# CATEGORY THEORY HW 1

ANDREJ KOLAR - POŽUN
27223000

1) LET US DEFINE THE CATEGORY OF PARTIAL MAPS $\mathcal{C}_{PAR}$.

$|\mathcal{C}_{PAR}| = |\mathcal{C}|$ $\quad \mathcal{C}_{PAR}(X,Y) =$ PARTIAL MAPS FROM $X$ TO $Y$.

LET'S DEFINE THE COMPOSITION OF MORPHISMS $(m_1,f_1): X \to Y$ $\quad ((m_1,f_1) \in \mathcal{C}_{PAR}(X,Y))$
$(m_2,f_2): Y \to Z$

NOTE: I WILL ALWAYS WORK WITH A SPECIFIC REPRESENTATIVE OF THE EQUIV. CLASS.

$(m_2,f_2) \circ (m_1,f_1) = ?$

$= \boxed{(m_1 \circ p, f_2 \circ g)}$

$$\begin{tikzcd} & & P \arrow[dl, "p"'] \arrow[dr, "g"] \arrow[dd, phantom, "\lrcorner", very near start] & & \\ & B_1 \arrow[dl, "m_1"'] \arrow[dr, "f_1"] & & B_2 \arrow[dl, "m_2"'] \arrow[dr, "f_2"] & \\ X & & Y & & Z \end{tikzcd}$$

$\mathcal{C}$ HAS PULLBACKS SO WE COMPOSE MAPS BY FORMING A PULLBACK $P$ OF THE COSPAN $B_1 \xrightarrow{f_1} Y \xleftarrow{m_2} B_2$

$p$ IS MONO BECAUSE $m_2$ IS MONO AND PULLBACKS PRESERVE MONOS.

$m_1 \circ p$ IS THEN ALSO MONO $\Rightarrow$ $(m_1 \circ p, f_2 \circ g): X \to Z$ IS INDEED A PARTIAL MAP REPRESENTATIVE.

IS THIS WELL DEFINED? LET'S CHOOSE ANOTHER REPRESENTATIVE $(m_1', f_1') \equiv (m_1, f_1)$

$$\begin{tikzcd} & & P' \arrow[dl, "p'"'] \arrow[dr, "g'"] \arrow[dd, phantom, "\lrcorner", very near start] & & \\ & B_1' \arrow[dl, "m_1'"'] \arrow[dr, "f_1'"] & & B_2 \arrow[dl, "m_2"'] \arrow[dr, "f_2"] & \\ X & & Y & & Z \end{tikzcd}$$

WE KNOW THIS COMMUTES:
WHERE $i: B_1 \to B_1'$ IS ISO SINCE $(m_1,f_1) \equiv (m_1',f_1')$

$$\begin{tikzcd} B_1 \arrow[rr, bend left, "f_1"] \arrow[dr, "i"] \arrow[dd, bend right, "m_1"'] & & Y \\ & B_1' \arrow[ur, "f_1'"] \arrow[dl, "m_1'"] & \\ X & & \end{tikzcd}$$

LOOK AT THE PULLBACK SQUARE:

$$\begin{tikzcd} P \arrow[dr, "e"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "i \circ p"'] & & \\ & P' \arrow[r, "g'"] \arrow[d, "p'"] \arrow[dr, phantom, "\lrcorner", very near start] & B_2 \arrow[d, "m_2"] \\ & B_1' \arrow[r, "f_1'"] & Y \end{tikzcd}$$

$m_2 \circ g = f_1 \circ p = (f_1' \circ i) \circ (p) = f_1' \circ (i \circ p)$

$\Rightarrow \exists!\, e: P \to P'$ S.T. $g = g' \circ e$
$i \circ p = p' \circ e$

$(m_2 \circ f_2) \circ (m_1', f_1') = (m_1' \circ p', f_2 \circ g') =$

WE HAVE:

$$\begin{tikzcd} P \arrow[d, "e"] \arrow[dd, bend right, "m_1 \circ p"'] \arrow[dr, bend left, "f_2 \circ g"] & \\ P' \arrow[r, "f_2 \circ g'"] \arrow[d, "m_1' \circ p'"] & Z \\ X & \end{tikzcd}$$

THIS COMMUTES: $f_2 \circ g' \circ e = f_2 \circ g$
$m_1' \circ p' \circ e = m_1' \circ i \circ p = m_1 \circ i^{-1} \circ i \circ p$
$= m_1 \circ p$

THEREFORE $(m_1 p, f_2 g) \sqsubset (m_1' p', f_2 g')$

WE CAN REPEAT THIS ARGUMENT SWITCHING THE ROLES OF $P$ AND $P'$ TO GET $(m_1' p', f_2 g') \sqsubset (m_1 p, f_2 g)$

AND THEREFORE $\boxed{(m_1' p', f_2 g') \equiv (m_1 p, f_2 g)}$

IF WE INSTEAD REPLACE $(m_2,f_2)$ BY $(m_2',f_2') \equiv (m_2,f_2)$ THE ARGUMENT IS ANALOGOUS.

COMPOSITION IS WELL DEFINED ✓

IDENTITY MORPHISM IS OBVIOUSLY $(1,1) \in \mathcal{C}_{PAR}(X,X)$ $\left(\begin{tikzcd} & X \arrow[dl, "1"'] \arrow[dr, "1"] & \\ X & & X \end{tikzcd}\right)$

$(1,1)\circ(m,f) =$

$$\begin{tikzcd}
 & & B \arrow[dl, "m"'] \arrow[dr, "1"] \arrow[dd, phantom, "\checkmark"] & & \\
 & X \arrow[dl, "1"'] \arrow[dr, "1"] & & B \arrow[dl, "m"] \arrow[dr, "f"] & \\
X & & X & & Y
\end{tikzcd}
=
\begin{tikzcd}
 & B \arrow[dl, "m"'] \arrow[dr, "f"] & \\
X & & Y
\end{tikzcd}
\quad (m,f)$$

(AND ALSO $(m,f)\circ(1,1) = (m,f)$ OF COURSE)

HERE WE USED THAT

$$\begin{tikzcd}
B \arrow[r, "1"] \arrow[d, "m"'] & B \arrow[d, "m"] \\
X \arrow[r, "1"'] & X
\end{tikzcd}$$

IS A PULLBACK

PROOF: ~~COM~~ IT COMMUTES ✓ $(m=m)$

$$\begin{tikzcd}
Z \arrow[dr, "f_2"] \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "f_1"'] & & \\
 & B \arrow[r, "1"] \arrow[d, "m"'] & B \arrow[d, "m"] \\
 & X \arrow[r, "1"'] & X
\end{tikzcd}$$

$mf_2 = f_1$ THEN $f_2 : Z \to B$ IS CLEARLY THE UNIQUE MAP ~~...~~ THAT MAKES THE TWO TRIANGLES COMMUTE □

LET'S ~~LEMMA~~ CHECK ASSOCIATIVITY

$(m_1,f_1) : X \to Y$

$(m_2,f_2) : Y \to Z$

$(m_3,f_3) : Z \to W$

$(m_3,f_3)\circ\left((m_2,f_2)\circ(m_1,f_1)\right) = (m_1 p_{12} p_{12}^3,\ f_3 g_{12}^3)$

$$\begin{tikzcd}
 & & & P_{12}^3 \arrow[dl, "p_{12}^3"'] \arrow[drr, "g_{12}^3"] & & & \\
 & & P_{12} \arrow[dl, "p_{12}"'] \arrow[dr, "g_{12}"] & & & & \\
 & B_1 \arrow[dl, "m_1"'] \arrow[dr, "f_1"] & & B_2 \arrow[dl, "m_2"'] \arrow[dr, "f_2"] & & B_3 \arrow[dl, "m_3"'] \arrow[dr, "f_3"] & \\
X & & Y & & Z & & W
\end{tikzcd}$$

$\left((m_3,f_3)\circ(m_2,f_2)\right)\circ(m_1,f_1) = (m_1 p_1^{23},\ f_3 g_{23} g_1^{23})$

$$\begin{tikzcd}
 & & P_1^{23} \arrow[dl, "p_1^{23}"'] \arrow[drr, "g_1^{23}"] & & & & \\
 & B_1 \arrow[dl, "m_1"'] \arrow[dr, "f_1"] & & & P_{23} \arrow[dl, "p_{23}"'] \arrow[dr, "g_{23}"] & & \\
X & & Y & B_2 \arrow[l, "m_2"'] \arrow[dr, "f_2"] & & B_3 \arrow[dl, "m_3"'] \arrow[dr, "f_3"] & \\
 & & & & Z & & W
\end{tikzcd}$$

I WILL SHOW THAT BOTH OF RESULTS ON THE LEFT ARE EQUIVALENT TO THE FOLLOWING PARTIAL MAP REPR. $(m_1 p_{12} p,\ f_3 g_{23} g)$

$$\begin{tikzcd}
 & & & P \arrow[dl, "p"'] \arrow[dr, "g"] & & & \\
 & & P_{12} \arrow[dl, "p_{12}"'] \arrow[dr, "g_{12}"] & & P_{23} \arrow[dl, "p_{23}"'] \arrow[dr, "g_{23}"] & & \\
 & B_1 \arrow[dl, "m_1"'] \arrow[dr, "f_1"] & & B_2 \arrow[dl, "m_2"'] \arrow[dr, "f_2"] & & B_3 \arrow[dl, "m_3"'] \arrow[dr, "f_3"] & \\
X & & Y & & Z & & W
\end{tikzcd}$$

THIS IS TRUE BECAUSE BY PULLBACK LEMMA THE FOLLOWING RECTANGLE IS A PULLBACK

$$\begin{tikzcd}
P \arrow[r, "p"] \arrow[d, "g"'] & P_{12} \arrow[r, "p_{12}"] \arrow[d, "g_{12}"] & B_1 \arrow[d, "f_1"] \\
P_{23} \arrow[r, "p_{23}"'] & B_2 \arrow[r, "m_2"'] & Y
\end{tikzcd}$$

PULLBACKS ARE UNIQUE SO THERE EXISTS AN ISO $e : P \to P_1^{23}$.

THIS COMMUTES:

$$\begin{tikzcd}
P \arrow[d, "e"] \arrow[dd, bend right=60, "m_1 p_{12} p"'] \arrow[ddrr, bend left, "f_3 g_{23} g"] & & \\
P_1^{23} \arrow[d, "m_1 p_1^{23}"'] \arrow[drr, "f_3 g_{23} g_1^{23}"] & & \\
X & & W
\end{tikzcd}$$

THIS SHOWS $(m_1 p_{12} p_1, f_3 g_{23} q) \equiv (m_1 p_1^{23}, f_3 g_{23} q_1^{23})$ $\equiv (m_1 p_{12} p_{12}^3, f_3 q_{12}^3)$

SAME ARGUMENT WITH OTHER 2 PULLBACKS ($P$ AND $P_{12}^3$) SHOWS THIS

SO ASSOCIATIVITY IS SATISFIED. $\mathcal{C}_{PAR}$ IS A CATEGORY

(2) $X \in \mathcal{C}$

$IX = X$

$$I(X \xrightarrow{f} Y) = \begin{tikzcd} X \arrow[d, "1"'] \arrow[dr, "f"] & \\ X & Y \end{tikzcd} = (1, f).$$

$I$ IS A FUNCTOR: $I(X \xrightarrow{1} X) = \begin{tikzcd} X \arrow[d, "1"'] \arrow[dr, "1"] & \\ X & X \end{tikzcd} = (1,1)$ ✓

$$I(X \xrightarrow{g \circ f} Z) = (1, g \circ f)$$

$g: Y \to Z$
$f: X \to Y$

$$I(Y \xrightarrow{g} Z) \circ I(X \xrightarrow{f} Y) = (1, g) \circ (1, f) = \begin{tikzcd} & & X \arrow[dl, "1"'] \arrow[dr, "f"] & & \\ & X \arrow[dl, "1"'] \arrow[dr, "f"] & & Y \arrow[dl, "1"'] \arrow[dr, "g"] & \\ X & & Y & & Z \end{tikzcd}$$

$$= (1, g \circ f) \checkmark$$

$I$ IS FAITHFUL:

$f, g: X \to Y$, $f \neq g$ THEN $I(f) = (1, f)$
$I(g) = (1, g)$

ASSUME $(1, f) \equiv (1, g)$:

$$\begin{tikzcd} X \arrow[d, "1"'] \arrow[dr, "i"] \arrow[rr, bend left, "f"] & & Y \\ X & X \arrow[l, "1"] \arrow[ur, "g"'] & \end{tikzcd}$$

$1 \circ i = 1 \Rightarrow i = 1$

$f = g \circ i = g \circ 1 = g$ CONTRADICTION!

$I$ IS INDEED FAITHFUL. $\square$

③ FIRST, LET US SHOW THAT $\eta_X$ IS MONO $(\forall X \in \mathcal{C})$

$$\begin{tikzcd} Z \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X \arrow[r, "\eta_X"] & \tilde{X} \end{tikzcd}$$

ASSUME $\eta_X \circ f = \eta_X \circ g$, WE NEED TO SHOW $f = g$

$(1,f)$ AND $(1,g)$ ARE ~~A~~ PARTIAL MAPS SO

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "1"'] & X \arrow[d, "\eta_X"] \\ Z \arrow[r, "t_{1f}"'] & \tilde{X} \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "1"'] & X \arrow[d, "\eta_X"] \\ Z \arrow[r, "t_{1g}"'] & \tilde{X} \end{tikzcd}$$

THESE TWO ARE PULLBACKS

$$t_{1f} = \eta_X \circ f = \eta_X \circ g = t_{1g}$$

$$\begin{tikzcd} Z \arrow[dr, "i"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "1"'] & & \\ & Z \arrow[r, "f"] \arrow[d, "1"'] & X \arrow[d, "\eta_X"] \\ & Z \arrow[r, "t_{1f} = t_{1g}"'] & \tilde{X} \end{tikzcd}$$

SO $i = 1$ AND $g = f \circ i = f \Rightarrow \eta_X$ IS MONO □

~~NOW~~

LET'S DEFINE THE FUNCTOR $\tilde{(-)}: \mathcal{C} \to \mathcal{C}$

$\tilde{(-)} X = \tilde{X}$ (THE OBJECT IN PART. MAP CLASSIFIER)

$\tilde{(-)}(X \xrightarrow{f} Y) \in \mathcal{C}(\tilde{X}, \tilde{Y})$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] \\ \tilde{X} \arrow[r, "t_{\eta_X, f}"'] & \tilde{Y} \end{tikzcd}$$

$\eta_X$ IS MONO SO $(\eta_X, f) \in \mathcal{C}_{PAR}(\tilde{X}, Y)$

$\exists!\, t_{\eta_X, f} \in \mathcal{C}(\tilde{X}, \tilde{Y})$

S.T. THIS IS A PULLBACK

SO LET'S DEFINE

$\tilde{(-)}(X \xrightarrow{f} Y) = t_{\eta_X, f}$

IT'S A FUNCTOR:

$\tilde{(-)}(X \xrightarrow{1} X) = 1$

$$\begin{tikzcd} X \arrow[r, "1"] \arrow[d, "\eta_X"'] & X \arrow[d, "\eta_X"] \\ \tilde{X} \arrow[r, "1"'] & \tilde{X} \end{tikzcd}$$

$f: X \to Y$

$g: Y \to Z$

$\tilde{(-)}(g \circ f)$

$$\begin{tikzcd} X \arrow[r, "g \circ f"] \arrow[d, "\eta_X"'] & Z \arrow[d, "\eta_Z"] \\ \tilde{X} \arrow[r, "t_{\eta_X, g \circ f}"'] & \tilde{Z} \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta_X"'] & Y \arrow[r, "g"] \arrow[d, "\eta_Y"] & Z \arrow[d, "\eta_Z"] \\ \tilde{X} \arrow[r, "t_{\eta_X, f}"'] & \tilde{Y} \arrow[r, "t_{\eta_Y, g}"'] & \tilde{Z} \end{tikzcd}$$

~~THIS BECAUSE OF~~

PULLBACK LEMMA

$$\begin{tikzcd} X \arrow[r, "g \circ f"] \arrow[d, "\eta_X"'] & Z \arrow[d, "\eta_Z"] \\ \tilde{X} \arrow[r, "t_{\eta_Y, g} \circ t_{\eta_X, f}"'] & \tilde{Z} \end{tikzcd}$$

~~PULLBACK LEMMA~~

BY UNIQUENESS OF $t_{m,f}$ WE HAVE $t_{\eta_X, g\circ f} = t_{\eta_Y g} \circ t_{\eta_X f} \Rightarrow \widetilde{(-)}$ IS A FUNCTOR ✓

$\widetilde{(-)}$ IS FAITHFUL: ~~Peg~~ $\rightsquigarrow f, g : \tilde{X} \to \tilde{Y}$, $f = g$

$f = \widetilde{(-)} f'$

$g = \widetilde{(-)} g'$

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] \\ \tilde{X} \arrow[r, "f"] & \tilde{Y} \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "g'"] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] \\ \tilde{X} \arrow[r, "g"'] & \tilde{Y} \end{tikzcd}$$

$f = g \Rightarrow f \circ \eta_X = g \circ \eta_X \Rightarrow \eta_Y \circ f' = \eta_Y \circ g' \Rightarrow f' = g'$ ✓

DIAGRAM COMMUTATIVITY

$\eta_Y$ MONO

$\Rightarrow \widetilde{(-)}$ IS FAITHFUL □

$X \in \mathcal{C}$

$f \in \mathcal{C}(X, Y)$

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & \tilde{X} \arrow[d, "t_{\eta_X, f}"] \\ Y \arrow[r, "\eta_Y"'] & \tilde{Y} \end{tikzcd}$$

NATURALITY CONDITION IS AUTOMATICALLY SATISFIED BY OUR DEFINITION OF $\widetilde{(-)}$

$(t_{\eta_X, f} \circ \eta_X = \eta_Y \circ f)$

④ ($\Rightarrow$)

$(m, f) \equiv (m', f')$

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\eta_Y"] \\ X \arrow[r, "t_{mf}"'] & \tilde{Y} \end{tikzcd}$$

$i$ ISO BECAUSE $(m, f) \equiv (m', f')$

$$\begin{tikzcd} B' \arrow[dr, "i"] \arrow[drr, bend left, "f'"] \arrow[ddr, bend right, "m'"'] & & \\ & B \arrow[r, "f"] \arrow[d, "m"'] & Y \arrow[d, "\eta_Y"] \\ & X \arrow[r, "t_{mf}"'] & \tilde{Y} \end{tikzcd}$$

$t_{mf} \circ m = \eta_Y \circ f$

$t_{mf} \circ m \circ i = \eta_Y \circ f \circ i$

$t_{mf} \circ m' = \eta_Y \circ f'$

SO
$$\begin{tikzcd} B' \arrow[r, "f'"] \arrow[d, "m'"'] & Y \arrow[d, "\eta_Y"] \\ X \arrow[r, "t_{mf}"'] & \tilde{Y} \end{tikzcd}$$
IS ALSO PULLBACK.

BY UNIQUENESS OF $t_{mf} \Rightarrow t_{mf} = t_{m'f'}$ □

($\Leftarrow$)

$$\begin{tikzcd} B' \arrow[dr, "e"] \arrow[drr, bend left, "f'"] \arrow[ddr, bend right, "m'"'] & & \\ & B \arrow[r, "f"] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\eta_Y"] \\ & X \arrow[r, "t_{mf}"'] & \tilde{Y} \end{tikzcd}$$

$$\begin{tikzcd} B' \arrow[r, "f'"] \arrow[d, "m'"'] & Y \arrow[d, "\eta_Y"] \\ X \arrow[r, "t_{mf}"'] & \tilde{Y} \end{tikzcd}$$

$t_{mf} \circ m' = \eta_Y \circ f'$

$\Rightarrow \exists\, e : B' \to B$ S.T. $f' = f \circ e$, $m' = m \circ e$

SO $(m', f') \sqsubset (m, f)$

SWITCH ROLES OF $B'$ AND $B$ AND WE ALSO GET $(m, f) \sqsubset (m', f')$ SO $(m, f) \equiv (m', f')$ □

5 $\quad \mathcal{C}(-,Z): \mathcal{C}^{op} \to \underline{\mathbf{Set}}$

$$\mathcal{C}(-,Z)X = \mathcal{C}(X,Z)$$

$$\mathcal{C}(X \xleftarrow{f} Y): \mathcal{C}(X,Z) \mapsto \mathcal{C}(Y,Z): \quad g \to g \circ f$$

$$F = \mathcal{C}_{PAR}(-,Y) \circ I^{op}: \mathcal{C}^{op} \to \underline{\mathbf{Set}}$$

$$FX = \mathcal{C}_{PAR}(X,Y)$$

$$F(X \xleftarrow{f} Z) = \mathcal{C}_{PAR}(X,Y) \to \mathcal{C}_{PAR}(Z,Y): \quad (m_1,f_1) \to (m_1,f_1)\circ(1,f)$$

$(\Rightarrow)$ WE WILL SHOW THAT $F$ IS NAT. ISO. TO $\mathcal{C}(-,\tilde{Y})$

$X \in |\mathcal{C}^{op}|$ $\quad f: X \leftarrow Z$

$$\begin{tikzcd}
\mathcal{C}_{PAR}(X,Y) \arrow[r, "t"] \arrow[d, "(m_1,f_1)\to(m_1,f_1)\circ(1,f)"'] & \mathcal{C}(X,\tilde{Y}) \arrow[d, "g \to g\circ f"] \\
\mathcal{C}_{PAR}(Z,Y) \arrow[r, "t"] & \mathcal{C}(Z,\tilde{Y})
\end{tikzcd}$$

$t$ IS DEFINED AS FOLLOWS: FOR $(m_1,f_1) \in \mathcal{C}_{PAR}(X,Y)$ USE CLASSIFIER

$$\begin{tikzcd}
B_1 \arrow[r, "f_1"] \arrow[d, "m_1"'] & Y \arrow[d, "\eta_Y"] \\
X \arrow[r, "t_{m_1 f_1}"'] & \tilde{Y}
\end{tikzcd}$$

AND SET $t((m_1,f_1)) = t_{m_1 f_1}$

NATURALITY CONDITION:

$$\begin{tikzcd}
P \arrow[r, "g"] \arrow[d, "p"'] & B_1 \arrow[r, "f_1"] \arrow[d, "m_1"] & Y \arrow[d, "\eta_Y"] \\
Z \arrow[r, "f"] \arrow[rr, bend right] & X \arrow[r, "t_{m_1 f_1}"] & \tilde{Y}
\end{tikzcd}$$

$$t_{p,(f_1\circ g)} = t((m_1,f_1)\circ(1,f)) = t_{m_1,f_1}\circ f$$

UNIQUENES OF $t$ BECAUSE IT'S A PART. MAP CLASSIFIER.

$t$ IS ISO IN $\underline{\mathbf{Set}}$ $\Rightarrow$ $t$ IS BIJECTIVE

$t$ IS INJECTIVE BY EXERCISE 4

$t$ IS SURJECTIVE: $g \in \mathcal{C}(X,\tilde{Y})$

$$\begin{tikzcd}
P \arrow[r, "g"] \arrow[d, "p"'] & Y \arrow[d, "\eta_Y"] \\
X \arrow[r, "g"] & \tilde{Y}
\end{tikzcd}$$

$t((p,g)) = g$ ✓

$\Rightarrow t$ IS NATURAL ISOMORPHISM □

($\Leftarrow$) AGAIN LET'S WRITE $\tilde{Y}$ INSTEAD OF $Z$.

WE HAVE NAT. ISO. $t$

$$
\begin{tikzcd}
X \\
Z \arrow[u, "f"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}_{PAR}(X,Y) \arrow[r, "t^X"] \arrow[d, "(m_1,f_1) \to (m_1,f_1)\circ(1,f)"'] & \mathcal{C}(X,\tilde{Y}) \arrow[d, "g \to g\circ f"] \\
\mathcal{C}_{PAR}(Z,Y) \arrow[r, "t^Z"'] & \mathcal{C}(Z,\tilde{Y})
\end{tikzcd}
$$

DEFINE $\eta_Y = t^Y(1) = t_{11}$

~~TAKE~~

TAKE $(m,f) \in \mathcal{C}_{PAR}(X,Y)$

LET'S SHOW THIS COMMUTES:

$$
\begin{tikzcd}
B \arrow[r, "f"] \arrow[d, "m"'] & Y \arrow[d, "t^Y_{11}"] \\
X \arrow[r, "t^X_{mf}"'] & \tilde{Y}
\end{tikzcd}
$$

BY NAT. CONDITION:

$$t^Y_{11} \circ f = t^B_{((1,1)\circ(1,f))} = t^B_{(1,f)}$$

$$t^X_{mf} \circ m = t^B_{((m,f)\circ(1,m))} = t^B_{(1,f)}$$

$$
\begin{tikzcd}
 & & B \arrow[dl, "1"'] \arrow[dr, "1"] & & \\
 & B \arrow[dl, "1"'] \arrow[drr, "m"] & & B \arrow[r, "f"] \arrow[d, "m"] & Y \\
B & & & X &
\end{tikzcd}
$$

COMMUTES ✓

THE PULLBACK OF COSPAN $X \xrightarrow{t^X_{mf}} \tilde{Y} \xleftarrow{t^Y_{11}} Y$ OBVIOUSLY ALSO COMMUTES

$$
\begin{tikzcd}
P \arrow[r, "g"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "t^Y_{11}"] \\
X \arrow[r, "t^X_{mf}"'] & \tilde{Y}
\end{tikzcd}
$$

SINCE $t^*$ IS ISO IT FOLLOWS $(m,f) \equiv (p,g)$

WHICH MEANS

$$
\begin{tikzcd}
B \arrow[r, "f"] \arrow[d, "m"'] & Y \arrow[d, "t^Y_{11}"] \\
X \arrow[r, "t^X_{mf}"'] & \tilde{Y}
\end{tikzcd}
$$

IS ALSO A PULLBACK SQUARE.

UNIQUENESS OF $t^X_{mf}$ IS CLEAR FROM THE FACT THAT $t^X$ IS ISO.

$\Rightarrow$ $Y$ HAS A PART. MAP CLASSIFIER GIVEN BY $\tilde{Y}$ AND $\eta_Y = t^Y_{11}$ $\Box$

An additional note:

I realise that at several points in the homework I have been using the same name (eg. function name f) for different things, sometimes in the same exercise (but for different, hopefully clearly seperated parts of the exercise). Since the instructions mention that “quality of exposition is an important concern” I’d just like to point out I am aware that this could be an issue in an actual paper and that I would of course not be so lazy in that case.
I hope my solution procedure is clear enough and my handwriting readable. Have a nice day!