AME 535a Introduction to Computational Fluid Dynamics
University of Southern California – Fall 2018

Project 2: Time-Dependent Problems and Finite Volume Method
Solitons and Traffic Flow

Handed Out: 20 Sept. 2018 *Due: 04 Oct. 2018*

Problem 1: Solitons (40 pts)

Problem Statement

J. Scott Russell, Scottish civil engineer and naval architect, wrote in 1844:

> "I believe I shall best introduce this phenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel."

In 1895, Korteweg and de Vries formulated the equation modeling Russell's observation, now known as the KdV equation:

$$u_t \;-\; 6\,u\,u_x \;+\; u_{xxx} \;=\; 0, \tag{1}$$

The term uu_x describes the sharpening of the wave and u_{xxx} the dispersion (*i.e.*, waves with different wave lengths propagate with different velocities). When these two terms balance, a propagating wave with unchanged form results. These waves with unchanged form are called *solitons*. They have a special shape, and have some very distinct properties. For example, a soliton with a larger amplitude will be 'thinner' than one with a smaller relative amplitude. In addition, the amplitude and velocity are interdependent.

Solitons are seen in several fields. The primary application of solitons today is in optical fibers, where the linear dispersion of the fiber provides smoothing of the wave, and the non-linear properties give the sharpening. The result is a very stable and long-lasting pulse that is free from dispersion, which is a problem with traditional optical communication techniques.

The KdV equation and solitons also have relevance to Tsunamis. The KdV equation is a governing

PDE for shallow water waves[1]. When a Tsunami propagates in the open ocean, it has a long wavelength in comparison to the depth of the water, and is hence considered a shallow water wave. A simple model for Tsunami propagation in the open ocean would thus be a soliton governed by the KdV equation.

Using direct substitution, we can show that the one-soliton solution

$$u_1(x,t) \;=\; -\,\frac{v}{2\cosh^2\left[\,{}^1\!/_2\,\sqrt{v}(x-vt-x_0)\,\right]} \tag{2}$$

solves the KdV equation (1). Here, $v > 0$ and x_0 are arbitrary parameters. Differentiate the one-soliton solution:

$$\begin{aligned}
u(x,t) &= -\frac{1}{2}\,\frac{v}{\left(\cosh\left[{}^1\!/_2\,\sqrt{v}\,(x-vt-x_0)\right]\right)^2} && (3)\\
u_x(x,t) &= \frac{1}{2}\,\frac{v^{3/2}\,\sinh\left[{}^1\!/_2\,\sqrt{v}\,(x-vt-x_0)\right]}{\left(\cosh\left[{}^1\!/_2\,\sqrt{v}\,(x-vt-x_0)\right]\right)^3} && (4)\\
u_{xx}(x,t) &= -\frac{1}{4}\,\frac{v^2\left(2\,\left(\cosh\left[{}^1\!/_2\,\sqrt{v}\,(x-vt-x_0)\right]\right)^2-3\right)}{\left(\cosh\left[{}^1\!/_2\,\sqrt{v}\,(x-vt-x_0)\right]\right)^4} && (5)\\
u_{xxx}(x,t) &= \frac{1}{2}\,\frac{v^{5/2}\,\sinh\left[{}^1\!/_2\,\sqrt{v}\,(x-vt-x_0)\right]\left[\left(\cosh\left[{}^1\!/_2\,\sqrt{v}\,(x-vt-x_0)\right]\right)^2-3\right]}{\left(\cosh\left[{}^1\!/_2\,\sqrt{v}\,(x-vt-x_0)\right]\right)^5} && (6)\\
u_t(x,t) &= -\frac{1}{2}\,\frac{v^{5/2}\,\sinh\left[{}^1\!/_2\,\sqrt{v}\,(x-vt-x_0)\right]}{\left(\cosh\left[{}^1\!/_2\,\sqrt{v}\,(x-vt-x_0)\right]\right)^3}\;. && (7)
\end{aligned}$$

Inserting into the KdV equation, and simplifying we get indeed

$$u_t(x,t)\;-\;6\,u(x,t)\,u_x(x,t)\;+\;u_{xxx}(x,t)\;=\;0\;. \tag{8}$$

Questions

1.1. (5 pts) We will solve the KdV equation numerically using the method of lines and finite difference approximations for the space derivatives. Rewrite the equation as

$$\frac{\partial u}{\partial t}\;=\;6\,u\,u_x\;-\;u_{xxx}\;, \tag{9}$$

and use a second-order accurate finite difference approximation for the right hand side (u_{xxx} is the only real new term here). In this case, you may derive or look up the finite difference approximation. Write the resulting semi-discrete equation.

<u>Hint</u>: use second-order accurate centered finite differences to discretize the right-hand-side of Eqn.(9), i.e. for the spatial discretization. The KdV equation is not the same as the wave

[1] see for instance Whitham, G.B, *Linear and Nonlinear Waves*, Wiley, New York, 1974

equation, andcentered difference is appropriate for this problem. You could of course use backward differences, but would find that the stability limit is very constraining, requiring a very small time step (more than 10 times smaller than with central differences).

1.2. (16 pts) For the time integration, we will use a fourth order Runge-Kutta scheme:

$$\underline{\alpha}^{(1)} = \Delta t\ f\Big(\underline{u}^{(n)}\Big) \tag{10}$$

$$\underline{\alpha}^{(2)} = \Delta t\ f\Big(\underline{u}^{(n)} + \alpha^{(1)}/2\Big) \tag{11}$$

$$\underline{\alpha}^{(3)} = \Delta t\ f\Big(\underline{u}^{(n)} + \alpha^{(2)}/2\Big) \tag{12}$$

$$\underline{\alpha}^{(4)} = \Delta t\ f\Big(\underline{u}^{(n)} + \alpha^{(3)}\Big) \tag{13}$$

$$\underline{u}^{(n+1)} = \underline{u}^{(n)} + \frac{1}{6}\Big(\underline{\alpha}^{(1)} + 2\,\underline{\alpha}^{(2)} + 2\,\underline{\alpha}^{(3)} + \underline{\alpha}^{(4)}\Big)\ , \tag{14}$$

in which f is the spatial discretization function. The stability region for this scheme consists of all z such that

$$\left|1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}\right| \le 1\ . \tag{15}$$

In particular, all points on the imaginary axis between $\pm i\,2\sqrt{2}$ are included.

Our equation is non-linear, and to make a stability analysis we first have to linearize it. In this case, the stability will be determined by the discretization of the third-derivative term u_{xxx}. Therefore, consider the simplified problem

$$\frac{\partial u}{\partial t} = -\,u_{xxx}\ , \tag{16}$$

and use von Neumann stability analysis to derive an expression for the maximum allowable time-step Δt in terms of Δx.

<u>Hint</u>: You do not need to make use of the specific Runge-Kutta steps (10)–(14) to do the stability analysis: all you need is the stability region (15).

1.3. (19 pts) Write a program that solves the KdV equation (9) using your 2^{nd} order spatial discretization and the four-stage Runge-Kutta time stepping scheme. Solve it in the region $-10 \le x \le 10$ with a grid size $\Delta x = 0.1$, and periodic boundary conditions $x(-10) = x(10)$.

Integrate from $t = 0$ to $t = 2$, using an appropriate time-step that satisfies the stability condition you derived above. For each of the initial conditions below, plot the initialization ($t = 0$) and the solution at $t = 2$. Unless otherwise specified, take $v = 20$ and $x_0 = 0$. Comment on the results.

Do this for all of the following initial conditions:

- **a.** To begin with, use a single soliton (2) as initial condition: $u(x,0) = u_1(x,0)$.
- **b.** The one-soliton solution looks almost like a Gaussian. Try $u(x,0) = -10e^{-x^2}$.
- **c.** Try the two-soliton solution $u(x,0) = -\frac{6}{\cosh^2(x)}$.
- **d.** Create "your own" two-soliton solution by superposing (adding) two one-soliton solutions with $v = 14$ and $v = 6$, both with $x_0 = 0$.

e. Same as before but with $(v, x_0) = (14, -3)$ and $(v, x_0) = (6, 3)$. Describe what happens when the two solitons cross (amplitudes, velocities), and after they have crossed.

f. Try $u(x, 0) = 2\ \sin(x/\pi)$.

Hint: To code the Runga-Kutta scheme, create the function f that returns the spatial discretization of a vector of nodal values, and call it to compute each of the four stages.

Problem 2: Traffic Flow (60 pts)

Problem Statement

The occurrence of car pile-up crashes on highways (often involving scores of cars) are often likened to a “domino effect”, wherein successive cars “fall off like dominoes”. Such a domino effect is often caused when a car decelerates abruptly, inducing a chain reaction that involves successive car drivers having to slam on their brakes and the decelerate abruptly. This chain reaction propagates through the highway and can lead to pile-up crashes. In this project, we explore some of the origins of this “domino effect.”

Mathematically, this domino effect can be likened to a shock wave occurring in the traffic. The notion of a shock wave has its origins in gas dynamics, wherein a compression wave passing through a gas can (under some conditions) become stronger and stronger and ultimately develop into a shock. Across this shock wave, near discontinuous transitions occur in density and velocity of the gas.

In the traffic scenario, we define density as the number of vehicles per unit length of highway. If a car decelerates abruptly, it sets up a compression wave behind it (with lots of cars behind getting bunched up together). This compression wave travels through the “vehicular fluid” on the highway and can lead to a pile-up crash (shock wave).

We shall first examine the formation of the shock wave using the non-linear hyperbolic equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0 \, , \tag{17}$$

in which $\rho = \rho(x,t)$ is the density of cars (vehicles/km) and $v = v(x,t)$ their (average) velocity. Let’s assume that the velocity is a function of density, namely

$$v = v_{\max} \left(1 - \frac{\rho}{\rho_{\max}} \right) \, , \tag{18}$$

where $v_{\max}$ is the maximum speed and $0 \leq \rho \leq \rho_{\max}$. The flux of cars is therefore given by

$$f(\rho) = \rho \, v_{\max} \left(1 - \frac{\rho}{\rho_{\max}} \right) . \tag{19}$$

We will investigate the use of first-order finite volume schemes to solve this problem.

Questions

2.1. (24 pts) Implement a first-order conservative finite volume scheme

$$\rho_i^{(n+1)} = \rho_i^{(n)} - \frac{\Delta t}{\Delta x}\left(F_{i+\frac{1}{2}}^{(n)} - F_{i-\frac{1}{2}}^{(n)}\right). \tag{20}$$

a. For the numerical flux function use Godunov's scheme, in which the flux is the exact solution to the Riemann problem at the interface between two volumes, *i.e.*,

$$F_{i+\frac{1}{2}}^{G} = f\left(\rho\left(x_{i+\frac{1}{2}}, t^{n+}\right)\right) = \begin{cases} \min\limits_{\rho\in[\rho_i,\rho_{i+1}]} f(\rho), & \rho_i < \rho_{i+1} \\ \max\limits_{\rho\in[\rho_i,\rho_{i+1}]} f(\rho), & \rho_i > \rho_{i+1} \end{cases}. \tag{21}$$

Analyzing the continuous function $f(\rho)$, determine the Godunov scheme flux in a manner that does not require a brute force search for the minimum and maximum of the flux between ρ_i and ρ_{i+1}.

b. Look at the problem of a traffic light turning green at time $t = 0$ and solve the continuity equation (17). Use the following problem parameters:

$$\rho_{\max} = 1.0, \quad v_{\max} = 1.0, \quad \Delta x = 0.001, \quad \Delta t = \frac{0.8\Delta x}{v_{\max}}. \tag{22}$$

The initial condition at the instant when the traffic light turns green is

$$\rho(x,0) = \begin{cases} 0.45, & 0 \le x \le 0.4 \\ 0.45 + 0.3\cos\left[5\pi(x-0.5)\right], & 0.4 < x \le 0.5 \\ 0.75, & 0.5 < x \le 0.65 \\ 0.45 + 0.3\cos\left[5\pi(x-0.65)\right], & 0.65 < x \le 0.75 \\ 0.45, & 0.75 < x \le 1. \end{cases} \tag{23}$$

Keep the boundary values equal to their initial conditions:

$$\rho(0,t) = \rho(0,0), \quad \rho(1,t) = \rho(1,0), \quad v(0,t) = v(0,0), \quad v(1,t) = v(1,0). \tag{24}$$

Describe what happens to the rising and falling edges of ρ as time evolves. How is this related to the domino effect?

2.2. (36 pts) Many researchers feel that using the continuity equation alone to model the traffic is not an accurate representation. They therefore use the momentum equation too, which has the form

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial(\rho\,\theta_0)}{\partial x} + \frac{V_e(\rho) - v}{\tau}, \tag{25}$$

where $V_e(\rho)$ represents the equilibrium velocity of the cars (in some sense, the desired velocity of the cars), and τ is a relaxation time that indicates the time it takes for u to attain the velocity V_e. The term θ_0 is assumed to be a constant.

We will now solve the system of continuity and momentum equations with attention to the role that θ_0 plays on the domino effect. Rewrite the momentum equation as

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho v^2 + \rho\,\theta_0)}{\partial x} = \frac{\rho\left[V_e(\rho) - v\right]}{\tau} \tag{26}$$

and solve equations (17) and (26) using the Lax method. Use the parameters

$$\tau = 10^{-2}\,, \qquad V_e(\rho) = v_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)\,, \tag{27}$$

the same initial and boundary conditions on ρ as in the previous question, and take

$$v(x,0) = 1 - \rho(x,0)\,, \quad v(0,t) = v(0,0)\,, \quad v(1,t) = v(1,0)\,. \tag{28}$$

> **Description of the Lax Scheme**
>
> Given a system of PDEs
>
> $$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = H\,, \tag{29}$$
>
> where U, F and H are vectors, the Lax scheme is
>
> $$U_i^{(n+1)} = U_i^{(n)} - \frac{\Delta t}{\Delta x}\left(F_{i+\frac{1}{2}}^{(n)} - F_{i-\frac{1}{2}}^{(n)}\right) + H\,\Delta t\,, \tag{30}$$
>
> where
>
> $$F_{i+\frac{1}{2}}^{(n)} = \frac{1}{2}\left(F_i^{(n)} + F_{i+1}^{(n)}\right) - \frac{\lambda_{i+\frac{1}{2}}}{2}\left(U_{i+1}^{(n)} - U_i^{(n)}\right) \tag{31}$$
>
> and
>
> $$\lambda_{i+\frac{1}{2}} = \max_{[i,i+1]} \lambda_{\mathbb{J}}\,. \tag{32}$$
>
> Here $\lambda_{i+\frac{1}{2}}$ represents the maximum eigenvalue of the Jacobian matrix $\mathbb{J}$ in the range of $i \in [i, i+1]$. The Jacobian is given by
>
> $$\{\mathbb{J}\}_{j,k} = \frac{\partial F_j}{\partial U_k}\,, \tag{33}$$
>
> in which the subscripts j and k refer to the respective elements of the vectors F and U.

a. Before we begin to solve the equations, determine the Δt required to ensure numerical stability of the Lax scheme for a given Δx. Recall that the von Neumann method is usable only for linear equations, whereas our equations are non-linear. Furthermore, the von Neumann method is used only for finite difference schemes, whereas we are using a finite volume scheme to solve our equations. Instead, determine stability by going through the following steps:

i. Linearize the continuity equation (17) and the momentum equation (26). To do this, assume small perturbations $\delta\rho$ and δv on ρ and v, respectively, such that

$$\rho = \rho_e + \delta\rho \tag{34}$$

and

$$v = V_e + \delta v\,, \tag{35}$$

where ρ_e and V_e represent a stationary and homogenous solution (its partial derivatives with respect to both x and t are zero). Substitute equations (34) and (35) into equations (17) and (26), and ignore terms containing products of the form $\delta\rho^2$, δv^2, $\delta\rho\,\delta v$, since they represent the products of small quantities and can therefore be neglected. Since stability does not depend on the source terms (right-hand-side of the equations), assume $H = 0$ for this analysis. We ultimately arrive at a system of linearized PDEs. Show your derivation and write the system of two linearized equations.

ii. Write the linearized system as

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0 \ ,$$

i.e., write down the expressions for

$$U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

in which the subscripts 1 and 2 refer to the respective variables and fluxes of the continuity and momentum equations. These will be in terms of $\{\rho_e, V_e, \theta_0, \delta\rho, \delta v\}$.

iii. Find the eigenvalues, λ, of the Jacobian

$$\mathbb{J} = \begin{bmatrix} \frac{\partial F_j}{\partial U_k} \end{bmatrix} = \begin{bmatrix} \partial F_1/\partial U_1 & \partial F_1/\partial U_2 \\ \partial F_2/\partial U_1 & \partial F_2/\partial U_2 \end{bmatrix}$$

and the determine largest one, $\lambda_{i+\frac{1}{2}}$.

<u>Hint</u>: To do this, first write F in terms of U, i.e. explicitly write down the functions $F_1(U_1, U_2)$ and $F_2(U_1, U_2)$. Then compute the Jacobian matrix $\mathbb{J}$ by differentiating F_1 and F_2 with respect to U_1 and U_2. Finally, compute the eigenvalues of $\mathbb{J}$, and determine which one is the largest.

iv. The Lax scheme is essentially a finite volume scheme. However, for the purpose of stability determination, we need to work with its finite difference representation. Write the Lax scheme in its corresponding finite difference representation, still assuming $H = 0$. In other words, write U^{n+1} at point i in terms of F's and U's evaluated at the other points grid (i, $i-1$, $i+1$, *etc.*) but not at the faces $i \pm \frac{1}{2}$.

v. Use the von Neumann Method to evaluate stability of the linearized system of PDEs when using the finite difference representation. After applying the principles of the von Neumann stability analysis, you eventually arrive at a system of two non-linear inequalities. containing the terms V_e, θ_0 and $\frac{\Delta t}{\Delta x}$. From these, you can determine the $\frac{\Delta t}{\Delta x}$ required to ensure stability. Plot the two inequalities on the complex plane for the $\frac{\Delta t}{\Delta x}$ that you have chosen.

<u>Hint</u>: In the linearized system, the F's can be written in terms of U's (as in iii), and $F = JU$ relates the Fourier representations for U and F.

b. Now solve equations (17) and (26), using $\Delta x = 0.001$ and the Δt that satisfies the stability criterion determined above. Solve the equations for $\theta_0 = 0.05, 0.1, 0.25, 0.5, 0.7$. For each of these, give plots of $\rho(x,t)$ and $v(x,t)$. What can you say about the role of θ_0 on the domino effect? Explain why you get the behavior you observe. How does this behavior compare with what you saw when you used the continuity equation alone?

Note: Even if you don't get through all of Question 2.2.a and find the $\Delta t/\Delta x$ value required for stability, you can still solve the system of equations by taking a very small Δt and so potentially get full points on Question 2.2.b.

Hint: The solution may "blow up" for some values of θ_0 that you are asked to use. This is the correct behavior. In your report, show the solution (density and velocity) up to the time when it blows up, and describe what happens. For the larger θ_0 values, you will be able to let your solution run for a longer time. Make sure you describe the behavior and the role of θ_0.