COSMOLOGY, HW 1

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(1)

a) $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = \frac{L^2}{y^2}(-dt^2+dx^2+dy^2)$

$$\boxed{g_{\mu\nu} = \frac{L^2}{y^2}\begin{pmatrix}-1&&\\&1&\\&&1\end{pmatrix}}\begin{array}{l}t\\x\\y\end{array}$$

$$g_{\mu\nu} g^{\nu\sigma} = \delta_\mu{}^\sigma \Rightarrow \boxed{g^{\mu\nu} = \frac{y^2}{L^2}\begin{pmatrix}-1&&\\&1&\\&&1\end{pmatrix}}$$

$$\Gamma^\sigma_{\mu\nu} = \frac12 g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) = \frac12 g^{\sigma\rho}(\delta_\mu^2 \partial_y g_{\nu\rho} + \delta_\nu^2 \partial_y g_{\rho\mu} - \delta_\rho^2 \partial_y g_{\mu\nu}) =$$

$$= \frac12 g^{\sigma\rho}\left(-\frac{2}{y}\right)(\delta_\mu^2 g_{\nu\rho} + \delta_\nu^2 g_{\rho\mu} - \delta_\rho^2 g_{\mu\nu}) = -\frac1y g^{\sigma\rho}(\delta_\mu^2 g_{\nu\rho}+\delta_\nu^2 g_{\rho\mu} - \delta_\rho^2 g_{\mu\nu})$$

$\mu=2,\nu\neq2:\quad \Gamma^\sigma_{2\nu} = -\frac1y g^{\sigma\rho}(g_{\nu\rho} - \delta_\rho^2 \cancel{g_{2\nu}}^{\,0}) = -\frac1y \delta^\sigma{}_\nu$, (g DIAG.) $\boxed{\Gamma^0_{20} = -\frac1y}$ $\boxed{\Gamma^1_{21}=-\frac1y}$

$\mu\neq2,\nu=2:\quad \Gamma^\sigma_{\mu2} = -\frac1y g^{\sigma\rho} g_{\rho\mu} = -\frac1y\delta^\sigma{}_\mu$, $\boxed{\Gamma^0_{02}=-\frac1y}$ $\boxed{\Gamma^1_{12} = -\frac1y}$

$\mu=\nu=2:\quad \Gamma^\sigma_{22} = -\frac1y g^{\sigma\rho}(g_{2\rho}+g_{\rho2}-\delta^2_\rho g_{22})$, $\boxed{\Gamma^2_{22} = -\frac1y}$

$\mu\neq2,\nu\neq2\quad \Gamma^\sigma_{\mu\nu} = -\frac1y g^{\sigma\rho}(-\delta^2_\rho g_{\mu\nu}) = \underbrace{\frac1y g^{22} g_{\mu\nu}}_{\sigma=2} \Rightarrow \boxed{\Gamma^2_{00} = -\frac1y}$ $\boxed{\Gamma^2_{11}=\frac1y}$

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

$\frac{1}{12}3^2(3^2-1) = 6$ NEODV. KOMP.

1) $\boxed{R^0{}_{202}} = -\partial_2\Gamma^0_{02} + \Gamma^0_{0\lambda}\Gamma^\lambda_{22} - \Gamma^0_{2\lambda}\Gamma^\lambda_{02} = -\frac1{y^2}+\frac1{y^2}-\frac1{y^2} = \boxed{-\frac1{y^2}}$

$\boxed{R^2{}_{002}} = g^{2\mu}R_{\mu002} = g^{2\mu}R_{0\mu02}(-1) = (-1)g^{2\mu}g_{0\nu}R^\nu{}_{\mu02} = (-1)g^{22}g_{00}R^0{}_{202} = \boxed{R^0{}_{202}}$

$R_{0202} = g_{00}R^0{}_{202}$ $\quad\boxed{R^0{}_{220}} = \boxed{-R^0{}_{202}}$

2) $\boxed{R^1{}_{212}} = -\partial_2\Gamma^1_{12} + \Gamma^1_{1\lambda}\Gamma^\lambda_{22} - \Gamma^1_{2\lambda}\Gamma^\lambda_{12} = -\frac1{y^2}+\frac1{y^2}-\frac1{y^2} = \boxed{-\frac1{y^2}}$ $\quad R_{1212} = g_{11}R^1{}_{212}$

$\boxed{R^2{}_{112}} = g^{22}(-g_{11})R^1{}_{212} = \boxed{-R^1{}_{212}}$ $\quad\boxed{R^1{}_{221}} = g^{11}(-g_{11})R^1{}_{212} = \boxed{-R^1{}_{212}}$

3) $\boxed{R^0{}_{102}} = -\partial_2\Gamma^0_{01} + \Gamma^0_{0\lambda}\Gamma^\lambda_{21} - \Gamma^0_{2\lambda}\Gamma^\lambda_{01} = \boxed{0} = \boxed{R^1{}_{002}} = \boxed{R^0{}_{120}} = \boxed{R^0{}_{210}} = \boxed{R^2{}_{010}} = \boxed{R^2{}_{001}} = \ldots$

4) $\boxed{R^0{}_{112}} = -\partial_2\Gamma^0_{21} + \Gamma^0_{1\lambda}\Gamma^\lambda_{21} - \Gamma^0_{2\lambda}\Gamma^\lambda_{11} = \boxed{0}$, 5) $\boxed{R^0{}_{212}} = -\partial_2\Gamma^0_{21}+\Gamma^0_{1\lambda}\Gamma^\lambda_{21} - \Gamma^0_{2\lambda}\Gamma^\lambda_{11} = \boxed{0}$

6) $\boxed{R^0{}_{101}} = \Gamma^0_{0\lambda}\Gamma^\lambda_{11} - \Gamma^0_{1\lambda}\Gamma^\lambda_{01} = -\frac1{y^2} - 0 = \boxed{-\frac1{y^2}}$, $\quad R_{0101} = g_{00}R^0{}_{101}$

$\boxed{R^1{}_{001}} = -g^{11}g_{00}R^0{}_{101} = \boxed{R^0{}_{101}}$ $\quad\boxed{R^0{}_{110}} = -g^{00}g_{00}R^0{}_{101} = \boxed{-R^0{}_{101}}$ $\quad\boxed{R^1{}_{010}} = +g^{11}R_{1010} = -g^{11}g_{00}R^0{}_{110} = \boxed{-R^0{}_{101}}$

: $\boxed{R^2{}_{020}} = \boxed{-R^0{}_{202}}$, $\boxed{R^0{}_{022}} = \boxed{R^2{}_{200}} = \boxed{0}$

POZABIL: →

$\boxed{R^2{}_{121}} = \boxed{R^1{}_{212}}$, $\boxed{R^1{}_{122}} = \boxed{R^2{}_{211}} = \boxed{0}$

$R_{0011}+R_{0110}+R_{0101} = 0$

$R_{0011} = -R_{0101}(1-1) = 0$

$\Rightarrow \boxed{R^0{}_{011} = R^1{}_{100} = 0}$

OSTALE NENAPISENE KOMPONENTE [PERMUTACIJE 3), 4), 5)] TUDI OČITNO NIČ

$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$ $\boxed{R_{00}} = R^0{}_{000} + R^1{}_{010} + R^2{}_{020} = \frac{1}{y^2} + \frac{1}{y^2} = \boxed{\frac{2}{y^2}}$

$\boxed{R_{11}} = R^0{}_{101} + R^2{}_{121} = -\frac{1}{y^2} - \frac{1}{y^2} = \boxed{-\frac{2}{y^2}}$

$\boxed{R_{22}} = R^0{}_{202} + R^1{}_{212} = \boxed{-\frac{2}{y^2}}$

$\boxed{R_{01}} = R^0{}_{001} + R^1{}_{011} + R^2{}_{021} = 0 = \boxed{R_{10}}$

$\boxed{R_{12}} = R^0{}_{102} + R^1{}_{112} + R^2{}_{122} = 0 = \boxed{R_{21}}$

$\boxed{R_{02}} = R^0{}_{002} + R^1{}_{012} + R^2{}_{022} = 0 = \boxed{R_{20}}$

$$\boxed{R_{\mu\nu} = +\frac{2}{y^2}\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}}$$

$\boxed{R} = g^{\mu\nu}R_{\mu\nu} = \frac{2}{y^2}\cdot\frac{y^2}{L^2}(-1-1-1) = \boxed{-\frac{6}{L^2}}$

b) $t = \text{CONST.} / dt = 0$ GEODETSKA EN.

$\left(\frac{dl}{d\lambda} = \dot{l}\right)$ $\boxed{\ddot{x}^\mu + \Gamma^\mu_{\rho\sigma}\dot{x}^\sigma\dot{x}^\rho = 0}$

NAŠ PRIMER: $\ddot{x} + \dot{x}\dot{y}(\Gamma^1_{21} + \Gamma^1_{12}) = \boxed{\ddot{x} + 2\cdot\left(-\frac{1}{y}\right)\dot{x}\dot{y} = 0}$ 1)

$\boxed{\ddot{y} + \frac{1}{y}(\dot{x}^2 - \dot{y}^2) = 0}$ 2)

$x^2+y^2 = 1$
$(\cos\lambda, \sin\lambda) = (x,y)$
NE GRE, KER $\cos\lambda$ IN $\sin\lambda$ NISTA DOBRO DEFINIRANA ZA $\lambda = \pm\infty$. ZATO POSKUSIMO Z HIPERBOLIČNIMI FUNKCIJAMI

POGOJI ZA NASTAVEK:
- $(x(\lambda=-\infty), y(\lambda=-\infty)) = (-l/2, 0)$
- $(x(\infty), y(\infty)) = (l/2, 0)$
- $x^2+y^2 = \frac{l^2}{4}$

$y>0$

$ch^2 x - sh^2 x = 1 \quad / : ch^2 x$

$1 - th^2 x = \frac{1}{ch^2 x}$

$1 = \frac{1}{ch^2(\lambda)} + th^2(\lambda)$

PROBAMO: $x(\lambda) = \frac{l}{2} th(\lambda)$, $x(\pm\infty) = \pm\frac{l}{2}$ ✓

$y(\lambda) = \frac{l}{2}\frac{1}{ch(\lambda)}$, $y(\pm\infty) = 0$ ✓

$x^2+y^2 = \frac{l^2}{4}$ ✓ $y>0$ ✓

ENAČBA 1): $\frac{l}{2}\left(\frac{\dot{ch^2(\lambda) - sh^2(\lambda)}}{ch^2(\lambda)}\right) + \left(-\frac{4}{l}ch(\lambda)\right)\frac{l^2}{4}\left[\frac{1}{ch^2(\lambda)}\cdot\left(-\frac{1}{ch^2(\lambda)}\right)sh(\lambda)\right] =$

$= \frac{l}{2}\left(\frac{\dot{1}}{ch^2(\lambda)}\right) + \frac{l}{2}\cdot 2\cdot\frac{sh(\lambda)}{ch^3(\lambda)} = \frac{l}{2}(-2)\frac{1}{ch^3(\lambda)}\cdot sh(\lambda) + l\frac{sh(\lambda)}{ch^3(\lambda)} = 0$ ✓

ENAČBA 2): $\frac{l}{2}\left(-\frac{\dot{sh(\lambda)}}{ch^2(\lambda)}\right) + \frac{2}{l}ch(\lambda)\left[\frac{l^2}{4}\frac{1}{ch^4(\lambda)} - \frac{l^2}{4}\frac{sh^2(\lambda)}{ch^4(\lambda)}\right] = \frac{l}{2}\left(-\frac{ch^3(\lambda) - 2sh^2(\lambda)ch(\lambda)}{ch^4(\lambda)}\right) + \frac{l}{2}\left(\frac{1}{ch^3(\lambda)} - \frac{sh^2(\lambda)}{ch^3(\lambda)}\right) =$

$= \frac{l}{2}\left[-\frac{1}{ch(\lambda)} + \frac{2sh^2(\lambda)}{ch^3(\lambda)} + \frac{1}{ch^3(\lambda)} - \frac{sh^2(\lambda)}{ch^3(\lambda)}\right] = \frac{l}{2}\left[\frac{1-ch^2(\lambda)}{ch^3(\lambda)} + \frac{sh^2(\lambda)}{ch^3(\lambda)}\right] = 0$ ✓

$1 - ch^2 = -sh^2$ □

c) $x = \text{CONST.}$ GEODESIC EQ:

$\ddot{y} + \Gamma^2_{22}\dot{y}^2 = 0 \quad \ddot{y} - \frac{1}{y}\dot{y}^2 = 0,$

$u = \dot{y}$

$\ddot{y} = \frac{du}{dy}\dot{y} = u'u$

$u'u - \frac{1}{y}u^2 = 0 \Rightarrow u' = \frac{u}{y} \Rightarrow \frac{du}{u} = \frac{dy}{y} \Rightarrow \ln u = \ln y + \tilde{A} \Rightarrow \boxed{u = Ay}$

$\frac{dy}{y} = A\,d\lambda \Rightarrow \ln y = A\lambda + \tilde{B} \Rightarrow \boxed{y = Be^{A\lambda}}$ A,B KONSTANTI

$\boxed{\Delta s} = \int_{\lambda_1}^{\lambda_2}\sqrt{g_{22}\left(\frac{dy}{d\lambda}\right)^2}\,d\lambda = L\int_{\lambda_1}^{\lambda_2}\frac{ABe^{A\lambda}}{Be^{A\lambda}}d\lambda$ $\frac{dy}{d\lambda} = ABe^{A\lambda}$ $\Rightarrow \lambda = \frac{1}{A}\ln\left(\frac{y}{B}\right)$

$= LA(\lambda_2 - \lambda_1) = \boxed{L\left(\ln\frac{y_2}{B} - \ln\frac{y_1}{B}\right)}$ $\boxed{y_1 \to 0 \Rightarrow \Delta s \to \infty}$

ČE $ds^2 = dx^2 + dy^2$ IMAMO OČITNO KOT SMO VAJENI $\boxed{\Delta s = y_2 - y_1}$

②

a) $S = \frac{1}{16\pi G}\underbrace{\int d^4x\sqrt{-g}\,(R-2\Lambda)}_{I_1} + \underbrace{\int d^4x\sqrt{-g}\,\mathcal{L}_M}_{I_2} = I_1 + I_2$

Variramo $\delta g^{\mu\nu}$. Želimo $\delta S = \int d^4x \underbrace{\ldots\ldots}_{\text{EINSTEIN EQ.}} \delta g^{\mu\nu} = 0$

$$\delta I_1 = \frac{1}{16\pi G}\int d^4x\, \delta\left(\sqrt{-g}(R-2\Lambda)\right) = \frac{1}{16\pi G}\Big[\underbrace{\int d^4x\,\delta\sqrt{-g}\,(R-2\Lambda)}_{S_1} + \underbrace{\int d^4x\sqrt{-g}\,\delta g^{\mu\nu}R_{\mu\nu}}_{S_2} + \underbrace{\int d^4x\sqrt{-g}\,g^{\mu\nu}\delta R_{\mu\nu}}_{S_3}\Big]$$

$\delta g_{\mu\nu} = ?:$ $\quad g^{\mu\nu}g_{\nu\lambda} = \underset{\text{KONST.}}{\delta^\mu_\lambda} \;/\,\delta \Rightarrow \delta g^{\mu\nu}g_{\nu\lambda} + g^{\mu\nu}\delta g_{\nu\lambda} = 0 \;/\cdot g_{\mu\rho}$

$\Rightarrow g_{\mu\rho}\,\delta g^{\mu\nu}g_{\nu\lambda} + \delta^\nu_\rho\,\delta g_{\nu\lambda} = 0 \Rightarrow \delta g_{\rho\lambda} \Rightarrow \boxed{\delta g_{\mu\nu} = -g_{\lambda\mu}\,g_{\rho\nu}\,\delta g^{\lambda\rho}}$

S_1: TRIK: $\ln(\det M) = \mathrm{Tr}(\ln M)\;/\,\delta \Rightarrow \frac{1}{\det M}\delta\det M = \mathrm{Tr}(M^{-1}\delta M)$

$g = \det g_{\mu\nu}$ $\qquad \delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g = +\frac{1}{2\sqrt{-g}}\,g\,(g_{\mu\nu}\delta g^{\mu\nu}) = -\frac{1}{2}\frac{(-g)}{\sqrt{-g}}(g_{\mu\nu}\delta g^{\mu\nu}) =$

$= -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu}$

$\delta g = g\,(g^{\mu\nu}\delta g_{\mu\nu}) = -g\,(g^{\mu\nu}g_{\lambda\mu}g_{\rho\nu}\,\delta g^{\lambda\rho}) = -g\,(g_{\lambda\rho}\,\delta g^{\lambda\rho})$

$\delta S_1 = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\,\frac{1}{2}g_{\mu\nu}(R+2\Lambda)\,\delta g^{\mu\nu}$

S_2: $\delta S_2 = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\,R_{\mu\nu}\,\delta g^{\mu\nu}$

S_3: $\delta R_{\mu\nu} = \delta R^\lambda{}_{\mu\lambda\nu} = \partial_\lambda\delta\Gamma^\lambda_{\nu\mu} - \partial_\nu\delta\Gamma^\lambda_{\lambda\mu} + \Gamma^\alpha_{\nu\mu}\delta\Gamma^\lambda_{\lambda\alpha} + \Gamma^\lambda_{\lambda\alpha}\delta\Gamma^\alpha_{\nu\mu} -$

$- \Gamma^\lambda_{\nu\alpha}\delta\Gamma^\alpha_{\lambda\mu} - \Gamma^\alpha_{\lambda\mu}\delta\Gamma^\lambda_{\nu\alpha}$

$\nabla_\alpha\delta\Gamma^\lambda_{\mu\nu} = \partial_\alpha\delta\Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\alpha\beta}\delta\Gamma^\beta_{\mu\nu} - \Gamma^\beta_{\alpha\mu}\delta\Gamma^\lambda_{\beta\nu} - \Gamma^\beta_{\alpha\nu}\delta\Gamma^\lambda_{\mu\beta}$

S KOV. ODVODOM LAHKO PREPIŠEMO $\delta R_{\mu\nu} = \nabla_\lambda(\delta\Gamma^\lambda_{\nu\mu}) - \nabla_\nu(\delta\Gamma^\lambda_{\lambda\mu}) - \Gamma^\alpha_{\nu\lambda}(\delta\Gamma^\lambda_{\alpha\mu}) + \Gamma^\alpha_{\lambda\nu}(\delta\Gamma^\lambda_{\alpha\mu})$

$(\Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\mu})$

$\Rightarrow \Gamma^\alpha_{\lambda\nu} = \Gamma^\alpha_{\nu\lambda} \Rightarrow \boxed{\delta R_{\mu\nu}} = \boxed{\nabla_\lambda(\delta\Gamma^\lambda_{\nu\mu}) - \nabla_\nu(\delta\Gamma^\lambda_{\lambda\mu})}$

$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \Rightarrow \delta\Gamma^\sigma_{\mu\nu} = \frac{1}{2}\delta g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) +$

$+ \frac{1}{2}g^{\sigma\rho}(\partial_\mu\delta g_{\nu\rho} - \partial_\nu\delta g_{\rho\mu} - \partial_\rho\delta g_{\mu\nu})$

$\delta S_3 = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\,g^{\mu\nu}\left(\nabla_\lambda(\delta\Gamma^\lambda_{\nu\mu}) - \nabla_\nu(\delta\Gamma^\lambda_{\lambda\mu})\right) \underset{\nabla_\lambda g^{\mu\nu}=0}{=} \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\left[\nabla_\lambda(g^{\mu\nu}\delta\Gamma^\lambda_{\nu\mu}) - \nabla_\nu(g^{\mu\nu}\delta\Gamma^\lambda_{\lambda\mu})\right]$

DOBILI SMO INTEGRALA TIPA $\int d^4x\,\nabla_\lambda(\ldots)$, KATERA LAHKO PO STOKESU PREVEDEMO NA INTEGRAL PO ROBU PROSTORČASA. KER NA ROBU VELJA $\delta g_{\mu\nu} = 0$, $\delta\partial_\alpha g_{\mu\nu} = 0$ JE TUDI VARIACIJA CHRISTOFFELOV NA ROBU NIČ. SLEDI $\boxed{\delta S_3 = 0}$

SKUPAJ:

$$\cancel{\delta S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} \, g_{\mu\nu}}$$

$$\delta S = \int d^4x \sqrt{-g} \left[\frac{1}{16\pi G_N} \left(\frac{1}{2} g_{\mu\nu}(-R+2\Lambda) + R_{\mu\nu} \right) \delta g^{\mu\nu} + \frac{1}{\sqrt{-g}} \frac{\delta I_2}{\delta g^{\mu\nu}} \right) \delta g^{\mu\nu}$$

$$I_2 = \int d^4x \sqrt{-g} \, \mathcal{L}_M \qquad \delta I_2 = \int d^4x \left(\frac{\delta I_2}{\delta g^{\mu\nu}} \right) \delta g^{\mu\nu}$$

TOREJ: $\frac{1}{2} g_{\mu\nu}(-R+2\Lambda) + R_{\mu\nu} = -16\pi G_N \frac{1}{\sqrt{-g}} \frac{\delta I_2}{\delta g^{\mu\nu}}$, $T_{\mu\nu} := -2 \frac{1}{\sqrt{-g}} \frac{\delta I_2}{\delta g^{\mu\nu}}$

$$\Rightarrow \boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = 8\pi G_N T_{\mu\nu}}$$

b) $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu} \quad / g^{\mu\nu}$

$R - \frac{1}{2} \cdot 4 R = 8\pi G_N T^\lambda{}_\lambda \Rightarrow -R = 8\pi G_N T^\lambda{}_\lambda$

$$\Rightarrow R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} (8\pi G_N T^\lambda{}_\lambda) = 8\pi G_N T_{\mu\nu} \Rightarrow \boxed{R_{\mu\nu} = 8\pi G_N (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\lambda{}_\lambda)} \quad \square$$

ZA KONFORMNO SNOV $T^\lambda{}_\lambda = 0 \Rightarrow \boxed{R_{\mu\nu} = 8\pi G_N T_{\mu\nu}}$

c) $S = \int d^4x \sqrt{-g} \, f(R) \qquad \delta S = \int d^4x \, \delta(\sqrt{-g}) f(R) + \int d^4x \sqrt{-g} \, f'(R) \delta g^{\mu\nu} R_{\mu\nu} + \int d^4x \sqrt{-g} \, f'(R) g^{\mu\nu} \delta R_{\mu\nu} =$

$= \delta S_1 + \delta S_2 + \delta S_3$

$f'(R) = \frac{df}{dR}$ $\quad \delta S_1$ SMO ŽE V, δS_2 ŽE V PRAVI OBLIKI

δS_3: $\delta S_3 = \int d^4x \sqrt{-g} \, f'(R) g^{\mu\nu} \delta R_{\mu\nu}$ OD PREJ: $\boxed{\delta R_{\mu\nu} = \nabla_\lambda (\delta \Gamma^\lambda_{\nu\mu}) - \nabla_\nu (\delta \Gamma^\lambda_{\lambda\mu})}$

OD PREJ: $\delta \Gamma^\sigma_{\mu\nu} = \frac{1}{2} \delta g^{\sigma\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) + \frac{1}{2} g^{\sigma\rho} (\partial_\mu \delta g_{\nu\rho} + \partial_\nu \delta g_{\rho\mu} - \partial_\rho \delta g_{\mu\nu})$

OPAZIMO, ČE $\Gamma^{\sigma\,\prime}_{\mu\nu} = \Gamma^\sigma_{\mu\nu} + \delta \Gamma^\sigma_{\mu\nu}$, JE $\delta \Gamma^\sigma_{\mu\nu}$ RAZLIKA DVEH CHRISTOFFELOV IN TOREJ TENZOR.
ZGORNJA ENAČBA ZA $\delta \Gamma^\sigma_{\mu\nu}$ NI ZAPISANA V ČISTO TENZORSKI OBLIKI, SAJ SE ∂_μ NE TRANSFORMIRA KOT TENZOR.
ZATO MORAMO NAREDITI TRANSFORMACIJO $\partial_\mu \to \nabla_\mu$

TOREJ $\boxed{\delta \Gamma^\sigma_{\mu\nu}} = \frac{1}{2} \delta g^{\sigma\rho} \underbrace{(\nabla_\mu g_{\nu\rho} + \nabla_\nu g_{\rho\mu} - \nabla_\rho g_{\mu\nu})}_{=0, \text{ KER } \nabla_\lambda g_{\mu\nu} = 0} + \boxed{\frac{1}{2} g^{\sigma\rho} (\nabla_\mu \delta g_{\nu\rho} + \nabla_\nu \delta g_{\rho\mu} - \nabla_\rho \delta g_{\mu\nu})}$

TOREJ $\delta R_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha} (\nabla_\lambda \nabla_\nu \delta g_{\mu\alpha} + \nabla_\lambda \nabla_\mu \delta g_{\alpha\nu} - \nabla_\lambda \nabla_\alpha \delta g_{\nu\mu}) - \frac{1}{2} g^{\lambda\alpha} (\nabla_\nu \nabla_\lambda \delta g_{\mu\alpha} + \nabla_\nu \nabla_\mu \delta g_{\alpha\lambda} - \nabla_\nu \nabla_\alpha \delta g_{\lambda\mu})$

PO PREJŠNJI FORMULI PRETVORIMO $\delta g_{\mu\nu} \to \delta g^{\mu\nu}$. OPAZIMO ŠE, DA JE $\delta g_{\mu\nu}$ SIMETRIČEN SAJ JE RAZLIKA SIM. „MATRIK".

$$\Rightarrow g^{\mu\nu} \delta R_{\mu\nu} = \frac{1}{2} g^{\mu\nu} g^{\lambda\alpha} \Big[-g_{\mu\gamma} g_{\alpha\delta} \nabla_\lambda \nabla_\nu \delta g^{\gamma\delta} - g_{\alpha\gamma} g_{\nu\delta} \nabla_\lambda \nabla_\mu \delta g^{\gamma\delta} + g_{\nu\gamma} g_{\mu\delta} \nabla_\lambda \nabla_\alpha \delta g^{\gamma\delta} +$$
$$+ g_{\mu\gamma} g_{\alpha\delta} \nabla_\nu \nabla_\lambda \delta g^{\gamma\delta} + g_{\alpha\gamma} g_{\lambda\delta} \nabla_\nu \nabla_\mu \delta g^{\gamma\delta} - g_{\lambda\gamma} g_{\mu\delta} \nabla_\nu \nabla_\alpha \delta g^{\gamma\delta} \Big]$$

ČLENA, PODČRTANA Z - - - - : $\frac{1}{2} g^{\mu\nu} g^{\lambda\alpha} [-g_{\mu\gamma} g_{\alpha\delta} \nabla_\lambda \nabla_\nu \delta g^{\gamma\delta} + g_{\mu\gamma} g_{\alpha\delta} \nabla_\nu \nabla_\lambda \delta g^{\gamma\delta}] =$

$$= \frac{1}{2} [-\delta^\nu_\gamma \delta^\lambda_\delta \nabla_\lambda \nabla_\nu \delta g^{\gamma\delta} + \delta^\nu_\gamma \delta^\lambda_\delta \nabla_\nu \nabla_\lambda \delta g^{\gamma\delta}] =$$

$$= \frac{1}{2} (-\nabla_\delta \nabla_\gamma \delta g^{\gamma\delta} + \nabla_\gamma \nabla_\delta \delta g^{\gamma\delta}] = \frac{1}{2} [-\nabla_\delta \nabla_\gamma \delta g^{\gamma\delta} + \nabla_\delta \nabla_\gamma \delta g^{\delta\gamma}] = \boxed{0}$$

NEMI INDEKSI

ČLENA, PODČRTANA Z ___ : $-\frac{1}{2} g^{\mu\nu} g^{\lambda\alpha} g_{\alpha\gamma} g_{\nu\delta} \nabla_\lambda \nabla_\mu \delta g^{\gamma\delta} - \frac{1}{2} g^{\mu\nu} g^{\lambda\alpha} g_{\lambda\gamma} g_{\mu\delta} \nabla_\nu \nabla_\alpha \delta g^{\gamma\delta} =$

$$= -\frac{1}{2} \delta^\mu_\delta \delta^\lambda_\gamma \nabla_\lambda \nabla_\mu \delta g^{\gamma\delta} - \frac{1}{2} \delta^\nu_\delta \delta^\alpha_\gamma \nabla_\nu \nabla_\alpha \delta g^{\gamma\delta} = -\frac{1}{2} \nabla_\gamma \nabla_\delta \delta g^{\gamma\delta} - \frac{1}{2} \nabla_\delta \nabla_\gamma \delta g^{\gamma\delta} =$$

NEMI INDEKS!
IN $\delta g^{\gamma\delta} = \delta g^{\delta\gamma}$

$$= \boxed{-\nabla_\mu \nabla_\nu \delta g^{\mu\nu}}$$

OSTALA ČLENA: $\frac{1}{2} g^{\mu\nu} g^{\lambda\alpha} g_{\nu\gamma} g_{\mu\delta} \nabla_\lambda \nabla_\alpha \delta g^{\gamma\delta} + \frac{1}{2} g^{\mu\nu} g^{\lambda\alpha} g_{\alpha\gamma} g_{\lambda\delta} \nabla_\nu \nabla_\mu \delta g^{\gamma\delta} =$

$$= \frac{1}{2} \delta^\mu_\gamma g_{\mu\delta} \Box \delta g^{\gamma\delta} + \frac{1}{2} \delta^\lambda_\gamma g^{\mu\nu} g_{\lambda\delta} \nabla_\nu \nabla_\mu \delta g^{\gamma\delta} = \frac{1}{2} g_{\gamma\delta} \Box \delta g^{\gamma\delta} \cdot 2 =$$

$\Box = g^{\mu\nu} \nabla_\mu \nabla_\delta$

$$= \boxed{g_{\mu\nu} \Box \delta g^{\mu\nu}}$$

TOREJ $\delta S_3 = \int d^4x \sqrt{-g}\, f'(R) \left(g_{\mu\nu} \Box \delta g^{\mu\nu} - \nabla_\mu \nabla_\nu \delta g^{\mu\nu} \right) = \int d^4x \sqrt{-g} \left(g_{\mu\nu} \Box f'(R) - \nabla_\mu \nabla_\nu f'(R) \right) \delta g^{\mu\nu}$

2X PER PARTES IN VARIACIJE NIČ NA ROBU IN $\Box g_{\mu\nu} = 0$

SKUPAJ $\delta S = \int d^4x \sqrt{-g} \Big[-\frac{1}{2} g_{\mu\nu} f(R) + R_{\mu\nu} f'(R) + g_{\mu\nu} \Box f'(R) - \nabla_\mu \nabla_\nu f'(R) \Big) \delta g^{\mu\nu} = 0$

$$\Rightarrow \boxed{g_{\mu\nu} \left(\Box f'(R) - \frac{1}{2} f(R) \right) + R_{\mu\nu} f'(R) - \nabla_\mu \nabla_\nu f'(R) = 0}$$

③

a) $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0 \;/\; g^{\mu\nu}$

$R - 2R + 4\Lambda = 0 \Rightarrow \boxed{R = 4\Lambda}$

$\boxed{R_{\mu\nu}} = -\Lambda g_{\mu\nu} + 2\Lambda g_{\mu\nu} = \boxed{\Lambda g_{\mu\nu}}$

b) $R_{\rho\sigma\mu\nu} = \frac{1}{\alpha^2}(g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu})$

$R^{\lambda}{}_{\sigma\mu\nu} = \frac{1}{\alpha^2} \underbrace{g^{\lambda\rho}(g_{\rho\mu}}_{\delta^\lambda_\mu} g_{\sigma\nu} - \underbrace{g_{\rho\nu}}_{\delta^\lambda_\nu} g_{\sigma\mu}) = \frac{1}{\alpha^2}(\cancel{g}\,\delta^\lambda_\mu g_{\sigma\nu} - \delta^\lambda_\nu g_{\sigma\mu})$

~~$R_{\sigma\nu}$~~ $R^{\lambda}{}_{\sigma\lambda\nu} = \frac{1}{\alpha^2}(\delta^\lambda_\lambda g_{\sigma\nu} - \delta^\lambda_\nu g_{\sigma\lambda}) = \frac{1}{\alpha^2}(4 g_{\sigma\nu} - g_{\sigma\nu}) = \boxed{\frac{1}{\alpha^2} 3 g_{\sigma\nu} = R_{\sigma\nu}}$

$\Rightarrow \frac{3}{\alpha^2} g_{\mu\nu} = \Lambda g_{\mu\nu} \Rightarrow \boxed{\alpha = \pm\sqrt{\frac{3}{\Lambda}}}$

c) $T_{\mu\nu} = \cancel{(\varepsilon+p)u^\mu u^\nu + p g}\;\; (\varepsilon + p) u_\mu u_\nu + p g_{\mu\nu}$

$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0$

$\Updownarrow$

$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} = 8\pi G_N T_{\mu\nu}$

$\Rightarrow 8\pi G_N T_{\mu\nu} = -\Lambda g_{\mu\nu}$

$\Rightarrow \boxed{p = -\frac{\Lambda}{8\pi G_N}}$ $\boxed{\varepsilon = -p}$

d) $ds^2_{(1)} = -(1 - \frac{r^2}{\ell^2}) dt^2 + \frac{dr^2}{(1 - \frac{r^2}{\ell^2})} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$

$ds^2_{(2)} = -d\tau^2 + e^{2\tau/\ell}[d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta \, d\phi^2]$

IZ ZADNJIH ČLENOV: $\boxed{r = e^{\tau/\ell}\rho}$

TRANSFORMACIJSKO PRAVILO ZA TENZORJE: $g_{\tau\tau} = \left(\frac{dt}{d\tau}\right)^2 g_{tt} + \left(\frac{dr}{d\tau}\right)^2 g_{rr} + \overset{0}{\cancel{\left(\frac{d\theta}{d\tau}\right)^2}} g_{\theta\theta} + \overset{0}{\cancel{\left(\frac{d\phi}{d\tau}\right)^2}} g_{\phi\phi}$

$\Rightarrow g_{\tau\tau} = -1 = -\left(\frac{dt}{d\tau}\right)^2 (1 - \frac{r^2}{\ell^2}) + \left(\frac{\rho}{\ell} e^{\tau/\ell}\right)^2 (1 - \frac{r^2}{\ell^2})^{-1} =$

$= -\left(\frac{dt}{d\tau}\right)^2 (1 - \frac{\rho^2}{\ell^2} e^{2\tau/\ell}) + \frac{\rho^2}{\ell^2} e^{2\tau/\ell} (1 - \frac{\rho^2}{\ell^2} e^{2\tau/\ell})^{-1}$

$\Rightarrow \left(\frac{dt}{d\tau}\right)^2 = \frac{1}{1 - \frac{\rho^2}{\ell^2} e^{2\tau/\ell}} + \frac{\rho^2}{\ell^2} \frac{e^{2\tau/\ell}}{(1 - \frac{\rho^2}{\ell^2} e^{2\tau/\ell})^2} = \frac{1}{(1 - \frac{\rho^2}{\ell^2} e^{2\tau/\ell})^2} \Rightarrow \boxed{\frac{dt}{d\tau} = \frac{1}{1 - \frac{\rho^2}{\ell^2} e^{2\tau/\ell}}}$

ŠE: $g_{\rho\rho} = e^{2\tau/\ell} = -\left(\frac{dt}{d\rho}\right)^2 (1 - \frac{\rho^2}{\ell^2} e^{2\tau/\ell}) + \underbrace{\left(\frac{dr}{d\rho}\right)^2}_{= e^{2\tau/\ell}} (1 - \frac{\rho^2}{\ell^2} e^{2\tau/\ell})^{-1}$

$\Rightarrow \left(\frac{dt}{d\rho}\right)^2 = -\frac{e^{2\tau/\ell}}{1 - \frac{\rho^2}{\ell^2} e^{2\tau/\ell}} + \frac{e^{2\tau/\ell}}{(1 - \frac{\rho^2}{\ell^2} e^{2\tau/\ell})^2} = \frac{e^{2\tau/\ell} \frac{\rho^2}{\ell^2} e^{2\tau/\ell}}{(1 - \frac{\rho^2}{\ell^2} e^{2\tau/\ell})^2}$

$\Rightarrow \boxed{\frac{dt}{d\rho} = \frac{\rho}{\ell} \cdot \frac{e^{2\tau/\ell}}{1 - \frac{\rho^2}{\ell^2} e^{2\tau/\ell}}}$

$$t = \frac{e^{2\tau/\ell}}{\ell} \int d\rho \, \frac{\rho}{1-\frac{\rho^2}{\ell^2}e^{2\tau/\ell}} + C(\tau) \,, \quad u = 1-\frac{\rho^2}{\ell^2}e^{2\tau/\ell}$$

$$\Rightarrow \boxed{t} = \frac{e^{2\tau/\ell}}{\ell} \cdot \frac{\ell^2}{2} e^{-2\tau/\ell} \int du \, \frac{1}{u} + C(\tau) = \boxed{\frac{\ell}{2} \ln\left(1-\frac{\rho^2}{\ell^2}e^{2\tau/\ell}\right) + C(\tau)}$$

OD PRE)

$$\frac{dt}{d\tau} = \frac{\ell}{2} \cdot \frac{1}{1-\frac{\rho^2}{\ell^2}e^{2\tau/\ell}} \left(-\frac{\rho^2}{\ell^2}\cdot\frac{2}{\ell}e^{2\tau/\ell}\right) + C'(\tau) = \frac{-\frac{\rho^2}{\ell^2}e^{2\tau/\ell}}{1-\rho^2/\ell^2 e^{2\tau/\ell}} + C'(\tau) = \frac{1}{1-\frac{\rho^2}{\ell^2}e^{2\tau/\ell}}$$

$$\Rightarrow C'(\tau) = \frac{1-\rho^2/\ell^2 e^{2\tau/\ell}}{1-\rho^2/\ell^2 e^{2\tau/\ell}} = 1 \Rightarrow \boxed{C(\tau) = \tau}$$

$$\boxed{t = \frac{\ell}{2} \ln\left(1-\frac{\rho^2}{\ell^2}e^{2\tau/\ell}\right) + \tau}$$

4) $$S_M = \int d^4x \sqrt{-g}\, F_{\mu\nu}F^{\mu\nu} = \int d^4x \sqrt{-g}\, F_{\mu\nu}\, g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}$$

V II ČLENU PRIMERNO PREIMENUJEMO INDEKSE

$$\delta S = \int d^4x \, \delta(\sqrt{-g})\, F_{\mu\nu}F^{\mu\nu} + \int d^4x \sqrt{-g}\, F_{\mu\nu}F_{\alpha\beta}\left(\delta g^{\mu\alpha} g^{\nu\beta} + g^{\mu\alpha}\delta g^{\nu\beta}\right) =$$

$$= \int d^4x \sqrt{-g}\left(-\frac{1}{2}g_{\mu\nu}\right) F_{\alpha\beta}F^{\alpha\beta}\delta g^{\mu\nu} + \int d^4x \sqrt{-g}\, F_{\mu\alpha}F_{\nu\beta}\, g^{\alpha\beta}\delta g^{\mu\nu} + \int d^4x \sqrt{-g}\, F_{\beta\nu}F_{\alpha\mu}\, g^{\alpha\beta}\delta g^{\mu\nu}) =$$

$$= \int d^4x \sqrt{-g}\left(-\frac{1}{2}g_{\mu\nu}\right) F_{\alpha\beta}F^{\alpha\beta}\delta g^{\mu\nu} + \int d^4x \sqrt{-g}\; 2 g^{\alpha\beta} F_{\mu\alpha}F_{\nu\beta}\,\delta g^{\mu\nu}$$

$(F_{\alpha\beta} = -F_{\beta\alpha})$

$$\boxed{T_{\mu\nu}} = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = \boxed{g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - 4F_{\mu}{}^{\beta}F_{\nu\beta}}$$