DG DN2 28172042

① $\mathbb{R}^3$, $g_L((x_1,y_1,t_1),(x_2,y_2,t_2)) = t_1t_2 - x_1x_2 - y_1y_2$

$R>0$, $D_R = \{(x,y) \mid x^2+y^2<R^2\}$, $H_R = \{(x,y,t) \mid t^2-x^2-y^2=R^2, t>0\}$

$\phi_R: D_R \to H_R$, ϕ_R PRESLIKA (x,y) V PRESEČIŠČE PREMICE, KI GRE SKOZI $(x,y,0)$, $(0,0,-R)$ IN H_R

a) PREDPIS $\phi_R(x,y)=?$ FIKSIRAJMO $(x_0,y_0)\in D_R$. ISKANA PREMICA JE

$$\boxed{(x,y,t)} = (x_0,y_0,0) + s(0-x_0, 0-y_0, -R-0) = \boxed{(x_0(1-s), y_0(1-s), -sR)} \quad s\in\mathbb{R}$$

PRESEČIŠČE Z H_R: $R^2 = t^2-x^2-y^2 = s^2R^2 - x_0^2(1-s)^2 - y_0^2(1-s)^2$

$\Rightarrow R^2(s^2-1) - x_0^2(1-s)^2 - y_0^2(1-s)^2 = 0$

$\Rightarrow \cancel{s^2(R^2-x_0^2-y_0^2)+s(2x_0+2y_0)+(R^2-x_0-y_0)=0 \quad D=4(x_0^2+y_0^2)\dots}$

$\Rightarrow (R^2-x_0^2-y_0^2)s^2 + (2x_0^2+2y_0^2)s + (-R^2-x_0^2-y_0^2) = 0$

$D = 4(x_0^2+y_0^2)^2 + 4(R^2-x_0^2-y_0^2)(R^2+x_0^2+y_0^2) = 4R^4 > 0$

$s = \dfrac{-2(x_0^2+y_0^2) \pm 2R^2}{2(R^2-x_0^2-y_0^2)}$, H_R USTREZA $t>0$, TOREJ $s<0$ ($t=-sR$), KER $x_0^2+y_0^2<R^2$ VZAMEMO

REŠITEV $\boxed{s = -\dfrac{x_0^2+y_0^2+R^2}{R^2-x_0^2-y_0^2}}$

TOREJ $\boxed{\phi_R(x,y)} = \left(\left(1+\dfrac{x^2+y^2+R^2}{R^2-x^2-y^2}\right)x, \left(1+\dfrac{x^2+y^2+R^2}{R^2-x^2-y^2}\right)y, \dfrac{x^2+y^2+R^2}{R^2-x^2-y^2}R\right) =$

$$= \boxed{\frac{1}{R^2-x^2-y^2}\left(2R^2x, 2R^2y, (x^2+y^2+R^2)R\right)} = \left(\phi_R^1(x,y), \phi_R^2(x,y), \phi_R^3(x,y)\right)$$

IZRAČUNAJ METRIKO $g_R = -\phi_R^* g_L$ V BAZI $\{\partial_x, \partial_y\}$

V MATRIČNI OBLIKI: $g_L = \begin{array}{c} \begin{matrix} x & y & t \end{matrix} \\ \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$

S FORMULO ZA POVLEK LAHKO DOBIMO KOMPONENTE

MATRIKE g_R:

$$\boxed{(g_R)_{ij} = -\sum_{k,l} \frac{\partial \phi_R^k}{\partial x_i} \cdot \frac{\partial \phi_R^l}{\partial x_j} (g_L)_{kl}} \quad (x_1 := x,\ x_2 := y)$$

$\boxed{\dfrac{\partial \phi_R^1}{\partial x}} = \dfrac{\partial}{\partial x}\left(\dfrac{2R^2x}{R^2-x^2-y^2}\right) = \dfrac{2R^2(R^2-x^2-y^2)+2R^2\cdot 2x^2}{(R^2-x^2-y^2)^2} = \boxed{\dfrac{2R^2(R^2+x^2-y^2)}{(R^2-x^2-y^2)^2}}$, $\boxed{\dfrac{\partial \phi_R^1}{\partial y} = \dfrac{4R^2xy}{(R^2-x^2-y^2)^2}}$

$\boxed{\dfrac{\partial \phi_R^2}{\partial x} = \dfrac{4R^2xy}{(R^2-x^2-y^2)^2}, \quad \dfrac{\partial \phi_R^2}{\partial y} = \dfrac{2R^2(R^2-x^2+y^2)}{(R^2-x^2-y^2)^2}}$

$$\boxed{\frac{\partial \varphi_R^3}{\partial X}} = \frac{2RX(R^2-X^2-Y^2)+R(X^2+Y^2+R^2)2X}{(R^2-X^2-Y^2)^2} = \boxed{\frac{4R^3X}{(R^2-X^2-Y^2)^2}}, \quad \boxed{\frac{\partial \varphi_R^3}{\partial Y} = \frac{4R^3Y}{(R^2-X^2-Y^2)^2}}$$

$$\cancel{(g_R)_{XX}}\ \boxed{(g_R)_{11}} = -\left[-\left(\frac{\partial \varphi_R^1}{\partial X}\right)^2 - \left(\frac{\partial \varphi_R^2}{\partial X}\right)^2 + \left(\frac{\partial \varphi_R^3}{\partial X}\right)^2\right] = \boxed{\frac{1}{(R^2-X^2-Y^2)^4}\left(4R^4(R^2+X^2-Y^2)^2 + 16R^4X^2Y^2 - 16R^6X^2\right)}$$

$$\boxed{(g_R)_{21}} = (g_R)_{12} = \frac{\partial \varphi_R^1}{\partial X}\frac{\partial \varphi_R^1}{\partial Y} + \frac{\partial \varphi_R^2}{\partial X}\frac{\partial \varphi_R^2}{\partial Y} - \frac{\partial \varphi_R^3}{\partial X}\frac{\partial \varphi_R^3}{\partial Y} = \boxed{\frac{1}{(R^2-X^2-Y^2)^4}\left(8R^4XY(R^2+X^2-Y^2) + 8R^4XY(R^2-X^2+Y^2) - 16R^6XY\right) = \boxed{0}}$$

$$\boxed{(g_R)_{22}} = \boxed{\frac{1}{(R^2-X^2-Y^2)^4}\left(16R^4X^2Y^2 + 4R^4(R^2-X^2+Y^2)^2 - 16R^6Y^2\right)}$$

(V b) NALOGI JE g_R POENOSTAVLJEN)

OD TU NAJPREJ $g := g_R$.

b) IZRAČUNAJ LEVI-CIVITA POVEZAVNO FORMO.

UPORABIL BOM $\Gamma^i_{kl} = \frac{1}{2}\sum_m g^{im}\left(\frac{\partial g_{mk}}{\partial X_l} + \frac{\partial g_{ml}}{\partial X_k} - \frac{\partial g_{kl}}{\partial X_m}\right)$

INVERZ METRIKE $g^{ij} = (g_{ij})^{-1}$ (KER DIAGONALNA)

PREDEN NADALJUJEMO POENOSTAVIMO METRIKO IZ a):

$$\boxed{g_{11}} = \frac{4R^4}{(R^2-X^2-Y^2)^4}\left(R^4+X^4+Y^4+2R^2X^2-2R^2Y^2-2X^2Y^2+4X^2Y^2-4R^2X^2\right) =$$

$$= \frac{4R^4}{(R^2-X^2-Y^2)^4}\left(R^4+X^4+Y^4-2R^2X^2-2R^2Y^2+2X^2Y^2\right) = \frac{4R^4}{(R^2-X^2-Y^2)^4}\left((R^2-X^2-Y^2)^2\right) = \boxed{\frac{4R^4}{(R^2-X^2-Y^2)^2}}$$

ZA g_{22} ZAMENJAMO $X \leftrightarrow Y$ TOREJ $g_{22} = g_{11}$

$$\Rightarrow \boxed{g = \frac{4R^4}{(R^2-X^2-Y^2)^2} \begin{array}{cc} X & Y \\ \multicolumn{2}{c}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \end{array}} \ *$$

$$\boxed{\Gamma^1_{11}} = \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial X} = \frac{1}{2}\frac{(R^2-X^2-Y^2)^2}{4R^4} 4R^4 \frac{4X}{(R^2-X^2-Y^2)^3} = \boxed{\frac{2X}{R^2-X^2-Y^2}}$$

$$\boxed{\Gamma^2_{22}} = \boxed{\frac{2Y}{R^2-X^2-Y^2}} \quad (X \leftrightarrow Y)$$

$$\boxed{\Gamma^1_{22}} = \frac{1}{2} g^{11}\left(-\frac{\partial g_{22}}{\partial X}\right) = \boxed{-\Gamma^1_{11}} \quad (g_{22} = g_{11})$$

$$\boxed{\Gamma^2_{11}} = \frac{1}{2} g^{22}\left(-\frac{\partial g_{11}}{\partial Y}\right) = \boxed{-\Gamma^2_{22}}$$

$$\boxed{\Gamma^1_{12} = \Gamma^1_{21}} = \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial y} = \boxed{\Gamma^2_{22}}$$

$$\boxed{\Gamma^2_{12} = \Gamma^2_{21}} = \frac{1}{2} g^{22} \frac{\partial g_{22}}{\partial x} = \boxed{\Gamma^1_{11}}$$

POVEZAVA $\omega_{ki} = \sum_j \Gamma^k_{ij}\, dx_j$ $\qquad \omega_{11} = \frac{2x}{R^2-x^2-y^2} dx + \frac{2y}{R^2-x^2-y^2} dy$

$$\omega_{12} = \frac{2y}{R^2-x^2-y^2} dx - \frac{2x}{R^2-x^2-y^2} dy \qquad \omega_{22} = \frac{2x}{R^2-x^2-y^2} dx + \frac{2y}{R^2-x^2-y^2} dy$$

$$\omega_{21} = -\frac{2y}{R^2-x^2-y^2} dx + \frac{2x}{R^2-x^2-y^2} dy$$

$$\boxed{\omega} = \boxed{\frac{2}{R^2-x^2-y^2} \begin{pmatrix} x\,dx + y\,dy & y\,dx - x\,dy \\ -y\,dx + x\,dy & x\,dx + y\,dy \end{pmatrix}}$$

$R = d\omega + \omega \wedge \omega$, $\qquad d\left(\frac{x}{R^2-x^2-y^2}\right) = \frac{R^2-x^2-y^2+2x^2}{(R^2-x^2-y^2)^2} dx + \frac{2xy}{(R^2-x^2-y^2)^2} dy = \frac{(R^2+x^2-y^2)dx + 2xy\,dy}{(R^2-x^2-y^2)^2}$

$$d\left(\frac{y}{R^2-x^2-y^2}\right) = \frac{(R^2-x^2+y^2)dy + 2xy\,dx}{(R^2-x^2-y^2)^2}$$

$$d\omega = \frac{2}{(R^2-x^2-y^2)^2} \begin{pmatrix} 2xy\, dy\wedge dx + 2xy\, dx\wedge dy, & (R^2-x^2+y^2)\, dy\wedge dx - (R^2+x^2-y^2)\, dx\wedge dy \\ -(R^2-x^2+y^2)\, dy\wedge dx + (R^2+x^2-y^2)\, dx\wedge dy, & 2xy\, dy\wedge dx + 2xy\, dx\wedge dy \end{pmatrix} =$$

$$= \frac{2}{(R^2-x^2-y^2)^2} \begin{pmatrix} 0 & (-2R^2)\, dx\wedge dy \\ 2R^2\, dx\wedge dy & 0 \end{pmatrix}$$

$$\omega\wedge\omega = \frac{4}{(R^2-x^2-y^2)^2} \begin{pmatrix} xy\, dx\wedge dy + xy\, dy\wedge dx + (yx\, dx\wedge dy + xy\, dy\wedge dx), & (-x^2 dx\wedge dy + y^2 dy\wedge dx) + (y^2 dx\wedge dy - x^2 dy\wedge dx) \\ (-y^2 dx\wedge dy + x^2 dy\wedge dx) + (x^2 dx\wedge dy - y^2 dy\wedge dx), & (xy\, dx\wedge dy + xy\, dy\wedge dx) + (xy\, dx\wedge dy + xy\, dy\wedge dx) \end{pmatrix}$$

$$= \frac{4}{(R^2-x^2-y^2)^2} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \boxed{R = \frac{4R^2}{(R^2-x^2-y^2)^2} \begin{pmatrix} 0 & -dx\wedge dy \\ dx\wedge dy & 0 \end{pmatrix}}$$

V 2D: GAUSSOVA UKRIVLJENOST JE $K = \langle e_x, R e_y \rangle (e_x, e_y)$ $\qquad$, $\|dx\|^2 = \frac{4R^4}{(R^2-x^2-y^2)^2} = \|dy\|^2$

$$e_x = \frac{R^2-x^2-y^2}{2R^2} dx$$

$$e_y = \frac{R^2-x^2-y^2}{2R^2} dy$$

$$Re_y = -\frac{2}{R^2-x^2-y^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} dx\wedge dy$$

$$\langle e_x, Re_y \rangle = -\frac{1}{R^2} (1, 0) \frac{4R^4}{(R^2-x^2-y^2)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} dx\wedge dy$$

$$= -\frac{4R^2}{(R^2-x^2-y^2)^2} dx\wedge dy$$

$$\boxed{K} = -\frac{1}{R^2} dx\wedge dy\, (\partial x, \partial y) = \boxed{-\frac{1}{R^2}}$$

c) $H = \{(x,y) \mid x \in \mathbb{R}, y>0\}$ z $g_H = \frac{1}{y^2} g_e$

MATRIČNO $\boxed{g_H = \frac{1}{y^2}\begin{array}{cc} x & y \\ \end{array}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}}$

POKAŽI, DA JE $\tau : H \to D_1$

$\tau(z) = \frac{iz+1}{z+i}$ IZOMETRIJA.

$\boxed{g_{D_1} = \frac{4}{(1-x^2-y^2)^2}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}}$

VELJATI MORA

$p \in H \quad g_{H_p}(v_1, v_2) = g_{D_1 \tau(p)}((D\tau)_p v_1, (D\tau)_p v_2)$

$p = (x,y)$

$v_1^T g_{H_p} v_2 = v_1^T (D\tau)_p^T \, g_{D_1 \tau(p)} (D\tau)_p v_2 \quad \forall v_1, v_2$

$\Rightarrow g_{H_p} = (D\tau)_p^T \, g_{D_1 \tau(p)} (D\tau)_p \quad (*)$

$\boxed{\tau'(z)} = \frac{i(z+i)-(iz+1)}{(z+i)^2} = \boxed{-\frac{2}{(z+i)^2}}$

$\boxed{\tau(z)} = \frac{i(x+iy)+1}{x+iy+i} =$

$= \frac{(i(x+iy)+1)(x-i(y+1))}{x^2+(y+1)^2} =$

$= \frac{i(x+iy)x+(x+iy)(y+1)+x-i(y+1)}{x^2+(y+1)^2} =$

$= \frac{1}{x^2+(1+y)^2}\left(-xy+x+xy+x+i(x^2+y^2+y-y-1)\right)$

$= \boxed{\frac{2x+(x^2+y^2-1)i}{x^2+(1+y)^2}}$

$\boxed{1-\mathrm{Re}(\tau(z))^2-\mathrm{Im}(\tau(z))^2} =$

$= 1 - \frac{4x^2+(x^2+y^2-1)^2}{(x^2+(1+y)^2)^2} =$

$= \frac{1}{(x^2+(1+y)^2)^2}\cdot\left(x^4+(1+y)^4+2x^2(1+y)^2-4x^2-x^4-y^4-1-2x^2y^2+2x^2+2y^2\right)$

$\frac{1}{(x^2+(1+y)^2)^2}\left(4y^3+8y^2+4y+4x^2y\right) = \frac{4y}{(x^2+(1+y)^2)^2}\left(y^2+2y+1+x^2\right)$

$= \boxed{\frac{4y}{(x^2+(1+y)^2)}} \Rightarrow \boxed{g_{D_1 \tau(p)} = \frac{(x^2+(1+y)^2)^2}{(4y)^2} 4 \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}}$

V KOMPLEKSNEM:

(*) V KOMPLEKSNEM

$\boxed{\frac{1}{y^2}} = \overline{\tau'(z)} \cdot \frac{1}{4} \cdot \frac{1}{y^2} \cdot (x^2+(1+y)^2)^2 \cdot \tau'(z) =$

$= \frac{1}{(x-iy-i)^2} \cdot \frac{1}{(x+iy+i)^2} (x^2+(1+y)^2)^2 \frac{1}{y^2} =$

$= \frac{1}{(x^2+(1+y)^2)^2}(x^2+(1+y)^2)^2 \frac{1}{y^2} =$

$= \boxed{\frac{1}{y^2}}$ ✓

d) POIŠČI PREDPISE GEODETK $\gamma_x, \gamma_y : \mathbb{R} \to D_1$ S POGOJI $\gamma_x(0) = \gamma_y(0) = (0,0)$

$\dot\gamma_x(0) = \partial_x$, $\dot\gamma_y(0) = \partial_y$

$\gamma = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$

GEODETSKA ENAČBA: $\ddot u_k + \sum_{i,j} \Gamma_{ij}^k \dot u_i \dot u_j = 0$

OZNAKA $\gamma(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \Rightarrow$

$$\boxed{\begin{aligned} \ddot x + \frac{2x}{1-x^2-y^2}(\dot x^2 - \dot y^2) + \frac{2y}{1-x^2-y^2} \cdot 2\dot x \dot y = 0 \\ \ddot y + \frac{2y}{1-x^2-y^2}(\dot y^2 - \dot x^2) + \frac{2x}{1-x^2-y^2} 2\dot x \dot y = 0 \end{aligned}}$$

γ_x: $(x(0), y(0)) = (0,0)$, $(\dot x(0), \dot y(0)) = (1,0)$

$y(t) = 0$ ZADOŠČA ZAČETNIM POGOJEM IN DRUGI ENAČBI

PRVA ENAČBA POSTANE $\ddot x(1-x^2) + 2x \cdot \dot x^2 = 0$ $\leftarrow$ t NE NASTOPA V ENAČBI, UVEDEMO $z(x) = \dot x$, $\ddot x = \frac{dz}{dx} \cdot z$

$\Rightarrow \frac{dz}{dx} z(1-x^2) + 2x z^2 = 0$

$\Rightarrow \frac{dz}{dx}(1-x^2) = -2xz$

$\ln z = -\int \frac{2x}{1-x^2} dx + A \Rightarrow \ln(z) = \ln(1-x^2) + A \;/e^{(\cdot)} \Rightarrow z = B(1-x^2)$

$\Rightarrow \frac{dx}{1-x^2} = B\,dt \Rightarrow \operatorname{arctanh}(x) = Bt + C \Rightarrow \boxed{x(t) = \operatorname{th}(Bt+C)}$

$x(0) = 0 \Rightarrow \boxed{C = 0}$

$\dot x(0) = 1$, $\dot x(t) = \frac{1}{\operatorname{ch}^2(Bt)} \cdot B \Rightarrow \boxed{\dot x(0) = B = 1}$

$\boxed{\gamma_x(t) = \begin{pmatrix} \operatorname{th}(t) \\ 0 \end{pmatrix}}$ EDINA REŠITEV PO EKSISTENČNEM IZREKU ZA NDE.

ZA $\gamma_y(t)$ ZAMENJAMO VLOGI X IN Y (SISTEM ENAČB JE INVARIANTEN NA $x \leftrightarrow y$)

IN DOBIMO $\boxed{\gamma_y(t) = \begin{pmatrix} 0 \\ \operatorname{th}(t) \end{pmatrix}}$

③ ∇ LEVI CIVITA KOV. ODVOD NA TS^2

a) DEF ∇^* NA T^*S^2 Z: $X(\alpha(Y)) = (\nabla^*_X \alpha)(Y) + \alpha(\nabla_X Y)$, $X,Y \in \Gamma(TS^2)$, $\alpha \in \Omega^1(S^2)$

$\underline{\nabla^* \text{ KOV. ODVOD}}$ $\quad (\nabla^*_X \alpha)(Y) = X(\alpha(Y)) - \alpha(\nabla_X Y)$

1) LINEARNOST VZAMEMO $\alpha = c_1\alpha_1 + c_2\alpha_2$ $\quad c_1, c_2 \in \mathbb{R}$ $\alpha_1, \alpha_2 \in \Omega^1(S^2)$

$$\boxed{\nabla^*_X (c_1\alpha_1 + c_2\alpha_2)(Y)} = X(c_1\alpha_1(Y) + c_2\alpha_2(Y)) - c_1\alpha_1(\nabla_X Y) - c_2\alpha_2(\nabla_X Y) =$$

$$= c_1 X(\alpha_1(Y)) + c_2 X(\alpha_2(Y)) - c_1\alpha_1(\nabla_X Y) - c_2\alpha_2(\nabla_X Y) = \boxed{c_1(\nabla^*_X \alpha_1)(Y) + c_2(\nabla^*_X \alpha_2)(Y)} \checkmark$$

2) LEIBNIZ

$f \in C^\infty(S^2)$

$\alpha \in \Omega^1(S^2)$

LEVA STRAN ENAČBE $\quad (\nabla^*_X (f\alpha))(Y) \overset{?}{=} (df(X) \otimes \alpha)(Y) + (\overset{\text{FUNKCIJA}}{f}\, \nabla^*_X \alpha)(Y) =$

$$= (df(X) \otimes \alpha)(Y) + f\,(X(\alpha(Y)) - \alpha(\nabla_X Y))$$

DESNA STRAN: $\quad X(f\alpha(Y)) - f\alpha(\nabla_X Y) \underset{\text{LEIBNIZ}}{=} X(f)\,\alpha(Y) + f\,X(\alpha(Y)) - f\alpha(\nabla_X Y)$

LEVA = DESNA: $\quad (df(X) \otimes \alpha)(Y) + \cancel{f(X(\alpha(Y)) - \alpha(\nabla_X Y))} = X(f)\alpha(Y) + \cancel{f X(\alpha(Y)) - f\alpha(\nabla_X Y)}$

$$(df(X) \otimes \alpha)(Y) = X(f)\alpha(Y)$$

V RESNICI POVSOD $\otimes$ TOREJ JE TO $X(f) \otimes \alpha(Y)$

$$\Rightarrow df(X) \otimes \alpha(Y) = X(f) \otimes \alpha(Y)$$

$$\sum_i \frac{df}{dx_i} X_i = \sum_i X_i \frac{df}{dx_i} \checkmark$$

∇^* JE KOV. ODVOD. $\square$

b) $\{e_1, e_2\}$ LOKALNO ON OGRODJE TS^2, $\{e_1^*, e_2^*\}$ DUALNO LOKALNO OGRODJE T^*S^2

$$\nabla_{e_j} e_i = \sum_{k=1}^{2} A_{ij}^k e_k \ , \quad \nabla^*_{e_j} e_i^* = \sum_{k=1}^{2} B_{ij}^k e_k^* \qquad \text{POKAŽI: } A_{ij}^k = B_{ij}^k$$

VERJETNO $A_{ij}^k = -B_{ij}^k$

V FORMULO IZ a) BOM VSTAVIL $X, Y \in \{e_1, e_2\}$ $\alpha \in \{e_1^*, e_2^*\}$

$X = Y = e_1$, $\alpha = e_1^*$: $\quad e_1(e_1^*(e_1)) = (\nabla^*_{e_1} e_1^*)(e_1) + e_1^*(\nabla_{e_1} e_1)$

$$e_1(1) \underset{\text{LASTNOST DERIVACIJ}}{=} \boxed{0 = B_{11}^1 + A_{11}^1}$$

PONOVIMO RAČUN ZA VSE IZBIRE X, Y, α:

X	α	Y	ENAČBA	
e_1	e_1^*	e_1	$0 = B_{11}^1 + A_{11}^1$	①
e_1	e_1^*	e_2	$0 = B_{11}^2 + A_{21}^1$	②
e_2	e_2^*	e_1	$0 = B_{22}^1 + A_{12}^2$	③
e_2	e_2^*	e_2	$0 = B_{22}^2 + A_{22}^2$	④
e_1	e_2^*	e_1	$0 = B_{21}^1 + A_{11}^2$	⑤
e_1	e_2^*	e_2	$0 = B_{21}^2 + A_{21}^2$	⑥
e_2	e_1^*	e_1	$0 = B_{12}^1 + A_{12}^1$	⑦
e_2	e_1^*	e_2	$0 = B_{12}^2 + A_{22}^1$	⑧

~~POGLEJMO ČE TE ENAČBE USTREZAJO~~ $\cancel{B_{ij}^k = A_{ij}^k}$

$\cancel{A_{11}^1 = A_{11}^1}$ ✓

$\cancel{A_{11}^2 = A_{21}^1}$

$\cancel{A_{22}^1 = A_{12}^2}$

$\cancel{A_{22}^2 = A_{22}^2}$ ✓

$\cancel{A_{21}^1 = A_{11}^2}$

$\cancel{A_{21}^2 = A_{21}^2}$ ✓

$\cancel{A_{12}^1 = A_{12}^1}$ ✓

$\cancel{A_{12}^2 = A_{22}^1}$

∇ LEVI CIVITA KOV. ODVOD ~~BR~~

$\Rightarrow$ BREZ TORZIJE $\Rightarrow$

$\Rightarrow A_{ij}^k = A_{ji}^k$

∇^* TUDI LEVI CIVITA

$\Rightarrow B_{ij}^k = B_{ji}^k$

① $A_{11}^1 = -B_{11}^1$ ✓ ④ $B_{22}^2 = -A_{22}^2$ ✓ ⑥ $B_{21}^2 = -A_{21}^2$ ✓ ⑦ $B_{12}^1 = -A_{12}^1$ ✓

②, ⑤ $\underset{②}{B_{11}^2} = -A_{21}^1 = \underset{\text{BREZ TORZIJE}}{-A_{12}^1} = B_{12}^1 = B_{21}^1 = \underset{⑤}{-A_{11}^2}$

③, ⑧ $\boxed{B_{22}^1} = -A_{12}^2 = -A_{21}^2 = B_{21}^2 = B_{12}^2 = \boxed{-A_{22}^1}$ TOREJ RES $A_{ij}^k = -B_{ij}^k$

c) ZA LEVI-CIVITOV ODVOD VEMO $\nabla_{e_j}(e_i) = \sum_k \Gamma_{ij}^k e_k \Rightarrow A_{ij}^k = \Gamma_{ij}^k$

$B_{ij}^k = \Gamma_{ij}^{k*}$ ← CHRISTOFFELI ZA ∇^*

~~ω~~ POVEZAVA $\omega_{ki}^* = \sum_j \Gamma_{ij}^{k*} dx_j = \sum_j B_{ij}^k dx_j$, $\omega^* = \begin{pmatrix} B_{11}^1 dx_1 + B_{12}^1 dx_2, & B_{21}^1 dx_1 + B_{22}^1 dx_2 \\ B_{11}^2 dx_1 + B_{12}^2 dx_2, & B_{21}^2 dx_1 + B_{22}^2 dx_2 \end{pmatrix}$

$F_{\nabla^*} = d\omega^* + \omega^* \wedge \omega^*$ $\quad d\omega^* = \begin{pmatrix} \frac{\partial B_{12}^1}{\partial x_1} - \frac{\partial B_{11}^1}{\partial x_2}, & \frac{\partial B_{22}^1}{\partial x_1} - \frac{\partial B_{21}^1}{\partial x_2} \\ \frac{\partial B_{12}^2}{\partial x_1} - \frac{\partial B_{11}^2}{\partial x_2}, & \frac{\partial B_{22}^2}{\partial x_1} - \frac{\partial B_{21}^2}{\partial x_2} \end{pmatrix} dx_1 \wedge dx_2$

$$\omega^* \wedge \omega^* = \begin{pmatrix} B_{21}^1 B_{12}^2 - B_{22}^1 B_{11}^2, & B_{11}^1 B_{22}^1 - B_{12}^1 B_{21}^1 + B_{21}^1 B_{22}^2 - B_{22}^1 B_{21}^2 \\ B_{11}^2 B_{12}^1 - B_{12}^2 B_{11}^1 + B_{21}^2 B_{12}^2 - B_{22}^2 B_{11}^2, & B_{11}^2 B_{22}^1 - B_{12}^2 B_{21}^1 \end{pmatrix} dx_1 \wedge dx_2$$

~~$F_{\nabla^*} =$~~ TEKOM NALOGE b) SMO VIDELI $B_{22}^1 = B_{21}^2$, $B_{11}^2 = B_{12}^1$ $\Rightarrow$ $\omega^* \wedge \omega^* = \begin{pmatrix} 0, & B_{11}^1 B_{22}^1 + B_{22}^2 B_{21}^1 - {B_{12}^1}^2 - {B_{22}^1}^2 \\ {B_{11}^2}^2 - B_{11}^1 B_{22}^1 + {B_{12}^2}^2 - B_{22}^2 B_{11}^2, & 0 \end{pmatrix} dx_1 \wedge dx_2$

TOREJ

$$F_{\nabla^*} = \begin{pmatrix} \frac{\partial B_{11}^2}{\partial x_1} - \frac{\partial B_{11}^1}{\partial x_2}, & \frac{\partial B_{22}^1}{\partial x_1} - \frac{\partial B_{11}^2}{\partial x_2} + B_{11}^1 B_{22}^1 + B_{22}^2 B_{11}^2 - {B_{11}^2}^2 - {B_{22}^1}^2 \\ \frac{\partial B_{22}^1}{\partial x_1} - \frac{\partial B_{11}^2}{\partial x_2} + {B_{11}^2}^2 - B_{11}^1 B_{22}^1 + {B_{22}^1}^2 - B_{22}^2 B_{11}^2, & \frac{\partial B_{22}^2}{\partial x_1} - \frac{\partial B_{22}^1}{\partial x_2} \end{pmatrix} dx_1 \wedge dx_2$$

③ $End(TS^2)$ VEKTORSKI SVEŽENJ ENDOMORFIZMOV TS^2.

$\{e_1, e_2\}$ ONB T_pS^2, $\mathcal{A}, \mathcal{B} \in End(T_pS^2)$ $\langle \mathcal{A}, \mathcal{B} \rangle = Tr(AB^T)$ A, B PRIPADATA $\mathcal{A}, \mathcal{B}$ V BAZI $\{e_1, e_2\}$

$E \subset End(TS^2)$, $E_p = \{\mathcal{A} \in End(T_pS^2) \mid \mathcal{A} = \mathcal{A}^*, Tr(\mathcal{A}) = 0\}$

KONSTRUIRAJ IZOMORFIZEM $\Psi : S^2 \times \mathbb{R}^2 \to E^\perp$

$Tr(\mathcal{A}) = 0 \iff Tr(A) = 0$ TOREJ LAHKO E_p OPIŠEMO Z $\{A \in Mat(2\times 2, \mathbb{C}) \mid Tr(A) = 0, A^\dagger = A\}$

$\mathcal{A} = \mathcal{A}^* \underset{\text{KER BAZA ON}}{\iff} A = A^\dagger$

$\Rightarrow$ TAKE MATRIKE SO OBLIKE $A = \begin{bmatrix} a & b \\ \bar{b} & -a \end{bmatrix}$, $a \in \mathbb{R}$, $b \in \mathbb{C}$

TOREJ $E_p = \{\begin{bmatrix} a & b \\ \bar{b} & -a \end{bmatrix} \mid a \in \mathbb{R}, b \in \mathbb{C}\}$

VZAMIMO $\mathcal{B} \in End(T_pS^2)$ $B = \begin{bmatrix} c & d \\ e & f \end{bmatrix}$

POGOJ ZA $\mathcal{B} \in E^\perp$: $0 \underset{\forall \mathcal{A} \in E}{=} \langle \mathcal{A}, \mathcal{B} \rangle = Tr\left(\begin{bmatrix} a & b \\ \bar{b} & -a \end{bmatrix} \begin{bmatrix} c & e \\ d & f \end{bmatrix}\right) =$

$= Tr\left(\begin{bmatrix} ac + bd & ae + bf \\ \bar{b}c - ad & \bar{b}e - af \end{bmatrix}\right) = ac + bd + \bar{b}e - af = 0 \quad \forall a \in \mathbb{R}, \forall b \in \mathbb{C}$

$\Rightarrow a(c-f) + (x+iy)d + (x-iy)e = 0 \Rightarrow a(c-f) + x(d+e) + iy(d-e) = 0$

$b = x+iy$
$x, y \in \mathbb{R}$

$x = y = 0, a = 1 \Rightarrow \boxed{c = f}$

$a = x = 0, y = 1 \Rightarrow d = e$

$a = y = 0, x = 1 \Rightarrow d = -e$

$\Rightarrow \boxed{d = e = 0}$

$\Rightarrow B = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$, $c \in \mathbb{C}$ $\quad E_p^\perp = \{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \mid a \in \mathbb{C}\}$

$\Rightarrow E^\perp = S^2 \times \{\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \mid a \in \mathbb{C}\}$

$\Psi : S^2 \times \mathbb{R}^2 \to E^\perp$

$\boxed{\Psi(p, (x,y)) \to (p, \varphi(x,y))}$

$p \in S^2$

$\varphi : \mathbb{R}^2 \to \{\mathcal{A} \in End(TS^2)$, KATERIH MATRIKE SO OBLIKE $A = a \cdot \mathbb{1}$, $a \in \mathbb{C}\}$

$\boxed{\varphi(x,y) = (x+iy)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$

$\Psi(p, (x,y)) = (p, (x+iy)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})$ INVERZ $\Psi^{-1}(p, a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = (p, (Re(a), Im(a)))$ TUDI OČITNO ZVEZNA

PRESLIKAVA JE OČITNO ~~GLADKA~~ ZVEZNA

$\Rightarrow \Psi$ IZOMORFIZEM.