LIEJEVE GRUPE DN

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1. $sl(2,\mathbb{R}) = \{A \in \mathbb{R}^{2\times 2} \mid tr(A) = 0\}$, ZA BAZO $sl(2,\mathbb{R})$ VZAMEMO $E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

E^L, F^L, H^L LEVO INV. POLJA Z $(E^L)_I = E$, $(F^L)_I = F$, $(H^L)_I = H$

IZRAČUNAJ PREDPISE TEH POLJ, TOKOVE IN KOMUTATORJE.

KER JE $SL(2,\mathbb{R})$ MATRIČNA LIEJEVA GRUPA, LEVO INV. POLJE X^L DOBIMO ~~ZAMAKN~~ Z LEVIMI TRANSLACIJAMI VEKTORJA $(X^L)_I$ TOREJ:

$$\boxed{(E^L)_X = XE, \quad (F^L)_X = XF, \quad (H^L)_X = XH} \qquad X \in SL(2,\mathbb{R})$$

TOK NEKEGA LEVO INV. POLJA X^L JE $\phi_t^{X^L}(X) = R_{\phi_t^{X^L}(e)}(X)$, ZA MATRIČNO GRUPO PA JE $\phi_t^{X^L}(e) = e^{tX} = \sum_{k=0}^{\infty} \frac{(tX)^k}{k!}$, $X = (X^L)_I$

TOREJ
$$\boxed{\begin{aligned} \phi_t^{E^L}(X) &= Xe^{tE} = X\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \\ \phi_t^{F^L}(X) &= Xe^{tF} = X\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \\ \phi_t^{H^L}(X) &= Xe^{tH} = X\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \end{aligned}}$$
$\leftarrow$ MATRIČNO MNOŽENJE

KOMUTATORJI SO DANI Z KOMUTATORJEM MATRIK V IDENTITETI

$$\boxed{\begin{aligned} [E^L, F^L] &= [E,F]^L = (EF - FE)^L = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)^L = H^L \\ [E^L, H^L] &= [E,H]^L = (EH - HE)^L = \left(\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)^L = -2E^L \\ [F^L, H^L] &= [F,H]^L = (FH - HF)^L = \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}\right) = 2F^L \end{aligned}}$$

2. IMAMO LIEJEVI GRUPI $U(n) = \{Q \in \mathbb{C}^{n\times n} \mid Q^H Q = I\}$
$SU(n) = \{Q \in \mathbb{C}^{n\times n} \mid Q^H Q = I, \det Q = 1\}$

POKAŽI, DA STA $U(1) \times SU(n)$ IN $U(n)$ DIFEOMORFNI

DEF $\alpha : U(1) \times SU(n) \to U(n)$, $\alpha(e^{i\varphi}, A)$ = MATRIKA A, KJER JE PRVA VRSTICA POMNOŽENA Z $e^{i\varphi}$

OČITNO JE $\det(\alpha(e^{i\varphi}, A)) = e^{i\varphi}$

α JE SURJEKTIVNA: ZA $Q \in U(n)$ LAHKO IZPOSTAVIMO $\det(Q)$ IZ PRVE VRSTICE, KOT JE TO VIDNO V α^{-1}

α JE TUDI OČITNO INJEKTIVNA.

INVERZ $\alpha^{-1} : U(n) \to U(1) \times SU(n)$ $\quad \alpha^{-1}(Q) = (\det(Q),$ MATRIKA Q, KJER PRVO VRSTICO DELIMO Z $\det(Q))$

OČITNO STA α, α^{-1} GLADKI $\Rightarrow$ α DIFEOMORFIZEM $\square$

POKAŽI $U(n) \not\cong U(1) \times SU(n)$ KOT GRUPA AMPAK $U(n) \cong U(1) \ltimes SU(n)$

ČE JE GRUPA $U(n)$ SEM DIREKTEN PRODUKT PODGRUP N, K $U(n) = N \rtimes K$ $N \lhd U(n)$, $K < U(n)$

VELJA, DA $\exists$ HOMOMORFIZEM GRUP $\psi : U(n) \to K$, DA $\ker(\psi) = N$, $\psi|_K = id|_K$

ZA HOMOMORFIZEM ψ VZAMEMO DETERMINANTO $\det : U(n) \to U(1)$,

KER MORA BITI SLIKA $im(\psi) = K$ PODGRUPA $U(n)$, DEFINICIJO ψ-JA MALCE SPREMENIMO

$$\psi(A) := \begin{bmatrix} \det(A) & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}_{n \times n}$$

OČITNO JE TO ŠE VEDNO HOMOMORFIZEM

IN VELJA $K = Im\,\psi < U(n)$ IN $K \cong U(1)$

VIDIMO ŠE $A \in \ker(\psi) \Leftrightarrow \det(A) = 1 \Rightarrow \ker(\psi) = SU(n)$

VZAMIMO $Q \in K$ $Q = \begin{bmatrix} e^{i\varphi} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}_{n \times n}$ $\det(Q) = e^{i\varphi}$ TOREJ $\psi(Q) = \begin{bmatrix} e^{i\varphi} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}_{n \times n}$

OZIROMA $\psi|_K = id_K$. SLEDI TOREJ $U(n) \cong SU(n) \rtimes Im(\psi) \cong SU(n) \rtimes U(1)$ $\square$

2) $SE(3)$ GRUPA VSEH IZOMETRIJ $\mathbb{R}^3$, KI OHRANJAJO ORIENT.

$\tau \in SE(3)$ $\tau : \mathbb{R}^3 \to \mathbb{R}^3$ $\tau(\vec{x}) = R\vec{x} + \vec{a}$ $R \in SO(3)$, $\vec{a} \in \mathbb{R}^3$

IMAMO BIJEKCIJO $\phi : SO(3) \times \mathbb{R}^3 \to SE(3)$ $\phi(R, \vec{a})(\vec{x}) = R\vec{x} + \vec{a}$ $\Rightarrow SE(3)$ 6 DIM. LIE. GRUPA.

POIŠČI BAZO LIE. ALGEBRE $\mathfrak{se}(3)$

OPAZIMO, DA OBSTAJA VLOŽITEV $\rho : SO(3) \times \mathbb{R}^3 \to GL(4, \mathbb{R})$ $\rho(R, \vec{a}) = \begin{bmatrix} R & \vec{a} \\ 0 & 1 \end{bmatrix}$

ρ JE RES HOMOMORFIZEM GRUP

$$\rho(R_1, \vec{a}_1)\rho(R_2, \vec{a}_2) = \begin{bmatrix} R_1 & \vec{a}_1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} R_2 & \vec{a}_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1 R_2 & R_1\vec{a}_2 + \vec{a}_1 \\ 0 & 1 \end{bmatrix}$$

$$\rho(\tau_1 \circ \tau_2) = \begin{bmatrix} R_1 R_2 & R_1\vec{a}_2 + \vec{a}_1 \\ 0 & 1 \end{bmatrix}$$ JE ENAKO $\square$

$(\tau_1 \circ \tau_2)(\vec{x}) = R_1 \tau_2 \vec{x} + \vec{a}_1 = R_1 R_2 \vec{x} + R_1 \vec{a}_2 + \vec{a}_1$

PIŠIMO $\vec{a} = (x, y, z)^T$ TOREJ $\rho(R, \vec{a}) = \begin{bmatrix} R & \begin{matrix} x \\ y \\ z \end{matrix} \\ 0 & 1 \end{bmatrix}$

ELEMENTI ALGEBRE SO POTEM

$$\boxed{Q_x} = \frac{\partial}{\partial x}\left(\begin{bmatrix} R & \begin{matrix} x \\ y \\ z \end{matrix} \\ 0 & 1 \end{bmatrix}\right)\Bigg|_I = \boxed{\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}, \quad \boxed{Q_y} = \frac{\partial}{\partial y}\left(\begin{bmatrix} R & \begin{matrix} x \\ y \\ z \end{matrix} \\ 0 & 1 \end{bmatrix}\right)\Bigg|_I = \boxed{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}, \quad \boxed{Q_z} = \frac{\partial}{\partial z}\left(\begin{bmatrix} R & \begin{matrix} x \\ y \\ z \end{matrix} \\ 0 & 1 \end{bmatrix}\right)\Bigg|_I = \boxed{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$$

ZA ALGEBRO $so(3)$ ŽE VEMO, DA ALGEBRO SESTAVLJAJO MATRIKE IZ

$$\left\{\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}, \begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}\right\}$$

V ALGEBRI $se(3)$ BODO TOREJ TE VLOŽENE V 4X4 MATRIKE:

$$\boxed{L_x=\begin{pmatrix}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{pmatrix} \quad L_y=\begin{pmatrix}0&0&1&0\\0&0&0&0\\-1&0&0&0\\0&0&0&0\end{pmatrix} \quad L_z=\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}}$$

IZRAČUN KOMUTATORJEV:

KOMUTATORJE GENERATORJEV ~~ZA~~ ~~ALGEBRO~~ $so(3)$ ŽE POZNAMO:

$$\boxed{\begin{aligned}[L_x,L_y]&=L_z\\ [L_y,L_z]&=L_x\\ [L_z,L_x]&=L_y\end{aligned}}$$

~~ZA~~ ~~NAPREJ~~ PREPROST RAČUN KOMUTATORJA MATRIK DA ŠE

$$\boxed{[a_x,a_y]=[a_y,a_z]=[a_x,a_z]=0} \quad \text{IN} \quad \boxed{\begin{matrix}[L_x,a_x]=0 & [L_y,a_x]=-a_z & [L_z,a_x]=a_y\\ [L_x,a_y]=a_z & [L_y,a_y]=0 & [L_z,a_y]=-a_x\\ [L_x,a_z]=-a_y & [L_y,a_z]=a_x & [L_z,a_z]=0\end{matrix}}$$

POKAŽI, DA $[se(3),se(3)]=se(3)$

TRDITEV OČITNO VELJA, SAJ NA DESNI STRANI ENAČB NASTOPAJO VSI BAZNI VEKTORJI ALGEBRE $se(3)$ □

ALI JE LIE. ALGEBRA $se(3)$ ENOSTAVNA?

ODGOVOR JE NE, KER IMA NETRIVIALNI IDEAL $Lin\{\{a_x,a_y,a_z\}\}$.

4) $\sigma_1=\begin{bmatrix}0&1\\1&0\end{bmatrix}$, $\sigma_2=\begin{bmatrix}0&-i\\i&0\end{bmatrix}$, $\sigma_3=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ $\quad \vec{s}=(s_1,s_2,s_3)\in\mathbb{R}^3 \quad \sigma(\vec{s}):=s_1\sigma_1+s_2\sigma_2+s_3\sigma_3$

POKAŽI, DA ZA $\forall t\in\mathbb{R}$, $\forall \vec{s}\in\mathbb{R}^3$ $|\vec{s}|=1$ VELJA: $e^{it\sigma(\vec{s})}=(\cos t)I+i\sin t\,\sigma(\vec{s})$

$$e^{it\sigma(\vec{s})}=I+it\sigma(\vec{s})+\frac{1}{2}(it\sigma(\vec{s}))^2+\frac{1}{3!}(it\sigma(\vec{s}))^3+\dots$$

Z MNOŽENJEM MATRIK LAHKO PREVERIMO, DA VELJA $\boxed{\sigma_i^2=I}$ $i=1,2,3$ $\quad \boxed{\sigma_i\cdot\sigma_j=\sqrt{(-1)}\sum_{k=1}^{3}\varepsilon^{ijk}\sigma_k \quad i\neq j}$

~~$\sigma_i\sigma_j = \dots i\varepsilon\sigma_k$~~

SLEDI $\boxed{\vec{\sigma}(\vec{s})^2}=(s_1\sigma_1+s_2\sigma_2+s_3\sigma_3)^2=I(s_1^2+s_2^2+s_3^2)=1\cdot I=\boxed{I}$

OZIROMA

$\boxed{\sigma_i\cdot\sigma_j=-\sigma_j\cdot\sigma_i}$ $i\neq j$

TOREJ $e^{it\sigma(\vec{s})} = I + it\sigma(\vec{s}) + \frac{1}{2}(it)^2 I + \frac{1}{3!}(it)^3\sigma(\vec{s}) + \dots =$

$= \left(1 - \frac{1}{2}t^2 + \frac{1}{4!}t^4 - \dots\right)I + i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots\right)\sigma(\vec{s}) =$

$= \boxed{\cos t \cdot I + i\sin(t)\,\sigma(\vec{s})}$

$e^{it\sigma(\vec{s})} \in SU(2)$

$\sigma_i^h = \sigma_i \quad i = 1,2,3$

$\left(e^{it\sigma(\vec{s})}\right)^h = \left(\cos t\, I + i\sin t\,\sigma(\vec{s})\right)^h = \cos t\, I - i\sin t\,\sigma(\vec{s})$

TOREJ $\left(e^{it\sigma(\vec{s})}\right)^h e^{it\sigma(\vec{s})} = (\cos t\, I - i\sin t\,\sigma(\vec{s}))(\cos t\, I + i\sin t\,\sigma(\vec{s})) =$

$= \cos^2 t\, I + \sin^2 t\,\sigma(\vec{s})^2 = I \Rightarrow e^{it\vec{\sigma}(s)} \in U(2)$

$\cos t\, I + i\sin t\,\vec{\sigma}(s) = \begin{bmatrix} \cos t + i\sin t \cdot s_3 & i\sin t(s_1 - is_2) \\ i\sin t(s_1 + is_2) & \cos t - i\sin t \cdot s_3 \end{bmatrix}$

$\det(\quad) = \cos^2 t + \sin^2 t\, s_3^2 + \sin^2 t(s_1^2 + s_2^2) = 1 \Rightarrow e^{it\sigma(\vec{s})} \in SU(2)$ □

2. KLASIFICIRAJ POVEZANE LIE. PODGRUPE HEISENBERGOVE LIEJEVE GRUPE

$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3\times 3} \middle| x, y, z \in \mathbb{R} \right\}$

VEMO, DA IMAMO BIJEKCIJO MED LIEJEVIMI PODALGEBRAMI IN POVEZANIMI LIEJEVIMI PODGRUPAMI.

POGLEJMO SI TOREJ RAJE PODALGEBRE $\mathfrak{h} = \left\{ \begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3} \middle| x, y, z \in \mathbb{R} \right\}$

BAZA ALGEBRE JE $X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $Z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $[X,Y] = Z \quad [X,Z] = [Y,Z] = 0$

1D PODALGEBRE SO NPR.

$\mathrm{Lin}(X) = \left\{ \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$ PRIPADAJOČA PODGRUPA JE $\exp\left(\begin{bmatrix} 0 & t & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\mathrm{Lin}\,Y = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} \right\}$ — " — $\exp\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$ — DIFEOMORFNE $\mathbb{R}$

$\mathrm{Lin}\,Z = \left\{ \begin{bmatrix} 0 & 0 & t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$ — " — $\exp\left(\begin{bmatrix} 0 & 0 & t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

SPLOŠNEJE ČE IZBEREMO $A = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}$ $(a,b,c \in \mathbb{R})$ JE $\boxed{\mathrm{Lin}(A) = \left\{ \begin{bmatrix} 0 & ta & tb \\ 0 & 0 & tc \\ 0 & 0 & 0 \end{bmatrix} \right\}}$

IN $\boxed{\exp\left(\begin{bmatrix} 0 & ta & tb \\ 0 & 0 & tc \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & ta & tb + \frac{t^2}{2}ac \\ 0 & 1 & tc \\ 0 & 0 & 1 \end{bmatrix}}$, KAR JE SPET DIFEOMORFNO $\mathbb{R}$

(t^2 SICER NI INJEKTIVNA FUNKCIJA, VENDAR LAHKO PRAVILNI PREDZNAK RAZBEREMO IZ OSTALIH KOMPONENT)

POGLEJMO SI ŠE 2D PODALGEBRE: ~~IMA~~

IMAMO $Lin(\{X,Z\})$ ALI $Lin(\{Y,Z\})$ ~~NAJA~~ ($Lin\{X,Y\}$ NI PODALGEBRA, KER $[X,Y]=Z$)

IMAMO TUDI NPR. $Lin(\{aX+bY,Z\})$, $Lin(\{aX+bZ,X\})$,... OZIROMA ~~IMAMO~~ VSAKO LIN. OGRINJAČO 2 NEODVISNIH VEKTORJEV RAZEN ~~...~~ MORDA TAKE OBLIKE $Lin[\{aX+bY,a'X+b'Y\}]$, KER

$[aX+bY,a'X+b'Y]=ab'Z-ba'Z.$ ~~...~~ $a,b,a',b'\in\mathbb{R}$

TO JE PODALGEBRA, ČEE $ab'=a'b$

TOREJ 2D PODALG. SO ~~VSE~~ ~~...~~ VSAKE Lin. OGRINJAČE NEODVISNIH VEKTORJEV RAZEN $Lin(\{aX+bY,a'X+b'Y\})$, ČE $ab'\neq a'b$

PRIPADAJOČE PODGRUPE SO OBLIKE

~~...~~ ZA FIKSNE $a,a',b,b',c,c'\in\mathbb{R}$ TAKE, DA JE $Lin\left(\left\{\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix},\begin{pmatrix}0&a'&b'\\0&0&c'\\0&0&0\end{pmatrix}\right\}\right)$ PODALGEBRA

$t,s\in\mathbb{R}$

$$\exp\left(\begin{bmatrix}0&ta+sa'&tb+sb'\\0&0&tc+sc'\\0&0&0\end{bmatrix}\right)=$$

$$=\boxed{\begin{pmatrix}1&ta+sa',&tb+sb'+\frac{1}{2}(ta+sa')(tc+sc')\\0,&1,&tc+sc'\\0&0&1\end{pmatrix}}$$

~~RAVNO~~ IDEALOM LIEJEVE ALGEBRE $\underline{h}$ USTREZAJO PODGRUPE EDINKE H

~~IMAMO~~ V 1D JE IDEAL $Lin(Z)$ TOREJ JE $\exp(tZ)$ EDINKA.

V 2D SO IDEALI ALGEBRE OBLIKE $Lin(\{aX+bY,Z\})$ EKSPONENT LE-TEH PA SO POTEM PODGRUPE EDINKE.

(6) DEFINIRAJ GRUPNO OPERACIJO NA $\mathbb{R}\times SU(n)$ IN HOMOMORFIZEM LIEJEVIH GRUP $\omega : \mathbb{R}\times SU(n) \to U(n)$, KI JE KROVNI HOMOMORFIZEM

~~OPERACIJA DEF G [illegible]~~

GRUPNI OPERACIJI NA $\mathbb{R}\times SU(n)$: $(\alpha, R)\cdot(\alpha', R') = (\alpha+\alpha', RR')$ OČITNO JE GRUPA ZAPRTA ZA TI OPERACIJI,

$(\alpha, R)^{-1} = (-\alpha, R^{-1})$ MNOŽENJE PA JE ASOCIATIVNO

DEFINIRAJMO $\omega(\alpha, R) = e^{i\alpha}R$, KI JE HOMOMORFIZEM GRUP

$\omega(\alpha,R)\,\omega(\alpha',R') = e^{i(\alpha+\alpha')}RR' = \omega(\alpha+\alpha', RR')$ IN GLADKA PRESLIKAVA

$\Rightarrow$ HOMOMORFIZEM LIEJEVIH GRUP

NJEGOVO JEDRO JE $\ker\omega = \{(2k\pi, I) \mid k\in\mathbb{Z}\} \cup \{(\frac{1}{2k\pi}, \begin{bmatrix} e^{i2k\pi} & & \\ & \ddots & \\ & & e^{i2k\pi}\end{bmatrix}) \mid k\in\mathbb{Z}\setminus\{0\}\}$

JE ZAPRTO IN DISKRETNO, TOREJ JE ω KROVNI HOMOMORFIZEM.

(7) G LIEJEVA GRUPA G JE MNT. $\mathbb{R}^3$ Z OPERACIJO $(x,y,z)(x',y',z') = (x+e^{z}x', x+e^{-z}y', z+z')$

OPIŠI ADJ. IN KOADJ. ORBITE.

OČITNO JE $e=(0,0,0)$ IZRAČUNAJMO $(x,y,z)^{-1} = (a,b,c)$

$(x,y,z)(a,b,c) = (x+e^{z}a, y+e^{-z}b, z+c) = (0,0,0)$

$\Rightarrow \boxed{(x,y,z)^{-1} = (-e^{-z}x, -e^{z}y, -z)}$

$p=(x,y,z)$, $L_p(x',y',z') = (x+e^{z}x', x+e^{-z}y', z+z')$

ZA BAZO LIEJEVE ALGEBRE VZAMIMO

$X = (1,0,0)$
$Y = (0,1,0)$
$Z = (0,0,1)$

V KOORDINATAH JE ODVOD JACOBIAN: $(dL_p)_e = \begin{bmatrix} e^{z} & 0 & 0 \\ 0 & e^{-z} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

IZRAČUNAJMO LEVO INV. POLJA
$$\boxed{\begin{aligned}(X^L)_p &= (dL_p)_e(X) = (e^{z}, 0, 0)\\ (Y^L)_p &= (0, e^{-z}, 0)\\ (Z^L)_p &= (0,0,1)\end{aligned}}$$

~~IZRAČUNAJMO PRIPADAJOČE TOKOVE~~ ZA POLJUBEN $W \in T_eG$ $W=(a,b,c)$

JE POTEM $\boxed{(W^L)_p = (e^{z}a, e^{-z}b, c)}$

PRIPADAJOČI TOK TEGA POLJA JE

$$\begin{aligned} \dot{x} &= ae^{z} \quad \Rightarrow \quad \dot{x} = a\,e^{ct+z(0)} \qquad x(t) = \frac{ae^{z(0)}}{c}e^{ct} - \frac{ae^{z(0)}}{c} + x(0) = \frac{ae^{z(0)}}{c}(e^{ct}-1) + x(0) \\ \dot{y} &= be^{-z} \qquad\qquad \dot{y} = b\,e^{-ct-z(0)} \qquad y(t) = \frac{be^{-z(0)}}{c}(1-e^{-ct}) + y(0) \\ \dot{z} &= c \quad \rightarrow \quad z(t) = ct + z(0) \end{aligned}$$

$$\Rightarrow \phi_t^{W^L}(p) = \left(\frac{ae^{z}}{c}(e^{ct}-1)+x,\ \frac{be^{-z}}{c}(1-e^{-ct})+y,\ ct+z\right)$$

$$\phi_t^{W^L}(e) = \boxed{\left(\frac{a}{c}(e^{ct}-1),\ \frac{b}{c}(1-e^{-ct}),\ ct\right) = exp(tW)}$$

VEMO $\boxed{Ad_p(W)} = \frac{d}{dt}\Big|_{t=0}\left(p\,exp(tW)\,p^{-1}\right) =$

$$= \frac{d}{dt}\Big|_{t=0}\left((x,y,z)\cdot\left(\frac{a}{c}(e^{ct}-1), \frac{b}{c}(1-e^{-ct}), ct\right)\cdot\left(-xe^{-z}, -ye^{z}, -z\right)\right) =$$

$$\cancel{= \frac{d}{dt}\Big|_{t=0}\left((x,y,z)\left(\frac{a}{c}(e^{ct}-1)-x, \frac{b}{c}(1-e^{-ct})-y, ct-z\right)\right) =}$$

$$\cancel{= \frac{d}{dt}\Big|_{t=0}\left(e^{z}\frac{a}{c}(e^{ct}-1), e^{-z}\frac{b}{c}(1-e^{-ct}), ct\right) = (ae^{z}, be^{-z}, c)}$$

$$= \frac{d}{dt}\Big|_{t=0}\left((x,y,z)\left(\frac{a}{c}(e^{ct}-1) - xe^{-z+ct}, \frac{b}{c}(1-e^{-ct}) - ye^{z-ct}, ct-z\right)\right) =$$

$$= \frac{d}{dt}\Big|_{t=0}\left(x + e^{z}\frac{a}{c}(e^{ct}-1) - xe^{ct}, y + e^{-z}\frac{b}{c}(1-e^{-ct}) - ye^{-ct}, ct\right) =$$

$$= \boxed{\left(e^{z}a - xc,\ e^{-z}b + yc,\ c\right)}$$

~~Orbite~~ $\cancel{O_{(x,y,z)} = \{(\alpha,\beta,0) \mid \alpha,\beta\in\mathbb{R}\} ; c=0}$

ORBITE:

$$\boxed{O_{(a,b,c)}} = \left\{ \cancel{\{(e^{\alpha}a, e^{-\alpha}b, 0) \mid \alpha\in\mathbb{R}\} ; c=0} \right. = \boxed{\begin{cases} \{(e^{\alpha}a, e^{-\alpha}b, 0) \mid \alpha\in\mathbb{R}\} & ; \ c=0 \\ \{(\alpha,\beta,c) \mid \alpha,\beta\in\mathbb{R}\} & ; \ c\neq 0 \end{cases}}$$

TOREJ ČE $c = a = 0$ JE ORBITA $Lin(\{(0,1,0)\})$

$c = b = 0$ JE ORBITA $Lin(\{(1,0,0)\})$

$W = 0$ JE ORBITA $\{\vec{0}\}$

KOADJ. ORBITE:

IZBERIMO DUALNO BAZO $(T_eG)^*$ $X^* = (1,0,0)$, $Y^* = (0,1,0)$, $Z^* = (0,0,1)$

$$f = (a^*, b^*, c^*)$$

$$(pf)(a,b,c) = f(p^{-1}(a,b,c)) = f((ae^{-z} + cxe^{-z}, be^{z} - cye^{z}, c)) =$$

$$= a^*(ae^{-z} + cxe^{-z}) + b^*(be^{z} - cye^{z}) + c^*c$$

$$= (a^*e^{-z})a + (b^*e^{z})b + \cancel{\ldots}(a^*xe^{-z} - b^*ye^{z} + c^*)c$$

$$\Rightarrow (x,y,z)(a^*,b^*,c^*) = (a^*e^{-z}, b^*e^{z}, a^*xe^{-z} - b^*ye^{z} + c^*)$$

$$O_{(a^*,b^*,c^*)} = \begin{cases} \{(0,0,c^*)\} \; ; a^* = b^* = 0 \\ \{(\alpha, 0, \beta) \mid \alpha, \beta \in \mathbb{R}\} \; ; a^* \neq 0, b^* = 0 \\ \{(0, \alpha, \beta) \mid \alpha, \beta \in \mathbb{R}\} \; ; a^* = 0, b^* \neq 0 \\ \{(e^{-\alpha}a^*, e^{\alpha}b^*, \beta)\} \; ; a^* \neq 0 \; b^* \neq 0 \end{cases}$$

$\alpha, \beta \in \mathbb{R}$

(8) $\mathbb{H} = \mathbb{R}^4$ ALGEBRA KVARTENIONOV $q \in \mathbb{H}$ $q = (t, \vec{r})$ $t \in \mathbb{R}$ ~~…~~, $\vec{r} \in \mathbb{R}^3$

LIE. GRUPA $S^3 = \{(t,\vec{r}) \in \mathbb{H} \mid t^2 + |\vec{r}|^2 = 1\}$

DEF. DELOVANJE $S^3 \times S^3$ NA $\mathbb{H}$ z $(q_1,q_2)p = q_1 p q_2^{-1}$ IN OZNAČIMO S π REALNO REPR.

$\pi : S^3 \times S^3 \to GL(4,\mathbb{R})$

$T \subset S^3$ PODGRUPA $T = \{(\cos\varphi, (\sin\varphi, 0, 0)) \mid \varphi \in [0, 2\pi)\} \cong S^1$

DEF: $\tau : S^3 \to S^3 \times S^3$ $\tau(q) = (q,q)$

POKAŽI, DA JE $\forall q \in S^3$ KONJ. NEKEMU ELEMENTU T.

ZA DAN KVARTENION $q = (t,x,y,z)$ DEF. $\alpha = t + ix$ ~~…~~ $\beta = y + iz$

MNOŽENJE IN INVERTIRANJE KVARTENIONOV POTEM USTREZA MNOŽENJU/INVERZU MATRIK $\begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}$

$q \in S^3$, $p \in \mathbb{H}$, $t \in T$ ŽELIMO $q = p t p^{-1}$

$t \in T$ USTREZA MATRIKA $\begin{bmatrix} \cos\varphi + i\sin\varphi & 0 \\ 0 & \cos\varphi - i\sin\varphi \end{bmatrix} \underset{\text{OZNAKA}}{=} \begin{bmatrix} \varphi & 0 \\ 0 & \bar{\varphi} \end{bmatrix}$

POGLEJMO PRODUKT $\begin{bmatrix}\alpha & \beta\\ -\bar\beta & \bar\alpha\end{bmatrix}\begin{bmatrix}\varphi & 0\\ 0 & \bar\varphi\end{bmatrix}\begin{bmatrix}\alpha & \beta\\ -\bar\beta & \bar\alpha\end{bmatrix}^{-1} =$

$$= \begin{bmatrix}\alpha & \beta\\ -\bar\beta & \bar\alpha\end{bmatrix}\cdot\begin{bmatrix}\varphi & 0\\ 0 & \bar\varphi\end{bmatrix}\frac{1}{|\alpha|^2+|\beta|^2}\begin{bmatrix}\bar\alpha & -\beta\\ \bar\beta & \alpha\end{bmatrix} = \frac{1}{|\alpha|^2+|\beta|^2}\begin{bmatrix}\alpha & \beta\\ -\bar\beta & \bar\alpha\end{bmatrix}\begin{bmatrix}\bar\alpha\varphi & -\beta\varphi\\ \bar\varphi\bar\beta & \bar\varphi\alpha\end{bmatrix} =$$

$$= \frac{1}{|\alpha|^2+|\beta|^2}\begin{bmatrix}|\alpha|^2\varphi+|\beta|^2\bar\varphi & -\alpha\beta\varphi+\alpha\beta\bar\varphi\\ -\bar\alpha\bar\beta\varphi+\bar\alpha\bar\beta\bar\varphi & +|\beta|^2\varphi+|\alpha|^2\bar\varphi\end{bmatrix}$$

KVARTERNION, PRIPADAJOČ TEJ MATRIKI IMA $t = Re\left(\frac{|\alpha|^2\varphi+|\beta|^2\bar\varphi}{|\alpha|^2+|\beta|^2}\right) = \cos\varphi$

$x = Im\,(\;-11-) = \sin\varphi\left(\frac{|\alpha|^2-|\beta|^2}{|\alpha|^2+|\beta|^2}\right)$

$y = Re\left(\frac{\alpha\beta(\bar\varphi-\varphi)}{|\alpha|^2+|\beta|^2}\right) = Re\left(\alpha\beta\cdot(-2)i\sin\varphi\right)\frac{1}{|\alpha|^2+|\beta|^2}$

$z = Im\left(\alpha\beta\,(-2)\,i\sin\varphi\right)\frac{1}{|\alpha|^2+|\beta|^2}$

PIŠIMO $\alpha = \alpha_1 + i\alpha_2 \quad \beta = \beta_1 + i\beta_2$

$\boxed{t} = \boxed{\cos\varphi}$

$\boxed{x} = \boxed{\frac{\alpha_1^2+\alpha_2^2-\beta_1^2-\beta_2^2}{\alpha_1^2+\alpha_2^2+\beta_1^2+\beta_2^2}\sin\varphi}$

$\boxed{y} = \frac{1}{|\alpha|^2+|\beta|^2}\left(-2\sin\varphi\, Re\left(i\,(\alpha_1+i\alpha_2)(\beta_1+i\beta_2)\right)\right) = \boxed{\frac{2\sin\varphi}{|\alpha|^2+|\beta|^2}\left(\alpha_1\beta_2+\alpha_2\beta_1\right)}$

$\boxed{z = -\frac{2\sin\varphi}{|\alpha|^2+|\beta|^2}\left(\alpha_1\beta_1-\alpha_2\beta_2\right)}$

~~PREVERIMO $t^2+x^2+y^2+z^2 = \cos^2\varphi + \sin^2\varphi\left[+\frac{4\sin^2\varphi}{(|\alpha|^2+|\beta|^2)^2}\right]\left(|\alpha_1|^2+\alpha_2^2-\beta_1^2-\beta_2^2\right)^2 + 4(\alpha_1\beta_2+\alpha_2\beta_1)^2 + 4(\alpha_1\beta_1-\alpha_2\beta_2)^2 =$~~

$=$

Po ~~KONSTRUKCIJI~~ ~~KVATERNIONA~~

$\det q = \det(p)\cdot\det(r)\cdot\det(p)^{-1} = \det(r) \Rightarrow |q| = |r| = 1$

KOT MATRIKA

REZULTAT JE TOREJ IZ S^3

$\Rightarrow$ ČE IMAMO DAN $q \in S^3$ JE TA KONJUGIRAN ELEMENTU $r \in T$, ZA KATEREGA VELJA $\cos\varphi$ = SKALARNA KOMPONENTA q

S PRIMERNO IZBIRO $\varphi, \alpha_1, \alpha_2, \beta_1, \beta_2$ LAHKO DOBIMO POLJUBEN EL. $q \in S^3$. □

(OBRNITI BI BILO TREBA ZGORNJE ENAČBE)

$(t, x, y, z) = F(\varphi, \alpha_1, \alpha_2, \beta_1, \beta_2)$ ← TREBA BI BILO PREVERITI, DA F OBRNLJIVA.

IZRAČUNAJ KARAKTER REPREZENTACIJE $(\pi\circ 2)|_T$

$$((\pi\circ 2)|_T(\varphi))(p) = (\varphi,\varphi)p = \varphi p \varphi^{-1} = \begin{bmatrix} \varphi & 0 \\ 0 & \bar\varphi \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{bmatrix}\begin{bmatrix} \bar\varphi & 0 \\ 0 & \varphi \end{bmatrix} =$$

$$= \begin{bmatrix} \varphi & 0 \\ 0 & \bar\varphi \end{bmatrix}\begin{bmatrix} \alpha\bar\varphi & \beta\varphi \\ -\bar\beta\bar\varphi & \bar\alpha\varphi \end{bmatrix} = \begin{bmatrix} \alpha & \beta\varphi^2 \\ -\bar\beta\bar\varphi^2 & \bar\alpha \end{bmatrix}$$

POZOR: OZNAKA φ POMENI $\varphi \in T$ $\varphi = (\cos\varphi, (\sin\varphi, 0, 0))$ IN $\varphi = \cos\varphi + i\sin\varphi$. KAJ JE MIŠLJENO JE JASNO IZ KONTEKSTA.

$$\varphi^2 = (\cos\varphi + i\sin\varphi)^2 = \cos^2\varphi - \sin^2\varphi + 2i\sin\varphi\cos\varphi = \cos 2\varphi + i\sin 2\varphi$$

$$\bar\varphi^2 = \cos 2\varphi - i\sin 2\varphi$$

ČE POGLEDAMO KAM SE S PRESLIKAVO $(\pi\circ 2)|_T(\varphi)$ PRESLIKAJO BAZNI VEKTORJI DOBIMO

$$\boxed{((\pi\circ 2)|_T(\varphi))} = \begin{array}{c|cccc} & 1 & i & j & k \\ \hline 1 & 1 & 0 & 0 & 0 \\ i & 0 & 1 & 0 & 0 \\ j & 0 & 0 & \cos 2\varphi & -\sin 2\varphi \\ k & 0 & 0 & \sin 2\varphi & \cos 2\varphi \end{array}$$

KARAKTER JE POTEM $\boxed{\chi(\varphi)} = \mathrm{tr}((\pi\circ 2)|_T(\varphi)) = \boxed{2 + 2\cos 2\varphi}$

ZAPIŠI REPR. $\pi\circ 2 : S^3 \to GL(4,\mathbb{R})$ KOT DIREKTNO VSOTO NERAZC. REPR. GRUPE S^3.

T JE TORUS V S^3. POVOLJ JE TOREJ GLEDATI RAZCEP KARAKTERJA $\pi\circ 2|_T$ NA KARAKTERJE IRR. REPR. S^3

IRR. REPR. S^3 POZNAMO, SAJ SO ENAKE KOT IRR. REPR. $SU(2)$ $\quad (S^3 \cong SU(2))$

TE REPR. IN PRIPADAJOČI KARAKTERJI SO

V_0, $\chi_{V_0}(\varphi) = 1$

V_1, $\chi_{V_1}(\varphi) = 2\cos\varphi$

V_2, $\chi_{V_2}(\varphi) = 2\cos(2\varphi) + 1$

$\vdots$

OPAZIMO, DA VELJA

$$\chi(\varphi) = \chi_{V_0}(\varphi) + \chi_{V_2}(\varphi)$$

TOREJ

$$\boxed{\pi\circ 2 \cong V_0 \oplus V_2}$$

(3)

a) KLASIFICIRAJ IRR. $\mathbb{C}$ REPR. $\mathbb{Z}_2 \times SO(2)$

$\mathbb{Z}_2$ IN $SO(2)$ STA KOMPAKTNI TOREJ BODO IRR. REPR. $\mathbb{Z}_2 \times SO(2)$ TENZORSKI PRODUKT IRR. REPR. $\mathbb{Z}_2$ IN $SO(2)$

~~INVERZIJA~~ ZRCALJENJE

$\mathbb{Z}_2 = \{I, C_2\}$ $(C_2)^2 = I$

$\mathbb{Z}_2$ IMA DVE IRR. REPR. TRIVIALNA: $\rho(I) = 1,\ \rho(C_2) = 1$

NETRIVIALNA $\rho(I) = 1,\ \rho(C_2) = -1$

REPR. $SO(2) \cong U(1)$ POZNAMO $\rho(e^{i\varphi}) = e^{in\varphi}$ $n \in \mathbb{Z}$

REPR. $\mathbb{Z}_2 \times SO(2)$ SO TOREJ

$$\boxed{\begin{matrix} \rho_n^+((I, e^{i\varphi})) = e^{in\varphi} & \rho_n^+((C_2, e^{i\varphi})) = e^{in\varphi} \\ \rho_n^-((I, e^{i\varphi})) = e^{in\varphi} & \rho_n^-((C_2, e^{i\varphi})) = -e^{in\varphi} \end{matrix}}$$

KER SO 1-DIMENZIONALNE SO ENAKE SVOJIM KARAKTERJEM

b) POKAŽI, DA $\exists$ NATANKO DVE NEIZOMORFNI ~~IRR~~ $\mathbb{C}$ REPR. $O(2)$ NA $\mathbb{C}$

SPOMNIMO SE $O(2) \cong \mathbb{Z}_2 \rtimes SO(2)$ ~~S PRODUKTOM~~ $\cancel{(Z_1, R_{\varphi_1})(Z_2, R_{\varphi_2}) =}$

$\cancel{=}$

~~S PRODUKTOM~~

$\cancel{(A_1, R_{\varphi_1})(A_2, R_{\varphi_2})} =$

$\cancel{= (A_1 A_2, R_{\varphi_1 + \varphi_2})}$

$\cancel{= (Z_1 Z_2, R_{(-1)^{k_2}\varphi_1 + \varphi_2})}$

$\cancel{Z_1 = Z_2 =}$

$Z_0 = I$ $Z_1 = C_2$

S PRODUKTOM $(A_1, R_{\varphi_1})(A_2, R_{\varphi_2}) = \begin{cases} (A_1 A_2, R_{\varphi_1 + \varphi_2}) & A_2 = I \\ (A_1 A_2, R_{-\varphi_1 + \varphi_2}) & A_2 = C_2 \end{cases}$

$\mathbb{C}$ REPR. $\mathbb{Z}_2 \times SO(2)$ ŽE POZNAMO, POGLEJMO SI, KATERE IZMED NJIH SO TUDI REPR. $\mathbb{Z}_2 \rtimes SO(2)$

ČE JE V ZGORNJEM PRODUKTU $A_2 = I$ JE VSE OK IN SE STVARI MNOŽIJO KOT V a) NALOGI.

ČE $A_2 = C_2$ DOBIMO $\rho_n^{\pm}((A_1, R_{\varphi_1}))\,\rho_n^{\pm}((C_2, R_{\varphi_2})) = \underset{\substack{\uparrow \\ \text{ODVISNO OD } A_1}}{\pm} e^{in\varphi_1} \cdot (-1) e^{in\varphi_2} = \mp e^{in(\varphi_1 + \varphi_2)}$

$\rho_n^{\pm}((A_1 C_2, R_{-\varphi_1 + \varphi_2})) = \cancel{\rho_n^{\pm}((A_1 C_2, R_{\varphi_1+\varphi_2}))} = \mp e^{in(\varphi_2 - \varphi_1)}$

REZULTAT NI ENAK, RAZEN, ČE JE $n=0$.

REPR. $\rho_n^{\pm}$ JE TOREJ HOMOMORFIZEM $O(2) \to GL(\mathbb{C})$ * LE, ČE $n=0$

IMAMO TOREJ LE DVE IRR. REPR. $\rho^+((A_1, R_\varphi)) = 1$

$\rho^-((I, R_\varphi)) = 1 \quad \rho^-((C_2, R_\varphi)) = -1$,

KI NISTA IZOMORFNI. $\square$

SKONSTRUIRAJ IRR. REPR. $O(2)$, KI NI ENODIMENZIONALNA $\mathbb{C}$

SKONSTRUIRALI BOMO $\rho: \mathbb{Z}_2 \ltimes SO(2) \to GL(2, \mathbb{C})$

SEVEDA ~~MO~~ MORA VELJATI $\rho(I, R_0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

INVERZIJA ~~INVERZIJA~~ BO ZAMENJALA BAZNA VEKTORJA $\rho(C_2, R_0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\rho(I, R_\varphi) = \begin{bmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{bmatrix}$$

$$\rho(C_2, R_\varphi) = \begin{bmatrix} 0 & e^{-i\varphi} \\ e^{i\varphi} & 0 \end{bmatrix}$$

OČITNO $\rho((I, R_{\varphi_1})) \cdot \rho((I, R_{\varphi_2})) = \rho((I, R_{\varphi_1+\varphi_2}))$

POGLEJMO $\rho((C_2, R_{\varphi_1}))\rho((I, R_{\varphi_2})) = \begin{bmatrix} 0 & e^{-i\varphi_1} \\ e^{i\varphi_1} & 0 \end{bmatrix}\begin{bmatrix} e^{i\varphi_2} & 0 \\ 0 & e^{-i\varphi_2} \end{bmatrix} = \begin{bmatrix} 0 & e^{-i(\varphi_1+\varphi_2)} \\ e^{i(\varphi_1+\varphi_2)} & 0 \end{bmatrix} = \rho((C_2, R_{\varphi_1+\varphi_2}))$

$\rho((I, R_{\varphi_1}))\rho((C_2, R_{\varphi_2})) = \begin{bmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{-i\varphi_1} \end{bmatrix}\begin{bmatrix} 0 & e^{-i\varphi_2} \\ e^{i\varphi_2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & e^{i(\varphi_1-\varphi_2)} \\ e^{i(\varphi_2-\varphi_1)} & 0 \end{bmatrix} = \rho((C_2, R_{\varphi_2-\varphi_1}))$

$\rho((C_2, R_{\varphi_1}))\rho((C_2, R_{\varphi_2})) = \begin{bmatrix} 0 & e^{-i\varphi_1} \\ e^{i\varphi_1} & 0 \end{bmatrix}\begin{bmatrix} 0 & e^{-i\varphi_2} \\ e^{i\varphi_2} & 0 \end{bmatrix} = \begin{bmatrix} e^{i(\varphi_2-\varphi_1)} & 0 \\ 0 & e^{i(\varphi_1-\varphi_2)} \end{bmatrix} = \rho((I, R_{\varphi_2-\varphi_1}))$

$\Rightarrow \rho$ RES REPR. GRUPE

JE IRREDUCIBILNA SAJ $\chi((I, R_\varphi)) = 2\cos\varphi$, KAR NE MORE BITI VSOTA KARAKTERJEV 1D IRR. REPREZENTACIJ. (SAJ SO TISTI LE ± 1)

10.

$$H = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3} \;\middle|\; x,y,z \in \mathbb{R} \right\}$$

a) POKAŽI, DA JE $\forall$ KONČNODIM. IRR. $\mathbb{C}$ REPR. H ENODIMENZIONALNA

LIE-KOLCHINOV TEOREM: ČE JE LIE. GRUPA G POVEZANA IN REŠLJIVA, SO NJENE KONČNE, $\mathbb{C}$ REPR. ENODIMENZIONALNE.

H JE POVEZANA, SAJ JE $\mathbb{R}^3$ POVEZANA.

REŠLJIVOST HITRO PREVERIMO: SPOMNIMO SE, DA ZA BAZO $\mathfrak{h}$ VELJA

$[X,Y]=Z$, $[X,Z]=0$, $[Y,Z]=0$

SLEDI $[\mathfrak{h},\mathfrak{h}] = \mathrm{Lin}(\{Z\})$ IN KER $[Z,Z]=0$ JE ALGEBRA $\mathfrak{h}$ IN TOREJ TUDI GRUPA H REŠLJIVA.

TRDITEV POTEM SLEDI IZ LIE-KOLCHINOVEGA TEOREMA $\square$

b) POIŠČI $\mathbb{C}$ REPR. H, KI NI IZOMORFNA DIREKTNI VSOTI PODREPREZENTACIJ

TAKA REPR. JE KAR INKLUZIJA $\rho: H \to GL(3,\mathbb{C})$.

ČE BI TA REPR. BILA REDUCIBILNA, BI $\exists$ BAZA, KJER PRIPADAJOČE MATRIKE ZAVZAMEJO ~~BLOČNO~~ DIAGONALNO OBLIKO (~~KJER PA SO BLOKI VELIKOSTI NAJVEČ 2x2, SEVEDA~~).
V SPLOŠNEM BI LAHKO IMELI TUDI BLOČNO DIAGONALNO OBLIKO, KJER BLOKI PREDSTAVLJAJO IRR. REPREZENTACIJE, VENDAR IZ NALOGE a) VIDIMO, DA SO TE ENODIMENZIONALNE.
TAKA BAZA NE $\exists$, SAJ NA PRIMER MATRIKE $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in H$ NE MOREMO DIAGONALIZIRATI (NJENA JORDANOVA FORMA JE $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$)

$\hookrightarrow$ NE MOREMO SE ZNEBITI 2x2 BLOKA.

ρ TOREJ NE MOREMO RAZCEPITI, ~~NI~~ SAMA PA NI NERAZCEPNA PO a), SAJ JE TRIDIMENZIONALNA.