

Directed hypergraph connectivity augmentation by hyperarc re-orientations

Combinatorial Optimization and Graph Theory

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Table of contents

- 1 Introduction
- 2 Principal results
- 3 Setting up the framework
- 4 Towards $(k + 1)$ -hyperarc connectivity
- 5 Finding *admissible* hyperpaths
- 6 Conclusion

State of the art, goal of the article

Nash-Williams (1960)

G is a $2k$ -edge connected undirected graph $\Leftrightarrow G$ admits a k -arc connected orientation.

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Ito et al (2023)

- Algorithmic proof of *Nash-Williams*, by flipping one arc at a time.
- Exhibiting a sequence of orientations such that :
 - ▶ The arc-connectivity does not decrease, and the arc-connectivity of the last element of the sequence is k .
 - ▶ The next orientation in the sequence can be obtained from the previous one by flipping exactly one arc.
 - ▶ The sequence can be obtained in polynomial time (in the size of the directed graph).

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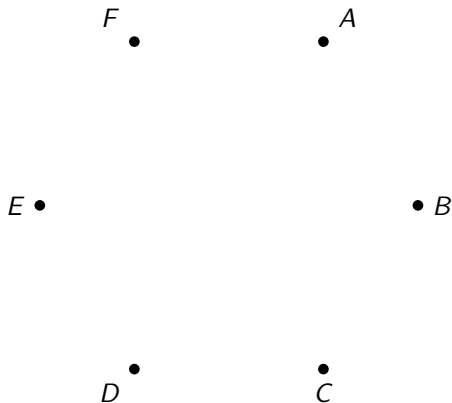
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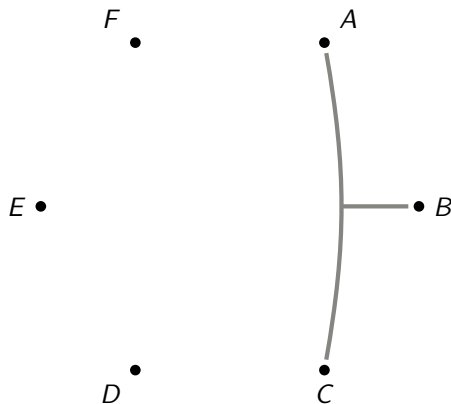
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Goal of the article : Expanding the result of **Ito and al.** to hypergraphs.

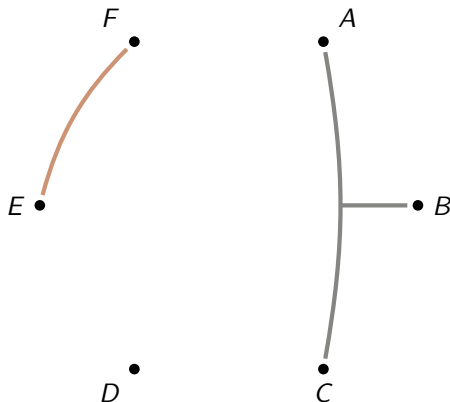
Hypergraphs



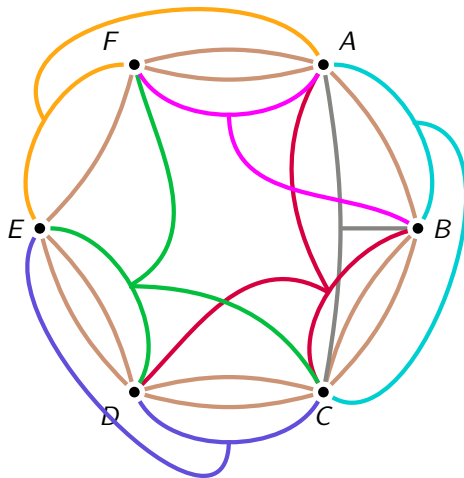
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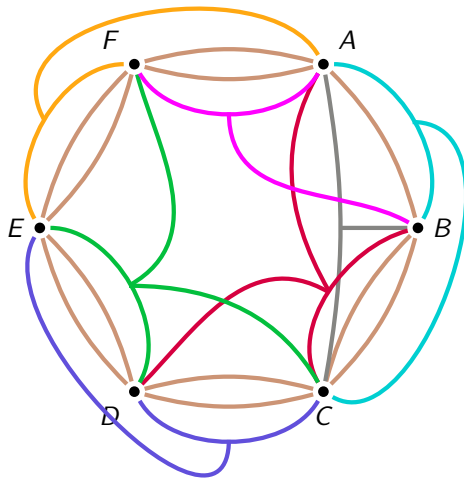


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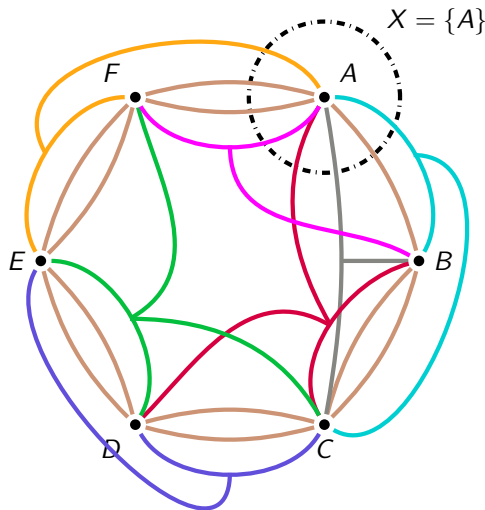
Degree of $\emptyset \neq X \subsetneq V$

$d_{\mathcal{H}}(X)$ is the number of hyperedges intersecting both X and $V \setminus X$.



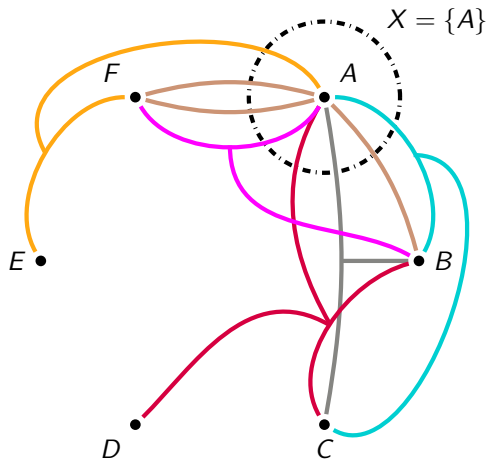
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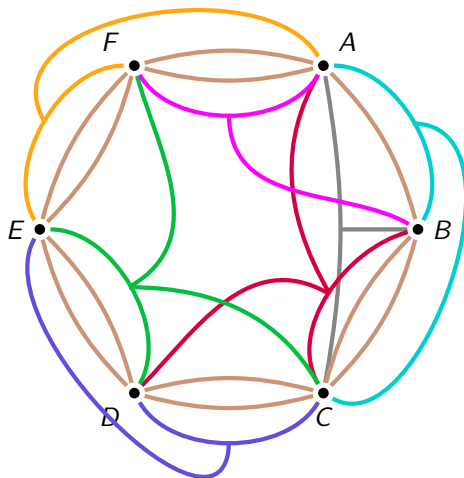


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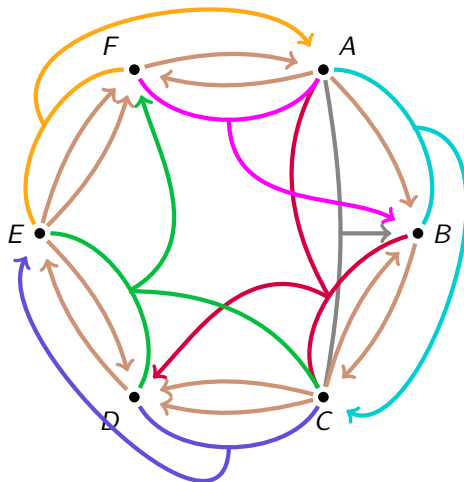
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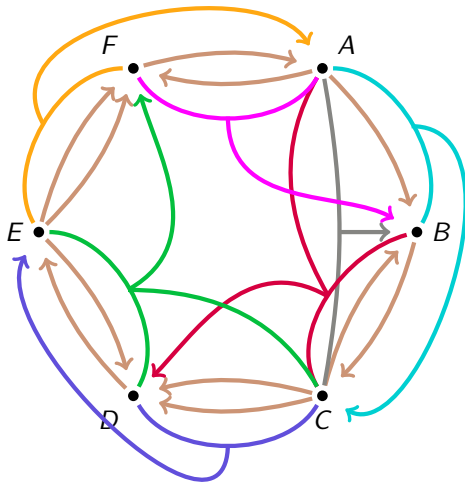


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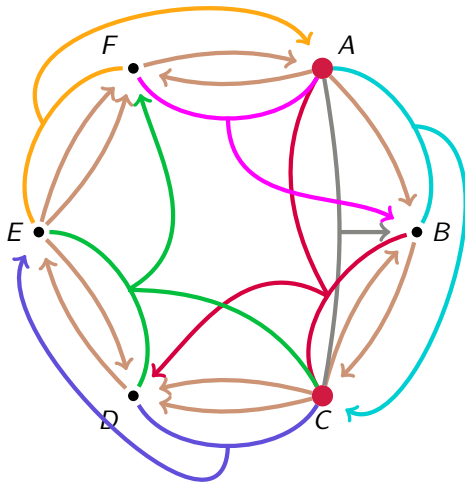
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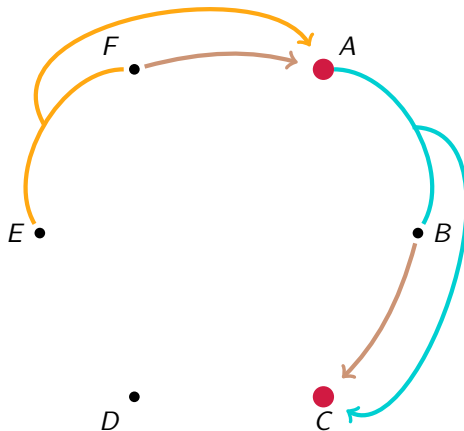
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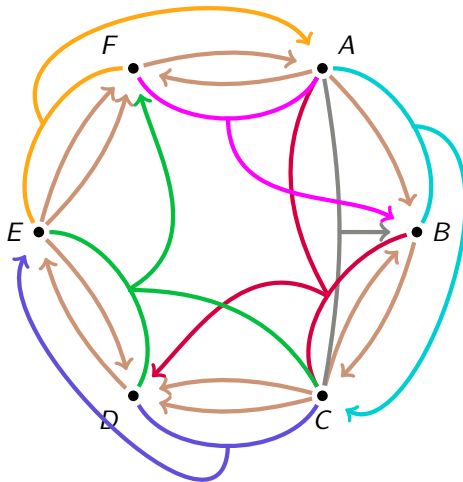
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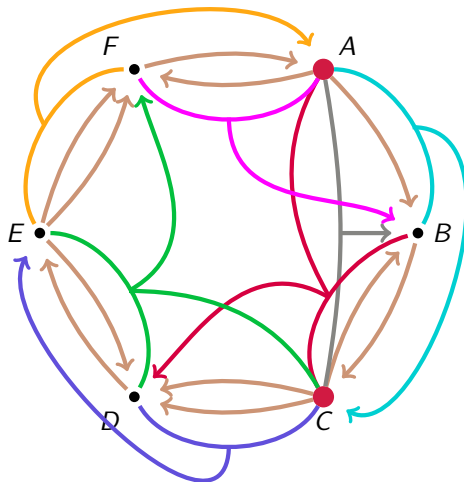
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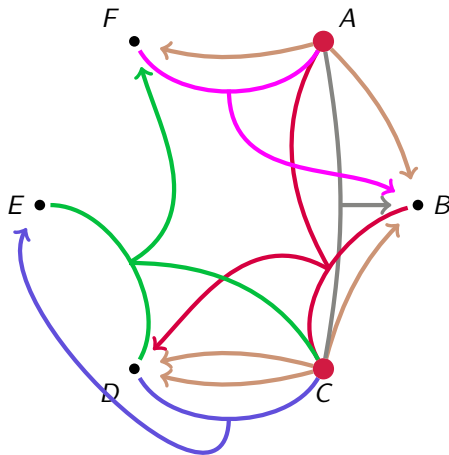
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Hyperarc-connectivity

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- The hyperarc-connectivity of a hypergraph, denoted $\lambda(\vec{\mathcal{H}})$, is the maximum value of k such that $\vec{\mathcal{H}}$ is k -hyperarc-connected.

Main result

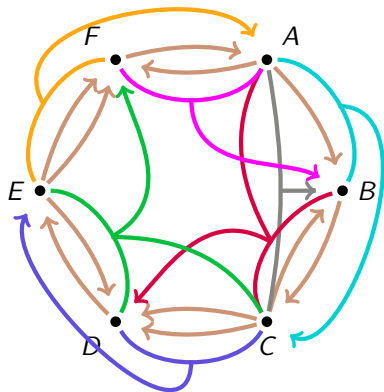
We use a result of Frank : \mathcal{H} is (k, k) -partition-connected if and only if it admits a k -hyperarc-connected orientation.

Main result (Theorem 7)

Let $\mathcal{H} = (V, E)$ be a $(k + 1, k + 1)$ -partition-connected hypergraph and $\vec{\mathcal{H}}$ is a k -hyperarc connected orientation of \mathcal{H} . Then there exists a sequence of hypergraphs $(\vec{\mathcal{H}}_i)_{i \in 0 \dots \ell}$ such that $\vec{\mathcal{H}}_{i+1}$ is obtained from $\vec{\mathcal{H}}_i$ by reorienting exactly one hyperarc and $\lambda(\vec{\mathcal{H}}_{i+1}) \geq \lambda(\vec{\mathcal{H}}_i)$ and $\lambda(\vec{\mathcal{H}}_\ell) = k + 1$. Such a sequence of orientations can be obtained with $\ell \leq |V|^3$ and found in polynomial time (in the size of \mathcal{H}).

"High-Level"-running of the algorithm

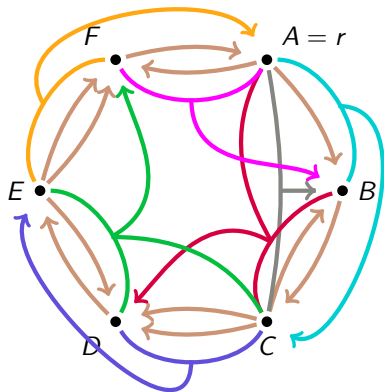
Our algorithm will provide a 3-hyperarc-connected orientation of \mathcal{H} , starting from a 2-hyperarc-connected.



- 1 Take r in $V(\mathcal{H})$.
- 2 Compute sets of vertices.
- 3 Stopping Criteria
- 4 Select a set R (cf. 2.)
- 5 Find an admissible (s, t) -hyperpath in R to reorient
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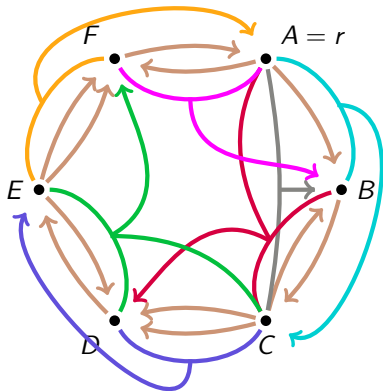
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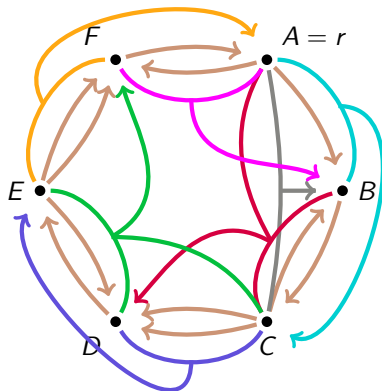
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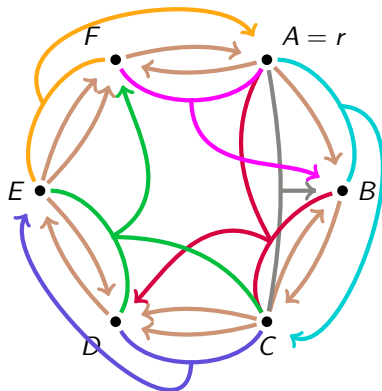
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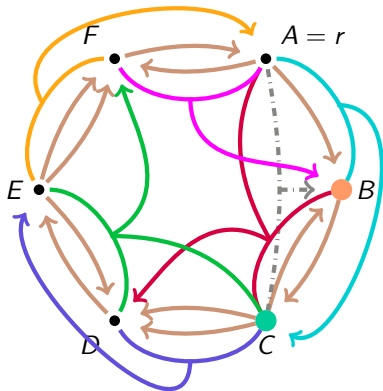
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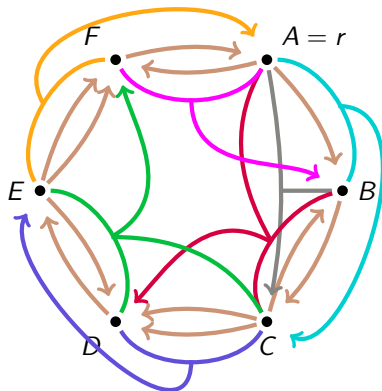
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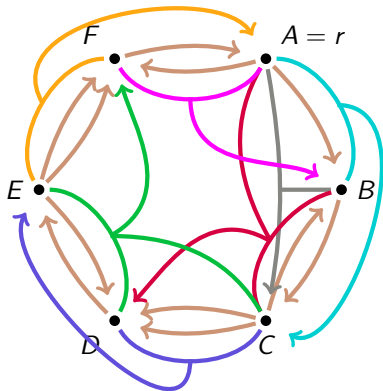
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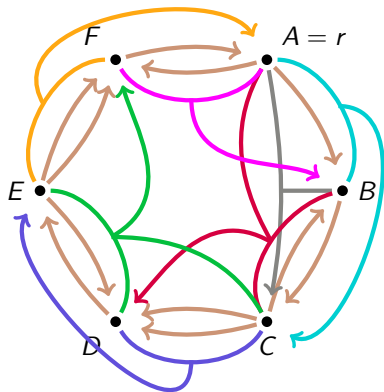
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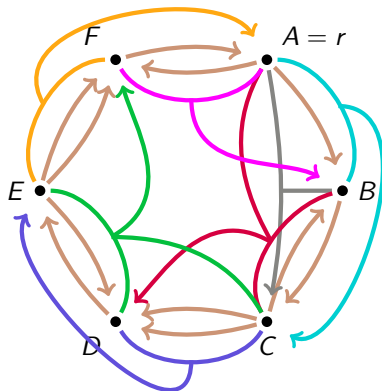
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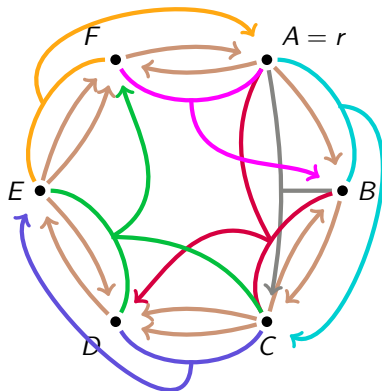
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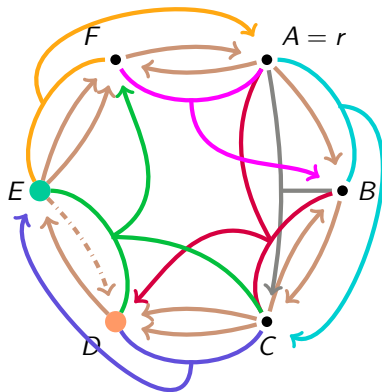
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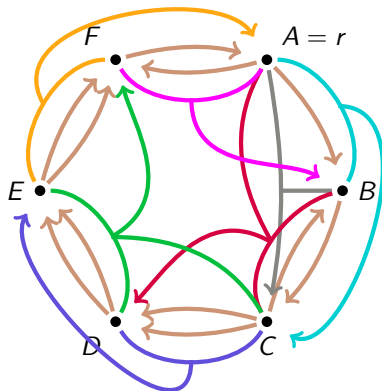
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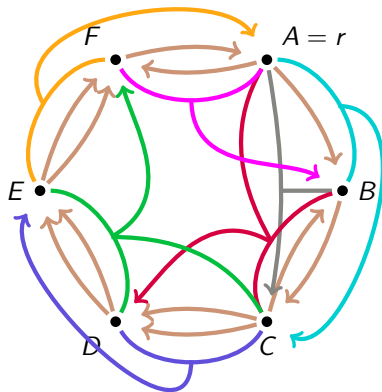
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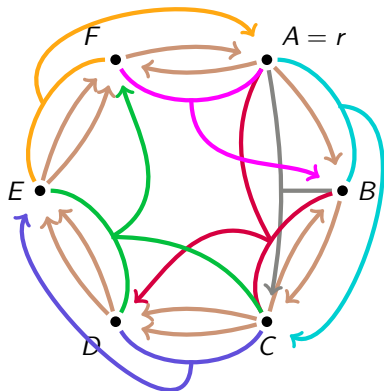
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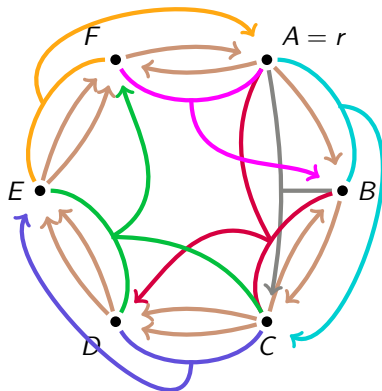
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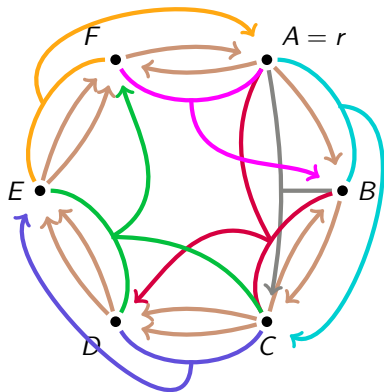
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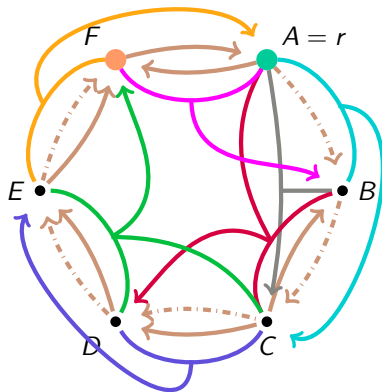
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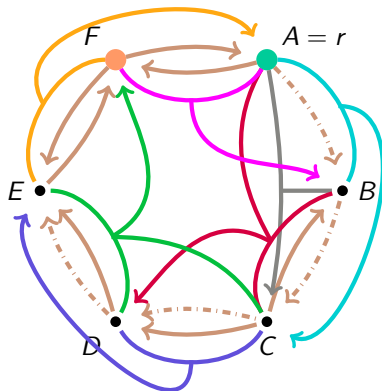
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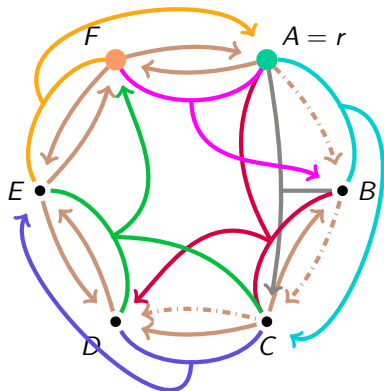
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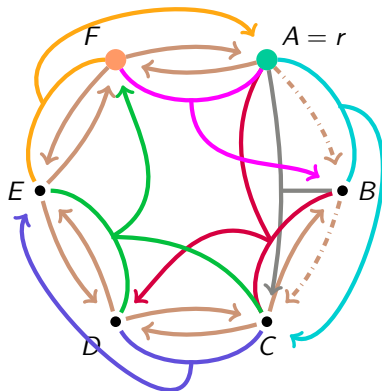
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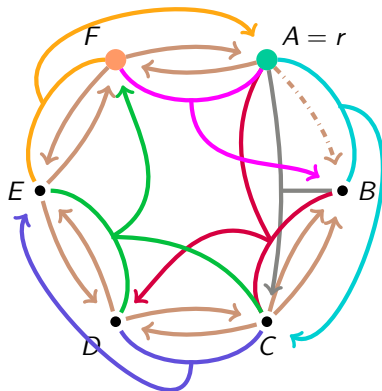
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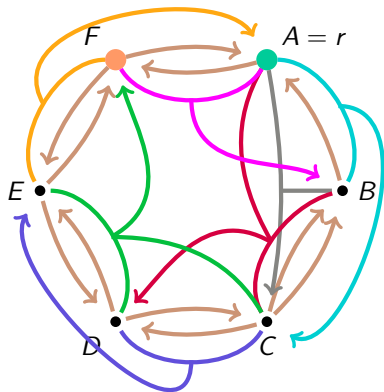
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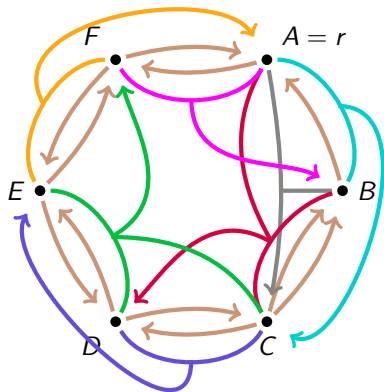
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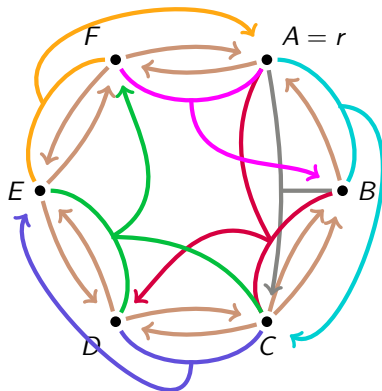
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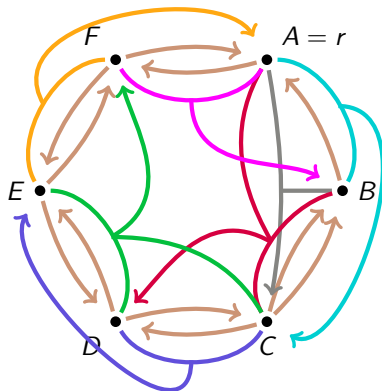
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Finding *admissible* (s, t) -hyperpaths in R

Admissible hyperpaths

- Three criterion for P to be an admissible (s, t) -hyperpath in R :

A brief detour...

Finding *admissible* (s, t) -hyperpaths in R

Admissible hyperpaths

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 1. *Stopping Criteria*-related argument

A brief detour...

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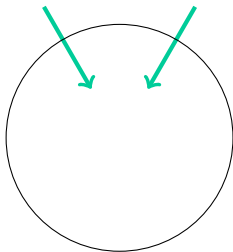
Finding *admissible* (s, t) -hyperpaths in R

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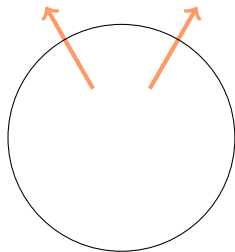
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A brief detour...

Tight and Minimal-tight sets



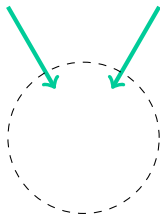
In-Tight sets



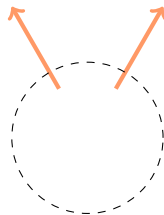
Out-Tight sets

- $\mathcal{T}_- = \{X \subseteq V - r, d^-(X) = k\} \cup \{V\}$
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Tight and Minimal-tight sets



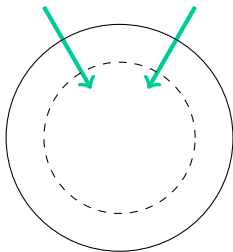
Minimal In-Tight sets



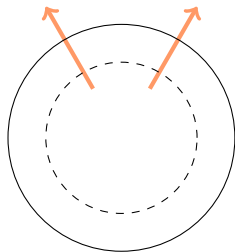
Minimal Out-Tight sets

- \mathcal{M}_- : Inclusion-wise minimal members of \mathcal{T}_-
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Tight and Minimal-tight sets



In-Tight sets



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Crossing sets and structural results

Claim 1(b)

Let X, Y two crossing sets in V .

If $X, Y \in \mathcal{T}_+$, then both $X \cup Y \in \mathcal{T}_+$ and $X \cap Y \in \mathcal{T}_+$.

Proof of Claim 1(b)

- Since X, Y are crossing, $X \cap Y \neq \emptyset$, $X \cup Y \neq V$.

Crossing sets and structural results

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- By submodularity, $d^+(X) + d^+(Y) \geq d^+(X \cup Y) + d^+(X \cap Y)$
- As $\lambda(\vec{\mathcal{H}}) = k$, we have $d^+(X \cup Y) \geq k$ and $d^+(X \cap Y) \geq k$

Crossing sets and structural results

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- Grouping these equations, we obtain :

$$k + k = d^+(X) + d^+(Y) \geq d^+(X \cup Y) + d^+(X \cap Y) \geq k + k.$$

Crossing sets and structural results

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Let X, Y two crossing sets in V .

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- Grouping these equations, we obtain :

$$k + k = d^+(X) + d^+(Y) \geq d^+(X \cup Y) + d^+(X \cap Y) \geq k + k.$$
- This implies $d^+(X \cup Y) = k = d^+(X \cap Y)$, i.e. $X \cap Y, X \cup Y \in \mathcal{T}_+$

Finding *admissible* (s, t) -hyperpaths in R

Admissible hyperpaths

Three criterion for P to be an admissible (s, t) -hyperpath in R :

1. Stopping criteria for the main algorithm :
2. s is a safe source in $S \subseteq R$, t is a safe sink in $T \subseteq R$.
3. Reorienting each hyperarc, **one by one**, does not decrease the hyperarc-connectivity

- Stopping criteria : $\mathcal{M}_- = \{V\}$ and $\mathcal{M}_+ = \{V\}$.
- $\mathcal{T}_- = \{X \subseteq V - r, d^-(X) = k\} \cup \{V\}$
- \mathcal{M}_- : Inclusion-wise minimal members of \mathcal{T}_-
- Finally, if $\lambda(\vec{\mathcal{H}}) \geq k$ and $\mathcal{T}_- = \mathcal{T}_+ = \{V\}$, $\vec{\mathcal{H}}$ is $(k + 1)$ -hyperarc-connected.

Existence of a safe source (*a safe sink*)

Lemma 10

$\forall S \in \mathcal{M}_-,$ there is a safe source $s \in S$.

Lemma 11

$\forall T \in \mathcal{M}_+,$ there is a safe sink $t \in T$.

Towards hyperarc connectivity augmentation

$\mathcal{R} : R \subseteq V - r$ inclusion-wise minimal such that either :

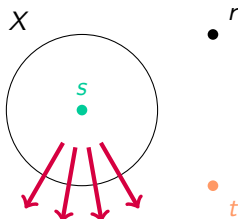
- $R \in \mathcal{T}_-$, and contains a member of \mathcal{T}_+
- or $R \in \mathcal{T}_+$, and contains a member of \mathcal{T}_- .

Lemma 13

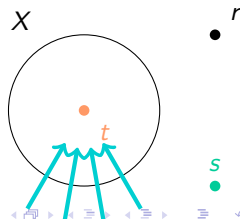
Let $R \in \mathcal{R}, S \in \mathcal{M}_-, T \in \mathcal{M}_+$ such that $S, T \subseteq R$. Let s be a safe source in S , t a safe sink in T .

- $\forall X \subseteq V - r$ such that $s \in X, t \notin X$, we have $d^+(X) \geq k + 1$.
- $\forall X \subseteq V - r$ such that $s \notin X, t \in X$, we have $d^-(X) \geq k + 1$.

a.



b.



Towards hyperarc connectivity augmentation

Lemma 13

Let $R \in \mathcal{R}, S \in \mathcal{M}_-, T \in \mathcal{M}_+$ such that $S, T \subseteq R$. Let s be a safe source in S , t a safe sink in T .

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Proof of Lemma 13

By contradiction, either :

- a. $\exists X \subseteq V - r, s \in X, t \notin X, d^+(X) = k$, i.e. $s \in X, t \notin X, X \in \mathcal{T}_+$.
 - a1. $R \in \mathcal{R} \cap \mathcal{T}_-$
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- b. $\exists X \subseteq V - r, s \notin X, t \in X, d^-(X) = k$, i.e. $s \notin X, t \in X, X \in \mathcal{T}_-$.
 - b1. $R \in \mathcal{R} \cap \mathcal{T}_-$
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Towards hyperarc connectivity augmentation

Lemma 13

Let $R \in \mathcal{R}$, $S \in \mathcal{M}_-$, $T \in \mathcal{M}_+$ such that $S, T \subseteq R$. Let s be a safe source in S , t a safe sink in T .

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Proof of Lemma 13

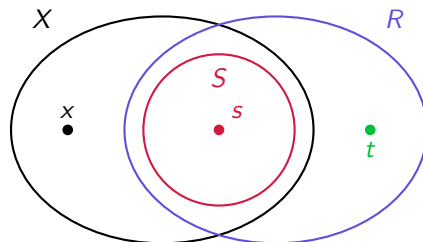
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 - b1. $R \in \mathcal{R} \cap \mathcal{T}_-$
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Proof of Lemma 13

$a : \exists X \subseteq V - r, s \in X, t \notin X, X \in \mathcal{T}_+$

- Since $s \in S$ is a **safe source** and $s \in X \in \mathcal{T}_+$, we have $S \subsetneq X$
- We also have $t \in R \setminus X$ by [a.], so $X \setminus R \neq \emptyset$.

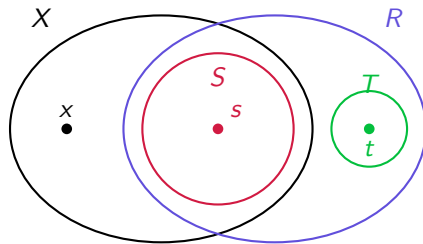


Proper representation of a

Proof of Lemma 13

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Proper representation of a

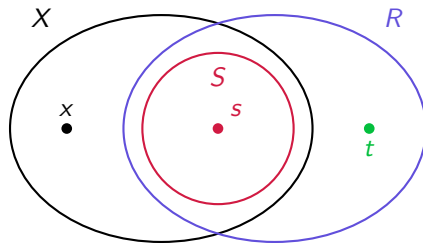
$a1. : R \in \mathcal{R} \cap \mathcal{T}_-, \exists X \subseteq V - r, s \in X, t \notin X, X \in \mathcal{T}_+$

- As $t \in R \setminus X \neq \emptyset$, and using Claim 1, we have $R \setminus X \in \mathcal{T}_-$.
- $T \cap X \neq \emptyset$ would contradict the minimality of T , so T and X are disjoint.
- As $R \setminus X \in \mathcal{T}_-$, $T \in \mathcal{T}_+$, and $T \subseteq R \setminus X$, this contradicts R minimal.

Proof of Lemma 13

$a : \exists X \subseteq V - r, s \in X, t \notin X, X \in \mathcal{T}_+$

- Since $s \in S$ is a **safe source** and $s \in X \in \mathcal{T}_+$, we have $S \subsetneq X$
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Proper representation of a

$a2. : R \in \mathcal{R} \cap \mathcal{T}_+, \exists X \subseteq V - r, s \in X, t \notin X, X \in \mathcal{T}_+.$

- $R \in \mathcal{T}_+, X \in \mathcal{T}_+$, and $X \cap R \neq \emptyset \implies X \cap R \in \mathcal{T}_+$
- $S \in \mathcal{T}_-, S \subseteq R \cap X$. Since $t \in R \setminus X, X \cap R \subsetneq R$.
- This contradicts the minimality of R .

Proof of Lemma 13

Lemma 13

Let $R \in \mathcal{R}, S \in \mathcal{M}_-, T \in \mathcal{M}_+$ such that $S, T \subseteq R$. Let s be a safe source in S , t a safe sink in T .

- a. $\forall X \subseteq V - r$ such that $s \in X, t \notin X$, we have $d^+(X) \geq k + 1$.
- b. $\forall X \subseteq V - r$ such that $s \notin X, t \in X$, we have $d^-(X) \geq k + 1$.

Proof of Lemma 13

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Proof of Lemma 13

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Finding *admissible* (s, t) -hyperpaths in $R \in \mathcal{R}$

Admissible hyperpaths

Three criterion for P to be an admissible (s, t) -hyperpath in R :

1. *Stopping criteria*-related argument.
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3. Reorienting each hyperarc, **one by one**, does not decrease the hyperarc-connectivity
 - ▶ How to proceed ?

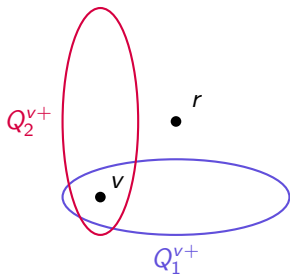
Introduction of Q_+^v

Definition of Q_+^v

Consider the sets of \mathcal{T}_+ containing v . Q_+^v is **the** minimal (inclusion-wise) one.

Unicity of Q_+^v :

Q_+^v is unique.



Let Q_1^{v+} , Q_2^{v+} verifying the above definition.

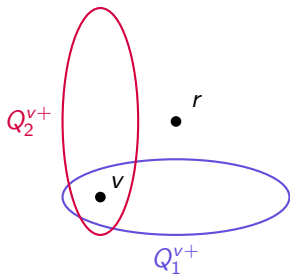
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By definition, $Q_1^{v+} \not\subseteq Q_2^{v+}$ and $Q_2^{v+} \not\subseteq Q_1^{v+}$.

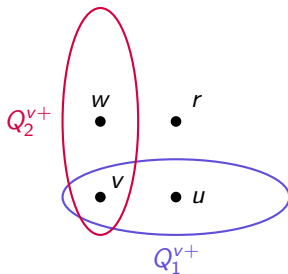
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Denote $u \in Q_1^{v+} \setminus Q_2^{v+}$, $w \in Q_2^{v+} \setminus Q_1^{v+}$.

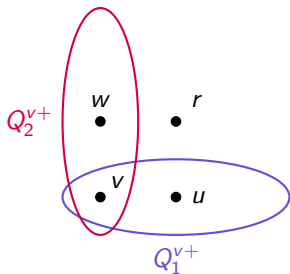
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As $r \notin Q_1^{v+}, Q_2^{v+}$, both are crossing sets.

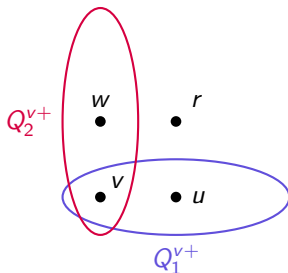
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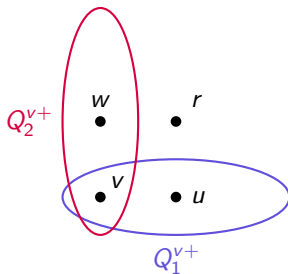
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$Q_1^{v+} \cap Q_2^{v+}$ is smaller (inclusion-wise) than Q_1^{v+} and Q_2^{v+} .

Existence of an hyperpath that does not leave Q_+^v

Lemma 12(a)

$\forall s \in V, \forall t \in Q_+^s$, there exists an (s, t) -hyperpath that does not leave Q_+^s .

Proof of Lemma 12 (a)

- By contradiction, assume that there is $s \in V, t \in Q_+^s$ such that any (s, t) -hyperpath leaves Q_+^s .

Existence of an hyperpath that does not leave Q_+^v

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Proof of Lemma 12 (a)

- By contradiction, assume that there is $s \in V, t \in Q_+^s$ such that any (s, t) -hyperpath leaves Q_+^s .
- There is $s \in Z \subseteq Q_+^s \setminus \{t\}$ such that any hyperarc leaving Z will also leave Q_+^s .

Existence of an hyperpath that does not leave Q_+^v

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- We have the following inequalities

Existence of an hyperpath that does not leave Q_+^v

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 - ▶ $d_{\vec{\mathcal{H}}}^+(Q_+^s) \geq d_{\vec{\mathcal{H}}}^+(Z)$

Existence of an hyperpath that does not leave Q_+^v

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Existence of an hyperpath that does not leave Q_+^v

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 - ▶ $k = d_{\vec{\mathcal{H}}}^+(Q_+^s)$ by definition.

Existence of an hyperpath that does not leave Q_+^v

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 - ▶ $d_{\vec{\mathcal{H}}}^+(Z) \geq k$, as $\vec{\mathcal{H}}$ is k -hyperarc-connected.
 - ▶ $k = d_{\vec{\mathcal{H}}}^+(Q_+^s)$ by definition.
- We can deduce that $d_{\vec{\mathcal{H}}}^+(Z) = k$, which automatically implies that $Z \in \mathcal{T}_+$.

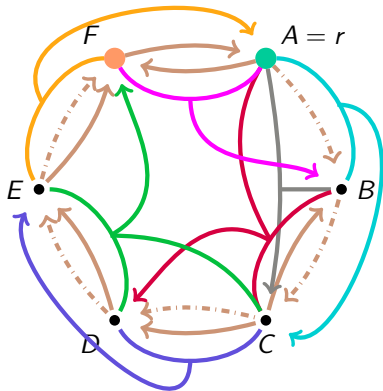
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- There is $s \in Z \subseteq Q_+^s \setminus \{t\}$ such that any hyperarc leaving Z will also leave Q_+^s .
- We have the following inequalities
 - ▶ $d_{\vec{\mathcal{H}}}^+(Q_+^s) \geq d_{\vec{\mathcal{H}}}^+(Z)$
 - ▶ $d_{\vec{\mathcal{H}}}^+(Z) \geq k$, as $\vec{\mathcal{H}}$ is k -hyperarc-connected.
 - ▶ $k = d_{\vec{\mathcal{H}}}^+(Q_+^s)$ by definition.
- We can deduce that $d_{\vec{\mathcal{H}}}^+(Z) = k$, which automatically implies that $Z \in \mathcal{T}_+$.
- Q_+^s is not minimal, hence the contradiction.

Finding an admissible (s, t) -hyperpath in $R \in \mathcal{R} \cap \mathcal{T}_-$ 

- 1 Take r in $V(\mathcal{H})$.
- 2 Compute sets of vertices.
- 3 Stopping Criteria
- 4 Select a set R (cf. 2.)
- 5 Find an admissible (s, t) -hyperpath in R to reorient
- 6 Reorient the corresponding hyperpath.
- 7 Goto (2.)

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Algorithm Admissible (s, t) -hyperpath in $R \in \mathcal{R} \cap \mathcal{T}_-$

- 1: Take a set $S \in \mathcal{M}_-$, with $S \subseteq R$, then a safe source $s \in S$.
 - 2: $Z = \{s\}$, $F = (Z, \emptyset)$, $V' = R$
 - 3: **while** $h = (X, v)$ exists such that $v \in V' - Z$ and $X \cap Z \neq \emptyset$ **do**
 - 4: Let $u \in X \cap Z$.
 - 5: $Z \leftarrow Z \cup \{v\}$
 - 6: $F \leftarrow F + uv$
 - 7: **if** $Q_+^v \subsetneq V'$ **then**
 - 8: $V' \leftarrow Q_+^v$
 - 9: **end if**
 - 10: **end while**
 - 11: $T = V'$
 - 12: Take a safe sink $t \in T$
 - 13: $P' = F[s, t]$
 - 14: P is the corresponding hyperpath in $\vec{\mathcal{H}}$, obtained with P' .
 - 15: **Return** S, T, s, t, P
-

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- First efficient algorithm for computing a k -hyperarc-connected orientation of a hypergraph
- Extensions of Main Theorem : Starting without conditions on $\lambda(\vec{\mathcal{H}})$.

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Thank you for your attention.

