# Directed hypergraph connectivity augmentation by hyperarc re-orientations

Combinatorial Optimization and Graph Theory

Benoît Bompol, Armand Grenier

Thursday, Nov 23rd 2023

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  - Algorithmic proof of *Nash-Williams*, by flipping one arc at a time.
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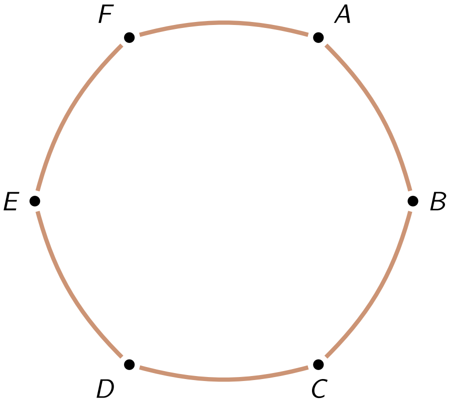
Goal of the article : Expanding the result of **Ito and al.** to hypergraphs.
Side note : This article generalise the results of **Ito and al.**, as directed graphs are special case of hypergraphs.

# Hypergraphs



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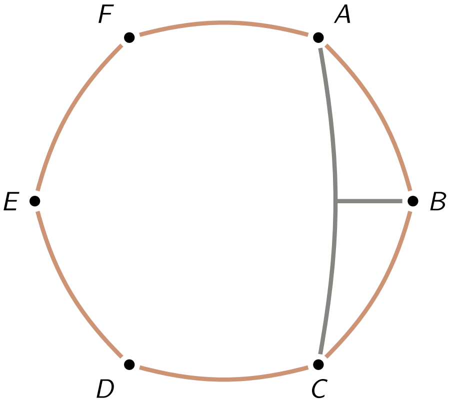

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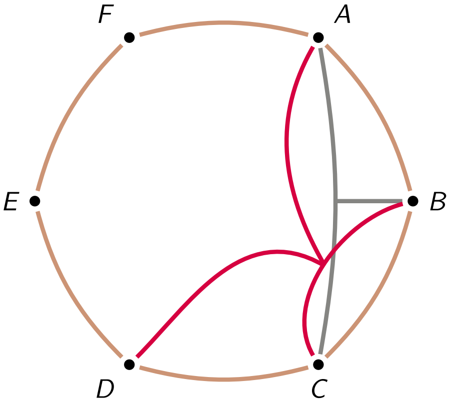

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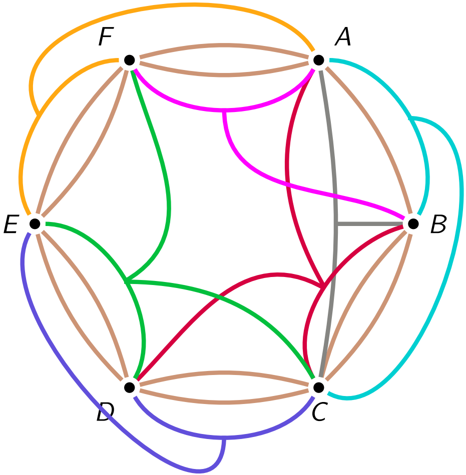

# Degree of $\varnothing \neq X \subsetneq V$

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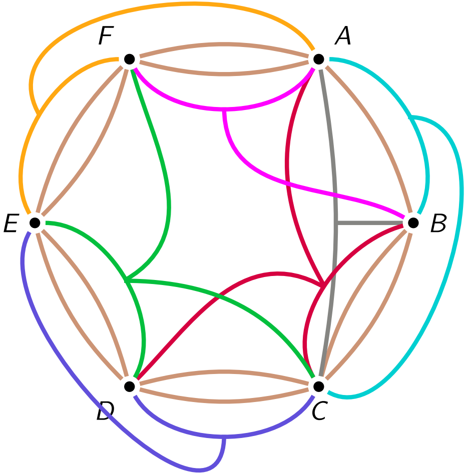

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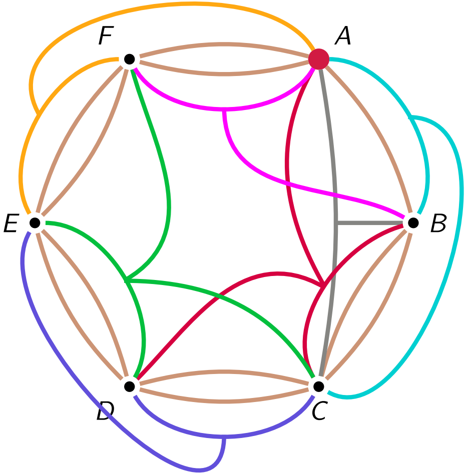

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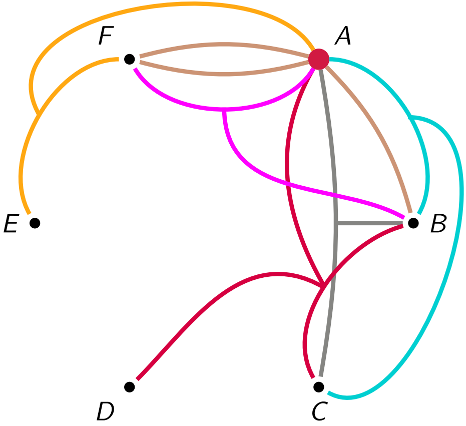

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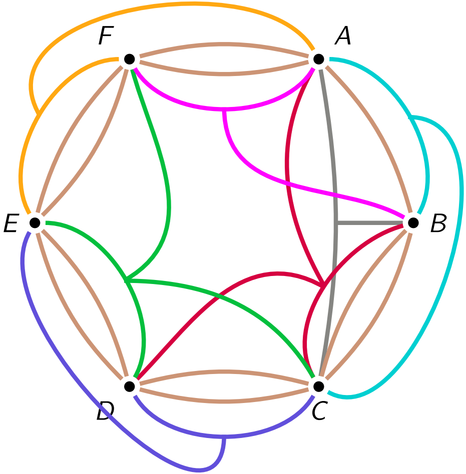

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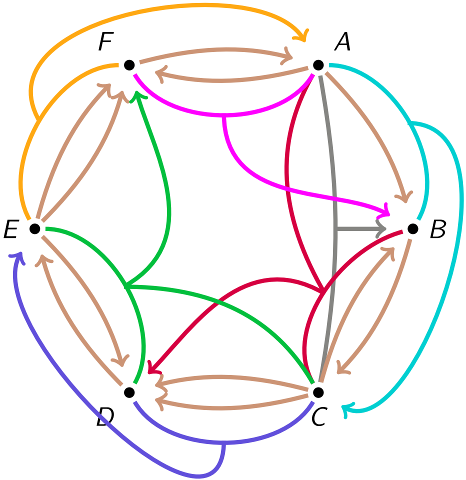

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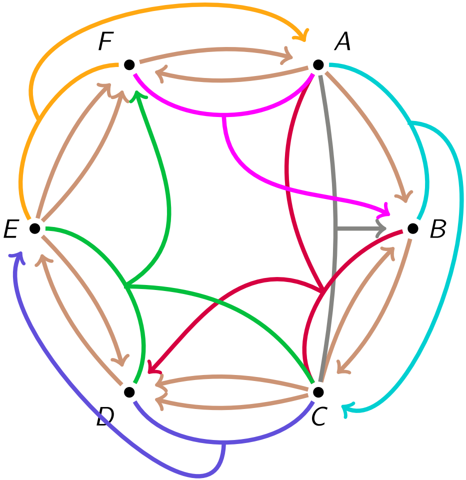

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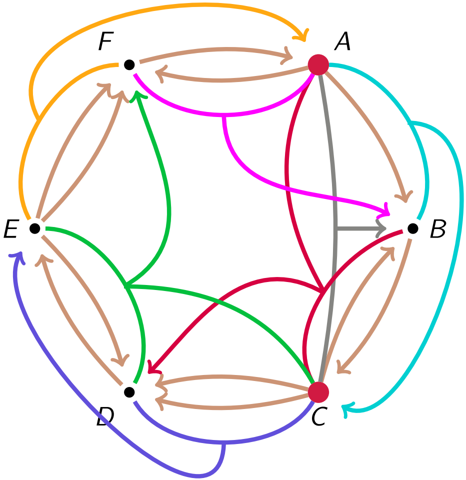

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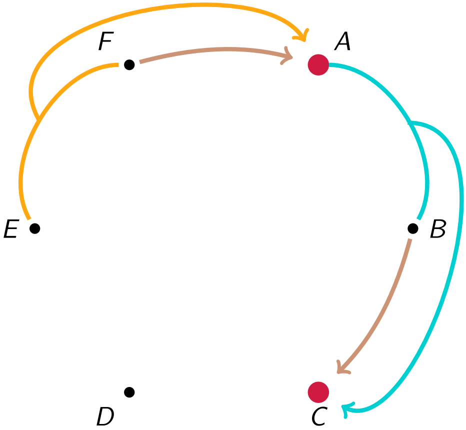

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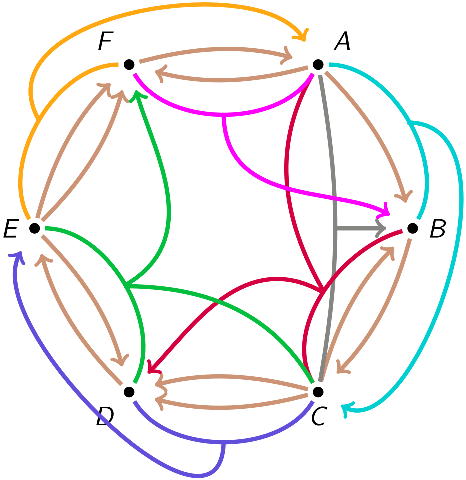

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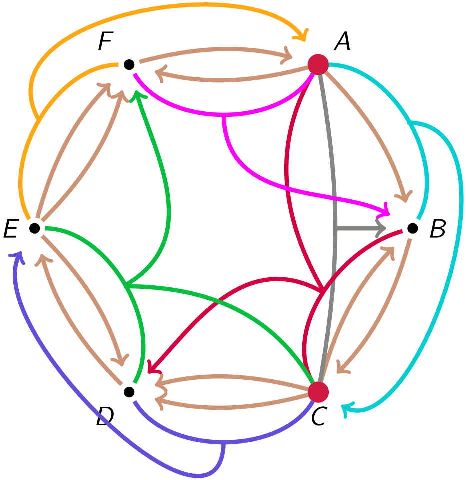

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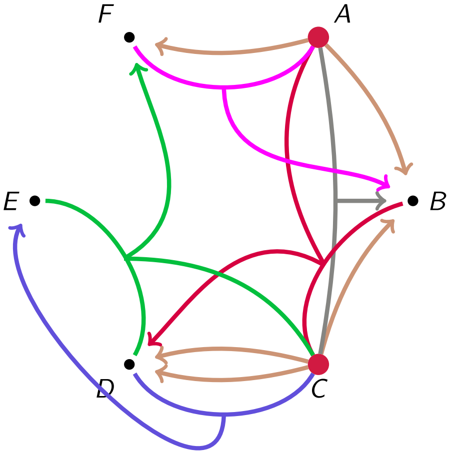

# Hyperarc-connectivity and $(k,k)$-partition connected hypergraphs

- $\vec{\mathcal{H}}$ is *k-hyperarc-connected*, if, $\forall \varnothing \neq X \subsetneq V$, $d^+_{\vec{\mathcal{H}}}(X) \geq k$.
- The hyperarc-connectivity of a hypergraph, denoted $\lambda(\vec{\mathcal{H}})$, is the maximum value of $k$ such that $\vec{\mathcal{H}}$ is *k-hyperarc-connected*.

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We use a result of Frank : $\mathcal{H}$ is $(k, k)$-partition-connected if and only if it admits a $k$-hyperarc-connected orientation.

# Main result

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Let $\mathcal{H} = (V, E)$ be a $(k+1, k+1)$-partition-connected hypergraph and $\vec{\mathcal{H}}$ is a k-hyperarc connected orientation of $\mathcal{H}$. Then there exists a sequence of hypergraphs $(\vec{\mathcal{H}}_i)_{i \in 0 \dots \ell}$ such that $\vec{\mathcal{H}}_{i+1}$ is obtained from $\vec{\mathcal{H}}_i$ by reorienting exactly one hyperarc and $\lambda(\vec{\mathcal{H}}_{i+1}) \geq \lambda(\vec{\mathcal{H}}_i)$ and $\lambda(\vec{\mathcal{H}}_\ell) = k+1$. Such a sequence of orientations can be obtained with $\ell \leq |V|^3$ and found in polynomial time (in the size of $\mathcal{H}$).

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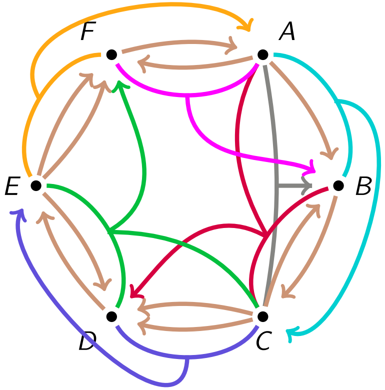
Generalization of **Ito and al.**, as digraphs are special cases of hypergraphs.

# "High-Level"-running of the algorithm

Our algorithm will provide a 3-hyperarc-connected orientation of $\mathcal{H}$, starting from a 2-hyperarc-connected.




1. Take $r$ in $V(\mathcal{H})$.
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3. Stopping Criteria
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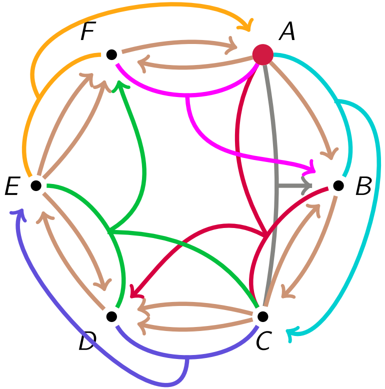


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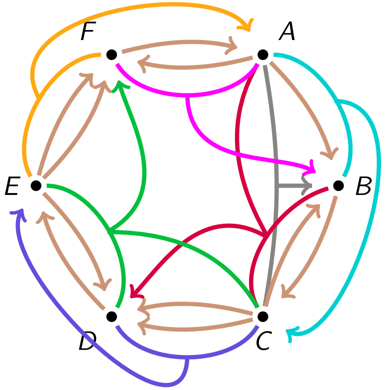


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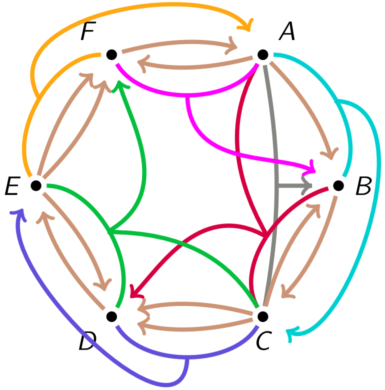


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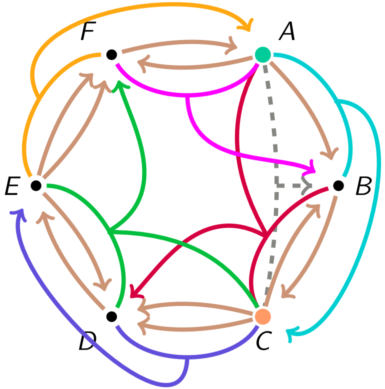


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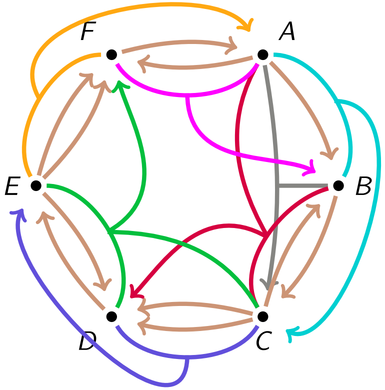


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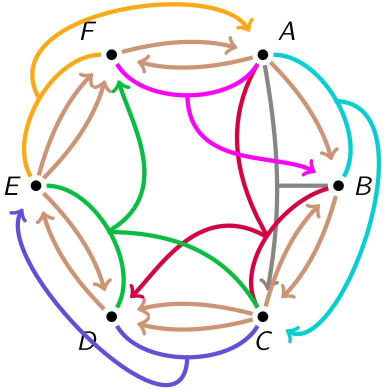


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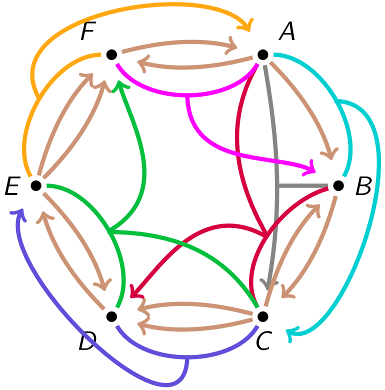


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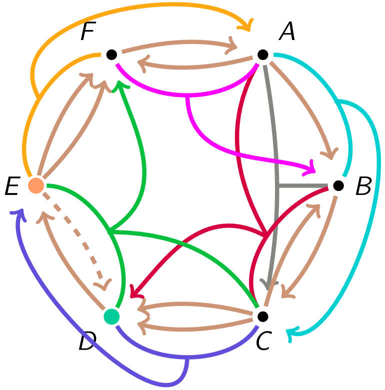


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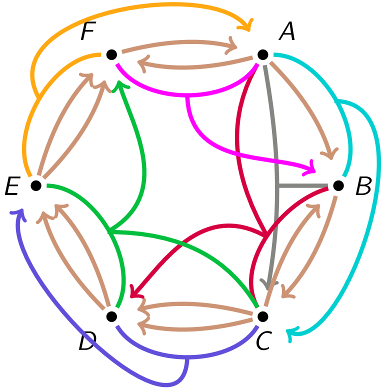


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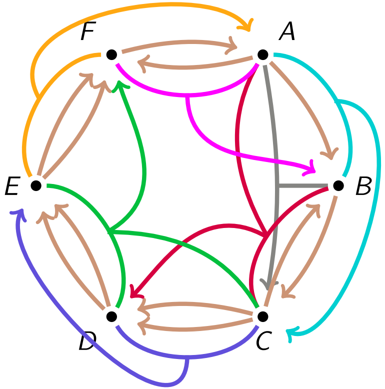


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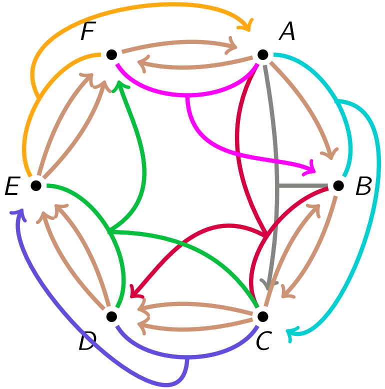


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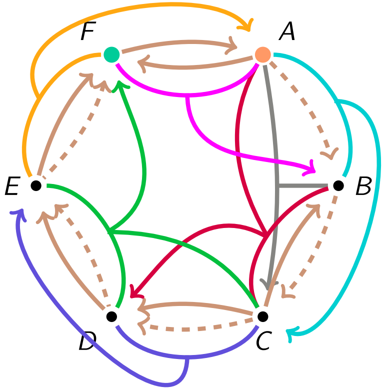


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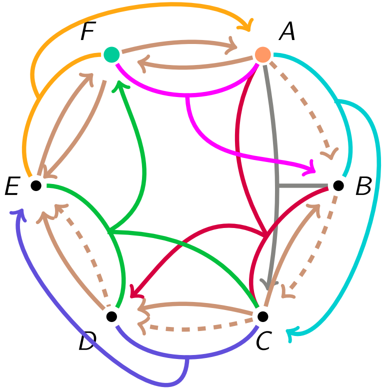


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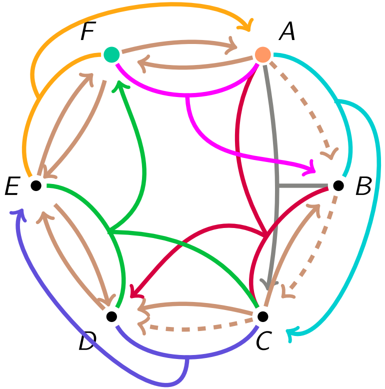


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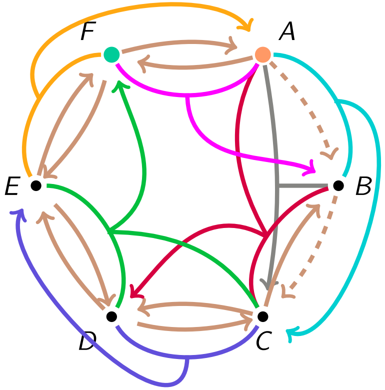


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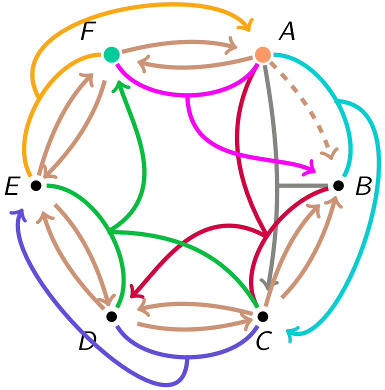


1. Take $r$ in $V(\mathcal{H})$.
2. Compute sets of vertices.
3. Stopping Criteria
4. Select a set $R$ (cf. 2.)
5. Find an admissible $(s, t)$-hyperpath in $R$ to reorient
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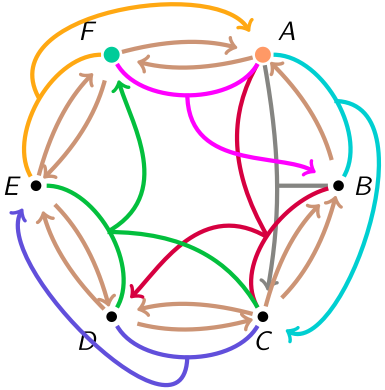


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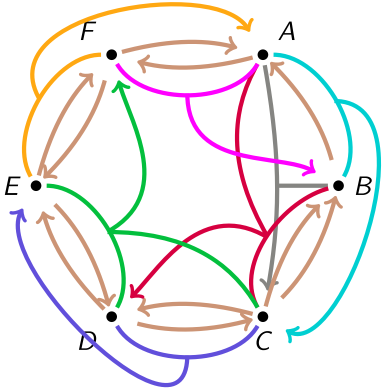


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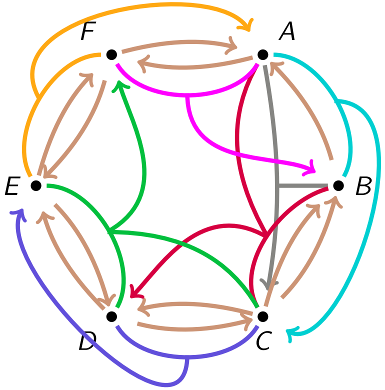


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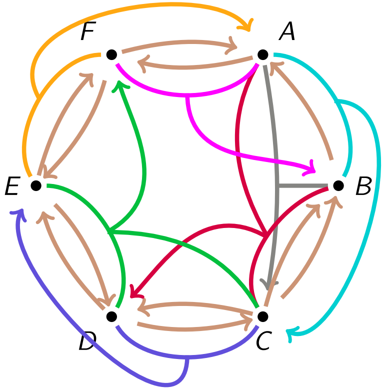


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# Finding *admissible* $(s, t)$-hyperpaths in $R$

- Crucial segment of the algorithm.
- Three criterion for $P$ to be an admissible $(s, t)$-hyperpath in $R$:
  1. $s$ is a safe source in $S \subseteq R$, $t$ is a safe sink in $T \subseteq R$.
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What are safe sources and safe sinks ?

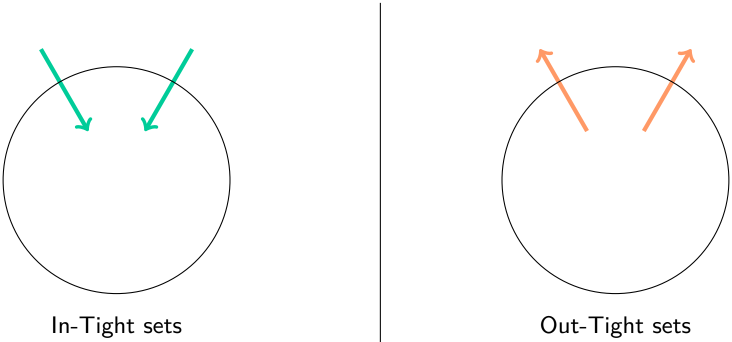
*A brief detour...*

# Tight and Dangerous sets

Remainder of the algorithm :

- `Input` : A $k$-hyperarc-connected orientation of a $(k+1, k+1)$-partition-connected hypergraph.
- `Output` : A $k+1$-hyperarc-connected hypergraph.

# Tight and Dangerous sets




- $\mathcal{T}_- = \{X \subseteq V - r, d^-(X) = k\} \cup \{V\}$
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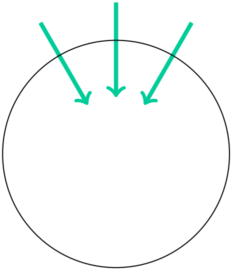
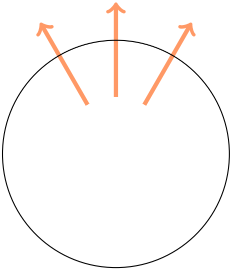


In-Dangerous sets



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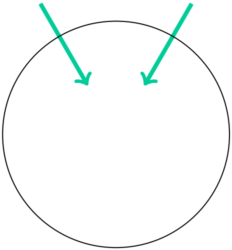

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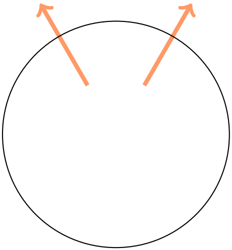

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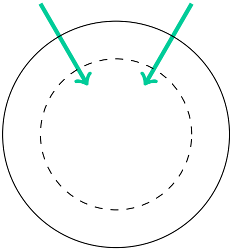
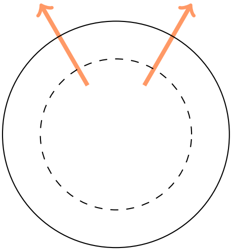


Minimal In-Tight sets



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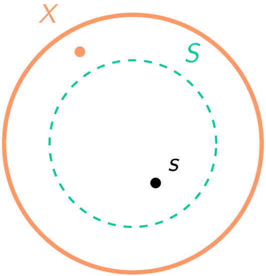
# Safe Sources and Safe Sinks

Definitions are symmetric (but proofs are not).

- For $S \in \mathcal{M}_-$, $s$ is a safe source in $S$ if :
  - a For every $s \in X \in \mathcal{T}_+$, we have $S \subsetneq X$.
  - b For every $s \in X \in \mathcal{D}_+$ such that $S \setminus X \neq \varnothing$, there exists $Y \in \mathcal{T}_+$ such that $s \notin Y \subsetneq X$.

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Finding a safe sink $t$ in $T \in \mathcal{M}_+$ can be done by checking each vertex if they correspond to the definition.

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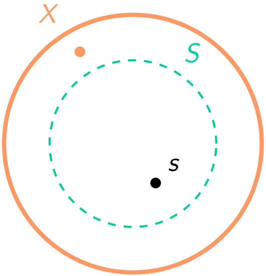


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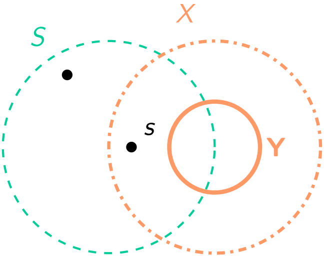


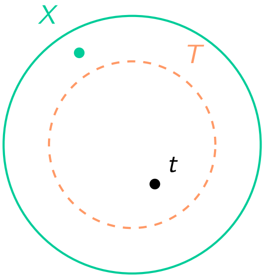

Condition (b)

Finding a safe sink $t$ in $T \in \mathcal{M}_+$ can be done by checking each vertex if they correspond to the definition.

# Safe Sources and Safe Sinks

Definitions are symmetric (but proofs are not).

- For $T \in \mathcal{M}_+$, $t$ is a safe sink in $T$ if :
  - c For every $t \in X \in \mathcal{T}_-$, we have $T \subsetneq X$.
  - d For every $t \in X \in \mathcal{D}_-$ such that $T \setminus X \neq \varnothing$, there exists $Y \in \mathcal{T}_-$ such that $t \notin Y \subsetneq X$.

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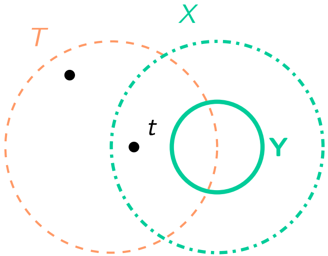



Condition (d)

Finding a safe sink $t$ in $T \in \mathcal{M}_+$ can be done by checking each vertex if they correspond to the definition.

# Existence of a safe source (*a safe sink*)

## Lemma 10

$\forall S \in \mathcal{M}_-,$ there is a safe source $s \in S$.

Likewise,

## Lemma 11

$\forall T \in \mathcal{M}_+,$ there is a safe sink $t \in T$.

## Quick outline of a proof for Lemma 10 :

- Let $S \in \mathcal{M}_-$.
- Considering a family of vertex sets $(\chi)$ that cover as many vertices of $S$ as possible, but using as little as vertex sets possible.
- We can prove that, under given assumptions, $\chi$ cannot cover every vertex of $S$.
- Vertices that are not covered by $\chi$ are "potential" safe sources, the last part of the proof is verifying that they are effectively safe sources.

# Finding *admissible* $(s,t)$-hyperpaths in $R \in \mathcal{R}$

$\mathcal{R}$ : $R \subseteq V - r$ inclusion-wise minimal such that either :

- $R \in \mathcal{T}_-$, and contains a member of $\mathcal{T}_+$
- or $R \in \mathcal{T}_+$, and contains a member of $\mathcal{T}_-$.

Three criterion for $P$ to be an admissible $(s,t)$-hyperpath in $R$:

1. $s$ is a safe source in $S \subseteq R$, $t$ is a safe sink in $T \subseteq R$.
2. Reorienting each hyperarc, **one by one**, does not decrease the hyperarc-connectivity
3. Let $\vec{\mathcal{H}}'$ the hypergraph obtained after reorientation of $P$.
   - $\mathcal{M}'$ : Inclusion-wise minimal members of $\mathcal{M}'_- \cup \mathcal{M}'_+$
   - Either $|\mathcal{M}'| < |\mathcal{M}|$, either $|\mathcal{M}'| = |\mathcal{M}|$ and $\mathcal{M}'$ covers more vertices than $\mathcal{M}$.

Point 3. is the stopping criteria for the main algorithm :

- $\mathcal{M} = \{V\}$ implies both $\mathcal{M}_- = \{V\}$ and $\mathcal{M}_+ = \{V\}$.
- $\mathcal{T}_- = \{X \subseteq V - r, d^-(X) = k\} \cup \{V\}$
- $\mathcal{M}_-$ : Inclusion-wise minimal members of $\mathcal{T}_-$
- Finally, if $\lambda(\vec{\mathcal{H}}) \geq k$ and $\mathcal{T}_- = \mathcal{T}_+ = \{V\}$, $\vec{\mathcal{H}}$ is $(k+1)$-hyperarc-connected.

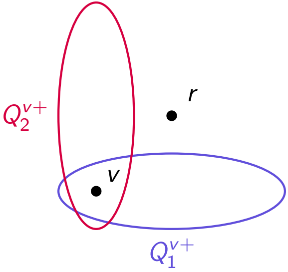
# Introduction of $Q_+^v$

## Definition of $Q_+^v$

Consider the sets of $\mathcal{T}_+$ containing $v$. $Q_+^v$ is **the** minimal (inclusion-wise) one.

## Unicity of $Q_+^v$ :

If it exists, $Q_+^v$ is unique.




Let $Q_1^{v+}$, $Q_2^{v+}$ verifying the above definition.

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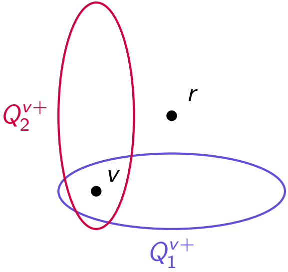


By definition, $Q_1^{v+} \not\subseteq Q_2^{v+}$ and $Q_2^{v+} \not\subseteq Q_1^{v+}$.

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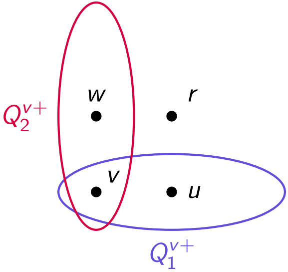


Denote $u \in Q_1^{v+} \setminus Q_2^{v+}$, $w \in Q_2^{v+} \setminus Q_1^{v+}$.

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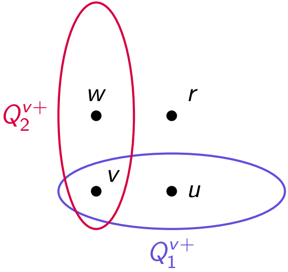


As $r \notin Q_1^{v+}, Q_2^{v+}$, both are are crossing sets.

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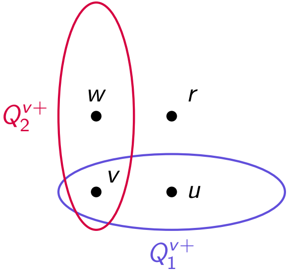


By submodularity, if $X, Y \in \mathcal{T}_+$, both $X \cup V \in \mathcal{T}_+$ and $X \cap V \in \mathcal{T}_+$.

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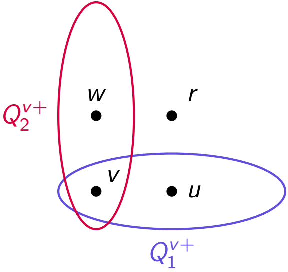


$Q_1^{v+} \cap Q_2^{v+}$ is smaller (inclusion-wise) than $Q_1^{v+}$ and $Q_2^{v+}$.

# Existence of an hyperpath that does not leave $Q_+^v$

## Lemma 12 (a)

$\forall s \in V, \forall t \in Q_+^s$, there exists an $(s,t)$-hyperpath that does not leave $Q_+^s$.

## Proof of Lemma 12 (a)

- By contradiction, assume that there is $s \in V, t \in Q_+^s$ such that any $(s,t)$-hyperpath leaves $Q_+^s$.
- There is $s \in Z \subseteq Q_+^s \setminus \{t\}$ such that any hyperarc leaving $Z$ will also leave $Q_+^s$.
- We have the following inequalities
  - $d_{\mathcal{H}}^+(Q_+^s) \geq d_{\mathcal{H}}^+(Z)$
  - $d_{\mathcal{H}}^+(Z) \geq k$, as $\mathcal{H}$ is $k$-hyperarc-connected.
  - $k = d_{\mathcal{H}}^+(Q_+^s)$ by definition.
- We can deduce that $d_{\mathcal{H}}^+(Z) = k$, which automatically implies that $Z \in \mathcal{T}_+$.
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  - $k = d_{\vec{\mathcal{H}}}^+(Q_+^s)$ by definition.
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- By contradiction, assume that there is $s \in V$, $t \in Q_+^s$ such that any $(s,t)$-hyperpath leaves $Q_+^s$.
- There is $s \in Z \subseteq Q_+^s \setminus \{t\}$ such that any hyperarc leaving $Z$ will also leave $Q_+^s$.
- We have the following inequalities
  - $d^+_{\vec{\mathcal{H}}}(Q_+^s) \geq d^+_{\vec{\mathcal{H}}}(Z)$
  - $d^+_{\vec{\mathcal{H}}}(Z) \geq k$, as $\mathcal{H}$ is $k$-hyperarc-connected.
  - $k = d^+_{\vec{\mathcal{H}}}(Q_+^s)$ by definition.
- We can deduce that $d^+_{\vec{\mathcal{H}}}(Z) = k$, which automatically implies that $Z \in \mathcal{T}_+$.
- $Q_+^s$ is not minimal, hence the contradiction.

# Existence of an hyperpath that does not leave $Q_+^v$

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# Finding an admissible $(s, t)$-hyperpath in $R \in \mathcal{R} \cap \mathcal{T}_-$

1. Only input of the algorithm $R \in \mathcal{R} \in \mathcal{T}_-$
   - $s, t$ are constrained (maybe not unique) by the choice of $R$.
2. Choosing $S \in \mathcal{M}_-$, then a safe source $s \in S$.
3. Main part of the algorithm : $s$-out arborescence
   - $F$ : (Directed) arborescence, rooted in $s$
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# Finding an admissible $(s,t)$-hyperpath in $R \in \mathcal{R} \cap \mathcal{T}_-$

**Algorithm** Admissible $(s,t)$-hyperpath in $R \in \mathcal{R} \cap \mathcal{T}_-$

Take a set $S \in \mathcal{M}_-$, with $S \subseteq R$, then a safe source $s \in S$.
$Z = \{s\}$, $F = (Z, \varnothing)$, $V' = R$
**while** $h = (X, v)$ exists such that $v \in V' - Z$ and $X \cap Z \neq \varnothing$ **do**
    Let $u \in X \cap Z$.
    $Z \leftarrow Z \cup \{v\}$
    $F \leftarrow F + uv$
    **if** $Q_+^v \subsetneq V'$ **then**
        $V' \leftarrow Q_+^v$
    **end if**
**end while**
$T = V'$
Take a safe sink $t \in T$
$P' = F[s,t]$
$P$ is the corresponding hypergraph in $\vec{\mathcal{H}}$ with respect to $P'$.
**Return** $S, T, s, t, P$