

Directed hypergraph connectivity augmentation by hyperarc re-orientations

Combinatorial Optimization and Graph Theory

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- Connectivity problems, characterisations
- Hypergraphs

State of the art, goal of the article

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 - ▶ Exhibiting a sequence of orientations such that :
 - The arc-connectivity does not decrease, and the arc-connectivity of the last element of the sequence is k .
 - The next orientation in the sequence can be obtained from the previous one by flipping exactly one arc.
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Side note : This article generalise the results of **Ito and al.**, as directed graphs are special case of hypergraphs.

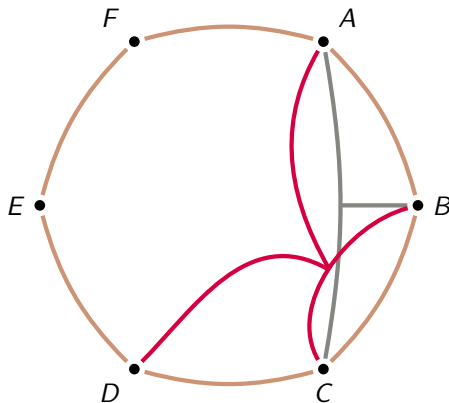
Hypergraphs



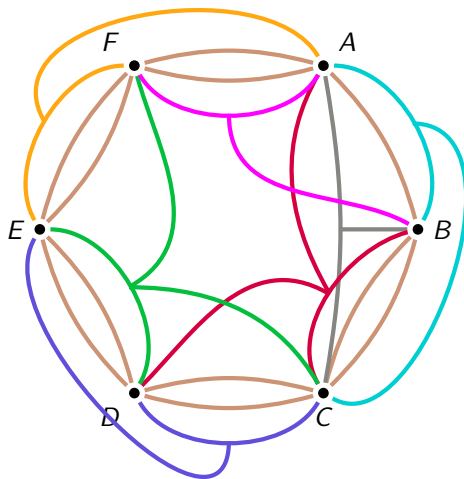
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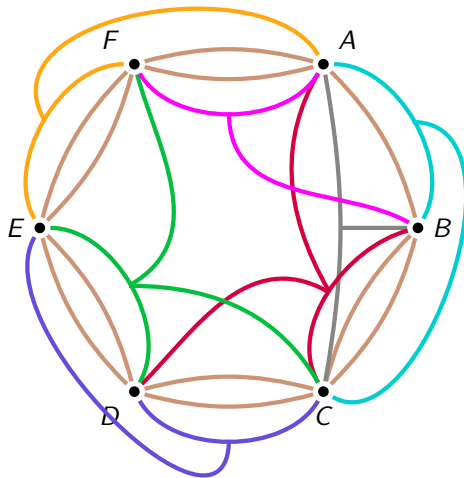


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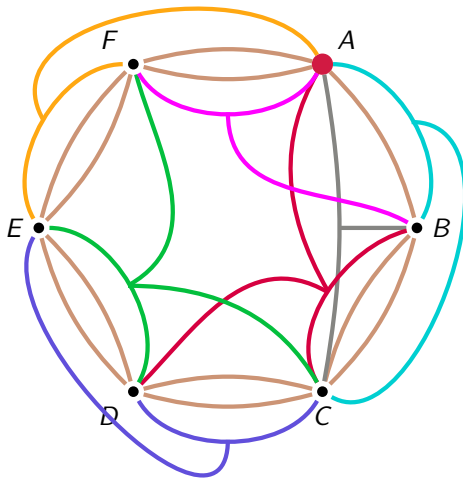
Degree of $\emptyset \neq X \subsetneq V$

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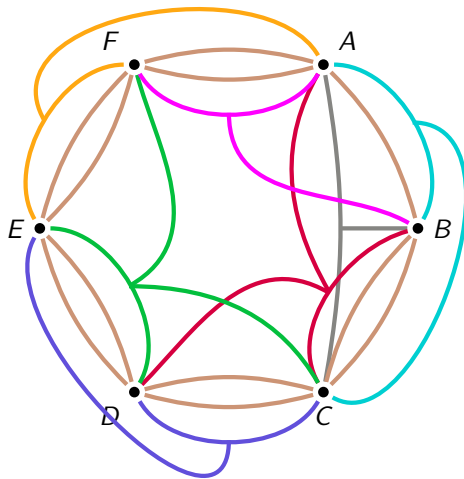


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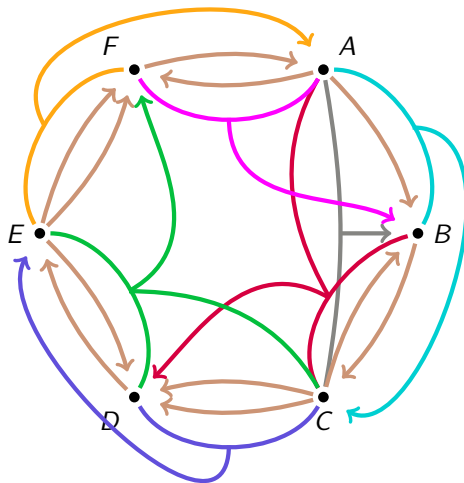
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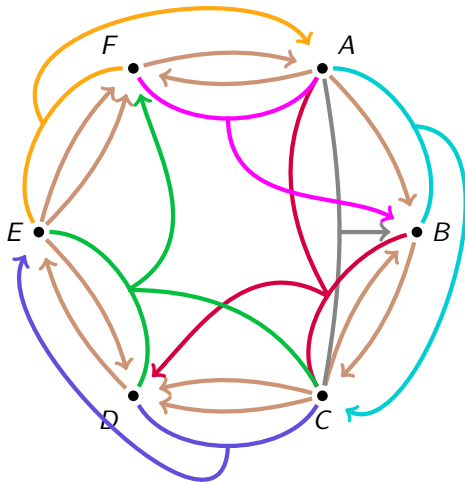


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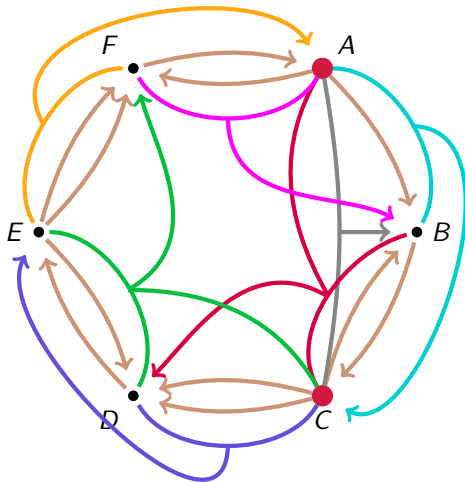
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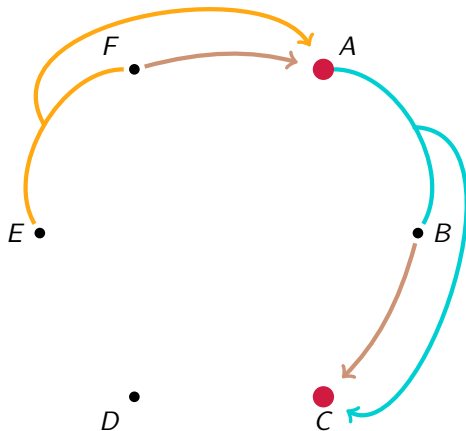
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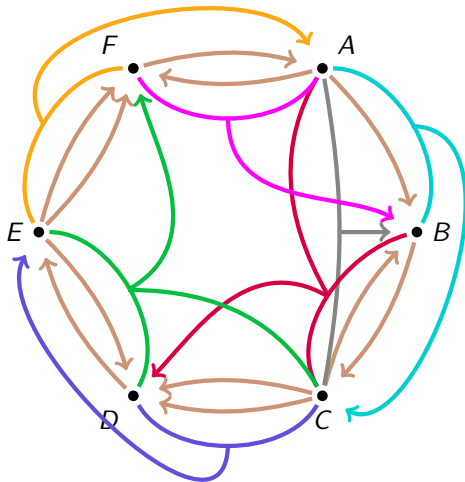
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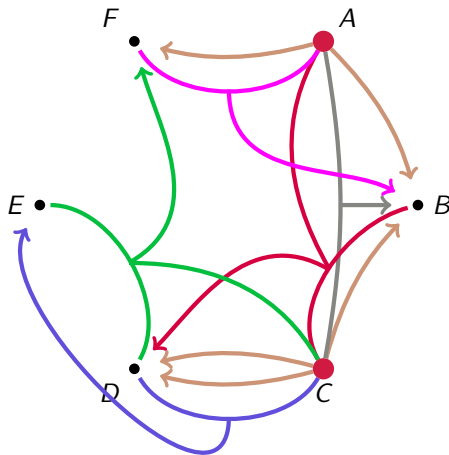
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Hyperarc-connectivity and (k, k) -partition connected hypergraphs

- $\vec{\mathcal{H}}$ is k -hyperarc-connected, if, $\forall \emptyset \neq X \subsetneq V$, $d_{\vec{\mathcal{H}}}^+(X) \geq k$.
- The hyperarc-connectivity of a hypergraph, denoted $\lambda(\vec{\mathcal{H}})$, is the maximum value of k such that $\vec{\mathcal{H}}$ is k -hyperarc-connected.

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We use a result of Frank : \mathcal{H} is (k, k) -partition-connected if and only if it admits a k -hyperarc-connected orientation.

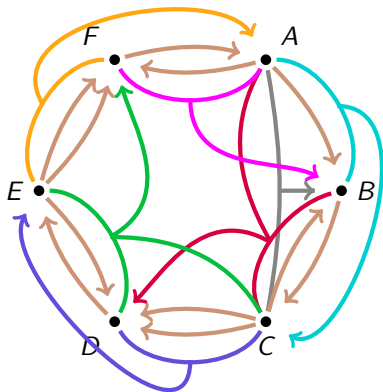
Main result

Main result (Theorem 7)

Let $\mathcal{H} = (V, E)$ be a $(k + 1, k + 1)$ -partition-connected hypergraph and $\vec{\mathcal{H}}$ is a k -hyperarc connected orientation of \mathcal{H} . Then there exists a sequence of hypergraphs $(\vec{\mathcal{H}}_i)_{i \in 0 \dots \ell}$ such that $\vec{\mathcal{H}}_{i+1}$ is obtained from $\vec{\mathcal{H}}_i$ by reorienting exactly one hyperarc and $\lambda(\vec{\mathcal{H}}_{i+1}) \geq \lambda(\vec{\mathcal{H}}_i)$ and $\lambda(\vec{\mathcal{H}}_\ell) = k + 1$. Such a sequence of orientations can be obtained with $\ell \leq |V|^3$ and found in polynomial time (in the size of \mathcal{H}).

"High-Level"-running of the algorithm

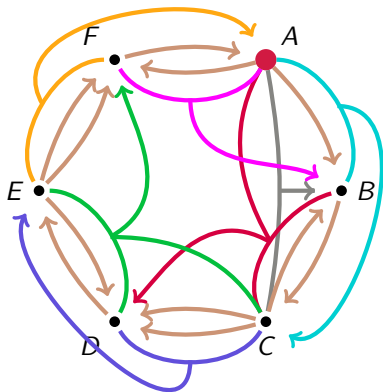
Our algorithm will provide a 3-hyperarc-connected orientation of \mathcal{H} , starting from a 2-hyperarc-connected.



- 1 Take r in $V(\mathcal{H})$.
- 2 Compute sets of vertices.
- 3 Stopping Criteria
- 4 Select a set R (cf. 2.)
- 5 Find an admissible (s, t) -hyperpath in R to reorient
- 6 Reorient the corresponding hyperpath.
- 7 Goto (2.)

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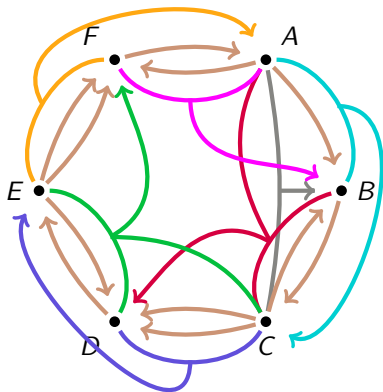
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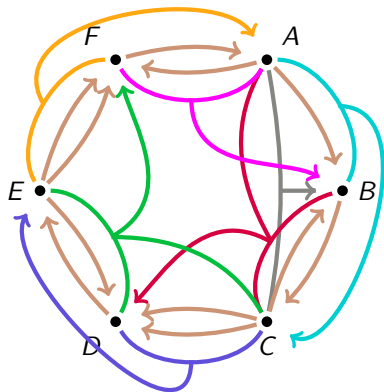
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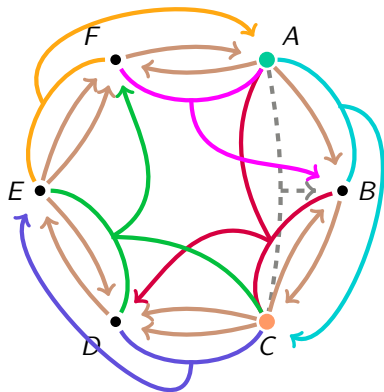
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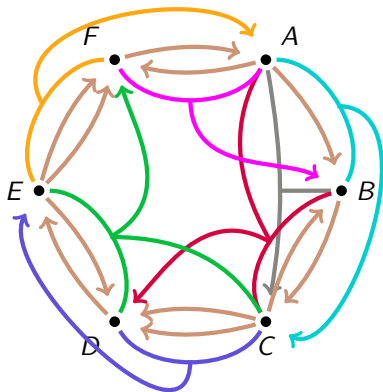
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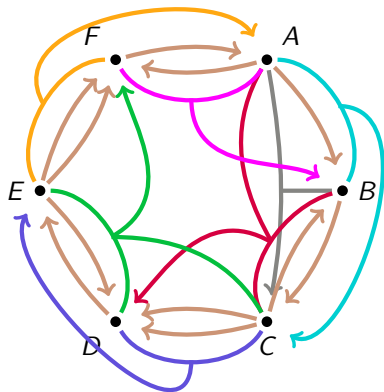
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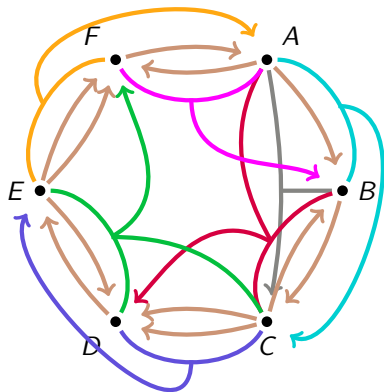
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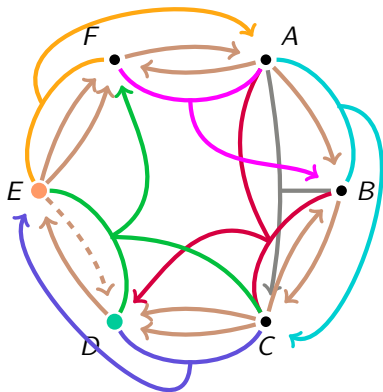
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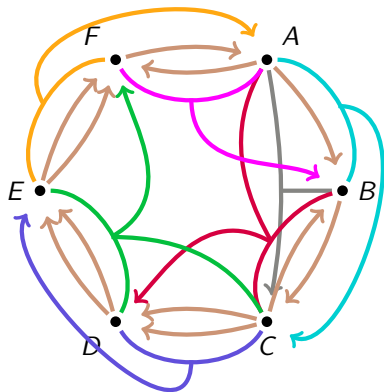
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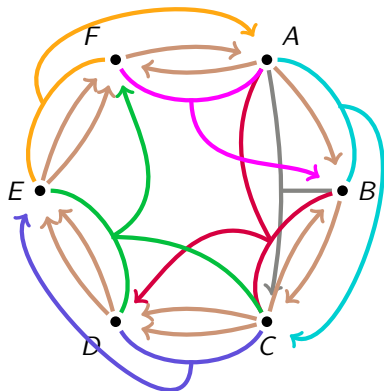
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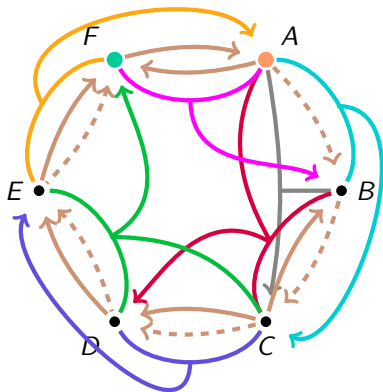
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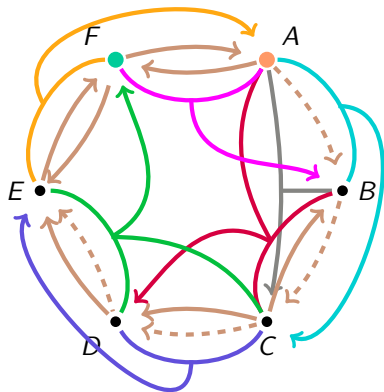
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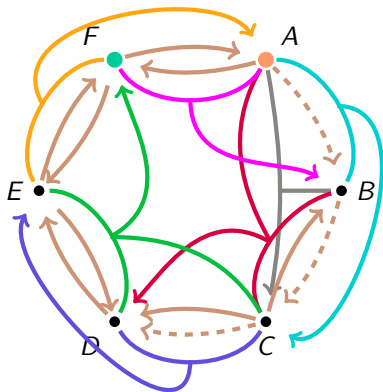
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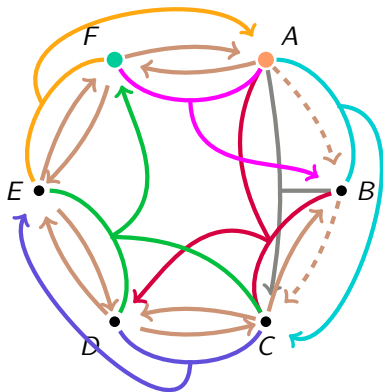
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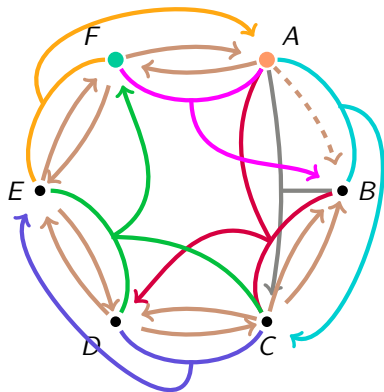
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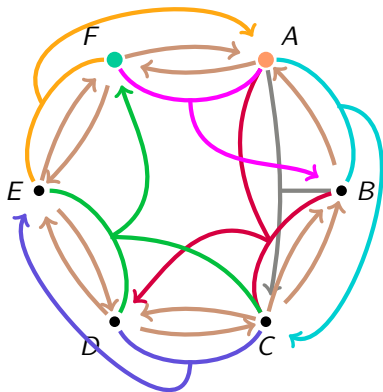
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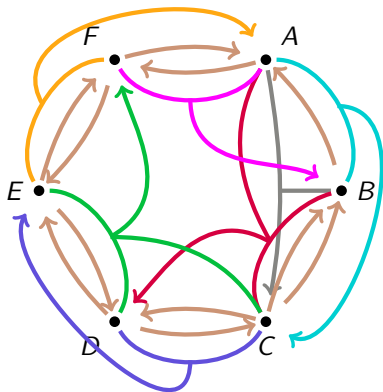
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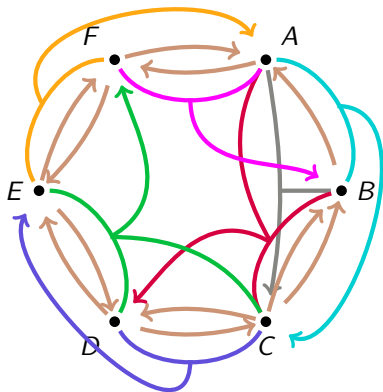
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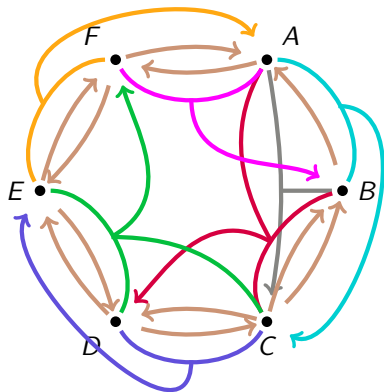
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Finding *admissible* (s, t) -hyperpaths in R

- Crucial segment of the algorithm.
- Three criterion for P to be an admissible (s, t) -hyperpath in R :
 - ① s is a *safe source* in $S \subseteq R$, t is a *safe sink* in $T \subseteq R$.
 - ② Reorient each hyperarc, *one by one*, does not decrease the hyperarc-connectivity.
 - ③ After reorientation of P , there is a set whose cardinality is a guarantee that the algorithm will stop.

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Finding *admissible* (s, t) -hyperpaths in R

- Crucial segment of the algorithm.
- Three criterion for P to be an admissible (s, t) -hyperpath in R :
 - ① s is a **safe source** in $S \subseteq R$, t is a **safe sink** in $T \subseteq R$.
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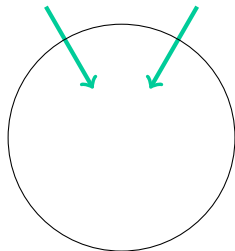
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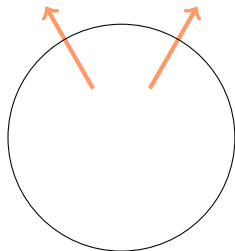
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A brief detour...

Tight and Dangerous sets



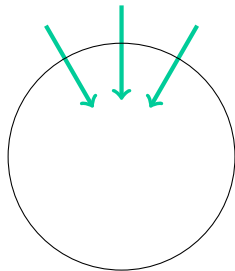
In-Tight sets



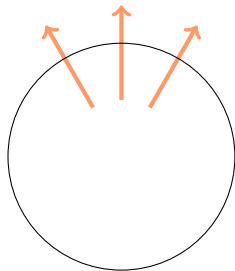
Out-Tight sets

- $\mathcal{T}_- = \{X \subseteq V - r, d^-(X) = k\} \cup \{V\}$
- $\mathcal{T}_+ = \{X \subseteq V - r, d^+(X) = k\} \cup \{V\}$
- $\mathcal{D}_- = \{X \subseteq V - r, d^-(X) = k + 1\}$
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- \mathcal{M}_- : Inclusion-wise minimal members of \mathcal{T}_-
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Tight and Dangerous sets



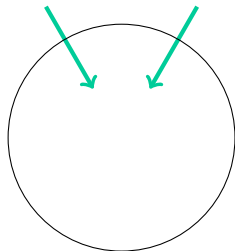
In-Dangerous sets



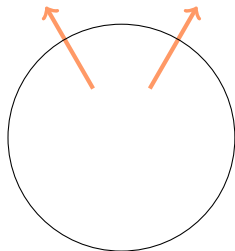
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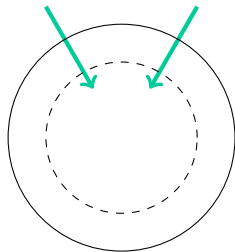
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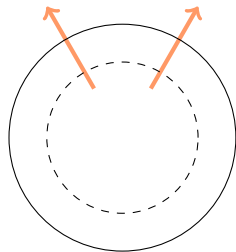
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Tight and Dangerous sets



Minimal In-Tight sets



Minimal Out-Tight sets

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Crossing sets and structural results

Let X, Y two crossing sets in V .

Claim 1(b)

If $X, Y \in \mathcal{T}_+$, then both $X \cup Y \in \mathcal{T}_+$ and $X \cap Y \in \mathcal{T}_+$.

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Proof of Claim 1(b)

- We have $\lambda(\vec{\mathcal{H}}) = k$
- Since X, Y are crossing, $X \cap Y \neq \emptyset$, $X \cup Y \neq V$.
- $k + k = d^+(X) + d^+(Y)$
- By submodularity, $d^+(X) + d^+(Y) \geq d^+(X \cup Y) + d^+(X \cap Y)$
- By $\lambda(\vec{\mathcal{H}}) = k$, $d^+(X \cup Y) \geq k$ and $d^+(X \cap Y) \geq k$
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Existence of a safe source (*a safe sink*)

Lemma 10

$\forall S \in \mathcal{M}_-$, there is a safe source $s \in S$.

Likewise,

Lemma 11

$\forall T \in \mathcal{M}_+$, there is a safe sink $t \in T$.

Quick outline of a proof for Lemma 10 :

- Let $S \in \mathcal{M}_-$.
- Considering a family of vertex sets (χ) that cover as many vertices of S as possible, but using as little as vertex sets possible.
- We can prove that, under given assumptions, χ cannot cover every vertex of S .
- Vertices that are not covered by χ are "potential" safe sources, the last part of the proof is verifying that they are effectively safe sources.

Towards hyperarc connectivity augmentation

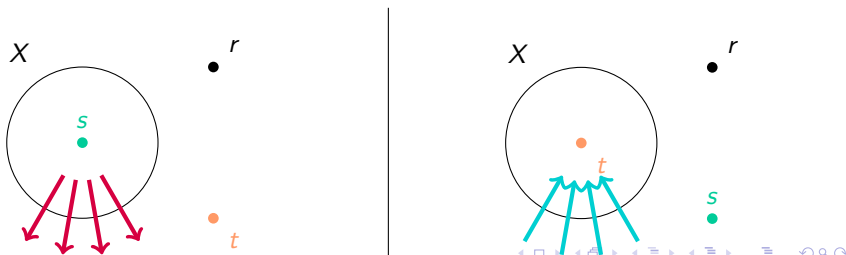
$\mathcal{R} : R \subseteq V - r$ inclusion-wise minimal such that either :

- $R \in \mathcal{T}_-$, and contains a member of \mathcal{T}_+
- or $R \in \mathcal{T}_+$, and contains a member of \mathcal{T}_- .

Lemma 13

Let $R \in \mathcal{R}$, $S \in \mathcal{M}_-$, $T \in \mathcal{M}_+$ such that $S, T \subseteq R$. Let s be a safe source in S , t a safe sink in T .

- $\forall X \subseteq V - r$ such that $s \in X$, $t \notin X$, we have $d^+(X) \geq k + 1$.
- $\forall X \subseteq V - r$ such that $s \notin X$, $t \in X$, we have $d^-(X) \geq k + 1$.



Towards hyperarc connectivity augmentation

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Proof of Lemma 13

By contradiction, either :

- a. $s \in X, t \notin X, d^+(X) = k$, i.e. $s \in X, t \notin X, X \in \mathcal{T}_+$.
 - a1. $R \in \mathcal{R} \cap \mathcal{T}_-$
 - a2. $R \in \mathcal{R} \cap \mathcal{T}_+$
- b. $s \notin X, t \in X, d^-(X) = k$, i.e. $s \notin X, t \in X, X \in \mathcal{T}_-$.
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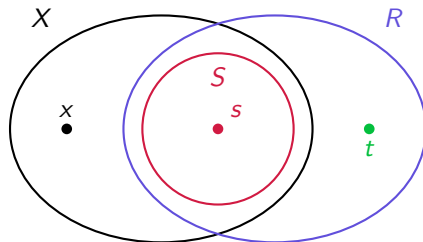
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Proof of Lemma 13

$a : \exists X \subseteq V - r, s \in X, t \notin X, X \in \mathcal{T}_+$

- Since $s \in S$ is a **safe source** and $s \in X \in \mathcal{T}_+$, we have $S \subsetneq X$
- We also have $t \in R \setminus X$ by [a.], so $X \setminus R \neq \emptyset$.

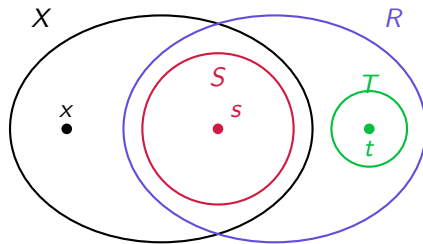


Proper representation of a

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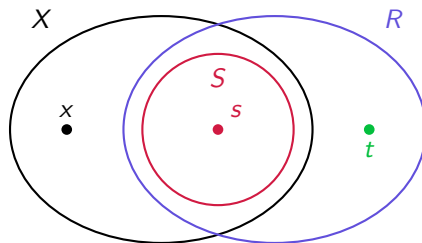
$a1. : R \in \mathcal{R} \cap \mathcal{T}_-, \exists X \subseteq V - r, s \in X, t \notin X, X \in \mathcal{T}_+$.

- As $t \in R \setminus X \neq \emptyset$, and using Claim 1, we have $R \setminus X \in \mathcal{T}_-$.
- $T \cap X \neq \emptyset$ would contradict the minimality of T , so T and X are disjoint.
- As $R \setminus X \in \mathcal{T}_-$, $T \in \mathcal{T}_+$, and $T \subseteq R \setminus X$, this contradicts R minimal.

Proof of Lemma 13

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Proper representation of a

$a2. : R \in \mathcal{R} \cap \mathcal{T}_+, \exists X \subseteq V - r, s \in X, t \notin X, X \in \mathcal{T}_+.$

- $R \in \mathcal{T}_+, X \in \mathcal{T}_+$, and $X \cap R \neq \emptyset \implies X \cap R \in \mathcal{T}_+$
- $S \in \mathcal{T}_-, S \subseteq R \cap X$. Since $r \in R \setminus X, X \cap R \subsetneq R$.
- This contradicts the minimality of R .

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By contradiction, either :

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Finding *admissible* (s, t) -hyperpaths in $R \in \mathcal{R}$

Three criterion for P to be an admissible (s, t) -hyperpath in R :

1. s is a safe source in $S \subseteq R$, t is a safe sink in $T \subseteq R$.
2. Reorienting each hyperarc, **one by one**, does not decrease the hyperarc-connectivity
3. Let $\vec{\mathcal{H}}'$ the hypergraph obtained after reorientation of P .
 - ▶ \mathcal{M}' : Inclusion-wise minimal members of $\mathcal{M}'_- \cup \mathcal{M}'_+$
 - ▶ Either $|\mathcal{M}'| < |\mathcal{M}|$, either $|\mathcal{M}'| = |\mathcal{M}|$ and \mathcal{M}' covers more vertices than \mathcal{M} .

Point 3. is the stopping criteria for the main algorithm :

- $\mathcal{M} = \{V\}$ implies both $\mathcal{M}_- = \{V\}$ and $\mathcal{M}_+ = \{V\}$.
- $\mathcal{T}_- = \{X \subseteq V - r, d^-(X) = k\} \cup \{V\}$
- \mathcal{M}_- : Inclusion-wise minimal members of \mathcal{T}_-
- Finally, if $\lambda(\vec{\mathcal{H}}) \geq k$ and $\mathcal{T}_- = \mathcal{T}_+ = \{V\}$, $\vec{\mathcal{H}}$ is $(k + 1)$ -hyperarc-connected.

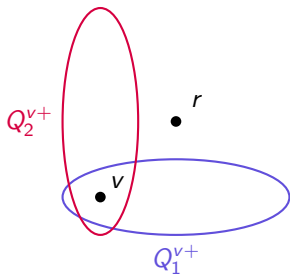
Introduction of Q_+^v

Definition of Q_+^v

Consider the sets of \mathcal{T}_+ containing v . Q_+^v is **the** minimal (inclusion-wise) one.

Unicity of Q_+^v :

If it exists, Q_+^v is unique.



Let Q_1^{v+} , Q_2^{v+} verifying the above definition.

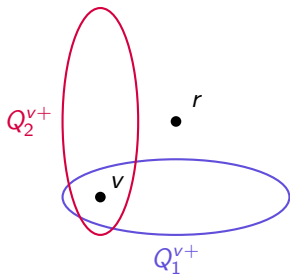
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By definition, $Q_1^{v+} \not\subseteq Q_2^{v+}$ and $Q_2^{v+} \not\subseteq Q_1^{v+}$.

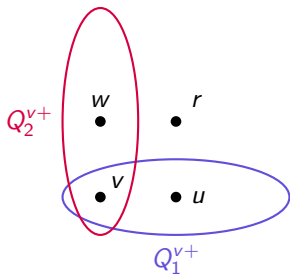
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Denote $u \in Q_1^{v+} \setminus Q_2^{v+}$, $w \in Q_2^{v+} \setminus Q_1^{v+}$.

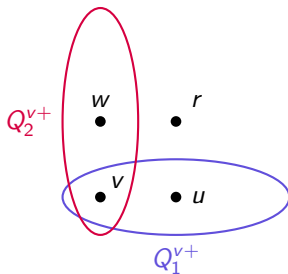
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As $r \notin Q_1^{v+}, Q_2^{v+}$, both are crossing sets.

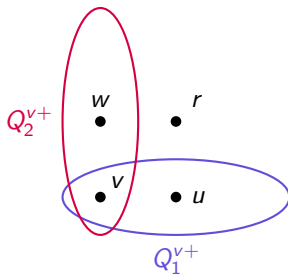
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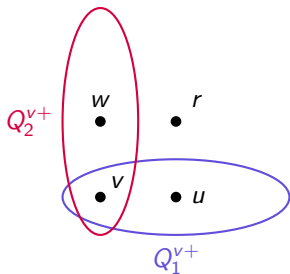
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$Q_1^{v+} \cap Q_2^{v+}$ is smaller (inclusion-wise) than Q_1^{v+} and Q_2^{v+} .

Existence of an hyperpath that does not leave Q_+^v

Lemma 12 (a)

$\forall s \in V, \forall t \in Q_+^s$, there exists an (s, t) -hyperpath that does not leave Q_+^s .

Proof of Lemma 12 (a)

- By contradiction, assume that there is $s \in V, t \in Q_+^s$ such that any (s, t) -hyperpath leaves Q_+^s .
- There is $s \in Z \subseteq Q_+^s \setminus \{t\}$ such that any hyperarc leaving Z will also leave Q_+^s .
- We have the following inequalities
 - ▶ $d_{\mathcal{H}}^+(Q_+^s) \geq d_{\mathcal{H}}^+(Z)$
 - ▶ $d_{\mathcal{H}}^+(Z) \geq k$, as \mathcal{H} is k -hyperarc-connected.
 - ▶ $k = d_{\mathcal{H}}^+(Q_+^s)$ by definition.
- We can deduce that $d_{\mathcal{H}}^+(Z) = k$, which automatically implies that $Z \in \mathcal{T}_+$.
- Q_+^s is not minimal, hence the contradiction.

Existence of an hyperpath that does not leave Q_+^v

Lemma 12 (a)

$\forall s \in V, \forall t \in Q_+^s$, there exists an (s, t) -hyperpath that does not leave Q_+^s .

Proof of Lemma 12 (a)

- By contradiction, assume that there is $s \in V, t \in Q_+^s$ such that any (s, t) -hyperpath leaves Q_+^s .
- There is $s \in Z \subseteq Q_+^s \setminus \{t\}$ such that any hyperarc leaving Z will also leave Q_+^s .
- We have the following inequalities
 - ▶ $d_{\mathcal{H}}^+(Q_+^s) \geq d_{\mathcal{H}}^+(Z)$
 - ▶ $d_{\mathcal{H}}^+(Z) \geq k$, as \mathcal{H} is k -hyperarc-connected.
 - ▶ $k = d_{\mathcal{H}}^+(Q_+^s)$ by definition.
- We can deduce that $d_{\mathcal{H}}^+(Z) = k$, which automatically implies that $Z \in \mathcal{T}_+$.
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Finding an admissible (s, t) -hyperpath in $R \in \mathcal{R} \cap \mathcal{T}_-$

1. Only input of the algorithm $R \in \mathcal{R} \in \mathcal{T}_-$
 - ▶ s, t are constrained (maybe not unique) by the choice of R .
2. Choosing $S \in \mathcal{M}_-$ such that $S \subseteq R$, then a safe source $s \in S$.
3. Main part of the algorithm : s -out arborescence
 - ▶ F : (Directed) arborescence, rooted in s
 - ▶ Z : Explored (yet) vertices
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Algorithm Admissible (s, t) -hyperpath in $R \in \mathcal{R} \cap \mathcal{T}_-$

- 1: Take a set $S \in \mathcal{M}_-$, with $S \subseteq R$, then a safe source $s \in S$.
 - 2: $Z = \{s\}$, $F = (Z, \emptyset)$, $V' = R$
 - 3: **while** $h = (X, v)$ exists such that $v \in V' - Z$ and $X \cap Z \neq \emptyset$ **do**
 - 4: Let $u \in X \cap Z$.
 - 5: $Z \leftarrow Z \cup \{v\}$
 - 6: $F \leftarrow F + uv$
 - 7: **if** $Q_+^v \subsetneq V'$ **then**
 - 8: $V' \leftarrow Q_+^v$
 - 9: **end if**
 - 10: **end while**
 - 11: $T = V'$
 - 12: Take a safe sink $t \in T$
 - 13: $P' = F[s, t]$
 - 14: P is the corresponding hyperpath in $\vec{\mathcal{H}}$, obtained with P' .
 - 15: **Return** S, T, s, t, P
-

A few words about complexity

- Computing \mathcal{R} , \mathcal{M}_- , \mathcal{M}_+ in polynomial time :
 - ▶ We can transform our hypergraph in a network (by trimming),
 - ▶ In which we can apply **Edmonds-Karp** to compute (s, t) -cuts.
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Final words

- Generalization to hypergraphs of a previous article (by **Ito and al.**).
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Thank you for your attention

