3. izpit iz Matematike 2, FMF, Aplikativna fizika

29. 8. 2024

Čas pisanja je 120 minut. Veliko uspeha!

Sedež (2.02)

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Ime in priimek

Vpisna številka

1. naloga (25 točk)

Dana je funkcija

$$f(x,y) = \begin{cases} y^2 \cos \frac{x}{x^2+y^2} & ,(x,y) \neq (0,0) \\ a & ,(x,y) = (0,0) \end{cases}$$

a) Določite konstanto a tako, da bo funckija f zvezna v točki $(0,0)$.

b) Določite funkcijska predpisa parcialnih odvodov $\frac{\partial f}{\partial x}$ in $\frac{\partial f}{\partial y}$

c) Preverite ali je funkcija f diferenciabilna v točki $(0,0)$

a) $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{r\to 0} r^2 \underbrace{\sin^2\varphi \cos\left(\frac{\cos\varphi}{r}\right)}_{\text{OMEJENO}} = 0 \Rightarrow a = 0$

$x = r\cos\varphi$
$y = r\sin\varphi$

b) $(x,y) \neq (0,0)$:

$$f_x = -y^2 \sin\left(\frac{x}{x^2+y^2}\right) \cdot \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = y^2 \sin\left(\frac{x}{x^2+y^2}\right) \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$f_y = 2y\cos\left(\frac{x}{x^2+y^2}\right) + y^2 \sin\left(\frac{x}{x^2+y^2}\right) \frac{x\cdot 2y}{(x^2+y^2)^2} = 2y\left(\cos\left(\frac{x}{x^2+y^2}\right) + \sin\left(\frac{x}{x^2+y^2}\right)\frac{xy^2}{(x^2+y^2)^2}\right)$$

$(x,y) = (0,0)$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{h^2}{h} = 0$$

c) $f(h,k) = f(0,0) + f_x(0,0)h + f_y(0,0)k + R(h,k) \Rightarrow R(h,k) = f(h,k)$

$$\lim_{(h,k)\to(0,0)} \frac{R(h,k)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{k^2 \cos\frac{h}{h^2+k^2}}{\sqrt{h^2+k^2}} = \lim_{r\to 0} r \underbrace{\sin^2\varphi \cos\left(\frac{\cos\varphi}{r}\right)}_{\text{OMEJENO}} = 0$$

$h = r\cos\varphi$
$k = r\sin\varphi$

$\Rightarrow$ JE DIFERENCIABILNA V $(0,0)$

2. naloga (25 točk)

Dana je funkcija $f : \mathbb{R}^2 \to \mathbb{R}$ s predpisom $f(x,y) = xye^{-(x^2+y^2)/2}$.

a) Poiščite in klasificirajte lokalne ekstreme funkcije f.

b) Dokažite, da so lokalni ekstremi iz točke a) celo globalni ekstremi funkcije f. Namig: Funkcijo f zapišite kot produkt dveh funkcij ene spremenljivke.

a) $f_x = e^{-(x^2+y^2)/2}(y - x^2y) = ye^{-(x^2+y^2)/2}(1-x^2)$

STAC. TOČKE: $f_x = f_y = 0$:

$(x,y) = (0,0)$

$(x,y) = (\pm 1, \pm 1)$

$f_y = xe^{-(x^2+y^2)/2}(1-y^2)$

$f_{xx} = ye^{-(x^2+y^2)/2}(-x(1-x^2) - 2x) = ye^{-(x^2+y^2)/2}(x^3 - 3x)$

$f_{yy} = xe^{-(x^2+y^2)/2}(y^3 - 3y)$

$f_{xy} = (1-x^2)e^{-(x^2+y^2)/2}(1-y^2)$

$$H = e^{-(x^2+y^2)/2}\begin{bmatrix} y(x^3-3x) & (1-x^2)(1-y^2) \\ (1-x^2)(1-y^2) & x(y^3-3y) \end{bmatrix}$$

$$\det(H) = e^{-(x^2+y^2)/2}\left(xy(y^3-3y)(x^3-3x) - (1-x^2)^2(1-y^2)^2\right)$$

$(x,y) = (0,0)$: $\det(H) < 0$ SEDLO, $(x,y) = (1,1)$: $\det(H) > 0$, $f_{xx} < 0$ LOK. MAKS.

$(x,y) = (-1,-1)$: $\det(H) > 0$, $f_{xx} < 0$ LOK. MAKS.

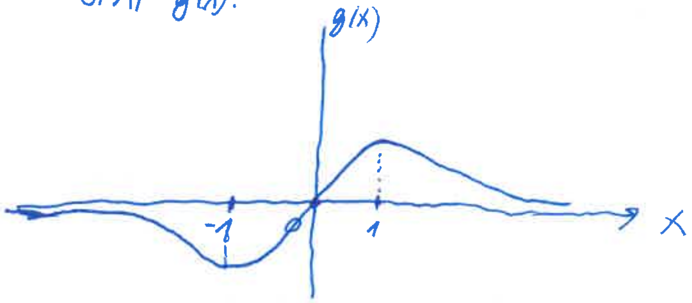
$(x,y) = (1,-1)$: $\det(H) > 0$, $f_{xx} > 0$ LOK. MIN.

$(x,y) = (-1,1)$ — // —

b) $f(x,y) = g(x)\cdot g(y)$, $g(x) = xe^{-x^2/2}$

$g'(x) = e^{-x^2/2}(1-x^2)$

SKICIRAMO GRAF $g(x)$:



IZ SKICE JE JASNO, DA VREDNOST $|f(x,y)|$ NE MORE PRESEČI $|g(1)|^2$, VREDNOST V LOKALNIH MAKS., MIN. PA JE PRAV TAKA.

KONKRETNEJE:

$\forall x,y \in \mathbb{R}$ $|f(x,y)| = |g(x)g(y)| \le |g(x)|\,|g(y)| \le$

$\le \|g\|_\infty^2 = e^{-1}$

$\uparrow$

$=$ VREDNOST $|f(x,y)|$ V LOK. MIN, MAX.

~~KONKRETNEJE: $\forall x,y$ $|f(x,y)| = |g(x)|\cdot|g(y)| \le |g(x)|^2 \le \|g\|_\infty^2 = e^{-1}$~~

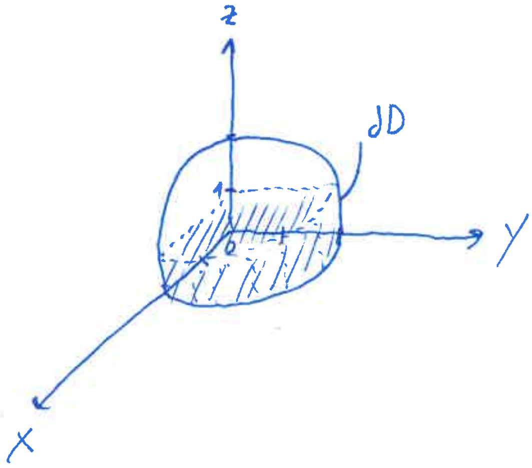
3. naloga (25 točk)

Dano je območje

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 4,\ x \geq 0,\ y \geq 0 \text{ in } 0 \leq z \leq 1\},$$

katerega rob ∂D je sestavljen iz petih ploskev. Skicirajte območje D in izračunajte ploskovni integral skalarnega polja $f(x, y, z) = xyz$ po robu območja ∂D, t.j.

$$\iint_{\partial D} f\, dS.$$



$F(x,y,z) = xyz = 0$ NA ROBOVIH $x=0,\ y=0,\ z=0$, OSTANE INTEGRAL PO

$x^2+y^2+z^2 \leq 4,\ x \geq 0,\ y \geq 0,\ z=1$ (S_1) IN $x^2+y^2+z^2=4,\ x\geq 0,\ y \geq 0,\ 0 \leq z \leq 1$ (S_2)

$$\iint_{\partial D} f\,dS = \iint_{S_1} f\,dS + \iint_{S_2} f\,dS = \int_0^{\sqrt{3}} dr \int_0^{\frac{\pi}{2}} d\varphi \cdot r\cos\varphi\, r\sin\varphi \cdot \underbrace{r}_{J} + \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{6}} d\theta\, 2\cos\theta\cos\varphi \cdot 2\cos\theta\sin\varphi \cdot 2\sin\theta \cdot \underbrace{2^2\cos\theta}_{J} =$$

S_1: $z=1 \Rightarrow x^2+y^2 \leq 3$

POLARNE KOORD.

$r \leq \sqrt{3}$

$\varphi \in [0, \frac{\pi}{2}]$

S_2: SFERIČNE KOORD.

$R=2$

$\varphi \in [0, \frac{\pi}{2}]$

$0 \leq z \leq 1$

$0 \leq 2\sin(\theta) \leq 1$

$\Rightarrow \theta \in [0, \frac{\pi}{6}]$

$$= \frac{R^4}{4}\Big|_0^{\sqrt{3}} \frac{u^2}{2}\Big|_0^1 + 2^5 \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{6}} d\theta \cos^3\theta \sin\theta \cos\varphi \sin\varphi =$$

$u = \cos\theta$

$du = -\sin\theta$

$$= \frac{9}{4} \cdot \frac{1}{2} + 2^5 \cdot \frac{1}{2} \cdot \int_1^{\frac{\sqrt{3}}{2}} (-1)\, u^3\, du =$$

$$= \frac{9}{8} + \frac{2^4}{4} \cdot u^4\Big|_{\frac{\sqrt{3}}{2}}^1 = \frac{9}{8} + 4\left(1 - \frac{9}{16}\right) =$$

$$= \frac{9}{8} + 4\,\frac{7}{16} = \frac{9}{8} + \frac{7}{4} = \frac{23}{8}$$

4. naloga (25 točk)

Dana je diferencialna enačba

$$y' + \frac{2x}{1+x^2}y = y^2.$$

a) Poiščite njeno splošno rešitev.

b) Poiščite tisto rešitev, ki zadošča pogoju $y(0) = 1$.

a) $y' + \frac{2x}{1+x^2}y = y^2$ BERNOULLI $u = y^{-1}$, $u' = -y^{-2}y'$

$-u' + \frac{2x}{1+x^2}u = 1$

HOM:

$u' = \frac{2x}{1+x^2}u$

$\int \frac{du}{u} = \int \frac{2x}{1+x^2}dx + C \Rightarrow$

$\ln u = \ln(1+x^2) + C \Rightarrow u_H = \tilde{C}(1+x^2)$

PART: $u(x) = \tilde{C}(x)(1+x^2) \Rightarrow -\tilde{C}'(x)(1+x^2) = 1 \Rightarrow \tilde{C}(x) = -\int \frac{1}{1+x^2}dx = -\operatorname{arctg}(x)$

$u_p = -\operatorname{arctg}(x)(1+x^2)$

$u(x) = (1+x^2)(C \mp \operatorname{arctg}(x))$

$\Rightarrow \boxed{y(x) = \frac{1}{(1+x^2)(C \mp \operatorname{arctg}(x))}}$

b) $y(0) = 1 \Rightarrow C = 1 \Rightarrow \boxed{y(x) = \frac{1}{(1+x^2)(1-\operatorname{arctg}(x))}}$