AME 535a Project 2.

Problem 1.

1.1 First, use central difference to the term U_x.

$$U_j = U_0$$
$$U_{j-1} = U_0 - \Delta x u' + \frac{\Delta x^2}{2} u'' - \frac{\Delta x^3}{6} u''' + O(\Delta x^4)$$
$$U_{j+1} = U_0 + \Delta x u' + \frac{\Delta x^2}{2} u'' + \frac{\Delta x^3}{6} u''' + O(\Delta x^4)$$

use method of undetermined coefficients

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} \delta'_{-1} \\ \delta'_0 \\ \delta'_1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} \quad \text{Thus } U_x = \frac{U_{j+1} - U_{j-1}}{2\Delta x}.$$

use central difference to term U_{xxx}.

$$U_{j-2} = U_0 - 2\Delta x u' + \frac{4\Delta x^2}{2} u'' - \frac{8\Delta x^3}{6} u''' + \frac{16\Delta x^4}{24} u'''' + \frac{32\Delta x^5}{120} u''''' + O(\Delta x^6)$$
$$U_{j-1} = U_0 - \Delta x u' + \frac{\Delta x^2}{2} u'' - \frac{\Delta x^3}{6} u''' + \frac{\Delta x^4}{24} u'''' + \frac{\Delta x^5}{120} u''''' + O(\Delta x^6)$$
$$U_j = U_0$$
$$U_{j+1} = U_0 + \Delta x u' + \frac{\Delta x^2}{2} u'' + \frac{\Delta x^3}{6} u''' + \frac{\Delta x^4}{24} u'''' + \frac{\Delta x^5}{120} u''''' + O(\Delta x^6)$$
$$U_{j+2} = U_0 + 2\Delta x u' + \frac{4\Delta x^2}{2} u'' + \frac{8\Delta x^3}{6} u''' + \frac{16\Delta x^4}{24} u'''' + \frac{32\Delta x^7}{120} u''''' + O(\Delta x^6)$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ -2 & -1 & 0 & 1 & 2 & 0 \\ 2 & \frac{1}{2} & 0 & \frac{1}{2} & 2 & 0 \\ -\frac{4}{3} & -\frac{1}{6} & 0 & \frac{1}{6} & \frac{4}{3} & 1 \\ \frac{2}{3} & \frac{1}{24} & 0 & \frac{1}{24} & \frac{2}{3} & 0 \end{array}\right] \Rightarrow \begin{bmatrix} \delta^3_{-2} \\ \delta^3_{-1} \\ \delta^3_0 \\ \delta^3_1 \\ \delta^3_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ -1 \\ \frac{1}{2} \end{bmatrix}$$

Therefore semi discretized equation is

$$\frac{\partial \bar{U}_j}{\partial t} = 6\bar{U}_j \cdot \frac{\bar{U}_{j+1} - \bar{U}_{j-1}}{2\Delta x} - \frac{-\bar{U}_{j-2} + 2\bar{U}_{j-1} - 2\bar{U}_{j+1} + \bar{U}_{j+2}}{2\Delta x^3}$$

1.2 for $\frac{\partial u}{\partial t} = -u_{xxx}$ $\xrightarrow[\text{discretized}]{\text{semi}}$ $\frac{\partial \bar{u}_j}{\partial t} = -\frac{-\bar{u}_{j-2}+2\bar{u}_{j-1}-2\bar{u}_{j+1}+\bar{u}_{j+2}}{2\Delta x^3}$

Use Fourier Analysis.

$$\frac{\partial}{\partial t}\sum \bar{U}_\theta \cancel{e^{ij\theta}} = -\sum \frac{-U_\theta \cancel{e^{ij\theta}} e^{-2i\theta} + 2U_\theta \cancel{e^{ij\theta}} e^{-i\theta} - 2U_\theta \cancel{e^{ij\theta}} e^{i\theta} + U_\theta \cancel{e^{ij\theta}} e^{2i\theta}}{2\Delta x^3}$$

then $-e^{-2i\theta}+2e^{-i\theta}-2e^{i\theta}+e^{2i\theta}$

$= -(\cos 2\theta - i\sin 2\theta) + 2(\cos\theta - i\sin\theta) - 2(\cos\theta + i\sin\theta) + \cos 2\theta + i\sin 2\theta$

$= -\cancel{\cos 2\theta} + i\sin 2\theta + \cancel{2\cos\theta} - 2i\sin\theta - \cancel{2\cos\theta} - 2i\sin\theta + \cancel{\cos 2\theta} + i\sin 2\theta$

$= 2i\sin 2\theta - 4i\sin\theta$

$= 4i\sin\theta\cos\theta - 4i\sin\theta$

$= 4i\sin\theta(\cos\theta - 1)$

then semi-discretized equation becomes

$$\frac{d\bar{U}_\theta}{dt} = \frac{-\cancel{4}^{2} i\sin\theta(\cos\theta-1)}{\cancel{2}\Delta x^3}\bar{U}_\theta$$

$$\Rightarrow \lambda_\theta = -\frac{2\sin\theta(\cos\theta-1)}{\Delta x^3}\cdot i$$

then $z_\theta = \Delta t\cdot\lambda_\theta = -\frac{\Delta t\cdot 2\sin\theta(\cos\theta-1)}{\Delta x^3}\cdot i$

$|z_\theta| \le 2\sqrt{2}\, i \Rightarrow \left|\cancel{-}\frac{\Delta t\, 2\sin\theta(\cos\theta-1)}{\Delta x^3}\cancel{i}\right| \le 2\sqrt{2}\cancel{i}$ set $C = \frac{\Delta t}{\Delta x^3}$.

$\Rightarrow |C\cdot 2\sin\theta(\cos\theta-1)| \le 2\sqrt{2} \Rightarrow C^2\cdot\cancel{4}\sin^2\theta(\cos\theta-1)^2 \le \cancel{8}\,2$

$C^2 \le \frac{2}{\sin^2\theta(\cos\theta-1)^2}$ let $f(\theta) = \sin^2\theta(\cos\theta-1)^2$ then $C^2 \le \left|\frac{2}{f(\theta)}\right|_{min} = \frac{2}{|f(\theta)|_{max}}$.

for $f(\theta)$ $f'(\theta) = 2\sin\theta\cdot\cos\theta(\cos\theta-1)^2 + \sin^2\theta\cdot 2(\cos\theta-1)\cdot -\sin\theta$

$= 2\sin\theta(\cos\theta-1)[\cos\theta(\cos\theta-1) - \sin^2\theta]$

$= 2\sin\theta(\cos\theta-1)(\cos 2\theta - \cos\theta)$

let $f'(\theta)=0 \Rightarrow \theta_1 = 0$ $\theta_2 = 120°$ $f(\theta_1) = 0$ $f(\theta_2) = (-\frac{\sqrt{3}}{2})^2(-\frac{1}{2}-1)^2 = \frac{27}{16}$

Thus $C^2 \le \frac{2}{\frac{27}{16}} = \frac{32}{27} \approx 1.185$ $\Rightarrow \frac{\Delta t}{\Delta x^3} \le 1.08866$.

1.3. chose $\frac{\Delta t}{\Delta x^3} = 1$ which satisfies stability then $\Delta t = 0.001$

See attached matlab program for detail

Comments for each case: (plots are attached below)

a. The soliton moves towards initial speed direction, merely changing shape and with small decay.

b. the initial wave split into two peaks and move toward opposite direction. ~~also note that~~.

c. the initial wave split into two peaks moving toward opposite direction one of which move slowly and another one moves relatively further away.

d. two exact solution superposed, the shape of the wave doesn't change too much, but we can see a another low peak following the high peak.

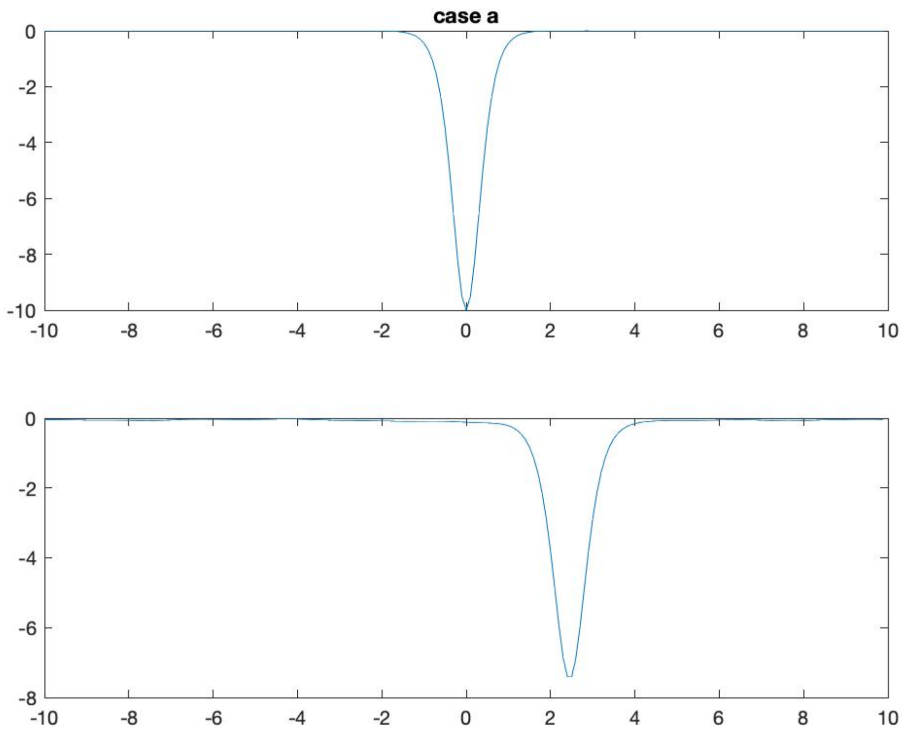
e. two soliton moves toward each other with small decay and change of shape, when they met, the magnitude sum up. and we keep their magnitude respectively after apart.

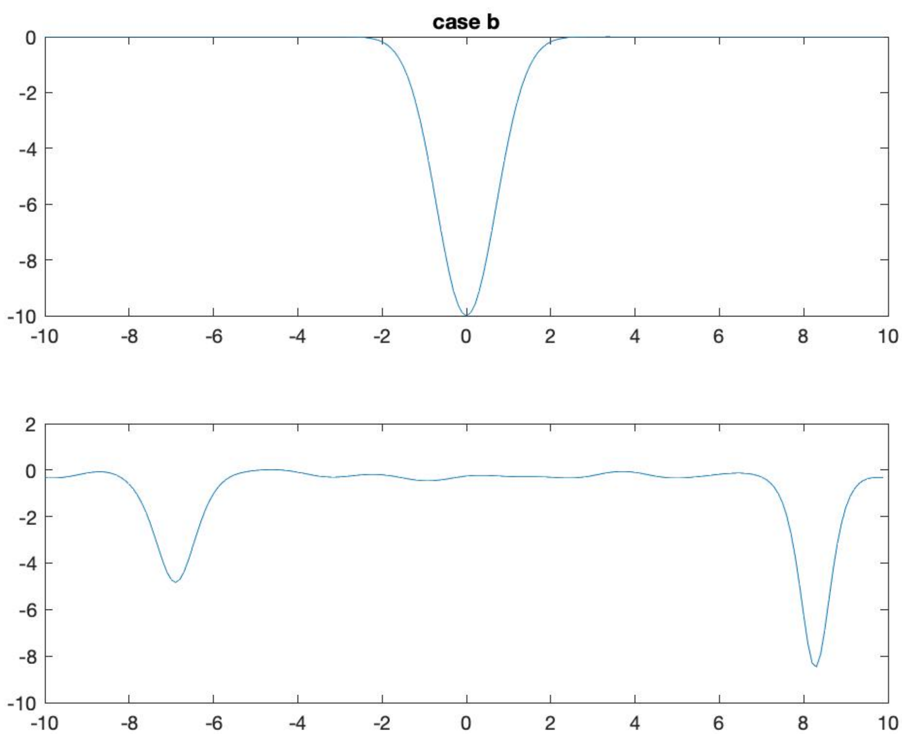
f. for $u(x,0) = 2\sin(x\pi)$, the result will become very chaotic with ~~multiply~~ mutiple peaks and fluctuation with respect to $u = 0$.

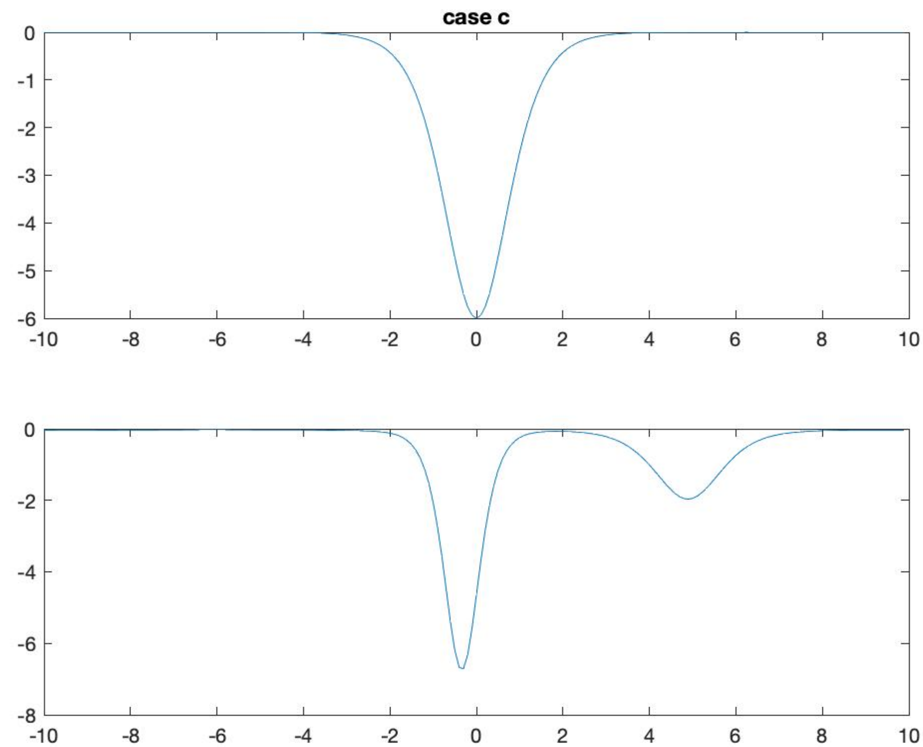
```
%solve and plot velocity vector in each initial condition
%creat velocity vector
N_j = 200;
u0 = zeros(1,N_j);

%case a single soliton u(x,0) = u1(x,0)
%initiate velocity vector
for j = 1:N_j
    x(j) = -10+(j-1)*0.1;
    u0(j) = -20/(2*(cosh(sqrt(20)*x(j)/2))^2);
end
%plot initial velocity vector
figure
subplot(2,1,1)
plot(x,u0)
title('case a')

%solve velocity vector by step
ua = solitonsolver(N_j,2,0.001,u0);
%plot final velocity vector
subplot(2,1,2)
plot(x,ua)

%case b single soliton u(x,0) = -10*exp(-x^2)
%initiate velocity vector
for j = 1:N_j
    u0(j) = -10*exp(-x(j)^2);
end
%plot initial velocity vector
figure
subplot(2,1,1)
plot(x,u0)
title('case b')

%solve velocity vector by step
ub = solitonsolver(N_j,2,0.001,u0);
%plot final velocity vector
subplot(2,1,2)
plot(x,ub)

%case c two_soliton soliton u(x,0) = -6/(cosh(x)).^2
%initiate velocity vector
for j = 1:N_j
    u0(j) = -6/(cosh(x(j)))^2;
end
%plot initial velocity vector
figure
subplot(2,1,1)
plot(x,u0)
title('case c')

%solve velocity vector by step
uc = solitonsolver(N_j,2,0.001,u0);
%plot final velocity vector
subplot(2,1,2)
plot(x,uc)
```

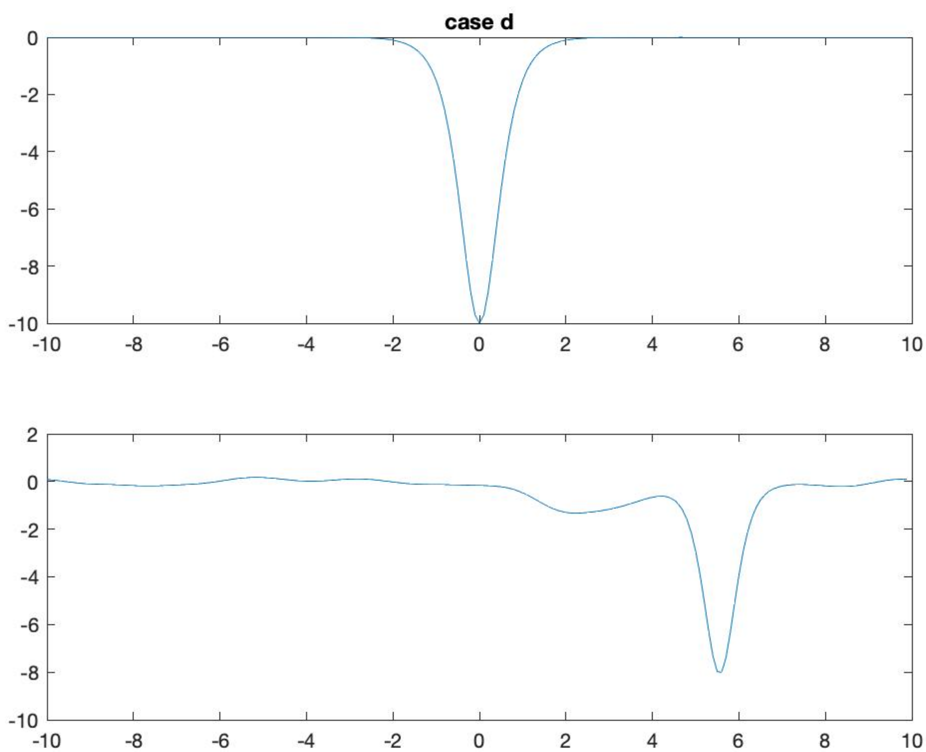
```
%case d own two_soliton soliton where v1=14,v2=6,both x0=0
%u(x,0) = -14/(2*(cosh(sqrt(14)*x/2))^2)-6/(2*(cosh(sqrt(6)*x/2))^2)
%initiate velocity vector
for j = 1:N_j
    u0(j) = -14/(2*(cosh(sqrt(14)*x(j)/2))^2)-6/(2*(cosh(sqrt(6)*x(j)/2))^2);
end
%plot initial velocity vector
figure
subplot(2,1,1)
plot(x,u0)
title('case d')

%solve velocity vector by step
ud = solitonsolver(N_j,2,0.001,u0);
%plot final velocity vector
subplot(2,1,2)
plot(x,ud)

%case e own two_soliton soliton where (v1,x0)=(14,-3),(v2,x0)=(6,3)
%u(x,0) = -14/(2*(cosh(sqrt(14)*(x+3)/2))^2)-6/(2*(cosh(sqrt(6)*(x-3)/2))^2)
%initiate velocity vector
for j = 1:N_j
    u0(j) = -14/(2*(cosh(sqrt(14)*(x(j)+3)/2))^2)-6/(2*(cosh(sqrt(6)*(x(j)-
3)/2))^2);
end
%plot initial velocity vector
figure
subplot(5,1,1)
plot(x,u0)
title('case e')

%solve velocity vector by step
for i = 1:4
    ue(i,:) = solitonsolver(N_j,i*0.5,0.001,u0);
end
%plot final velocity vector
subplot(5,1,2)
plot(x,ue(1,:))
subplot(5,1,3)
plot(x,ue(2,:))
subplot(5,1,4)
plot(x,ue(3,:))
subplot(5,1,5)
plot(x,ue(4,:))

%case f u0 = 2*sin(x/pi)
%initiate velocity vector
for j = 1:N_j
    u0(j) = 2*sin(x(j)/pi);
end
%plot initial velocity vector
figure
subplot(2,1,1)
plot(x,u0)
title('case f')

%solve velocity vector by step
```

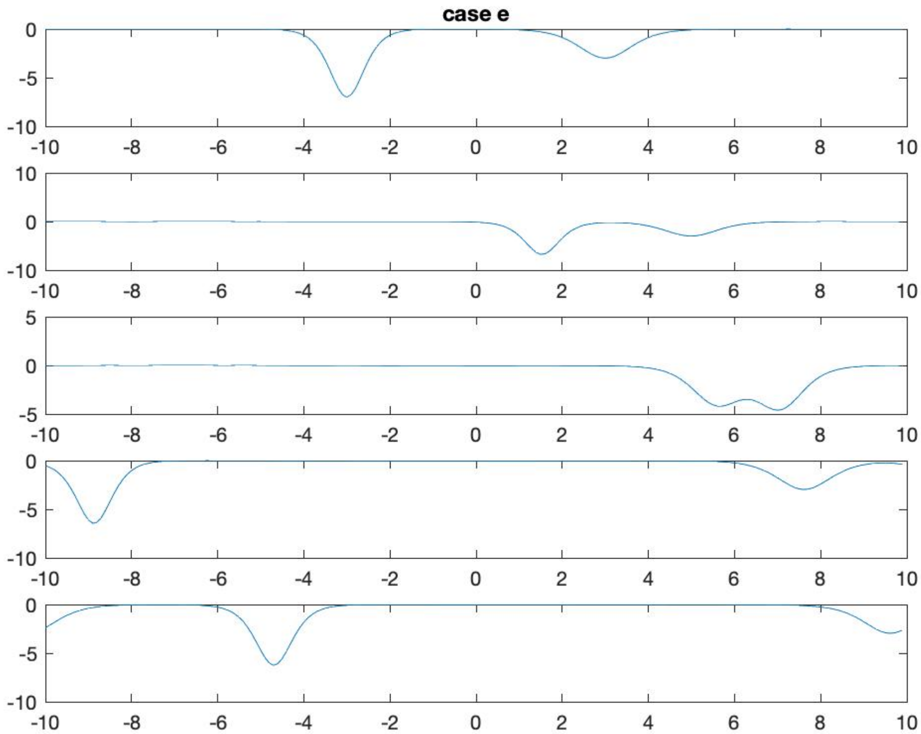
```
uf = solitonsolver(N_j,2,0.001,u0);
%plot final velocity vector
subplot(2,1,2)
plot(x,uf)


%solver of the governing equation
function [solution] = solitonsolver(J,t,dt,u)
%calculate number of time steps
N_t = t/dt;

%creat spatial discretize function f
function [sp] = f(un)
for j = 1:J
    if(j == 1)
        sp(j) = -(-un(J-1)+2*un(J)-
2*un(j+1)+un(j+2))/(2*0.1^3)+6*un(j)*(un(j+1)-un(J))/(2*0.1);
    elseif(j == 2)
        sp(j) = -(-un(J)+2*un(1)-
2*un(j+1)+un(j+2))/(2*0.1^3)+6*un(j)*(un(j+1)-un(j-1))/(2*0.1);
    elseif(j == J-1)
        sp(j) = -(-un(j-2)+2*un(j-1)-
2*un(j+1)+un(1))/(2*0.1^3)+6*un(j)*(un(j+1)-un(j-1))/(2*0.1);
    elseif(j == J)
        sp(j) = -(-un(j-2)+2*un(j-1)-2*un(1)+un(2))/(2*0.1^3)+6*un(j)*(un(1)-
un(j-1))/(2*0.1);
    else
        sp(j) = -(-un(j-2)+2*un(j-1)-
2*un(j+1)+un(j+2))/(2*0.1^3)+6*un(j)*(un(j+1)-un(j-1))/(2*0.1);
    end
end
end

%time integration using Runge-Kutta scheme
for n = 1:N_t
    a1 = dt*f(u);
    a2 = dt*f(u+a1/2);
    a3 = dt*f(u+a2/2);
    a4 = dt*f(u+a3);
    u = u+(a1+a2+a3+a4)/6;
end

solution = u;%return velocity field

end %end solver
```

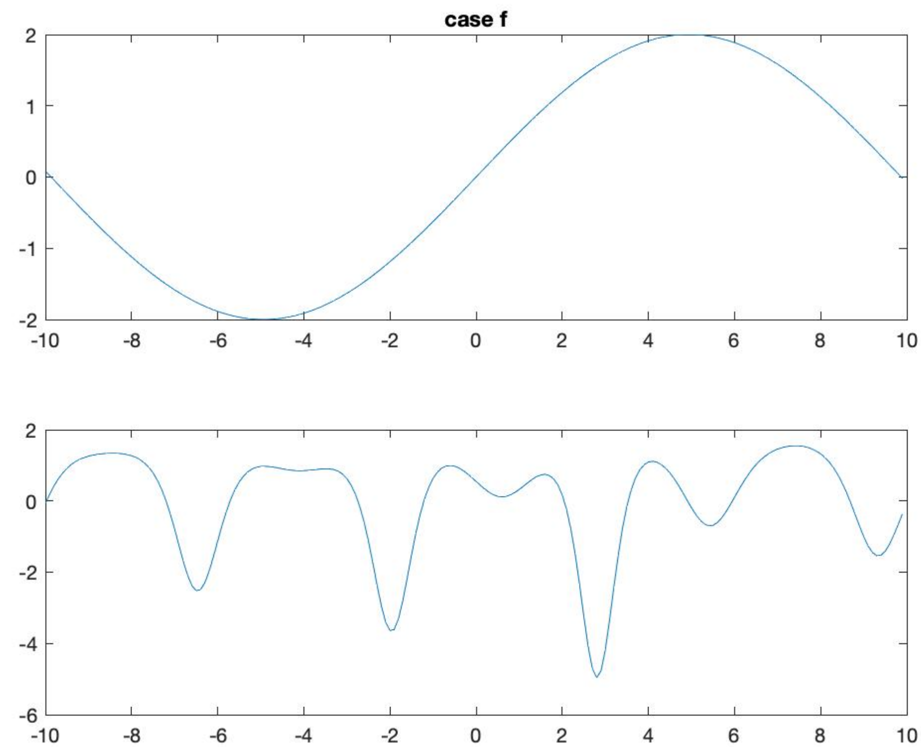

case a


case b


case c


case d


case e

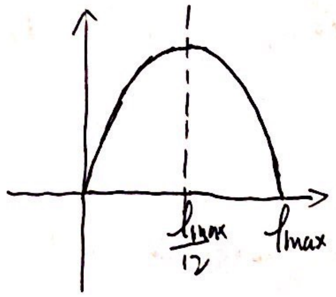

case f

Problem 2.

2.1.

a. $f(\rho) = \rho V_{max} (1 - \frac{\rho}{\rho_{max}}) = \rho V_{max} - \frac{\rho^2}{\rho_{max}} V_{max}$.

draw the diagram of $f(\rho)$



given :

$$\bar{F}^{G}_{i+\frac{1}{2}} = f(\rho(x_{i+\frac{1}{2}}, t^{n+1})) = \begin{cases} \min\limits_{\rho \in [\rho_i, \rho_{i+1}]} f(\rho) & \rho_i < \rho_{i+1} \\ \max\limits_{\rho \in [\rho_i, \rho_{i+1}]} f(\rho) & \rho_i > \rho_{i+1} \end{cases}$$

analyze each case :

① $\frac{\rho_m}{2} < \rho_i < \rho_{i+1}$ $f(\rho_{i+1})$ $\quad$ $\rho_i < \frac{\rho_m}{2} < \rho_{i+1}$ $\begin{cases} ② \text{ if } \rho_i \text{ closer to } \frac{\rho_m}{2} & f(\rho_{i+1}) \\ ③ \text{ if } \rho_{i+1} \text{ closer to } \frac{\rho_m}{2} & f(\rho_i) \end{cases}$

④ $\rho_i < \rho_{i+1} < \frac{\rho_m}{2}$ $f(\rho_i)$ $\quad$ ⑤ $\frac{\rho_m}{2} > \rho_i > \rho_{i+1}$ $f(\rho_i)$

$\rho_i > \frac{\rho_m}{2} > \rho_{i+1}$ $\begin{cases} ⑥ \text{ if } \rho_{i+1} \text{ closer to } \frac{\rho_m}{2} & f(\rho_{i+1}) \\ ⑦ \text{ if } \rho_i \text{ closer to } \frac{\rho_m}{2} & f(\rho_i) \end{cases}$

⑧ $\rho_i > \rho_{i+1} > \frac{\rho_m}{2}$ $f(\rho_{i+1})$

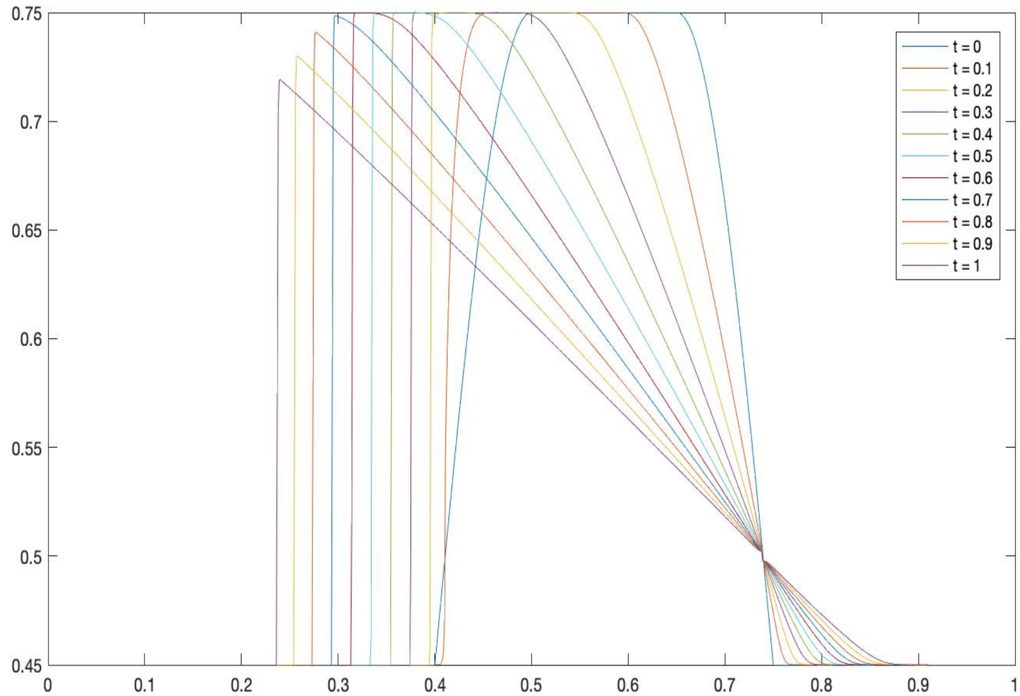
We can reorganize the cases stated above and simplify into :

$$\begin{cases} \rho_i + \rho_{i+1} \geq \rho_{max} & ①②⑥⑧ & f(\rho_{i+1}) & \bar{F}^{G}_{i+\frac{1}{2}} = \rho_{i+1} V_{max} (1 - \frac{\rho_{i+1}}{\rho_{max}}) \\ \rho_i + \rho_{i+1} \leq \rho_{max} & ③④⑤⑦ & f(\rho_i) & \bar{F}^{G}_{i+\frac{1}{2}} = \rho_i V_{max} (1 - \frac{\rho_i}{\rho_{max}}) \end{cases}$$

b. see attached matlab code and plot for details

Comments : the rising ~~ege~~ edge move backwards and falling edge becomes less steep as time evolves. To expain it, as car start moving the density of cars become smaller as time evolves. However, cars that behind. the cars that's still not moving will pack together which explains why the 'density' shock moving backward.

```
%parameters specification
roumax = 1;
vmax = 1;
d_x = 0.001;
d_t = 0.8*d_x/vmax;
%initiate density vector
%space domain [0,1]
N_x = 1/0.001+1;
rou0 = zeros(1,N_x);
for i = 1:N_x
    x(i) = d_x*(i-1);
    if(x(i) <= 0.4) rou0(i) = 0.45;
    elseif(x(i) > 0.4 && x(i) <= 0.5) rou0(i) = 0.45+0.3*cos(5*pi*(x(i)-
0.5));
    elseif(x(i) > 0.5 && x(i) <= 0.65) rou0(i) = 0.75;
    elseif(x(i) > 0.65 && x(i) <= 0.75) rou0(i) = 0.45+0.3*cos(5*pi*(x(i)-
0.65));
    else rou0(i) = 0.45;
    end
end

%solve continuity equation
for j = 1:11
rout(j,:) = trafficflow(rou0,N_x,d_x,0.1*(j-1),d_t,roumax,vmax);
end
plot(x,rout(1,:),...
     x,rout(2,:),...
     x,rout(3,:),...
     x,rout(4,:),...
     x,rout(5,:),...
     x,rout(6,:),...
     x,rout(7,:),...
     x,rout(8,:),...
     x,rout(9,:),...
     x,rout(10,:),...
     x,rout(11,:))
legend('t = 0','t = 0.1','t = 0.2','t = 0.3','t = 0.4','t = 0.5','t = 0.6','t
= 0.7','t = 0.8','t = 0.9','t = 1')

%creat solver for finite volume scheme of traffic flow
function [rou] = trafficflow(rou,N_x,d_x,t,d_t,roumax,vmax)
N_t = t/d_t;
for n = 1:N_t
    for i = 2:N_x-1
        if(rou(i+1)+rou(i) >= roumax)
            rou(i) = rou(i)-d_t*(rou(i+1)*vmax*(1-rou(i+1)/roumax)-
rou(i)*vmax*(1-rou(i)/roumax))/d_x;%f(rou) = rou*vmax*(1-rou/roumax)
        elseif(rou(i+1)+rou(i) <= roumax)
            rou(i) = rou(i)-d_t*(rou(i)*vmax*(1-rou(i)/roumax)-rou(i-
1)*vmax*(1-rou(i-1)/roumax))/d_x;
        end
    end
end
end
```


0.75
0.7
0.65
0.6
0.55
0.5
0.45
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t = 0
t = 0.1
t = 0.2
t = 0.3
t = 0.4
t = 0.5
t = 0.6
t = 0.7
t = 0.8
t = 0.9
t = 1

2.2.

a.

i. $\rho = \rho_e + \delta\rho \qquad V = V_e + \delta V$

eqt. (17) $\frac{\partial \rho}{\partial t} + \frac{\partial(\rho V)}{\partial x} = 0$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} V + \rho \frac{\partial V}{\partial x} = 0 \Rightarrow \frac{\partial(\cancel{\rho_e} + \delta\rho)}{\partial t} + \frac{\partial(\cancel{\rho_e} + \delta\rho)}{\partial x}(V_e + \cancel{\delta V}) + (\rho_e + \cancel{\delta\rho})\frac{\partial(\cancel{V_e} + \delta V)}{\partial x} = 0$$

$$\Rightarrow \frac{\partial \delta\rho}{\partial t} + V_e \frac{\partial \delta\rho}{\partial x} + \rho_e \frac{\partial \delta V}{\partial x} = 0 . \quad (17)$$

eqt. (26) $\frac{\partial \rho V}{\partial t} + \frac{\partial(\rho V^2 + \rho\theta_0)}{\partial x} = \frac{\rho[V_e(\rho) - V]}{T}$

$$\Rightarrow \frac{\partial[(\rho_e + \delta\rho)(V_e + \delta V)]}{\partial t} + \frac{\partial[(\rho_e + \delta\rho)(V_e + \delta V)^2 + (\cancel{\rho_e} + \delta\rho)\cdot\theta_0]}{\partial x} = \frac{(\rho_e + \cancel{\delta\rho})[\cancel{V_e(\rho_e + \delta\rho)} - \cancel{V_e} - \delta V]}{T}$$

$$\frac{\partial(\cancel{\rho_e V_e} + \rho_e \delta V + \delta\rho V_e + \cancel{\delta\rho\delta V})}{\partial t} + \frac{\partial(\cancel{\rho_e V_e^2} + 2\rho_e V_e \delta V + V_e^2 \delta\rho + \cancel{2V_e\delta\rho\delta V})}{\partial x} + \theta_0 \frac{\partial \delta\rho}{\partial x} = RHS$$

$$\Rightarrow \rho_e \frac{\partial \delta V}{\partial t} + V_e \frac{\partial \delta\rho}{\partial t} + 2\rho_e V_e \frac{\partial \delta V}{\partial x} + V_e^2 \frac{\partial \delta\rho}{\partial x} + \theta_0 \frac{\partial \delta\rho}{\partial x} = \cancel{-\frac{-\rho_e \delta V}{T}} \text{ source.}$$

0

(26)

ii (17) $\frac{\partial \delta\rho}{\partial t} + \frac{\partial(V_e\delta\rho + \rho_e \delta V)}{\partial x} = 0$

(26) $\frac{\partial(\rho_e\delta V + V_e\delta\rho)}{\partial t} + \frac{\partial(2\rho_e V_e \delta V + (V_e^2 + \theta_0)\delta\rho)}{\partial x} = 0$

therefore $U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} \delta\rho \\ \rho_e\delta V + V_e\delta\rho \end{bmatrix}, \quad \bar{F} = \begin{bmatrix} \bar{F}_1 \\ \bar{F}_2 \end{bmatrix} = \begin{bmatrix} V_e\delta\rho + \rho_e\delta V \\ 2\rho_e V_e \delta V + (V_e^2 + \theta_0)\delta\rho \end{bmatrix}$

iii $\frac{\partial \bar{F}_1}{\partial U_1} = 0 \qquad \frac{\partial \bar{F}_1}{\partial U_2} = 1 \qquad \frac{\partial \bar{F}_2}{\partial U_1} = \theta_0 - V_e^2 \qquad \frac{\partial \bar{F}_2}{\partial U_2} = 2V_e$

$\therefore J = \begin{bmatrix} 0 & 1 \\ \theta_0 - V_e^2 & 2V_e \end{bmatrix}$ find eigenvalues $\begin{vmatrix} -\lambda & 1 \\ \theta_0 - V_e^2 & 2V_e - \lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - 2V_e\lambda + V_e^2 - \theta_0 = 0 \qquad$ for $V_e, \theta_0 > 0 \quad \lambda_{i+\frac{1}{2}} = |\lambda|_{max} = V_e + \sqrt{\theta_0}$

$\Rightarrow \lambda = V_e \pm \sqrt{\theta_0}$

iv. assume $H=0$ use semi discretization. and we have.

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n\right)$$

where $U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ $F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$ $F_{j+\frac{1}{2}} = \frac{1}{2}(F_j^n + F_{j+1}^n) - \frac{\lambda_{j+\frac{1}{2}}}{2}(U_{j+1}^n - U_j^n)$

$F = JU$ $\lambda_{j+\frac{1}{2}} = \text{const} = Ve + i\theta_0$

$$\Rightarrow U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left[\frac{1}{2}(\cancel{F_j^n} + F_{j+1}^n) - \frac{\lambda_{j+1}}{2}(U_{j+1}^n - U_j^n) - \frac{1}{2}(F_{j-1}^n + \cancel{F_j^n}) + \frac{\lambda_{j-1}}{2}(U_j^n - U_{j-1}^n)\right]$$

$$\Rightarrow U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left[\frac{1}{2}(F_{j+1}^n - F_{j-1}^n) + \frac{Ve+i\theta_0}{2}(-U_{j+1}^n + 2U_j^n - U_{j-1}^n)\right]$$

V. (stability analysis)

Fourier analysis $\Longrightarrow$

$$\sum \left\{ U_\theta^{n+1}\cancel{e^{ij\theta}} = U_\theta^n \cancel{e^{ij\theta}} - \frac{\Delta t}{\Delta x}\left[\frac{1}{2}(JU_\theta^n \cancel{e^{ij\theta}} e^{i\theta} - JU_\theta^n \cancel{e^{ij\theta}} e^{-i\theta}) + \frac{Ve+i\theta_0}{2}(-U_\theta^n \cancel{e^{ij\theta}} e^{i\theta} + 2U_\theta^n \cancel{e^{ij\theta}} - U_\theta^n \cancel{e^{ij\theta}} e^{-i\theta})\right]\right\}$$

$$\Rightarrow U_\theta^{n+1} = U_\theta^n - \frac{\Delta t}{\Delta x}\left[\cancel{\frac{1}{2}} J U_\theta^n \underbrace{(e^{i\theta} - e^{-i\theta})}_{2i\sin\theta} + \frac{Ve+i\theta_0}{\cancel{2}} U_\theta^n \underbrace{(-e^{i\theta} + 2 - e^{-i\theta})}_{\cancel{2} - \cancel{2}\cos\theta}\right]$$

$$\Rightarrow U_\theta^{n+1} = U_\theta^n \left[I - \frac{\Delta t}{\Delta x} i\sin\theta\, J - \frac{\Delta t}{\Delta x}(Ve + i\theta_0)(1 - \cos\theta) \cdot I\right]$$

$$\text{matrix } A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - b\begin{bmatrix} 0 & 1 \\ \theta_0 - Ve^2 & 2Ve \end{bmatrix} + a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where $b = \frac{\Delta t}{\Delta x} \cdot i\sin\theta$ $a = -\frac{\Delta t}{\Delta x}(Ve + i\theta_0)(1 - \cos\theta)$

$$= \begin{bmatrix} 1+a & -b \\ (Ve^2 - \theta_0)b & 1 - 2Veb + a \end{bmatrix}$$

to get eigenvalue to matrix A.

$$A - \lambda I = \begin{bmatrix} 1+a-\lambda & -b \\ (Ve^2 - \theta_0)b & 1 - 2Veb + a - \lambda \end{bmatrix}$$

$$|A-\lambda I| = 1-2V_e b+a-\lambda+a-2V_e ab+a^2-a\lambda-\lambda+2V_e b\lambda-a\lambda+\lambda^2 + (V_e^2-\theta_0)b^2 = 0$$

$$LHS = \lambda^2+(-1-a-1+2V_e b-a)\lambda+(1-2V_e b+a+a-2V_e ab+a^2+V_e^2 b^2-\theta_0 b^2)$$

$$= \lambda^2+2(V_e b-a-1)\lambda+(a^2+2a+1-2V_e b-2V_e ab+V_e^2 b^2-\theta_0 b^2)$$

$$\Delta = b^2-4ac = 4(\cancel{V_e^2 b^2}+\cancel{a^2}+\cancel{1}-\cancel{2V_e ab}-\cancel{2V_e b}+\cancel{2a})-4(\cancel{a^2}+\cancel{2a}+\cancel{1}-\cancel{2V_e b}-\cancel{2V_e ab}+\cancel{V_e^2 b^2}-\theta_0 b^2)$$

$$= 4\theta_0 b^2$$

thus $\lambda = \dfrac{-2(V_e b-a-1)\pm\sqrt{4\theta_0 b^2}}{2} = \dfrac{2(a+1-V_e b)\pm 2b\sqrt{\theta_0}}{2} = (a+1-V_e b)\pm b\sqrt{\theta_0}$

$$= (a+1)-(V_e\pm\sqrt{\theta_0})b.$$

plug in a, b then $\lambda_1 = -(V_e+\sqrt{\theta_0})\cdot c+1+(V_e+\sqrt{\theta_0})\cos\theta\, c - i(V_e+\sqrt{\theta_0})c\cdot\sin\theta$

and let $c=\frac{\Delta t}{\Delta x}$ $\quad\lambda_2 = -(V_e+\sqrt{\theta_0})\cdot c+1+(V_e+\sqrt{\theta_0})\cos\theta\, c - i(V_e-\sqrt{\theta_0})c\cdot\sin\theta$

to ensure stability. $|\lambda_1|<1$, $|\lambda_2|<1$

$x = 1+(V_e+\sqrt{\theta_0})(\cos\theta-1)\cdot c \qquad y = (V_e+\sqrt{\theta_0})\cdot c\sin\theta.$

$x^2+y^2\le 1 \Rightarrow \cancel{1}+(V_e+\sqrt{\theta_0})^2(\cos^2\theta+1-2\cos\theta)c^2+2c(V_e+\sqrt{\theta_0})(\cos\theta-1)+(V_e+\sqrt{\theta_0})^2\cdot c^2\sin^2\theta \le \cancel{1}\ 0$

$\Rightarrow (V_e+\sqrt{\theta_0})^2\cdot\cos^2\theta\, c^2+(V_e+\sqrt{\theta_0})^2c^2-2\cos\theta(V_e+\sqrt{\theta_0})^2\cdot c^2+(V_e+\sqrt{\theta_0})^2c^2\sin^2\theta+2c(V_e\pm\sqrt{\theta_0})(\cos\theta-1)\le 0$

$\Rightarrow 2(V_e+\sqrt{\theta_0})^2c^2-2\cos\theta(V_e+\sqrt{\theta_0})^2\cdot c^2+2c(V_e+\sqrt{\theta_0})(\cos\theta-1)\le 0$

$\Rightarrow \cancel{2}(V_e+\sqrt{\theta_0})^{\cancel{2}}c^{\cancel{2}}\cancel{(1-\cos\theta)}\cancel{+2c(V_e+\sqrt{\theta_0})(\cos\theta-1)}\le 0 \Rightarrow c\le\dfrac{1}{V_e+\sqrt{\theta_0}}$

$V_{e\,max}=0.55 \quad \theta_{0\,max}=0.7 \quad$ thus $\dfrac{\Delta t}{\Delta x}=c\le\dfrac{1}{0.55+\sqrt{0.7}}=0.721$

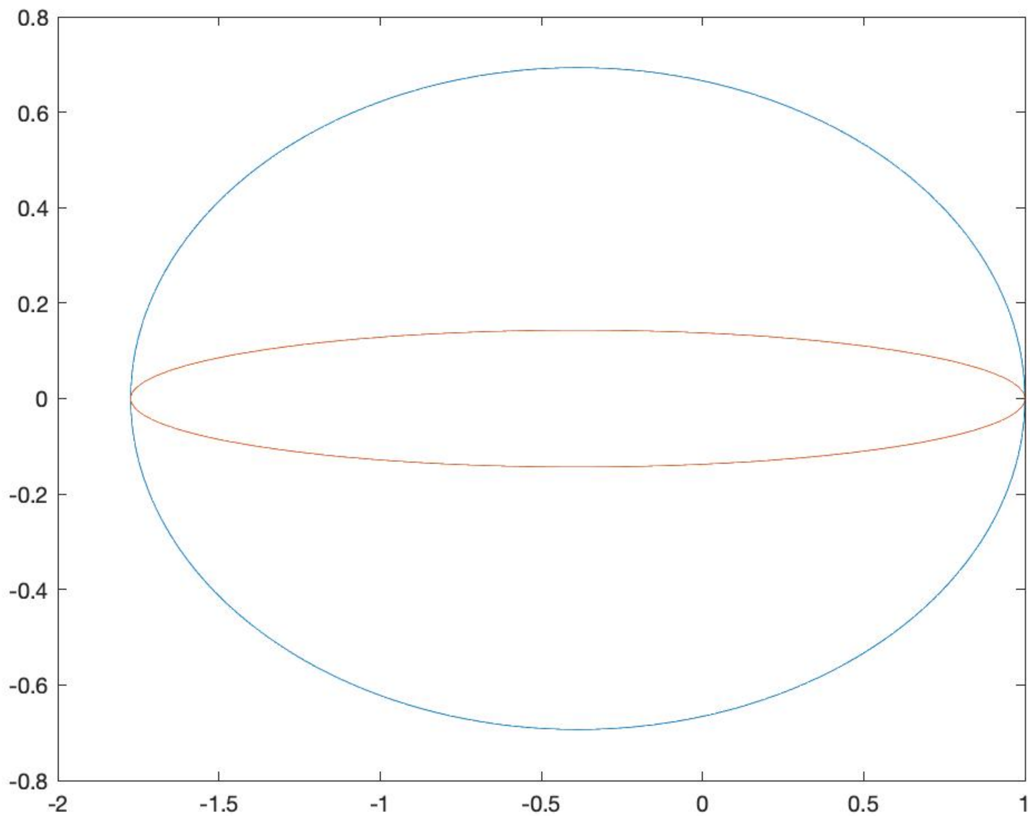
therefore $\Delta t\le 0.721\,\Delta x$. choose $\dfrac{\Delta t}{\Delta x}=0.5$.

plot inequalities on complex plane

See attached code and plot.

Scanned with CamScanner

```
ve = 0.55;
t0 = 0.7;
c = 0.5;
t = 0:0.01:2*pi;
x = -(ve+sqrt(t0))+1+(ve+sqrt(t0))*cos(t);
y1 = -0.5*(ve+sqrt(t0))*sin(t);
y2 = -0.5*(ve-sqrt(t0))*sin(t);
plot(x,y1,x,y2)
```



b. set $\Delta t = 0.5\,\Delta x = 0.0005$

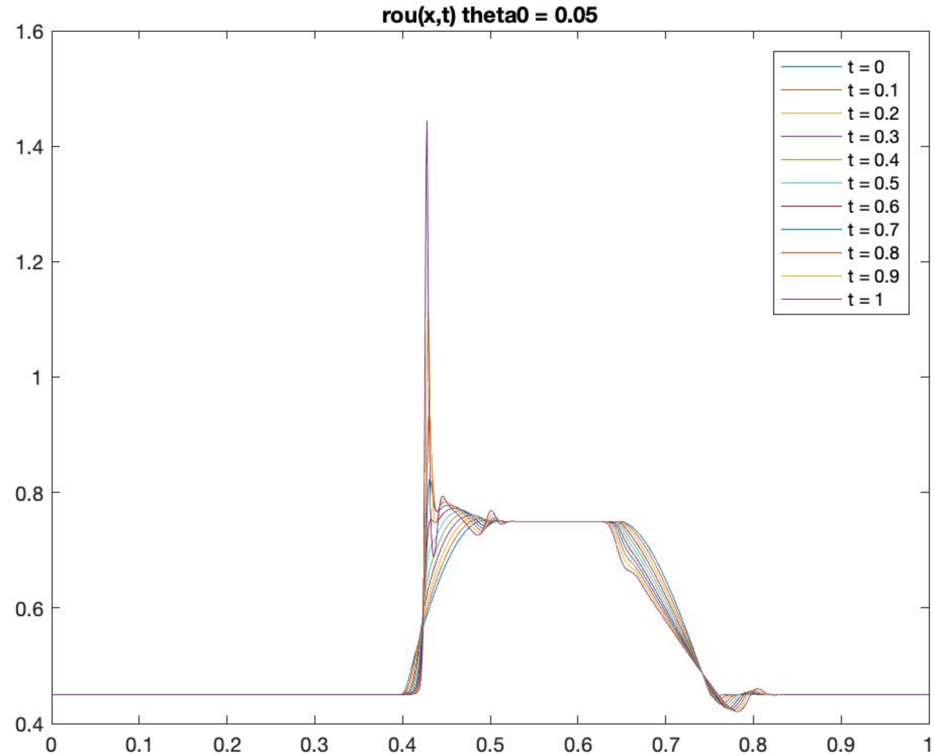
See attached matlab program and plots for details

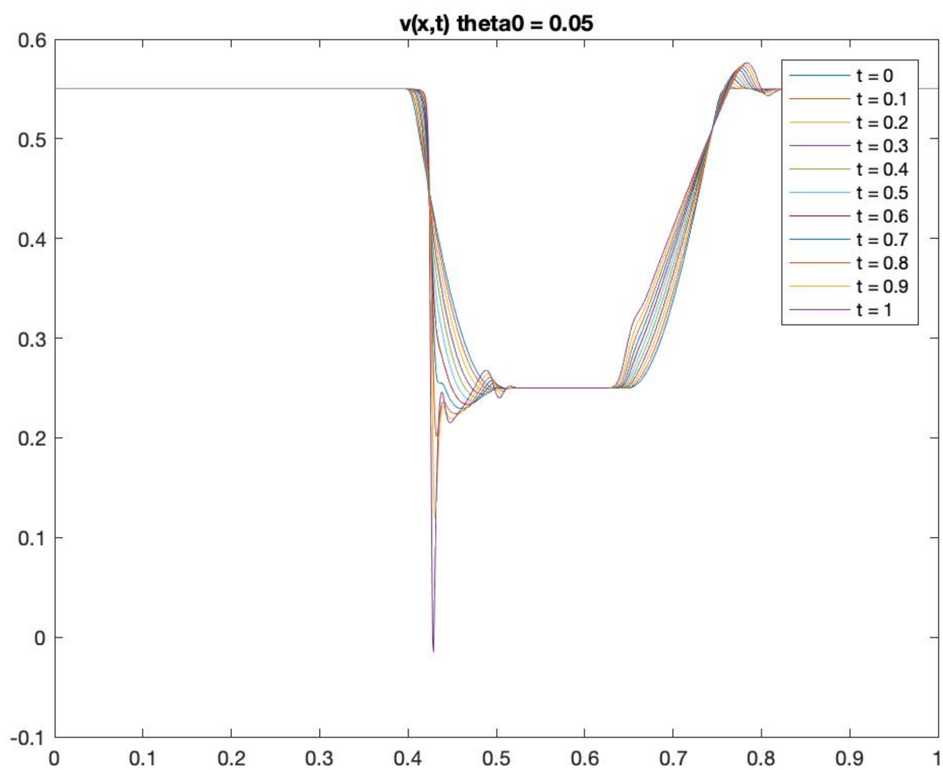
Comment: θ_0 plays the role of decay (dispersion) on the domino effect. Smooth the equation.

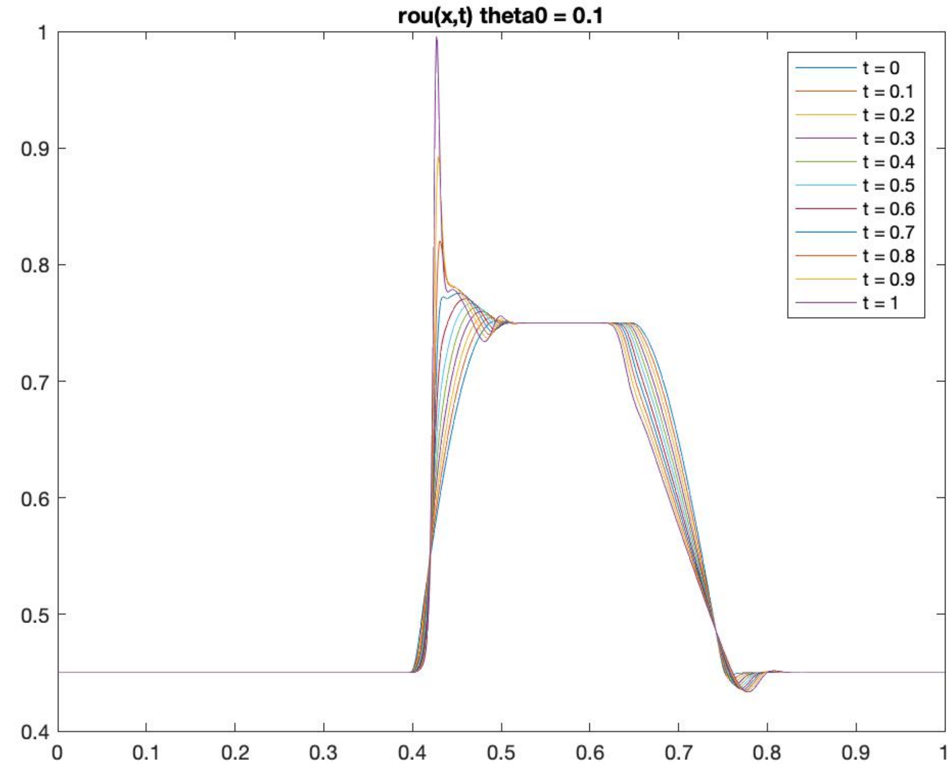
From the solution solved by using only continuity equation. The 'density' shock is more steep (discontinue) where as the solution by momentum eqt is more smooth. Another thing is that. with θ_0, the magnitude of density shock decays faster than the solution by using only continuity equation.

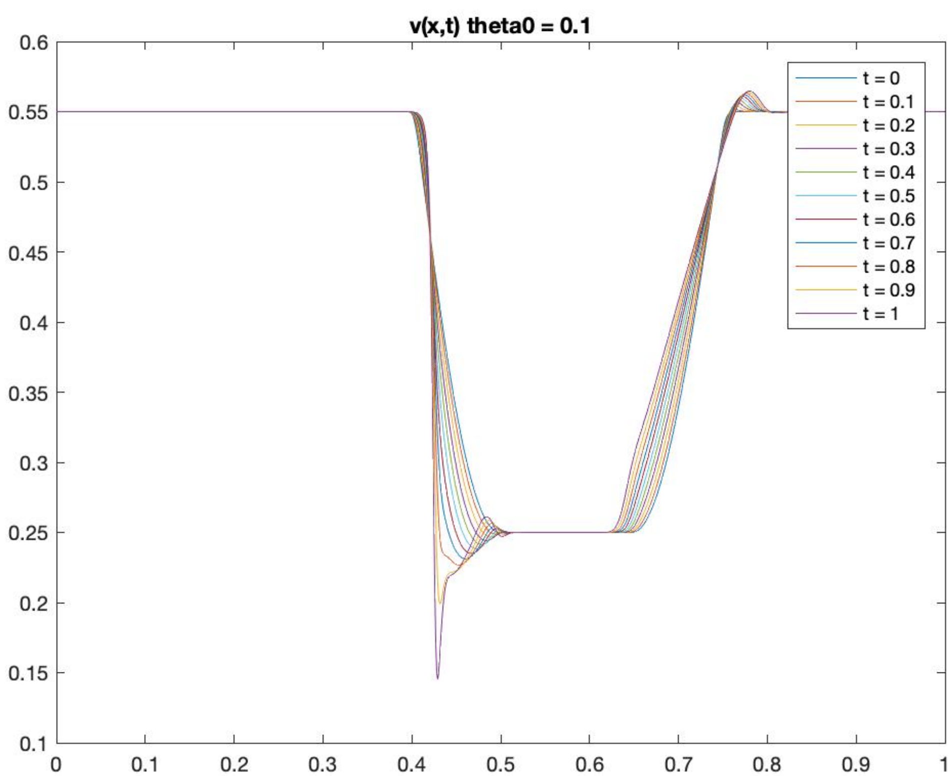
```
%specify parameters
roumax = 1;
vmax = 1;
d_x = 0.001;
d_t = 0.0005;
tao = 0.01;
theta0 = 0.7;
%initialize rou0 and v0 vector
N_x = 1/d_x+1;
rou0 = zeros(1,N_x);
v0 = zeros(1,N_x);
for i = 1:N_x
    x(i) = d_x*(i-1);
    if(x(i) <= 0.4) rou0(i) = 0.45;
    elseif(x(i) > 0.4 && x(i) <= 0.5) rou0(i) = 0.45+0.3*cos(5*pi*(x(i)-
0.5));
    elseif(x(i) > 0.5 && x(i) <= 0.65) rou0(i) = 0.75;
    elseif(x(i) > 0.65 && x(i) <= 0.75) rou0(i) = 0.45+0.3*cos(5*pi*(x(i)-
0.65));
    else rou0(i) = 0.45;
    end
end
for i = 1:N_x
    v0(i) = 1-rou0(i);
    %construct U(i+1/2),U(i-1/2),F(i+1/2),F(i-1/2),H vector
    U0(:,i) = [rou0(i);rou0(i)*v0(i)];
    F0(:,i) = [rou0(i)*v0(i);(v0(i)^2+theta0)*rou0(i)];
    H0(:,i) = [0;rou0(i)*(vmax*(1-rou0(i)/roumax)-v0(i))/tao];%ve = vmax(1-
rou/roumax)
end
for i = 1:11
    [rou(i,:),v(i,:)] = laxsolver(U0,F0,H0,N_x,d_x,i*0.1,d_t,theta0,tao);
end

plot(x,rou(1,:),...
     x,rou(2,:),...
     x,rou(3,:),...
     x,rou(4,:),...
     x,rou(5,:),...
     x,rou(6,:),...
     x,rou(7,:),...
     x,rou(8,:),...
     x,rou(9,:),...
     x,rou(10,:),...
     x,rou(11,:))
legend('t = 0','t = 0.1','t = 0.2','t = 0.3','t = 0.4','t = 0.5','t = 0.6','t
= 0.7','t = 0.8','t = 0.9','t = 1')
title('rou(x,t) theta0 = 0.7')
```

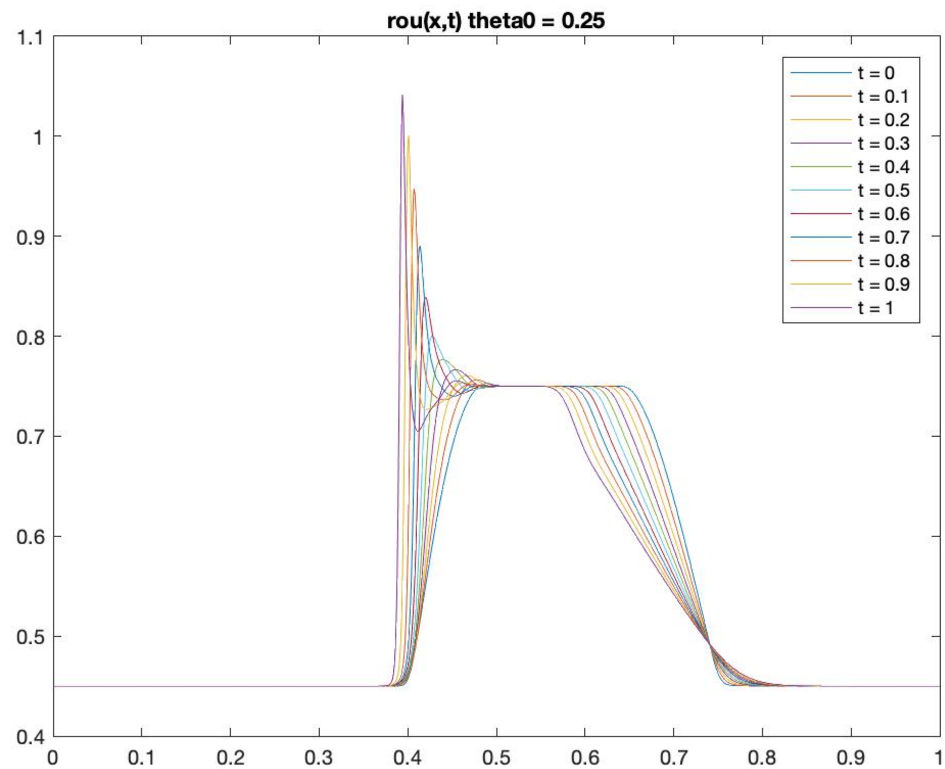
```
figure
plot(x,v(1,:),...
     x,v(2,:),...
     x,v(3,:),...
     x,v(4,:),...
     x,v(5,:),...
     x,v(6,:),...
     x,v(7,:),...
     x,v(8,:),...
     x,v(9,:),...
     x,v(10,:),...
     x,v(11,:))
legend('t = 0','t = 0.1','t = 0.2','t = 0.3','t = 0.4','t = 0.5','t = 0.6','t
= 0.7','t = 0.8','t = 0.9','t = 1')
title('v(x,t) theta0 = 0.7')


%creat solver using Lax Scheme
function [rou,v] = laxsolver(U,F,H,N_x,d_x,t,d_t,theta0,tao)
N_t = t/d_t;
for n = 1:N_t
    %determine lamda,lamda = v+sqrt(theta0)
    for i = 1:N_x-1
        %v = U2/U1
        if(U(2,i)/U(1,i)+sqrt(theta0) >= U(2,i+1)/U(1,i+1)+sqrt(theta0))
            lamda(i) = U(2,i)/U(1,i)+sqrt(theta0);
        else lamda(i) = U(2,i+1)/U(1,i+1)+sqrt(theta0);
        end
    end
    %lamda(N_x) = lamda(1) by periodic
    %lax scheme
    for j = 2:N_x-1
        U(:,j) = U(:,j)-d_t/d_x*((F(:,j+1)-F(:,j-1))/2-lamda(j)*(U(:,j+1)-
U(:,j))/2+lamda(j-1)*(U(:,j)-U(:,j-1))/2)...
                 +d_t*H(:,j);
        F(:,j) =
[U(1,j)*U(2,j)/U(1,j);U(1,j)*((U(2,j)/U(1,j))^2+theta0)];%get F at n+1
        H(:,j) = [0;U(1,j)*(1-U(1,j)-U(2,j)/U(1,j))/tao];%get H at n+1
    end
end
rou = U(1,:);
v = U(2,:)./U(1,:);
end
```

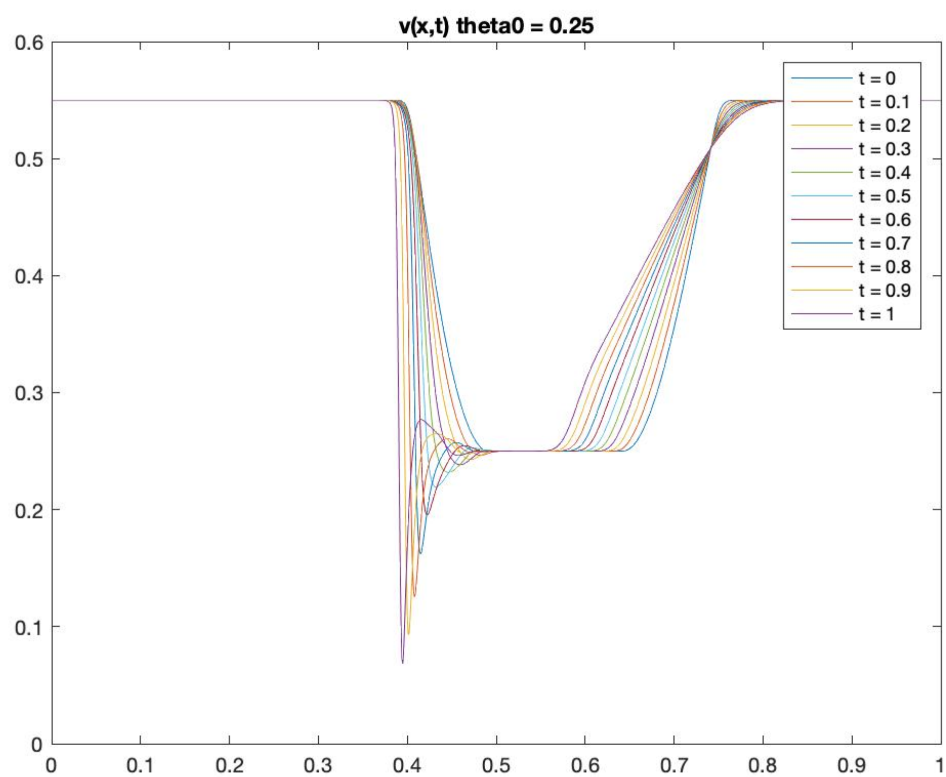

rou(x,t) theta0 = 0.05
t = 0
t = 0.1
t = 0.2
t = 0.3
t = 0.4
t = 0.5
t = 0.6
t = 0.7
t = 0.8
t = 0.9
t = 1

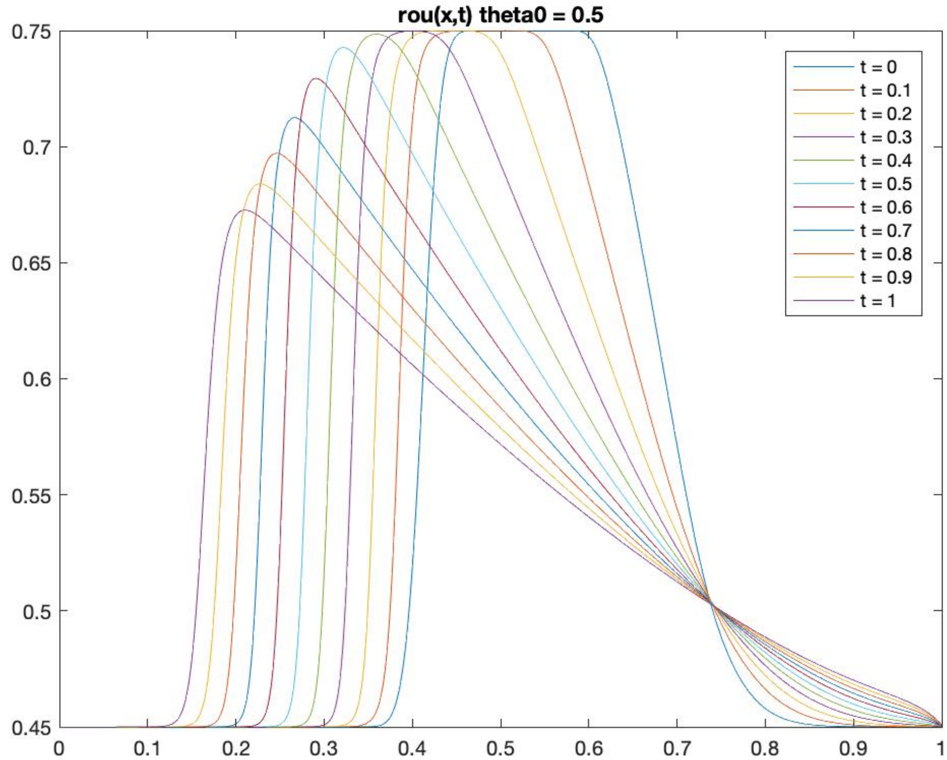

v(x,t) theta0 = 0.05
t = 0
t = 0.1
t = 0.2
t = 0.3
t = 0.4
t = 0.5
t = 0.6
t = 0.7
t = 0.8
t = 0.9
t = 1


rou(x,t) theta0 = 0.1
t = 0
t = 0.1
t = 0.2
t = 0.3
t = 0.4
t = 0.5
t = 0.6
t = 0.7
t = 0.8
t = 0.9
t = 1

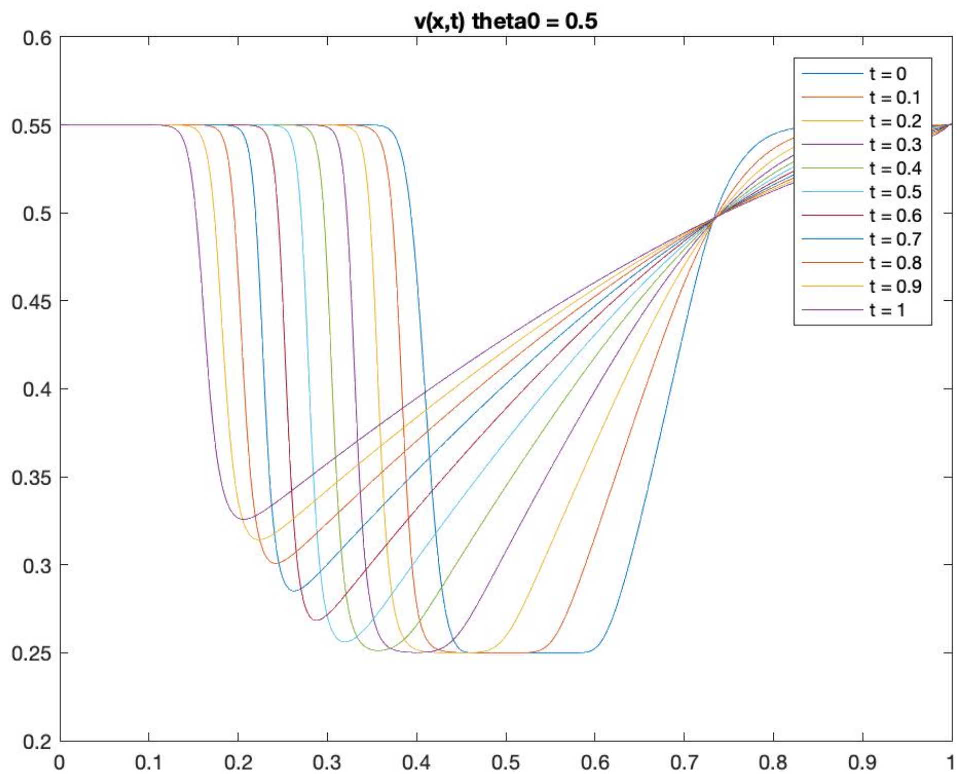

v(x,t) theta0 = 0.1
t = 0
t = 0.1
t = 0.2
t = 0.3
t = 0.4
t = 0.5
t = 0.6
t = 0.7
t = 0.8
t = 0.9
t = 1


rou(x,t) theta0 = 0.25
t = 0
t = 0.1
t = 0.2
t = 0.3
t = 0.4
t = 0.5
t = 0.6
t = 0.7
t = 0.8
t = 0.9
t = 1
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

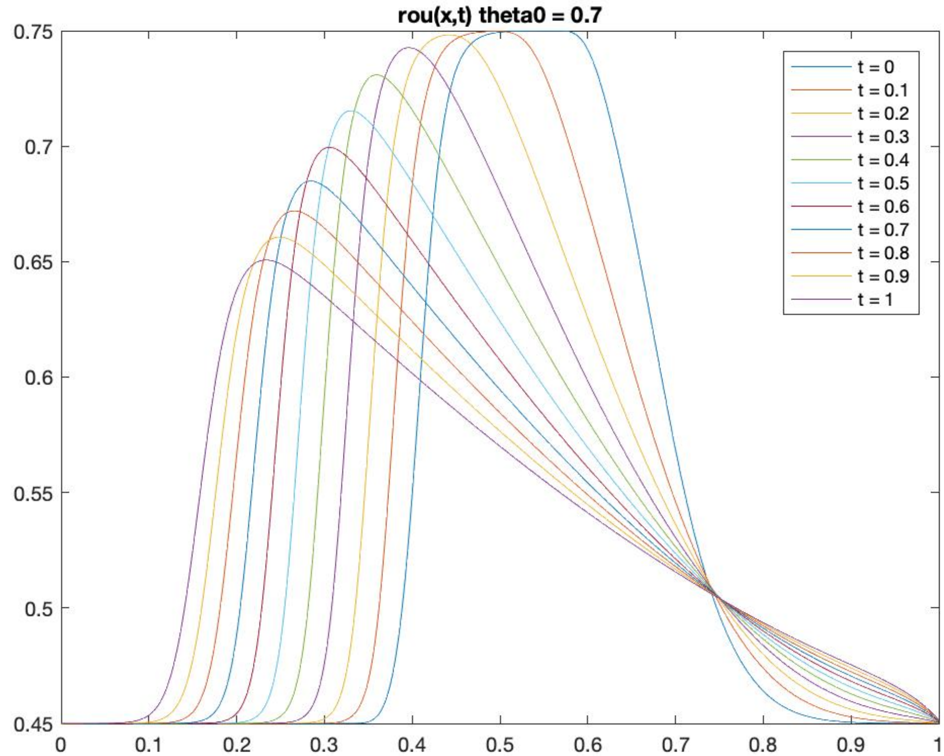

v(x,t) theta0 = 0.25
t = 0
t = 0.1
t = 0.2
t = 0.3
t = 0.4
t = 0.5
t = 0.6
t = 0.7
t = 0.8
t = 0.9
t = 1
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

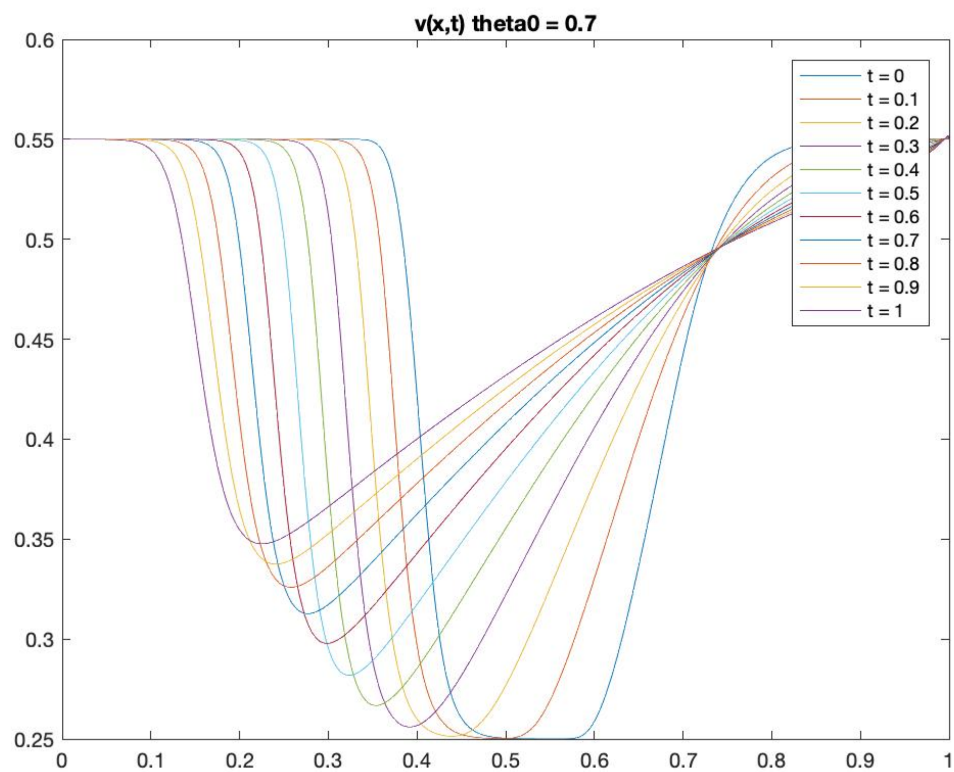


rou(x,t) theta0 = 0.5
t = 0
t = 0.1
t = 0.2
t = 0.3
t = 0.4
t = 0.5
t = 0.6
t = 0.7
t = 0.8
t = 0.9
t = 1



v(x,t) theta0 = 0.5
t = 0
t = 0.1
t = 0.2
t = 0.3
t = 0.4
t = 0.5
t = 0.6
t = 0.7
t = 0.8
t = 0.9
t = 1


rou(x,t) theta0 = 0.7
t = 0
t = 0.1
t = 0.2
t = 0.3
t = 0.4
t = 0.5
t = 0.6
t = 0.7
t = 0.8
t = 0.9
t = 1


v(x,t) theta0 = 0.7
t = 0
t = 0.1
t = 0.2
t = 0.3
t = 0.4
t = 0.5
t = 0.6
t = 0.7
t = 0.8
t = 0.9
t = 1