Directed hypergraph connectivity augmentation by hyperarc re-orientations

Combinatorial Optimization and Graph Theory

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Thursday, Nov 23rd 2023

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- *Ito et al.*, 2023 :
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  - Exhibiting a sequence of orientations such that :
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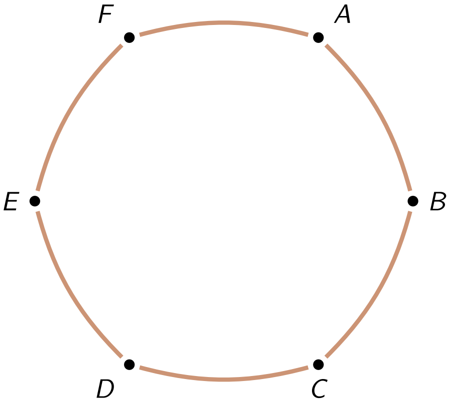
Side note : This article generalise the results of **Ito et al.**, as directed graphs are special case of hypergraphs.

Hypergraphs



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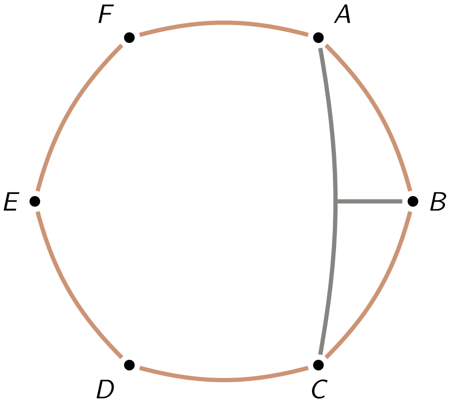

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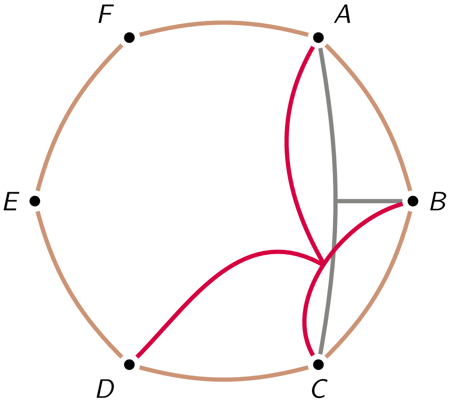

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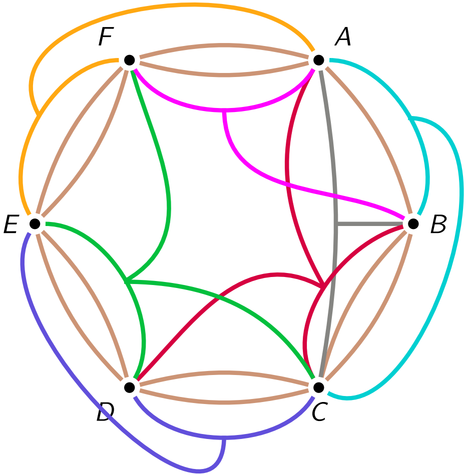

Degree of $\varnothing \neq X \subsetneq V$

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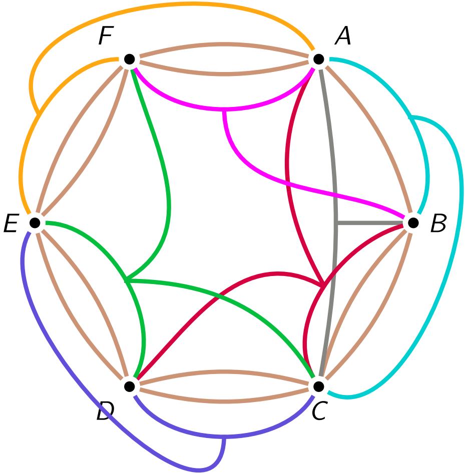

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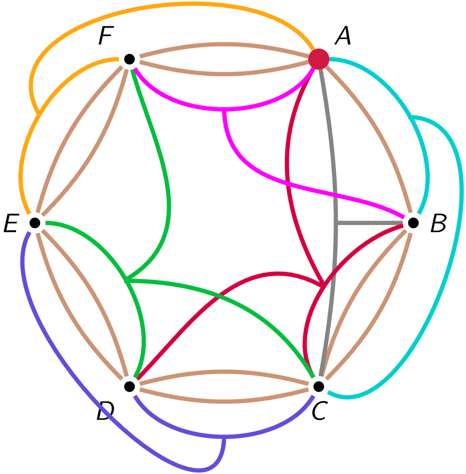

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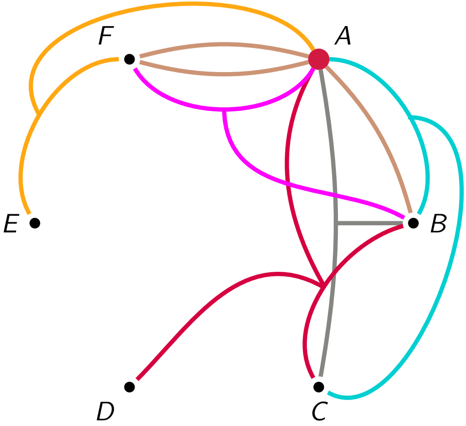

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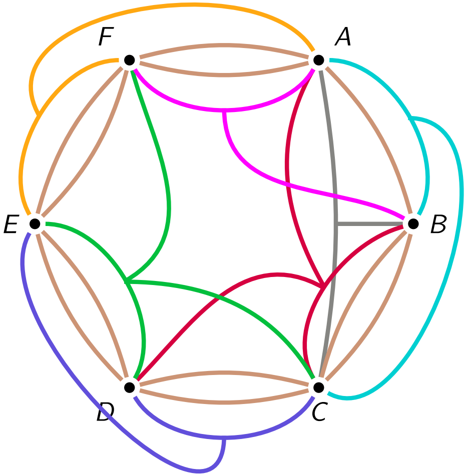

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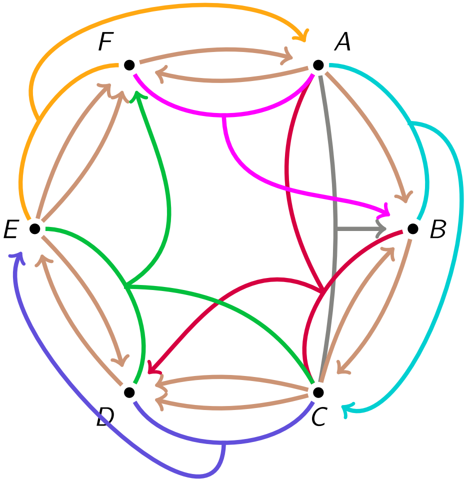

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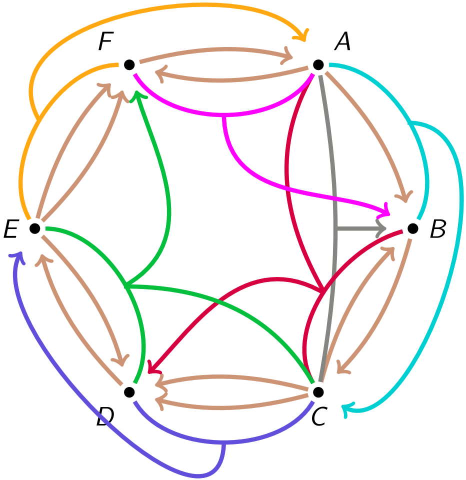

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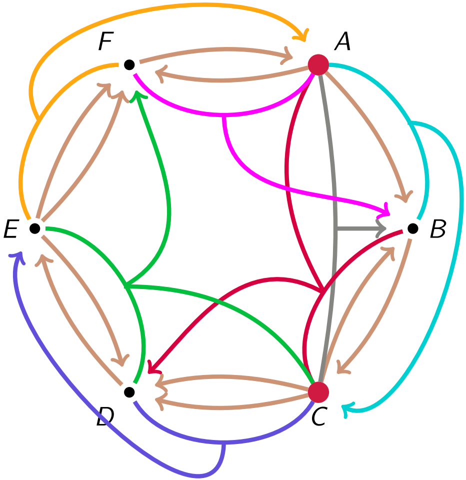

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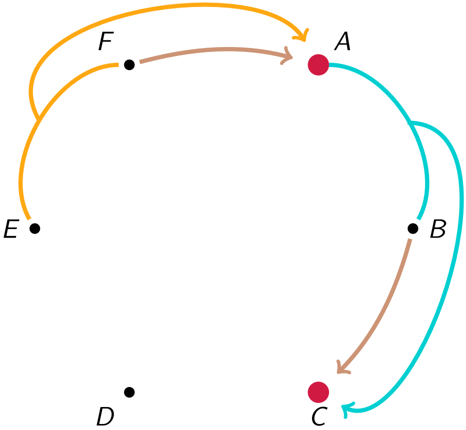

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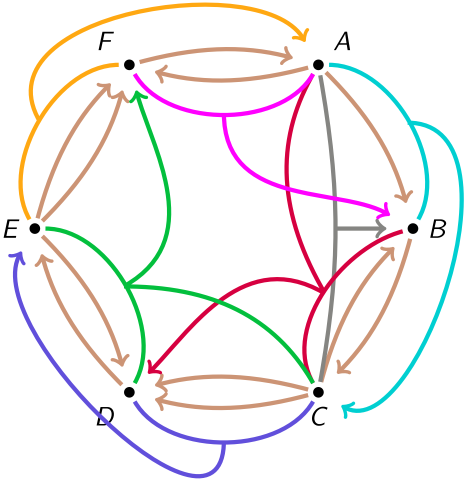

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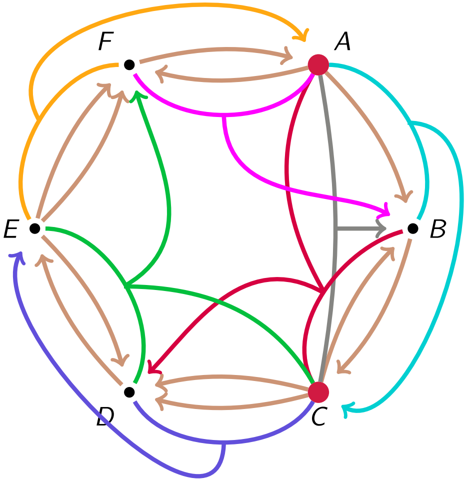

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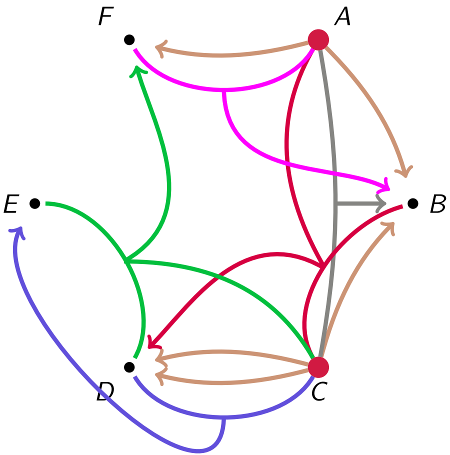

Hyperarc-connectivity and (k, k)-partition connected hypergraphs

- $\vec{\mathcal{H}}$ is *k-hyperarc-connected*, if, $\forall e \in \mathcal{E}$, $d^+_{\vec{\mathcal{H}}}(e) \geq k$.
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We use a result of Frank : $\mathcal{H}$ is (k, k)-partition-connected if and only if it admits a k-hyperarc-connected orientation.

Main result

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Let $\mathcal{H} = (V, E)$ be a $(k+1, k+1)$-partition-connected hypergraph and $\vec{\mathcal{H}}$ is a k-hyperarc orientation of $\mathcal{H}$. Then there exists a sequence of hyperarcs $(\vec{\mathcal{H}}_i)_{i \in 0 \ldots \ell}$ such that $\vec{\mathcal{H}}_{i+1}$ is obtained from $\vec{\mathcal{H}}_i$ by reorienting exactly one hyperarc and $\lambda(\vec{\mathcal{H}}_{i+1}) \geq \lambda(\vec{\mathcal{H}}_i)$ and $\lambda(\vec{\mathcal{H}}_\ell) = k+1$. Such a sequence of orientations can be obtained with $\ell \leq |V|^3$ and found in polynomial time (in the size of $\mathcal{H}$).

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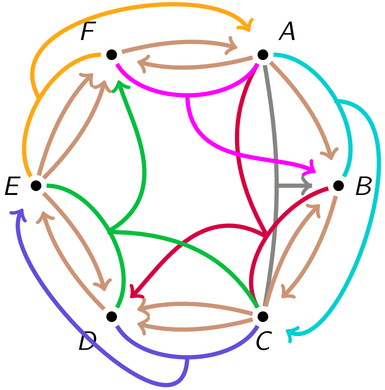
Generalization of **Ito et al.**, as digraphs are special cases of hypergraphs.

"High-Level"-running of the algorithm

Our algorithm will provide a 3-hyperarc-connected orientation of $\mathcal{H}$, starting from a 2-hyperarc-connected.



1. Take r in $V(\mathcal{H})$.
2. Compute sets of vertices.
3. Stopping Criteria
4. Select a set R (cf. 2.)
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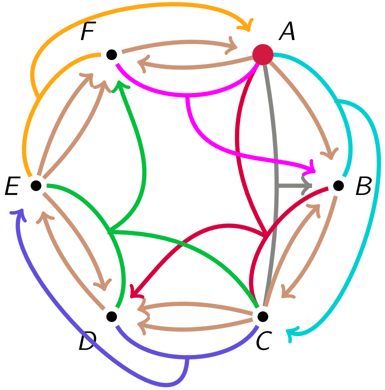

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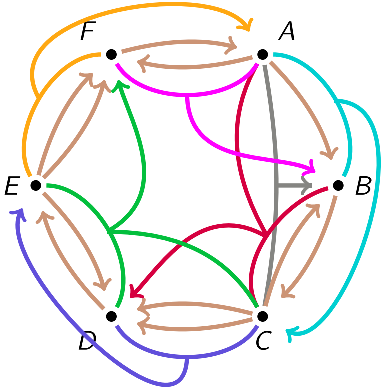

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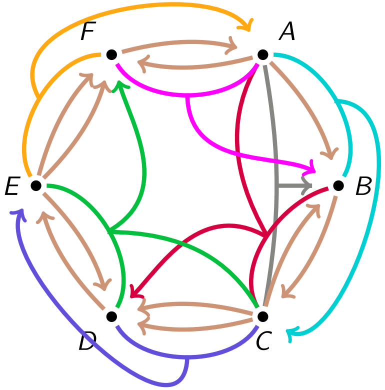

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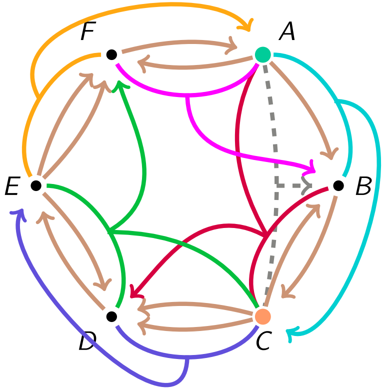

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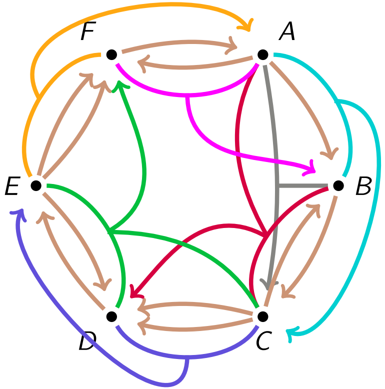

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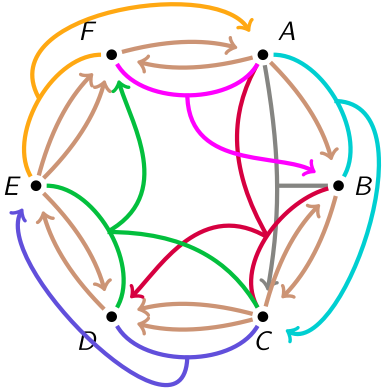

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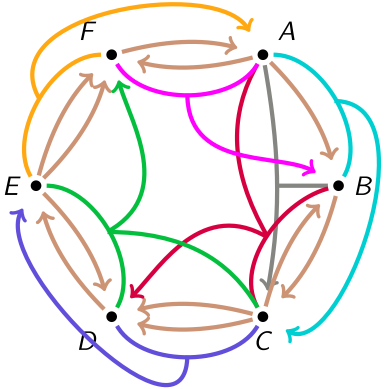

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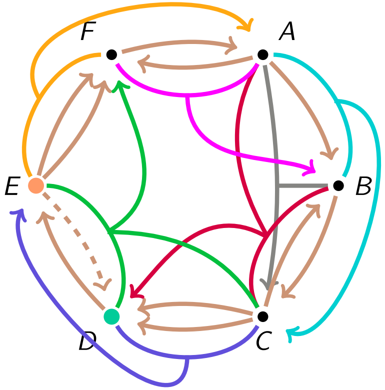

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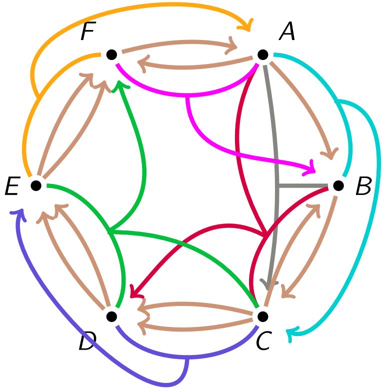

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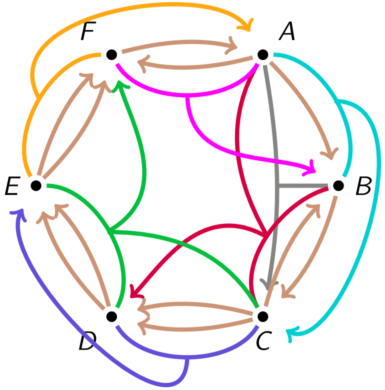

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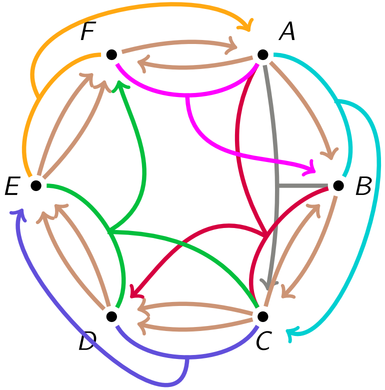

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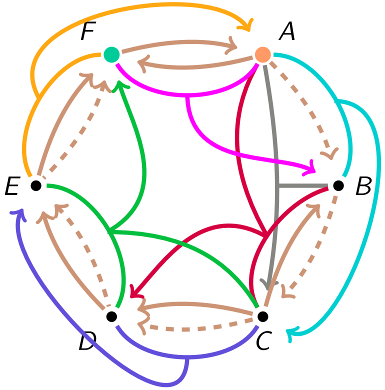

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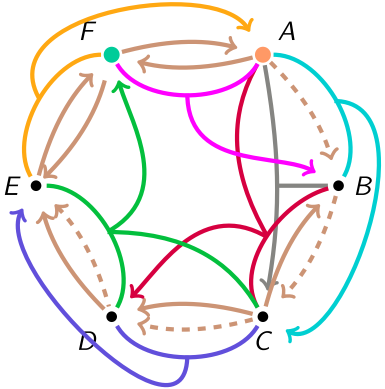

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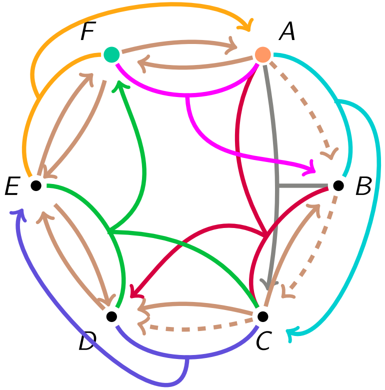

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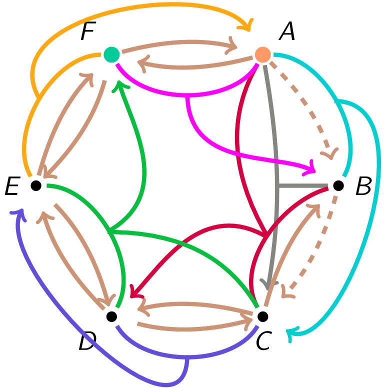

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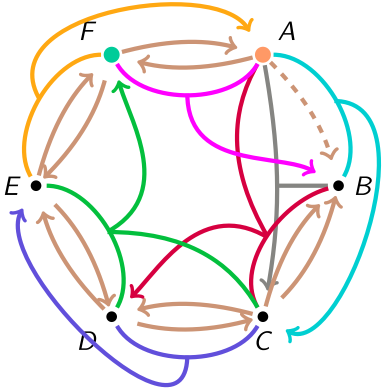

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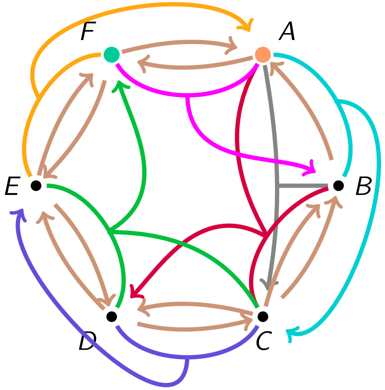

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3. Stopping Criteria
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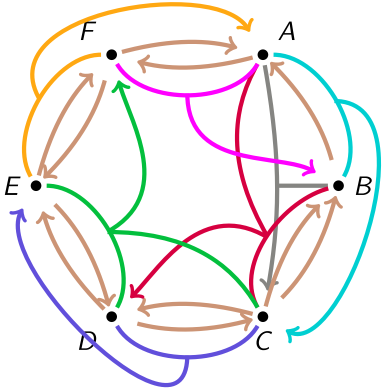

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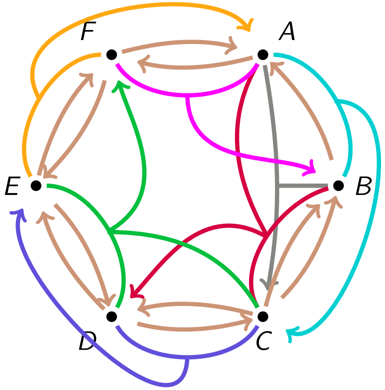

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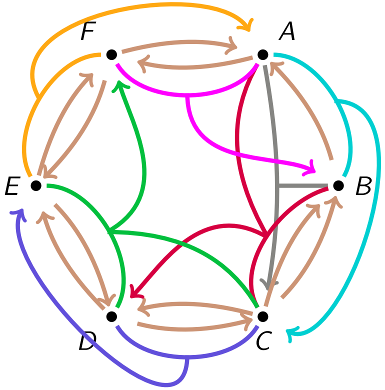

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Finding *admissible* (s,t)-hyperpaths in R

- Crucial segment of the algorithm.
- Three criterion for P to be an admissible (s,t)-hyperpath in R:
  1. s is a safe source in $S \subseteq R$, t is a safe sink in $T \subseteq R$.
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What are safe sources and safe sinks ?

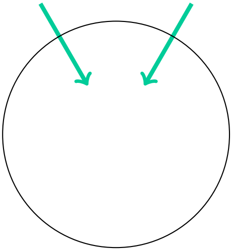
A brief detour...

Tight and Dangerous sets

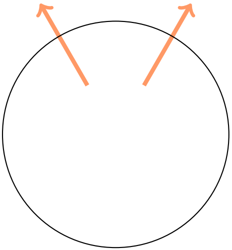
Remainder of the algorithm :

- `Input` : A k-hyperarc-connected orientation of a $(k+1, k+1)$-partition-connected hypergraph.
- `Output` : A $k+1$-hyperarc-connected hypergraph.

Tight and Dangerous sets



In-Tight sets



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- $\mathcal{T}_- = \{X \subseteq V - r, d^-(X) = k\} \cup \{V\}$
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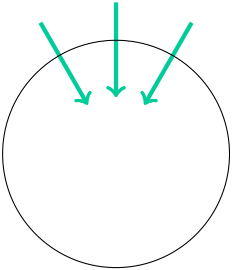

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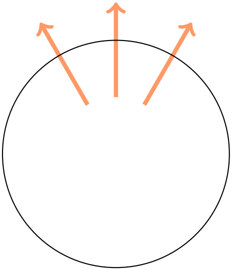

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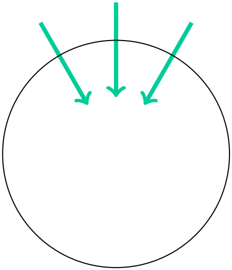

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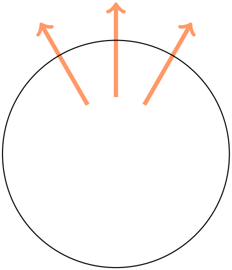

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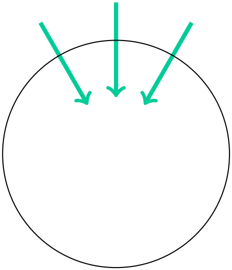

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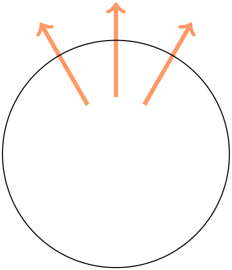

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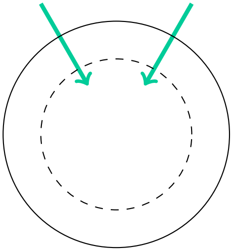

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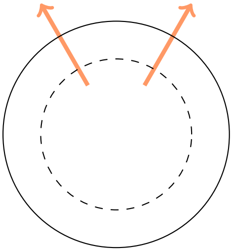

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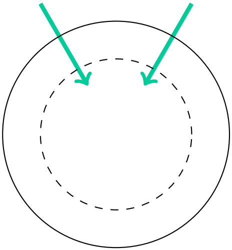
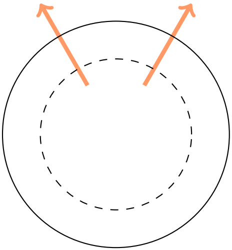


Minimal In-Tight sets



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