Directed hypergraph connectivity augmentation by hyperarc re-orientations

Combinatorial Optimization and Graph Theory

Benoît Bompol, Armand Grenier

Thursday, Nov 23rd 2023

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State of the art, goal of the article

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- *Ito and al.*, 2023 :
  - ▶ Algorithmic proof of *Nash-Williams*, by flipping one arc at a time.
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    - The next orientation in the sequence can be obtained from the previous one by flipping exactly one arc.
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Goal of the article : Expanding the result of **Ito and al.** to hypergraphs.
Side note : This article generalise the results of **Ito and al.**, as directed graphs are special case of hypergraphs.
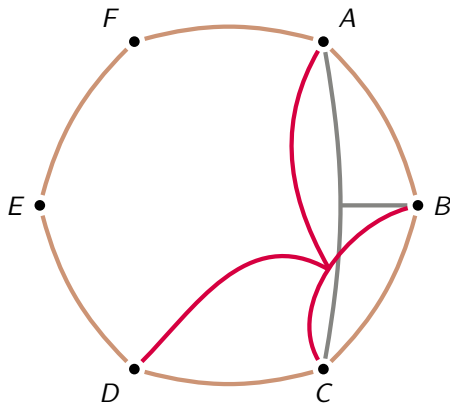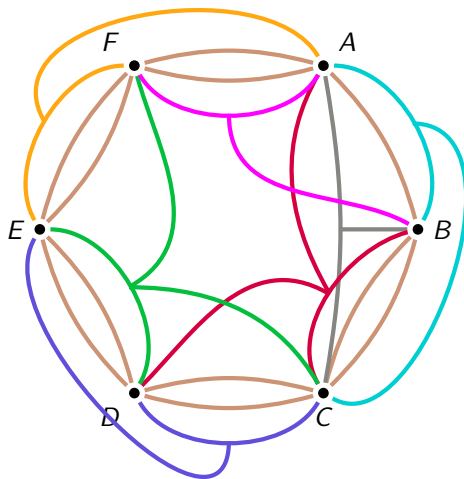
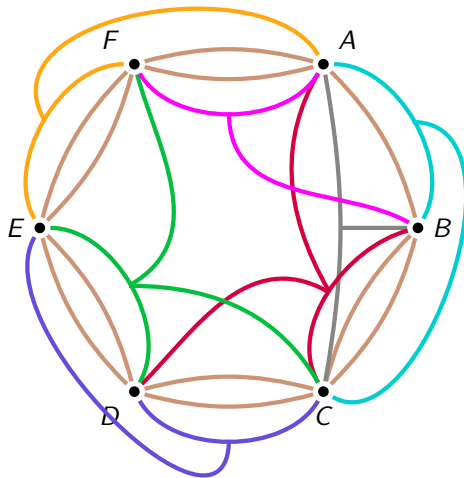# Hypergraphs

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Degree of $\varnothing \neq X \subsetneq V$

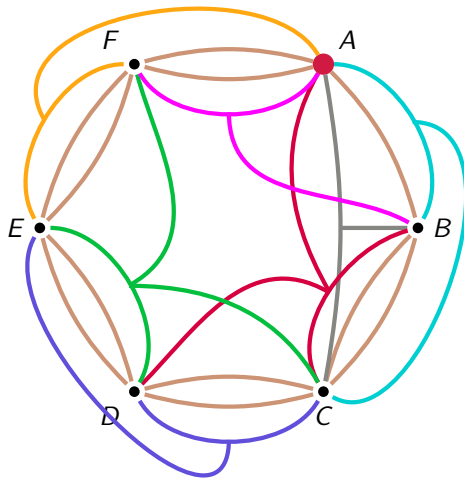$d_{\mathcal{H}}(X)$ is the number of hyperedges intersecting both X and $V \setminus X$.

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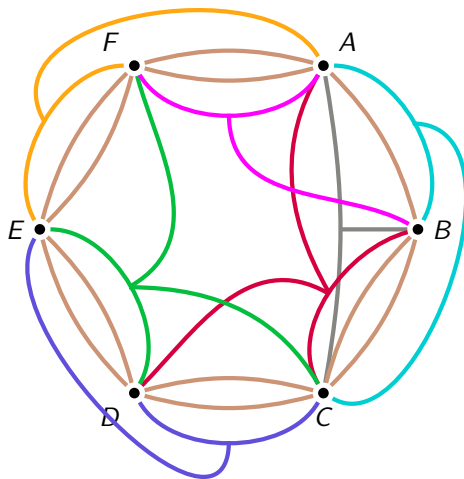$d_{\mathcal{H}}(X)$ is the number of hyperedges intersecting both X and $V \setminus X$.
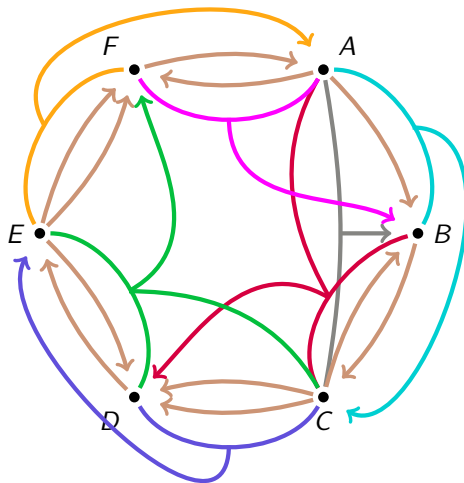
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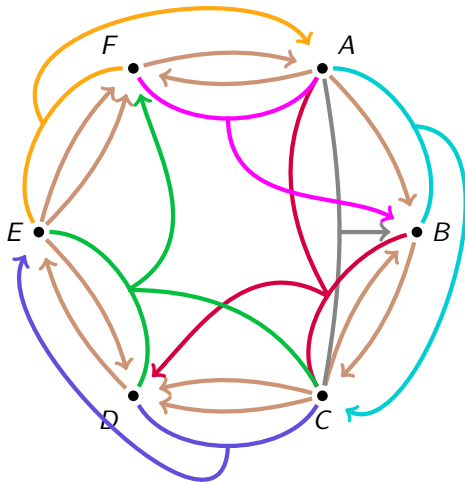
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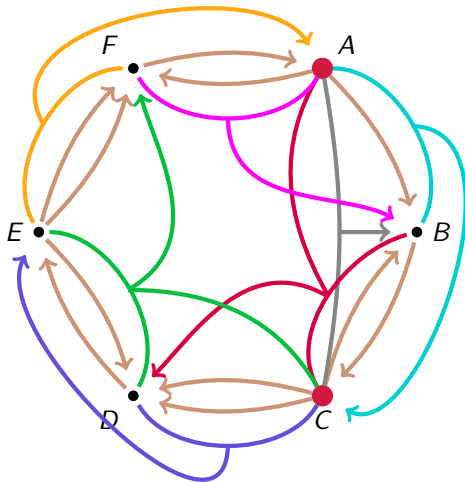
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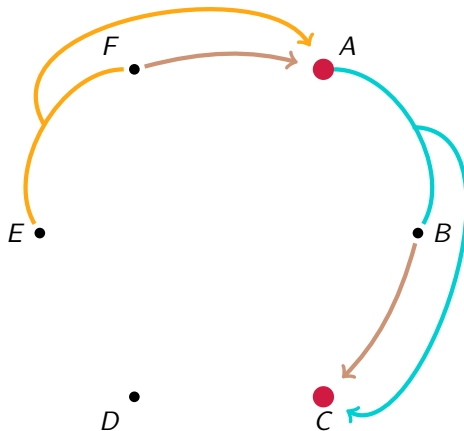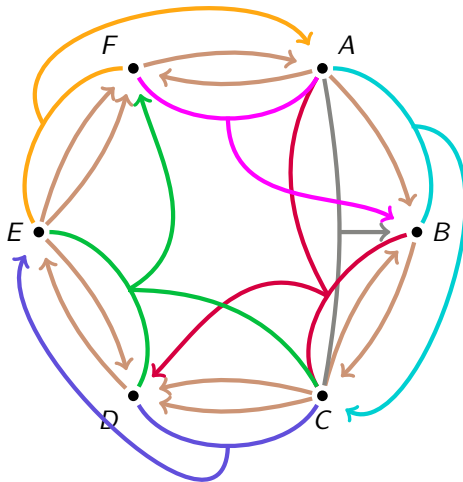
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Hyperarc-connectivity and (k, k)-partition connected hypergraphs

- $\vec{\mathcal{H}}$ is *k-hyperarc-connected*, if, $\forall \varnothing \neq X \subsetneq V$, $d_{\vec{\mathcal{H}}}^+(X) \geq k$.
- The hyperarc-connectivity of a hypergraph, denoted $\lambda(\vec{\mathcal{H}})$, is the maximum value of k such that $\vec{\mathcal{H}}$ is *k-hyperarc-connected*.

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We use a result of Frank : $\mathcal{H}$ is (k, k)-partition-connected if and only if it admits a k-hyperarc-connected orientation.
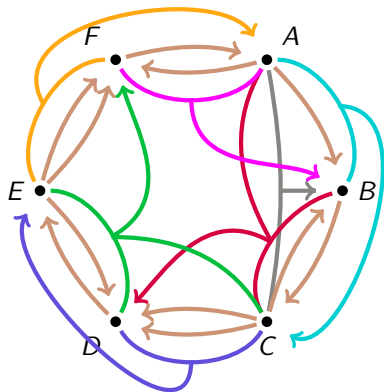
Main result

Main result (Theorem 7)

Let $\mathcal{H} = (V, E)$ be a $(k + 1, k + 1)$-partition-connected hypergraph and $\vec{\mathcal{H}}$ is a k-hyperarc connected orientation of $\mathcal{H}$. Then there exists a sequence of hypergraphs $(\vec{\mathcal{H}}_i)_{i \in 0 \ldots \ell}$ such that $\vec{\mathcal{H}}_{i+1}$ is obtained from $\vec{\mathcal{H}}_i$ by reorienting exactly one hyperarc and $\lambda(\vec{\mathcal{H}}_{i+1}) \geq \lambda(\vec{\mathcal{H}}_i)$ and $\lambda(\vec{\mathcal{H}}_\ell) = k + 1$. Such a sequence of orientations can be obtained with $\ell \leq |V|^3$ and found in polynomial time (in the size of $\mathcal{H}$).

"High-Level"-running of the algorithm

Our algorithm will provide a 3-hyperarc-connected orientation of $\mathcal{H}$, starting from a 2-hyperarc-connected.

1. Take r in $V(\mathcal{H})$.
2. Compute sets of vertices.
3. Stopping Criteria
4. Select a set R (cf. 2.)
5. Find an admissible (s, t)-hyperpath in R to reorient
6. Reorient the corresponding hyperpath.
7. Goto (2.)

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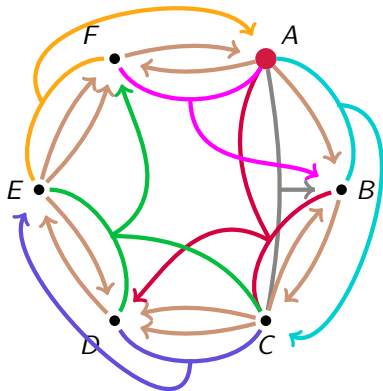7. Goto (2.)

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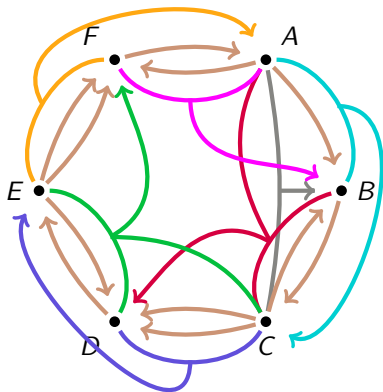7. Goto (2.)

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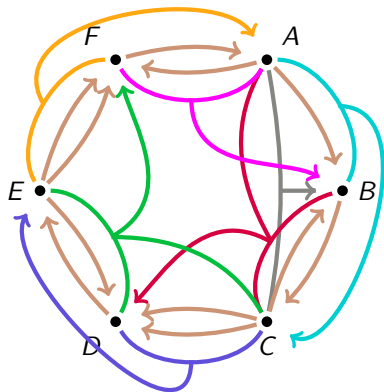7. Goto (2.)

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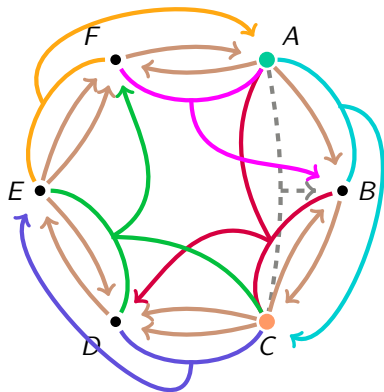7. Goto (2.)

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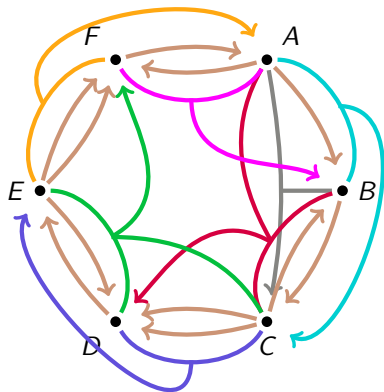7. Goto (2.)

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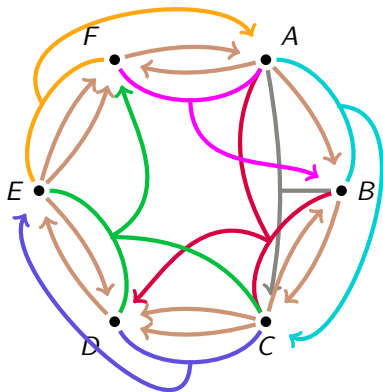7. Goto (2.)

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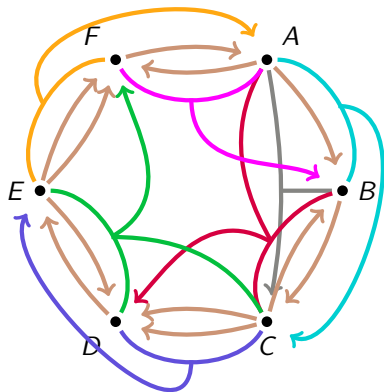7. Goto (2.)

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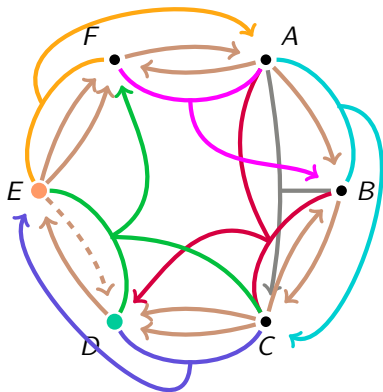7. Goto (2.)

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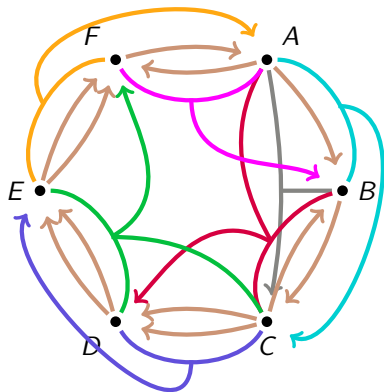7. Goto (2.)

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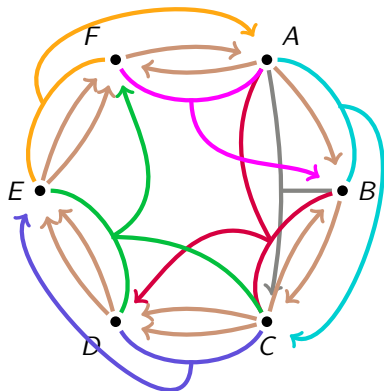7. Goto (2.)

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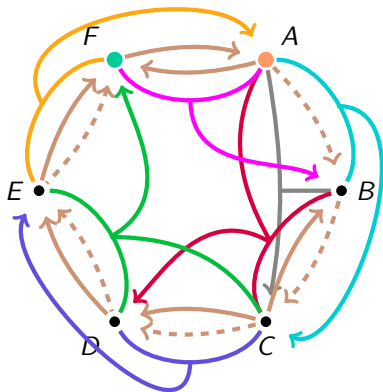7. Goto (2.)

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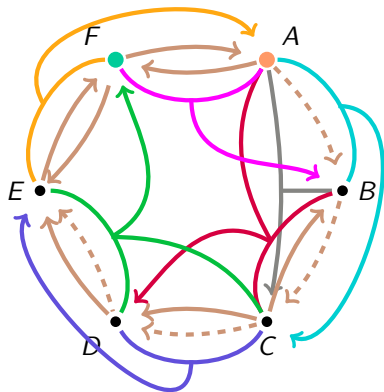7. Goto (2.)

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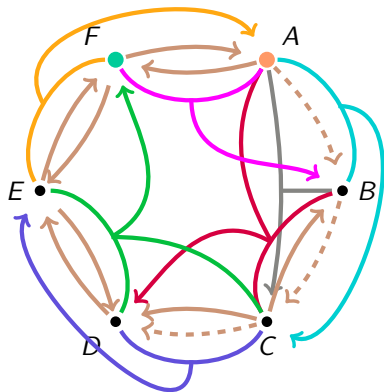7. Goto (2.)

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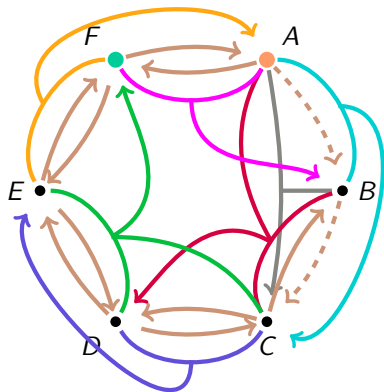7. Goto (2.)

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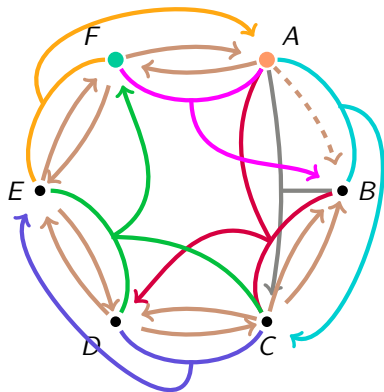7. Goto (2.)
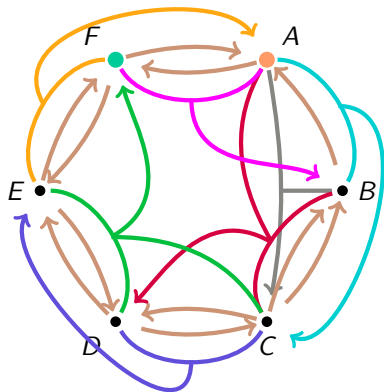
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Finding *admissible* (s, t)-hyperpaths in R

- Crucial segment of the algorithm.
- Three criterion for P to be an admissible (s, t)-hyperpath in R:
    1. s is a safe source in $S \subseteq R$, t is a safe sink in $T \subseteq R$.
    2. Reorient each hyperarc, one by one, does not decrease the hyperarc-connectivity.
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A brief detour...

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Tight and Dangerous sets
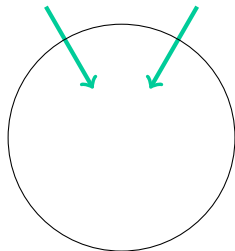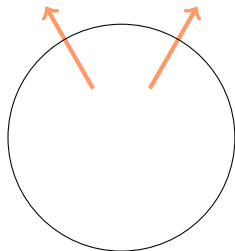
In-Tight sets   Out-Tight sets

- $\mathcal{T}_- = \{X \subseteq V - r, d^-(X) = k\} \cup \{V\}$
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Tight and Dangerous sets
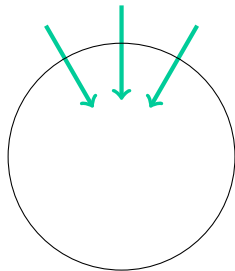
In-Dangerous sets
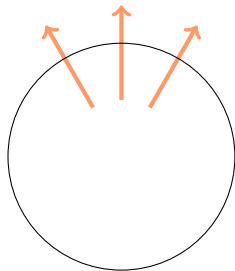
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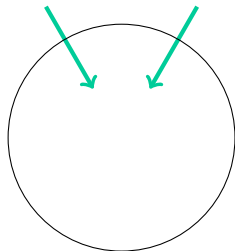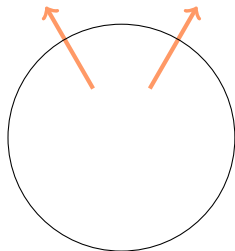
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Tight and Dangerous sets

Minimal In-Tight sets
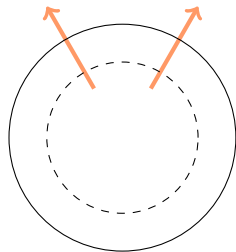
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Crossing sets and structural results

Let X, Y two crossing sets in V.

Claim 1(b)

If $X, Y \in \mathcal{T}_+$, then both $X \cup Y \in \mathcal{T}_+$ and $X \cap Y \in \mathcal{T}_+$.

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Proof of Claim 1(b)

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- $k + k = d^+(X) + d^+(Y)$
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Existence of a safe source (*a safe sink*)

Lemma 10
$\forall S \in \mathcal{M}_-$, there is a safe source $s \in S$.

Likewise,

Lemma 11
$\forall T \in \mathcal{M}_+$, there is a safe sink $t \in T$.

Quick outline of a proof for Lemma 10 :

- Let $S \in \mathcal{M}_-$.
- Considering a family of vertex sets (χ) that cover as many vertices of S as possible, but using as little as vertex sets possible.
- We can prove that, under given assumptions, χ cannot cover every vertex of S.
- Vertices that are not covered by χ are "potential" safe sources, the last part of the proof is verifying that they are effectively safe sources.

Towards hyperarc connectivity augmentation

$\mathcal{R}$: $R \subseteq V - r$ inclusion-wise minimal such that either :
- $R \in \mathcal{T}_-$, and contains a member of $\mathcal{T}_+$
- or $R \in \mathcal{T}_+$, and contains a member of $\mathcal{T}_-$.

Lemma 13

Let $R \in \mathcal{R}, S \in \mathcal{M}_-, T \in \mathcal{M}_+$ such that $S, T \subseteq R$. Let s be a safe source in S, t a safe sink in T.

a. $\forall X \subseteq V - r$ such that $s \in X$, $t \notin X$, we have $d^+(X) \geq k + 1$.

b. $\forall X \subseteq V - r$ such that $s \notin X$, $t \in X$, we have $d^-(X) \geq k + 1$.

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Proof of Lemma 13

By contradiction, either :

  a. $s \in X, t \notin X, d^+(X) = k$, i.e. $s \in X, t \notin X, X \in \mathcal{T}_+$.

    a1. $R \in \mathcal{R} \cap \mathcal{T}_-$

    a2. $R \in \mathcal{R} \cap \mathcal{T}_+$

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$a : \exists X \subseteq V - r, s \in X, t \notin X, X \in \mathcal{T}_+$

. Since $s \in S$ is a **safe source** and $s \in X \in \mathcal{T}_+$, we have $S \subsetneq X$

. We also have $t \in R \setminus X$ by [a.], so $X \setminus R \neq \varnothing$.

Proper representation of a

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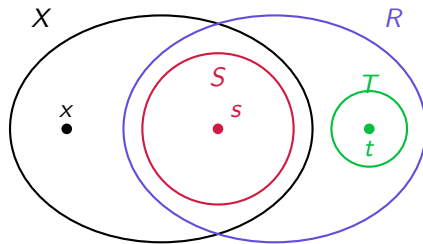
Proper representation of a

$a1. : R \in \mathcal{R} \cap \mathcal{T}_-, \exists X \subseteq V - r, s \in X, t \notin X, X \in \mathcal{T}_+$.

. As $t \in R \setminus X \neq \varnothing$, and using Claim 1, we have $R \setminus X \in \mathcal{T}_-$.

. $T \cap X \neq \varnothing$ would contradict the minimality of T, so T and X are disjoint.

. As $R \setminus X \in \mathcal{T}_-$, $T \in \mathcal{T}_+$, and $T \subseteq R \setminus X$, this contradicts R minimal.

Proof of Lemma 13

$a : \exists X \subseteq V - r, s \in X, t \notin X, X \in \mathcal{T}_+$

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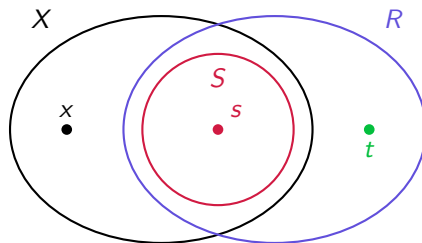
Proper representation of a

$a2. : R \in \mathcal{R} \cap \mathcal{T}_+, \exists X \subseteq V - r, s \in X, t \notin X, X \in \mathcal{T}_+$.

　. $R \in \mathcal{T}_+, X \in \mathcal{T}_+$, and $X \cap R \neq \varnothing \implies X \cap R \in \mathcal{T}_+$

　. $S \in \mathcal{T}_-, S \subseteq R \cap X$. Since $r \in R \setminus X, X \cap R \subsetneq R$.

　. This contradicts the minimality of R.
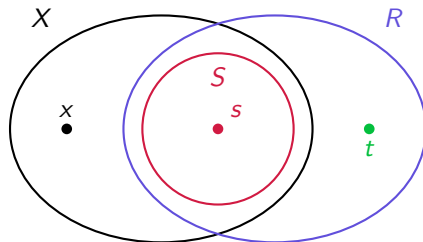
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Finding *admissible* (s, t)-hyperpaths in $R \in \mathcal{R}$

Three criterion for P to be an admissible (s, t)-hyperpath in R:

1. s is a safe source in $S \subseteq R$, t is a safe sink in $T \subseteq R$.
2. Reorienting each hyperarc, **one by one**, does not decrease the hyperarc-connectivity
3. Let $\vec{\mathcal{H}}'$ the hypergraph obtained after reorientation of P.
   - $\mathcal{M}'$: Inclusion-wise minimal members of $\mathcal{M}'_- \cup \mathcal{M}'_+$
   - Either $|\mathcal{M}'| < |\mathcal{M}|$, either $|\mathcal{M}'| = |\mathcal{M}|$ and $\mathcal{M}'$ covers more vertices than $\mathcal{M}$.

Point 3. is the stopping criteria for the main algorithm :

- $\mathcal{M} = \{V\}$ implies both $\mathcal{M}_- = \{V\}$ and $\mathcal{M}_+ = \{V\}$.
- $\mathcal{T}_- = \{X \subseteq V - r, d^-(X) = k\} \cup \{V\}$
- $\mathcal{M}_-$: Inclusion-wise minimal members of $\mathcal{T}_-$
- Finally, if $\lambda(\vec{\mathcal{H}}) \geq k$ and $\mathcal{T}_- = \mathcal{T}_+ = \{V\}$, $\vec{\mathcal{H}}$ is $(k+1)$-hyperarc-connected.

Introduction of Q_+^v

Definition of Q_+^v

Consider the sets of $\mathcal{T}_+$ containing v. Q_+^v is **the** minimal (inclusion-wise) one.

Unicity of Q_+^v :

If it exists, Q_+^v is unique.

Let Q_1^{v+}, Q_2^{v+} verifying the above definition.

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By definition, $Q_1^{v+} \not\subseteq Q_2^{v+}$ and $Q_2^{v+} \not\subseteq Q_1^{v+}$.

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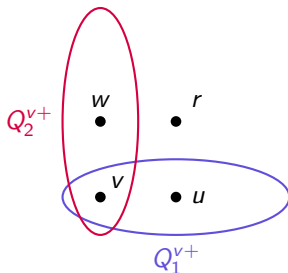
Denote $u \in Q_1^{v+} \setminus Q_2^{v+}$, $w \in Q_2^{v+} \setminus Q_1^{v+}$.

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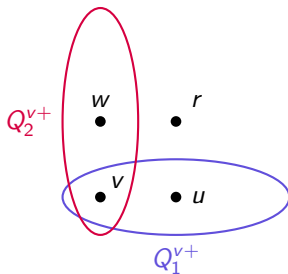
As $r \notin Q_1^{v+}, Q_2^{v+}$, both are are crossing sets.

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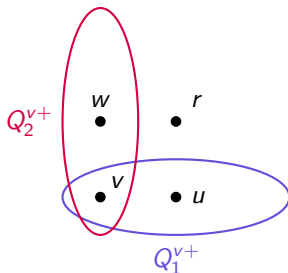
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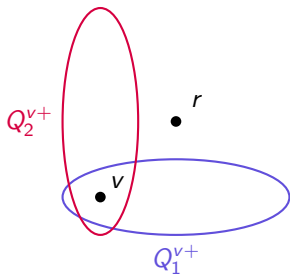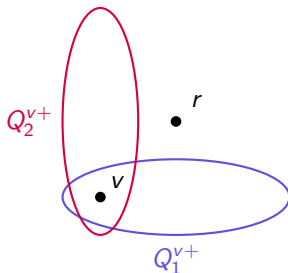
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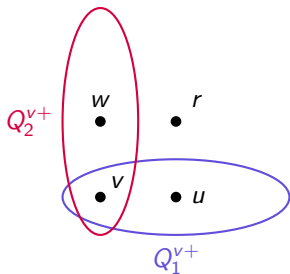
$Q_1^{v+} \cap Q_2^{v+}$ is smaller (inclusion-wise) than Q_1^{v+} and Q_2^{v+}.

Existence of an hyperpath that does not leave Q_+^v

Lemma 12 (a)

$\forall s \in V, \forall t \in Q_+^s$, there exists an (s,t)-hyperpath that does not leave Q_+^s.

Proof of Lemma 12 (a)

- By contradiction, assume that there is $s \in V, t \in Q_+^s$ such that any (s,t)-hyperpath leaves Q_+^s.
- There is $s \in Z \subseteq Q_+^s \setminus \{t\}$ such that any hyperarc leaving Z will also leave Q_+^s.
- We have the following inequalities
  - $d_{\mathcal{H}}^+(Q_+^s) \geq d_{\mathcal{H}}^+(Z)$
  - $d_{\mathcal{H}}^+(Z) \geq k$, as $\mathcal{H}$ is k-hyperarc-connected.
  - $k = d_{\mathcal{H}}^+(Q_+^s)$ by definition.
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Existence of an hyperpath that does not leave Q_+^v

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$\forall s \in V, \forall t \in Q_+^s$, there exists an (s, t)-hyperpath that does not leave Q_+^s.

Proof of Lemma 12 (a)

- By contradiction, assume that there is $s \in V, t \in Q_+^s$ such that any (s, t)-hyperpath leaves Q_+^s.
- There is $s \in Z \subseteq Q_+^s \setminus \{t\}$ such that any hyperarc leaving Z will also leave Q_+^s.
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  - ▶ $d_{\mathcal{H}}^+(Q_+^s) \geq d_{\mathcal{H}}^+(Z)$
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Finding an admissible (s, t)-hyperpath in $R \in \mathcal{R} \cap \mathcal{T}_-$

1. Only input of the algorithm $R \in \mathcal{R} \in \mathcal{T}_-$
   - s, t are constrained (maybe not unique) by the choice of R.
2. Choosing $S \in \mathcal{M}_-$ such that $S \subseteq R$, then a safe source $s \in S$.
3. Main part of the algorithm : s-out arborescence
   - F : (Directed) arborescence, rooted in s
   - Z : Explored (yet) vertices
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Algorithm Admissible (s, t)-hyperpath in $R \in \mathcal{R} \cap \mathcal{T}_-$

1: Take a set $S \in \mathcal{M}_-$, with $S \subseteq R$, then a safe source $s \in S$.
2: $Z = \{s\}$, $F = (Z, \varnothing)$, $V' = R$
3: **while** $h = (X, v)$ exists such that $v \in V' - Z$ and $X \cap Z \neq \varnothing$ **do**
4:     Let $u \in X \cap Z$.
5:     $Z \leftarrow Z \cup \{v\}$
6:     $F \leftarrow F + uv$
7:     **if** $Q_+^v \subsetneq V'$ **then**
8:         $V' \leftarrow Q_+^v$
9:     **end if**
10: **end while**
11: $T = V'$
12: Take a safe sink $t \in T$
13: $P' = F[s, t]$
14: P is the corresponding hyperpath in $\vec{\mathcal{H}}$, obtained with P'.
15: **Return** S, T, s, t, P

A few words about complexity

- Computing $\mathcal{R}$, $\mathcal{M}_-$, $\mathcal{M}_+$ in polynomial time :
  - ▶ We can transform our hypergraph in a network (by trimming),
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Final words

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Thank you for your attention.