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Permutation groups and combinatorial structures

Homework 2

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Problem statement

Let G be a transitive permutation group on Ω and let $\Delta \subset \Omega$ be a block for G. Let $K = G_\Delta^\Delta \leq \mathrm{Sym}(\Delta)$. and let $H = G^\mathcal{B}$ ($\mathcal{B} = \{\Delta^g | g \in G\}$). Prove that then there exists a pair $\phi : \Omega \to \Delta \times \mathcal{B}$ a bijection and $\iota : G \to K \wr H$ a monomorphism such that (ϕ, ι) is an isomorphism of actions of G on Ω and $\iota(G)$ on $\Delta \times \mathcal{B}$. I will add an additional assumption that G is finite, as we have already established that in this course we are interested in actions of finite groups (in turn, $\mathcal{B}$ is then finite as well.)

Solution

Preliminary definitions / notation

Let us first enumerate the elements of the partition $\mathcal{B} = \{\Delta_1, \ldots \Delta_m\}$, where $\Delta_1 = \Delta$ and $\Delta_2, \ldots, \Delta_m$ other elements of the partition. Furthermore let us choose $g_1, \ldots, g_m \in G$ such that $\Delta_i = \Delta^{g_i}$. Note that such a choice may not be unique and that different choices might yield a different isomorphism of actions, but this is fine, since we are only interested in proving one exists - from here on we will fix our choice of $g_1, \ldots, g_m$. Since $\mathcal{B}$ is a partition of Ω, we know that each $\omega \in \Omega$ belongs to a unique Δ_i for some $i \in \{1, \ldots, m\}$. Additionally, when considering action of group H on $\mathcal{B}$ we will write $i^h \in \{1, \ldots, m\}$ for some $h \in H$. This will mean that Ω_i is mapped to Ω_{i^h} under h. Similarly, g_i would be mapped to g_{i^h}.

The group $K \wr H$

The group $H = G^\mathcal{B}$ acts on $\mathcal{B}$. As mentioned in the lectures, since $\mathcal{B}$ is finite we can consider K^m to be the space of all functions from $\mathcal{B}$ to K. The "function composition"is then just componentwise multiplication in K^m. Since H acts on $\mathcal{B}$, we also know we have an action of H on K^m by permuting the components. For $h \in H$ we have: $(k_1, \ldots, k_m)^h = (k_{1^{h^{-1}}}, \ldots, k_{m^{h^{-1}}})$ As we know from the lectures, this gives rise to a homomorphism $\theta : H \to \mathrm{Aut}(K^m)$. We can use it to form a semi-direct product.

$K \wr H$ is the semi direct product $K^m \rtimes_\theta H$. The elements of $K \wr H$ are therefore simply elements of the set $K^m \times H$, while the product operation is $(k_1, k_2, \ldots, k_m, h)(k'_1, k'_2, \ldots, k'_m, h') = (k_1 k'_{1^h}, \ldots, k_m k'_{m^h}, hh')$.

The map ϕ

Definition of ϕ

We have already established that for every $\omega \in \Omega$ there is a unique $\Delta_i \in \mathcal{B}$ such that $\omega \in \Delta_i$. Using same notation as before, we can say $\omega \in \Delta_i = \Delta^{g_i}$. It then follows that there is a unique $\delta \in \Delta$ such that $\delta^{g_i} = \omega$, namely $\omega^{g_i^{-1}} \in \Delta$. Here we have used the known fact that the map $\delta \to \delta^{g_i}$ gives a bijection between Δ and Δ^{g_i}.
We define a map $\phi : \Omega \to \Delta \times \mathcal{B}$ by assigning to each ω the just described unique pair (δ, Δ_i).

ϕ is a bijection

The fact that ϕ is a bijection is clear: It is clearly surjective, as for each $(\delta, \Delta_i) \in \Delta \times \mathcal{B}$, we have $\phi(\delta^{g_i}) = (\delta, \Delta_i)$. It is also injective: Take $\phi(\omega_1) = \phi(\omega_2) = (\delta, \Delta_i)$. This means ω_1, ω_2 lie in the same block $\Delta_i = \Delta^{g_i}$. By the definition of ϕ it holds that $\omega_1 = \delta^{g_i} = \omega_2$. ϕ is therefore really also injective and in turn a bijection.

The group morphism ι

definition of ι and well-definedness

We are looking for a monomorphism $\iota : G \to K \wr H$. Therefore, we need a map from G to $K^m \times H$. Denote with g^K an image of $g \in G_\Delta$ under the action $G \rightsquigarrow K$ and g^H an image of $g \in G$ under the action $G \rightsquigarrow H$.

Define a map $\iota(g) = ((g_1 g g_{1^g}^{-1})^K, (g_2 g g_{2^g}^{-1})^K, \ldots, (g_m g g_{m^g}^{-1})^K, g^H)$, where we have abused notation and wrote, for example 1^g as a shorthand for 1^{g^H}. In order for this to be well defined we must check that $g_i g g_{i^g}^{-1} \in G_\Delta$ for each $g \in G$ and $i \in \{1, \ldots, m\}$. Take an arbitrary $\delta \in \Delta$. We would like to check that

$\delta^{g_i g g_{i^g}^{-1}} \in \Delta$. Observe that $\delta^{g_i} \in \Delta_i$. The block Δ_i gets mapped to Δ_{i^g} by g. Therefore $\delta^{g_i g} \in \Delta_{i^g}$. It then follows that $\delta^{g_i g g_{i^g}^{-1}} \in \Delta$, which proves that ι is well defined.

ι is a group homomorphism

ι is a homomorphism: First, observe that, since H and K are group actions, we have $(id_G)^H = id_H$ and $(id_G)^K = id_K$. Using this fact, plugging id_G in ι immediately gives: $\iota(id_G) = (id_K, \ldots, id_K, id_H)$.
Now consider $g, g' \in G$:
Observe that $\iota(g)\iota(g') = ((g_1 g g_{1^g}^{-1})^K, \ldots, (g_m g g_{m^g}^{-1})^K, g^H)((g_1 g' g_{1^{g'}}^{-1})^K, \ldots, (g_m g' g_{m^{g'}}^{-1})^K, (g')^H)$ The product on the right hand side is in $K \wr H$. Denoting $k'_i = (g_i g' g_{i^{g'}}^{-1})^K$ observe that $k'_{i^h} = (g_{i^h} g' g_{i^{hg'}}^{-1})^K$. This observation allows us to easily compute the product in $K \wr H$.
We get $\iota(g)\iota(g') = ((g_1 g g_{1^g}^{-1})^K (g_{1^g} g' g_{1^{gg'}}^{-1})^K, \ldots, (g_m g g_{m^g}^{-1})^K (g_{m^g} g' g_{m^{gg'}}^{-1})^K, g^H g'^H)$, which can be further simplified into $((g_1 g g' g_{1^{gg'}}^{-1})^K, \ldots, (g_m g g' g_{m^{gg'}}^{-1})^K, (gg')^H) = \iota(gg')$. ι therefore really is a homomorphism.

ι is a monomorphism

It is also a monomorphism: Assume $\iota(g) = (id_K, \ldots, id_K, id_H)$. From $g^H = id_H$ it follows that $g \in G$ fixes the blocks (as in, does not map one block to another). We also have for every $i \in \{1, \ldots, m\}$ that $(g_i g g_i^{-1})^K = id_K$. it follows that $g_i g g_i^{-1} \in G_{(\Delta)}$. This implies that g fixes every element from Δ_i: We know every such element can be expressed as $\delta_1^{g_i}$ for some $\delta_1 \in \Delta$. Assume that g maps it to some $\delta_2^{g_i} \in \Delta_i$ (we know it must map it to the same block). Therefore, $\delta_1^{g_i g} = \delta_2^{g_i}$. Equivalently this means $\delta_1^{g_i g g_i^{-1}} = \delta_2$. But $g_i g g_i^{-1} \in G_{(\Delta)}$ and therefore $\delta_1 = \delta_2$. Naturally then also $\delta_1^{g_i} = \delta_2^{g_i}$ and we see that g fixes every $\delta^{g_i} \in \Delta_i$. As this holds for every $i \in \{1, \ldots, m\}$ and every $\delta \in \Delta$ it holds that g fixes every $\omega \in \Omega$. As our initial action G is faithful, this implies $g = id_G$. Therefore, ι is a monomorphism. Restricting the codomain we get an isomorphism $\iota : G \to \iota(G)$.

Action isomorphism

First let us remember how we defined the imprimitive action of wreath product: $(\delta, \Delta_i)^{(k_1, \ldots km)h} = (\delta^{k_i}, \Omega_{i^h})$.
We have already proved that $\phi : \Omega \to \Delta \times \mathcal{B}$ is a bijection and that $\iota : G \to \iota(G) \subset K \wr H$ is an isomorphism. What we now need to prove is that (ϕ, ι) really is an action isomorphism between the action of G on Ω and the action of $\iota(G)$ on $\Delta \times \mathcal{B}$. $\iota(G) \subset K \wr H$ will act in the same manner as the imprimitive action of the wreath product.
We must prove that $\forall g \in G, \forall \omega \in \Omega$: $\phi(\omega^g) = \phi(\omega)^{\iota(g)}$.

Left hand side

Let us denote $\phi(\omega) = (\delta, \Omega_i)$, where $\delta^{g_i} = \omega$. Let's decipher what $\phi(\omega^g)$ will be. As ω is in the block Ω_i, ω^g will be in the block Ω_{i^g} and it holds $\delta^{g_i g} = \omega^g \in \Delta^{g_{i^g}}$. It then follows that $\delta^{g_i g g_{i^g}^{-1}} \in \Delta$ is the unique element of Δ that gets mapped to ω^g under g_{i^g}. Therefore, $\phi(\omega^g) = (\delta^{g_i g g_{i^g}^{-1}}, \Omega_{i^g})$.

Right hand side

On the other hand $\phi(\omega)^{\iota(g)} = (\delta, \Omega_i)^{((g_1 g g_{1^g}^{-1})^K, \ldots, (g_m g g_{m^g}^{-1})^K, g^H)} = ((\delta^{g_i g g_{i^g}^{-1}}, \Omega_{i^g})$.

The two sides are equal and we can conclude we have found an isomorphism of actions.