the Balayage of A.

Note that R_A^v is sometimes called a hyperparabolic function, and let $\hat{R}_A^v$ denote the l.s.c. regularization of R_A^v, also called the *smooth reduction*. Let $K \subset E$ be a compact set, then via the Riesz representation theorem there exists a measure μ_K such that

$$Hu = \mu_K$$

in E.

Definition 1.3. $\hat{R}_K$ is called the *thermal capacitary potential*, and μ_K is the *thermal capacitary distribution*. Denote

$$C(K) = \mu_K(E)$$

as the *thermal capacity*.

With this at hand we can define the inner capacity for $A \subset E$ and open set

$$C_*(A) = \sup\{C(K); K \subset A\}$$

and the outer capacity for $A \subset E$ an arbitrary set

$$C^*(A) = \inf\{C(O); O \supset A,\ O \text{ open}\} \tag{1.1}$$

Date: February 26, 2015.

1.1.2. *Another definition.* Define capacity w.r.t measures, originally done at about the same time by

[KM] Kaiser, W.; Müller, B. Removable sets for the heat equation, Vestnik Moskov. Univ. Ser. I Mat. Meh. 28 (1973). and

[L] Lanconelli, Sul problema di Dirichlet per l'equazione del calore, Annali di Matematica Pura ed Applicata. Series IV, 1973, for the reference set R^{n+1}.

$$C(A) = \sup\{\mu(E), 0 \le u_\mu \le 1, \text{ in } E, \operatorname{supp}\mu \subset A\} \tag{1.2}$$

coincides with the previous definition at least for compact sets. Note that in the following we will denote instead of E use a space time cylinder of the type $\Omega \times (0,T) = \Omega_T$.

1.1.3. *Wiener criterion.* Let

$$F(x,t) = (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}}$$

then we can define the heat balls as

$$\Omega(z_0, c) = \{z \in \mathbb{R}^{n+1} : F(z_0 - z) > (4\pi c)^{-n/2}\}$$

Lanconelli, 1973 preliminary result.

Theorem 1.4 (Evans and Gariepy ARMA -82, [EG]). *A point $z_0 \in \partial E$ is regular iff*

$$\sum_{i=0}^{\infty} 2^{kn/2} C((E^C \cap \Omega(z_0, 2^{-k})) \setminus \Omega(z_0, 2^{-(k+1)})) = +\infty$$

1.1.4. *Variational capacity.*

$$W = \{v \in L^2(0,T; H_0^1(\Omega)); v_t \in L^2(0,T; H^{-1}(\Omega))\}$$

Smooth functions are dense in W, and we can thus define

$$C_{var}(K, \Omega_T) = \inf\{\|u\|^2_{W(\Omega_T)} : u \ge 1_K; u \in C_0^\infty(\Omega \times \mathbb{R})\}$$

Theorem 1.5 (Pierre, SIAM, -83, [P]). *There exists constants c_1, c_2 such that*

$$c_1 C_{var}(K, \Omega_T) \le C(K) \le c_2 C_{var}(K, \Omega_T)$$

1.2. **Nonlinear heat equation.** Degenerate parabolic p-Laplace, $p > 2$

$$u_t - \nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0.$$

1.2.1. *Nonlinear capacity.* The definition of the nonlinear parabolic capacity was done by Kinnunen, Korte, Kuusi and Parviainen, 2013, Math. Ann. [KKKP]. The definition is same as in (1.2) but now u_μ solves

$$u_t - \nabla \cdot (|\nabla u|^{p-2} \nabla u) = \mu.$$

in Ω_T, we call the capacity C^p. It has much of the good properties of a capacity, see [KKKP], for example it is a Choquet capacity.

1.2.2. *Variational capacity.* It was remarked by Pierre (SIAM, -83, [P]) that the space for

$$V(\Omega_T) = L^p(0,T;W_0^{1,p}(\Omega))$$
$$W_p(\Omega_T) = \{v \in V(\Omega_T), v_t \in V'(\Omega_T)\}$$

would be suitable for a variational capacity for the parabolic p-Laplacian. In a paper by

[DPP] Droniou, J., Porretta, A., and Prignet, A. Pot. An. 2003

$$C_{DPP}(K) = \inf\{\|v\|_{W_p(\Omega_\infty)} : u \geq 1_K; u \in C_0^\infty(\Omega \times \mathbb{R})\}$$

They studied precisely this space and defined a capacity in terms of its norm. With this at hand they could prove certain good qualities, for example that it is a reasonable definition of capacity and that zero sets are removable for the equation. That is there is a unique solution to the bounded Cauchy measure data problem given that the measure does not charge sets of zero capacity and the initial data is in L^1.

We took a slightly different point of view, in that instead of minimizing the norm we will minimize the following anisotropic quantity

$$C_{var}^p(K) = \inf\{\|v\|^p_{V(\Omega_\infty)} + \|v_t\|^{p'}_{V'(\Omega_\infty)} : v \geq 1_K; v \in C_0^\infty(\Omega \times \mathbb{R})\}$$

The downside of the above definition is the same downside as with the variational quantity introduced by Pierre C_{var}, and that it is very hard to prove any capacitary properties for this quantity. Pierre solves it by also studying C_{DPP} for $p = 2$ and then noting that $C_{var} \approx C^2_{DPP}$, but this only works if $p = p' = 2$.

Theorem 1.6 (B.A, T6, M. Parviainen, to appear DCDS-A)**.**

$$C^p_{var}(K) \approx C^p(K).$$

This allows us to estimate the capacity of certain simple sets, but as we will se later the result we prove is actually stronger and allows for more estimates.

The relation to the capacity in C_{DPP} obviously becomes

$$\min\{C_{DPP}(K)^p, C_{DPP}(K)^{p'}\} \leq C^p_{var}(K) \leq \max\{C_{DPP}(K)^p, C_{DPP}(K)^{p'}\}$$

2. Part II

2.1. **Local definitions.** We need to introduce a local version of the variational quantity we had before, as such we define

$$C_{var}(K,\Omega_T) = \inf\{\lambda^2 : \lambda^2 = \|v\|^p_{V(\Omega_{\lambda^{2-p}T})} + \|v_t\|^{p'}_{V'(\Omega_{\lambda^{2-p}T})},$$
$$v \geq 1_K; v \in C_0^\infty(\Omega \times \mathbb{R})\}$$

Let us also introduce an energy based quantity

$$C_{en}(K,\Omega_T) = \inf\Big\{ \sup_{0<t<T} \int_\Omega v(x,t)dx + \int_{\Omega_T} |\nabla v|^p dxdt,$$

$$v \geq 1_K,\ v \text{ is superparabolic}\Big\}$$

We have the following two theorems

Theorem 2.1 (B.A, T6, M. Parviainen). *Let $K \subset \Omega_T$ be a compact set consisting of a finite collection of space time boxes, then*

$$C_p(K, \Omega_T) \approx C_{en}(K, \Omega_T).$$

Theorem 2.2 (B.A, T6, M. Parviainen). *Let $K \subset \Omega_T$ be a compact set consisting of a finite collection of space time boxes, let $\lambda^2 = C_{var}(K, \Omega_T)$ and assume that $K \subset \Omega_{\lambda^{2-p}T}$, then*

$$C_{var}(K, \Omega_T) \approx C_{en}(K, \Omega_T).$$

To conclude the equivalence between the parabolic capacity and the variational capacity we need to consider decreasing sequences of compact sets. The problem is now that we cannot consider the energy capacity in the limit since we do not know how to take a limit of the energy capacity. Thus the best we can do is the equivalence between C_p and C_{var}. Letting $T \to \infty$ together with a convergence result for C_{var} gives our equivalence.

Note that in the paper by Gariepy and Ziemer JDE 1982, [GZ], they show for example that if we consider the simpler variational quantity

$$C_1(K) = \inf\left\{\left[\int_0^T \left[\int_{\mathbb{R}^n} (|\nabla u|^p + |u_t|^{p/2})dx\right]^{q/p} dt\right]^{1/q} : \right.$$
$$\left. v \geq 1_K; v \in C_0^\infty(\Omega \times \mathbb{R})\right\}$$

where $p, q \geq 2$, this is too weak to capture the behavior of the capacity, at least if $n = 2$. What they show more specifically is that if $v(t) : [0,1] \to \mathbb{R}^2$ is Peano's classical space filling curve, then the set $K = \{(v(t), t) : t \in [0,1]\}$ satisfies $C_1(K) > 0$ and $C^2(K) = 0$. Another energy type capacity was studied by Ziemer in JDE 1980, [Z], where the only difference is that they have removed the restriction of superparabolicity, this is also too weak as proved again by [GZ]. This begs the question, what does the L^1 norm of u_t miss that the dual norm of H^{-1} catches? It would be nice to understand this on a deeper level.

2.2. Estimating capacities of explicit sets.

2.2.1. *Curves.*

Theorem 2.3 (B.A, T6, M. Parviainen). *Let $\phi : [t_1, t_2] \to \Omega$ be a Lipschitz curve and let $K \subset \Omega$ be a set with elliptic p-capacity 0, then the set*

$$K_\phi = \{(x + \phi(t), t) : x \in K, t \in [t_1, t_2]\}$$

has parabolic capacity 0, if $2 \leq p \leq n$.

As we mentioned before, regularity is essential, since if we consider a Hölder curve (Peano), then the capacity can be positive.

2.2.2. *Slicing.*

Theorem 2.4 (B.A, T6, M. Parviainen). *Let $K \subset \Omega_\infty$ be a compact set, then*
$$\int_0^\infty C_e(\phi_t(K), \Omega)dt \leq C_{var}(K, \Omega_\infty)$$
where $\pi_t(x, s) = x1_t(s)$.

First done by Watson 1978, [W] for the heat equation.

2.2.3. *Cylinders with proof.*

Theorem 2.5 (B.A, T6, M. Parviainen). *Let $Q_r = B(x_0, r) \times (t_0 - \hat{T}, t_0)$ where $\hat{T} < t_0$ and $Q_{2r} \subset \Omega_T$, then*
$$C_p(\overline{Q}_r, \Omega_\infty) \approx r^n + \hat{T} r^{n-p}$$

Proof. The lower bound follows from Theorem 2.4 and the fact that the disc has capacity approximately r^n. To prove the upper bound, let us do the following construction. Consider the function $u : \Omega \to \mathbb{R}$, $-\Delta_p u = 0$, $u = 1$ on $\overline{B}(x_0, r)$ and $u = 0$ on $\partial\Omega$. Let now $h(x, t)$ solve
$$\begin{cases} h = 0, & \text{on } \partial\Omega \times [t_0, \infty), \\ h = u, & \text{on } t = t_0, x \in \Omega, \\ h_t - \Delta_p h = 0, & \text{in } \Omega \times (t_0, \infty) \end{cases}$$
Now the following function is almost right,
$$v(x, t) \begin{cases} h(x, t), & t \geq t_0 \\ u(x), & t_0 - \hat{T} < t < t_0 \\ 0, & t \leq t_0 - \hat{T} \end{cases}$$
we only need to do this by extending it below the cylinder with a small ϵ and take a limit. Now calculating the energy of v actually gives us the upper bound via our equivalence theorem, Theorem 2.1 and Theorem 2.2 □

2.2.4. *Polar sets.* In his paper
Watson 1978, [W]
he defined a polar set as follows

Definition 2.6. We call a set E polar if there is a superparabolic function u defined in a neighborhood of E such that $u = +\infty$ on E.

However this might not be the most natural definition for $p > 2$. In Lindqvist, Kuusi, Parviainen (preprint, [LKP]) it was defined as backwards limits instead.

So the question remains, what relation is there between the capacity and the points of infinities for superparabolic functions.

With the above definition of polar sets, Watson Proc. London Math. Soc. (3) 37 (1978), proved that polar sets have capacity zero. [W].

2.2.5. *Some open questions.*

- Consider a domain $E \subset \mathbb{R}^{n+1}$, and a set $A \subset \partial E$. Consider two functions g_1, g_2 on ∂E such that they are continuous on $\partial E \setminus A$, does the Perron solution with boundary datum g_1 coincide with the Perron solution with boundary datum g_2?
- Is there a characterization of sets of capacity zero in terms of the sets where the smooth reduction vanishes identically.
- If the set is strictly contained in Ω_T, is polar sets the same as sets of capacity zero. In [KKKP] it was shown that a polar set strictly contained in Ω_T has capacity zero.

References

[DPP] J. Droniou, A. Porretta and A. Prignet. Parabolic capacity and soft measures for nonlinear equations. *Potential Anal.* 19 (2003), no. 2, 99–161.

[EG] L. C. Evans and R. F. Gariepy. Wiener's test for the heat equation. *Arch. Rational Mech. Anal.*, 78 (1982), 293–314.

[GZ] R. Gariepy and W. P. Ziemer. Thermal capacity and boundary regularity. *J. Differential Equations.*, 45 (1982), 374–388.

[KM] V. Kaizer and B Müller. Removable sets for the heat equation. *Vestnik Moskov. Univ. Ser. I Mat. Meh.*, 28 (1973).

[KKKP] J. Kinnunen, R. Korte, T. Kuusi and M. Parviainen, Nonlinear parabolic capacity and polar sets of superparabolic functions. *Math. Ann.* 355 (2013), no. 4, 1349–1381.

[L] E. Lanconelli. Sul problema di Dirichlet per l'equazione del calore. *Ann. Mat. Pura Appl.*, 97 (1973), 83–114.

[LKP] P. Lindqvist and T. Kuusi and M. Parviainen. Shadows of infinities. preprint.

[P] M. Pierre. Parabolic capacity and Sobolev spaces. *SIAM J. Math. Anal.* 14 (1983), no. 3, 522–533.

[S1] L.M.R. Saraiva, Removable singularities and quasilinear parabolic equations. *Proc. London Math. Soc.* (3) 48 (1984), no. 3, 385–400.

[S2] L.M.R. Saraiva, Removable singularities of solutions of degenerate quasilinear equations. *Ann. Mat. Pura Appl.* (4) 141 (1985), 187–221.

[W] N. A. Watson. Thermal capacity. *Proc. London Math. Soc.*, 37 (1978), 342–362.

[Z] W. P. Ziemer. Behavior at the boundary of solutions of quasilinear parabolic equations. *J. Differential Equations.*, 35 (1980), 291–305.