Cosmology: Problem Set 1
General Relativity

This problem set reviews the main tools from General Relativity for describing curved spacetimes; Section 1 of the course. All relevant materials are discussed in Carroll's book in Chapters 1–4. Solutions to this Problem Set are due on Sunday 28.4.2019.

1. Three dimensional Anti-de Sitter space and geodesics (30 points)

(a) Consider the following line element of a three dimensional Anti-de Sitter space with coordinates $x^\mu = (t, x, y)$:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{L^2}{y^2}\left(-dt^2 + dx^2 + dy^2\right), \tag{1}$$

where L is some constant length scale. The coordinate t is time and x and y are two spatial coordinates with the following ranges: $x \in (-\infty, \infty)$ and $y \in [0, \infty)$. The spatial part of the spacetime is known as the Poincaré half-plane. Compute the Christoffel symbols $\Gamma^\mu{}_{\rho\sigma}$, the Riemann tensor $R^\mu{}_{\nu\rho\sigma}$, the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R for this spacetime.

(b) Now, we use the geodesic equation to analyse the shape of the geodesics in the spacetime described by the metric (1) at constant time $t = const.$. Note that this implies that the purely spatial geodesics extend in the two dimensional subspace ($dt = 0$):

$$ds^2\Big|_{dt=0} = \frac{L^2}{y^2}\left(dx^2 + dy^2\right). \tag{2}$$

In particular, show that a geodesic starting from $(x, y) = (-\ell/2, 0)$ and ending at $(x, y) = (+\ell/2, 0)$ is a semi-circle.

(c) What are the geodesics with constant x? Compute the length of such a geodesic starting at y_1 and ending at y_2. What is its length if we take $y_1 \to 0$? What would be the length of a geodesic with constant x that ran between y_1 and y_2 in flat space with the metric

$$ds^2 = dx^2 + dy^2. \tag{3}$$

2. Einstein's equations (30 points)

(a) Derive the Einstein's equations (equations of motion) from the Einstein-Hilbert action

$$S = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\,(R - 2\Lambda) + \int d^4x\sqrt{-g}\,\mathcal{L}_M, \tag{4}$$

where $\mathcal{L}_M$ is some matter Lagrangian by performing the metric variation $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ and extremising the resulting first-order variation of the action $\delta S = 0$. Make sure that you understand how various terms in S transform under the metric variation. In particular, show explicitly what is the variation of the Ricci tensor $\delta R_{\mu\nu}$, which was the step that we skipped in the lectures.

(b) By taking the trace (contraction with $g^{\mu\nu}$) of Einstein's equations with $\Lambda = 0$,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu}, \tag{5}$$

show that the trace-reversed version of Einstein's equations can be written as

$$R_{\mu\nu} = 8\pi G_N \left(T_{\mu\nu} - \frac{1}{2} T^{\lambda}{}_{\lambda} g_{\mu\nu} \right). \tag{6}$$

What does this result simplify to for conformal matter such as pure radiation?

(c) Derive the Einstein's equations (equations of motion) from the action

$$S = \int d^4x \sqrt{-g}\, f(R) \tag{7}$$

by performing the same metric variation as in (a) and ensuring that $\delta S = 0$. The function $f(R)$ is an arbitrary scalar function of the Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$.

3. The cosmological constant and de Sitter space (25 points)

(a) Use Einstein's equations in vacuum ($T_{\mu\nu} = 0$) with a non-zero cosmological constant Λ

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0 \tag{8}$$

to compute the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R in terms of Λ and the metric $g_{\mu\nu}$.

(b) Using the Riemann tensor for de Sitter space

$$R_{\rho\sigma\mu\nu} = \frac{1}{\alpha^2} \left(g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu} \right), \tag{9}$$

with α a constant, verify that your results from (a) are correct and fix α in terms of Λ.

(c) Assume that $T_{\mu\nu}$ has the form of an ideal fluid. What kind of energy density ε and pressure P with $\Lambda = 0$ give the same Einstein's equations as in Eq. (8)? In other words, what kind of matter $T_{\mu\nu}$ gives rise to the same equations of motion for the metric $g_{\mu\nu}$ as in Eq. (8) if $\Lambda = 0$?

(d) There are many ways to write the metric of de Sitter space. Two examples are

$$ds^2_{(1)} = -\left(1 - \frac{r^2}{\ell^2}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{r^2}{\ell^2}\right)} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{10}$$

and

$$ds^2_{(2)} = -d\tau^2 + e^{2\tau/\ell}\left[d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta d\phi^2\right]. \tag{11}$$

Find the coordinate transformation from $ds^2_{(1)}$ to $ds^2_{(2)}$—that is, from (t, r, θ, ϕ) to $(\tau, \rho, \theta, \phi)$—which amounts to finding two functions $t(\tau, \rho)$ and $r(\tau, \rho)$. What are they?

4. Energy-momentum tensor (15 points)

Derive the energy-momentum tensor $T_{\mu\nu}$ for the Maxwell action

$$S = \int d^4x \sqrt{-g} F_{\mu\nu} F^{\mu\nu}, \tag{12}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ by varying the metric.