CS111, Winter 24
ASSIGNMENT 1

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Problem 1: Give an asymptotic estimate for the number $h(n)$ of "Hello"s printed by Algorithm PRINT-HELLOS below. Your solution *must* consist of the following steps:

(a) First express $h(n)$ using the summation notation $\sum$.

(b) Next, give a closed-form expression[1] for $h(n)$.

(c) Finally, give the asymptotic value of $h(n)$ using the Θ-notation.

Show your work and include justification for each step.

Algorithm PRINTHELLOS (n : **integer**)
  for $i \leftarrow 1$ **to** n **do**
    for $j \leftarrow 1$ **to** $3i^2 + i$ **do** print("Hello")
  for $i \leftarrow 1$ **to** $2n^2$ **do**
    for $j \leftarrow 1$ **to** i **do** print("Hello")

Note: If you need any summation formulas for this problem, you are allowed to look them up. You do not need to prove them, you can just state in the assignment when you use them.

Solution 1:

1. $\sum_{i=1}^{n} 3i^2 + i + \sum_{i=1}^{2n^2} i$ utilizing example 11 on asymptotic notation.

2. $\sum_{i=1}^{n} 3i^2 + \sum_{i=1}^{n} i + \sum_{i=1}^{2n^2} i$ Using the formula of summation.
$= 3\frac{(n(n+1)(2n+1))}{6} + \frac{n(n+1)}{2} + \frac{2n^2(2n^2+1)}{2}$
$= 3\frac{(n(n+1)(2n+1))}{6} + \frac{n^2+n}{2} + \frac{4n^4+2n^2}{2}$
$= 3\frac{(2n^3+n^2+2n^2+n)}{6} + \frac{n^2+n}{2} + \frac{4n^4+2n^2}{2}$
$= 1\frac{(2n^3+n^2+2n^2+n)}{2} + \frac{n^2+n}{2} + \frac{4n^4+2n^2}{2}$ Cancel the 3 and 6 to reduce it.
$= \frac{2n^3+n^2+2n^2+n}{2} + \frac{n^2+n}{2} + \frac{4n^4+2n^2}{2}$
$= \frac{2n^3+n^2+2n^2+n+n^2+n+4n^4+2n^2}{2}$ Combine the like terms into one fraction.
$= \frac{4n^4+2n^3+3n^2+2n^2+n^2+n+n}{2}$
$= \frac{4n^4+2n^3+6n^2+2n}{2}$
$= \frac{2(2n^4+n^3+3n^2+n)}{2}$ Cancel the 2 out.
$= 2n^4 + n^3 + 3n^2 + n$

3. (From example 9) Consider the function $h(n) = 2n^4 + n^3 + 3n^2 + n$. We start with an upper bound: $h(n) = 2n^4 + n^3 + 3n^2 + n \leq 2n^4 + n^4 + 3n^4 + n^4 = 7n^4$, so $h(n) = O(n^4)$. Next we get n lower bound estimate. For $n \geq 4$ we have $2n^4 + n^4 + 3n^4 \geq 2n^4$, So, $h(n) = 2n^4 + n^3 + 3n^2 + n \geq 2n^4$. Therefore $h(n) = \Omega(n^4)$ Putting those two bounds together, we obtain that $h(n) = \Theta(n^4)$

Problem 2:

[1] A closed-form expression is a formula that can be evaluated in some fixed number of arithmetic operations, independent of n. For example, $3n^5 + n - 1$ and $n2^n + 5n^2$ are closed-form expressions, but $\sum_{i=1}^{n} i^2$ is not, as it involves $n - 1$ additions.

(a) Use properties of quadratic functions to prove that $3x^2 \geq (x+1)^2$ for all real $x \geq 4$.
(b) Use mathematical induction and the inequality from part (a) to prove that $3^n \geq 2^n + 3n^2$ for all integers $n \geq 4$.
(c) Let $g(n) = 2^n + 3n^2$ and $h(n) = 3^n$. Using the inequality from part (b), prove that $g(n) = O(h(n))$. You need to give a rigorous proof derived directly from the definition of the O-notation, without using any theorems from class. (First, give a complete statement of the definition. Next, show how $g(n) = O(h(n))$ follows from this definition.)

Solution 2:

1. Given:
   $3x^2 \geq (x+1)^2$
   To Prove:
   $x \geq 4$
   Proof:
   $3x^2 \geq (x+1)^2$
   $3x^2 \geq x^2 + 1 + 2x$
   $2x^2 - 2x - 1 \geq 0$
   We then use the quadratic formula.
   $a = 2, b = -2, c = -1$
   $= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
   $= \frac{-(-2) \pm \sqrt{-2^2 - 4(2)(-1)}}{2(2)}$
   $= \frac{2 \pm \sqrt{4+8}}{4}$
   $= \frac{2 \pm \sqrt{12}}{4} = \frac{2 \pm 2\sqrt{3}}{4}$
   $x = \frac{2+2\sqrt{3}}{4}, x = \frac{2-2\sqrt{3}}{4}$ Simplify both of the equations.
   $x = \frac{1}{2} + \frac{\sqrt{3}}{2}$ or $x = \frac{1}{2} + \frac{\sqrt{3}}{2}$
   We can test $x \geq 4$ by plugging the numbers into the $2x^2 - 2x - 1 \geq 0$
   $2(4)^2 - (24) - 1 \geq 0$
   $32 - 8 - 1 \geq 0$
   $23 \geq 0$
   If we select any numbers outside of $x | x \not\subset (\frac{1}{2} - \frac{\sqrt{3}}{2}, \frac{1}{2}, + \frac{\sqrt{3}}{2})$ based on $2x^2 - 2x - 1 \geq 0$ where $x \geq 4$ and if we plug any numbers we would get a positive number hence it goes to infinity. Then it proves that $3x^2 \geq (x+1)^2$

2. Base case: $n \geq 4$
   $3^4 \geq 2^4 + (3(4)^2$
   $81 \geq 16 + 48$
   $81 \geq 64$
   Induction Hypothesis: $n = k$
   $3^{k+1} = 3^k * 3 \geq 3(2^k + 3k^2)$
   Now we say:
   $3(2^k + 3k^2) \geq 2^{k+1} + 3(k+1)^2$
   Isolate Terms:
   $3 * 2^k \geq 2^{k+1}$
   $3 * 2^k \geq 2^k * 2$ Cancel 2^k out.
   $3 \geq 2$
   From Part (a) we can say that:
   $3k^2 \geq 3(k+1)^2$
   So we can say that:
   $3^{k+1} \geq 3(2^k + 3k^2) \geq 2^{k+1} + 3(k+1)^2$

3. Let $g(n)$ and $h(n)$; $\mathbb{Z} \rightarrow \mathbb{Z}$ be true functions. We say that $g(n)$ is of order (at most) $h(n)$, denoted $h(n) = O(g(n))$, if these are constants c and n_0 such that $|h(n)| \leq c * g(n)$ for all $n \geq n_0$. Based on part B we have proved that $3^n > 2^n + 3n^2$ if $n > 4$.

Problem 3: Give asymptotic estimates, using the Θ-notation, for the following functions:

(a) $3n^3 - 15n^2 + 2n + 4$

(b) $3n^2 \log n + 2n^2\sqrt{n} + n^2$

(c) $n \log^3 n - 5n + \dfrac{n^2}{\log n}$

(d) $7 \cdot n^5 + n^3 \log^2 n + 2^n$

(e) $\log^9 n + n^3 4^n + n5^n$

Solution 3:

1. We have $3n^3 - 15n^2 + 2n + 4 \leq 3n^3 + 2n^3 + 4n^3 = 9n^3$ (We dropped $15n^2$ because it is negative)
   For $9n^3 \leq n^4$(9 is the c_1 aka coefficient)
   $9 \leq n$
   $n \geq 1 \Rightarrow O(n^3)$
   We also have $3n^3 - 15n^2 + 2n + 4 \geq 3n^3 - 15n^2$
   $3n^3 - 15n^2 + 2n + 4 \geq 3n^3 - 15n^2$
   $3n^3 - 15n^2 + 2n + 4 \geq -12n^3, (-12n^3$ is wrong so to remove it take a variable with a higher degree.)
   $3n^3 - 15n^2 + 2n + 4 \geq 3n^3 - 3$
   $3n^3 - 15n^2 + 2n + 4 \geq \Omega(3n^3 - n^3)$
   $3n^3 - 15n^2 + 2n + 4 \geq \Omega(2n^3)$
   $3n^3 - 15n^2 + 2n + 4 \geq 2\Omega(n^3)$ for $n \geq 15 \Rightarrow \Omega(n^3)$ (2 is c_2)
   Pick $c_1 = 9, c_2 = 2, n_0 = max(1, 15) = 15$
   We have: $n^3 \leq f(n) \leq 9n^3$ for $n \geq 15 \Rightarrow f(n) = \Theta(n^3)$

2. $3n^2 \log n + 2n^2\sqrt{n} + n^2$
   We have $3n^2 \log n + 2n^2 n^{0.5} + n^2 \leq 3n^{2.5} + 2n^{2.5} + n^{2.5}$ multiply the term with $n^{0.5}$ to make them the highest power $\rightarrow = 6n^{2.5}$ for n $\geq 1 \Rightarrow O(n^{2.5})$
   We also have $3n^2 + 2n^2 n^{0.5} + n^2 \geq 2n^{2.5}$
   $3n^2 + 2n^2 n^{0.5} + n^2 \geq \Omega(2n^{2.5})$
   $3n^2 + 2n^2 n^{0.5} + n^2 \geq 2\Omega(n^{2.5})$, (2 is c_2) for
   $n \geq 1 \Rightarrow \Omega(n^{2.5})$
   Pick $c_1 = 6, c_2 = 2, n_0 = max(1, 1) = 1$
   We have; $n^{2.5} \leq f(n) \leq 6n^{2.5}$ for $n \geq 1 \Rightarrow f(n) = \Theta(n^{2.5})$

3. $nlog^3 n - 5n\frac{n^2}{\log n}$
   We first remove the negative when calculating Big O.
   $= n \log^3 n + \frac{n^2}{\log n}$
   We multiple it with $\log n$ we get
   $n \log^3 n + \frac{n^2}{\log n} \leq \frac{1}{\log n}(n \log^4 n + n^2)$
   Now we raise power for all and drop by $\log n$ because $n^2 > \log n$
   $n \log^3 n + \frac{n^2}{\log n} \leq \frac{1}{\log n}(n * n + n^2)$
   $n \log^3 n + \frac{n^2}{\log n} \leq \frac{1}{\log n}(n^2 + n^2)$
   $\Rightarrow \frac{1}{\log n}(2n^2)$ for $n \geq 2 \Rightarrow 0(\frac{n^2}{\log n})$

We also have $n\log^3 n - 5n + \frac{n^2}{\log n} \geq \frac{n^2}{\log n}$ for
$n \geq 4 \Rightarrow f(n) = 1\Omega(\frac{n^2}{\log n})$
Pick $c_1 = 2, c_2 = 2, n_0 = max(2, 4) = 4$
We have: $\frac{n^2}{\log n} \leq f(n) \leq \frac{2n^2}{\log n}$ for $n \geq 4 \Rightarrow f(n) = \Theta\frac{n^2}{\log n}$

4. $7 * n^5 + n^3 \log^2 n + 2n$
We have $7n^5 + n^3 \log^2 n \leq 2^n + 7O(2^n) + O(2^n)$
$n^3 \log^2 n \Rightarrow n^3 O(n) \Rightarrow n^4 = O(2^n)$(Log breaks down)
$= 9 * 2^n$ for $n \geq 1 \Rightarrow O(2^n)$[Our $c_1 = 9, n_0 = 1$]
We also have $7 * n^5 + n^3 \log^2 n + 2n \geq 1 * 2^n$
$7 * n^5 + n^3 \log^2 n + 2n \geq 1\Omega(2^n)$ for
$n \geq 1 \Rightarrow \Omega(2^n)$
Pick $c_1 = 9, c_2 = 1, n_0 = max(1, 1) = 1$
We have: $2^n \leq f(n) \leq 9 * 2^n$ for $n \geq 1 \Rightarrow f(n) = \Theta(2^n)$

5. $\log^9 n + n^3 4^n + n5^n$
We have $\log^9 n + n^3 + 4^n + n5^n \leq n5^n + O(n5^n) + O(n5^n)$
$= 3n5^n$ for $n \geq 1 \Rightarrow O(n5^n)$
We also have $\log^9 n + n^3 + 4^n + n5^n \geq 1 * n5^n$
$\log^9 n + n^3 + 4^n + n5^n \geq 1\Omega(n5^n) \log n \geq 1 \Rightarrow \Omega(n5^n)$
Pick $c_1 = 3, c_2 = 1, n_0 = max(1, 1) = 1$
We have: $n5^n \leq f(n) \leq 3n5^n$ for $n \geq 1 \Rightarrow f(n) = \Theta(n5^n)$

Academic integrity declaration. The homework papers must include at the end an academic integrity declaration. This should be a brief paragraph where you state *in your own words* (1) whether you did the homework individually or in collaboration with a partner student (if so, provide the name), and (2) whether you used any external help or resources.

Submission. To submit the homework, you need to upload the pdf file to Gradescope and the cpp file on canvas (in Assignments). Before uploading your code to canvas, test it through Codeforces. The score that you received on Codeforces will be your score for the RSA code. If you submit with a partner, you need to put two names on the assignment and submit it as a group assignment.

Reminders. Remember that only LaTeX papers are accepted.