Teorija Kategorij 2022–23
Homework Exercise 2

Monads, Kleisli categories and string diagrams

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In this homework you are asked to provide definitions, characterisations and examples relating to category-theoretic properties associated with a mathematically important family of monads on the category **Set***. In all questions you will be marked purely on whether the definitions, characterisations and examples you provide have the properties required of them. You are not asked to provide any proof of correctness, although you should of course convince yourself of the correctness of your answers before submitting them.*

Let $(S, 0, +, 1, \cdot)$ be a *semiring* (see Wikipedia for the definition). Let $V \colon \mathbf{Set} \to \mathbf{Set}$ be the endofunctor defined below:

$$\begin{aligned} V(X) &:= \{\mathbf{v} \colon X \to S \mid \mathbf{v} \text{ has finite support}\} \\ V(f \colon X \to Y) &:= \mathbf{v} \mapsto \left(y \mapsto \sum_{x \in f^{-1}(y)} \mathbf{v}_x \right) . \end{aligned}$$

(The *support* of $\mathbf{v} \colon X \to S$ is the set $\{x \in X \mid \mathbf{v}_x \neq 0\}$, where $\mathbf{v}_x$ denotes the function application $\mathbf{v}(x)$.)

Question 1 *Define natural transformations* $\eta \colon 1_{\mathbf{Set}} \Rightarrow V$ *and* $\mu \colon V^2 \Rightarrow V$ *such that* (V, η, μ) *is a monad.*

For this question it is enough to produce explicit definitions of the *components* of the two natural transformations.

Read about the *Kleisli category* $\mathcal{C}_T$ of a monad (T, η, μ) on a category $\mathcal{C}$, and the associated adjunction $F \dashv G \colon \mathcal{C}_T \to \mathcal{C}$. (The basic definitions are on Wikipedia. The Wikipedia article can be supplemented, if desired, by one of the course textbooks.)

Question 2 *Give an explicit description of the Kleisli category* $\mathbf{Set}_V$ *for the monad* V *above. That is, define this category explicitly in direct mathematical terms, exploiting properties of the specific monad* V*, rather than generically in terms of an arbitrary monad structure.*

For this exercise you are expected to produce explicit definitions of the *hom-sets* $\mathbf{Set}_V(X, Y)$, the *identities* and the *composition* operation in the Kleisli category. It is sufficient to give these definitions in such a way that the category so defined is *isomorphic to* the generically constructed Kleisli category.

In the remaining questions you should work with the Kleisli category $\mathbf{Set}_V$ in the form explicitly defined in your answer to Question 2.

In the case that the semiring S is *commutative*, the binary product operation on $\mathbf{Set}$ extends to a bifunctor $\times : \mathbf{Set}_V \times \mathbf{Set}_V \to \mathbf{Set}_V$ whose action on objects (sets) maps X, Y to the product set $X \times Y$. This bifunctor is part of a symmetric monoidal structure on the Kleisli category $\mathbf{Set}_V$.

Question 3 *Define the morphism action of the bifunctor* $\times : \mathbf{Set}_V \times \mathbf{Set}_V \to \mathbf{Set}_V$*.*

Question 4 *Give an example showing that the definition you gave in Question 3 does not give rise to a bifunctor in the case of a non-commutative semiring* S*.*

Henceforth, the semiring S is assumed to be commutative.

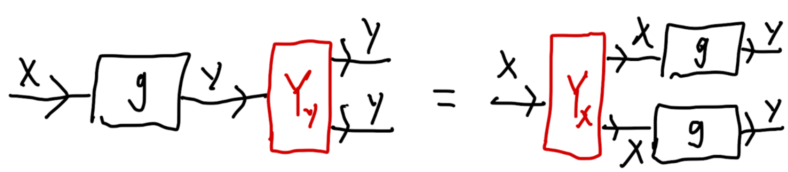
Let $F\colon \mathbf{Set} \to \mathbf{Set}_V$ be the left adjoint in the adjunction associated with the Kleisli category. For any set X define $\mathsf{Y}_X \in \mathbf{Set}_V(X, X \times X)$ and $\mathsf{o}_X \in \mathbf{Set}_V(X, 1)$ (where 1 is a chosen terminal object in $\mathbf{Set}$) by:

$$\mathsf{Y}_X \ := \ F(\Delta_X) \qquad\qquad \mathsf{o}_X \ := \ F(!_X) \ ,$$

where $\Delta_X = x \mapsto (x, x) : X \to X \times X$ and $!_X \colon X \to 1$ is the unique map. We can write these maps as boxes for use in *string diagrams* based on the monoidal structure on $\mathbf{Set}_V$. (For an introduction to string diagrams, read the paper *"A survey of graphical languages for monoidal categories"*, by Peter Selinger, as far as the send of Section 3.1. There is a link to the paper from the course webpage.)

Question 5 *Explicitly characterise those Kleisli maps* $g \in \mathbf{Set}_V(X, Y)$ *for which the following string-diagram equation holds.*

Question 6 *Explicitly characterise those Kleisli maps* $g \in \mathbf{Set}_V(X, Y)$ *for which the following string-diagram equation holds.*



Can you simplify your characterisation in the case in which the semiring S *satisfies the property:* $x \cdot y = 0$ *only if (at least) one of* x *or* y *is* 0*?*

(There is no particular importance to the colour coding in the diagrams above, which is used only to highlight the boxes representing the Kleisli maps Y_X and o_X.)