Cosmology: Problem Set 2
FRW Universe

This problem set focuses on the FRW Universe; Section 2 of the course. The relevant materials are in Weinberg's book in Chapter 1. Solutions to this Problem Set are due on Sunday 12.5.2019. The total number of points is 100 + 40 additional optional points.

1. The FRW metric and the Friedmann equations (40 points)

The FRW metric with $K = \{-1, 0, 1\}$ can be written as

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right] \\ &\equiv -dt^2 + a^2(t)\tilde{g}_{ij}dx^i dx^j. \end{aligned} \tag{1}$$

We use Latin indices $i, j, \ldots$ to denote purely spatial components and Greek indices $\mu, \nu, \ldots$ to denote spacetime indices. For example, in spherical coordinates, $x^\mu = (t, r, \theta, \phi)$ and $x^i = (r, \theta, \phi)$. The second line of equation (1) defines the purely spatial metric $\tilde{g}_{ij}$.

(a) Compute the Christoffel symbols, the Riemann tensor, the Ricci tensor and the Ricci scalar of the corresponding spacetime. In comparing your results to Weinberg, don't forget that his book uses the opposite '−' sign convention for the Riemann tensor compared to the one used in our lectures.

(b) Using your results from (a), along with the assumption that the energy-momentum tensor is described by an ideal fluid with energy density $\varepsilon(t)$ and pressure $P(t)$, such that its components satisfy

$$T_{00} = \varepsilon(t), \qquad T_{i0} = 0, \qquad T_{ij} = a^2(t)P(t)\tilde{g}_{ij}, \tag{2}$$

use Einstein's equations (either the standard version involving the Einstein tensor or the trace-reversed version) to derive the two Friedmann equations

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G_N}{3}\varepsilon, \tag{3}$$

$$\dot{\varepsilon} = -\frac{3\dot{a}}{a}(\varepsilon + P). \tag{4}$$

(c) Show that the energy-momentum conservation equation

$$\nabla_\mu T^{\mu\nu} = 0 \tag{5}$$

is compatible with equation (4). For this reason, equations (3) and (4) form the full set of dynamical equations of motion that describe the evolution of the FRW universe — this means that they can be used to solve for the two unknown functions $a(t)$ and $\varepsilon(t)$ provided that the equation of state $P(\varepsilon)$ is known.

(d) Setting $K = 0$ and using the equation of state $P(\varepsilon) = w\varepsilon$, solve the Friedmann equations to find $\varepsilon(t)$ and $a(t)$. Compute also the proper distance to the particle and event horizons in this universe.

2. A universe without matter (20 points + 10 optional points)
Consider the case of a universe with zero energy density $\varepsilon = 0$ and zero pressure $P = 0$.

(a) Solve the Friedmann equations (3) and (4) to find $a(t)$ and K.

(b) Imagine that a source of light is emitted by a comoving source in this universe at initial time $t = t_1$ and that this source of light is received somewhere at time $t = t_2$. Compute the redshift z between these two events.

(c) Now imagine that this light source was emitted at the origin of the FRW coordinate system (1): at $r = 0$. After $t = t_1$, light traveled "outwards" (towards increasing r) along a null geodesic which satisfies $ds^2 = 0$, with $d\theta = d\phi = 0$. Compute the coordinate distance $r(t_1, t_2)$ travelled by this light source as a function of t_1 and t_2. Compute also the proper distance $d(t_1, t_2)$.

(d) [Optional]: Given that this calculations took place in an empty universe with no matter, how can you reconcile these (non-trivial) results with the fact that one would expect an empty universe to contain no gravity?

3. Two-component universe I (20 points + 20 optional points)
Consider a universe with $\Omega_R = \Omega_K = 0$, and $\Omega_M \neq 0$ and $\Omega_\Lambda \neq 0$.

(a) Find the value of the redshift $z(t^*)$ as a function of the ratio Ω_Λ/Ω_M at time t^* when the universe transitioned from an initially decelerating matter-dominated universe to an accelerating universe dominated by the cosmological constant. This means that you have to find $z(t^*)$ when $a(t)$ transitioned from $\ddot{a} < 0$ to $\ddot{a} > 0$ at $t = t^*$ — in other words, when $\ddot{a}(t^*) = 0$.

(b) [Optional]: If the universe is 13.7 billion years old, how long after the Big Bang did the acceleration begin assuming that $\Omega_M = 0.25$ and $\Omega_\Lambda = 0.75$.

4. Two-component universe II (20 points)

Consider a universe with $\Omega_K = \Omega_\Lambda = 0$, and $\Omega_M \neq 0$ and $\Omega_R \neq 0$.

(a) Using conformal time η defined as

$$d\eta = \frac{dt}{a(t)}, \tag{6}$$

write the two Friedmann equations (3) and (4) in terms of η instead of t. By using the resulting two equations, show that

$$\frac{d^2 a}{d\eta^2} = \frac{4\pi G}{3} \left(\varepsilon - 3P\right) a^3. \tag{7}$$

(b) Let us write the total energy density, which is a combination of contributions from matter and radiation, as

$$\varepsilon(\eta) \equiv \varepsilon_M(\eta) + \varepsilon_R(\eta), \tag{8}$$

where

$$\varepsilon_M(\eta) = \frac{\varepsilon_{eq}}{2} \left(\frac{a_{eq}}{a(\eta)}\right)^3, \qquad \varepsilon_R(\eta) = \frac{\varepsilon_{eq}}{2} \left(\frac{a_{eq}}{a(\eta)}\right)^4, \tag{9}$$

with constant ε_{eq} and a_{eq}. They are defined as follows. At conformal time $\eta = \eta_{eq}$, we define the scale factor $a(\eta)$ to be $a(\eta_{eq}) \equiv a_{eq}$. Hence, at time η_{eq}, both contributions from matter and radiation to the total energy density were equal, and $\varepsilon(\eta_{eq}) \equiv \varepsilon_{eq}$. By solving the second-order differential equation (7) and fixing the two integration constants with appropriate initial conditions, show that $a(\eta)$ of this two-component universe is

$$a(\eta) = a_{eq} \left[c_1 \left(\frac{\eta}{\eta_*}\right)^2 + c_2 \left(\frac{\eta}{\eta_*}\right) \right], \tag{10}$$

where c_1, c_2 and η^* are constants. Find c_1 and c_2 and express η_* in terms of η_{eq}.

(c) What would $a(\eta)$ be for matter- and radiation-dominated (single-component) universes? How can you understand these scalings by taking two different limits of equation (10)?