# KOZMOLOGIJA DN 2

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(1) $ds^2 = -dt^2 + a^2(t)\,\tilde g_{ij}\, dx^i dx^j = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta\, d\varphi^2\right]$

$$\begin{array}{cc} & \begin{array}{cccc} t & r & \theta & \varphi \end{array} \\ g_{\mu\nu} = & \begin{pmatrix} -1 & & & \\ & \frac{a^2}{1-kr^2} & & \\ & & a^2 r^2 & \\ & & & a^2 r^2 \sin^2\theta \end{pmatrix} \end{array}$$

$$g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & \frac{1-kr^2}{a^2} & & \\ & & (ar)^{-2} & \\ & & & (ar\sin\theta)^{-2} \end{pmatrix}$$

$\Gamma^\sigma_{\mu\nu} = \frac12 g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$

$\sigma=0$: $\boxed{\Gamma^0_{00} = 0}$ $\boxed{\Gamma^0_{i0}} = \frac12 g^{00}(\partial_i g_{00} + \partial_0 g_{0i} - \partial_0 g_{i0}) = \boxed{0 = \Gamma^0_{0i}}$

$\boxed{\Gamma^0_{ij}} = \frac12 g^{00}(\partial_i g_{j0} + \partial_j g_{0i} - \partial_0 g_{ij}) = \boxed{\frac12 \partial_0 g_{ij}}$

$\Rightarrow \boxed{\Gamma^0_{11} = \frac{a\dot a}{1-kr^2}}$ $\boxed{\Gamma^0_{22} = a\dot a r^2}$ $\boxed{\Gamma^0_{33} = a\dot a r^2 \sin^2\theta}$

$\sigma=k$: $\boxed{\Gamma^k_{00} = 0}$ $\Gamma^k_{i0} = \frac12 g^{kk}(\cancel{\partial_i g_{0k}} + \partial_0 g_{ki} - \cancel{\partial_k g_{i0}}) = \frac12 g^{kk}\partial_0 g_{ki} \Rightarrow \boxed{\Gamma^1_{10}} = \frac12 \frac{2a\dot a}{a^2} = \boxed{\frac{\dot a}{a}} = \boxed{\Gamma^1_{01}}$

$\boxed{\Gamma^2_{20} = \frac{\dot a}{a} = \Gamma^2_{02}}$ $\boxed{\Gamma^3_{30} = \Gamma^3_{03} = \frac{\dot a}{a}}$

$\Gamma^k_{ij} = \frac12 g^{kk}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij})$

$k=1$: $\Gamma^1_{ij} = \frac12 g^{11}(\partial_i g_{j1} + \partial_j g_{1i} - \partial_1 g_{ij})$ $\boxed{\Gamma^1_{11}} = \frac12 \frac{1-kr^2}{a^2}\frac{a^2}{(1-kr^2)^2}(+2kr) = \boxed{\frac{+kr}{1-kr^2}}$ $\boxed{\Gamma^1_{13} = \Gamma^1_{31} = 0}$

$k=2$: $\Gamma^2_{ij} = \frac12 g^{22}(\partial_i g_{j2} + \partial_j g_{2i} - \partial_2 g_{ij})$ $\boxed{\Gamma^1_{21}} = \boxed{\Gamma^1_{12}} = \frac12 g^{11}\partial_2 g_{11} = \boxed{0}$ $\boxed{\Gamma^1_{22}} = -\frac12 g^{11}\partial_1 g_{22} = \boxed{-r(1-kr^2)}$

$\boxed{\Gamma^2_{11} = 0}$ $\boxed{\Gamma^2_{12}} = \frac12 g^{22}\partial_1 g_{22} = \boxed{\frac1r = \Gamma^2_{21}}$ $\boxed{\Gamma^1_{23} = 0 = \Gamma^1_{32}}$ $\boxed{\Gamma^1_{33}} = -\frac12 g^{11}\partial_1 g_{33} = \boxed{-r(1-kr^2)\sin^2\theta}$

$\boxed{\Gamma^2_{22} = 0}$ $\boxed{\Gamma^2_{13} = \Gamma^2_{31} = 0}$ $\boxed{\Gamma^2_{33}} = -\frac12 g^{22}\partial_2 g_{33} = -\frac12 2\sin\theta\cos\theta = \boxed{-\sin\theta\cos\theta}$ $\boxed{\Gamma^2_{23} = \Gamma^2_{32} = 0}$

$k=3$: $\Gamma^3_{ij} = \frac12 g^{33}(\partial_i g_{j3} + \partial_j g_{3i} - \cancel{\partial_3 g_{ij}})$ $\boxed{\Gamma^3_{11} = 0 = \Gamma^3_{12} = \Gamma^3_{21}}$ $\boxed{\Gamma^3_{13}} = \frac12 g^{33}\partial_1 g_{33} = \boxed{\frac1r = \Gamma^3_{31}}$

$\boxed{\Gamma^3_{22} = 0}$ $\boxed{\Gamma^3_{23}} = \frac12 g^{33}\partial_2 g_{33} = \boxed{\mathrm{ctg}\,\theta = \Gamma^3_{32}}$ $\boxed{\Gamma^3_{33} = 0}$

$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$

$\rho=0$: $\boxed{R^0{}_{112}} = \Gamma^0_{1\lambda}\Gamma^\lambda_{21} - \Gamma^0_{2\lambda}\Gamma^\lambda_{11} = \boxed{0}$ ← ZARADI SIM. LAST. SO NIČ TUDI PERMUTACIJE TEH INDEKSOV (*)

$\boxed{R^0{}_{212}} = \partial_1 \Gamma^0_{22} + \Gamma^0_{1\lambda}\Gamma^\lambda_{22} - \Gamma^0_{2\lambda}\Gamma^\lambda_{12} = 2ra\dot a + \frac{a\dot a}{1-kr^2}(kr^2-1)r - a\dot a r^2 \frac1r = \boxed{a\dot a \frac{kr^2}{1-kr^2}}$

PODOBNO KOT V 1. DN LAHKO PERMUTACIJE DOBIMO IZ SIMETRIJE NPR. $\boxed{R^0{}_{221}} = g^{00} R_{0221} = -g^{00} R_{0212} = -g^{00} g_{00} R^0{}_{212} = \boxed{-R^0{}_{212}}$

$\boxed{R^0{}_{122} = 0} = \boxed{R^1{}_{022}}$ $\boxed{R^1{}_{220}} = \frac{kr^2-1}{a^2} R^0{}_{212} = \boxed{-\frac{\dot a}{a} kr^2}$ $\boxed{R^1{}_{202} = \frac{\dot a}{a} kr^2}$

$\boxed{R^2{}_{012}} = -\frac{a^2}{1-kr^2}\frac{1}{a^2r^2} R^1{}_{202} = \boxed{-\frac{\dot a}{a}\frac{k}{1-kr^2}}$ $\boxed{R^2{}_{021} = \frac{\dot a}{a}\frac{k}{1-kr^2}}$ $\boxed{R^2{}_{102} = R^2{}_{012}}$ $\boxed{R^2{}_{210} = R^2{}_{201} = 0}$

$\boxed{R^0{}_{113}} = \Gamma^0_{1\lambda}\Gamma^\lambda_{31} - \Gamma^0_{3\lambda}\Gamma^\lambda_{11} = \boxed{0}$ SPET (*) $\boxed{R^0{}_{313}} = \partial_1 \Gamma^0_{33} + \Gamma^0_{1\lambda}\Gamma^\lambda_{33} - \Gamma^0_{3\lambda}\Gamma^\lambda_{13} = 2a\dot a r\sin^2\theta + (-1)a\dot a r \sin^2\theta - a\dot a r\sin^2\theta = \boxed{0}$ SPET (*)

$\boxed{R^0{}_{223}} = \Gamma^0_{2\lambda}\Gamma^\lambda_{32} - \Gamma^0_{3\lambda}\Gamma^\lambda_{22} = \boxed{0}$ SPET (*), $\boxed{R^0{}_{323}} = \partial_2 \Gamma^0_{33} + \Gamma^0_{2\lambda}\Gamma^\lambda_{33} - \Gamma^0_{3\lambda}\Gamma^\lambda_{23} = 2a\dot a r^2 \sin\theta\cos\theta + a\dot a r^2(-\sin\theta\cos\theta) - a\dot a r^2 \sin^2\theta\,\mathrm{ctg}\,\theta = \boxed{0}$ SPET (*)

$\boxed{R^0{}_{101}} = \partial_0 \Gamma^0_{11} + \Gamma^0_{0\lambda}\Gamma^\lambda_{11} - \Gamma^0_{1\lambda}\Gamma^\lambda_{01} = \frac{1}{1-kr^2}(\dot a^2 + a\ddot a) - \frac{a\dot a}{1-kr^2}\cdot\frac{\dot a}{a} = \boxed{\frac{a\ddot a}{1-kr^2}}$ $\boxed{R^0{}_{110} = -R^0{}_{101}}$, $\boxed{R^0{}_{011} = 0} = \boxed{R^1{}_{100}}$

$\boxed{R^1{}_{001} = \frac{\ddot a}{a}}$ $\boxed{R^1{}_{010} = -\frac{\ddot a}{a}}$

$\boxed{R^0{}_{202}} = \partial_0 \Gamma^0_{22} + \Gamma^0_{0\lambda}\Gamma^\lambda_{22} - \Gamma^0_{2\lambda}\Gamma^\lambda_{02} = (\dot a^2 + a\ddot a) r^2 - a\dot a r^2 \frac{\dot a}{a} = \boxed{a\ddot a r^2}$ $\boxed{R^0{}_{220} = -R^0{}_{202}}$

$\boxed{R^0{}_{022}=0}$ $\boxed{R^2{}_{002} = \frac{\ddot a}{a}}$ $\boxed{R^2{}_{020} = -\frac{\ddot a}{a}}$ $\boxed{R^2{}_{200}=0}$

$\boxed{R^0{}_{303}} = \partial_0 \Gamma^0_{33} + \Gamma^0_{0\lambda}\Gamma^\lambda_{33} - \Gamma^0_{3\lambda}\Gamma^\lambda_{03} = (\dot a^2 + a\ddot a) r^2 \sin^2\theta - a\dot a r^2 \sin^2\theta \frac{\dot a}{a} = \boxed{a\ddot a r^2 \sin^2\theta} = \boxed{-R^0{}_{330}}$

$\boxed{R^0{}_{033} = R^3{}_{300} = 0}$ $\boxed{R^3{}_{003}} = \boxed{\frac{\ddot a}{a} = -R^3{}_{030}}$

$\boxed{R^0{}_{102}} = \Gamma^0_{0\lambda}\Gamma^\lambda_{21} - \Gamma^0_{2\lambda}\Gamma^\lambda_{01} = \boxed{0}$ SPET (K) $\boxed{R^0{}_{203}} = \Gamma^0_{0\lambda}\Gamma^\lambda_{32} - \Gamma^0_{3\lambda}\Gamma^\lambda_{02} = \boxed{0}$

$\boxed{R^0{}_{103}} = \Gamma^0_{0\lambda}\Gamma^\lambda_{31} - \Gamma^0_{3\lambda}\Gamma^\lambda_{01} = \boxed{0}$ —||— OSTALE NENAPISANE $R^\sigma{}_{\sigma\mu\nu}$ Z KAKIM IZMED INDEKSOV 0 SO ZARADI SIMETRIJ TENZORJA TUDI NIČELNE.

ČE V INDEKSIH NI ČASOVNEGA (0): VEMO, DA JE $\tilde g_{ij}$ MAX. SIMETRIČEN. ZATO VELJA:

$$R_{ijkl} = C\cdot(\tilde g_{ik}\tilde g_{jl} - \tilde g_{il}\tilde g_{jk}) \quad \text{OZ.} \quad R^i{}_{jkl} = \tilde C\,(\delta^i_k \tilde g_{jl} - \delta^i_l \tilde g_{jk})$$

IZRAČUNAJMO $\tilde C$: $R^1{}_{212} \underset{\text{PO DEF.}}{=} \partial_1\Gamma^1_{22} + \Gamma^1_{1\lambda}\Gamma^\lambda_{22} - \Gamma^1_{2\lambda}\Gamma^\lambda_{12} = \partial_r(r(Kr^2-1)) + \cancel{\ldots} + \frac{Kr}{1-Kr^2}(-r(1-Kr^2)) + \frac{\dot a}{a} a\dot a r^2 + r(1-Kr^2)\,1/r =$

$$= Kr^2 - 1 + 2Kr^2 - Kr^2 + \dot a^2 r^2 + 1 - Kr^2 = (K+\dot a^2) r^2$$

$$R^1{}_{212} = \tilde C \tilde g_{22} = \tilde C r^2 \Rightarrow \boxed{\tilde C = K + \dot a^2} \Rightarrow \boxed{R^i{}_{jkl} = (K+\dot a^2)(\delta^i_k \tilde g_{jl} - \delta^i_l \tilde g_{jk})}$$

$\boxed{R_{00}} = -\frac{\ddot a}{a} - \frac{\ddot a}{a} - \frac{\ddot a}{a} = \boxed{-\frac{3\ddot a}{a}}$ $\boxed{R_{11}} = \frac{a\ddot a}{1-Kr^2} + R^2{}_{121} + R^3{}_{131} = \frac{a\ddot a}{1-Kr^2} + (K+\dot a^2)[2\tilde g_{11}] = \boxed{\frac{a\ddot a + 2K + 2\dot a^2}{1-Kr^2}}$

$\boxed{R_{22}} = a\ddot a r^2 + R^1{}_{212} + R^3{}_{232} = a\ddot a r^2 + (K+\dot a^2)[2\tilde g_{22}] = \boxed{(a\ddot a + 2K + 2\dot a^2) r^2}$

$\boxed{R_{33}} = a\ddot a r^2 \sin^2\theta + (K+\dot a^2)(2\cdot r^2\sin^2\theta) = \boxed{(a\ddot a + 2K + 2\dot a^2) r^2\sin^2\theta}$

$\boxed{R_{01}} = R^0{}_{001} + R^1{}_{011} + R^2{}_{021} + R^3{}_{031} = \boxed{0} = \boxed{R_{10}}$ $\boxed{R_{02} = R_{20} = R_{03} = R_{30} = 0}$

$\boxed{R_{12} = R_{21} = R_{13} = R_{31} = R_{23} = R_{32} = 0}$

$$R_{\mu\nu} = -\frac{3\ddot a}{a}\begin{pmatrix}1&&&\\&0&&\\&&0&\\&&&0\end{pmatrix} + (a\ddot a + 2K + 2\dot a^2)\begin{pmatrix}0&&&\\&(1-Kr^2)^{-1}&&\\&&r^2&\\&&&r^2\sin^2\theta\end{pmatrix}$$

$$\boxed{R} = g^{\mu\nu}R_{\mu\nu} = 3\frac{\ddot a}{a} + (a\ddot a + 2K + 2\dot a^2)\left(\frac{1}{a^2}+\frac{1}{a^2}+\frac{1}{a^2}\right) = \boxed{\left(\frac{6\ddot a}{a} + \frac{6K}{a^2} + \frac{6\dot a^2}{a^2}\right)} = \boxed{\frac{6}{a}\left(\ddot a + \frac{K}{a} + \frac{\dot a^2}{a}\right)}$$

b) $\cancel{T_{00} = \varepsilon(t),\ T_{i0} = 0,\ T_{ij} = a^2(t) p(t) \tilde g_{ij}}$, $\cancel{R_{00} = -\frac{3\ddot a}{a},\ R_{ij} = (a\ddot a + 2K + 2\dot a^2)\tilde g_{ij}}$

$R_{\mu\nu} \cancel{- \frac{1}{2} R g_{\mu\nu} = 8\pi G_N T_{\mu\nu}}$

$\mu=\nu=0$: $\cancel{-\frac{3\ddot a}{a} + \frac{1}{2}\frac{6}{a}\left(\ddot a + \frac{K}{a} + \frac{\dot a^2}{a}\right) = 8\pi G_N \varepsilon(t)}$ $\cancel{\Rightarrow R = \left(8\pi G_N \varepsilon + \frac{3\ddot a}{a}\right)2}$

$$\boxed{\cancel{\frac{3K}{a^2} + \frac{3\dot a^2}{a^2} = 8\pi G_N \varepsilon}} \iff \boxed{\cancel{\left(\frac{\dot a}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G_N}{3}\varepsilon}}$$

b) $T_{00} = \varepsilon(t),\ T_{0i} = 0,\ T_{ij} = a^2(t)P(t)\tilde{g}_{ij}$ $\quad R_{00} = -3\frac{\ddot a}{a},\ R_{ij} = (a\ddot a + 2k + 2\dot a^2)\tilde{g}_{ij}$

EINSTEIN: $R_{\mu\nu} = 8\pi G_N\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\lambda}{}_{\lambda}\right)$ $\quad T^{\lambda}{}_{\lambda} = g^{\lambda\mu}T_{\mu\lambda} = -\varepsilon(t) + 3P(t)$

$\mu=\nu=0$: $\boxed{-3\frac{\ddot a}{a}} = 8\pi G_N\left(\varepsilon + \frac{1}{2}(3P-\varepsilon)\right) = \boxed{4\pi G_N(3P+\varepsilon)}$ (1)

$\mu=i,\nu=j$: $(a\ddot a + 2k + 2\dot a^2)\tilde{g}_{ij} = 8\pi G_N\left(a^2P\tilde{g}_{ij} - \frac{1}{2}a^2\tilde{g}_{ij}(3P-\varepsilon)\right)$

$$\boxed{a\ddot a + 2k + 2\dot a^2 = a^2 4\pi G_N(\varepsilon - P)} \quad (2)$$

$3\cdot\frac{(2)}{a^2} + (1)$: $\frac{6k}{a^2} + \frac{6\dot a^2}{a^2} = 12\pi G_N(\varepsilon - P) + 4\pi G_N(3P+\varepsilon) = 16\pi G_N\varepsilon \quad /:6$

$$\boxed{\frac{k}{a^2} + \frac{\dot a^2}{a^2} = \frac{8\pi G_N}{3}\varepsilon}$$

ODVAJAMO ZADNJO EN. PO ČASU: $-2\frac{k}{a^3}\dot a + 2\frac{\dot a}{a}\left(\frac{\ddot a a - \dot a^2}{a^2}\right) = \frac{8\pi G_N}{3}\dot\varepsilon$

$$-\frac{2\dot a}{a}\left(\frac{k}{a^2} + \frac{\dot a^2}{a^2}\right) + \frac{2\dot a\ddot a}{a^2} = \frac{8\pi G_N}{3}\dot\varepsilon$$

$$-\frac{2\dot a}{a}\cdot\frac{8\pi G_N}{3}\varepsilon + \frac{2\dot a\ddot a}{a^2} = \frac{8\pi G_N}{3}\dot\varepsilon$$

$$-\frac{2\dot a}{a}\cdot\frac{8\pi G_N}{3}\varepsilon + 2\frac{\dot a}{a}\left(-\frac{4}{3}\pi G_N\right)(3P+\varepsilon) = \frac{8\pi G_N}{3}\dot\varepsilon$$

$$-\frac{2\dot a}{a}\varepsilon - \frac{\dot a}{a}(3P+\varepsilon) = \dot\varepsilon \Rightarrow \boxed{\dot\varepsilon = -\frac{3\dot a}{a}(\varepsilon + P)}$$

c) $\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^{\mu}_{\mu\lambda}T^{\lambda\nu} + \Gamma^{\nu}_{\mu\lambda}T^{\mu\lambda}$ $\quad T^{00} = \varepsilon(t),\ T^{0i} = 0,\ T^{ij} = \tilde{g}^{ij}\frac{P(t)}{a^2(t)}$

$$\nabla_\mu T^{\mu 0} = \partial_\mu T^{\mu 0} + \Gamma^{\mu}_{\mu\lambda}T^{\lambda 0} + \Gamma^{0}_{\mu\lambda}T^{\mu\lambda} = \partial_0 T^{00} + \frac{\dot a}{a}(3T^{00}) + \frac{a\dot a}{1-kr^2}T^{11} + a\dot a r^2 T^{22} + a\dot a r^2\sin^2\theta\, T^{33} =$$

$$= \dot\varepsilon + 3\frac{\dot a}{a}\varepsilon + \frac{\dot a}{a}(P+P+P) = 0 \Rightarrow \boxed{\dot\varepsilon = -\frac{3\dot a}{a}(\varepsilon + P)}$$

d) $P(\varepsilon) = w\varepsilon,\ k=0$: $\frac{\dot a^2}{a^2} = \frac{8\pi G_N}{3}\varepsilon,\ \dot\varepsilon = -\frac{3\dot a}{a}(1+w)\varepsilon \quad /^2$

$$\dot\varepsilon^2 = 9\left(\frac{8\pi G_N}{3}\right)\varepsilon^3(1+w)^2 \Rightarrow \dot\varepsilon = 8\pi G_N(1+w)\varepsilon^{3/2}$$

$$\frac{d\varepsilon}{\varepsilon^{3/2}} = 8\pi G_N(1+w)dt$$

$$-2\varepsilon^{-\frac{1}{2}} = 8\pi G_N(1+w)t + A \quad /^{-2}$$

$$\boxed{\varepsilon = \frac{4}{(8\pi G_N(1+w)t + A)^2}}$$

$\Rightarrow \frac{\dot a}{a} = \sqrt{\frac{8\pi G_N}{3}}\frac{2}{(8\pi G_N(1+w)t + A)}$

$\ln a = 2\sqrt{\frac{8\pi G_N}{3}}\int\frac{dt}{(8\pi G_N(1+w)t + A)} + \tilde{B}$

$\ln a = 2\sqrt{\frac{8\pi G_N}{3}}\frac{1}{8\pi G_N(1+w)}\ln[8\pi G_N(1+w)t + A] + \tilde{B}$

$\boxed{a(t)} = B\cdot(8\pi G_N(1+w)t + A)^{2/(\sqrt{3\cdot 8\pi G_N}(1+w))} = \boxed{B\,(8\pi G_N(1+w)t + A)^{\frac{2}{\sqrt{24\pi G_N}(w+1)}}}$

PARTICLE HORIZON: $\boxed{d_{MAX}(t)} = a(t)\int_0^t\frac{dt'}{a(t')} = (8\pi G_N(1+w)t + A)^{2/(\sqrt{24\pi G_N}(w+1))}\int_0^t dt'\,(8\pi G_N(1+w)t' + A)^{-2/(\sqrt{24\pi G_N}(1+w))} =$

$$= \frac{(8\pi G_N(1+w)t + A)^D}{8\pi G_N(1+w)}\cdot\frac{1}{-D+1}\left((8\pi G_N(1+w)t + A)^{-D+1} - A^{-D+1}\right) =$$

$$\boxed{D := \frac{2}{\sqrt{24\pi G_N}(1+w)}}$$

$$= \boxed{\frac{8\pi G_N(1+w)t + A}{8\pi G_N(1+w)(1-D)} - \frac{(8\pi G_N(1+w)t + A)^D A^{-D+1}}{8\pi G_N(1+w)(1-D)}}$$

d) $p = w\varepsilon,\ K=0:\quad \frac{\dot a^2}{a^2} = \frac{8\pi G_N}{3}\varepsilon,\quad \dot\varepsilon = -\frac{3\dot a}{a}(1+w)\varepsilon$ $\leftarrow w \neq -1$

$$\Rightarrow \dot\varepsilon^2 = 9\cdot\frac{8\pi G_N}{3}\varepsilon^3(1+w)^2 \Rightarrow \dot\varepsilon = \sqrt{24\pi G_N}(1+w)\varepsilon^{3/2} \Rightarrow \frac{d\varepsilon}{\varepsilon^{3/2}} = \sqrt{24\pi G_N}(1+w)\,dt$$

$$\Rightarrow -2\varepsilon^{-1/2} = \sqrt{24\pi G_N}(1+w)t + A \Rightarrow \boxed{\varepsilon(t) = \frac{4}{(\sqrt{24\pi G_N}(1+w)t+A)^2}}$$

$$\frac{\dot a}{a} = \sqrt{\frac{8\pi G_N}{3}}\cdot\frac{2}{(\sqrt{24\pi G_N}(1+w)t+A)} \Rightarrow \ln a = 2\sqrt{\frac{8\pi G_N}{3}}\int dt'\,\frac{1}{(\sqrt{24\pi G_N}(1+w)t+A)} + \tilde B$$

$$\Rightarrow \ln a = \frac{2}{3(1+w)}\ln(\sqrt{24\pi G_N}(1+w)t+A) + \tilde B \Rightarrow \boxed{a(t) = B\cdot(\sqrt{24\pi G_N}(1+w)t+A)^{\frac{2}{3(1+w)}}}$$

PARTICLE HORIZON:

EVENT HORIZON

(2) a) $\varepsilon = p = 0:\quad \left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} = 0 \Rightarrow \dot a^2 = -k$

$\boxed{\begin{array}{l} k=1 \text{ NI REŠITVE} \\ k=0\quad a(t)=a_0 \\ k=-1\quad a(t)=a(0)+t \end{array}}$ $\Rightarrow$ $k=0:\ a(0)=1 \to \boxed{a(t)=a_0}=a_0$ (STATIČNO VESOLJE)

$k=-1:\ a(t) = a(t=0)+t$

$a(0)=0$ (a KONKAVNA)

$\Rightarrow \boxed{a(t)=t}$

b) $z = \frac{a(t_1)}{a(t_2)} - 1\quad \boxed{k=0:\ z=0}\quad \boxed{k=-1:}\ \boxed{z} = \frac{a(0)+t_1}{a(0)+t_2} - 1 = \boxed{\frac{t_1-t_2}{a(0)+t_2}} = \boxed{\frac{t_1-t_2}{t_2}}$

c) $ds^2=0,\ d\theta=d\varphi=0 \Rightarrow dt^2 = a^2(t)\frac{dr^2}{1-kr^2} \Rightarrow dt = a(t)\frac{dr}{\sqrt{1-kr^2}}$

$$\int_{t_1}^{t_2}\frac{dt}{a(t)} = \int_0^{r(t_1,t_2)}\frac{dr}{\sqrt{1-kr^2}}$$

$\mathrm{sh}(\ln x) = \frac{e^{\ln x} - e^{-\ln x}}{2} = \frac{x - 1/x}{2} = \frac{x^2-1}{2x}$

$k=0:\quad \boxed{\frac{t_2-t_1}{a_0} = r(t_1,t_2)}$

$$k=-1\quad \int_{t_1}^{t_2}\frac{dt}{a(0)+t} = \int_0^{r(t_1,t_2)}\frac{dr}{\sqrt{1+r^2}} \Rightarrow \ln\frac{a(0)+t_2}{a(0)+t_1} = \mathrm{arsinh}(r(t_1,t_2))$$

$$\Rightarrow \boxed{r(t_1,t_2)} = \sinh\left(\ln\frac{a(0)+t_2}{a(0)+t_1}\right) = \boxed{\frac{\left(\frac{a_0+t_2}{a_0+t_1}\right)^2-1}{2\left(\frac{a_0+t_2}{a_0+t_1}\right)}} =$$

$$a_0=0\quad \frac{(t_2/t_1)^2-1}{2t_2/t_1} = \boxed{\frac{t_2^2-t_1^2}{2t_2t_1}}$$

② c) (NADALJEVANJE)

$k=0$: $t(\lambda) = t_1 + \lambda$, $r(\lambda) = \lambda / a_0$

$$\boxed{\Delta S} = \int_0^{t_2 - t_1} \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda = \int_0^{t_2 - t_1} \sqrt{\underbrace{g_{00}}_{-1} \cdot 1 + \underbrace{\underbrace{g_{11}}_{a_0^2} \cdot \frac{1}{a_0^2}}_{1}}\, d\lambda = \boxed{0}$$

$g_{11} = a_0^2$

$k=-1$: $t(\lambda) = \lambda$, $r(\lambda) = \frac{\lambda^2 - t_1^2}{2\lambda t_1}$

$$\boxed{\Delta S} = \int_{t_1}^{t_2} \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda = \int_{t_1}^{t_2} \sqrt{-1 + \underbrace{g_{rr}\, r'(\lambda)^2}_{4}}\, d\lambda = \boxed{\sqrt{3}\,(t_2 - t_1)}$$

$$\boxed{r'(\lambda)} = \frac{(2\lambda)^2 t_1 - 2t_1(\lambda^2 - t_1^2)}{4\lambda^2 t_1^2} = \frac{1}{t_1} - \frac{\lambda^2 - t_1^2}{2\lambda^2 t_1} = \frac{1}{t_1} - \frac{1}{2t_1} + \frac{t_1}{2\lambda^2} = \boxed{\frac{1}{2t_1} + \frac{t_1}{2\lambda^2}}$$

$$\boxed{g_{11}} = \lambda^2 \frac{1}{1 + \frac{(\lambda^2 - t_1^2)^2}{4\lambda^2 t_1^2}} = \frac{\lambda^2}{1 + \frac{\lambda^4 - 2\lambda^2 t_1^2 + t_1^4}{4\lambda^2 t_1^2}} = \boxed{\frac{\lambda^2\, 4\lambda^2 t_1^2}{\lambda^4 + 2\lambda^2 t_1^2 + t_1^4}}$$

$$\boxed{g_{11} \cdot r'(\lambda)^2} = \frac{\lambda^2\, 4\lambda^2 t_1^2}{\lambda^4 + 2\lambda^2 t_1^2 + t_1^4} \cdot \left( \frac{2\lambda^2 + 2t_1^2}{2\lambda^2 \cdot 2t_1} \right)^2 = \frac{(2\lambda^2 + 2t_1^2)^2}{(\lambda^2 + t_1^2)} = \boxed{4}$$

d) ~~GRAVITACIJA JE POSLEDICA NETRIVIALNE METRIKE (GIBAMO SE PO GEODETKAH) ČEPRAV JE RES, DA NIMAMO SNOVI (ALI RES $T_{\mu\nu}=0$) SMO V NALOGI PREDPOSTAVILI NEKO NETRIVIALNO METRIKO, GEODETKE TAKO NISO VEČ "RAVNE PREMICE", KAR RAČUNAMO KOT GRAVITACIJO.~~

REZULTATI SO POSLEDICA NETRIVIALNE METRIKE. ČEPRAV NIMAMO SNOVI, NISO VSE REŠITVE EINSTEINOVIH ENAČB NETRIVIALNE ($R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 0$). METRIKA Z $k=-1$, KI SMO JO IZBRALI JE TAKA.

$$\boxed{k=0}$$

$$r(1) = \qquad \Delta s = \int_1^{\lambda_2} \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda$$

$$t(1) = \lambda$$

$$a_0 = 0 \qquad r(1) =$$

③ $\Omega_R = \Omega_k = 0$

$$t(z) = \frac{1}{H_0} \int_0^{1/1+z} \frac{dx}{x\sqrt{\Omega_\Lambda + \Omega_M/x^3}} = \frac{1}{H_0} \int_0^{(1+z)^{-1}} \frac{\sqrt{x}\, dx}{\sqrt{\Omega_\Lambda x^3 + \Omega_M}} \overset{u=\sqrt{x}}{=}$$

$$= \frac{1}{H_0} \cdot 2 \cdot \int_0^{(1+z)^{-1/2}} \frac{u^2\, du}{\sqrt{\Omega_\Lambda u^6 + \Omega_M}} \overset{v=u^3}{=} \frac{1}{H_0} \frac{2}{3} \int_0^{(1+z)^{-3/2}} \frac{dv}{\sqrt{\Omega_\Lambda v^2 + \Omega_M}} = \frac{2}{3H_0} \frac{1}{\sqrt{\Omega_M}} \int_0^{(1+z)^{-3/2}} \frac{dv}{\sqrt{1 + \frac{\Omega_\Lambda}{\Omega_M} v^2}} \overset{w = \sqrt{\frac{\Omega_\Lambda}{\Omega_M}}\, v}{=}$$

$$= \frac{2}{3H_0} \cdot \frac{1}{\sqrt{\Omega_\Lambda}} \int_0^{(1+z)^{-3/2}\sqrt{\frac{\Omega_\Lambda}{\Omega_M}}} \frac{dw}{\sqrt{1+w^2}} = \frac{2}{3H_0} \cdot \frac{1}{\sqrt{\Omega_\Lambda}} \operatorname{arcsinh}\left(\sqrt{\frac{\Omega_\Lambda}{\Omega_M}} \left(\frac{a}{a_0}\right)^{3/2}\right) \qquad 1+z = \frac{a_0}{a}$$

$$\boxed{\frac{a(t)}{a_0} = \left(\sqrt{\frac{\Omega_M}{\Omega_\Lambda}}\, sh\left(\frac{3H_0}{2}\sqrt{\Omega_\Lambda}\, t\right)\right)^{2/3}}$$

NEPOMEMBNO, KER JE POD ULOMKOM

$$= 0$$

$$\Rightarrow th^2\left(\frac{3H_0}{2}\sqrt{\Omega_\Lambda}\, t^*\right) = \frac{1}{3}$$

$$a(t) = \text{KONST} \cdot sh\left(\frac{3H_0}{2}\sqrt{\Omega_\Lambda}\, t\right)^{2/3} \qquad \dot{a}(t) = \text{KONST} \cdot \frac{ch\left(\frac{3H_0}{2}\sqrt{\Omega_\Lambda}\, t\right)}{sh\left(\frac{3H_0}{2}\sqrt{\Omega_\Lambda}\, t\right)^{1/3}}$$

NI VAŽNA, KER BOMO ODVODE ENAČILI Z 0.

$$0 = \ddot{a}(t) = sh\left(\frac{3H_0}{2}\sqrt{\Omega_\Lambda}\, t^*\right)^{4/3} \frac{3H_0}{2}\sqrt{\Omega_\Lambda} - ch\left(\frac{3H_0}{2}\sqrt{\Omega_\Lambda}\, t^*\right)^2 \frac{1}{3} sh\left(\frac{3H_0}{2}\sqrt{\Omega_\Lambda}\, t^*\right)^{-2/3} \frac{3H_0}{2}\sqrt{\Omega_\Lambda}$$

PIŠEMO SAMO ŠTEVEC, BREZ KONSTANT

$$\Rightarrow sh^2\left(\frac{3H_0}{2}\sqrt{\Omega_\Lambda}\, t^*\right) = ch^2\left(\frac{3H_0}{2}\sqrt{\Omega_\Lambda}\, t^*\right) \cdot \frac{1}{3}$$

$$th^2\left(\frac{3H_0}{2}\sqrt{\Omega_\Lambda}\, t^*\right) = \frac{1}{3} \qquad \text{DEF: } B \text{ z } th^2(B) = \frac{1}{3} \qquad \Rightarrow \boxed{t^* = \frac{2}{3H_0} \frac{B}{\sqrt{\Omega_\Lambda}}}$$

$$\Rightarrow \boxed{a_0 \left(\sqrt{\frac{\Omega_M}{\Omega_\Lambda}}\, sh(B)\right)^{2/3} = a(t^*)}$$

$$\Rightarrow \boxed{z(t^*) = \left[\sqrt{\frac{\Omega_M}{\Omega_\Lambda}}\, sh(B)\right]^{-2/3} - 1}$$

~~b) STAROST VESOLJA~~

$$\boxed{t(0)} = \frac{2}{3H_0} \frac{1}{\sqrt{\Omega_\Lambda}} \operatorname{arcsinh}\left(\sqrt{\frac{\Omega_\Lambda}{\Omega_M}}\right) = \frac{2}{3} \cdot \sqrt{\frac{4}{3}} \operatorname{arcsinh}(\sqrt{3}) \cdot 14.4 \cdot 10^9 \text{ yrs} = \boxed{14.6 \cdot 10^9 \text{ yrs}}$$

V TEM MODELU SE VESOLJE ZAČNE ŠIRITI TAKOJ, t(0) TOREJ PREDSTAVLJA ČAS OD ZAČETKA ŠIRJENJA, KER JE TA DALJŠI OD STAROSTI VESOLJA, KI JE PODANA, SE JE VESOLJE ZAČELO ŠIRITI TAKOJ PO NASTANKU.

ČE BI NAPR. VZELI $\Omega_M = 0.75$, $\Omega_\Lambda = 0.25$, BI DOBILI $t(0) = 13.5 \cdot 10^9$ yrs, IN VESOLJE BI SE ZAČELO ŠIRITI

PO

(4) a) $\Omega_K = \Omega_\Lambda = 0 \quad d\eta = \frac{dt}{a(t)}$

$\Downarrow$

$k=0$

$$\frac{da}{dt} = \frac{da}{d\eta}\frac{d\eta}{dt} = \frac{da}{d\eta}\frac{1}{a(t)}$$

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G_N}{3}\varepsilon$$

$$\Rightarrow \boxed{\left(\frac{da}{d\eta}\right)^2 = \frac{8\pi G_N}{3}\varepsilon a^4} \quad (1)$$

$$\dot\varepsilon = -\frac{3\dot a}{a}(\varepsilon+P) \Rightarrow \boxed{\frac{d\varepsilon}{d\eta} = -\frac{3}{a}\frac{da}{d\eta}(\varepsilon+P)} \quad (2)$$

$$\frac{d}{d\eta}(1):\ 2\frac{da}{d\eta}\frac{d^2a}{d\eta^2} = \frac{8\pi G_N}{3}\left(\frac{d\varepsilon}{d\eta}a^4 + \varepsilon\, 4a^3\frac{da}{d\eta}\right)$$

$$\Rightarrow 2\frac{d^2a}{d\eta^2} = \frac{8\pi G_N}{3}\left(-3a^3(\varepsilon+P) + 4a^3\varepsilon\right) \Rightarrow \boxed{\frac{d^2a}{d\eta^2} = \frac{4\pi G_N}{3}(\varepsilon-3P)a^3}$$

b) $\varepsilon(\eta) = \varepsilon_M(\eta) + \varepsilon_R(\eta)$, $\varepsilon_M(\eta) = \frac{\varepsilon_{eq}}{2}\left(\frac{a_{eq}}{a(\eta)}\right)^3$, $\varepsilon_R(\eta) = \frac{\varepsilon_{EQ}}{2}\left(\frac{a_{EQ}}{a(\eta)}\right)^4$

$f' = \frac{df}{d\eta}$ $\Rightarrow$ $a'' = \frac{4\pi G_N}{3}a^3(\varepsilon-3P)$ $\quad \cancel{P = -\frac{a}{3}\frac{\varepsilon'}{a'} - \varepsilon}$

$$\varepsilon' = -\frac{3}{a}a'(\varepsilon+P) \Rightarrow \boxed{P = -\frac{a}{3}\frac{\varepsilon'}{a'} - \varepsilon}$$

$$\Rightarrow a'' = \frac{4\pi G_N}{3}a^3\left(4\varepsilon + \frac{a}{a'}\varepsilon'\right)$$

$$\varepsilon' = \varepsilon_M' + \varepsilon_R' = \frac{\varepsilon_{EQ}}{2}\left(a_{eq}^3(-3)\frac{1}{a^4}a' - 4a_{eq}^4\frac{1}{a^5}a'\right)$$

$$\Rightarrow a'' = \frac{4\pi G_N}{3}a^3\frac{\varepsilon_{EQ}}{2}\left(4\frac{a_{EQ}^3}{a^3} + \cancel{4\frac{a_{EQ}^4}{a^4}} - 3\frac{a_{EQ}^3}{a^3} - \cancel{4\frac{a_{EQ}^4}{a^4}}\right)$$

$$\Rightarrow a'' = \frac{4\pi G_N\varepsilon_{EQ}}{6}a_{EQ}^3$$

$$a' = \frac{2\pi G_N\varepsilon_{EQ}a_{EQ}^3}{3}\eta + A \Rightarrow a = \frac{\pi G_N\varepsilon_{EQ}a_{EQ}^3}{3}\eta^2 + A\eta + B$$

$a(\eta=0) = 0 \Rightarrow B = 0$

$$\Rightarrow \boxed{a(\eta) = \frac{\pi G_N\varepsilon_{EQ}a_{EQ}^3}{3}\eta^2 + A\eta} = \cancel{a_{EQ}\left[C_1\left(\frac{\eta}{\eta_*}\right)^2 + C_2\left(\frac{\eta}{\eta_*}\right)\right]}$$

$$\cancel{\Rightarrow \frac{C_1}{\eta_*^2} = \frac{a_{EQ}^2\pi G_N\varepsilon_{EQ}}{3}, \quad \frac{C_2}{\eta_*} = \frac{A}{a_{EQ}}}$$

$\cancel{a(\eta_{EQ}) = a_{EQ}}$ $\Rightarrow$ $\cancel{A = \frac{a_{EQ}}{\eta_{EQ}} - \frac{\pi G_N\varepsilon_{EQ}a_{EQ}^3\eta_{EQ}}{3}}$

$$\cancel{\Rightarrow a(\eta) = a_{EQ}\left[\frac{\pi G_N\varepsilon_{EQ}a_{EQ}^2}{3}\eta^2 + \left(\frac{1}{\eta_{EQ}} - \frac{\pi G_N\varepsilon_{EQ}a_{EQ}^2\eta_{EQ}}{3}\right)\eta\right] = a_{EQ}\left[C_1\left(\frac{\eta}{\eta_*}\right)^2 + C_2\left(\frac{\eta}{\eta_*}\right)\right]}$$

$$\cancel{a(\eta_{EQ}) = a_{EQ} \Rightarrow C_1\left(\frac{\eta_{EQ}}{\eta_*}\right)^2 + C_2\left(\frac{\eta_{EQ}}{\eta_*}\right) = 1}$$

~~Vstavimo v (1):~~

$$\cancel{\left(\frac{2\pi G_N\varepsilon_{EQ}a_{EQ}^3}{3}\eta + A\right)^2 = \frac{8\pi G_N}{3}\frac{\varepsilon_{EQ}}{2}\left(\frac{a_{EQ}^3}{a^3} + \frac{a_{EQ}^4}{a^4}\right)\left(\frac{\pi G_N\varepsilon_{EQ}a_{EQ}^3}{3}\eta^2 + A\eta\right)^4}$$

4 b) (NADALJEVANJE) $a(\eta) = \frac{\pi G_N \varepsilon_{EQ} a_{EQ}^3}{3}\eta^2 + A\eta \qquad \varepsilon = \frac{\varepsilon_Q}{2}\left(\frac{a_{EQ}^3}{a^3} + \frac{a_{EQ}^4}{a^4}\right)$

VSTAVIMO V $a'^2 = \frac{8\pi G_N}{3}\varepsilon a^4$:

$$\Rightarrow \left(\frac{2\pi G_N \varepsilon_{EQ} a_{EQ}^3}{3}\eta + A\right)^2 = \frac{4\pi G_N \varepsilon_{EQ}}{3}\left(a_{EQ}^3 a + a_{EQ}^4\right)$$

$$\Rightarrow \frac{4\pi^2 G_N^2 \varepsilon_{EQ}^2 a_{EQ}^6}{9}\eta^2 + \frac{4\pi G_N \varepsilon_{EQ} a_{EQ}^3 A\eta}{3} + A^2 = \frac{4\pi G_N \varepsilon_{EQ}}{3}\left(a_{EQ}^4 + \frac{\pi G_N \varepsilon_{EQ} a_{EQ}^6}{3}\eta^2 + A\eta a_{EQ}^3\right)$$

$$\Rightarrow A^2 = \frac{4\pi G_N \varepsilon_{EQ} a_{EQ}^4}{3} \Rightarrow \boxed{A = \sqrt{\frac{4\pi G_N \varepsilon_{EQ} a_{EQ}^4}{3}}}$$

$$\Rightarrow a(\eta) = a_{EQ}\left[\frac{\pi G_N \varepsilon_{EQ} a_{EQ}^2}{3}\eta^2 + 2a_{EQ}\sqrt{\frac{\pi G_N \varepsilon_{EQ}}{3}}\eta\right] = \boxed{a_{EQ}\left(C_1\left(\frac{\eta}{\eta_*}\right)^2 + C_2\left(\frac{\eta}{\eta_*}\right)\right)}$$

$$\boxed{\eta_* = \sqrt{\frac{3}{\pi G_N \varepsilon_{EQ} a_{EQ}^2}}} \quad \boxed{C_1 = 1} \quad \boxed{C_2 = 2}$$

$$a(\eta_{EQ}) = a_{EQ} \Rightarrow C_1\left(\frac{\eta_{EQ}}{\eta_*}\right)^2 + C_2\left(\frac{\eta_{EQ}}{\eta_*}\right) = 1 \quad / \cdot \eta_*^2 \Rightarrow \eta_{EQ}^2 + 2\eta_{EQ}\eta_* = \eta_*^2$$

$$\Rightarrow \eta_*^2 - 2\eta_{EQ}\eta_* - \eta_{EQ}^2 = 0 \Rightarrow \boxed{\eta_*} = \frac{2\eta_{EQ} \pm \sqrt{4\eta_{EQ}^2 + 4\eta_{EQ}^2}}{2} =$$

$$= \eta_{EQ} \pm \sqrt{2}\,\eta_{EQ} \underset{\eta_* > 0}{=} \boxed{\eta_{EQ}(1+\sqrt{2})}$$

c) MATTER DOM. $w = 0 \Rightarrow$ 1 d) NALOGA $a(t) = B\left(\sqrt{24\pi G_N}\, t\right)^{2/3} = \text{CONST.} \cdot t^{2/3} = D \cdot t^{2/3}$

$$d\eta = \frac{dt}{a(t)} \quad / \int \quad \Rightarrow \eta = \int \frac{dt}{D t^{2/3}} + E \Rightarrow \eta = \frac{1}{D}\cdot 3t^{1/3} + \cancel{E}^{\,0}$$

$$\Rightarrow \boxed{a(\eta)} = D\left[\left(\frac{D}{3}\eta\right)^3\right]^{2/3} \boxed{\propto \eta^2}$$

RADIATION DOM: $w = 1/3 \Rightarrow a(t) = B\left(\sqrt{24\pi G_N}\,\frac{4}{3}t\right)^{1/2} = D\,t^{1/2}$

$$\Rightarrow \eta = \int \frac{dt}{D t^{1/2}} + E = \frac{1}{D}\cdot 2\cdot t^{1/2} + \cancel{E}^{\,0}$$

$$\Rightarrow \boxed{a(\eta)} = D\left[\left(\frac{D}{2}\eta\right)^2\right]^{1/2} \boxed{\propto \eta}$$

IZ b): $\boxed{a(\eta) = a_{EQ}\left[\frac{1}{(1+\sqrt{2})^2}\left(\frac{\eta}{\eta_{EQ}}\right)^2 + \frac{2}{1+\sqrt{2}}\left(\frac{\eta}{\eta_{EQ}}\right)\right]}$

$\eta \ll \eta_{EQ}$: $a(\eta) = \frac{2a_{EQ}}{1+\sqrt{2}}\frac{\eta}{\eta_{EQ}} \propto \eta$ RADIATION DOMINATED ✓

$\eta \gg \eta_{EQ}$: $a(\eta) = \frac{a_{EQ}}{(1+\sqrt{2})^2}\left(\frac{\eta}{\eta_{EQ}}\right)^2 \propto \eta^2$ MATTER DOMINATED ✓

① d) (NADALJEVANJE) $\quad a(0)=0 \Rightarrow A=0 \Rightarrow \boxed{a(t)=B\left(\sqrt{24\pi G_N}\,(1+w)t\right)^{\frac{2}{3(1+w)}}}$

$\boxed{\varepsilon(t)=\frac{4}{24\pi G_N(1+w)^2 t^2}}$ $\quad \Big\}\ w\neq -1$

PARTICLE HORIZON: $\boxed{d_{MAX}(t)} = a(t)\int_0^t \frac{dt'}{a(t')} = t^{\frac{2}{3(1+w)}}\int_0^t dt'\,(t')^{-\frac{2}{3(1+w)}} = \frac{t^{\frac{2}{3(1+w)}}}{-\frac{2}{3(1+w)}+1}\,(t')^{-\frac{2}{3(1+w)}+1}\Big|_0^t =$

$$= \frac{t^{\frac{2}{3(1+w)}}}{\frac{1+3w}{3(1+w)}}\left(t^{\frac{1+3w}{3(1+w)}} - \lim_{t'\to 0} t'^{\frac{1+3w}{3(1+w)}}\right) = \boxed{\frac{3(1+w)}{1+3w}\,t}$$

~~$w\neq -1$~~ ~~$w\neq -\frac{1}{3}$~~

$w > -\frac{1}{3}$ ALI $w<-1$

$w \downarrow -\frac{1}{3}$

ČE $w=-\frac{1}{3}$: $\boxed{d_{MAX}(t)=\infty}$

ČE $-1 \leq w \leq -\frac{1}{3}$ $\boxed{d_{MAX}(t)=+\infty}$ $\leftarrow$ ~~TAKO~~ NISO FIZIKALNI ~~PARTICLE HORIZON~~ SLABO DEFINIRAN

EVENT HORIZON: $\boxed{\tilde{d}_{MAX}(t)} = a(t)\int_t^\infty \frac{dt'}{a(t')} = \frac{t^{\frac{2}{3(1+w)}}}{\frac{1+3w}{3(1+w)}}\left(\lim_{t'\to\infty} t'^{\frac{1+3w}{3(1+w)}} - t^{\frac{1+3w}{3(1+w)}}\right) =$

$-1<w<-\frac{1}{3}$: $\tilde{d}_{MAX}(t) = \frac{3(1+w)}{1+3w}(-t) = \boxed{-\frac{3(1+w)}{1+3w}t}$

$= \longrightarrow$ $w \downarrow -\frac{1}{3}$ $\tilde{d}_{MAX}(t) = \boxed{+\infty}$

$\longrightarrow$ SICER: $\tilde{d}_{MAX}(t) = \boxed{\infty}$

$w=-1$: $\quad \dot{\varepsilon}=0 \Rightarrow \varepsilon=\varepsilon_0 \quad \frac{\dot{a}}{a}=\sqrt{\frac{8\pi G_N}{3}\varepsilon_0} \Rightarrow \ln a = \sqrt{\frac{8\pi G_N}{3}\varepsilon_0}\,t + \tilde{C}$

$\Rightarrow \boxed{a = C\exp\left(\sqrt{\frac{8\pi G_N}{3}\varepsilon_0}\,t\right)} = \boxed{a(0)\exp\left(\sqrt{\frac{8\pi G_N\varepsilon_0}{3}}\,t\right)}$

$\boxed{d_{MAX}(t)} = \exp\left(\sqrt{\frac{8\pi G_N\varepsilon_0}{3}}\,t\right)\int_0^t dt'\exp\left(-\sqrt{\frac{8\pi G_N\varepsilon_0}{3}}\,t'\right) =$

$$= \boxed{-\sqrt{\frac{3}{8\pi G_N\varepsilon_0}}\left(1-\exp\left(\sqrt{\frac{8\pi G_N\varepsilon_0}{3}}\,t\right)\right)}$$

$$\boxed{\tilde{d}_{MAX}(t) = +\sqrt{\frac{3}{8\pi G_N\varepsilon_0}}}$$

③ b) STAROST VESOLJA: $t(0) = \frac{2}{3H_0}\frac{1}{\sqrt{\Omega_\Lambda}}\,\mathrm{arcsinh}\left(\sqrt{\frac{\Omega_\Lambda}{\Omega_M}}\cdot 1\right) = 13.7\cdot 10^9$ yrs

$\Omega_M=\frac{1}{4},\ \Omega_\Lambda=\frac{3}{4}$ $\qquad \Rightarrow \boxed{H_0 = 7.4\cdot 10^{-11}\ \text{yrs}^{-1}}$

IMAMO $a(t)$. VELIKI POK: $t=0$ $\quad$ POSPEŠEVATI ZAČNE PRI $\ddot{a}(t^*)=0$.

TO PA SMO SE IZRAČUNALI: $\boxed{t^*} = \frac{2}{3H_0}\frac{B}{\sqrt{\Omega_\Lambda}} = \boxed{6.8\cdot 10^9\ \text{yrs}}$

$\mathrm{th}^2(B) = \frac{1}{3} \Rightarrow B = 0.658$