2. izpit iz Matematike 2, FMF, Aplikativna fizika

28. 6. 2024

Čas pisanja je 120 minut. Veliko uspeha!

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Sedež (2.01)

Ime in priimek

Vpisna številka

1. naloga (25 točk)

Dana je funkcija $f(x, y) = 4y - yx^2 - y^3$.

a) Poiščite in klasificirajte njene stacionarne točke.

b) Poiščite največjo in najmanjšo vrednost funkcije f na krogu $x^2 + y^2 \leq 1$.

a) $f_x = -2xy = 0$

$f_y = 4 - x^2 - 3y^2 = 0$

$x=0$: $4 = 3y^2 \Rightarrow y = \pm\sqrt{\frac{4}{3}} = \pm\frac{2\sqrt{3}}{3}$

$y=0$: $4 = x^2 \Rightarrow x = \pm 2$

STAC. TOČKE: $T_1(0, \frac{2\sqrt{3}}{3})$, $T_2(0, -\frac{2\sqrt{3}}{3})$, $T_3(2, 0)$, $T_4(-2, 0)$

$f_{xx} = -2y$

$f_{yy} = -6y$

$f_{xy} = -2x$

$H = \begin{bmatrix} -2y & -2x \\ -2x & -6y \end{bmatrix}$ $\det(H) = 12y^2 - 4x^2$

T_1: $\det H > 0$, $f_{xx} < 0$ LOKALNI MAKSIMUM

T_2: $\det H > 0$, $f_{xx} > 0$ LOKALNI MINIMUM

T_3: $\det H < 0$ SEDLO

T_4: $\det H < 0$ SEDLO

b) VSI KANDIDATI $T_1, \ldots, T_4$ SO IZVEN $x^2 + y^2 \leq 1$

OSTANE ROB: $x^2 + y^2 = 1 \Rightarrow x^2 = 1 - y^2$

$g(y) = 4y - y(1-y^2) - y^3 = 4y - y = \cancel{4}\, 3y$

$g'(y) = 0 \Rightarrow y = 0$

KANDIDATI: $y = 0$ $\rightarrow f(1-0, 0) = 0$

$y = \pm 1$ (ROB ROBA) $\rightarrow f(1-1, 1) = 3$ $\leftarrow$ MAKSIMUM

$\rightarrow f(1-1, -1) = -3$ $\leftarrow$ MINIMUM

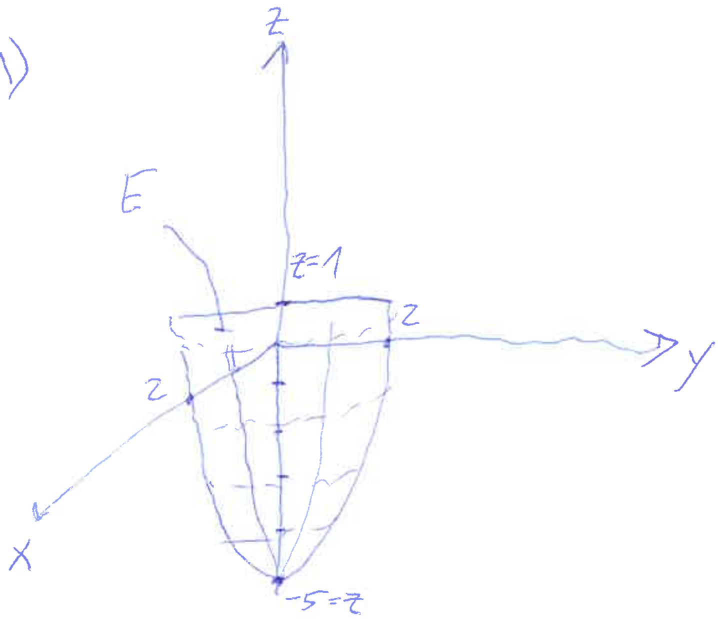
2. naloga (25 točk)

Naj bo E območje v $\mathbb{R}^3$ omejeno s ploskvijo $z = 2y^2 + 2x^2 - 5$ ter ravnino $z = 1$.

a) Skicirajte območje E.

b) Izračunajte $\iiint_E y^2 \, \mathrm{d}V$.

a)



b) CILINDRIČNE KOORDINATE

$x = r\cos\varphi \qquad \varphi \in [0, 2\pi)$

$y = r\sin\varphi$

$z = z \qquad z \in [-5, 1]$

$z = 2y^2 + 2x^2 - 5 \Rightarrow z = 2r^2 - 5 \Rightarrow r = \sqrt{\frac{z+5}{2}}$

$\Rightarrow r \leq \sqrt{\frac{z+5}{2}}$

$$\iiint_E y^2 dV = \int_0^{2\pi} d\varphi \int_{-5}^{1} dz \int_0^{\sqrt{\frac{z+5}{2}}} r^2 \sin^2\varphi \, r \, dr = \pi \cdot \int_{-5}^{1} \frac{\left(\sqrt{\frac{z+5}{2}}\right)^4}{4} dz =$$

$$= \frac{\pi}{4 \cdot 4} \int_{-5}^{1} (\underbrace{z+5}_{u})^2 dz = \frac{\pi}{16} \int_0^6 u^2 du = \frac{\pi}{16 \cdot 3} 6^3 = \frac{6\pi \cdot 3}{4} = \frac{18\pi}{4} = \frac{9\pi}{2}$$

3. naloga (25 točk)

Dano imamo krivuljo $\vec{r}(t) = (t, t^2/2, t^3/3)$, $t \in \mathbb{R}$.

a) Pokažite, da je podana parametrizacija regularna.

b) Naj bo $\vec{F}(x,y,z) = (4y^2, 3z, x^2)$ vektorsko polje. Izračunajte krivuljni integral $\int_\gamma \vec{F} \cdot d\vec{s}$, kjer je γ del krivulje $\vec{r}(t)$ med točkama $(0,0,0)$ in $(2,2,8/3)$.

c) Izračunajte spremljajoči trieder v točki $(2,2,8/3)$.

d) Kolikšni sta fleksijska in torzijska ukrivljenost krivulje v odvisnosti od parametra t?

a) $\dot{\vec{r}}(t) = (1, t, t^2) \neq \vec{0}$ ✓

b) $$\int_\gamma \vec{F}\cdot d\vec{s} = \int_0^2 dt\, (t^4, t^3, t^2)\cdot(1,t,t^2) = \int_0^2 dt\, 3t^4 = 3\,\frac{t^5}{5}\Big|_0^2 = \boxed{\frac{3\cdot 2^5}{5}}$$

$\vec{r}(0) = (0,0,0)$

$\vec{r}(2) = (2,2,8/3)$

c)

$1 \quad t \quad t^2$

$0 \quad 1 \quad 2t$

$\dot{\vec{r}}(2) = (1,2,4)$ $\quad \ddot{\vec{r}}(t) = (0,1,2t)$ $\quad \ddot{\vec{r}}(2) = (0,1,4)$ $\quad \dot{\vec{r}}(t)\times\ddot{\vec{r}}(t) = (t^2, -2t, 1)$

$$\vec{T} = \frac{\dot{\vec{r}}(2)}{\|\dot{\vec{r}}(2)\|} = \frac{(1,2,4)}{\sqrt{1+4+16}} = \boxed{\frac{(1,2,4)}{\sqrt{21}}}$$

$$\vec{B} = \frac{\dot{\vec{r}}(2)\times\ddot{\vec{r}}(2)}{\|\dot{\vec{r}}(2)\times\ddot{\vec{r}}(2)\|} = \boxed{\frac{(4,-4,1)}{\sqrt{33}}}$$

$$\vec{N} = \vec{B}\times\vec{T} = \frac{1}{\sqrt{33\cdot 21}}\,(4,-4,1)\times(1,2,4) = \frac{(-18,-15,12)}{\sqrt{396}} = \frac{(-18,-15,12)}{6\sqrt{11}} = \boxed{\frac{(-6,-5,4)}{2\sqrt{11}}}$$

d) $$\kappa = \frac{\|\dot{\vec{r}}(t)\times\ddot{\vec{r}}(t)\|}{\|\dot{\vec{r}}(t)\|^3} = \boxed{\frac{\sqrt{t^4+4t^2+1}}{\sqrt{1+t^2+t^4}^{\,3}}}$$

$$\omega = \frac{[\dot{\vec{r}}(t), \ddot{\vec{r}}(t), \dddot{\vec{r}}(t)]}{\|\dot{\vec{r}}(t)\times\ddot{\vec{r}}(t)\|^2} = \boxed{\frac{2}{t^4+4t^2+1}}$$

$\dddot{\vec{r}}(t) = (0,0,2)$

4. naloga (25 točk)

Dana je diferencialna enačba

$$y'' - 3y' + 2y = e^x + e^{2x}$$

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a) Poiščite njeno splošno rešitev.

b) Poiščite tisto rešitev, ki zadošča pogojema $y(0) = 0, y'(0) = 1$.

a) HOMOGEN DEL: $y'' - 3y' + 2y = 0$

$\Rightarrow \lambda^2 - 3\lambda + 2 = 0 \Rightarrow (\lambda-2)(\lambda-1) = 0$

~~$(\lambda+3)(\lambda-4)=0$~~

$\lambda_1 = 2$

$\lambda_2 = 1$

$\Rightarrow y_H(x) = Ae^x + Be^{2x}$

PARTIKULARNI DEL: NASTAVEK: $y_P(x) = Axe^x + Bxe^{2x}$

~~$Ae^x + Axe^x + Be^{2x} + 2Bxe^{2x}$~~ $y_P(x)' = Ae^x + Axe^x + Be^{2x} + 2Bxe^{2x} = Ae^x(1+x) + Be^{2x}(1+2x)$

$y_P(x)'' = Ae^x(1+x) + Ae^x + 2Be^{2x}(1+2x) + 2Be^{2x}$

$y'' - 3y' + 2y = e^x + e^{2x} \Rightarrow Ae^x(2+x) + 2Be^{2x}(2+2x) - 3Ae^x(1+x) - 3Be^{2x}(1+2x) + 2Axe^x + 2Bxe^{2x} =$

$= e^x + e^{2x}$

KOEF. e^x: $A(2+x) - 3A(1+x) + 2Ax = 1$

$-A = 1 \Rightarrow A = -1$

KOEF. e^{2x}: $2B(2+2x) - 3B(1+2x) + 2Bx = 1$

$B = 1$ $\Rightarrow y_P(x) = x(e^{2x} - e^x)$

$$\boxed{y(x) = Ae^x + Be^{2x} + x(e^{2x} - e^x)}$$

b) $y(0) = 0 \Rightarrow A + B = 0 \Rightarrow A = -B$

$y'(0) = 1 \Rightarrow A + 2B + 0 = 1 \Rightarrow B = 1, A = -1$

$$\Rightarrow \boxed{y(x) = e^{2x} - e^x + x(e^{2x} - e^x)}$$