Directed hypergraph connectivity augmentation by hyperarc re-orientations

Combinatorial Optimization and Graph Theory

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Thursday, Nov 23rd 2023

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- *Ito et al.*, 2023 :
  - Algorithmic proof of *Nash-Williams*, by flipping one edge at a time.
  - Exhibiting a sequence of orientations such that :
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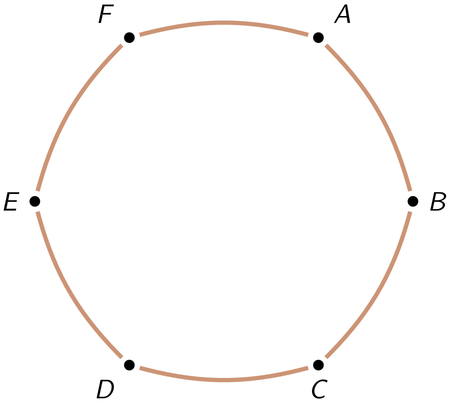
Side note : This article generalise the results of **Ito et al.**, as directed graphs are special case of hypergraphs.

Hypergraphs



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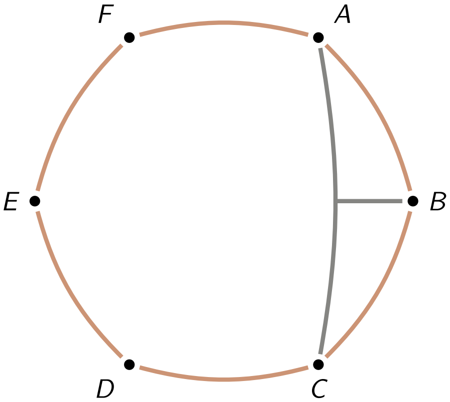

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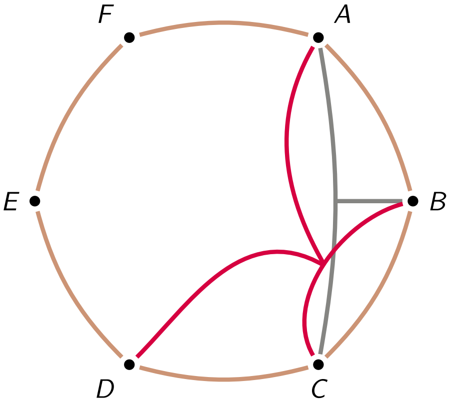

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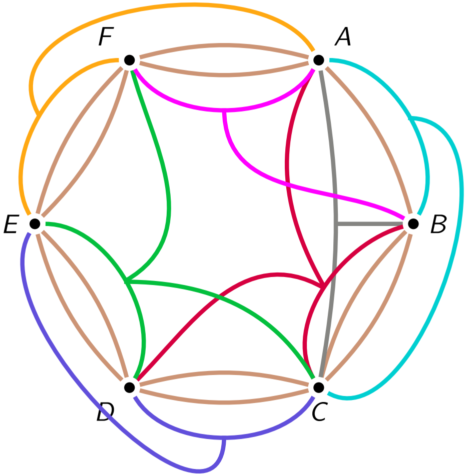
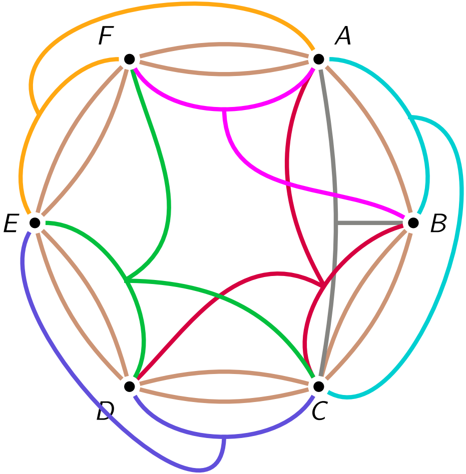


Degree of $\varnothing \neq X \subsetneq V$

$d_{\mathcal{H}}(X)$ is the number of hyperedges intersecting both X and $V \setminus X$.



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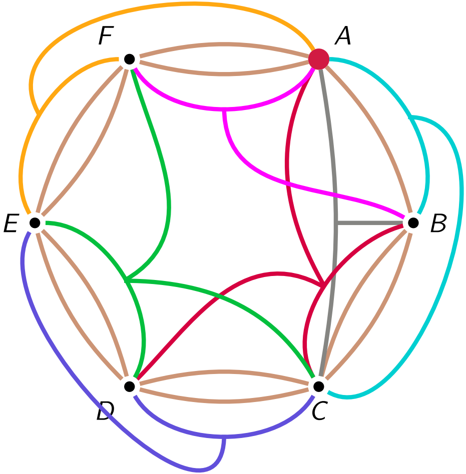

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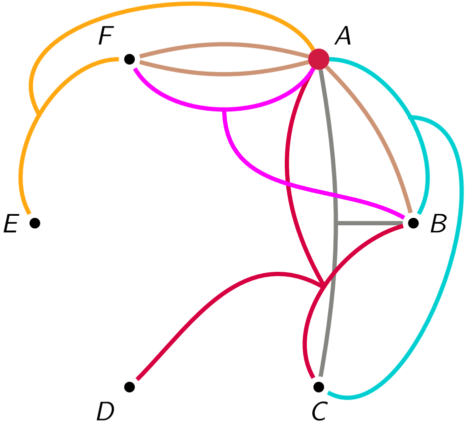

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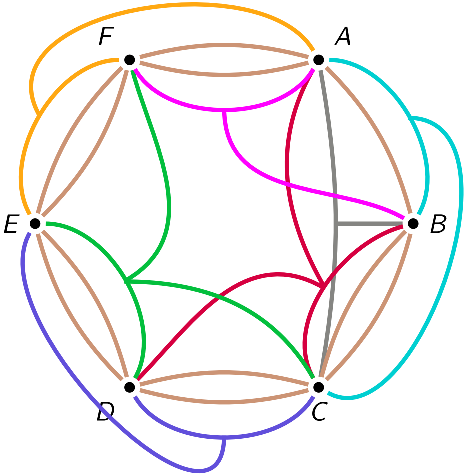

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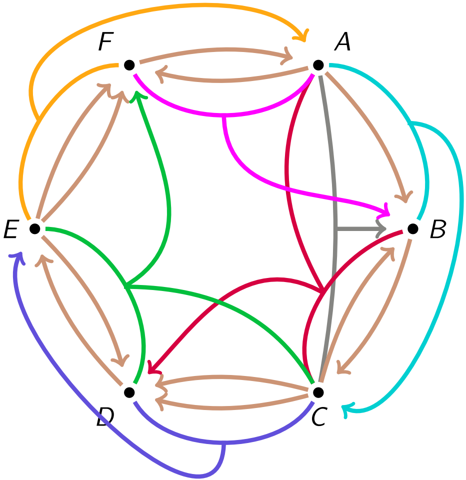

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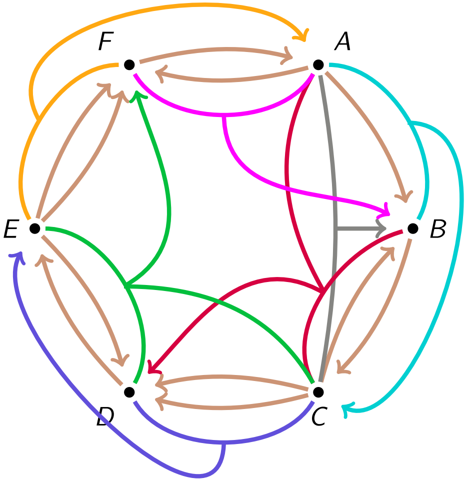

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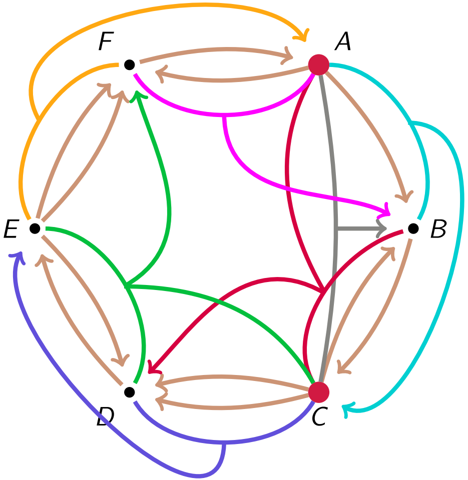

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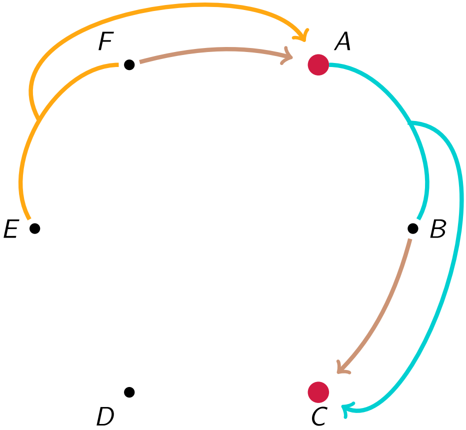

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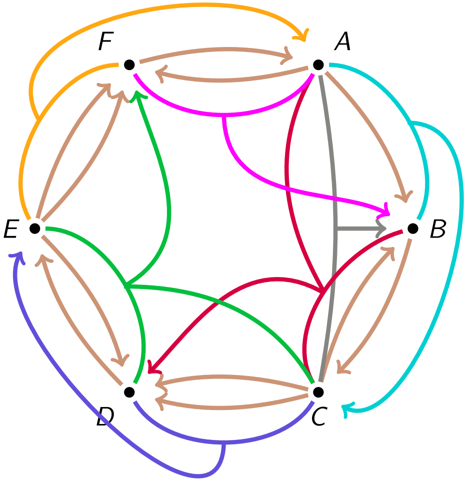

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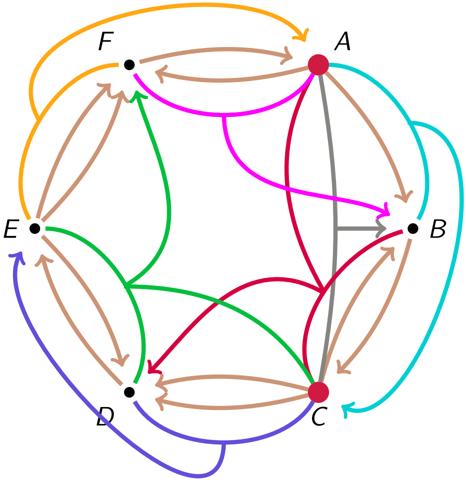

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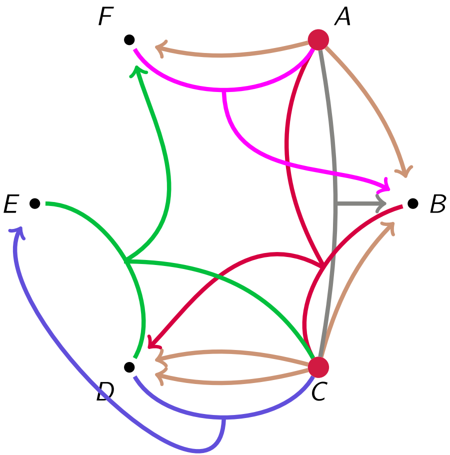

Hyperarc-connectivity and (k, k)-partition connected hypergraphs

- $\vec{\mathcal{H}}$ is *k-hyperarc-connected*, if, $\forall e \in \mathcal{E}$, $d^+_{\vec{\mathcal{H}}}(e) \geq k$.
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We use a result of Frank : $\mathcal{H}$ is (k, k)-partition-connected if and only if it admits a k-hyperarc-connected orientation.

Main result

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Let $\mathcal{H} = (V, E)$ be a $(k+1, k+1)$-partition-connected hypergraph and $\vec{\mathcal{H}}$ is a k-hyperarc orientation of $\mathcal{H}$. Then there exists a sequence of hyperarcs $(\vec{\mathcal{H}}_i)_{i \in 0 \dots \ell}$ such that $\vec{\mathcal{H}}_{i+1}$ is obtained from $\vec{\mathcal{H}}_i$ by reorienting exactly one hyperarc and $\lambda(\vec{\mathcal{H}}_{i+1}) \geq \lambda(\vec{\mathcal{H}}_i)$ and $\lambda(\vec{\mathcal{H}}_\ell) = k + 1$. Such a sequence of orientations can be obtained with $\ell \leq |V|^3$ and found in polynomial time (in the size of $\mathcal{H}$).

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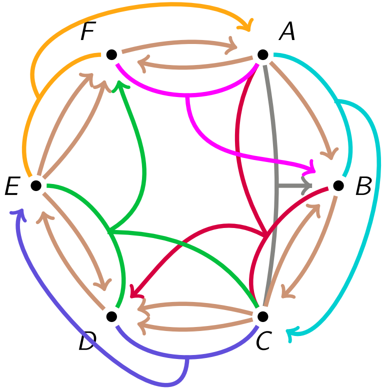
Generalization of **Ito et al.**, as digraphs are special cases of hypergraphs.

"High-Level"-running of the algorithm

Our algorithm will provide a 3-hyperarc-connected orientation of $\mathcal{H}$, starting from a 2-hyperarc-connected.



1. Take r in $V(\mathcal{H})$.
2. Compute sets of vertices.
3. Stopping Criteria
4. Select a set R (cf. 2.)
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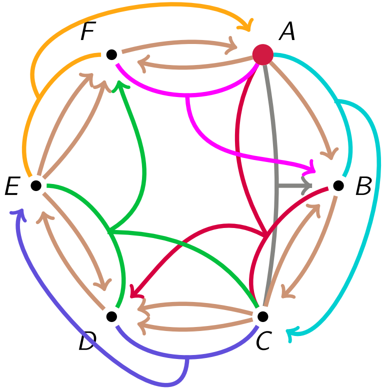

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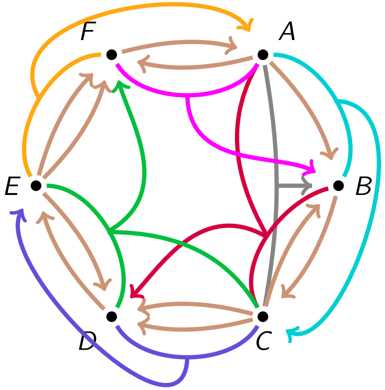

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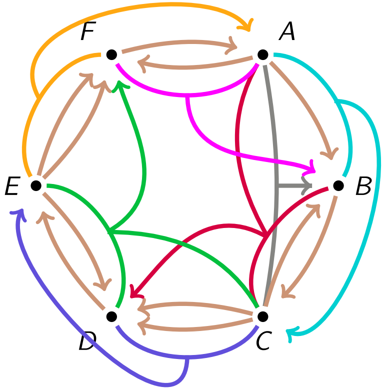

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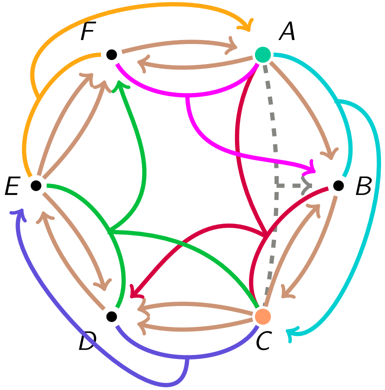

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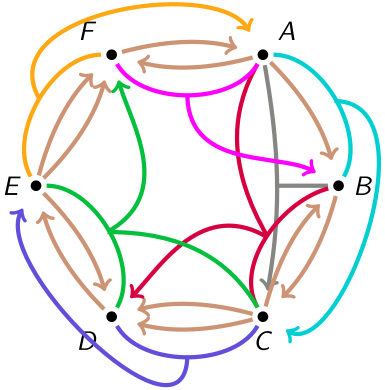

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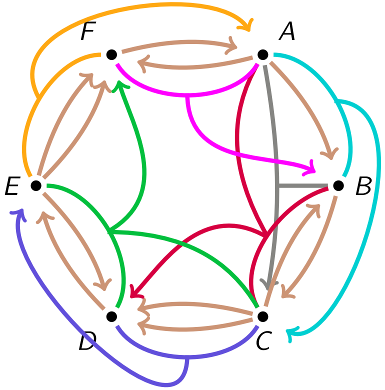

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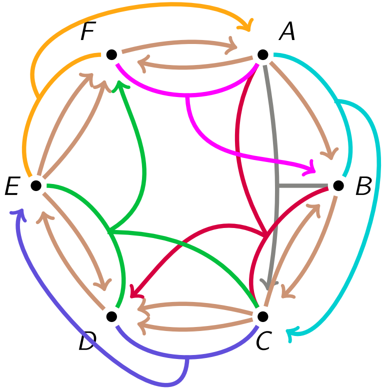

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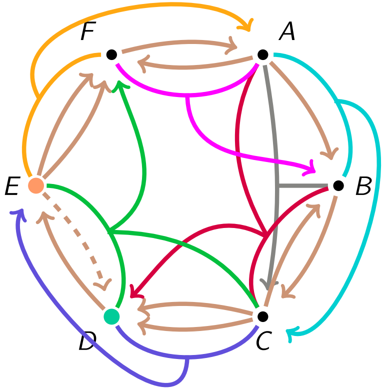

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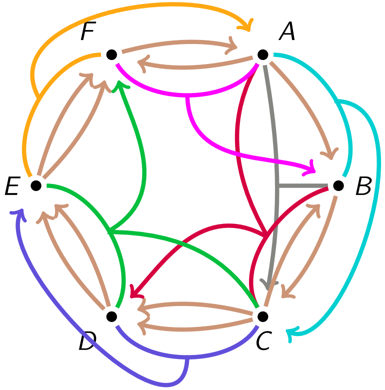

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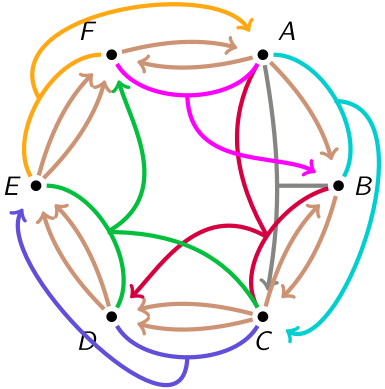

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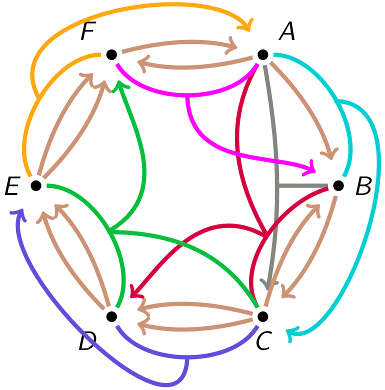

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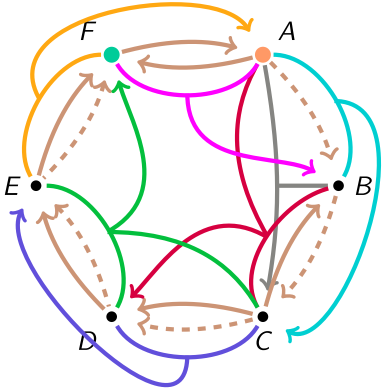

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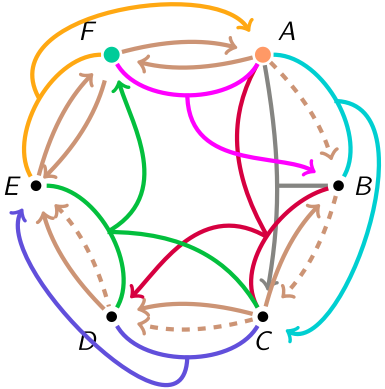

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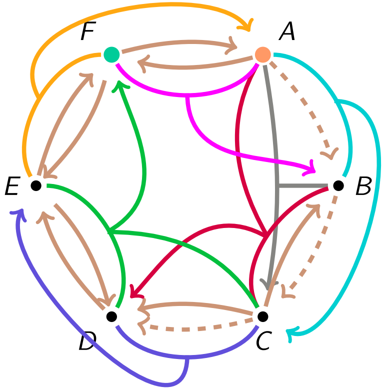

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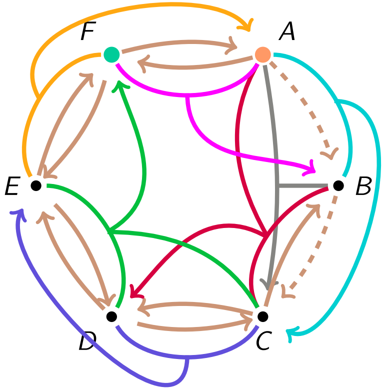

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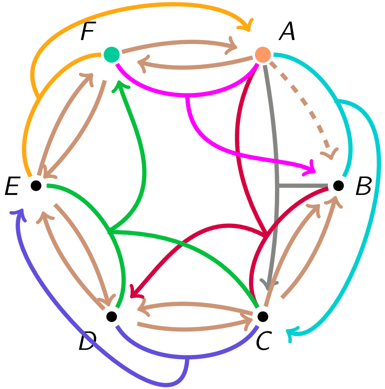

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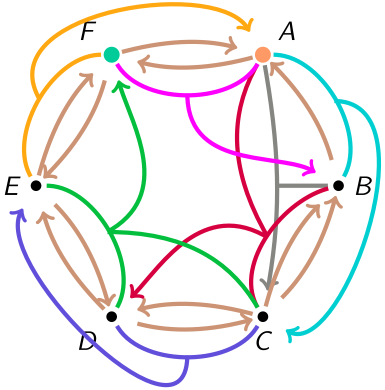

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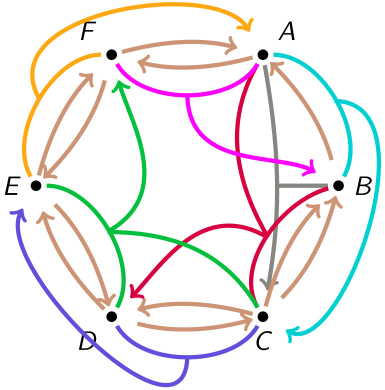

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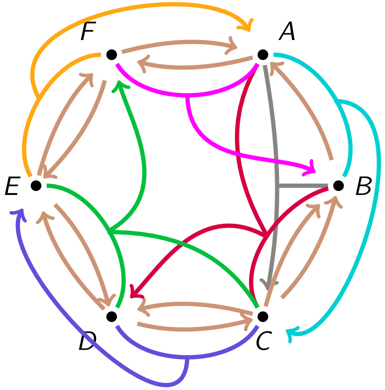

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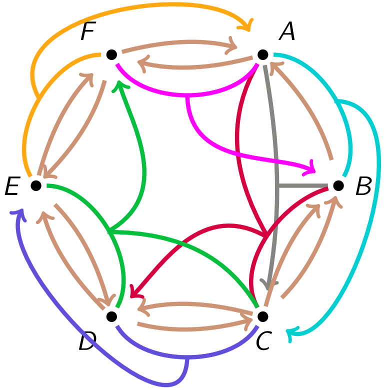

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Finding *admissible* (s, t)-hyperpaths in R

- Crucial segment of the algorithm.
- Three criterion for P to be an admissible (s, t)-hyperpath in R:
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What are safe sources and safe sinks ?

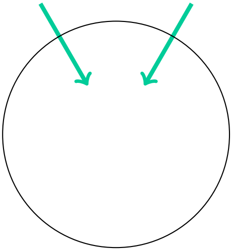
A brief detour...

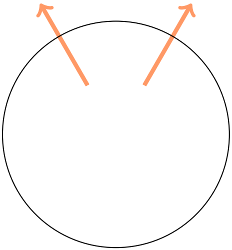
Tight and Dangerous sets

Remainder of the algorithm :

- `Input` : A k-hyperarc-connected orientation of a $(k+1, k+1)$-partition-connected hypergraph.
- `Output` : A $k+1$-hyperarc-connected hypergraph.

Tight and Dangerous sets


In-Tight sets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- $\mathcal{T}_- = \{X \subseteq V - r, d^-(X) = k\} \cup \{V\}$
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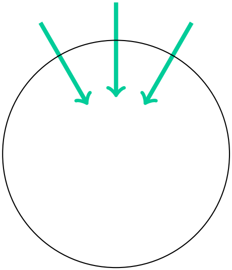

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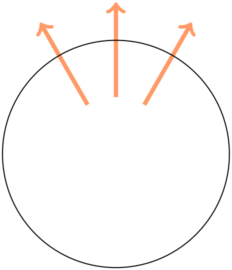

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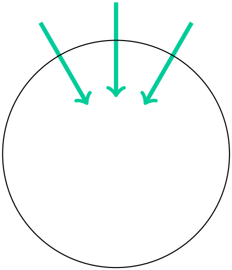

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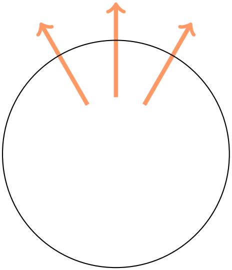

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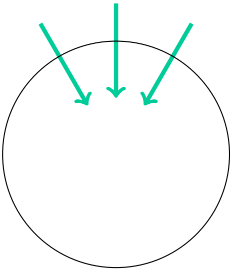

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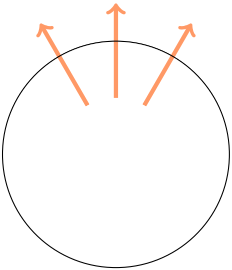

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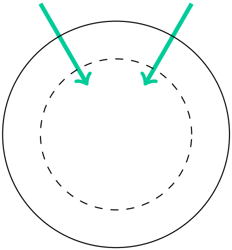

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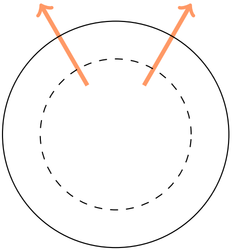

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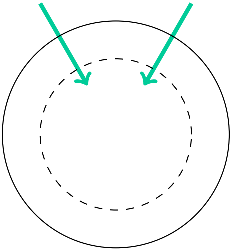
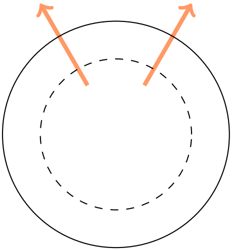


Minimal In-Tight sets



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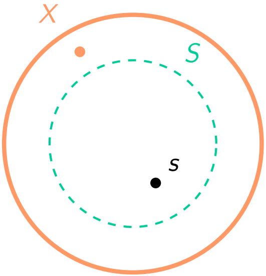
Safe Sources and Safe Sinks

Definitions are symmetric (but proofs are not).

- For $S \in \mathcal{M}_-$, s is a safe source in S if :
  - a For every $s \in X \in \mathcal{T}_+$, we have $S \subsetneq X$.
  - b For every $s \in X \in \mathcal{D}_+$ such that $S \setminus X \neq \varnothing$, there exists $Y \in \mathcal{T}_+$ such that $s \notin Y \subsetneq X$.



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Finding a safe sink t in $T \in \mathcal{M}_+$ can be done by checking each vertex if they correspond to the definition.

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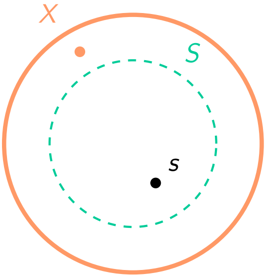

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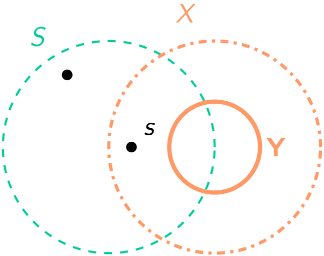

Condition (b)

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Safe Sources and Safe Sinks

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- For $T \in \mathcal{M}_+$, t is a safe sink in T if :
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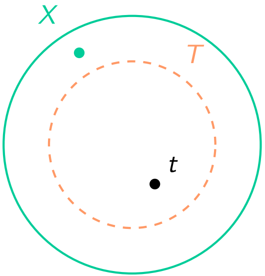

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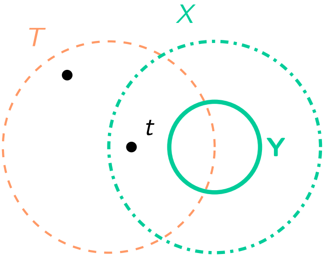

Condition (d)

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Existence of a safe source (*a safe sink*)

Lemma 10

$\forall S \in \mathcal{M}_-,$ there is a safe source $s \in S$.

Likewise,

Lemma 11

$\forall T \in \mathcal{M}_+,$ there is a safe sink $t \in T$.

Quick sketch of a proof for Lemma 10 :

- Let $S \in \mathcal{M}_-$.
- Considering a family of vertex sets (χ) that cover as many vertices of S as possible, but using as little as vertex sets possible.
- We can prove that, under given assumptions, χ cannot cover every vertex of S.
- Vertices that are not covered by χ are "potential" safe sources, the last part of the proof is verifying that they are effectively safe sources.