Indicies:

- $i = (1, n)$: $packing\ orders$
- $j = (1, m)$: $line\ numbers$
- $t = (1, T)$: $time\ slots, given\ in\ terms\ of\ a\ base\ time\ interval\ \Delta T\ e.g. 15\ minutes$

Decision Variables:

- $x(i, j, t, ) \coloneqq packing\ order\ i\ starts\ on\ line\ j\ at\ time\ t$ (binary)
- $y(i, k, j) \coloneqq setup\ between\ orders\ i\ and\ k\ on\ line\ j$ (binary)
- $b(i, k) \coloneqq batch\ indicator, order\ i\ and\ k\ batched\ together$ (binary)
- $prod(i, t) \coloneqq number\ of\ units\ produced\ at\ time\ t$ (positive integer)
- $u(j) \coloneqq line\ in\ use$ (binary)

Decision Variables (OTIF tracking)

- $late(i) \coloneqq$ (binary)
- $lateness(i) \coloneqq$ (positive integer)

Decision Variables (Workforce tracking)

- $workers_{used}(t) \coloneqq total\ workers\ actually\ working\ at\ time\ t$ (integer $[0, W]$ )
- $\mathrm{workers_{max}} \coloneqq maximal\ workers\ used\ in\ any\ time\ slot$ (integer $[0, W]$ )
- $\mathrm{workers}_{min} \coloneqq minimum\ workers\ used\ in\ any\ time\ slot$ (integer $[0, W]$ )

Decision Variables (WIP tracking)

- $inv(i) \coloneqq number\ of\ packing\ order\ i\ in\ stock$ (positive integer)
- $ship(i, t) \coloneqq packing\ order\ i\ is\ shipped\ at\ time\ t$ (binary)
- $time_{ship}(i) \coloneqq time\ slot\ when\ order\ i\ is\ shipped$(integer $[0, T]$)
- $has_{inv}(i, t) \coloneqq order\ i\ has\ available\ inventory\ at\ time\ t$ (binary)
- $time_{start}(i) \coloneqq actual\ start\ time\ for\ packing\ order\ i$ (integer $[1, T]$ )
- $time_{completion}(i) \coloneqq actual\ completion\ time\ for\ packing\ order\ i$ (integer $[1, T]$ )
- $time_{flow}(i) \coloneqq time\ from\ start\ to\ completion\ for\ packing\ order\ i$ (non-negative integer)

Input parameters:

- $p(i,j) \coloneqq processing\ time\ for\ packing\ order\ i\ on\ line\ j$ (positive real, dividable by base time interval)
- $s(i,k,j) \coloneqq setup\ time\ for\ packing\ order\ j\ and\ k\ on\ line\ j$ (positive real , dividable by base time interval)
- $a(w,t) \coloneqq worker\ w\ is\ available\ at\ time\ t$ (binary)
- $inv_0(i) \coloneqq initial\ inventory\ stock\ for\ packing\ order\ i$ (positive integer)
- $\alpha \coloneqq reserved\ capacity$ (positive real less than 1)

Input parameters (OTIF):

- $due(i) \coloneqq due\ date\ for\ packing\ order\ i$ (integer $[1,T]$)
- $priority(i) \coloneqq priority\ weight\ for\ order\ i$ (positive integer, e.g. $[1{,}100]$)

# Constraints

- One assignment: Each order assigned to only one line, one time

$$\sum_{j}\sum_{t} x(i,j,t) = 1 \quad \forall i$$

- Line capacity: No overlap of order on the same line

$$\sum_{i}\ \sum_{t\le\tau<t+p(i,j)+\sum_k y(i,k,j)*s(i,k,j)*(1-b(i,k))} x(i,j,t) \le u(j) \quad \forall j, \forall \tau$$

- Reserved line capacity

$$\sum_{i}\sum_{j}\sum_{\tau}\ \sum_{t\le\tau<t+p(i,j)+\sum_k y(i,k,j)*s(i,k,j)*(1-b(i,k))} x(i,j,t) < (1-\alpha) * m * T$$

# Constraints

- Line in use

$$\sum_{i} \sum_{t \leq \tau < t+p(i,j)} x(i,j,t) \leq u(j) \quad \forall \tau$$

- Each order ships exactly once

$$\sum_{t} ship(i,t) = 1 \quad \forall i$$

- Shipping time

$$time_{ship}(i) = \sum_{t} t * ship(i,t) \quad \forall i$$

# Constraints (OTIF)

- Start time

$$\text{time}_{\text{start}}(i) = \sum_j \sum_t t \, * x(i,j,t) \quad \forall i$$

- Completion time

$$\text{time}_{complete}(i) = \sum_j \sum_t (t + p(i,j)) \, * x(i,j,t) \quad \forall i$$

- Lateness

$$lateness(i) \geq time_{complete}(i) - due(i) \quad \forall i$$
$$lateness(i) \geq 0 \quad \forall i$$

- Identify late orders (forces $late(i) = 1$ if order is late)

$$lateness(i) \leq T * late(i) \quad \forall i$$
$$time_{complete}(i) > due(i) - T\big(1 - late(i)\big) \quad \forall i$$

# Constraints (Shipping)

- Cannot ship before order complete

$$time_{ship}(i) \geq time_{complete}(i) \quad \forall i$$

- Has inventory

$$inv(i,t) \geq due(i,t) - M * \left(1 - has_{inv}(i,t)\right) \ \forall i \ \forall t$$
$$inv(i,t) \leq M * has_{inv}(i,t) \quad \forall i \ \forall t$$

- Can only ship when inventory is sufficient

$$ship(i,t) \leq has_{inv}(i,t) \quad \forall i \ \forall t$$

# Constraints (WIP)

- Flow time

$$time_{flow}(i) = time_{ship}(i) - time_{start}(i) \quad \forall \mathrm{i}$$

- Number of packing orders produced

$$prod(i,t) = \sum_{j} x(i,j,t - p(i,j))$$

- Inventory balance equation

$$inv(i,t) = inv(i,t-1) + prod(i,t) - ship(i,t) \quad \forall i, \forall t > 0$$
$$inv(i,0) = inv_0(i) \quad \forall i$$

# Constraints (Workforce)

- Active workers per time slot

$$\mathrm{workers}_{\mathrm{used}}(\tau) = \sum_{i} \sum_{j} \sum_{t \le \tau < t + p(i,j)} x(i,j,t) \;\; \forall \tau$$

- Maximum workforce

$$workers_{max} \ge workers_{used}(t) \;\; \forall t$$

- Minium workforce

$$\mathrm{workers}_{\mathrm{used}}(\mathrm{t}) \ge workers_{min}$$

- Workforce range

$$workforce_{range} = \mathrm{workers}_{\mathrm{max}} - \mathrm{workers}_{\mathrm{min}}$$

# Objective function

Minimization objective function:

$$f = \alpha * otif + \beta * wip_{obj} + \gamma * workforce + \delta * total_{not_{utilized}}$$

where

- OTIF term (heavy penalty for late order but also penalizes according to lateness)

$$\text{otif} = \sum_i priority(i) * (7 * late(i) + 3 * lateness(i))$$

- WIP term

$$wip_{obj} = \sum_t inv(t)$$

- Workforce term (reduces worker variance, deviations from target workforce and changes from one time instant to another)

$$workforce = workers_{range}$$

- Number of lines fully utilized

$$total_{not\_utilized} = \sum_j u(j)$$