Directed hypergraph connectivity augmentation by hyperarc re-orientations

Combinatorial Optimization and Graph Theory

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State of the art, goal of the article

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Ito et al (2023)

- Algorithmic proof of *Nash-Williams*, by flipping one arc at a time.
- Exhibiting a sequence of orientations such that :
  - The arc-connectivity does not decrease, and the arc-connectivity of the last element of the sequence is k.
  - The next orientation in the sequence can be obtained from the previous one by flipping exactly one arc.
  - The sequence can be obtained in polynomial time (in the size of the directed graph).

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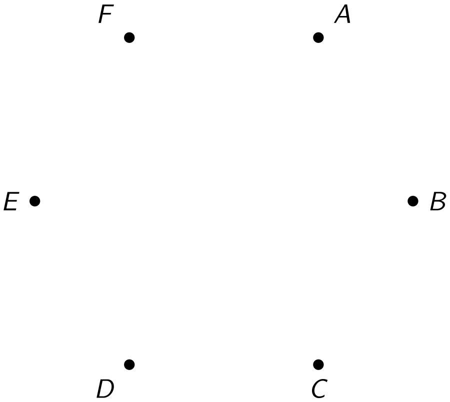
Goal of the article : Expanding the result of **Ito and al.** to hypergraphs.

Hypergraphs



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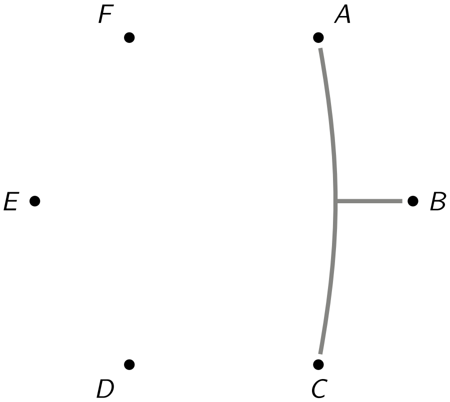

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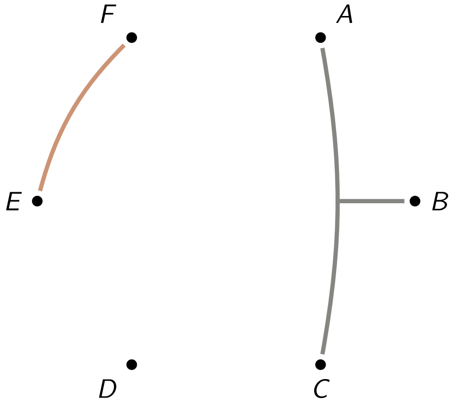

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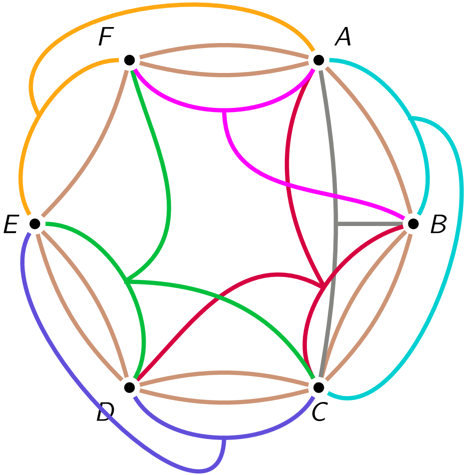
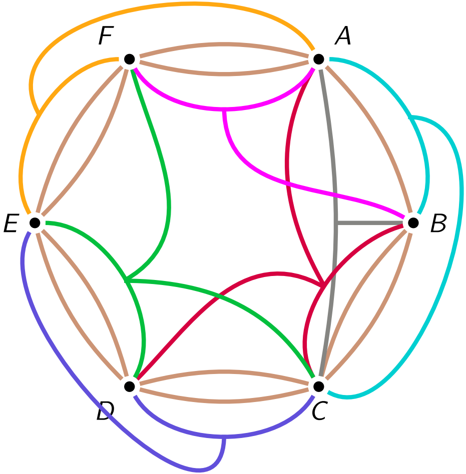


Degree of $\varnothing \neq X \subsetneq V$

$d_{\mathcal{H}}(X)$ is the number of hyperedges intersecting both X and $V \setminus X$.



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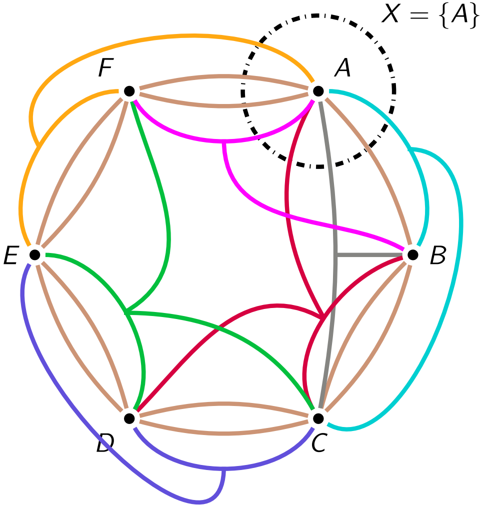

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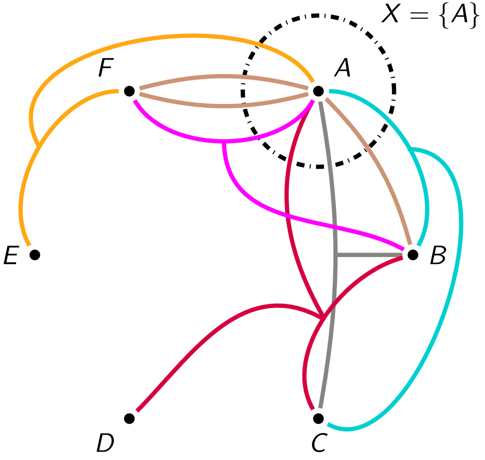

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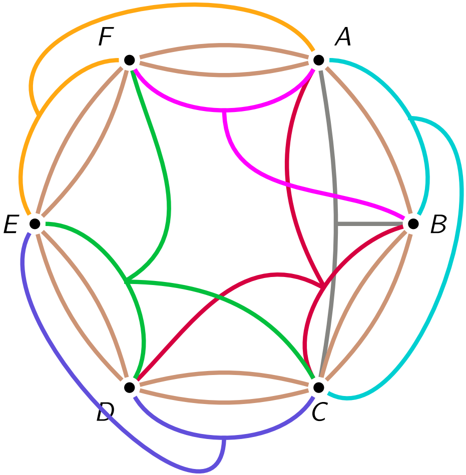

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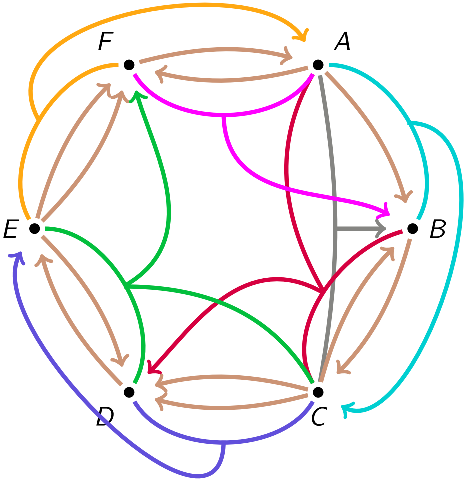

In-Degree of $\varnothing \neq X \subsetneq V$

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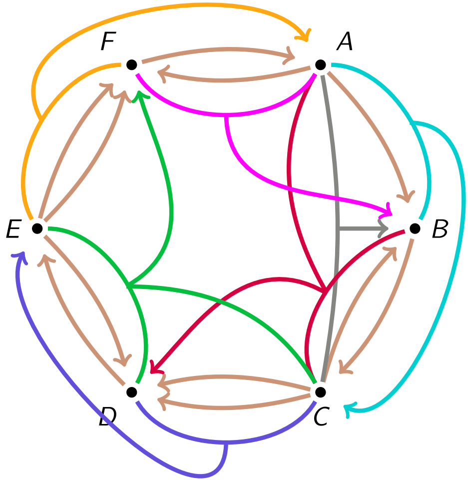

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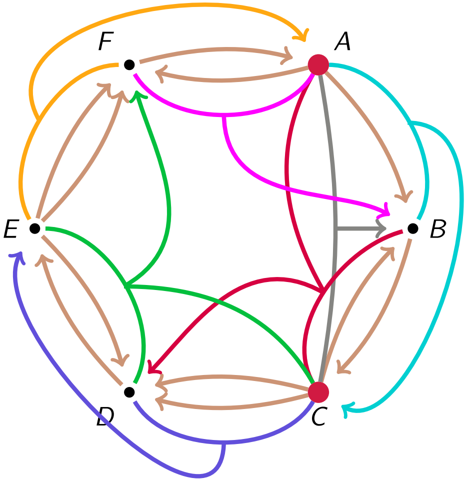

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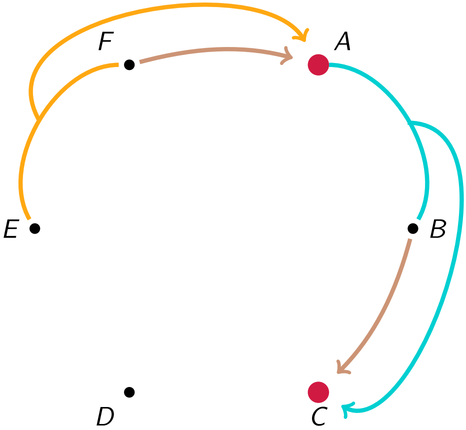

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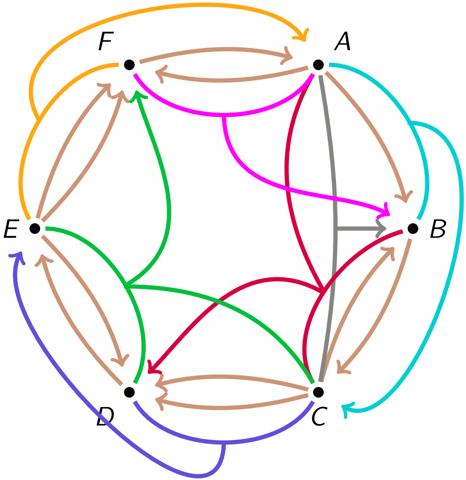

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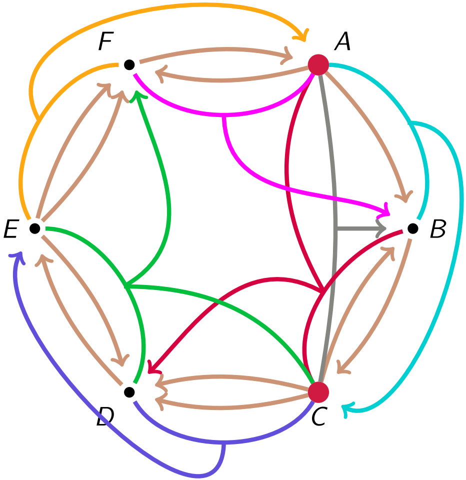

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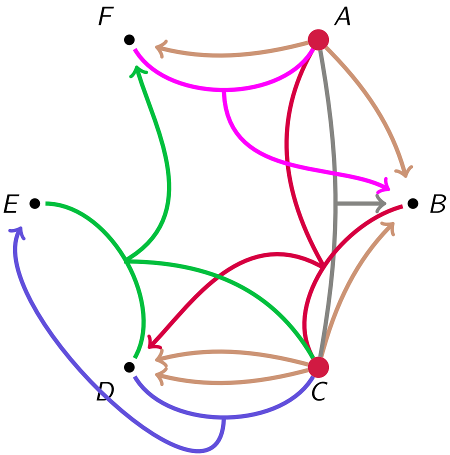

Hyperarc-connectivity

- $\vec{\mathcal{H}}$ is *k-hyperarc-connected*, if, $\forall \varnothing \neq X \subsetneq V$, $d^+_{\vec{\mathcal{H}}}(X) \geq k$.

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- $\vec{\mathcal{H}}$ is *k-hyperarc-connected*, if, $\forall \varnothing \neq X \subsetneq V$, $d^+_{\vec{\mathcal{H}}}(X) \geq k$.
- The hyperarc-connectivity of a hypergraph, denoted $\lambda(\vec{\mathcal{H}})$, is the maximum value of k such that $\vec{\mathcal{H}}$ is *k-hyperarc-connected*.

Main result

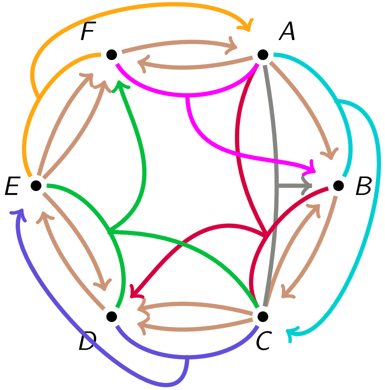
We use a result of Frank : $\mathcal{H}$ is (k,k)-partition-connected if and only if it admits a k-hyperarc-connected orientation.

Main result (Theorem 7)

Let $\mathcal{H} = (V,E)$ be a $(k+1,k+1)$-partition-connected hypergraph and $\vec{\mathcal{H}}$ is a k-hyperarc connected orientation of $\mathcal{H}$. Then there exists a sequence of hypergraphs $(\vec{\mathcal{H}}_i)_{i\in 0\ldots\ell}$ such that $\vec{\mathcal{H}}_{i+1}$ is obtained from $\vec{\mathcal{H}}_i$ by reorienting exactly one hyperarc and $\lambda(\vec{\mathcal{H}}_{i+1}) \geq \lambda(\vec{\mathcal{H}}_i)$ and $\lambda(\vec{\mathcal{H}}_\ell) = k+1$. Such a sequence of orientations can be obtained with $\ell \leq |V|^3$ and found in polynomial time (in the size of $\mathcal{H}$).

"High-Level"-running of the algorithm

Our algorithm will provide a 3-hyperarc-connected orientation of $\mathcal{H}$, starting from a 2-hyperarc-connected.



1. Take r in $V(\mathcal{H})$.
2. Compute sets of vertices.
3. Stopping Criteria
4. Select a set R (cf. 2.)
5. Find an admissible (s, t)-hyperpath in R to reorient
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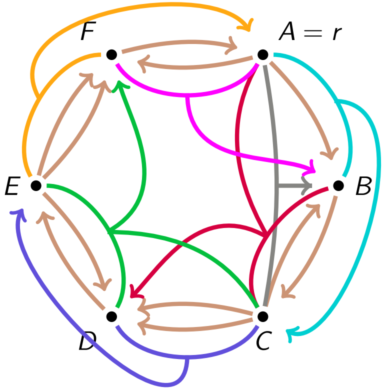

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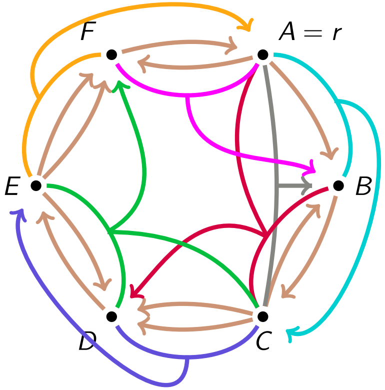

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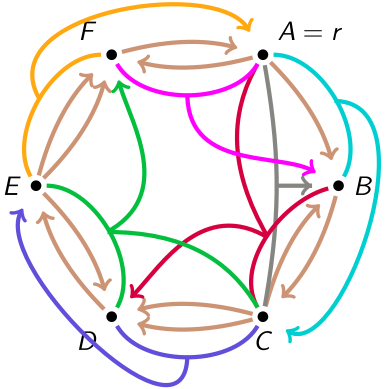

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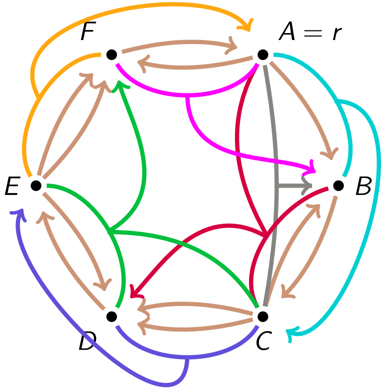

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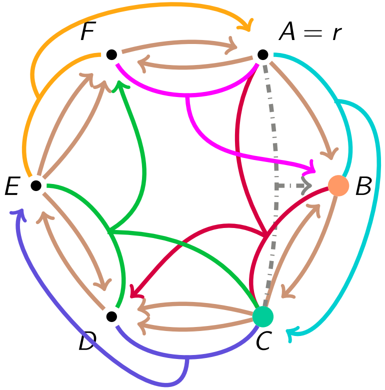

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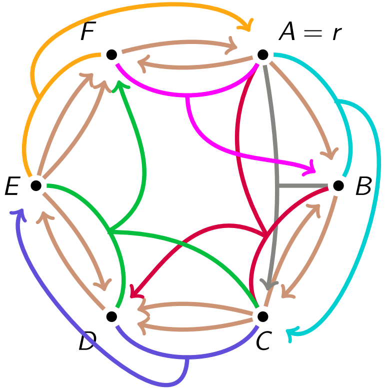

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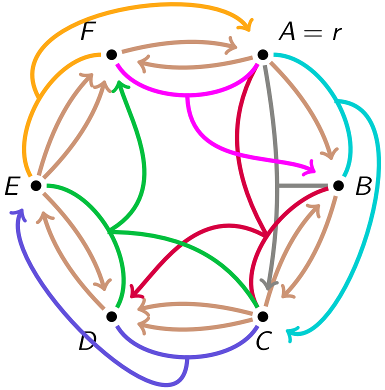

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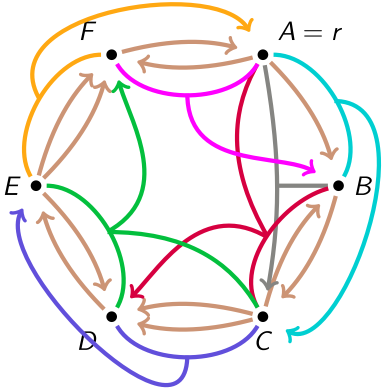

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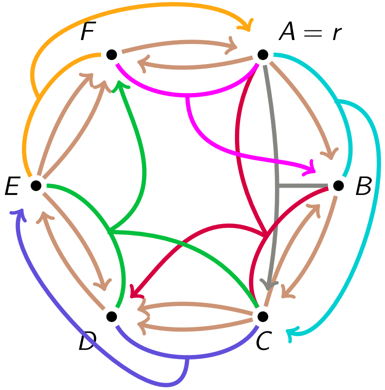

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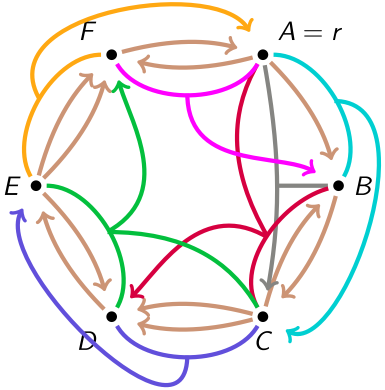

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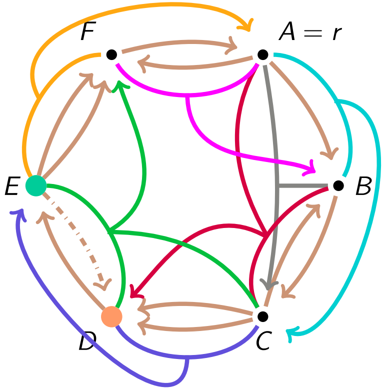

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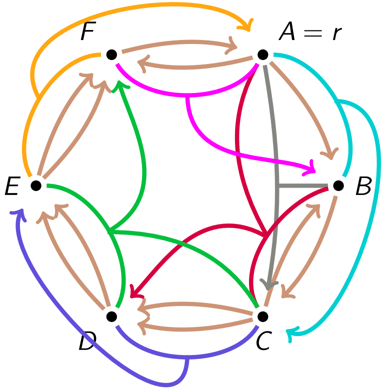

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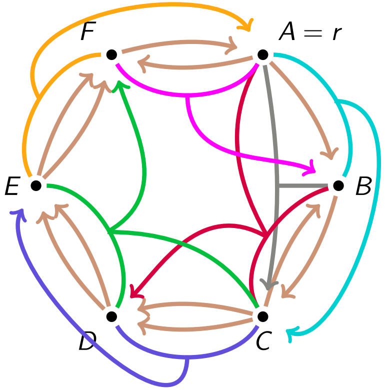

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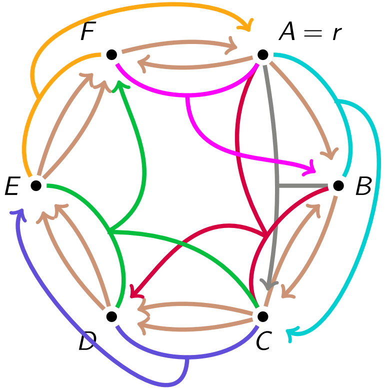

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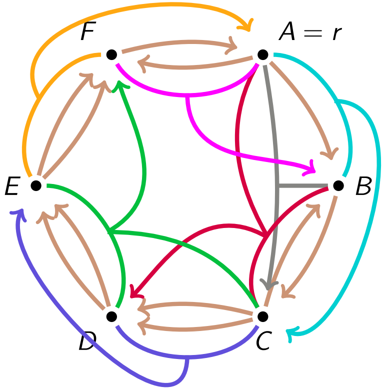

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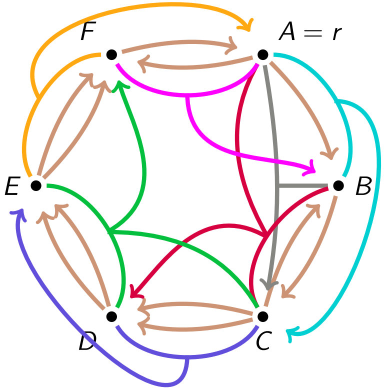

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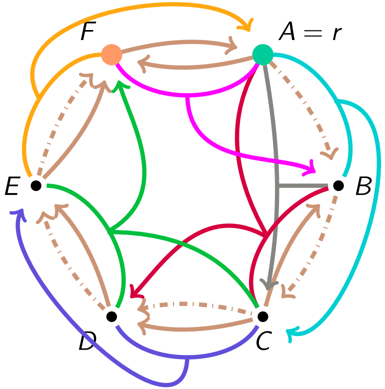

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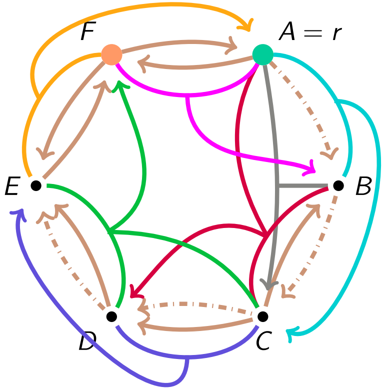

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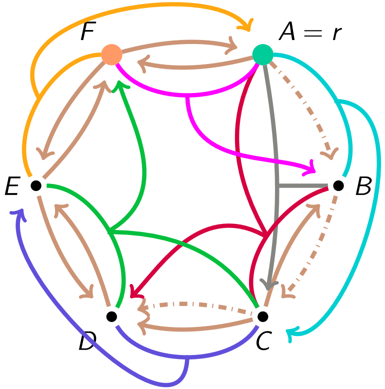

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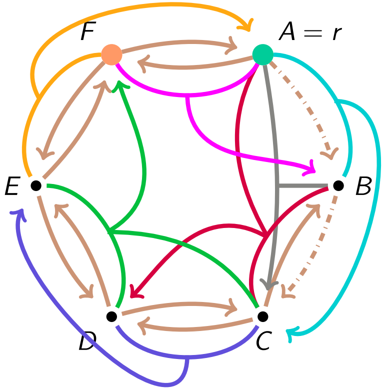

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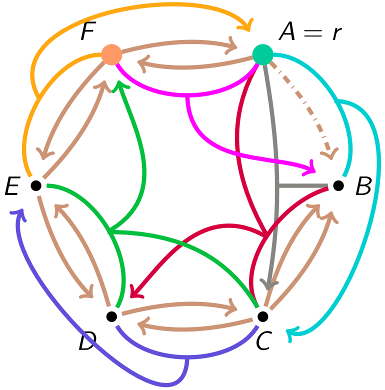

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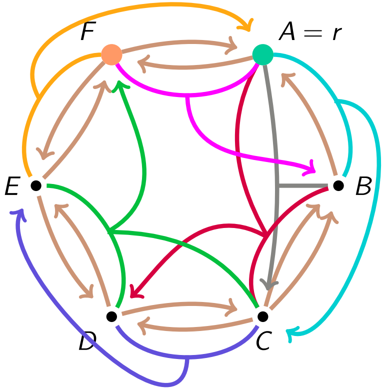

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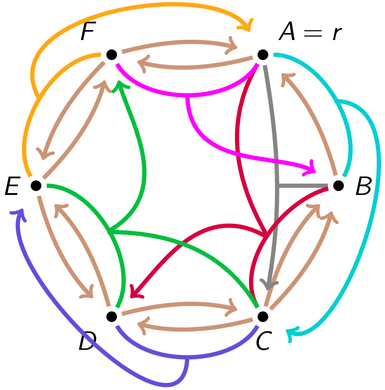
7. `Goto` (2.)

"High-Level"-running of the algorithm

Our algorithm will provide a 3-hyperarc-connected orientation of $\mathcal{H}$, starting from a 2-hyperarc-connected.



1. Take r in $V(\mathcal{H})$.
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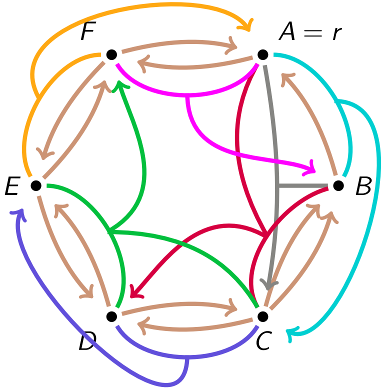

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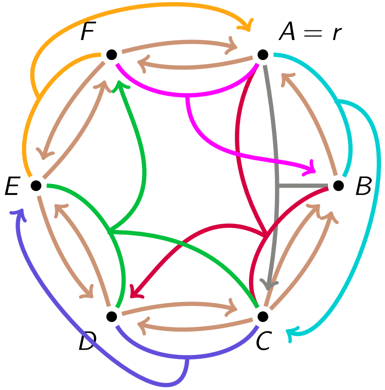

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Finding *admissible* (s,t)-hyperpaths in R

Admissible hyperpaths

- Three criterion for P to be an admissible (s,t)-hyperpath in R:

A brief detour...

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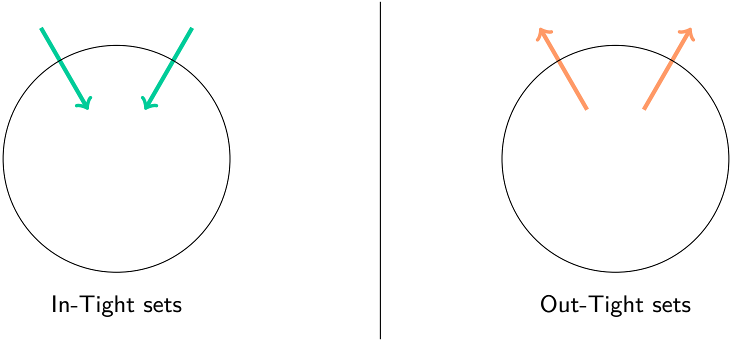
A brief detour...

Tight and Minimal-tight sets



- $\mathcal{T}_- = \{X \subseteq V - r, d^-(X) = k\} \cup \{V\}$
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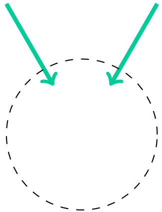
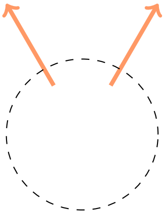
Tight and Minimal-tight sets



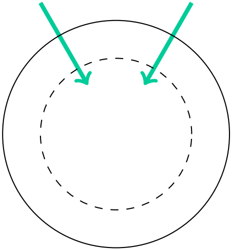
Minimal In-Tight sets



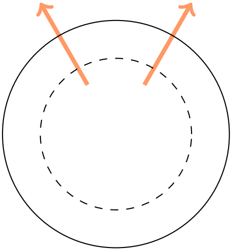
Minimal Out-Tight sets

- $\mathcal{M}_-$: Inclusion-wise minimal members of $\mathcal{T}_-$
- $\mathcal{M}_+$: Inclusion-wise minimal members of $\mathcal{T}_+$

Tight and Minimal-tight sets



In-Tight sets



Out-Tight sets

- $\mathcal{T}_- = \{X \subseteq V - r, d^-(X) = k\} \cup \{V\}$
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Crossing sets and structural results

Claim 1(b)

Let X, Y two crossing sets in V.
If $X, Y \in \mathcal{T}_+$, then both $X \cup Y \in \mathcal{T}_+$ and $X \cap Y \in \mathcal{T}_+$.

Proof of Claim 1(b)

- Since X, Y are crossing, $X \cap Y \neq \varnothing$, $X \cup Y \neq V$.

Crossing sets and structural results

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- As $\lambda(\vec{\mathcal{H}}) = k$, we have $d^+(X \cup Y) \geq k$ and $d^+(X \cap Y) \geq k$

Crossing sets and structural results

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- Grouping these equations, we obtain :
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Crossing sets and structural results

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- Grouping these equations, we obtain :
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- This implies $d^+(X \cup Y) = k = d^+(X \cap Y)$, i.e. $X \cap Y, X \cup Y \in \mathcal{T}_+$

Finding *admissible* (s, t)-hyperpaths in R

Admissible hyperpaths

Three criterion for P to be an admissible (s, t)-hyperpath in R:

1. Stopping criteria for the main algorithm :
2. s is a safe source in $S \subseteq R$, t is a safe sink in $T \subseteq R$.
3. Reorienting each hyperarc, **one by one**, does not decrease the hyperarc-connectivity

- Stopping criteria : $\mathcal{M}_- = \{V\}$ and $\mathcal{M}_+ = \{V\}$.
- $\mathcal{T}_- = \{X \subseteq V - r, d^-(X) = k\} \cup \{V\}$
- $\mathcal{M}_-$: Inclusion-wise minimal members of $\mathcal{T}_-$
- Finally, if $\lambda(\vec{\mathcal{H}}) \geq k$ and $\mathcal{T}_- = \mathcal{T}_+ = \{V\}$, $\vec{\mathcal{H}}$ is $(k+1)$-hyperarc-connected.

Existence of a safe source (*a safe sink*)

Lemma 10

$\forall S \in \mathcal{M}_-$, there is a safe source $s \in S$.

Lemma 11

$\forall T \in \mathcal{M}_+$, there is a safe sink $t \in T$.

Towards hyperarc connectivity augmentation

$\mathcal{R}$: $R \subseteq V - r$ inclusion-wise minimal such that either :

- $R \in \mathcal{T}_-$, and contains a member of $\mathcal{T}_+$
- or $R \in \mathcal{T}_+$, and contains a member of $\mathcal{T}_-$.

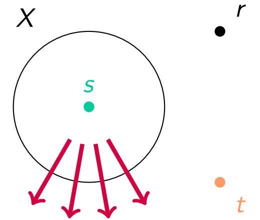
Lemma 13

Let $R \in \mathcal{R}, S \in \mathcal{M}_-, T \in \mathcal{M}_+$ such that $S, T \subseteq R$. Let s be a safe source in S, t a safe sink in T.

a. $\forall X \subseteq V - r$ such that $s \in X$, $t \notin X$, we have $d^+(X) \geq k+1$.

b. $\forall X \subseteq V - r$ such that $s \notin X$, $t \in X$, we have $d^-(X) \geq k+1$.

a.



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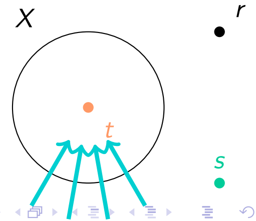

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Proof of Lemma 13

By contradiction, either :

a. $\exists X \subseteq V - r, s \in X, t \notin X, d^+(X) = k$, i.e. $s \in X, t \notin X, X \in \mathcal{T}_+$.
   - a1. $R \in \mathcal{R} \cap \mathcal{T}_-$
   - a2. $R \in \mathcal{R} \cap \mathcal{T}_+$

b. $\exists X \subseteq V - r, s \notin X, t \in X, d^-(X) = k$, i.e. $s \notin X, t \in X, X \in \mathcal{T}_-$.
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- b1. $R \in \mathcal{R} \cap \mathcal{T}_-$
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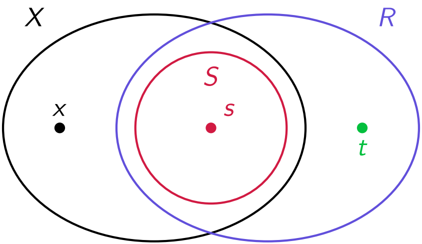
Proof of Lemma 13

$a : \exists X \subseteq V - r, s \in X, t \notin X, X \in \mathcal{T}_+$

- Since $s \in S$ is a **safe source** and $s \in X \in \mathcal{T}_+$, we have $S \subsetneq X$
- We also have $t \in R \setminus X$ by [a.], so $X \setminus R \neq \varnothing$.



Proper representation of a

Proof of Lemma 13

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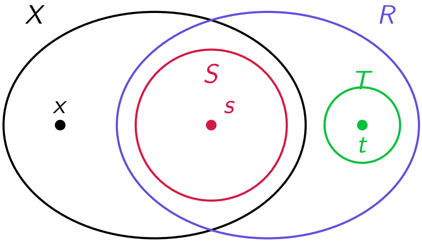

Proper representation of a

$a1.$: $R \in \mathcal{R} \cap \mathcal{T}_-, \exists X \subseteq V - r, s \in X, t \notin X, X \in \mathcal{T}_+$.

- As $t \in R \setminus X \neq \varnothing$, and using Claim 1, we have $R \setminus X \in \mathcal{T}_-$.
- $T \cap X \neq \varnothing$ would contradict the minimality of T, so T and X are disjoint.
- As $R \setminus X \in \mathcal{T}_-$, $T \in \mathcal{T}_+$, and $T \subseteq R \setminus X$, this contradicts R minimal.

Proof of Lemma 13

$a : \exists X \subseteq V - r, s \in X, t \notin X, X \in \mathcal{T}_+$

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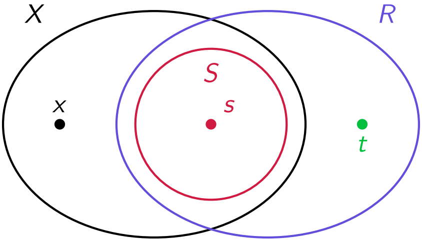

Proper representation of a

$a2.$: $R \in \mathcal{R} \cap \mathcal{T}_+, \exists X \subseteq V - r, s \in X, t \notin X, X \in \mathcal{T}_+$.

- $R \in \mathcal{T}_+, X \in \mathcal{T}_+$, and $X \cap R \neq \varnothing \implies X \cap R \in \mathcal{T}_+$
- $S \in \mathcal{T}_-, S \subseteq R \cap X$. Since $t \in R \setminus X, X \cap R \subsetneq R$.
- This contradicts the minimality of R.

Proof of Lemma 13

Lemma 13

Let $R \in \mathcal{R}, S \in \mathcal{M}_-, T \in \mathcal{M}_+$ such that $S, T \subseteq R$. Let s be a safe source in S, t a safe sink in T.

a. $\forall X \subseteq V - r$ such that $s \in X$, $t \notin X$, we have $d^+(X) \geq k+1$.

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Finding *admissible* (s,t)-hyperpaths in $R \in \mathcal{R}$

Admissible hyperpaths

Three criterion for P to be an admissible (s,t)-hyperpath in R:

1. *Stopping criteria*-related argument.
2. s is a **safe source** in $S \subseteq R$, t is a **safe sink** in $T \subseteq R$.
3. Reorienting each hyperarc, **one by one**, does not decrease the hyperarc-connectivity
   - How to proceed ?

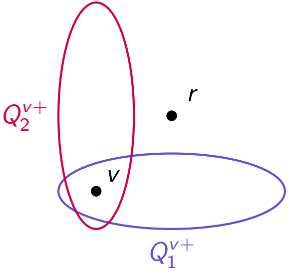
Introduction of Q_+^v

Definition of Q_+^v

Consider the sets of $\mathcal{T}_+$ containing v. Q_+^v is **the** minimal (inclusion-wise) one.

Unicity of Q_+^v :

Q_+^v is unique.



Let Q_1^{v+}, Q_2^{v+} verifying the above definition.

Introduction of Q_+^v

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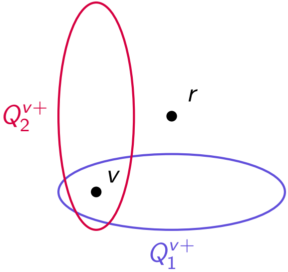

By definition, $Q_1^{v+} \not\subseteq Q_2^{v+}$ and $Q_2^{v+} \not\subseteq Q_1^{v+}$.

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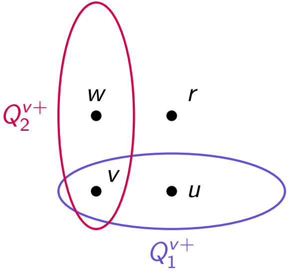

Denote $u \in Q_1^{v+} \setminus Q_2^{v+}$, $w \in Q_2^{v+} \setminus Q_1^{v+}$.

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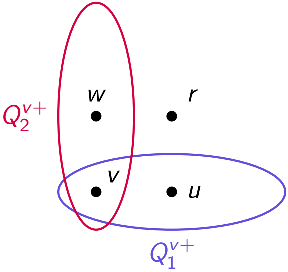

As $r \notin Q_1^{v+}, Q_2^{v+}$, both are are crossing sets.

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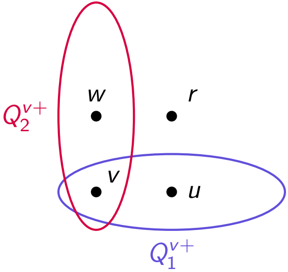

By submodularity, if $X, Y \in \mathcal{T}_+$, both $X \cup V \in \mathcal{T}_+$ and $X \cap V \in \mathcal{T}_+$.

Introduction of Q_+^v

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Consider the sets of $\mathcal{T}_+$ containing v. Q_+^v is **the** minimal (inclusion-wise) one.

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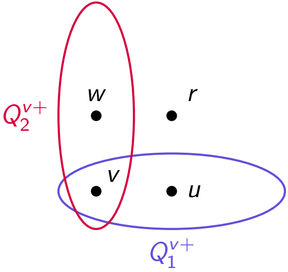

$Q_1^{v+} \cap Q_2^{v+}$ is smaller (inclusion-wise) than Q_1^{v+} and Q_2^{v+}.

Existence of an hyperpath that does not leave Q_+^v

Lemma 12(a)

$\forall s \in V, \forall t \in Q_+^s$, there exists an (s,t)-hyperpath that does not leave Q_+^s.

Proof of Lemma 12 (a)

- By contradiction, assume that there is $s \in V, t \in Q_+^s$ such that any (s,t)-hyperpath leaves Q_+^s.

Existence of an hyperpath that does not leave Q_+^v

Lemma 12(a)

$\forall s \in V, \forall t \in Q_+^s$, there exists an (s,t)-hyperpath that does not leave Q_+^s.

Proof of Lemma 12 (a)

- By contradiction, assume that there is $s \in V$, $t \in Q_+^s$ such that any (s,t)-hyperpath leaves Q_+^s.
- There is $s \in Z \subseteq Q_+^s \setminus \{t\}$ such that any hyperarc leaving Z will also leave Q_+^s.

Existence of an hyperpath that does not leave Q_+^v

Lemma 12(a)

$\forall s \in V, \forall t \in Q_+^s$, there exists an (s,t)-hyperpath that does not leave Q_+^s.

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- By contradiction, assume that there is $s \in V, t \in Q_+^s$ such that any (s,t)-hyperpath leaves Q_+^s.
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- We have the following inequalities

Existence of an hyperpath that does not leave Q_+^v

Lemma 12(a)

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Proof of Lemma 12 (a)

- By contradiction, assume that there is $s \in V, t \in Q_+^s$ such that any (s,t)-hyperpath leaves Q_+^s.
- There is $s \in Z \subseteq Q_+^s \setminus \{t\}$ such that any hyperarc leaving Z will also leave Q_+^s.
- We have the following inequalities
  - $d^+_{\vec{\mathcal{H}}}(Q_+^s) \geq d^+_{\vec{\mathcal{H}}}(Z)$

Existence of an hyperpath that does not leave Q_+^v

Lemma 12(a)

$\forall s \in V, \forall t \in Q_+^s$, there exists an (s,t)-hyperpath that does not leave Q_+^s.

Proof of Lemma 12 (a)

- By contradiction, assume that there is $s \in V, t \in Q_+^s$ such that any (s,t)-hyperpath leaves Q_+^s.
- There is $s \in Z \subseteq Q_+^s \setminus \{t\}$ such that any hyperarc leaving Z will also leave Q_+^s.
- We have the following inequalities
  - $d_{\vec{\mathcal{H}}}^+(Q_+^s) \geq d_{\vec{\mathcal{H}}}^+(Z)$
  - $d_{\vec{\mathcal{H}}}^+(Z) \geq k$, as $\vec{\mathcal{H}}$ is k-hyperarc-connected.

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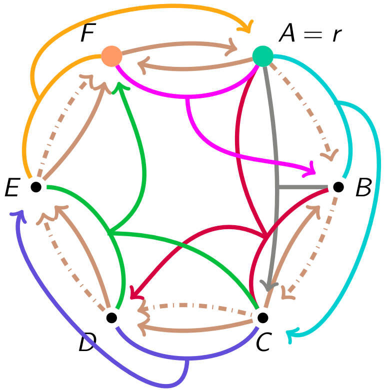
- Q_+^s is not minimal, hence the contradiction.

Finding an admissible (s, t)-hyperpath in $R \in \mathcal{R} \cap \mathcal{T}_{-}$



1. Take r in $V(\mathcal{H})$.
2. Compute sets of vertices.
3. Stopping Criteria
4. Select a set R (cf. 2.)
5. Find an admissible (s, t)-hyperpath in R to reorient
6. Reorient the corresponding hyperpath.
7. `Goto` (2.)

Finding an admissible (s,t)-hyperpath in $R \in \mathcal{R} \cap \mathcal{T}_-$

Algorithm Admissible (s,t)-hyperpath in $R \in \mathcal{R} \cap \mathcal{T}_-$

Take a set $S \in \mathcal{M}_-$, with $S \subseteq R$, then a safe source $s \in S$.
$Z = \{s\}$, $F = (Z, \varnothing)$, $V' = R$
while $h = (X, v)$ exists such that $v \in V' - Z$ and $X \cap Z \neq \varnothing$ **do**
    Let $u \in X \cap Z$.
    $Z \leftarrow Z \cup \{v\}$
    $F \leftarrow F + uv$
    if $Q_+^v \subsetneq V'$ **then**
        $V' \leftarrow Q_+^v$
    end if
end while
$T = V'$
Take a safe sink $t \in T$
$P' = F[s, t]$
P is the corresponding hyperpath in $\vec{\mathcal{H}}$, obtained with P'.
Return S, T, s, t, P

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Thank you for your attention.