FMF

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Permutation groups and combinatorial structures

Homework 1

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Problem statement

Let G be a permutation group on a finite set X and let C be the centraliser of G in $\mathrm{Sym}\,(X)$. Prove that if G is semiregular, then C is transitive.

Solution

What we need to show transitivity is that for each $x_1, x_2 \in X$ there exists an $c \in C$, such that $x_2 = x_1^c$.

For the case that $x_1 = x_2$ such a $c \in C$ obviously exists and is equal to the identity $id \in C$. For the case that $x_1 \neq x_2$ remember that the action of G on X partitions X into disjoint orbits. We consider two possible cases:

x_1 and x_2 in the same orbit of G

Let us first consider the case that $x_1, x_2 \in X$ both lie in the same orbit x_0^G for some $x_0 \in X$. Therefore $x_1 = x_0^{g_1}$ and $x_2 = x_0^{g_2}$ for some $g_1, g_2 \in G$. First, notice that every element $x \in x_0^G$ can be uniquely expressed as x_0^g for some $g \in G$: Assume that $x_0^g = x_0^{g'}$ for $g, g' \in G$. Then $x_0 = x_0^{g^{-1}g'}$ and therefore $g^{-1}g' \in G_{x_0}$. As the action is semiregular, all the stabilisers are trivial and therefore $g^{-1}g' = id$ or $g = g'$.

Let us define a map $c \in \mathrm{Sym}\,(X)$ in the following way. For $x \in X \backslash x_0^G$, we define $x^c = x$. For each $x_0^g \in x_0^G$ we define $\left(x_0^g\right)^c = x_0^{g_2 g_1^{-1} g}$. Note that with this definition we have $x_1^c = x_2$.

Bijectivity of c

We must verify that such a map c is really in $\mathrm{Sym}\,(X)$ - that it is a bijection. Since X is finite it suffices to prove surjectivity. For every $x \in X \backslash x_0^G$ surjectivity is clear as c fixes these points. Furthermore every $x_0^g \in x_0^G$ is in the image: Direct calculation shows $\left(x_0^{g_1 g_2^{-1} g}\right)^c = x_0^g$. Therefore, $c \in \mathrm{Sym}\,(X)$.

c is in the centraliser

It also holds that $c \in C$: Pick any $g \in G$. It holds that $(x_0^c)^g = \left(\left(x_0^{\mathrm{id}}\right)^c\right)^g = \left(x_0^{g_2 g_1^{-1}}\right)^g = x_0^{g_2 g_1^{-1} g} = (x_0^g)^c$. This can be rewritten as $x_0 = x_0^{cgc^{-1}g^{-1}}$. Since the action is semiregular, only the identity fixes x_0 therefore we have $cgc^{-1}g^{-1} = id$ and therefore $cg = gc$. Since $g \in G$ was arbitrary it holds that $c \in C$.

For each two elements of X that are in the same orbit, we have therefore found an element of the centraliser that maps one to another.

x_1 and x_2 in different orbits of G

Consider now the case where x_1 and x_2 are in different orbits. Similarly as before we can uniquely represent these two elements as $x_1 = x_{01}^{g_1}$ and $x_2 = x_{02}^{g_2}$ for some x_{01}, x_{02} ($x_{01}^G \cap x_{02}^G = \emptyset$).

Define $c \in \mathrm{Sym}\,(X)$ in the following way: For $x \in X \backslash \left(x_{01}^G \cup x_{02}^G\right)$ set $x^c = x$, for $x \in x_{01}^G$ define $\left(x_{01}^g\right)^c = x_{02}^{g_2 g_1^{-1} g}$ and for $x \in x_{02}^G$ define $\left(x_{02}^g\right)^c = x_{01}^{g_1 g_2^{-1} g}$. Note that again it holds that $x_1^c = x_2$.

Bijectivity of c

Such a map c is again really a bijection. As before, it suffices to prove surjectivity. A chosen $x \in X \backslash \left(x_{01}^G \cup x_{02}^G\right)$ is obviously in the image as before. Also, $x_{01}^g \in x_{01}^G$ is in the image: $\left(x_{02}^{g_2 g_1^{-1} g}\right)^c = x_{01}^g$. Similarly $x_{02}^g \in x_{02}^G$ is in the image as $\left(x_{01}^{g_1 g_2^{-1} g}\right)^c = x_{02}^g$. Therefore $c \in \mathrm{Sym}(X)$.

c is in the centraliser

It also holds that $c \in C$: for any $g \in G$ it holds that $x_{01}^{gc} = x_{02}^{g_2 g_1^{-1} g} = x_{01}^{cg}$, which is equivalent to $x_{01} = x_{01}^{cgc^{-1}g^{-1}}$. As before, using the fact that G is semiregular we have $cgc^{-1}g^{-1} = id$ and therefore $cg = gc$ and $c \in C$.

So also for each two elements of X from different orbits, we have found an element of the centraliser that maps one of another. This completes the proof that C is transitive.