AME 535a project 4.

1. $\forall v \in X \equiv H^1(\Omega)$ To find weak statement.

Strong formulation is $-k^i \nabla^2 u^i = 0$ in Ω^i, $i = 0, 1, \dots 4$.

multiply strong formulation by v and integrate.

$$\int_\Omega v \cdot -k^i \nabla^2 u^i \, dA = 0 \Rightarrow \sum_{i=0}^{4} -k^i \int_{\Omega^i} v \nabla^2 u^i \, dA = 0$$

integrate by part $v \nabla^2 u = \nabla(v \nabla u) - \nabla v \nabla u$

plug back in $\sum_{i=0}^{4} -k^i \underbrace{\left[\int_{\Omega^i} \nabla(v \nabla u) \, dA - \int_{\Omega^i} \nabla v \nabla u \, dA \right]}_{①} = 0$

for ① use Gauss Theorem.

$$\underbrace{\int_{\Gamma^N} + \int_{\Gamma^R}} v \nabla u \hat{n} \, dS - \int_\Omega \nabla v \nabla u \, dA = 0$$

for Neumann BC only applys to the root

which is $-(\nabla u^0 \cdot \hat{n}^0) = -1$

which means on other Boundary $\nabla u^i \cdot \hat{n}^i = 0$

for Robin B.C

$$-k^i (\nabla u^i \cdot \hat{n}^i) = Bi\, u^i \quad \text{on } \Gamma^i_{ext},\ i=0,1,\dots,4.$$

then plug B.C back to original eqt.

$$-k^0 \cdot \int_{\Gamma_{root}} v\, ds + \sum_{i=0}^{4} \cancel{-k^i} \int_{\Gamma/\Gamma_{root}} Bi\, u^i \cdot v / \cancel{-k^i}\, ds$$

$$+ \sum_{i=0}^{4} \cancel{-}k^i \int_{\Omega} \nabla u^i \cdot \nabla v\, dA = 0 \xrightarrow{\text{sort out.}} k^0 \equiv 1$$

$$\underbrace{\sum_{i=0}^{4} k^i \int_{\Omega} \nabla u^i \nabla v\, dA + Bi \int_{\Gamma/\Gamma_{root}} u^i v\, ds}_{a} = \underbrace{\int_{\Gamma_{root}} v\, ds}_{l}.$$

$$a(u(\mu), v; \mu) = l(v)$$

$$a(w, v; \mu) = \sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla w \cdot \nabla v\, dA + Bi \int_{\Gamma/\Gamma_{root}} w v\, ds$$

$$l(v) = \int_{\Gamma_{root}} v\, ds.$$

2.

proof: let $w = u+v$

then $J(w) = J(u+v) = \frac{1}{2}\sum_{i=0}^{4} k^i \int_{\Omega_i} D(u+v)\cdot D(u+v)\, dA$

$$+ \frac{B_i}{2}\int_{\Gamma/\Gamma_{root}} (u+v)^2\, ds - \int_{\Gamma_{root}} (u+v)\, ds$$

$$= \frac{1}{2}\sum_{i=0}^{4} k^i \int_{\Omega_i} (Du\cdot Du + 2Du\,Dv + Dv\cdot Dv)\, dA$$

$$+ \frac{B_i}{2}\int_{\Gamma/\Gamma_{root}} u^2 + v^2 + 2uv\, ds - \int_{\Gamma_{root}} u\, ds - \int_{\Gamma_{root}} v\, ds.$$

$$= \frac{1}{2}\sum_{i=0}^{4} k^i \int_{\Omega_i} Du\cdot Du\, dA + \frac{B_i}{2}\int_{\Gamma/\Gamma_{root}} u^2\, ds - \int_{\Gamma_{root}} u\, ds \quad \underline{J(u)}$$

$$+ \sum_{i=0}^{4} k^i \int_{\Omega_i} Du\cdot Dv\, dA + B_i \int_{\Gamma/\Gamma_{root}} uv\, ds - \int_{\Gamma_{root}} v\, ds \quad \delta J_v(u)$$

$$+ \frac{1}{2}\sum_{i=0}^{4} k^i \int_{\Omega_i} Dv\cdot Dv\, dA + \frac{B_i}{2}\int_{\Gamma/\Gamma_{root}} v^2\, ds \quad > 0$$

for $\delta J_v(u) = \sum_{i=0}^{4} k^i \int_{\Omega_i} Du\cdot Dv\, dA + B_i \int_{\Gamma/\Gamma_{root}} u\cancel{v}^{0}\, ds - \int_{\Gamma_{root}} \cancel{v}^{0}\, ds$

$$= \sum_{i=0}^{4} k^i \left[\int_{\Omega_i} D(vDu)\, dA - \int_{\Omega_i} v\underbrace{D^2 u}_{=f=0}\, dA\right]$$

$$= \sum_{i=0}^{4} k^i \int_{\Gamma} \cancel{v}_{0}\, Du\cdot\hat{n}\, dA \quad = 0$$

therefore $J(w) = J(u+v) = J(u) + \frac{1}{2}\sum_{i=0}^{4} k^i \int_{\Omega_i} \nabla v \cdot \nabla v \, dA$

$$+ \underbrace{\frac{Bi}{2} \int_{\Gamma / \Gamma_{root}} v^2 \, ds}_{> 0}$$

thus $J(w) > J(u) \quad \forall\, w \in X,\ w \neq u$

$u\ (u)$ is the minimizer of $J(w)$.

3. finite element space

$$X_h = \{ v \in H^1(\Omega) \mid v|_{T_h} \in P^1(T_h),\ \forall T_h \in \mathcal{T}_h \}$$

then expression for local basis function $\mathcal{H}^k_\alpha,\ \alpha = 1, 2, 3.$

$$\mathcal{H}^k_\alpha = C^k_\alpha + C^k_{x\alpha} x + C^k_{y\alpha} y$$

$$\mathcal{H}^k_\beta = C^k_\beta + C^k_{x\beta} x + C^k_{y\beta} \cdot y$$

$$\begin{bmatrix} 1 & x^k_1 & y^k_1 \\ 1 & x^k_2 & y^k_2 \\ 1 & x^k_3 & y^k_3 \end{bmatrix} \begin{bmatrix} C^k_\alpha \\ C^k_{x\alpha} \\ C^k_{y\alpha} \end{bmatrix} = \underset{\alpha=1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \text{ or } \underset{\alpha=2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \text{ or } \underset{\alpha=3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

$\Uparrow$

$\alpha = 1, 2, 3$ same for $\beta = 1, 2, 3$.

elemental matrices $\underline{A}^k \in \mathbb{R}^{3\times 3}$

$$A^k_{\alpha\beta} = \int_{T_n^k} \frac{\partial H^k_\alpha}{\partial x}\frac{\partial H^k_\beta}{\partial x} + \frac{\partial H^k_\alpha}{\partial y}\frac{\partial H^k_\beta}{\partial y}\, dA$$

$$= \text{Area}^k \left(G^k_{x\alpha} G^k_{x\beta} + G^k_{y\alpha} G^k_{y\beta} \right)$$

thus for domain interior

$$A^k = k^i \cdot \text{Area}^k \begin{bmatrix} G^k_{x1}G^k_{x1} + G^k_{y1}G^k_{y1} & G^k_{x1}G^k_{x2} + G^k_{y1}G^k_{y2} & G^k_{x1}G^k_{x3} + G^k_{y1}G^k_{y3} \\ G^k_{x2}G^k_{x1} + G^k_{y2}G^k_{y1} & G^k_{x2}G^k_{x2} + G^k_{y2}G^k_{y2} & G^k_{x2}G^k_{x3} + G^k_{y2}G^k_{y3} \\ G^k_{x3}G^k_{x1} + G^k_{y3}G^k_{y1} & G^k_{x3}G^k_{x2} + G^k_{y3}G^k_{y2} & G^k_{x3}G^k_{x3} + G^k_{y3}G^k_{y3} \end{bmatrix}$$

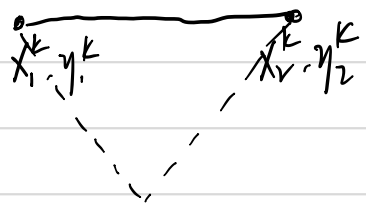
where k_i is thermal conductivity.

$$\text{Area}^k = \left| \frac{1}{2} \begin{vmatrix} 1 & x_1^k & y_1^k \\ 1 & x_2^k & y_2^k \\ 1 & x_3^k & y_3^k \end{vmatrix} \right|$$

for Boundary element we analyze one 1D element on the boundary

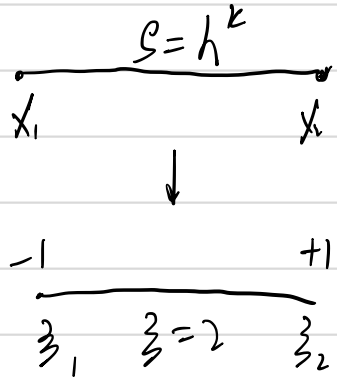
length of the element:

$$h^k = \sqrt{(x_2^k - x_1^k)^2 + (y_2^k - y_1^k)^2}$$



for boundary with Robin B.C.

Affine mapping:



$$A^k = a(w, v; u) = B_i \int_{\Gamma \setminus \Gamma_{root}} w \cdot v \, dS$$

then $$A^k_{\alpha\beta} = B_i \int_{\Gamma \setminus \Gamma_{root}} H_\alpha H_\beta \left(d\xi \frac{h^k}{2}\right)$$

$\alpha, \beta = 1$ or 2 $\quad H_1(\xi) = \frac{1-\xi}{2} \qquad H_2(\xi) = \frac{1+\xi}{2}$

plug into elemental matrices eqt

$$A^k_{11} = B_i \int_{-1}^{1} \left(\frac{1-\xi}{2}\right)^2 \frac{h^k}{2} \, d\xi = \frac{1}{3} B_i h^k$$

$$A^k_{12} = B_i \int_{-1}^{1} \left(\frac{1-\xi}{2}\right)\left(\frac{1+\xi}{2}\right) \frac{h^k}{2} \, d\xi = \frac{1}{6} B_i h^k$$

$$A^k_{21} = B_i \int_{-1}^{1} \left(\frac{1+\xi}{2}\right)\left(\frac{1-\xi}{2}\right) \frac{h^k}{2} \, d\xi = \frac{1}{6} B_i h^k$$

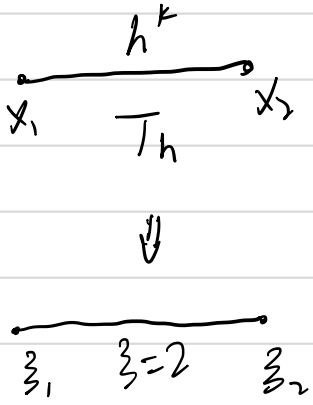
$$A^k_{22} = B_i \int_{-1}^{1} \left(\frac{1+\xi}{2}\right)^2 \frac{h^k}{2} \, d\xi = \frac{1}{3} B_i h^k$$

thus $$A^k = B_i h^k \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix}$$

for Robin B.C $\Gamma \setminus \Gamma_{root}$ $\quad k = 0, 1, 2, 3, 4$

for boundary with Neumann B.C. which is root element.

elemental load function:



$$F_{h_i} = \int_{T_{root}} v \, ds$$

$$\Rightarrow \quad F_{h_\alpha} = \int_{-1}^{1} \underline{H} \, \frac{h_k}{2} \, d\xi .$$

where $\underline{H_\alpha} = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$

$$H_1 = \frac{1-\xi}{2} \qquad H_2 = \frac{1+\xi}{2}.$$

then $F_{h_1} = \int_{-1}^{1} \frac{1-\xi}{2} \frac{h_k}{2} \, d\xi = \frac{1}{2} h_k$

$$F_{h_2} = \int_{-1}^{1} \frac{1+\xi}{2} \frac{h_k}{2} \, d\xi = \frac{1}{2} h_k$$

therefore on the root $\quad F_h = h_k \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \quad k=0.$

for output vector $\quad T_{root_h}(u) = (\underline{L_h})^T u_h(u)$

$$T_{root_h}(u) = l^0(u_h(u)) = l^0\left(\sum u_i \varphi_i\right)$$

$$= \sum u_i \, l^0(\varphi_i)$$

by definition $\sum u_i \, l^0(\varphi_i) = (\underline{F_h})^T \cdot \underline{u}$

then $(\underline{F_h})^T \cdot \underline{u} = (\underline{L_h})^T \cdot \underline{u} \quad \Rightarrow \quad \underline{L_h} = \underline{F_h} = h_k \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

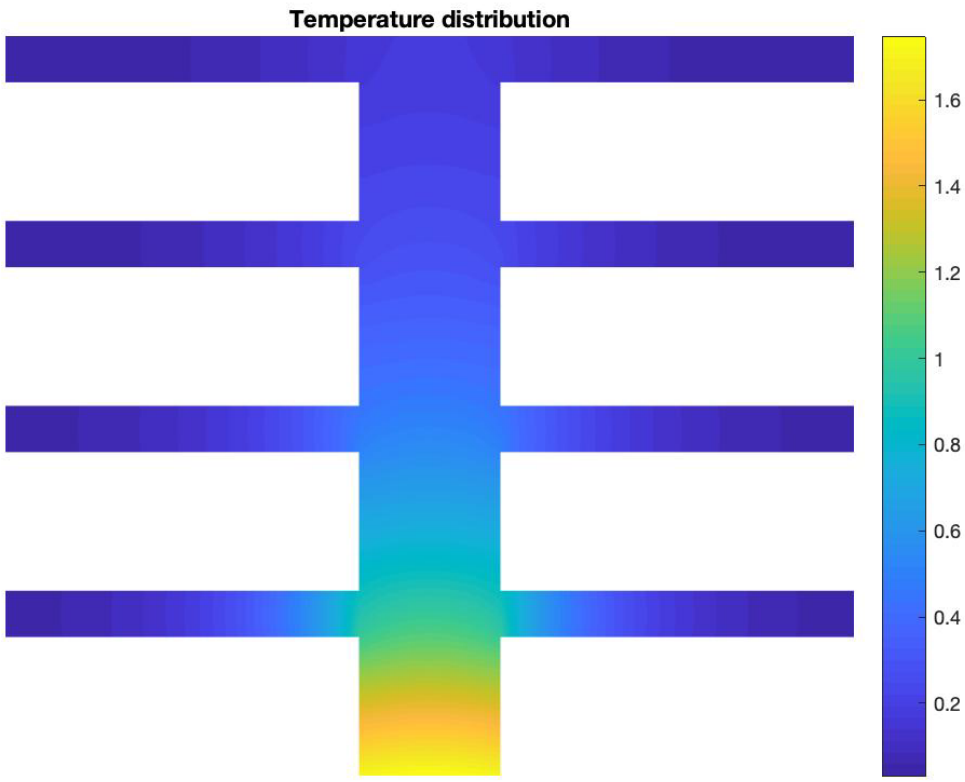
4. Code the elemental matrices, boundary conditions and loading vector and assemble them by using 'grids.mat'

solve the problem by $U = A \backslash F$.

and plot the solution

See attached code and plot

to get $T_{root}(M_0)$ do $(\bar{F}_h)^T \cdot U_{h_0}$ which in this case, we yield 1.7346.



```
% load grids
load 'grids.mat'

% specify parameters
K = [0.4,0.6,0.8,1.1,1];
Bi = 0.1;

% read numbers of elements from different mesh
for i = 1:7
    [M,N] = size(coarse.theta{i});
    coa_nofe(i) = M;
    [M,N] = size(medium.theta{i});
    med_nofe(i) = M;
    [M,N] = size(fine.theta{i});
    fin_nofe(i) = M;
end

% ***** medium mesh case *****

% initialization
Num = medium.nodes;
A = sparse(Num,Num);
F = sparse(Num,1);
Alocal = zeros(3);
Alocal_ex = zeros(2);
Flocal_root = zeros(2,1);

% fill out matrix of interior domain
for i = 1:5
    for k = 1:med_nofe(i)
        n1 = medium.theta{i}(k,1);
        n2 = medium.theta{i}(k,2);
        n3 = medium.theta{i}(k,3);
        x1 = medium.coor(n1,1);
        y1 = medium.coor(n1,2);
        x2 = medium.coor(n2,1);
        y2 = medium.coor(n2,2);
        x3 = medium.coor(n3,1);
        y3 = medium.coor(n3,2);
        coef = [1 x1 y1;1 x2 y2;1 x3 y3];
        rhs = [1;0;0];
        c1 = coef\rhs;
        cx1 = c1(2);
        cy1 = c1(3);
        rhs = [0;1;0];
        c2 = coef\rhs;
        cx2 = c2(2);
        cy2 = c2(3);
        rhs = [0;0;1];
        c3 = coef\rhs;
        cx3 = c3(2);
        cy3 = c3(3);
        Alocal(1,1) = cx1*cx1+cy1*cy1;
        Alocal(1,2) = cx1*cx2+cy1*cy2;
        Alocal(1,3) = cx1*cx3+cy1*cy3;
        Alocal(2,1) = cx2*cx1+cy2*cy1;
        Alocal(2,2) = cx2*cx2+cy2*cy2;
```

```
        Alocal(2,3) = cx2*cx3+cy2*cy3;
        Alocal(3,1) = cx3*cx1+cy3*cy1;
        Alocal(3,2) = cx3*cx2+cy3*cy2;
        Alocal(3,3) = cx3*cx3+cy3*cy3;
        area = abs(det(coef)/2);
        Alocal = K(i)*area*Alocal;
        for a = 1:3
            for b = 1:3
                I = medium.theta{i}(k,a);
                J = medium.theta{i}(k,b);
                A(I,J) = A(I,J)+Alocal(a,b);
            end
        end
    end
end

% fill out boundary with robin bc
for k = 1:med_nofe(6)
    n1 = medium.theta{6}(k,1);
    n2 = medium.theta{6}(k,2);
    x1 = medium.coor(n1,1);
    y1 = medium.coor(n1,2);
    x2 = medium.coor(n2,1);
    y2 = medium.coor(n2,2);
    h = sqrt((x1-x2)^2+(y1-y2)^2);
    Alocal_ex = [1/3 1/6;1/6 1/3];
    Alocal_ex = Bi*h*Alocal_ex;
    for a = 1:2
        for b = 1:2
            I = medium.theta{6}(k,a);
            J = medium.theta{6}(k,b);
            A(I,J) = A(I,J)+Alocal_ex(a,b);
        end
    end
end

% fill out load vector
for k = 1:med_nofe(7)
    n1 = medium.theta{7}(k,1);
    n2 = medium.theta{7}(k,2);
    x1 = medium.coor(n1,1);
    y1 = medium.coor(n1,2);
    x2 = medium.coor(n2,1);
    y2 = medium.coor(n2,2);
    h = sqrt((x1-x2)^2+(y1-y2)^2);
    Flocal_root = [1/2;1/2];
    Flocal_root = h*Flocal_root;
    for a = 1:2
        I = medium.theta{7}(k,a);
        F(I) = F(I)+Flocal_root(a);
    end
end

% solve for u
u = A\F;
% get mean temperature at root
T = F'*u;
plotsolution(medium,u)
```

5.

que 1. $T_{root} = \int_{T_{root}} u = l^0(u)$

$T_{rooth} = \int_{T_{root}} u_h = l^0(u_h)$

$l^0(u) = l^0(\sum u_i \varphi_i) = \sum u_i \, l^0(\varphi_i) = u^T F_h$

$l^0(u_h) = l^0(\sum \hat{u}_i \varphi_i) = \sum \hat{u}_i \, l^0(\varphi_i) = u_h^T F_h$

$T_{root}(u) - T_{rooth}(u) = u^T F_h - u_h^T F_h$

$= e^T \cdot F_h = l^0(e)$

$a(u, v) = a(u_h, v) = l(v)$

then $a(u, v) - a(u_h, v) = 0 = a(u - u_h, v) = a(e, v)$

take $v = e$ then $a(e, e) = l(e)$

therefore $a(e(u), e(u)) = T_{root}(u) - T_{rooth}(u)$

que 2. code for fine mesh and coarse mesh and work the solution use the equation given to get convergence.

for finest mesh $T_{root} = 1.7354$

for medium mesh $T_{root} = 1.7346$

for coarse mesh $T_{root} = 1.7317$

$$T_{root\,h} - T_{root\,2h} = C(2h)^b \quad ①$$

$$T_{root\,h} - T_{root\,4h} = C(4h)^b \quad ②$$

$$①/② \Rightarrow \frac{1.7354 - 1.7346}{1.7354 - 1.7317} = \frac{\cancel{C}2^b \cancel{h^b}}{\cancel{C}4^b \cancel{h^b}}$$

$$\Rightarrow 0.2162 = \left(\frac{1}{2}\right)^b$$

$$\Rightarrow b = 2.21.$$

Since $T_{root}(u) - T_{root\,h}(u) = a(e, e)$

and $e = u - u_h$ thus we can write e as $\sum e_i \varphi_i$

then $a(e, e) \rightarrow \cdots$ multiply by A_h

which we know is 2nd order accurate.

in the practice, we observe a convergence rate of 2.21 order of h

reason: We set U_{fine} as exact solution

which makes error smaller, thus converges faster.