Directed hypergraph connectivity augmentation by hyperarc re-orientations

Combinatorial Optimization and Graph Theory

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Thursday, Nov 23rd 2023

Table of contents

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  - Exhibiting a sequence of orientations such that :
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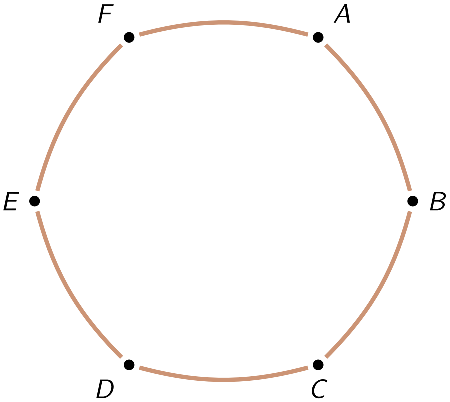
Goal of the article : Expanding the result of **Ito and al.** to hypergraphs.
Side note : This article generalise the results of **Ito and al.**, as directed graphs are special case of hypergraphs.

Hypergraphs



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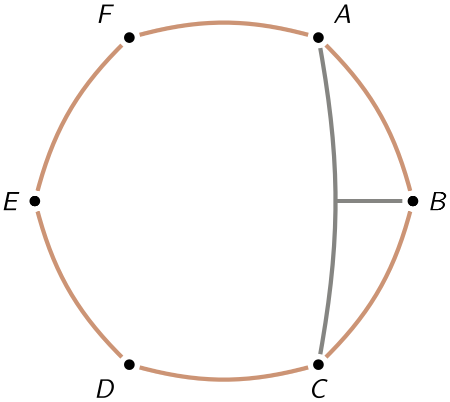

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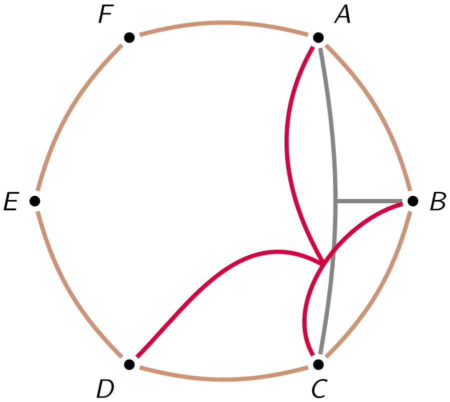

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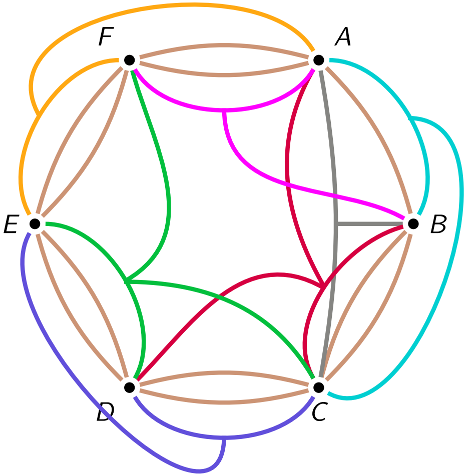
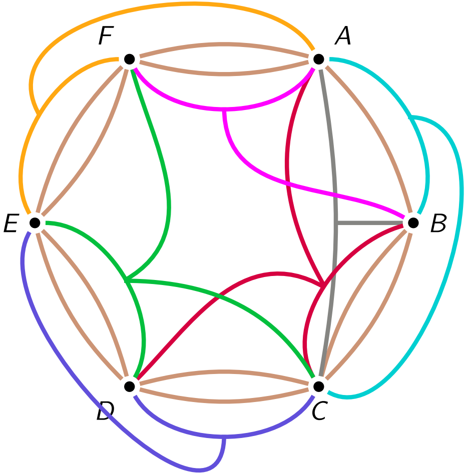


Degree of $\varnothing \neq X \subsetneq V$

$d_{\mathcal{H}}(X)$ is the number of hyperedges intersecting both X and $V \setminus X$.



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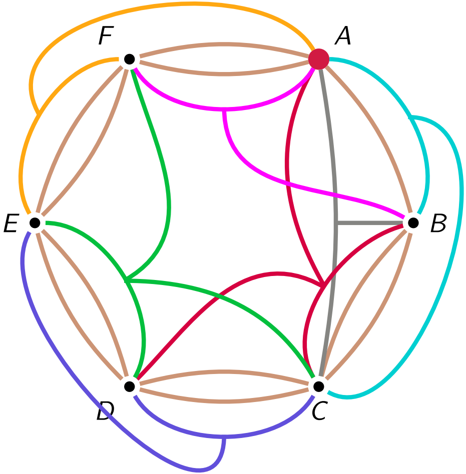

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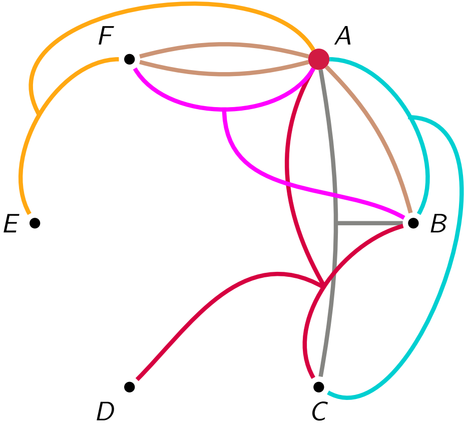

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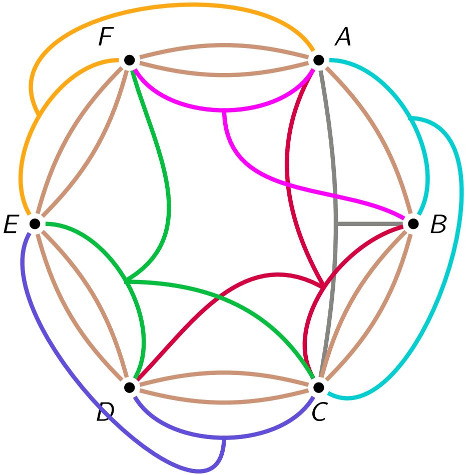

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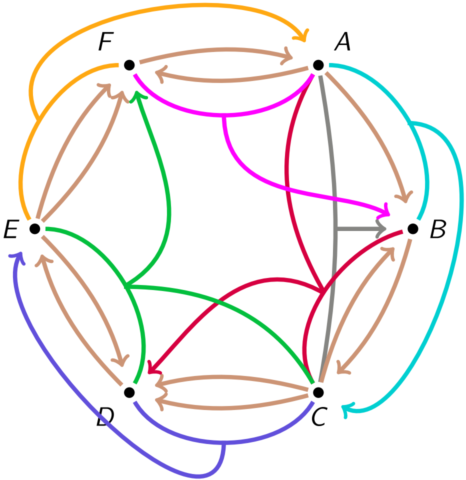

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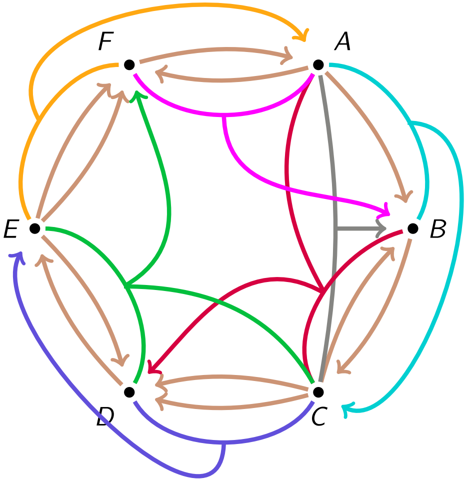

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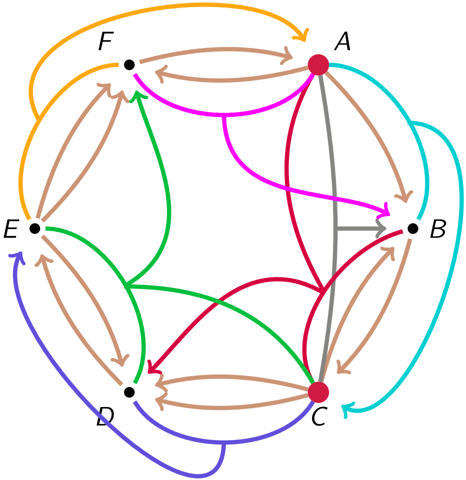

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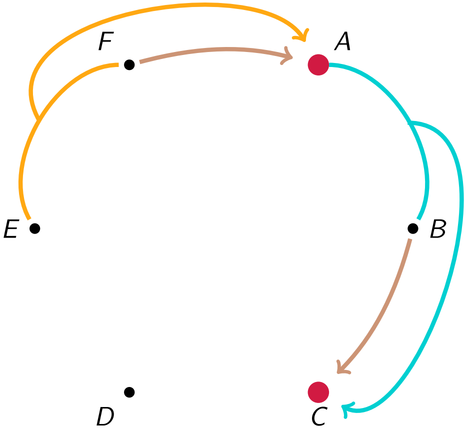

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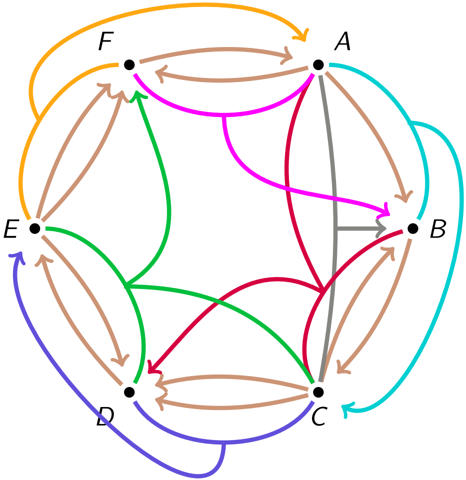

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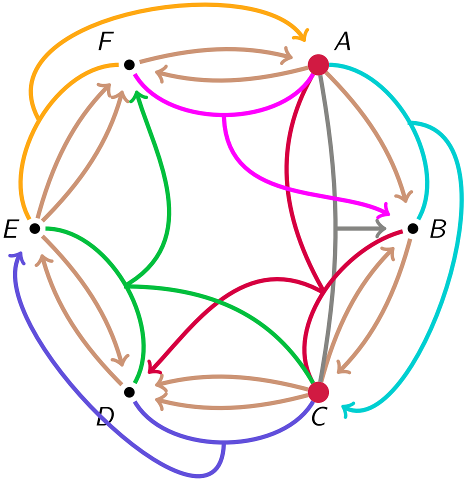

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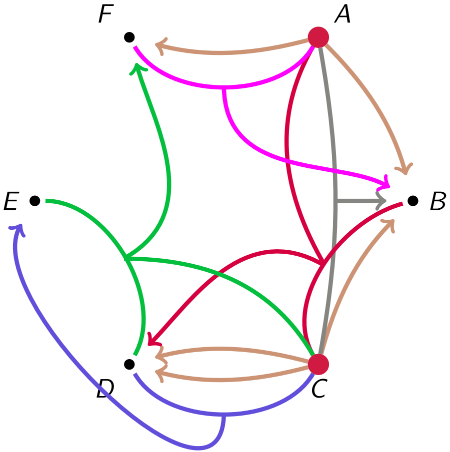

Hyperarc-connectivity and (k, k)-partition connected hypergraphs

- $\vec{\mathcal{H}}$ is *k-hyperarc-connected*, if, $\forall \varnothing \neq X \subsetneq V$, $d^+_{\vec{\mathcal{H}}}(X) \geq k$.
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We use a result of Frank : $\mathcal{H}$ is (k, k)-partition-connected if and only if it admits a k-hyperarc-connected orientation.

Main result

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Let $\mathcal{H} = (V, E)$ be a $(k+1, k+1)$-partition-connected hypergraph and $\vec{\mathcal{H}}$ is a k-hyperarc connected orientation of $\mathcal{H}$. Then there exists a sequence of hypergraphs $(\vec{\mathcal{H}}_i)_{i \in 0 \ldots \ell}$ such that $\vec{\mathcal{H}}_{i+1}$ is obtained from $\vec{\mathcal{H}}_i$ by reorienting exactly one hyperarc and $\lambda(\vec{\mathcal{H}}_{i+1}) \geq \lambda(\vec{\mathcal{H}}_i)$ and $\lambda(\vec{\mathcal{H}}_\ell) = k+1$. Such a sequence of orientations can be obtained with $\ell \leq |V|^3$ and found in polynomial time (in the size of $\mathcal{H}$).

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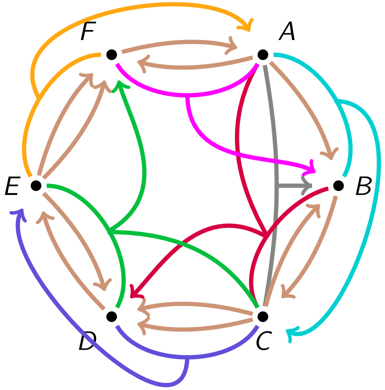
Generalization of **Ito and al.**, as digraphs are special cases of hypergraphs.

"High-Level"-running of the algorithm

Our algorithm will provide a 3-hyperarc-connected orientation of $\mathcal{H}$, starting from a 2-hyperarc-connected.



1. Take r in $V(\mathcal{H})$.
2. Compute sets of vertices.
3. Stopping Criteria
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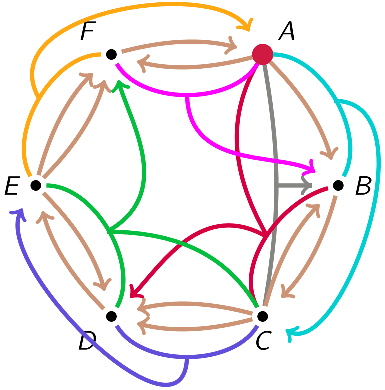

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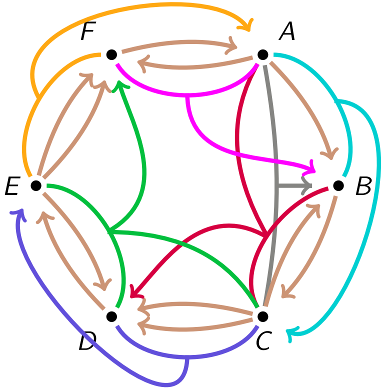

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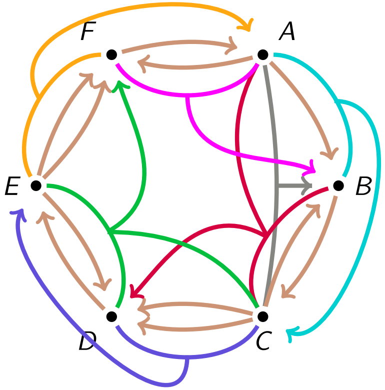

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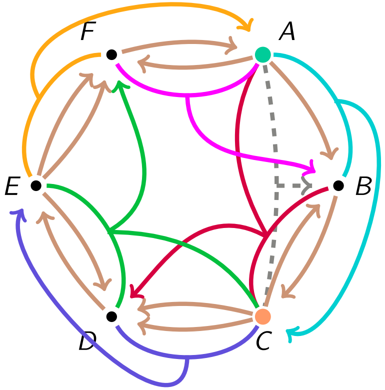

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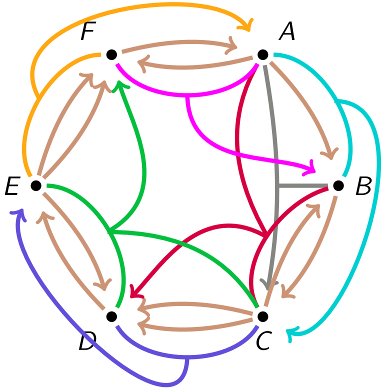

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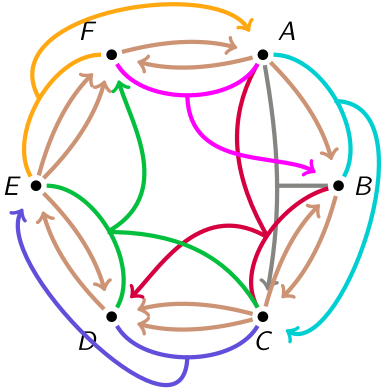

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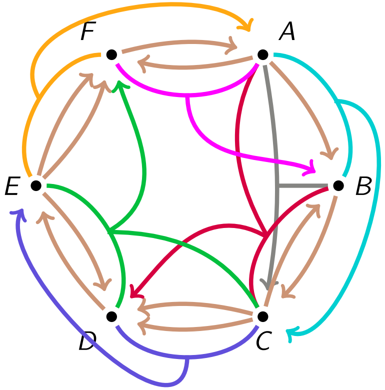

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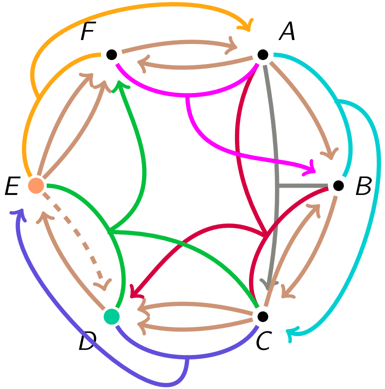

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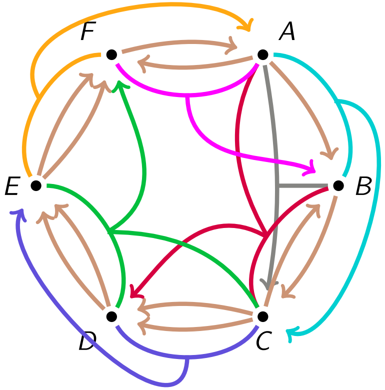

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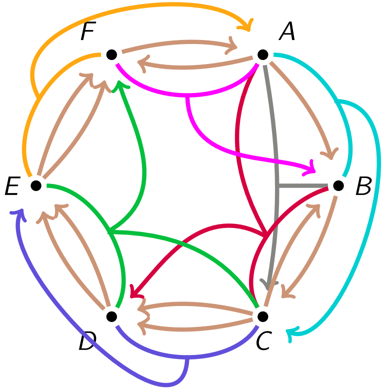

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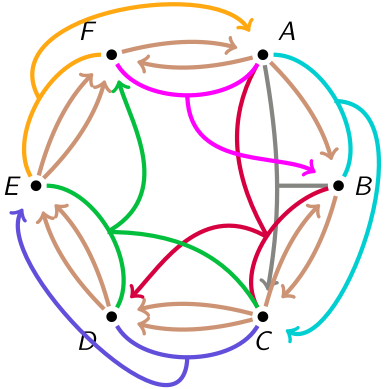

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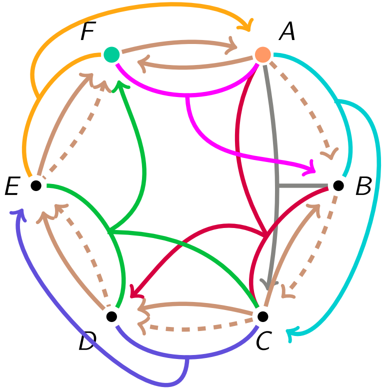

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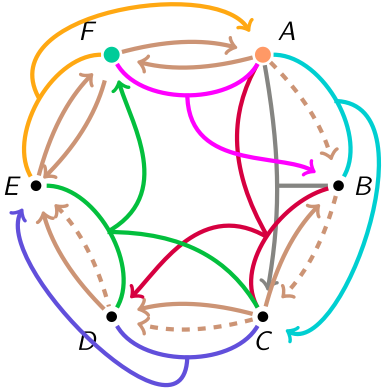

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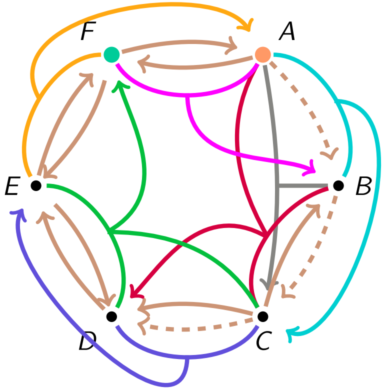

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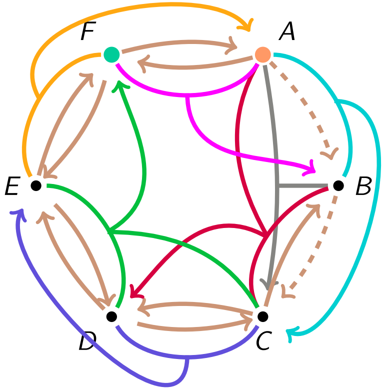

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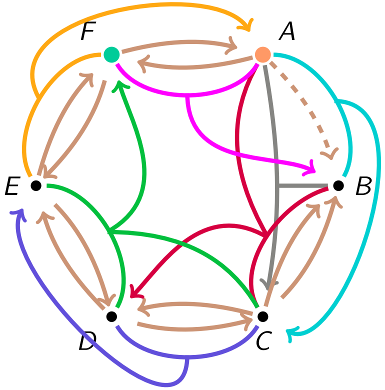

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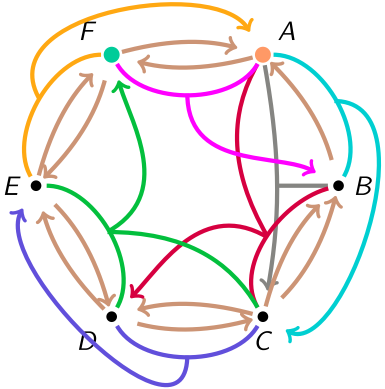
Our algorithm will provide a 3-hyperarc-connected orientation of $\mathcal{H}$, starting from a 2-hyperarc-connected.



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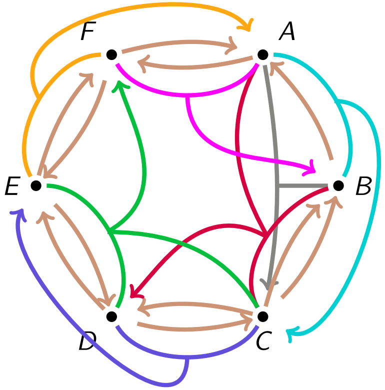

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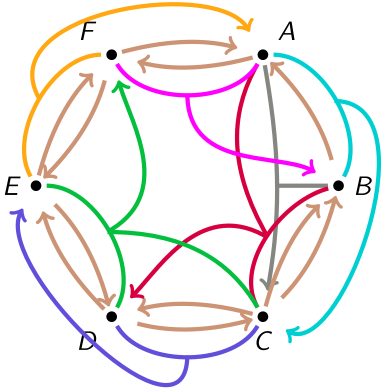

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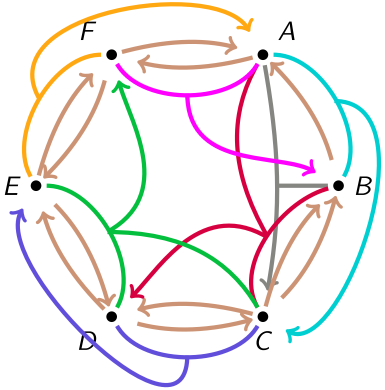

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Finding *admissible* (s, t)-hyperpaths in R

- Crucial segment of the algorithm.
- Three criterion for P to be an admissible (s, t)-hyperpath in R:
  1. s is a safe source in $S \subseteq R$, t is a safe sink in $T \subseteq R$.
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What are safe sources and safe sinks ?

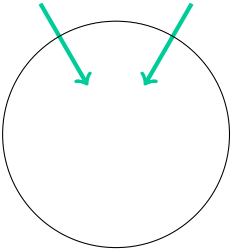
A brief detour...

Tight and Dangerous sets

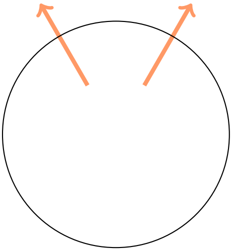
Remainder of the algorithm :

- `Input` : A k-hyperarc-connected orientation of a $(k+1, k+1)$-partition-connected hypergraph.
- `Output` : A $k+1$-hyperarc-connected hypergraph.

Tight and Dangerous sets



In-Tight sets



Out-Tight sets

- $\mathcal{T}_- = \{X \subseteq V - r, d^-(X) = k\} \cup \{V\}$
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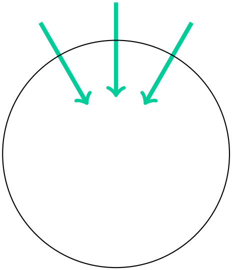

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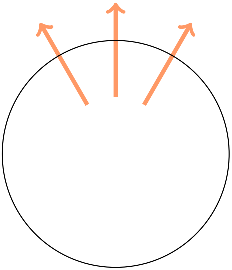

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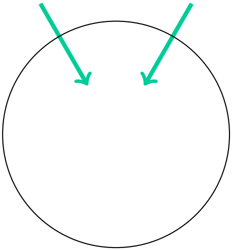

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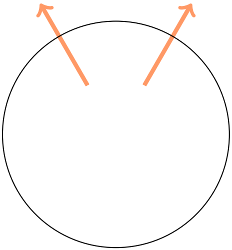

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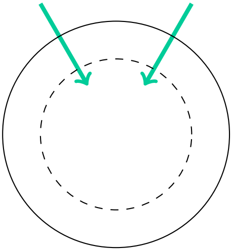
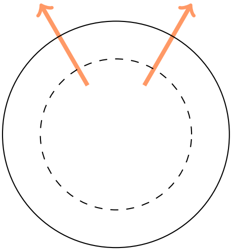


Minimal In-Tight sets



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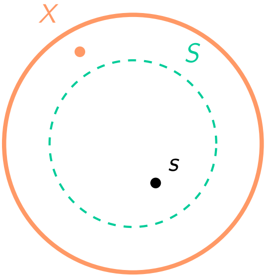
Safe Sources and Safe Sinks

Definitions are symmetric (but proofs are not).

- For $S \in \mathcal{M}_-$, s is a safe source in S if :
  a For every $s \in X \in \mathcal{T}_+$, we have $S \subsetneq X$.
  b For every $s \in X \in \mathcal{D}_+$ such that $S \setminus X \neq \varnothing$, there exists $Y \in \mathcal{T}_+$ such that $s \notin Y \subsetneq X$.



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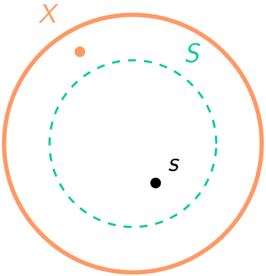

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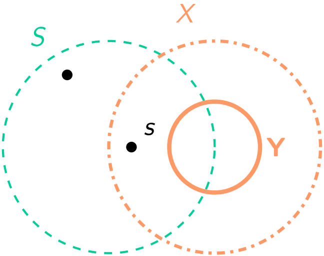


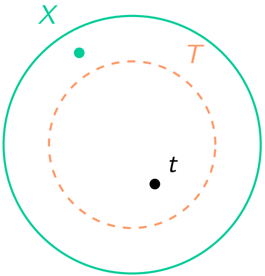
Condition (b)

Finding a safe sink t in $T \in \mathcal{M}_+$ can be done by checking each vertex if they correspond to the definition.

Safe Sources and Safe Sinks

Definitions are symmetric (but proofs are not).

- For $T \in \mathcal{M}_+$, t is a safe sink in T if :
  - c For every $t \in X \in \mathcal{T}_-$, we have $T \subsetneq X$.
  - d For every $t \in X \in \mathcal{D}_-$ such that $T \setminus X \neq \varnothing$, there exists $Y \in \mathcal{T}_-$ such that $t \notin Y \subsetneq X$.



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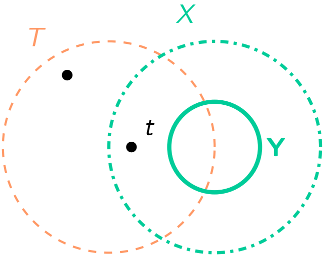


Condition (d)

Finding a safe sink t in $T \in \mathcal{M}_+$ can be done by checking each vertex if they correspond to the definition.

Existence of a safe source (*a safe sink*)

Lemma 10

$\forall S \in \mathcal{M}_-,$ there is a safe source $s \in S$.

Likewise,

Lemma 11

$\forall T \in \mathcal{M}_+,$ there is a safe sink $t \in T$.

Quick sketch of a proof for Lemma 10 :

- Let $S \in \mathcal{M}_-$.
- Considering a family of vertex sets (χ) that cover as many vertices of S as possible, but using as little as vertex sets possible.
- We can prove that, under given assumptions, χ cannot cover every vertex of S.
- Vertices that are not covered by χ are "potential" safe sources, the last part of the proof is verifying that they are effectively safe sources.

Finding *admissible* (s,t)-hyperpaths in $R \in \mathcal{R}$

$\mathcal{R}$: $R \subseteq V - r$ inclusion-wise minimal such that either :

- $R \in \mathcal{T}_-$, and contains a member of $\mathcal{T}_+$
- or $R \in \mathcal{T}_+$, and contains a member of $\mathcal{T}_-$.

Three criterion for P to be an admissible (s,t)-hyperpath in R:

1. s is a safe source in $S \subseteq R$, t is a safe sink in $T \subseteq R$.
2. Reorienting each hyperarc, **one by one**, does not decrease the hyperarc-connectivity
3. Let $\vec{\mathcal{H}}'$ the hypergraph obtained after reorientation of P.
   - $\mathcal{M}'$: Inclusion-wise minimal members of $\mathcal{M}'_- \cup \mathcal{M}'_+$
   - Either $|\mathcal{M}'| < |\mathcal{M}|$, either $|\mathcal{M}'| = |\mathcal{M}|$ and $\mathcal{M}'$ covers more vertices than $\mathcal{M}$.

Point 3. is the stopping criteria for the main algorithm :

- $\mathcal{M} = \{V\}$ implies both $\mathcal{M}_- = \{V\}$ and $\mathcal{M}_+ = \{V\}$.
- $\mathcal{T}_- = \{X \subseteq V - r, d^-(X) = k\} \cup \{V\}$
- $\mathcal{M}_-$: Inclusion-wise minimal members of $\mathcal{T}_-$
- Finally, if $\lambda(\vec{\mathcal{H}}) \geq k$ and $\mathcal{T}_- = \mathcal{T}_+ = \{V\}$, $\vec{\mathcal{H}}$ is $(k+1)$-hyperarc-connected.

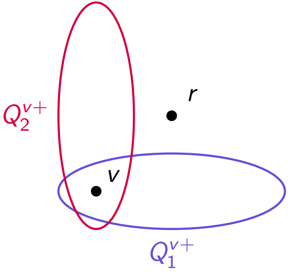
Introduction of Q_+^v

Definition of Q_+^v

Consider the sets of $\mathcal{T}_+$ containing v. Q_+^v is **the** minimal (inclusion-wise) one.

Unicity of Q_+^v :

If it exists, Q_+^v is unique.



Let Q_1^{v+}, Q_2^{v+} verifying the above definition.

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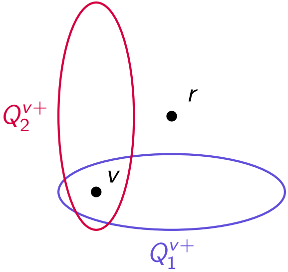

By definition, $Q_1^{v+} \not\subseteq Q_2^{v+}$ and $Q_2^{v+} \not\subseteq Q_1^{v+}$.

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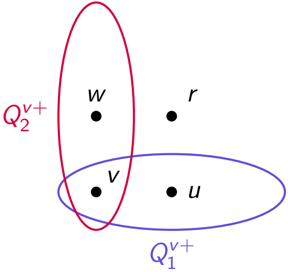

Denote $u \in Q_1^{v+} \setminus Q_2^{v+}$, $w \in Q_2^{v+} \setminus Q_1^{v+}$.

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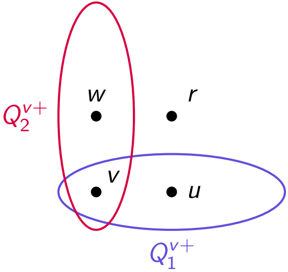

As $r \notin Q_1^{v+}, Q_2^{v+}$, both are are crossing sets.

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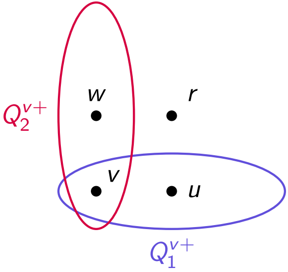

By submodularity, if $X, Y \in \mathcal{T}_+$, both $X \cup V \in \mathcal{T}_+$ and $X \cap V \in \mathcal{T}_+$.

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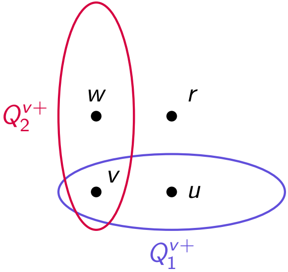

$Q_1^{v+} \cap Q_2^{v+}$ is smaller (inclusion-wise) than Q_1^{v+} and Q_2^{v+}.

Existence of an hyperpath that does not leave Q_+^v

Lemma 12 (a)

$\forall s \in V, \forall t \in Q_+^s$, there exists an (s,t)-hyperpath that does not leave Q_+^s.

Proof of Lemma 12(a)

- By contradiction, assume that there is $s \in V, t \in Q_+^s$ such that any (s,t)-hyperpath leaves Q_+^s.
- There is $s \in Z \subseteq Q_+^s \setminus \{t\}$ such that any hyperarc leaving Z will also leave Q_+^s.
- We have the following inequalities
  - $d_{\mathcal{H}}^+(Q_+^s) \geq d_{\mathcal{H}}^+(Z)$
  - $d_{\mathcal{H}}^+(Z) \geq k$, as $\mathcal{H}$ is k-hyperarc-connected.
  - $k = d_{\mathcal{H}}^+(Q_+^s)$ by definition.
- We can deduce that $d_{\mathcal{H}}^+(Z) = k$, which automatically implies that $Z \in \mathcal{T}_+$.
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  - $d^+_{\vec{\mathcal{H}}}(Z) \geq k$, as $\mathcal{H}$ is k-hyperarc-connected.
  - $k = d^+_{\vec{\mathcal{H}}}(Q_+^s)$ by definition.
- We can deduce that $d^+_{\vec{\mathcal{H}}}(Z) = k$, which automatically implies that $Z \in \mathcal{T}_+$.
- Q_+^s is not minimal, hence the contradiction.

Existence of an hyperpath that does not leave Q_+^v

Lemma 12 (a)

$\forall s \in V, \forall t \in Q_+^s$, there exists an (s,t)-hyperpath that does not leave Q_+^s.

Proof of Lemma 12(a)

- By contradiction, assume that there is $s \in V, t \in Q_+^s$ such that any (s,t)-hyperpath leaves Q_+^s.
- There is $s \in Z \subseteq Q_+^s \setminus \{t\}$ such that any hyperarc leaving Z will also leave Q_+^s.
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