

Indices:

- $u = (1, U)$: unique packing types (used to track demanded packing orders)
- $i = (1, I)$: production orders (decision to produce a product)
- $d = (1, D)$: demand (shipping requirements)
- $j = (1, J)$: line numbers
- $e \in E$ (see below)

Decision Variables:

- $x(i, j) :=$ order i is assigned to line j (binary)
- $s(i) :=$ start time of order i (non-negative real)
- $c(i) :=$ completion time of order i (non-negative real)
- $u(j) :=$ line in use (binary)

Events tracked:

$$E = \{s_1, \dots, s_n, c_1, \dots, c_n\}$$

Decision Variables (OTIF tracking)

- $late(i) :=$ (binary)
- $lateness(i) :=$ (positive integer)

Decision Variables (Workforce tracking)

- $started(i, e) :=$ order i has started before event e (binary)
- $notcompete(i, e) :=$ order i is not complete before event e (binary)
- $workers_{used}(t) :=$ total workers actually working at time t (relaxed to real $[0, W]$)
- $workers_{max} :=$ maximal workers used in any time slot (relaxed to real $[0, W]$)
- $workers_{min} :=$ minimum workers used in any time slot (relaxed to real $[0, W]$)

Decision Variables (WIP tracking)

- $prodbefore(u, d) :=$ number of units of product u produced and ready before demand d ships (non negative integer)
- $prodorder(i, d) :=$ order i is produced before demand d ships (non negative integer)
- $inv(u, d) :=$ number of item type u in stock after fulfilling demand d (non negative integer)
- $ship(d) :=$ shipping time of demand d (non negative real)
- $shipped(d_1, d) :=$ demand d_1 is shipped before or at the same time as demand d

Input parameters:

- $p(u, j) :=$ processing time for item type u on line j (positive real)
- $s(u, v) :=$ setup time for changing from item type u to item type v (positive real)
- $inv_0(u) :=$ initial inventory stock for packing unit type u (positive integer)
- $T_{max} :=$ the planning horizon (positive real)
- $type(i) :=$ unit type of order i

Input parameters (OTIF):

- $due(d) :=$ due date for demand d
- $prodtype(d) :=$ unit type for demand d
- $qty(d) :=$ quantity for demand d
- $priority(i) :=$ priority weight for order i (positive integer, e.g. [1,100])

Constraints

- One assignment: Each order at most assigned to only one line

$$\sum_j x(i,j) \leq 1 \quad \forall i$$

- Processing time

$$c(i) = s(i) + \sum_j p(i,j) * x(i,j) \quad \forall i$$

- Line capacity: No overlap of order on the same line

$$s(k) \geq c(i) + s(i,k) - T_{max} * (3 - x(i,j) - x(k,j) - y(i,k)) \quad \forall i < k \forall j$$

$$s(i) \geq c(k) + s(k,i) - T_{max} * (2 - x(i,j) - x(k,j) + y(i,k)) \quad \forall i < k \forall j$$

- Time horizon

$$s(i) \geq 0 \quad \forall i$$

$$c(i) \leq T_{max} \quad \forall i$$

- Line utilization

$$[u(j) = 1] \Rightarrow \sum_i x(i,j) > 0 \quad \forall j$$

$$[u(j) = 0] \Rightarrow \sum_i x(i,j) = 0 \quad \forall j$$

Constraints (OTIF)

- Lateness

$$\text{lateness}(d) \geq \text{ship}(d) - \text{due}(i) \quad \forall d$$

$$\text{lateness}(d) \geq 0 \quad \forall d$$

- Identify late orders (forces $\text{late}(d) = 1$ if order is late)

$$[\text{late}(d) = 1] \Rightarrow [\text{lateness} > 0]$$

$$[\text{late}(d) = 0] \Rightarrow [\text{lateness} = 0]$$

Constraints (Demand)

- Tracking produced items for demand

$$prodbefore(u, d) = \sum_{i: type(i)=u} prodorder(i, d)$$
$$[prodorder(i, d) = 1] \Rightarrow [c(i) \leq ship(d) \wedge x(i, j) = 1]$$
$$[prodorder(i, d) = 0] \Rightarrow [c(i) \geq ship(d) + \epsilon]$$

for small ϵ .

- Each order ships exactly once

$$ship(d) \leq T_{max} \quad \forall d$$

- Ship no earlier than due time

$$ship(d) \geq due(d) \quad \forall d$$

Constraints (WIP)

- Inventory balance equation

$$inv(u, d) = inv_0(u) + prodbefore(u, d) - \sum_{d_1: prodtype(d_1)=u} shipped(d_1, d) qty(d_1) \quad \forall u \forall d$$

$$inv(u, d) \geq 0 \quad \forall u \forall d$$

$$[shipped(d_1, d) = 1] \Rightarrow [ship(d_1) \leq ship(d)]$$

$$[shipped(d_1, d) = 0] \Rightarrow [ship(d_1) > ship(d)]$$

Constraints (Workforce)

- Active workers per time slot

$$\text{workers}_{\text{used}}(e) = \sum_i \text{active}(i, e) \quad \forall e$$

$$\text{active}(i, e) \leq \text{started}(i, e)$$

$$\text{active}(i, e) \leq \text{notcomplete}(i, e)$$

$$\text{active}(i, e) \geq \text{started}(i, e) + \text{notcomplete}(i, e) - 1$$

This is $\text{active}(i, e) = \text{started}(i, e) \wedge \text{notcomplete}(i, e)$

$$[\text{started}(i, e) = 1] \Rightarrow [s(i) \leq t(e)]$$

$$[\text{started}(i, e) = 0] \Rightarrow [s(i) > t(e)]$$

$$[\text{notcomplete}(i, e) = 1] \Rightarrow [c(i) \leq t(e)]$$

$$[\text{notcomplete}(i, e) = 0] \Rightarrow [c(i) > t(e)]$$

- Maximum workforce

$$\text{workers}_{\text{max}} \geq \text{workers}_{\text{used}}(e) \quad \forall e$$

- Minimum workforce

$$\text{workers}_{\text{used}}(e) \geq \text{workers}_{\text{min}}$$

- Workforce range

$$\text{workforce}_{\text{range}} = \text{workers}_{\text{max}} - \text{workers}_{\text{min}}$$

Objective function

Minimization objective function:

$$f = \alpha * otif + \beta * wip_{obj} + \gamma * workforce + \delta * total_{not_utilized}$$

where

- OTIF term (heavy penalty for late order but also penalizes according to lateness)

$$otif = \sum_i priority(i) * (7 * late(i) + 3 * lateness(i))$$

- WIP term

$$wip_{obj} = \sum_u \sum_d inv(u, d)$$

- Workforce term (reduces worker variance, deviations from target workforce and changes from one time instant to another)

$$workforce = workers_{range}$$

- Number of lines fully utilized

$$total_{not_utilized} = \sum_j u(j)$$