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Homework Exercise 1

Partial map categories

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Throughout this homework, it may help you to consider the example $\mathcal{C} = \mathbf{Set}$, and to work out what the various constructions (partial maps, the partial map category, the functor I, partial map classifiers) amount to in that particular case. Nevertheless, your solutions should be given in the general case of an arbitrary category $\mathcal{C}$ with pullbacks.

In a category $\mathcal{C}$, a *partial map representative* from X to Y (where $X, Y \in |\mathcal{C}|$) is a pair (m, f) where $m \colon X_0 \rightarrowtail X$ and $f \colon X_0 \longrightarrow Y$ are morphisms in $\mathcal{C}$ with a common domain X_0 and with m a monomorphism. Given two such representatives (m, f) and (m', f') define $(m, f) \sqsubseteq (m', f')$ to hold if there exists a morphism l in $\mathcal{C}$ making both triangles below commute.

$$
\begin{tikzcd}
X_0 \arrow[dr, dotted, "l"] \arrow[drr, "f"] \arrow[ddr, tail, "m"'] & & \\
& X_0' \arrow[r, "f'"] \arrow[d, tail, "m'"] & Y \\
& X &
\end{tikzcd}
$$

Verify for yourself that there is at most one map l making the above diagram commute and it is necessarily a monomorphism. Verify also that the relation $\sqsubseteq$ on partial map representatives is a preorder (i.e., reflexive and transitive). Write $(m, f) \equiv (m', f')$ to mean that both $(m, f) \sqsubseteq (m', f')$ and $(m', f') \sqsubseteq (m, f)$ hold. Verify for yourself that this is an equivalence relation, and that in the case that $(m, f) \equiv (m', f')$ it follows that the unique map l in the diagram above is an isomorphism.

The above verification is an important preliminary part of the homework to help your understanding of the material. However, you are not expected to include this verification in your submitted solution, which should rather answer only the five questions below. In your answers to these, you may make use of any of the facts mentioned above, but when doing so you should clearly state which fact(s) you are using. Claims should be justified. (So, for example, when asked to define an A satisfying property B, you should prove that your definition indeed produces an A, and that this also satisfies B.)

A *partial map* from X to Y is an equivalence class of partial map representatives from X to Y under the equivalence relation $\equiv$.

For all questions below, assume that the category $\mathcal{C}$ has pullbacks.

1. Show that partial maps in $\mathcal{C}$ form a category. That is, define a category $\mathcal{C}_{\text{par}}$ with $|\mathcal{C}_{\text{par}}| = |\mathcal{C}|$ and $\mathcal{C}_{\text{par}}(X, Y) =$ the collection of partial maps from X to Y.

2. Define an identity-on-objects faithful functor $I\colon \mathcal{C} \to \mathcal{C}_{\text{par}}$.

A *partial map classifier* for an object Y is given by an object $\widetilde{Y}$ and map $\eta_Y : Y \longrightarrow \widetilde{Y}$ in $\mathcal{C}$ such that, for every partial map representative (m, f) from X to Y, there exists a unique map $t_{m,f} \colon X \longrightarrow \widetilde{Y}$ for which the square below is a pullback.

$$\begin{tikzcd}
X_0 \arrow[r, "f"] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\eta_Y"] \\
X \arrow[r, "t_{m,f}"'] & \widetilde{Y}
\end{tikzcd}$$

3. Suppose that every object X of $\mathcal{C}$ has a partial map classifier. Show that the operation $X \mapsto \widetilde{X}$ extends to a faithful functor $\widetilde{(-)}\colon \mathcal{C} \to \mathcal{C}$ with respect to which the maps $(\eta_X)_{X \in |\mathcal{C}|}$ form the components of a natural transformation $\eta\colon 1_{\mathcal{C}} \Rightarrow \widetilde{(-)}$.

4. Suppose that Y has a partial map classifier. Show that, for any two representatives (m, f) and (m', f') of partial maps from X to Y, it holds that $(m, f) \equiv (m', f')$ if and only if $t_{m,f} = t_{m',f'}$.

Suppose that $\mathcal{C}$ is locally small. A functor $F\colon \mathcal{C}^{\text{op}} \to \mathbf{Set}$ is said to be *representable* if there exists an object $Z \in |\mathcal{C}|$ such that the functors F and $\mathcal{C}(-, Z)$ are naturally isomorphic.

5. Show that an object Y has a partial map classifier if and only if the functor $\mathcal{C}_{\text{par}}(-, Y) \circ I^{\text{op}}\colon \mathcal{C}^{\text{op}} \to \mathbf{Set}$ is representable. (Here $I^{\text{op}} : \mathcal{C}^{\text{op}} \to \mathcal{C}_{\text{par}}^{\text{op}}$ is the opposite-category version of $I\colon \mathcal{C} \to \mathcal{C}_{\text{par}}$.)