

Directed hypergraph connectivity augmentation by hyperarc re-orientations

Combinatorial Optimization and Graph Theory

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Thursday, Nov 23rd 2023

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- Connectivity problems, characterisations
- Hypergraphs

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 - ▶ Exhibiting a sequence of orientations such that :
 - The arc-connectivity does not decrease, and the arc-connectivity of the last element of the sequence is k .
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Side note : This article generalise the results of **Ito and al.**, as directed graphs are special case of hypergraphs.

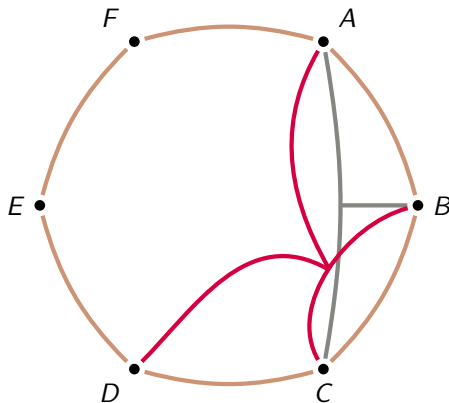
Hypergraphs



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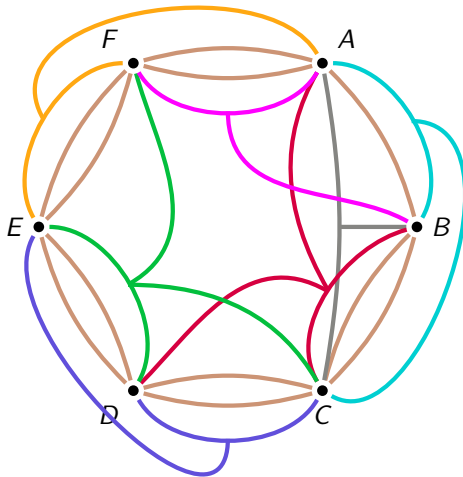


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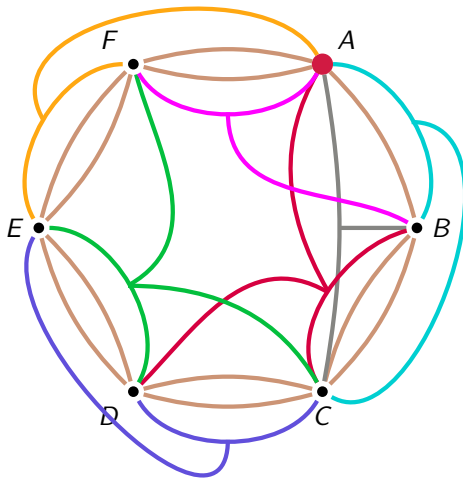
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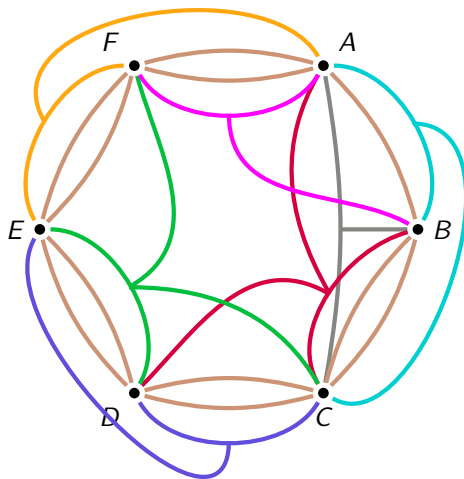


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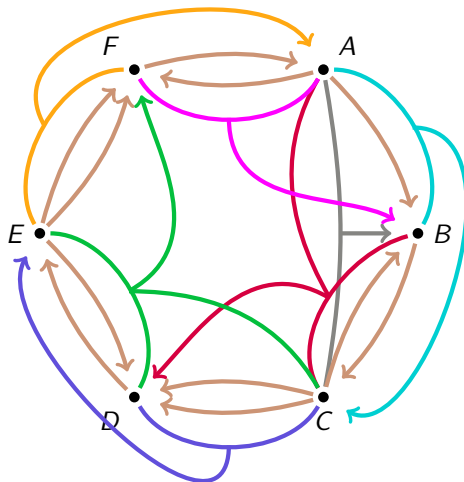
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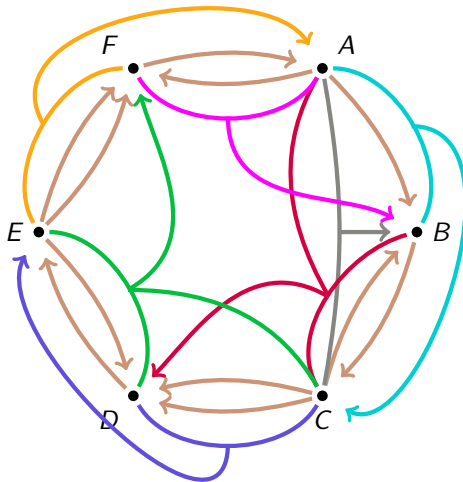


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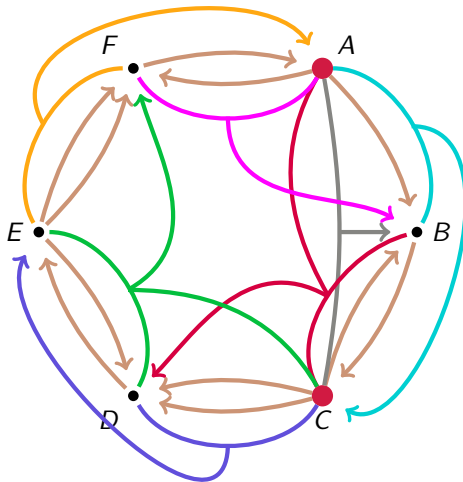
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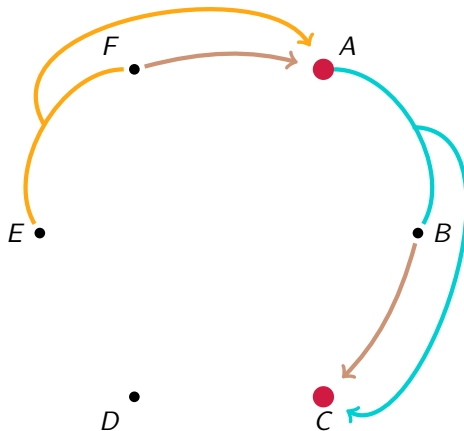
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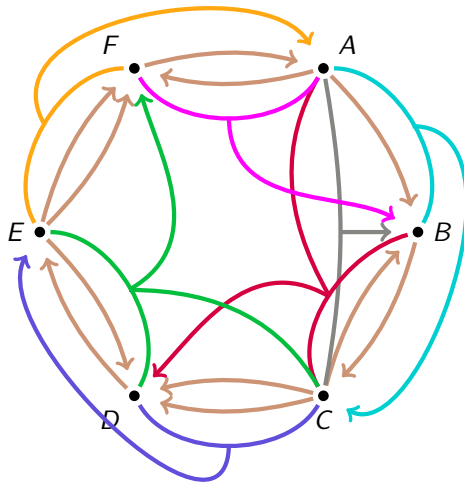
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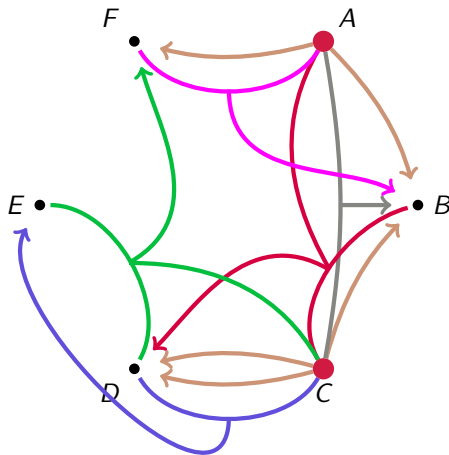
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Hyperarc-connectivity and (k, k) -partition connected hypergraphs

- $\vec{\mathcal{H}}$ is k -hyperarc-connected, if, $\forall \emptyset \neq X \subsetneq V$, $d_{\vec{\mathcal{H}}}^+(X) \geq k$.
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We use a result of Frank : \mathcal{H} is (k, k) -partition-connected if and only if it admits a k -hyperarc-connected orientation.

Main result

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Let $\mathcal{H} = (V, E)$ be a $(k + 1, k + 1)$ -partition-connected hypergraph and $\vec{\mathcal{H}}$ is a k -hyperarc connected orientation of \mathcal{H} . Then there exists a sequence of hypergraphs $(\vec{\mathcal{H}}_i)_{i \in 0 \dots \ell}$ such that $\vec{\mathcal{H}}_{i+1}$ is obtained from $\vec{\mathcal{H}}_i$ by reorienting exactly one hyperarc and $\lambda(\vec{\mathcal{H}}_{i+1}) \geq \lambda(\vec{\mathcal{H}}_i)$ and $\lambda(\vec{\mathcal{H}}_\ell) = k + 1$. Such a sequence of orientations can be obtained with $\ell \leq |V|^3$ and found in polynomial time (in the size of \mathcal{H}).

Main result

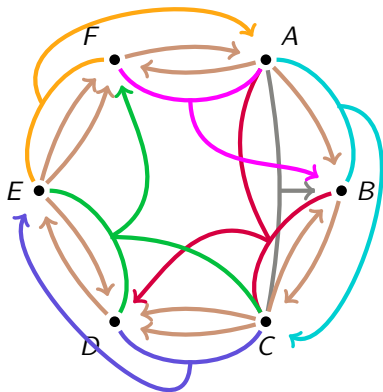
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Generalization of **Ito and al.**, as digraphs are special cases of hypergraphs.

"High-Level"-running of the algorithm

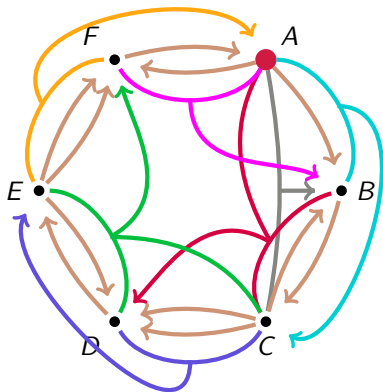
Our algorithm will provide a 3-hyperarc-connected orientation of \mathcal{H} , starting from a 2-hyperarc-connected.



- 1 Take r in $V(\mathcal{H})$.
- 2 Compute sets of vertices.
- 3 Stopping Criteria
- 4 Select a set R (cf. 2.)
- 5 Find an admissible (s, t) -hyperpath in R to reorient
- 6 Reorient the corresponding hyperpath.
- 7 Goto (2.)

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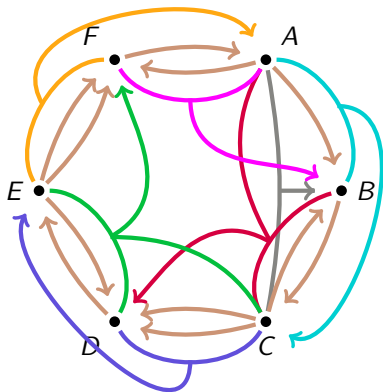
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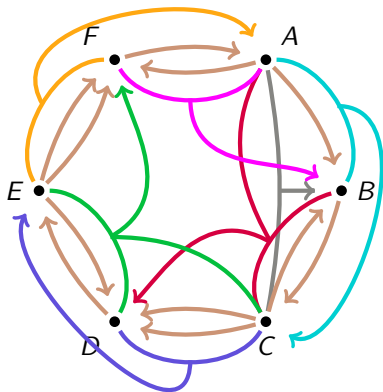
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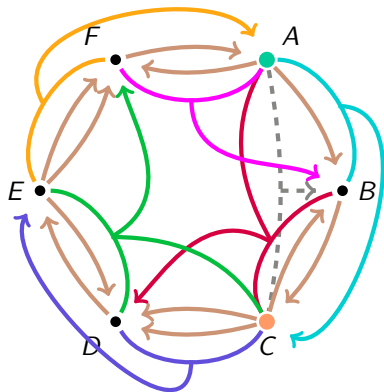
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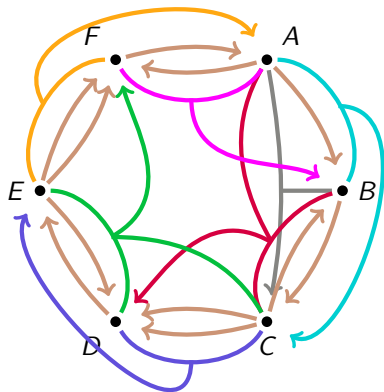
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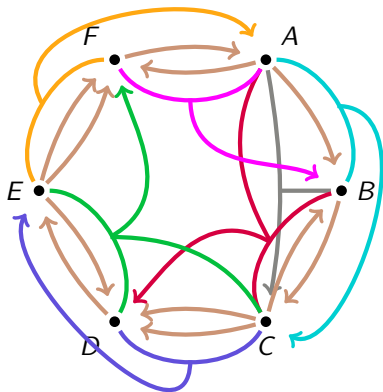
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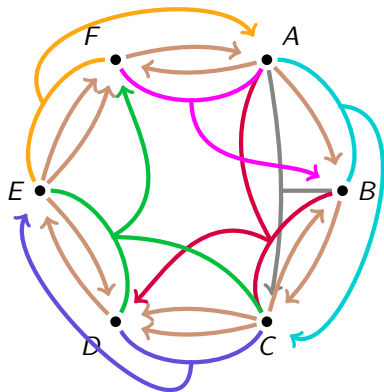
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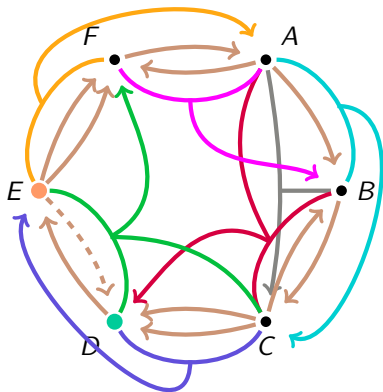
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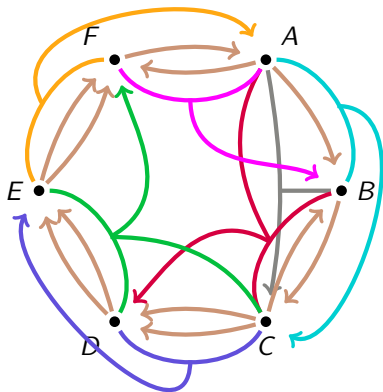
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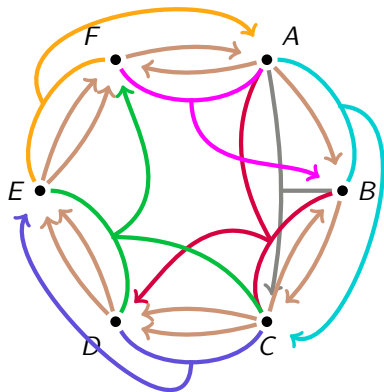
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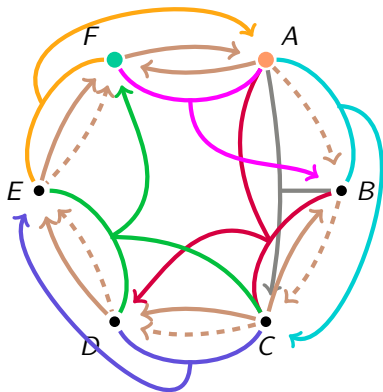
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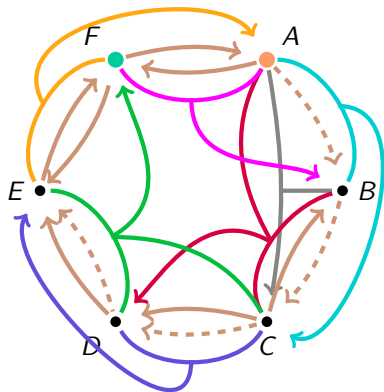
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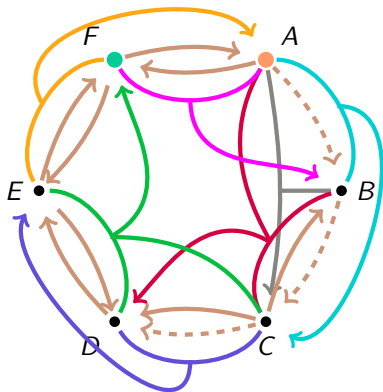
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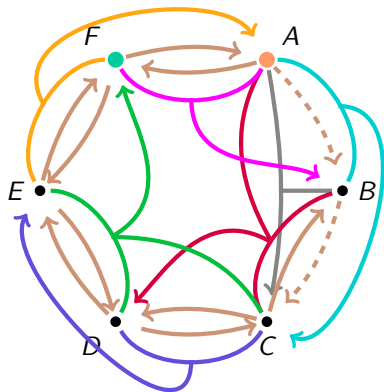
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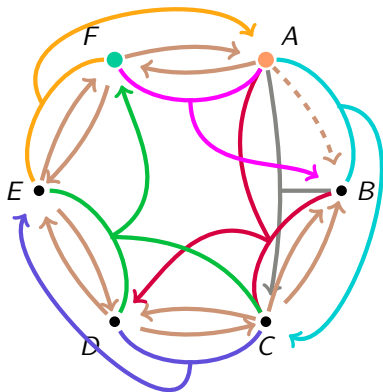
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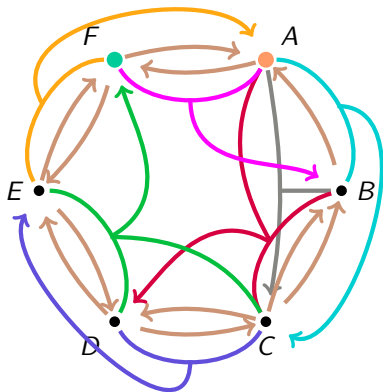
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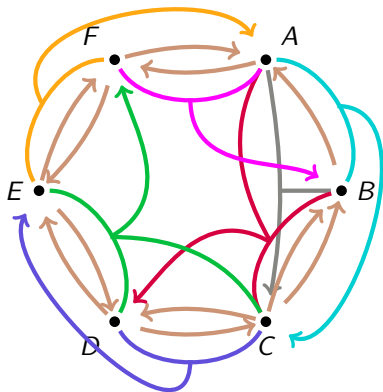
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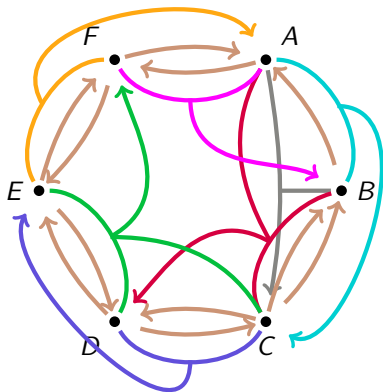
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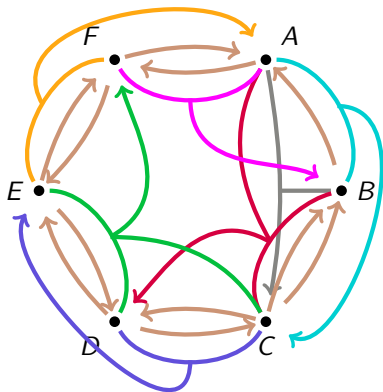
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Finding *admissible* (s, t) -hyperpaths in R

- Crucial segment of the algorithm.
- Three criterion for P to be an admissible (s, t) -hyperpath in R :
 - ① s is a safe source in $S \subseteq R$, t is a safe sink in $T \subseteq R$.
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- Three criterion for P to be an admissible (s, t) -hyperpath in R :
 - ① s is a safe source in $S \subseteq R$, t is a safe sink in $T \subseteq R$.
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What are **safe sources** and **safe sinks** ?

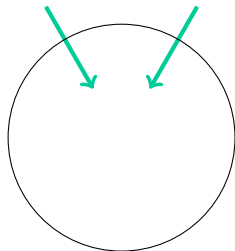
A brief detour...

Tight and Dangerous sets

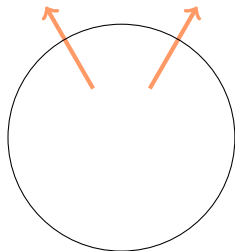
Remainder of the algorithm :

- Input : A k -hyperarc-connected orientation of a $(k + 1, k + 1)$ -partition-connected hypergraph.
- Output : A $k + 1$ -hyperarc-connected hypergraph.

Tight and Dangerous sets



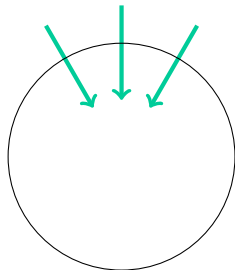
In-Tight sets



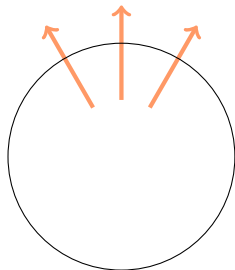
Out-Tight sets

- $\mathcal{T}_- = \{X \subseteq V - r, d^-(X) = k\} \cup \{V\}$
- $\mathcal{T}_+ = \{X \subseteq V - r, d^+(X) = k\} \cup \{V\}$
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Tight and Dangerous sets



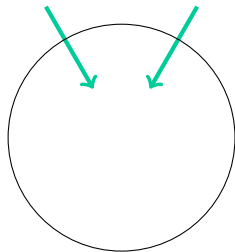
In-Dangerous sets



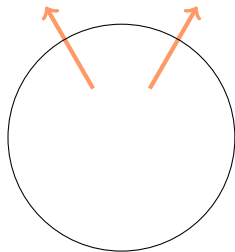
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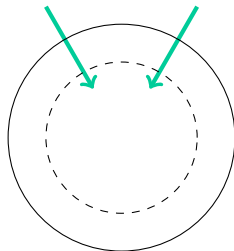
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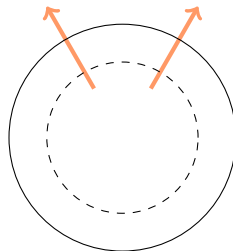
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Tight and Dangerous sets



Minimal In-Tight sets



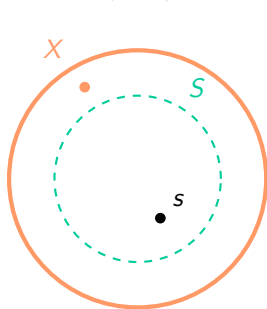
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Safe Sources and Safe Sinks

Definitions are symmetric (but proofs are not).

- For $S \in \mathcal{M}_-$, s is a safe source in S if :
 - For every $s \in X \in \mathcal{T}_+$, we have $S \subsetneq X$.
 - For every $s \in X \in \mathcal{D}_+$ such that $S \setminus X \neq \emptyset$, there exists $Y \in \mathcal{T}_+$ such that $s \notin Y \subsetneq X$.



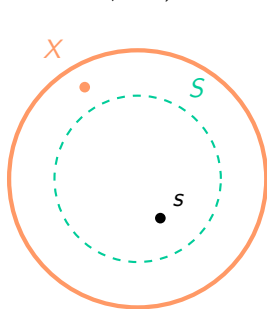
Condition (a)

Finding a safe sink t in $T \in \mathcal{M}_+$ can be done by checking each vertex if they correspond to the definition.

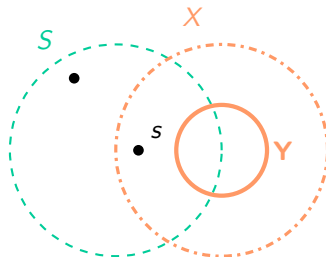
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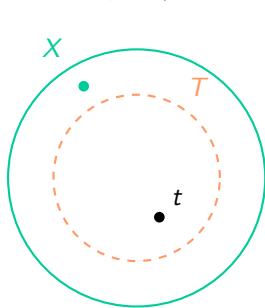
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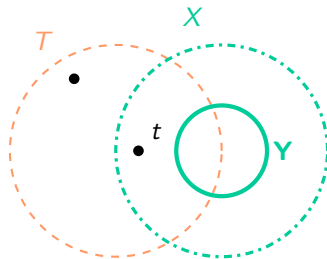
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Condition (c)



Condition (d)

Finding a safe sink t in $T \in \mathcal{M}_+$ can be done by checking each vertex if they correspond to the definition.

Existence of a safe source (*a safe sink*)

Lemma 10

$\forall S \in \mathcal{M}_-$, there is a safe source $s \in S$.

Likewise,

Lemma 11

$\forall T \in \mathcal{M}_+$, there is a safe sink $t \in T$.

Quick sketch of a proof for Lemma 10 :

- Let $S \in \mathcal{M}_-$.
- Considering a family of vertex sets (χ) that cover as many vertices of S as possible, but using as little as vertex sets possible.
- We can prove that, under given assumptions, χ cannot cover every vertex of S .
- Vertices that are not covered by χ are "potential" safe sources, the last part of the proof is verifying that they are effectively safe sources.

Finding *admissible* (s, t) -hyperpaths in $R \in \mathcal{R}$

$\mathcal{R} : R \subseteq V - r$ inclusion-wise minimal such that either :

- $R \in \mathcal{T}_-$, and contains a member of \mathcal{T}_+
- or $R \in \mathcal{T}_+$, and contains a member of \mathcal{T}_- .

Three criterion for P to be an admissible (s, t) -hyperpath in R :

1. s is a safe source in $S \subseteq R$, t is a safe sink in $T \subseteq R$.
2. Reorienting each hyperarc, **one by one**, does not decrease the hyperarc-connectivity
3. Let $\vec{\mathcal{H}}'$ the hypergraph obtained after reorientation of P .
 - ▶ \mathcal{M}' : Inclusion-wise minimal members of $\mathcal{M}'_- \cup \mathcal{M}'_+$
 - ▶ Either $|\mathcal{M}'| < |\mathcal{M}|$, either $|\mathcal{M}'| = |\mathcal{M}|$ and \mathcal{M}' covers more vertices than \mathcal{M} .

Point 3. is the stopping criteria for the main algorithm :

- $\mathcal{M} = \{V\}$ implies both $\mathcal{M}_- = \{V\}$ and $\mathcal{M}_+ = \{V\}$.
- $\mathcal{T}_- = \{X \subseteq V - r, d^-(X) = k\} \cup \{V\}$
- \mathcal{M}_- : Inclusion-wise minimal members of \mathcal{T}_-
- Finally, if $\lambda(\vec{\mathcal{H}}) \geq k$ and $\mathcal{T}_- = \mathcal{T}_+ = \{V\}$, $\vec{\mathcal{H}}$ is $(k+1)$ -hyperarc-connected.

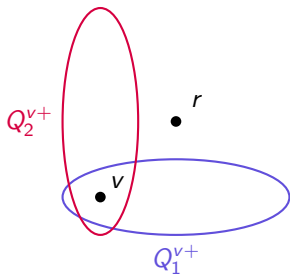
Introduction of Q_+^v

Definition of Q_+^v

Consider the sets of \mathcal{T}_+ containing v . Q_+^v is **the** minimal (inclusion-wise) one.

Unicity of Q_+^v :

If it exists, Q_+^v is unique.



Let Q_1^{v+} , Q_2^{v+} verifying the above definition.

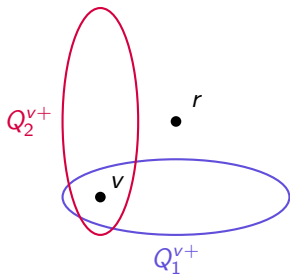
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By definition, $Q_1^{v+} \not\subseteq Q_2^{v+}$ and $Q_2^{v+} \not\subseteq Q_1^{v+}$.

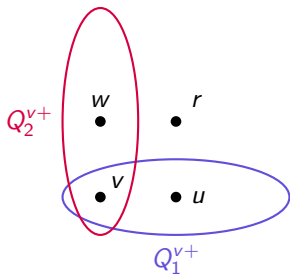
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Denote $u \in Q_1^{v+} \setminus Q_2^{v+}$, $w \in Q_2^{v+} \setminus Q_1^{v+}$.

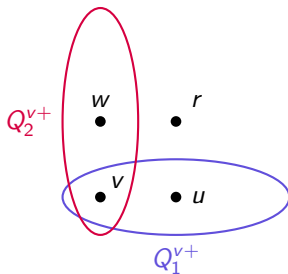
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As $r \notin Q_1^{v+}, Q_2^{v+}$, both are crossing sets.

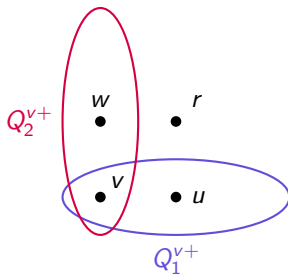
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By submodularity, if $X, Y \in \mathcal{T}_+$, both $X \cup Y \in \mathcal{T}_+$ and $X \cap Y \in \mathcal{T}_+$.

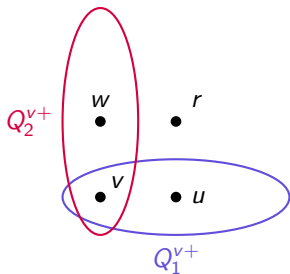
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$Q_1^{v+} \cap Q_2^{v+}$ is smaller (inclusion-wise) than Q_1^{v+} and Q_2^{v+} .

Existence of an hyperpath that does not leave Q_+^v

Lemma 12 (a)

$\forall s \in V, \forall t \in Q_+^s$, there exists an (s, t) -hyperpath that does not leave Q_+^s .

Proof of Lemma 12 (a)

- By contradiction, assume that there is $s \in V, t \in Q_+^s$ such that any (s, t) -hyperpath leaves Q_+^s .
- There is $s \in Z \subseteq Q_+^s \setminus \{t\}$ such that any hyperarc leaving Z will also leave Q_+^s .
- We have the following inequalities
 - ▶ $d_{\mathcal{H}}^+(Q_+^s) \geq d_{\mathcal{H}}^+(Z)$
 - ▶ $d_{\mathcal{H}}^+(Z) \geq k$, as \mathcal{H} is k -hyperarc-connected.
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Finding an admissible (s, t) -hyperpath in $R \in \mathcal{R} \cap \mathcal{T}_-$

Algorithm Admissible (s, t) -hyperpath in $R \in \mathcal{R} \cap \mathcal{T}_-$

- 1: Take a set $S \in \mathcal{M}_-$, with $S \subseteq R$, then a safe source $s \in S$.
 - 2: $Z = \{s\}$, $F = (Z, \emptyset)$, $V' = R$
 - 3: **while** $h = (X, v)$ exists such that $v \in V' - Z$ and $X \cap Z \neq \emptyset$ **do**
 - 4: Let $u \in X \cap Z$.
 - 5: $Z \leftarrow Z \cup v$
 - 6: $F \leftarrow F + uv$
 - 7: **if** $Q_+^v \subsetneq V'$ **then**
 - 8: $V' \leftarrow Q_+^v$
 - 9: **end if**
 - 10: **end while**
 - 11: $T = V'$
 - 12: Take a safe sink $t \in T$
 - 13: $P' = F[s, t]$
 - 14: P is the corresponding hypergraph in $\vec{\mathcal{H}}$ with respect to P' .
 - 15: **Return** S, T, s, t, P
-