2. kolokvij iz Matematike 2, FMF, Aplikativna Fizika

31. 1. 2023

Čas pisanja je 120 minut. Veliko uspeha!

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Sedež (2.01)

Ime in priimek

Vpisna številka

1. naloga (25 točk)

a) Dan je dvakratni integral

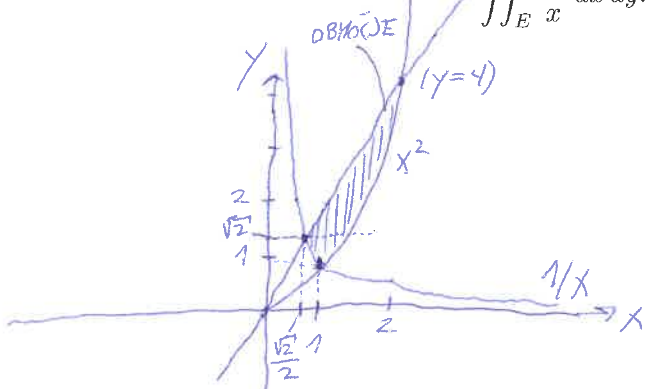
$$\int_{\frac{\sqrt{2}}{2}}^{1}\left(\int_{\frac{1}{x}}^{2x} f(x,y)\,dy\right)dx+\int_{1}^{2}\left(\int_{x^2}^{2x} f(x,y)\,dy\right)dx.$$

Skicirajte integracijsko območje ter obrnite vrstni red integracije.

b) Naj bo $E \subset \mathbb{R}^2$ območje, ki ga omejujeta krivulji $x = 1 - y^2$ in $x = 3(1 - y^2)$. Z uvedbo novih spremenljivk v in u, za katere velja $x = v(1 - u^2)$ in $y = u$, izračunajte dvojni integral

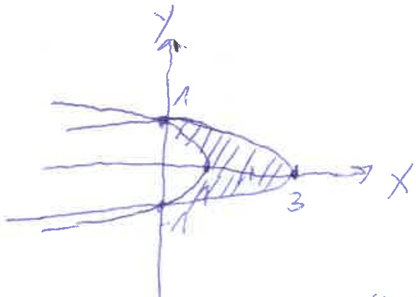
$$\iint_E \frac{y^2}{x}\,dx\,dy.$$

a)



$$\int_1^{\sqrt{2}} dy \int_{1/y}^{\sqrt{y}} f(x,y)\,dx + \int_{\sqrt{2}}^{4} dy \int_{y/2}^{\sqrt{y}} f(x,y)\,dx$$

b)



$$J = \begin{bmatrix} -2uv & (1-u^2) \\ 1 & 0 \end{bmatrix}$$

$$\det J = -(1-u^2)$$

$$u = y \Rightarrow u \in [-1,1] \qquad |\det J| = 1-u^2$$

$$v = \frac{x}{1-u^2} \Rightarrow v \in [1,3]$$

$$\Rightarrow \iint_E \frac{y^2}{x}\,dx\,dy = \int_{-1}^{1} du \int_{1}^{3} dv \frac{u^2}{v(1-u^2)}\cdot(1-u^2) = \int_{-1}^{1} du \int_{1}^{3} dv \frac{u^2}{v} = \left(\frac{u^3}{3}\Big|_{-1}^{1}\right)\cdot\left(\ln|v|\Big|_{1}^{3}\right) =$$

$$= \boxed{\frac{2}{3}\cdot \ln 3}$$

2. naloga (25 točk)

a) Naj bo $B(x,y)$ Eulerjeva beta funkcija. Poenostavite izraz

$$\frac{2^{19}\pi}{33}\cdot B\left(\frac{7}{2},2\right).$$

b) Dokažite, da je vrednost integrala

$$\iiint_D (x^2+y^2)z^6 \, dxdydz,$$

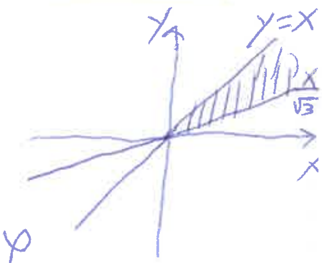
kjer je $D=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2\le 16, \quad z\ge 0, \quad \frac{x}{\sqrt{3}}\le y\le x\}$, enaka rezultatu iz točke a).

a) $$\frac{2^{19}\pi}{33} B\left(\frac{7}{2},2\right) = \frac{2^{19}\pi}{33}\,\frac{\Gamma(\frac{7}{2})\Gamma(2)}{\Gamma(\frac{11}{2})} = \frac{2^{19}\pi}{33}\cdot\frac{1}{\frac{9}{2}\cdot\frac{7}{2}} = \boxed{\frac{2^{21}\pi}{33\cdot 9\cdot 7}}$$

b) SFERIČNE: $r^2\le 16 \Rightarrow r\le 4$

$z\ge 0 \Rightarrow \theta\in[0,\frac{\pi}{2}]$

$\frac{x}{\sqrt{3}}\le y\le x$



$\frac{\cos\varphi}{\sqrt{3}}\le \sin\varphi\le\cos\varphi$

$\frac{1}{\sqrt{3}}\le tg\varphi\le 1 \Rightarrow \varphi\in[\frac{\pi}{6},\frac{\pi}{4}]$

$$\iiint_D (x^2+y^2)z^6\,dxdydz =$$

$$= \int_0^4 dr\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} d\varphi\int_0^{\frac{\pi}{2}} d\theta\; \overbrace{r^2\cos^2\theta}^{x^2+y^2}\; \overbrace{r^6\sin^6\theta}^{z^6}\; \overbrace{r^2\cos\theta}^{|J|} =$$

$$=\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\int_0^4 r^{10}dr\int_0^{\frac{\pi}{2}} d\theta\, \sin^6\theta\cos^3\theta = \frac{\pi}{12}\cdot\frac{2^{22}}{11}\cdot\int_0^{\frac{\pi}{2}}\sin^6\theta\cos^3\theta\, d\theta =$$

$$= \frac{\pi}{33}2^{20}\cdot\frac{1}{2}B\left(\frac{7}{2},2\right) = \boxed{\frac{2^{19}\pi}{33}B\left(\frac{7}{2},2\right)}$$

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3. naloga (25 točk)

Naj bo $a \in \mathbb{R}$. Dan je integral

$$F(a) = \int_0^\infty e^{-x^2} \cos(ax)\, dx. = \int_0^\infty f(a,x)\,dx$$

a) Pokažite, da je $F(0) = \frac{\sqrt{\pi}}{2}$ (Namig: Γ funkcija).

b) Natančno utemeljite, da lahko odvajate po parametru a.

c) S pomočjo integracije po delih pokažite, da je $F'(a) = -\frac{1}{2}aF(a)$.

d) Rešitev enačbe $F'(a) = -\frac{1}{2}aF(a)$ ima obliko $F(a) = Ce^{-\frac{a^2}{4}}$, kjer $C \in \mathbb{R}$. Določite C.

a) $F(0)=\int_0^\infty e^{-x^2}dx = \frac{1}{2}\int_0^\infty e^{-u}u^{-\frac{1}{2}}du = \frac{1}{2}\Gamma(\frac{1}{2}) = \boxed{\frac{\sqrt{\pi}}{2}}$

$u=x^2 \quad (x=\sqrt{u})$
$du=2x\,dx$

b) $F(0)$ OBSTAJA ✓

$\frac{\partial f(a,x)}{\partial a} = -e^{-x^2}\sin(ax)\cdot x$ ZVEZNA ✓

ENAK. KONV.: $|f_a| = |e^{-x^2}x\sin(ax)| \leq |e^{-x^2}x|$ $\quad g(x)=xe^{-x^2}$

$\int_0^\infty g(x)dx = \int_0^\infty xe^{-x^2}dx = \frac{1}{2}\cdot(-1)\int_0^{-\infty}e^u du = \frac{1}{2}$ ✓

c) $F'(a) = -\int_0^\infty e^{-x^2}x\sin(ax)dx = -\left(\cancel{\sin(ax)\cdot(-\tfrac{1}{2})e^{-x^2}\Big|_0^\infty} + \int_0^\infty \frac{a}{2}e^{-x^2}\cos(ax)dx\right)$

$u=\sin(ax) \qquad dv=e^{-x^2}x\,dx$
$du=\cos(ax)\cdot a\,dx \qquad v=-\frac{1}{2}e^{-x^2}$

$= -\frac{a}{2}F(a)$ ✓

d) $F(0)=\frac{\sqrt{\pi}}{2} \Rightarrow \boxed{C=\frac{\sqrt{\pi}}{2}}$

4. naloga (25 točk)

a) Za dan predpis

$$d(x,y) = \begin{cases} |x-y|, & (x,y \in \mathbb{Q}) \text{ ali } (x,y \in \mathbb{R}\backslash\mathbb{Q}) \\ |x-y|+1, & (x \in \mathbb{Q} \text{ in } y \in \mathbb{R}\backslash\mathbb{Q}) \text{ ali } (x \in \mathbb{R}\backslash\mathbb{Q} \text{ in } y \in \mathbb{Q}) \end{cases}$$

dokažite, da določa metriko na $\mathbb{R}$.

b) Določite odprto kroglo $K(0,2)$, ki je določena z metriko d.

c) Ugotovite ali tudi predpis

$$\tilde{d}(x,y) = \begin{cases} |x-y|+1, & (x,y \in \mathbb{Q}) \text{ ali } (x,y \in \mathbb{R}\backslash\mathbb{Q}) \\ |x-y|, & (x \in \mathbb{Q} \text{ in } y \in \mathbb{R}\backslash\mathbb{Q}) \text{ ali } (x \in \mathbb{R}\backslash\mathbb{Q} \text{ in } y \in \mathbb{Q}) \end{cases}$$

določa metriko na $\mathbb{R}$.

a) • $d(x,y) \geq 0$ ✓

• $d(x,y) = d(y,x)$ ✓

• $d(x,x) = 0$ ✓ $d(x,y) = 0 \Rightarrow |x-y| = 0 \Rightarrow x = y$ ✓

• $d(x,y) \leq d(x,z) + d(y,z)$

$x,y \in \mathbb{Q}$ ALI $x,y \in \mathbb{R}\backslash\mathbb{Q}$: $d(x,y) = |x-y| \leq |x-z| + |z-y| \leq d(x,z) + d(z,y)$

$x \in \mathbb{Q}$, $y \in \mathbb{R}\backslash\mathbb{Q}$ ALI OBRATNO: $d(x,y) = |x-y|+1 \leq |x-z| + |y-z| + 1 \underset{\uparrow}{=} d(x,z) + d(y,z)$

✓ JE METRIKA z JE ALI $\in \mathbb{Q}$ ALI $\in \mathbb{R}\backslash\mathbb{Q}$

b) $K(0,2) = \{y \in \mathbb{R} \mid d(\overset{\mathbb{Q}}{0}, y) < 2\}$

$y \in \mathbb{Q}$: $d(0,y) < 2 \Rightarrow |y| < 2$

$y \in \mathbb{R}\backslash\mathbb{Q}$: $d(0,y) < 2 \Rightarrow |y| < 1$

$\Rightarrow \boxed{K(0,2) = ((-2,2) \cap \mathbb{Q}) \cup (-1,1)}$

c) $d(x,x) = 1$ NI METRIKA!