1. kolokvij iz Matematike II, FMF, Aplikativna fizika

27. 11. 2024

Čas pisanja je 120 minut. Veliko uspeha!

Ime in priimek

Sedež (2.01)

Vpisna številka

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1. naloga

Dani sta funkciji $f(x,y) = 4x^2 + y^2 - 1$ in $g(x,y) = \sqrt{1 - \ln(x) - \ln(y)}$.

a) Določite definicijsko območje in zalogo vrednosti funkcije f. S pomočjo nivojnic ali presekov v $x = 0$, $y = 0$ ravninah skicirajte graf funkcije f.

b) V ravnini skicirajte definicijsko območje funkcije g in določite, ali gre za odprto ali zaprto množico. Odgovor utemeljite.

c) Ali lahko funkcijo zvezno razširimo v točkah $(x, 0)$ za $x > 0$?

a) $D_F = \mathbb{R}^2$, $Z_F = [-1, \infty)$ 1 3

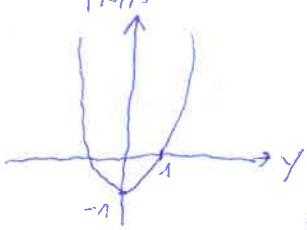
NIVOJNICE: $C = 4x^2 + y^2 - 1 \Rightarrow 4x^2 + y^2 = 1 + C$

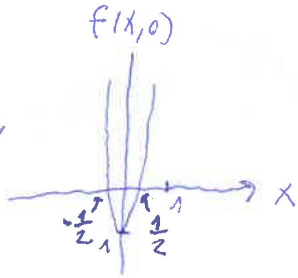
$C = -1 \Leftrightarrow x = y = 0$

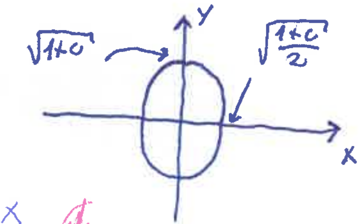
$C > -1$: $\frac{4x^2}{1+C} + \frac{y^2}{1+C} = 1 \leftarrow$ ELIPSE

PRESEKI: $F(0,y) = y^2 - 1$

$F(x,0) = 4x^2 - 1$







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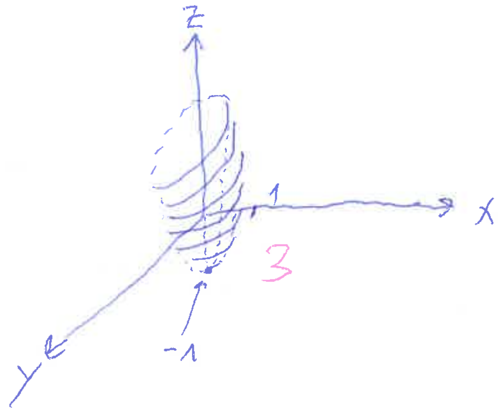

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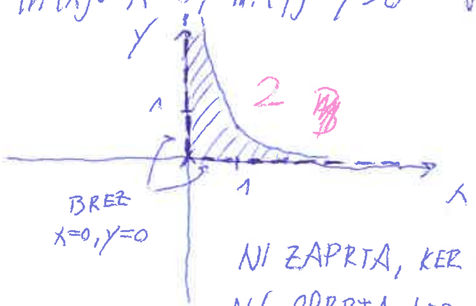
b) D_g: $\ln(x)$: $x > 0$, $\ln(y)$: $y > 0$ $\quad \sqrt{1 - \ln x - \ln y} \Rightarrow 1 - \ln x - \ln y \geq 0$ 5

$\ln x + \ln y \leq 1 \;/e$

$e^{\ln x + \ln y} \leq e$

$x \cdot y \leq e$

$y \leq e/x$ 3



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NI ZAPRTA, KER NE VSEBUJE ROBA

NI ODPRTA, KER NI ENAKA SVOJI NOTRANJOSTI 2

c) NE, KAJTI VREDNOST TE FUNKCIJE NA TOČKI $(x, 0)$ BI MORALA BITI $\lim_{y \to 0} \sqrt{1 - \ln(x) - \ln(y)}$ TA LIMITA PA NE OBSTAJA (GRE PROTI $+\infty$) 2

2. naloga

Dana je funkcija

$$f(x,y)=\begin{cases}\frac{x^2(y+1)+y^2}{x^2+y^2}, & (x,y)\neq(0,0)\\ 1, & (x,y)=(0,0).\end{cases}$$

a) Dokažite, da je funkcija f zvezna v $(0,0)$.

b) Določite funkcijska predpisa parcialnih odvodov $\frac{\partial f}{\partial x}$ in $\frac{\partial f}{\partial y}$.

c) Ali je f v izhodišču diferenciabilna?

a) $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{r\to 0} \frac{r^2\cos^2\varphi\,(r\sin\varphi+1)+r^2\sin^2\varphi}{r^2} = \lim\limits_{r\to 0}\big(\underbrace{\cos^2\varphi+\sin^2\varphi}_{1} + \overset{0}{r}\sin\varphi\cos^2\varphi\big) = 1$ ✓ 4

$x = r\cos\varphi$
$y = r\sin\varphi$

Do TU JE BILO NA KOLOKVIJU DOVOLJ

b) $(x,y)\neq 0$: $\frac{\partial f}{\partial x} = \frac{2x(y+1)(x^2+y^2)-(x^2(y+1)+y^2)2x}{(x^2+y^2)^2} = \frac{2x\left[\cancel{x^2y}+y^3+\cancel{y^2}+\cancel{x^2}-\cancel{x^2y}-\cancel{x^2}-\cancel{y^2}\right]}{(x^2+y^2)^2} = \frac{2xy^3}{(x^2+y^2)^2}$

3 $\frac{\partial f}{\partial y} = \frac{(x^2+2y)(x^2+y^2)-(x^2(y+1)+y^2)(2y)}{(x^2+y^2)^2} = \frac{x^4+\cancel{x^2y^2}+2x^2y+\cancel{2y^3}-\cancel{2x^2y^2}\ldots-\cancel{2yx^2}-\cancel{2y^3}}{(x^2+y^2)^2} = \frac{x^4-x^2y^2}{(x^2+y^2)^2}$ 3

$(x,y)=(0,0)$:

3 $\frac{\partial f}{\partial x} = \lim\limits_{h\to 0}\frac{f(h,0)-1}{h} = \lim\limits_{h\to 0}\frac{1-1}{h} = 0$ $\qquad \frac{\partial f}{\partial x} = \begin{cases}\frac{2xy^3}{(x^2+y^2)^2}; & (x,y)\neq(0,0)\\ 0; & (x,y)=(0,0)\end{cases}$

$\frac{\partial f}{\partial y} = \lim\limits_{h\to 0}\frac{f(0,h)-1}{h} = \lim\limits_{h\to 0}\frac{1-1}{h} = 0$ 3 $\qquad \frac{\partial f}{\partial y} = \begin{cases}\frac{x^2(x^2-y^2)}{(x^2+y^2)^2}; & (x,y)\neq(0,0)\\ 0; & (x,y)=(0,0)\end{cases}$

c) $f(h,k) = \overset{f(0,0)}{1} + \overset{0}{f_x(0,0)}h + \overset{0}{f_y(0,0)}k + R(h,k)$ 1

$\Rightarrow R(h,k) = f(h,k)-1 = \frac{h^2(k+1)+k^2}{h^2+k^2}-1 = \frac{h^2k}{h^2+k^2}$ 3

$\lim\limits_{(h,k)\to(0,0)}\frac{R(h,k)}{\sqrt{h^2+k^2}} = \lim\limits_{(h,k)\to(0,0)}\frac{h^2k}{(h^2+k^2)^{3/2}} = \lim\limits_{r\to 0}\frac{r^3\cos^2\varphi\sin\varphi}{r^3} = \cos^2\varphi\sin\varphi$ 3

$h = r\cos\varphi$
$k = r\sin\varphi$

$\Rightarrow$ LIMITA NE OBSTAJA

$\Rightarrow$ NI DIFERENCIABILNA

3. naloga

Dana je funkcija f s predpisom $f(x,y) = ye^{x^3y^2} + \sin(x^2)$.

a) S pomočjo linearnega približka približno izračunajte vrednost funkcije f v točki $(-0.1, 1.1)$.

b) Razvijte f v Taylorjevo vrsto okoli točke $(0,0)$ in določite vrednosti odvodov $\frac{\partial^{18} f}{\partial x^{18}}(0,0)$ in $\frac{\partial^{11} f}{\partial x^6 \partial y^5}(0,0)$.

a) $x_0 = 0,\ y_0 = 1 \qquad f(0,1) = 1$

$h = -0.1 \quad k = 0.1$

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$$\frac{\partial f}{\partial x} = e^{x^3y^2} 3x^2y^3 + \cos(x^2)2x \qquad \frac{\partial f}{\partial x}(0,1) = 0$$

$$\frac{\partial f}{\partial y} = e^{x^3y}(1 + 2y^2x^3), \qquad \frac{\partial f}{\partial y}(0,1) = 1$$

3

$$f(-0.1, 1.1) \approx 1 + 1 \cdot k = 1 + 1 \cdot 0.1 = 1.1$$

2

b) $$f(x,y) = ye^{x^3y^2} + \sin(x^2) = y \sum_{k=0}^{\infty} \frac{x^{3k}y^{2k}}{k!} + \sum_{k=0}^{\infty} (-1)^k \frac{(x^2)^{2k+1}}{(2k+1)!} =$$

$$= \boxed{\sum_{k=0}^{\infty} \frac{x^{3k}y^{2k+1}}{k!} + \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+2}}{(2k+1)!}}$$

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(x^{18} PRI $k=4$ V DRUGI VSOTI)

$$\frac{\partial^{18} f}{\partial x^{18}}(0,0) = 18! \cdot (-1)^4 \cdot \frac{1}{9!} = \boxed{\frac{18!}{9!}}$$

3

$$\frac{\partial^{11} f}{\partial x^6 \partial y^5}(0,0) = 6! \cdot 5! \cdot (\text{KOEF. PRED } x^6y^5) = \boxed{6!\, 5!\, \tfrac{1}{2}}$$

$k=2$ V PRVI VSOTI

4. naloga

a) Klasificirajte lokalne ekstreme funkcije $f(x,y) = x^3 + x^2y + xy^2 - \cancel{2}x$. $-9x$

b) Določite največjo in najmanjšo vrednost funkcije $g(x,y) = xy$ na elipsi $\frac{x^2}{2} + \frac{y^2}{8} = 1$.

a) $f_x = 3x^2 + 2xy + y^2 - 9 = 0$

$f_y = x^2 + 2xy = 0 \Rightarrow x(x+2y) = 0$ 4

$x = 0$: $y^2 = 9, y = \pm 3$

$x = -2y$: $12y^2 - 4y^2 + y^2 - 9 = 0$, $9y^2 = 9$, $y = \pm 1$

$f_{xx} = 6x + 2y$

$f_{yy} = 2x$

$f_{xy} = 2x + 2y$

$$H = \begin{bmatrix} 6x+2y & 2(x+y) \\ 2(x+y) & 2x \end{bmatrix}, \quad \det H = 2x(6x+2y) - 4(x+y)^2$$ 3

$x = 0, y = 3$: $\det H < 0$ SEDLO 6

$x = 0, y = -3$: $\det H < 0$ SEDLO

$x = -2, y = 1$: $\det H = -4(-\cancel{8}^{12} + 2) - 4 = 36 > 0$; $f_{xx} < 0$ LOKALNI MAKSIMUM

$x = 2, y = -1$: $\det H = 4(12-2) - 4 = 36 > 0$, $f_{xx} > 0$ LOKALNI MINIMUM 2

b) VEZ: $\frac{x^2}{2} + \frac{y^2}{8} - 1 = 0$

$L(x,y,\lambda) = xy - \lambda\left(\frac{x^2}{2} + \frac{y^2}{8} - 1\right)$ 5

$L_x = y - x\lambda = 0 \Rightarrow y = x\lambda$

$L_y = x - \frac{\lambda y}{y} = 0 \Rightarrow x - \frac{\lambda^2 x}{y} = 0 \Rightarrow x\left(1 - \frac{\lambda^2}{4}\right) = 0 \rightarrow \lambda^2 = 4 \Rightarrow \lambda = \pm 2$

$\downarrow$ $x = 0 \Rightarrow y = 0 \Rightarrow$ NI NA ELIPSI $\Rightarrow y = \pm 2x$

$-L_\lambda = \frac{x^2}{2} + \frac{y^2}{8} - 1 = 0$

$\frac{x^2}{2} + \frac{4x^2}{8} = 1$

$x^2 = 1 \Rightarrow x = \pm 1$

$y = \pm 2x$

KANDIDATI:

$T_1(1,2)$: $g = 2$

$T_2(1,-2)$: $g = -2$

$T_3(-1,2)$: $g = -2$

$T_4(-1,-2)$: $g = 2$

MAKSIMUMA

MINIMUMA 5