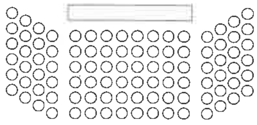
3. kolokvij iz Matematike II, FMF, Aplikativna fizika

16. 4. 2024

Čas pisanja je 120 minut. Veliko uspeha!



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Ime in priimek

Vpisna številka

1. naloga (25 točk)

Dana je funkcija

$$f(x) = \begin{cases} x & ; -\pi \le x \le 0 \\ 2x & ; 0 \le x \le \pi \end{cases}$$

a) Funkcijo f razvijte v Fourierovo vrsto na intervalu $[-\pi, \pi]$.

b) Skicirajte graf dobljenega Fourierovega razvoja na $\mathbb{R}$.

c) S pomočjo dobljenega Fourierovega razvoja določite

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}.$$

a) $\boxed{a_0} = \frac{1}{\pi}\left(\int_{-\pi}^{0} x\,dx + 2\int_{0}^{\pi} x\,dx\right) = \frac{1}{\pi}\left(-\frac{\pi^2}{2} + \pi^2\right) = \boxed{\frac{\pi}{2}}$

$u = x \quad dv = \cos(nx)\,dx$

$du = dx \quad v = \frac{1}{n}\sin(nx)$

$$\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \int_{-\pi}^{0} x\cos(nx)\,dx + 2\int_{0}^{\pi} x\cos(nx)\,dx = \frac{x\sin(nx)}{n}\Big|_{-\pi}^{0} - \frac{1}{n}\int_{-\pi}^{0}\sin(nx)\,dx +$$

$$+ \frac{2x\sin(nx)}{n}\Big|_{0}^{\pi} - \frac{2}{n}\int_{0}^{\pi}\sin(nx)\,dx = 0 + \frac{\cos(nx)}{n^2}\Big|_{-\pi}^{0} + 0 + \frac{2\cos(nx)}{n^2}\Big|_{0}^{\pi} =$$

$$= \frac{1}{n^2}\left(1 - \cos(n\pi) + 2\cos(n\pi) - 2\right) = \frac{1}{n^2}\left(\cos(n\pi) - 1\right) = \frac{1}{n^2}\left((-1)^n - 1\right) \Rightarrow \boxed{a_n = \frac{(-1)^n - 1}{\pi n^2}}$$

$$\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \int_{-\pi}^{0} x\sin(nx)\,dx + 2\int_{0}^{\pi} x\sin(nx)\,dx = -\frac{x\cos(nx)}{n}\Big|_{-\pi}^{0} + \frac{1}{n}\int_{-\pi}^{0}\cos(nx)\,dx - \frac{2x\cos(nx)}{n}\Big|_{0}^{\pi} +$$

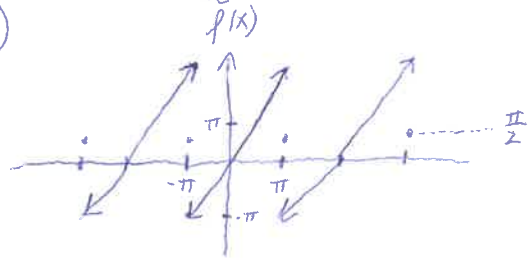
$dv = \sin(nx)\,dx \quad u = x$

$v = -\frac{\cos(nx)}{n} \quad du = dx$

$$+ \frac{2}{n}\int_{0}^{\pi}\cos(nx)\,dx = -\frac{\pi\cos(n\pi)}{n} + 0 - \frac{2\pi\cos(n\pi)}{n} + 0 = \frac{3\pi}{n}(-1)^{n+1} \Rightarrow \boxed{b_n = \frac{3(-1)^{n+1}}{n}}$$

$$\boxed{\tilde{f}(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{\pi n^2}\cos(nx) + \frac{3(-1)^{n+1}}{n}\sin(nx)\right)}$$

b)



c) $0 = \tilde{f}(0) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{(-1)^n - 1}{\pi n^2} =$

$\begin{cases} 0 & ; n \text{ sodo} \\ -2 & ; n \text{ liho} \end{cases}$ $\quad n = 2k - 1$

$$= \frac{\pi}{4} + \sum_{k=1}^{\infty}\frac{(-2)}{\pi(2k-1)^2}$$

$$\Rightarrow \boxed{\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}} = -\frac{\pi}{4}\cdot\left(-\frac{\pi}{2}\right) = \boxed{\frac{\pi^2}{8}}$$

2. naloga (25 točk)

Naj bo K krivulja v $\mathbb{R}^3$, ki jo dobimo kot presek ploskev $x^2 + y^2 = 16$ in $z = -2y$.

a) Parametrizirajte krivuljo K.

b) Določite spremljajoči trieder krivulje K v točki $(0, 4, -8)$.

c) Izračunajte še fleksijsko in torzijsko ukrivljenost krivulje K v isti točki.

a) $x = r\cos\varphi$ $\quad x^2+y^2=16 \Rightarrow r=4$

$y = r\sin\varphi$

$z = z$ $\quad z=-2y \Rightarrow z=-8\sin\varphi$

$$\boxed{\vec{r}(\varphi) = (4\cos\varphi, 4\sin\varphi, -8\sin\varphi)}$$

b) TOČKA $\varphi_0 = \frac{\pi}{2}$ $\quad \vec{r}(\varphi_0) = (0,4,-8)$

$$\dot{\vec{r}}(\varphi) = (-4\sin\varphi, 4\cos\varphi, -8\cos\varphi) \Rightarrow \dot{\vec{r}}(\varphi_0) = (-4,0,0)$$

$$\ddot{\vec{r}}(\varphi) = (-4\cos\varphi, -4\sin\varphi, 8\sin\varphi) \Rightarrow \ddot{\vec{r}}(\varphi_0) = (0,-4,8)$$

$$\vec{T} = \frac{\dot{\vec{r}}(\varphi_0)}{\|\dot{\vec{r}}(\varphi_0)\|} = (-1,0,0)$$

$$\dot{\vec{r}}(\varphi_0) \times \ddot{\vec{r}}(\varphi_0) = (0, 32, 16)$$

$$\|\dot{\vec{r}}(\varphi_0) \times \ddot{\vec{r}}(\varphi_0)\| = \sqrt{32^2+16^2} = 16\sqrt{5}$$

$$\vec{B} = \frac{\dot{\vec{r}}(\varphi_0) \times \ddot{\vec{r}}(\varphi_0)}{\| - \| - \|} = \frac{1}{\sqrt{5}}(0,2,1)$$

$$\vec{N} = \vec{B} \times \vec{T} = \frac{1}{\sqrt{5}}(0,-1,2)$$

c) $\dddot{\vec{r}}(\varphi) = (4\sin\varphi, -4\cos\varphi, 8\cos\varphi) \Rightarrow \dddot{\vec{r}}(\varphi_0) = (4,0,0)$

$$\boxed{\kappa} = \frac{\|\dot{\vec{r}}(\varphi_0) \times \ddot{\vec{r}}(\varphi_0)\|}{\|\dot{\vec{r}}(\varphi_0)\|^3} = \frac{16\sqrt{5}}{4^3} = \boxed{\frac{\sqrt{5}}{4}}$$

$$\boxed{\omega} = \frac{[\dot{\vec{r}}(\varphi_0), \ddot{\vec{r}}(\varphi_0), \dddot{\vec{r}}(\varphi_0)]}{\|\dot{\vec{r}}(\varphi_0) \times \ddot{\vec{r}}(\varphi_0)\|^2} = \boxed{0}$$

3. naloga (25 točk)

Ploskev S parametriziramo z

$$x(u,v) = (2+\cos u)\cos v,$$
$$y(u,v) = (2+\cos u)\sin v,$$
$$z(u,v) = \sin u,$$

kjer je $u, v \in [0, 2\pi)$.

a) Pokažite, da so koordinatne krivulje v podani parametrizaciji ploskve S, paroma pravokotne.

b) S pomočjo ploskovnega integrala izračunajte površino ploskve S.

c) Izračunajte še Gaussovo ukrivljenost ploskve S v točki, ki ustreza $u = 0, v = \pi/2$

a) $\vec{r}(u,v) = \left((2+\cos u)\cos v,\ (2+\cos u)\sin v,\ \sin u\right)$

a) $\vec{r}_u = (-\sin u \cos v, -\sin u \sin v, \cos u)$

$\vec{r}_v = (-(2+\cos u)\sin v, (2+\cos u)\cos v, 0)$

$\boxed{\vec{r}_u \cdot \vec{r}_v} = (2+\cos u)\sin u \sin v \cos v - (2+\cos u)\sin u \sin v \cos v = \boxed{0}$

b) $\vec{r}_u \times \vec{r}_v = \left(-(2+\cos u)\cos u \cos v, -(2+\cos u)\sin v \cos u, -(2+\cos u)\sin u\right)$

$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{(2+\cos u)^2} = 2+\cos u$

$\boxed{S} = \int_0^{2\pi} du \int_0^{2\pi} dv\, (2+\underbrace{\cos u}_{0}) = (2\pi)^2 \cdot 2 = \boxed{8\pi^2}$

c) $E = r_u^2 = 1$, $F \overset{\vec{r}_u \cdot \vec{r}_v}{=} 0$, $G \overset{r_v^2}{=} (2+\cos u)^2$ $G(u=0) = 9$

$\vec{r}_{uu} = (-\cos u \cos v, -\cos u \sin v, -\sin u) \overset{u=0,\ v=\frac{\pi}{2}}{\Rightarrow} (0, -1, 0)$

$\vec{r}_{uv} = (+\sin u \sin v, -\sin u \cos v, 0) \overset{-\|-}{\Rightarrow} (0, 0, 0)$

$\vec{r}_{vv} = (-(2+\cos u)\cos v, -(2+\cos u)\sin v, 0) \overset{-\|-}{\Rightarrow} (0, -3, 0)$

$\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v\ (u=0, v=\frac{\pi}{2})}{\| - \| - \|} = \dfrac{(0,-3,0)}{3} = (0,-1,0)$

$L = \vec{r}_{uu} \cdot \vec{n} = +1$, $M = 0$, $N = \vec{r}_{vv} \cdot \vec{n} = +3$

$\boxed{K} = \dfrac{LN - M^2}{EG - F^2} = \dfrac{3}{9} = \boxed{\dfrac{1}{3}}$

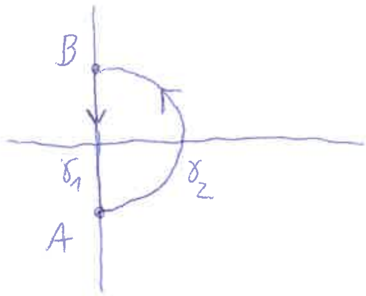
4. naloga (25 točk)

a) Dani sta točki $A(0,-1)$ in $B(0,1)$. Naj bo K_1 sklenjena krivulja, ki je sestavljena iz daljice BA ter krivulje, podane z enačbama $x^2+y^2=1, x \geq 0$. Izračunajte krivuljni integral vektorskega polja $\vec{F_1}(x,y)=(y^2,y^2)$ po krivulji K_1, kjer je krivulja orientirana v nasprotni smeri urinega kazalca.

b) Pokaži, da je rotor polja $\vec{F_2}(x,y,z)=\left(\frac{2xy}{x^4+y^2}, \frac{-x^2}{x^4+y^2}, 0\right)$ enak nič.

c) Določi še vrednost krivuljnega integrala $\int_{K_2} \vec{F_2}\cdot d\vec{s}$, kjer je K_2 sklenjena krivulja s parametrizacijo $\vec{r}(t)=(\cos^2(t), \sin(t), \cos(t)-\sin(t))$ in $t \in [0,2\pi)$. (Namig: Pomagaj si z lastnostjo polja iz točke b))

a)



$\gamma_1: \vec{r}(t) = \vec{B} + t\vec{BA} = (0,1)+t(0,-2) = (0, 1-2t)$

$\dot{\vec{r}}(t) = (0,-2)$

$$\int_{\gamma_1} \vec{F_1}\cdot d\vec{s} = \int_0^1 dt\, \left((1-2t)^2,(1-2t)^2\right)\cdot(0,-2) = -2\int_0^1 (1-2t)^2 dt =$$

$$= -2\int_0^1 (1-4t+4t^2)dt = -2\left(1-2+\frac{4}{3}\right) = -\frac{2}{3}$$

$$\int_{K_1} \vec{F_1}\cdot d\vec{s} = \int_{\gamma_1}\vec{F_1}\cdot d\vec{s} + \int_{\gamma_2}\vec{F_1}\cdot d\vec{s}$$

$\gamma_2: \vec{r}(\varphi) = (\cos\varphi, \sin\varphi), \quad \varphi\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\dot{\vec{r}}(\varphi) = (-\sin\varphi, \cos\varphi)$

0 LIHA $u=\sin\varphi$

$$\int_{\gamma_2} \vec{F_1}\cdot d\vec{s} = \int_{-\frac{\pi}{2}}^{\pi/2} d\varphi\, (\sin^2\varphi, \sin^2\varphi)\cdot(-\sin\varphi,\cos\varphi) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi\,(\cancel{-\sin^3\varphi} + \sin^2\varphi\cos\varphi)\,d\varphi =$$

$$= \int_{-1}^{1} u^2 du = \frac{2}{3} \quad \Rightarrow \boxed{\int_{K_1} \vec{F_1}\cdot d\vec{s} = 0}$$

b) $$\boxed{\vec{\nabla}\times\vec{F_2}} = \left(0, 0, \frac{\partial}{\partial x}\left(\frac{-x^2}{x^4+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{2xy}{x^4+y^2}\right)\right) = \left(0,0,\frac{-2x(x^4+y^2)+x^2\cdot 4x^3 - 2x(x^4+y^2)+2xy\cdot 2y}{(x^4+y^2)^2}\right) =$$

$$= \boxed{(0,0,0)}$$

c) PO b), $\vec{\nabla}\times\vec{F_2}=0 \Rightarrow \vec{F_2}$ JE POTENCIALNO POLJE

$$\boxed{\int_{K_2} \vec{F_2}\cdot d\vec{s} = 0}$$ (INTEGRAL POTENCIALNEGA POLJA PO SKLENJENI KRIVULJI.