Cosmology: Problem Set 3
Inflation and cosmological perturbations

This problem set focuses on the theory of inflation and cosmological perturbations; Sections 3 and 5 of the course. The relevant materials are in Weinberg's book in Chapters 4, 5 and 10. Solutions to this Problem Set are due on Sunday 2.6.2019. The total number of points is 100 (each sub-question is worth 10 points).

The exponential model of inflation (100 points)

In this exercise, you will work our various details of the inflationary theory with the exponential potential discussed in lectures.

(1.) Let us begin by studying a single scalar field model of inflation coupled to a dynamical gravitational field with the metric $g_{\mu\nu}$. The action of the scalar field is

$$S_\varphi = -\int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + V(\varphi)\right], \tag{1}$$

where $V(\varphi)$ is the (for now unspecified) scalar potential. First, derive the equation of motion for the inflaton φ.

(2.) Next, derive the energy-momentum tensor $T^{\mu\nu}$ for the scalar field and write it in the form of an ideal fluid:

$$T^{\mu\nu} = (\varepsilon + P)\,u^\mu u^\nu + Pg^{\mu\nu}, \tag{2}$$

where ε is the energy density, P the pressure and u^μ is the fluid velocity field normalised as $u^\mu u_\mu = -1$. Show that

$$\varepsilon = -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + V(\varphi), \tag{3}$$

$$P = -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi), \tag{4}$$

$$u^\mu = -\left(-\partial_\sigma\varphi\partial^\sigma\varphi\right)^{-1/2}\partial^\mu\varphi. \tag{5}$$

(3.) Assume that the solution for φ is spatially isotropic and homogeneous: $\varphi(t)$ depends only on time. Argue why the solution of Einstein's equations for $g_{\mu\nu}$ (with zero cosmological constant), $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_N T_{\mu\nu}$, can be written in the form of the FRW metric.

(4.) Using the fact that $g_{\mu\nu}$ is of the FRW type, show that the only two independent equations of motion in the system are the Friedmann equation and the scalar field equation:

$$H = \sqrt{\frac{8\pi G_N}{3}\left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right)}, \tag{6}$$

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0, \tag{7}$$

where $H(t) \equiv \dot{a}(t)/a(t)$ and $\dot{a}$ denotes a derivative of $a(t)$ with respect to time. The prime $'$ in $V'(\varphi)$ denotes a derivative of V with respect to the field φ. You need to show explicitly that the conservation of energy and momentum is ensured by equations (6) and (7).

(5.) By assuming that

$$|\dot{H}| \ll H^2, \tag{8}$$

$$|\ddot{\varphi}| \ll H|\dot{\varphi}|, \tag{9}$$

derive the two slow-roll flatness conditions on the potential

$$\left|\frac{V'(\varphi)}{V(\varphi)}\right| \ll \sqrt{16\pi G_N}, \tag{10}$$

$$\left|\frac{V''(\varphi)}{V(\varphi)}\right| \ll 24\pi G_N. \tag{11}$$

(6.) Assume the exponential potential

$$V(\varphi) = g e^{-\lambda\varphi}, \tag{12}$$

where g and λ are arbitrary constants. Show that

$$\varphi(t) = \frac{1}{\lambda}\ln\left(\frac{8\pi G_N g \epsilon^2 t^2}{3-\epsilon}\right), \tag{13}$$

where $\epsilon \equiv \lambda^2/(16\pi G_N)$ (and not the energy density ε) satisfies the equations of motion (6) and (7) and compute the scale factor $a(t)$ and $H(t)$. What constraints do the slow-roll conditions (10) and (11) impose on g and λ (or ϵ)?

(7.) Read Chapter 10.1 of Weinberg's book and write a summary of the steps involved in the derivation of equations (10.1.15), (10.1.16) and (10.1.17) for dynamical scalar field perturbations Ψ_q and $\delta\varphi_q$.

(8.) Discuss the importance of the quantity

$$\mathcal{R}_q \equiv -\Psi_q + H\delta\varphi_q/\dot{\bar{\varphi}} \tag{14}$$

and what it means for it be conserved outside the horizon (for $q/a \ll H$) during inflation. Note that $\bar{\varphi}$ denotes the unperturbed value of φ.

(9.) By using the exponential model of inflation with $V(\varphi)$ from Eq. (12), show that the Mukhanov-Sasaki equation (equation (10.1.35) in Weinberg)

$$\frac{d^2\mathcal{R}_q}{d\tau^2} + \frac{2}{z}\frac{dz}{d\tau}\frac{d\mathcal{R}_q}{d\tau} + q^2\mathcal{R}_q = 0, \tag{15}$$

where τ is the conformal time (note that we usually called it η)

$$\tau \equiv \int_{t_*}^{t} \frac{dt'}{a(t')}, \tag{16}$$

with t_* an arbitrary time, which we take to be $t_* = \infty$, and

$$z \equiv \frac{a\dot{\bar{\varphi}}}{H}, \tag{17}$$

takes the form

$$\frac{d^2\mathcal{R}_q}{d\tau^2} - \frac{2}{(1-\epsilon)\tau}\frac{d\mathcal{R}_q}{d\tau} + q^2\mathcal{R}_q = 0. \tag{18}$$

Solve this differential equation and discuss the behaviour of the two solutions at early time ($t \approx 0$). Note that you first need to figure out what regime of the conformal time τ corresponds to the limit of $t \to 0$.

(10.) Using the initial condition (see equation (10.1.38) of Weinberg)

$$\lim_{t\to 0} \mathcal{R}_q(t) \to -\frac{H(t)}{(2\pi)^{3/2}\sqrt{2q}\,a(t)\dot{\bar{\varphi}}(t)} e^{-iq\tau}, \tag{19}$$

explicitly derive the expression for $\mathcal{R}_q$ outside the horizon.