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① $S_\varphi = -\int d^4x \sqrt{-g}\left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi)\right] = -\int d^4x \sqrt{-g}\left[\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + V(\phi)\right]$

ZA SKALAR $\partial_\mu \phi = \nabla_\mu \phi$

$\phi \to \phi + \delta\phi:\ S_\varphi \to -\int d^4x \sqrt{-g}\left[\frac{1}{2} g^{\mu\nu}\left(\nabla_\mu \phi \nabla_\nu \phi + \nabla_\mu \delta\phi \nabla_\nu \phi + \nabla_\mu \phi \nabla_\nu \delta\phi + \mathcal{O}(\delta\phi^2)\right) + V(\phi) + V'(\phi)\delta\phi\right]$

$\Rightarrow 0 = \delta S_\varphi = -\int d^4x \sqrt{-g}\left[\frac{1}{2} g^{\mu\nu}\left(\nabla_\mu \delta\phi \nabla_\nu \phi + \nabla_\mu \phi \nabla_\nu \delta\phi\right) + V'(\phi)\delta\phi\right] = -\int d^4x \sqrt{-g}\left[g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \delta\phi + V'(\phi)\delta\phi\right] =$

NEMI INDEKSI + $g^{\mu\nu} = g^{\nu\mu}$

$= -\int d^4x \sqrt{-g}\left[-g^{\mu\nu} \nabla_\nu \nabla_\mu \phi + V'(\phi)\right]\delta\phi$

PER PARTES

$\Rightarrow \boxed{\nabla^\mu \nabla_\mu \phi - V'(\phi) = 0}$

② $S_\varphi = -\int d^4x \sqrt{-g}\left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi)\right] \Rightarrow \delta S_\varphi = -\int d^4x\, \delta\sqrt{-g}\left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi)\right] +$
$+ \sqrt{-g}\left[\frac{1}{2} \partial_\mu \phi \partial_\nu \phi\, \delta g^{\mu\nu}\right]$

IZ PREJŠNJIH NALOG VEMO: $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\, \delta g^{\mu\nu}$

$\Rightarrow \delta S_\varphi = -\int d^4x \sqrt{-g}\left[-\frac{1}{2} g_{\mu\nu}\left(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + V(\phi)\right) + \frac{1}{2}\partial_\mu \phi \partial_\nu \phi\right]\delta g^{\mu\nu}$

$\boxed{T_{\mu\nu}} = -\frac{2}{\sqrt{-g}}\frac{\delta S_\varphi}{\delta g^{\mu\nu}} = \boxed{\partial_\mu \phi \partial_\nu \phi - g_{\mu\nu}\left(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + V(\phi)\right)}$ $\boxed{T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu}\left(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + V(\phi)\right)}$

$\varepsilon := -\frac{1}{2}\partial_\mu \phi \partial^\mu \phi + V(\phi),\quad P := -\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi),\quad u^\mu = -\left(-\partial_\sigma \phi \partial^\sigma \phi\right)^{-1/2}\partial^\mu \phi$

$\tilde{T}^{\mu\nu} = (\varepsilon + P) u^\mu u^\nu + P g^{\mu\nu} = \left(-\partial_\alpha \phi \partial^\alpha \phi\right)\left(-\partial_\sigma \phi \partial^\sigma \phi\right)^{-1} \partial^\mu \phi \partial^\nu \phi + g^{\mu\nu}\left(-\frac{1}{2}\partial_\alpha \phi \partial^\alpha \phi - V(\phi)\right) =$

$= \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu}\left(\frac{1}{2}\partial_\alpha \phi \partial^\alpha \phi + V(\phi)\right) \Rightarrow \boxed{T^{\mu\nu} = \tilde{T}^{\mu\nu}}$

③ $\varphi = \varphi(t)$. $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_N T_{\mu\nu}$

1) NAŠ PROSTOR-ČAS JE PROSTORSKO HOMOGEN IN IZOTROPEN, SAJ JE INFLATON $\varphi(t)$ PROSTORSKO HOMOGEN IN IZOTROPEN, TOREJ NIMAMO NIČESAR, KAR BI NAM DALO NEKO PREFERENČNO SMER OZIROMA NEHOMOGENOST.

2) DA INFLACIJA REŠI FLATNESS PROBLEM SE MORA VESOLJE TEKOM INFLACIJE ŠIRITI. TO ŽELIMO UPOŠTEVATI V NAŠI METRIKI.

3) VEMO, DA JE METRIKA, KI PREDSTAVLJA PROSTORSKO HOMOGENO, IZOTROPNO VESOLJE, KI SE ŠIRI FRW METRIKA.

④ FRW $k=0$, $\varphi = \varphi(t)$. V PROBLEM SET 2 SMO POKAZALI, DA IZ FRW METRIKE IN MATERIJE V OBLIKI IDEALNE TEKOČINE SLEDIJO FRIEDMAN. ENAČBE:

① $H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G_N}{3}\varepsilon = \frac{8\pi G_N}{3}\left(-\frac{1}{2}\partial_\mu \phi \partial^\mu \phi + V(\phi)\right) \Rightarrow \boxed{H = \sqrt{\frac{8\pi G_N}{3}\left(\frac{1}{2}\dot\phi^2 + V(\phi)\right)}}$

$\to -\frac{1}{2}(-1)\dot\phi^2$

② $\dot\varepsilon = -3H(\varepsilon + P) \Rightarrow \frac{d}{dt}\left(\frac{1}{2}\dot\phi^2 + V(\phi)\right) = -3H\left(-\partial_\mu \phi \partial^\mu \phi\right)$

$\Rightarrow \dot\phi \ddot\phi + V'(\phi)\dot\phi = -3H\dot\phi^2$

$\Rightarrow \boxed{\ddot\phi + 3H\dot\phi + V'(\phi) = 0}$

EOM ZA ϕ:

(4) CONT.

EOM ZA ϕ: $\nabla_\mu \nabla^\mu \phi - V'(\phi) = 0$, $\nabla_\mu \nabla^\mu \phi \overset{\phi \text{ SKALAR}}{=} \nabla_\mu \partial^\mu \phi = \partial_\mu \partial^\mu \phi + \Gamma^\mu_{\mu\lambda} \partial^\lambda \phi =$

$\partial^0 \phi = -\partial_0 \phi$, VEMO: $\Gamma^i_{i0} = H,\ i = 1,2,3$ $\quad = -\ddot{\phi} - 3H\dot{\phi} = 0$

$$\Rightarrow \boxed{\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0} \quad \text{SPET ISTA ENAČBA}$$

V PS2 SMO POKAZALI, DA $\nabla_\mu T^{\mu 0}$ USTREZA ENAČBI $\dot{\varepsilon} = -3H(\varepsilon + P)$, TOREJ TUDI IZ TU DOBIMO SPET ENO OD PREJŠNJIH ENAČB.

(5) $|\dot{H}| \ll H^2$, $|\ddot{\phi}| \ll H|\dot{\phi}|$ POKAŽI: $\left|\frac{V'(\phi)}{V(\phi)}\right| \ll \sqrt{16\pi G_N}$, $\left|\frac{V''(\phi)}{V(\phi)}\right| \ll 24\pi G_N$

$$H = \sqrt{\frac{8\pi G_N}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)} \quad /^2 \quad \Rightarrow \quad H^2 = \frac{8\pi G_N}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \quad / \frac{d}{dt}$$

$$\Rightarrow 2H\dot{H} = \frac{8\pi G_N}{3}\left(\dot{\phi}\ddot{\phi} + V'(\phi)\dot{\phi}\right), \quad \text{VEMO: } \ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

$$\Rightarrow 2H\dot{H} = \frac{8\pi G_N}{3}\left(\dot{\phi}(-3H\dot{\phi})\right) \Rightarrow \boxed{\dot{H} = -4\pi G_N \dot{\phi}^2}$$

$$|\dot{H}| \ll H^2$$

$$\longrightarrow +4\pi G_N \dot{\phi}^2 \ll \frac{8\pi G_N}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \longleftarrow$$

$$\frac{8\pi G_N}{3}\dot{\phi}^2 \ll \frac{8\pi G_N}{3} V(\phi) \quad \Rightarrow \boxed{\dot{\phi}^2 \ll V(\phi)}$$

$$\Rightarrow \boxed{\varepsilon} = \frac{1}{2}\dot{\phi}^2 + V(\phi) \approx \boxed{V(\phi)}, \quad \boxed{P} = \frac{1}{2}\dot{\phi}^2 - V(\phi) \approx \boxed{-V(\phi)}, \quad \boxed{H \approx \sqrt{\frac{8\pi G_N}{3} V(\phi)}}$$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \underset{|\ddot{\phi}| \ll H|\dot{\phi}|}{\Rightarrow} 3H\dot{\phi} + V'(\phi) = 0 \Rightarrow \boxed{\dot{\phi}} = -\frac{V'(\phi)}{3H} = \boxed{-\frac{V'(\phi)}{\sqrt{24\pi G_N V(\phi)}}}$$

$$\frac{|\dot{H}|}{H^2} = \frac{4\pi G_N \dot{\phi}^2}{\frac{8\pi G_N}{3} V(\phi)} = \frac{3}{2}\frac{\dot{\phi}^2}{V(\phi)} = \frac{3}{2V(\phi)} \cdot \frac{V'(\phi)^2}{24\pi G_N V(\phi)} = \frac{1}{16\pi G_N}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \ll 1, \text{ KER } \frac{|\dot{H}|}{H^2} \ll 1$$

$$\Rightarrow \boxed{\left|\frac{V'(\phi)}{V(\phi)}\right| \ll \sqrt{16\pi G_N}} \ \checkmark$$

$$\dot{\phi} = -\frac{V'(\phi)}{3H} \quad / \frac{d}{dt} \Rightarrow$$

$$\Rightarrow \ddot{\phi} = -\frac{V''(\phi)\dot{\phi}}{3H} + \frac{V'(\phi)\dot{H}}{3H^2} = \frac{V''(\phi)V'(\phi)}{9H^2} - V'(\phi)\frac{1}{3}\frac{1}{16\pi G_N}\left|\frac{V'(\phi)}{V(\phi)}\right|^2 =$$

$$\dot{H} = -|\dot{H}|$$

$$= \frac{V''(\phi)V'(\phi)}{9H^2} - \frac{V'(\phi)^3}{48\pi G_N V(\phi)^2} =$$

$$= V'(\phi)\left(\frac{V''(\phi)}{9H^2} - \frac{1}{48\pi G_N}\left|\frac{V'(\phi)}{V(\phi)}\right|^2\right),$$

5) NADALJEVANJE:

$$\ddot\phi = V'(\phi)\left(\frac{V''(\phi)}{9H^2} - \underbrace{\frac{1}{48\pi G_N}\left(\frac{V'(\phi)}{V(\phi)}\right)^2}_{\ll \frac{1}{48\pi G_N}16\pi G_N = \frac13}\right)$$

$$\boxed{|\ddot\phi|} \ll H|\dot\phi| = \sqrt{\frac{8\pi G_N}{3}V(\phi)}\,\frac{V'(\phi)}{\sqrt{24\pi G_N V(\phi)}} = \boxed{\frac13 V'(\phi)}$$

DRUGI ČLEN V $\ddot\phi$ JE ZANEMARLJIVO MAJHEN TOREJ

$$\Rightarrow V'(\phi)\left|\frac{V''(\phi)}{9H^2}\right| \ll \frac13 V'(\phi)$$

$$\Rightarrow |V''(\phi)| \ll 9H^2$$

$$|V''(\phi)| \ll 9\,\frac{8\pi G_N}{3}V(\phi) \Rightarrow \boxed{\left|\frac{V''(\phi)}{V(\phi)}\right| \ll 24\pi G_N} \checkmark$$

6) $V(\phi) = g e^{-\lambda\phi}$ POKAŽI, DA $\phi(t) = \frac{1}{\lambda}\ln\left(\frac{8\pi G_N g\varepsilon^2 t^2}{3-\varepsilon}\right)$, $\varepsilon = \frac{\lambda^2}{16\pi G_N}$ ZADOŠČA ENAČBAM V 4. NALOGI

$\lambda^2 = 16\pi G_N\varepsilon$

$$\boxed{\dot\phi} = \frac{1}{\lambda}\frac{3-\varepsilon}{8\pi G_N g\varepsilon^2 t^2}\left(\frac{3-\varepsilon}{8\pi G_N \varepsilon^2}\right)^{-1}2t = \boxed{\frac{2}{\lambda t}},\quad \boxed{H} = \sqrt{\frac{8\pi G_N}{3}\left(\frac12\dot\phi^2 + V(\phi)\right)} =$$

$$= \sqrt{\frac{8\pi G_N}{3}\left(\frac{2}{\lambda^2 t^2} + g\,\frac{3-\varepsilon}{8\pi G_N g\varepsilon^2 t^2}\right)} = \sqrt{\frac{8\pi G_N}{3}\left(\frac{1}{8\pi G_N\varepsilon t^2} + \frac{3-\varepsilon}{8\pi G_N\varepsilon^2 t^2}\right)} = \sqrt{\frac{8\pi G_N}{3}\cdot\frac{3}{8\pi G_N\varepsilon^2 t^2}} =$$

$$= \boxed{\frac{1}{\varepsilon t}}$$

$$\ddot\phi + 3H\dot\phi + V'(\phi) = 0 \Rightarrow -\frac{2}{\lambda t^2} + \frac{3}{\varepsilon t}\cdot\frac{2}{\lambda t} - \lambda\frac{3-\varepsilon}{8\pi G_N\varepsilon^2 t^2} = 0 \quad /\cdot t^2\lambda$$

$$\Rightarrow -2 + \frac{6}{\varepsilon} - 16\pi G_N\varepsilon\,\frac{3-\varepsilon}{8\pi G_N\varepsilon^2} = 0 \quad /\cdot\varepsilon$$

$$\Rightarrow -2\varepsilon + 6 - 2(3-\varepsilon) = 0 \Rightarrow \boxed{0=0} \checkmark$$

$$\boxed{H(t) = \frac{1}{\varepsilon t}} \Rightarrow \ln a(t) = \frac{1}{\varepsilon t} \quad d(\ln a) = \frac{1}{\varepsilon}\frac{dt}{t} \Rightarrow \ln a = \frac{\ln(t)}{\varepsilon} + \tilde C \quad / e^{\cdot}$$

$$\Rightarrow \boxed{a(t) = C\sqrt[\varepsilon]{t}} = \boxed{C\,t^{1/\varepsilon}}$$

SLOW ROLL: $\left|\frac{V'(\phi)}{V(\phi)}\right| = \boxed{|\lambda| \ll \sqrt{16\pi G_N}}$ $\quad\left|\frac{V''(\phi)}{V(\phi)}\right| = \boxed{|\lambda|^2 \ll 24\pi G_N}$

NA g NI POGOJA, KAR JE SMISELNO, SAJ NE VPLIVA NA HITROST SPREMINJANJA $V(\phi)$.

(7) RECIMO, DA IMAMO MAJHNO PERTURBACIJO, KI SPREMENI INFLATONSKO POLJE IN METRIKO: $\phi(x,t) = \bar{\phi}(t) + \delta\phi(x,t)$

$$g_{\mu\nu}(x,t) = \underbrace{\bar{g}_{\mu\nu}(t)}_{\text{PRED PERTURBACIJO}} + h_{\mu\nu}(x,t) \quad , \quad \bar{g}_{\mu\nu} = FRW$$

MAJHNE SPREMEMBE METRIKE $\delta g_{\mu\nu} = h_{\mu\nu}$ POVZROČIJO MAJHNE SPREMEMBE V CHRISTOFFELIH $\delta\Gamma^{\lambda}_{\mu\nu}$ IN S TEM SPREMEMBE V RIEMANNOVEM IN TUDI RICCIJEVEM TENZORJU $\delta R_{\mu\nu}$.

ČE $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ IN $R_{\mu\nu} = \bar{R}_{\mu\nu} + \delta R_{\mu\nu}$ VSTAVIMO V EINSTEINOVO ENAČBO, DOBIMO ENAČBO ZA PERTURBACIJO $h_{\mu\nu}$. ZARADI INVARIANCE NA KOORDINATNO TRANSFORMACIJO LAHKO NAREDIMO TRANSFORMACIJO $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$, ZA NEKO FUNKCIJO ξ_μ.

S PRIMERNO TRANSFORMACIJO GREMO LAHKO V NEWTNOVO UMERITEV:

$$\boxed{\begin{aligned} h_{00} &= -2\Psi \\ h_{0i} &= 0 \\ h_{ij} &= -2a^2\delta_{ij}\Psi \end{aligned}} \to 1 \text{ PARAMETER: } \Psi$$

V 2. NALOGI SMO VIDELI, DA LAHKO INFLATON POSPRAVIMO V STRESS TENZOR OBLIKE $T^{\mu\nu} = (\varepsilon+p)u^\mu u^\nu + p g^{\mu\nu}$

ČE UPOŠTEVAMO ŠE PERTURBACIJO INFLATONA, DOBIMO POPRAVKE:

$$\delta\varepsilon = \dot{\bar\phi}\,\delta\dot\phi + V'(\bar\phi)\,\delta\phi - \Psi\dot{\bar\phi}^2 \qquad \delta u_i = \frac{\partial \delta u}{\partial x^i},\ \delta u = -\delta\phi/\dot{\bar\phi}$$

$$\delta p = \dot{\bar\phi}\,\delta\dot\phi - V'(\bar\phi)\,\delta\phi - \Psi\dot{\bar\phi}^2$$

S TEM EINSTEINOVE ENAČBE POSTANEJO:

$$\dot\Psi + H\Psi = 4\pi G_N \dot{\bar\phi}\,\delta\phi$$

$$\delta\ddot\phi + 3H\delta\dot\phi + \frac{\partial^2 V(\bar\phi)}{\partial\bar\phi^2}\delta\phi - \left(\frac{\nabla^2}{a^2}\right)\delta\phi = -2\Psi\frac{\partial V(\bar\phi)}{\partial\bar\phi} + 4\dot\Psi\dot{\bar\phi}$$

$$\left(\dot H - \frac{\nabla^2}{a^2}\right)\Psi = 4\pi G_N\left(-\dot{\bar\phi}\,\delta\dot\phi + \ddot{\bar\phi}\,\delta\phi\right)$$

GREMO Z NASTAVKOM: $\delta\phi = e^{i\vec{q}\vec{x}}\delta\phi_q(t)$, $\Psi = e^{i\vec{q}\vec{x}}\Psi_q(t)$ IN GA VSTAVIMO V ZGORNJE ENAČBE

DOBIMO:

$$\dot\Psi_q + H\Psi_q = 4\pi G_N\dot{\bar\phi}\,\delta\phi_q$$

$$\delta\ddot\phi_q + 3H\delta\dot\phi_q + \frac{\partial^2 V(\bar\phi)}{\partial\bar\phi^2}\delta\phi_q + \left(\frac{q^2}{a^2}\right)\delta\phi_q = -2\Psi_q\frac{dV(\bar\phi)}{d\bar\phi} + 4\dot\Psi_q\dot{\bar\phi}$$

$$\left(\dot H + \frac{q^2}{a^2}\right)\Psi_q = 4\pi G_N\left(-\dot{\bar\phi}\,\delta\dot\phi_q + \ddot{\bar\phi}\,\delta\phi_q\right)$$

KJER SMO UPOŠTEVALI $\nabla^2(e^{i\vec{q}\vec{x}} f(t)) = -q^2 f(t) e^{i\vec{q}\vec{x}}$ IN POKRAJŠALI EKSPONENTE.

⑧ ~~NEKJE MED~~ OD OBDOBJA PROTI KONCU INFLACIJE IN VSE DO OBDOBIJ, KO PREVLADUJETA ~~SEVANJE~~ SEVANJE IN MATERIJA ZA FLUKTUACIJE VELJA, DA JE NJIHOVA VALOVNA DOLŽINA VEČJA OD PARTICLE HORIZONTA, OZIROMA $\frac{q}{a} << H$, KJER JE q VALOVNI VEKTOR PERTURBACIJE.

OBSTAJA IZREK, KI PRAVI, DA V LIMITI $\frac{q}{a} << H$ OBSTAJAJO TAKO IMENOVANE ADIABATSKE REŠITVE EINSTEINOVIH ENAČB, KATERE ZNAČILNOST JE, DA JE KOLIČINA $R_q = -\Psi_q + H\delta\phi_q / \dot{\phi}_q$ (KJER SMO SPET V NEWTNOVI UMERITVI) OHRANJENA. TO JE SEVEDA POMEMBNO, SAJ NAM OHRANJENE KOLIČINE V FIZIKI ZELO POENOSTAVIJO ALI CELO REŠIJO PROBLEM. V TEM PRIMERU SE IZKAŽE, DA LAHKO S POMOČJO R_q DOBIMO:

$R_q \neq 0$:

$$\Psi_q = R_q\left(-1 + \frac{H}{a}\int_T^t a(t')\,dt'\right),$$

$$\delta s_q = -\frac{R_q}{a}\dot{s}\int_T^t a(t')\,dt',$$

$$\delta u_q = \frac{R_q}{a}\int_T^t a(t')\,dt',$$

$R_q = 0$:

$$\Psi_q = C_q\frac{H}{a},$$ Ψ_q … PERTUR. METRIKE

$$\delta s_q = -C_q\frac{\dot{s}}{a},$$ δs_q … PERTUR. SKALARJA $(\varepsilon, p, \phi, \ldots)$

$$\delta u_q = C_q\frac{1}{a},$$ δu_q … PERTUR. HITROST. ~~...~~ POTENCIALA

$$C_q \neq C_q(t)$$

TOREJ NAM V TEM PRIMERU R_q SKUPAJ Z NEPERTURBIRANIMI KOLIČINAMI DA PERTURBIRANE KOLIČINE.

⑨ POKAŽI, DA $\frac{d^2R_q}{d\tau^2} + \frac{2}{z}\frac{dz}{d\tau}\frac{dR_q}{d\tau} + q^2R_q = 0$ POSTANE $\frac{d^2R_q}{d\tau^2} - \frac{2}{1-\varepsilon}\cdot\frac{1}{\tau}\frac{dR_q}{d\tau} + q^2R_q = 0$

OB POTENCIALU V ⑥ NALOGI. $z = \frac{a\dot{\phi}}{H}$, $\tau = \int_\infty^t \frac{dt'}{a(t')}$

NAJPREJ, VEMO $a(t) = C\,t^{1/\varepsilon}$ IN $\varepsilon > 0$, $\varepsilon << 1$ TOREJ $\boxed{\tau} = \frac{1}{C}\int_\infty^t t'^{-\frac{1}{\varepsilon}}\,dt' =$

$$= \frac{1}{C}\left(\frac{t^{-\frac{1}{\varepsilon}+1}}{-\frac{1}{\varepsilon}+1}\right)\Bigg|_\infty^t = \frac{1}{C}\frac{\varepsilon}{\varepsilon-1}t^{\frac{\varepsilon-1}{\varepsilon}}\Big|_\infty^t = \boxed{\frac{1}{C}\frac{\varepsilon}{\varepsilon-1}t^{\frac{\varepsilon-1}{\varepsilon}}}$$, KER $\lim_{t\to\infty} t^{\frac{\varepsilon-1}{\varepsilon}} = 0$, ČE $0<\varepsilon<1$

DOVOLJ JE POKAZATI $\frac{1}{z}\frac{dz}{d\tau} = -\frac{1}{(1-\varepsilon)\tau}$

$$\boxed{z} = a\frac{\dot{\phi}}{H} = a\frac{2}{\lambda t}\varepsilon t = \boxed{a\frac{2\varepsilon}{\lambda}} \qquad \frac{dz}{dt} = \dot{a}\frac{2\varepsilon}{\lambda} \qquad \boxed{\frac{dz}{d\tau}} = a\frac{dz}{dt} = \boxed{a\dot{a}\frac{2\varepsilon}{\lambda}}$$

NALOGA ⑥

$$\boxed{\frac{1}{z}\frac{dz}{d\tau}} = \frac{\lambda}{2\varepsilon a}a\dot{a}\frac{2\varepsilon}{\lambda} = \dot{a} = C\frac{1}{\varepsilon}t^{\frac{1}{\varepsilon}-1} = C\frac{1}{\varepsilon}t^{\frac{1-\varepsilon}{\varepsilon}} =$$

$$= \left(\frac{1}{C}\varepsilon\,t^{\frac{\varepsilon-1}{\varepsilon}}\right)^{-1}\frac{\varepsilon-1}{\varepsilon-1} = \frac{1}{\tau(\varepsilon-1)} = \boxed{-\frac{1}{(1-\varepsilon)\tau}} \quad \square$$

↓

(9) NADALJEVANJE REŠUJEMO $\ddot{R}_q - \frac{2}{(1-\varepsilon)\tau}\dot{R}_q + q^2 R_q = 0$, $\dot{f} = \frac{df}{d\tau}$

POMNOŽIMO ENAČBO S τ^2: $\tau^2 \ddot{R}_q - \frac{2}{1-\varepsilon}\tau \dot{R}_q + \tau^2 q^2 R_q = 0$

DOBIMO ENO IZMED OBLIK BESSELOVE ENAČBE. VEMO, DA ENAČBO $x^2\frac{d^2y}{dx^2} + (2p+1)x\frac{dy}{dx} + a^2x^2y = 0$

REŠI $y = x^{-p}(C_1 J_p(ax) + C_2 Y_p(ax))$, J_p, Y_p BESSELOVI FUNKCIJI PRVE IN DRUGE VRSTE.

PRI NAS: $2p+1 = -\frac{2}{1-\varepsilon} \Rightarrow$ ~~$p=$~~ $2p = -\frac{3-\varepsilon}{1-\varepsilon} \Rightarrow \boxed{p = -\frac{3-\varepsilon}{(1-\varepsilon)2}}$, $\boxed{a = q}$

$$\boxed{R_q = \tau^{\frac{3-\varepsilon}{(1-\varepsilon)2}}\left(C_1 J_{\frac{3-\varepsilon}{2(\varepsilon-1)}}(q\tau) + C_2 Y_{\frac{3-\varepsilon}{2(\varepsilon-1)}}(q\tau)\right)}$$

$$\tau = \frac{1}{c}\frac{\varepsilon}{\varepsilon-1}t^{\frac{\varepsilon-1}{\varepsilon}} \Rightarrow \boxed{t = 0 \Leftrightarrow \tau = -\infty}$$ ZVIO

Z OZNAKO $\boxed{\alpha = \frac{3-\varepsilon}{(1-\varepsilon)2}} > 0$: $\boxed{R_q = \tau^\alpha(C_1 J_{-\alpha}(q\tau) + C_2 Y_{-\alpha}(q\tau))}$

ZA ASIMPTOTIKO JE LAŽJE DELAT Z HANKEL FUNKCIJAMI: $H^{(1)}_{-\alpha} = J_{-\alpha} \pm i Y_{-\alpha}$ (zgoraj (1), spodaj (2))

Z NJIMI DOBIMO $\boxed{R_q = \tau^\alpha\left(D_1 H^{(1)}_{-\alpha}(q\tau) + D_2 H^{(2)}_{-\alpha}(q\tau)\right)}$

ASIMPTOTSKI RAZVOJ $|x| \to \infty$ $H^{(1)}_{-\alpha}(x) \asymp \sqrt{\frac{2}{\pi x}}\exp\left(i\left(x + \frac{\alpha\pi}{2} - \frac{\pi}{4}\right)\right)$, $H^{(2)}_{-\alpha}(x) = H^{1}_{-\alpha}(x)^*$

$$\Rightarrow \boxed{R_q(t \approx 0) = \tau^\alpha\sqrt{\frac{2}{\pi q\tau}}\left(D_1 e^{i\left(q\tau + \frac{\alpha\pi}{2} - \frac{\pi}{4}\right)} + D_2 e^{-i\left(q\tau + \frac{\alpha\pi}{2} - \frac{\pi}{4}\right)}\right)}$$

$$= \boxed{\tau^\alpha \frac{1}{\sqrt{q\tau}}\left(E_1 e^{iq\tau} + E_2 e^{-iq\tau}\right)}$$

(10) $R_q(t \approx 0) = -\frac{H}{(2\pi)^{3/2}\sqrt{2q}\,a\dot{\phi}}e^{-iq\tau}$

KO PRIMERJAMO S PREJŠNJO NALOGO VIDIMO $E_1 = 0$, $\frac{\tau^\alpha}{\sqrt{q\tau}}E_2 = -\frac{H}{(2\pi)^{3/2}\sqrt{2q}\,a\dot{\phi}}$

$\tau^{\alpha-\frac{1}{2}}E_2 = -\frac{1}{(2\pi)^{3/2}\sqrt{2}} \cdot \frac{1}{\varepsilon t}\frac{t^{\frac{1}{\varepsilon}}}{c\,t^{1/\varepsilon}2} = -\frac{1}{(2\pi)^{3/2}\sqrt{2}}\cdot\frac{1}{2\varepsilon c}t^{-1/\varepsilon}$

$$\Rightarrow \boxed{E_2 = -\frac{1}{(2\pi)^{3/2}2\sqrt{2}\,c\cdot\varepsilon}\left(\frac{1}{c}\frac{\varepsilon}{\varepsilon-1}\right)^{\frac{1}{2}-\alpha}}$$

ČASI SE MORAJO POKRAJŠATI $\sqrt{\frac{2}{\pi}}e^{-i\frac{\alpha\pi}{2}+i\frac{\pi}{4}}D_2 = E_2 \Rightarrow \boxed{D_2 = \sqrt{\frac{\pi}{2}}e^{i\left(\frac{\alpha\pi}{2}-\frac{\pi}{4}\right)}E_2}$

$$\Rightarrow \boxed{R_q(\tau) = \tau^\alpha D_2 H^{(2)}_{-\alpha}(q\tau)}$$

$$⑩\quad R_q(t\to 0) = -\frac{H}{(2\pi)^{3/2}\sqrt{2q}\, a\dot\phi}\, e^{-iq\tau}$$

KO PRIMERJAMO Z NALOGO ⑨ VIDIMO, DA MORA VELJATI: $\boxed{E_1=0}$, $\dfrac{\tau^\alpha}{\sqrt{q\tau}}E_2 = -\dfrac{H}{(2\pi)^{3/2}\sqrt{2q}\, a\dot\phi}$

$$\Rightarrow \boxed{E_2} = -\frac{H}{(2\pi)^{3/2}\sqrt2}\cdot\frac{1}{a\dot\phi}\,\tau^{\frac12-\alpha} = -\frac{1}{(2\pi)^{3/2}\sqrt2}\cdot\frac{1}{\varepsilon t}\cdot\frac{\lambda t}{2}\cdot\frac{1}{c\,t^{1/\varepsilon}}\left(\frac1c\,\frac{\varepsilon}{\varepsilon-1}\,t^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac12-\alpha} =$$

$$= -\frac{\lambda/\varepsilon}{(2\pi)^{3/2}2\sqrt2\, c}\left(\frac1c\,\frac{\varepsilon}{\varepsilon-1}\right)^{-\frac{1}{1-\varepsilon}} t^{+\frac1\varepsilon}t^{-\frac1\varepsilon} = \boxed{-\frac{\lambda/\varepsilon}{(2\pi)^{3/2}2\sqrt2\, c}\cdot\left(\frac1c\,\frac{\varepsilon}{\varepsilon-1}\right)^{-\frac{1}{1-\varepsilon}}}$$

$$\frac12-\alpha = \frac{1-\varepsilon-3+\varepsilon}{2(1-\varepsilon)} = -\frac{1}{1-\varepsilon}$$

$$\sqrt{\frac{2}{\pi}}\,D_2\, e^{i\left(-\frac{\alpha\pi}{2}+\frac{\pi}{4}\right)} = E_2 \Rightarrow \boxed{D_2 = \sqrt{\frac{\pi}{2}}\, e^{-i\left(\frac{\pi}{4}-\frac{\alpha\pi}{2}\right)}E_2}\ ,\ D_1=0$$

$$\Rightarrow \boxed{R_q(\tau) = \tau^\alpha D_2\, H^{(2)}_{-\alpha}(q\tau)}$$

ZUNAJ HORIZONTA VELJA $q/a \ll H \Rightarrow q \ll c\,t^{1/\varepsilon}\,\dfrac{1}{\varepsilon t}$

$$\Rightarrow q \ll \frac{c}{\varepsilon}\,t^{\frac1\varepsilon-1} \Rightarrow q \ll \frac{c}{\varepsilon}\,t^{\frac{1-\varepsilon}{\varepsilon}} \Rightarrow q \ll \frac{c}{\varepsilon}\left(\text{KONST}\cdot\tau^{\frac{\varepsilon}{\varepsilon-1}}\right)^{\frac{1-\varepsilon}{\varepsilon}}$$

$$\Rightarrow q \ll \text{KONST}\cdot\tau^{-1} \Rightarrow q\tau \ll \text{KONST.} \Rightarrow q \ll 1$$

HANKELOVE FUNKCIJE LAHKO RAZVIJEMO. VELJA $H^{(2)}_{-\alpha}(x) = \dfrac{i\Gamma(-\alpha)}{\pi}\left(\dfrac{x}{2}\right)^\alpha + \mathcal{O}(x^2)$

VELJA ŠE: $H^{(2)}_{-\alpha}(x) = e^{-\alpha\pi i}H^{(2)}_{\alpha}(x)$

$$\Rightarrow \boxed{H^{(2)}_{-\alpha}(x) = e^{-\alpha\pi i}\,\frac{i\Gamma(\alpha)}{\pi}\left(\frac{x}{2}\right)^{-\alpha}}$$

DOBIMO

$$\cancel{R^0_q(\tau) = \lim_{q\tau\to0} R_q(\tau) = \tau^\alpha D_2\,\frac{i\Gamma(-\alpha)}{\pi}\left(\frac{q\tau}{2}\right)^\alpha =}$$

$$\cancel{= \tau^\alpha\,\frac{i\Gamma(-\alpha)}{\pi}\sqrt{\frac{\pi}{2}}\,e^{-i\left(\frac{\pi}{4}-\frac{\alpha\pi}{2}\right)}}$$

$$\cancel{= -\tau^\alpha\,\frac{i\Gamma(-\alpha)}{\pi}\sqrt{\frac{\pi}{2}}\,e^{-i\left(\frac{\pi}{4}-\frac{\alpha\pi}{2}\right)}\frac{\lambda/\varepsilon}{(2\pi)^{3/2}2\sqrt2\,c}\left(\frac1c\,\frac{\varepsilon}{\varepsilon-1}\right)^{-\frac{1}{1-\varepsilon}}\left(\frac{q\tau}{2}\right)^\alpha}$$

DOBIMO

$$\boxed{R^0_q} = \lim_{q\tau\to0} R_q(\tau) = \tau^\alpha D_2\, e^{-\alpha\pi i}\,\frac{i\Gamma(\alpha)}{\pi}\left(\frac{q\tau}{2}\right)^{-\alpha} =$$

$$= \boxed{-\sqrt{\frac{\pi}{2}}\, e^{-i\left(\frac{\pi}{4}+\frac{\alpha\pi}{2}\right)}\,\frac{\lambda/\varepsilon}{(2\pi)^{3/2}\,2\sqrt2\,c}\left(\frac1c\,\frac{\varepsilon}{\varepsilon-1}\right)^{-\frac{1}{1-\varepsilon}}\frac{i\Gamma(\alpha)}{\pi}\left(\frac{q}{2}\right)^{-\alpha}}$$