CS 111 ASSIGNMENT 3

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Problem 1: a) Consider the following linear homogeneous recurrence relation: $R_n = 4R_{n-1} - 3R_{n-2}$. It is known that: $R_0 = 1$, $R_2 = 5$. Find R_3.

b) Determine the general solution of the recurrence equation if its characteristic equation has the following roots: 1, -2, -2, 2, 7, 7.

c) Determine the general solution of the recurrence equation $A_n = 256A_{n-4}$.

d) Find the general form of the particular solution of the recurrence $B_n = 3B_{n-2} - 2B_{n-3} + 2$.

Solution 1:
(a) We know that: $R_0 = 1$, $R_2 = 5$
But we need first to put it into a characteristic equation and find its roots:
$x^2 - 4x + 3 = 0$
$\Rightarrow (x-1)(x-3)$ Factored it out.
General Form of the Equation:
$R_n = \alpha_1 1^n + \alpha_2 3^n$
Initial Condition Equation and Their solution:
$R_0 = \alpha_1 + \alpha_2 = 1$
$R_2 = \alpha_1 + \alpha_2 * 3^2 = 5$
We know that these are our current equations, so we need to solve these through a system of equations, I am going to be doing this by doing the substitution and elimination method mostly. I replace α_1 with x and α_2 with y.
Substitution:
$x + y = 1$
$-(x + 9y = 5)$
Put negatives in to cancel
$x + y = 1$
$-x - 9y = -5$
This should become: $-8y = -4$ Then divide both side by -8
$y = \frac{1}{2}$
Substitute it back into R_2:
$x + 9(\frac{1}{2}) = 5$
$x + \frac{9}{2} = 5$
$x = \frac{1}{2}$
Now we know our general formula, and our α_1 and α_2
$\frac{1}{2} * (1)^3 + \frac{1}{2}3^3 = 14$
So this means that $R_3 = 14$

(b) To find the general solution of this problem, we need to take the root and then change it into the general form, which is the following:
$(r-1)(r+2)^2 r(-2)(r-7)^2$
$A_n = \alpha_1 1^n + \alpha_2 2^n + \alpha_3(-2)^n + \alpha_4 n(-2)^n + \alpha_5(7)^n + \alpha_6 n(7)^n$

(c) $A_n = 256A_{n-4}$

$x^4 + 256 = 0$
$(x^2 - 16)(x^2 + 16) = 0$ We can break down -16 but for $+16$ we need to break them down into imaginary numbers as they are not perfect.
$(x - 4)(x + 4)(x - 4i)(x + 4i) = 0$
$A_n = \alpha_1 4^n + \alpha_2(-4)^n + \alpha_3(4i)^n + \alpha_4(-4i)^n$

(d) $B_n = 3B_{n-2} - 2B_{n-3} + 2$
2 is In-homogeneous so it becomes:
$B''_n = \beta * g(n) \Rightarrow B'$
β and B' are constant.
Then for homogeneous:
$B'n = 3B_{n-2} - 2B_{n-3}$
$B_n = B'_n + B''_n$
$B_{n-2} = B'_{n-2} + B''_{n-2}$
$B_{n-3} = B'_{n-2} + B''_{n-2}$
$B_{n-3} = B'_{n-3} + B''_{n-3}$
Plug the homogeneous and in-homogeneous back into an equation.
$B'_n + B''_n = 3(B'_{n-2} + B''_n n - 2) - 2(B'_{n-3} + B''_{n-3}) + 2$
$B'_n + B''_n = 3B'_{n-2} + 3B''_n n - 2 - 2B'_{n-3} + 2B''_{n-3} + 2$
Let start finding the homogeneous solution fully first:
$B'_n = 3B'_{n-2} - 2B'_{n-3}$ Let $B_n = x^n$
$x^n = 3x^{n-2} - 2x^{n-3}$
$x^3 = 3x - 2$
$x^3 - 3x + 2 = 0$
$\pm 1, \pm 2$ When doing the equation only +1 and -2 work as if we do -1 and +2 it wouldn't be equivalent to 0 so: $1^3 - 3(1) + 2 = 0, (-2)^3 - 3(-2) + 2 = 0$
This means that:
$x_1 = x_2 = 1, x_3 = -2$
$\alpha_1 * x_1^n + \alpha_2 * x_2^n$
Now let solve for the in-homogeneous solution next:
$B''_n = 3B''_{n-2} - 2B''_{n-3} + 2$
$B' = 3 * \beta' - 2\beta' + 2$
$\beta = \beta' + 2$
$0 \neq 2$
Let $B_n = \beta n^2$
$\beta n^2 = 3\beta(n-2)^2 - 2\beta(n-3)^2 + 2$
$= 3\beta(n^2 - 4n + 4) - 2\beta(n^2 - 6n + 9) + 2$
$= 3\beta n^2 - 12\beta n + 12\beta - 2\beta n^2 + 12\beta n$
$- 18\beta + 2$
$\beta n^2 = \beta n^2 - 6\beta + 2$
$0 = -6\beta + 2$
$6\beta = 2$
$\beta = \frac{1}{3}$ This means that the general form of the equation is:
$B_n = \alpha_1(1)^n + \alpha_2 n(1)^n + \alpha_3(-2)^n + \frac{1}{3}n^2$

Problem 2: Solve the following recurrence equations:

a)

$$\begin{aligned} f_n &= f_{n-1} + 4f_{n-2} + 2f_{n-3} \\ f_0 &= 0 \\ f_1 &= 1 \\ f_2 &= 4 \end{aligned}$$

Show your work (all steps: the characteristic polynomial and its roots, the general solution, using the initial conditions to compute the final solution.)

b)

$$\begin{aligned} t_n &= t_{n-1} + 2t_{n-2} + 2^n \\ t_0 &= 0 \\ t_1 &= 2 \end{aligned}$$

Show your work (all steps: the associated homogeneous equation, the characteristic polynomial and its roots, the general solution of the homogeneous equation, computing a particular solution, the general solution of the non-homogeneous equation, using the initial conditions to compute the final solution.)

Solution 2:

(a)

So in the problem we are given:

$$\begin{aligned} f_n &= f_{n-1} + 4f_{n-2} + 2f_{n-3} \\ f_0 &= 0 \\ f_1 &= 1 \\ f_2 &= 4 \end{aligned}$$

If we covert this to polynomial form it would be:

$r^3 = r^2 + 4r + 2$ Which becomes:

$r^3 - r^2 - 4r - 2 = 0$

Set $r = 1$ because this appears in the recurrence and with synthetic division we get: $(r+1)(r^2 - 2r - 2) = 0$ With that, the polynomial can be factored as:

$$(r+1)(r^2 - 2r - 2) = 0$$

This gives us roots $r = 1$, and from $r^2 - 2r - 2 = 0$, we find the other roots using the quadratic formula:

$$r = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)}$$

$$r = \frac{2 \pm \sqrt{4+8}}{2}$$

$$r = \frac{2 \pm \sqrt{12}}{2}$$

$$r = -1 \pm \sqrt{3}$$

So, the roots are $r = 1$, $r = 1 + \sqrt{3}$, and $r = 1 - \sqrt{3}$.

General Solution:

$f_n = f_1 * -1^n + f_2 * (1+\sqrt{3})^n + f_3 * (1-\sqrt{3})^n$
$f_0 = f_1 + f_2 + f_3 = 0$
$f_1 = f_1 * (-1)^1 + f_2(1+\sqrt{3})^1 + f_3(1-\sqrt{3})^1 = 1$
$f_2 = f_2 * (-1)^2 + f_2(1+\sqrt{3})^2 + f_3(1-\sqrt{3})^2 = 4$
Now let's get started on the systems of equations but let replace the f's with x,y, and z to make it easier for the eyes.

$x + y + z = 0$
$-x + (1+\sqrt{3}y + (1-\sqrt{z})z = 1$
$x + (4+2\sqrt{3})z = 4$
Now we move y and z to the other side of the x in the first equation: $x = -y - z$
Next, let's put it back into the other two other equations.
$-(-y-z) + (1+\sqrt{(3)})y + 1(1-\sqrt{3})z = 1$
$3y + 2\sqrt{3} + 3z - 2\sqrt{3}z = 4$
Next Simplify (normalized):
$2y + \sqrt{3}y + 2z - \sqrt{3}z = 1$
$y = (2-\sqrt{3})(1-2z+\sqrt{3}z)$
Simplify again to find z:
$[3(2-\sqrt{3})(1-2z+\sqrt{3}+2\sqrt{3}(2-\sqrt{3}))(1-2z-\sqrt{3}z)+3z-2\sqrt{3}z = 4]$
$[6z - 4\sqrt{3}z + \sqrt{3} = 4]$
$z = \frac{-6+5\sqrt{3}}{6}$
Now plug z in for y:
$y = (2-\sqrt{3})(1-2(-\frac{-6+5\sqrt{3}}{6}) + \sqrt{3}(\frac{-6+5\sqrt{3}}{6}))$
$y = \frac{5-6\sqrt{3}}{2\sqrt{3}} + 2$ Now next we solve for x:
$x = -(\frac{5-6\sqrt{3}}{2\sqrt{3}} + 2) - (\frac{-6+5\sqrt{3}}{6})$
$x = 2$
The final solution is:
$f_n = 2 * -1^n + -(\frac{5-6\sqrt{3}}{2\sqrt{3}} + 2)(1+\sqrt{3})^n - \frac{-6+5\sqrt{3}}{6} * (1-\sqrt{3})^n$

(b)
So in the problem we are given:
b)

$$\begin{aligned} t_n &= t_{n-1} + 2t_{n-2} + 2^n \\ t_0 &= 0 \\ t_1 &= 2 \end{aligned}$$

First, let us divide the solution into two parts where one is homogeneous and the other part is non-homogeneous.
$t_n = T_n^h + t_n^P$ Here:
$t_n^h =$ Homogeneous Part of the equation
$t_n^p =$ Non-Homogeneous Part of the equation which is the Particular solution
First we solve t_n^h:
Let us assume $t_n = r^n$
So $t_n - t_{n-1} - 2t_{n-2} = 0$
$r^n - r^{n-1} - 2r^{n-2} = 0$
Now we divide the entire equation by r^{n-2}, the smallest exponent, and we get:

$$\frac{r^n - r^{n-1} - 2r^{n-2}}{r^{n-2}} = 0$$

$r^2 - r - 2 = 0$
$(r-2)(r+1) = 0$
$r = 2$ or $r = -1$
$t_n^h = \alpha_1(2)^n + \alpha_2(-1)^n$
Now we find t_n^p:
$t_n = t_{n-1} + 2t_{n-2} + 2^n$
$Cn2^n = C(n-1)2^{n-1} + 2C(n-2)2^{n-2} + 2^n$
$-Cn2^n + C(n-1)2^{n-1} + 2C(n-2)2^{n-2} + 2^n = 0$
Now we solve for Constant (C) by dividing the entire equation by 2^{n-2}, the smallest exponent, and we get:

$$\frac{Cn2^n + C(n-1)2^{n-1} + 2C(n-2)2^{n-2} + 2^n}{2^{n-2}}$$

$-4Cn + 4Cn - 6C + 4 = 0$
$-6C + 4 = 0$
$-6C = -4$
$C = \frac{4}{6}$
$C = \frac{2}{3}$
Now we go back to the problem where:
$t_n = T_n^h + t_n^P$
$tn = \alpha_1(2)^n + \alpha_2(-1)^n + Cn2^n$
Let n = 0:
$0 = \alpha_1(2)^0 + \alpha_2(-1)^0 + \frac{2}{3}(0)2^0$
$0 = \alpha_1(1) + \alpha_2(1) + 0$
$0 = \alpha_1 + \alpha_2$
So we can say:
$\alpha_1 = -\alpha_2$
Now we substitute it back to the General Equation to find α_2
$t_n = -\alpha_2(2)^n + \alpha_2(-1)^n + Cn2^n$
$t_1 = -\alpha_2(2)^n + \alpha_2(-1)^n + Cn2^n$
Here $n = 1$ and $t_1 = 2$
$2 = -\alpha_2(2)^1 + \alpha_2(-1)^1 + \frac{2}{3}(1)(2)^1$
$2 = -2\alpha_2 - \alpha_2 + \frac{4}{3}$
$2 = -3\alpha_2 + \frac{4}{3}$
$-3\alpha_2 = 2 - \frac{4}{3}$ $-3\alpha_2 = \frac{2}{3}$
$3\alpha_2 = -\frac{2}{3}$
$\alpha_2 = -\frac{2}{9}$
$\alpha_1 = -\alpha_2$
$\alpha_1 = \frac{2}{9}$
Now we test the answer with T_1 to see if our answer works.
General form:
$T_n = \alpha_1(2)^n + \alpha_2(-1)^n + Cn2^n$
Now we plug $n = 1$
$2 = \frac{2}{9}(2)^1 - \frac{2}{9}(-1)^1 + \frac{2}{3}(1)(2)^1$
$2 = \frac{4}{9} + \frac{2}{9} + \frac{4}{3}$
$2 = \frac{4+2}{9} + \frac{4}{3}$
$2 = \frac{6}{9} + \frac{4}{3}$
$2 = \frac{6+12}{9}$
$2 = \frac{18}{9}$
$2 = 2$
Hence it is proved that our answer works

Problem 3: We want to tile an $n \times 1$ strip with 1×1 tiles that are green (G), blue (B), and red (R), 2×1 purple (P) and 2×1 orange (O) tiles. Green, blue and purple tiles cannot be next to each other, and there should be no two purple or three blue or green tiles in a row (for ex., GGOBR is allowed, but GGGOBR, GROPP and PBOBR are not). Give a formula for the number of such tilings. Your solution must include a recurrence equation (with initial conditions!), and a full justification. You do not need to solve it.

Solution 3:
So from the problem, we can tell that there are a couple of restrictions to take note of, first green, blue, and purple tiles cannot be next to each other, and there should be no two purple or three blue or green tiles in a row.

Let's start with Case G (Green), where it can end with one G, in which it cases would be RG (n-2) and OG (n-3) as Red is a 1×1 tile and Orange is a 2×1 time and we know that G is a 1×1 tile. Then there is the case if Case G starts with two G's so it would be like GG, in which it can become RGG (n-3) and OGG (n-4)

Next case we have the blue cases in which it can be represented by "B", so if we do the cases that have B at the end it should be something like RB (n-2) and OB (n-3) if we have the cases where it ends in double B which is "BB", then it would be RBB (n-3), OBB (n-4).

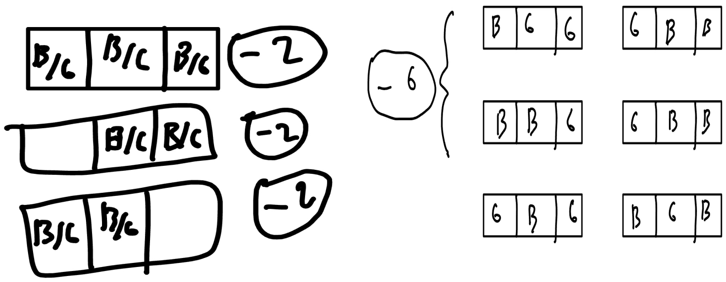
Next we have the case of Purple, we know in the restrictions that G, B, and P can not be next to each other, which means they can only be next to orange and red, so we have RP (n-3) and OP(n-4).

Also we also know that red and orange have no restriction whatsoever, so O(n-2) and R(n-1).

Through these cases we know the following initial conditions:
$T_0 = 1$
$T_1 = 3$
$T_2 = 9$
$T_3 = 23$
How did we get these? For the first one, we know an empty tile is just 1, and for the second initial condition it is just our one tile that is possible which is just Red, Green, and Blue combine them and you would get 3 and it becomes $T_1 = 3$.

Then for the other initial condition, we know that $T_2 = 9$ because first, we have P and O which are two of the tiles that are 2×1, and then 9 other tile combinations without any restriction, which in total would be 11 total tiles with two tiles combination, however we would subtract it by 2 again because of the restriction that Green and Blue can't be together, in which the combinations are GB and BG, which makes our $S_2 = 9$.

Then for our last initial condition, we take the amount of possible purple tiles which is 2 because Purple can't be next to G, B, and itself and it can't be 2P in a row, and the possibilities of orange color with restrictions, which is 6. Next, we count the possibilities of the combinations of the 3×1 which is just $3 \times 3 = 27$, so in total, we have $27 + 2 + 6 = 35$ but now we need to also include the restriction that is evident in the problem so we know that we have these following restriction depicted:



With these two following restrictions in total, we would subtract 12 from 35. Which in total would be 23. This means that it is $S_3 = 23$ which is our other condition.

Our general form of the solution knowing all of these facts would be (due to the restrictions):
$1T_{n-1} + 3T_{n-2} + 5T_{n-3} + 3T_{n-4}$

Academic integrity declaration. The homework papers must include at the end an academic integrity declaration. This should be a brief paragraph where you state *in your own words* (1) whether you did the homework individually or in collaboration with a partner student (if so, provide the name), and (2) whether you used any external help or resources.

We did this homework as partners, to get the basic concepts we used discussion notes and lecture notes to understand the material. To refresh ourselves on old concepts like systems of equations we took to the help of Youtube to gain a broader understanding of the material. We also got help from Ezekiel Pogue and Alice Thai, who are the TAs for the class on the homework. Lastly, we did use a scientific calculator to make sure we compute well.

Submission. To submit the homework, you need to upload the pdf file to Gradescope. If you submit with a partner, you need to put two names on the assignment and submit it as a group assignment.