Category theory, Homework 2

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Let $(S, 0, +, 1, \cdot)$ be a *semiring*. Let $V : \mathbf{Set} \to \mathbf{Set}$ be the endofunctor defined below:

$$V(X) := \{\mathbf{v} : X \to S | \mathbf{v} \text{ has finite support}\}$$
$$V(f : X \to Y) := v \mapsto \left(y \mapsto \sum_{x \in f^{-1}(y)} \mathbf{v}(x) \right)$$

1 First question

Define natural transformations $\eta : 1_{\mathbf{Set}} \Rightarrow V$ and $\mu : V^2 \Rightarrow V$ such that (V, η, μ) is a monad.

Let's first define the components $\eta_X : X \to V(X)$ as:

$$\eta_X(x) = 1_x,$$

where 1_x is the "indicator function":

$$1_x(y) = \begin{cases} 1 & y = x \\ 0 & y \neq x \end{cases}$$

This obviously has finite support.

Now for the multiplication $\mu_X : V^2(X) \to V(X)$ we can define:

$$\mu_X(k) = x \mapsto \sum_{v \in V(X)} k(v) \cdot v(x)$$

The sum on the right is actually a finite sum (as in, only a finite number of terms are non-zero), since $k \in V^2(X)$ has finite support. We could equivalently write $\sum_{v \in \mathrm{Supp}(k)}$, but I've opted for the above option for readability. Since the sum is finite, and for each term in the sum the correspodning $v \in V(X)$ has finite support, the resulting function $\mu_X(k)$ has finite support as well. I will make similar arguments in the future, but won't always be explicit about it. I will also not be explicit about when certain properties of a semiring are used, such as $0 \cdot x = 0$ or $0 + 0 = 0$, since I assume the reader is familiar with this and I would not like to clutter the text.

2 Second question

Give an explicit description of the Kleisli category $\mathbf{Set}_V$ *for the monad* V *above.*

The objects of $\mathbf{Set}_V$ are the same as objects of $\mathbf{Set}$ - sets. As for the morphisms it holds that: $\mathbf{Set}_V(X,Y) = \mathbf{Set}(X, V(Y))$.

So for each map $f : X \to V(Y)$ in $\mathbf{Set}$ we have a corresponding map $f^* : X \to Y$ in $\mathbf{Set}_V$.

In the following, we want to describe the category $\mathbf{Set}_V$. Nevertheless I will work with maps in $\mathbf{Set}$ (like $f : X \to V(Y)$), because they are more familiar. The appropriate maps in $\mathbf{Set}_V$ can then always be obtained by the above correspondence.

The identity map for object X is the monad unit $\eta_X : X \to V(X)$ as defined in the previous exercise.

Let's work out the composition. Given

$$f : X \to V(Y)$$
$$g : Y \to V(Z)$$

we would like to define

$$g \circ f : X \to V(Z)$$

According to Wikipedia this is equal to $\mu_Z \circ V(g) \circ f$. Let's work out what that is exactly. First part is $V(g) \circ f : X \to V^2(Z)$:

$$(V(g) \circ f)(x) = v \mapsto \sum_{y \in g^{-1}v} f(x)(y)$$

If we compose μ_Z with that we get:

$$(\mu_z \circ V(g) \circ f)(x) = z \mapsto \sum_{v \in V(Z)} \sum_{y \in g^{-1}v} f(x)(y) \cdot v(z)$$

Since multiplication is distributive in a semiring, we don't have to worry about having to put any parantheses here. The previous formula can be simplified even a bit more:

$$(\mu_z \circ V(g) \circ f)(x) = z \mapsto \sum_{y \in Y} f(x)(y) \cdot g(y)(z)$$

Again as before, this sum has only a finite number of non zero terms as $f(x)$ has finite support.

For clarity let us repeat the main result: Given:

$$f : X \to V(Y)$$
$$g : Y \to V(Z)$$

we define

$$g \circ f : X \to V(Z)$$

by

$$(g \circ f)(x) = z \mapsto \sum_{y \in Y} f(x)(y) \cdot g(y)(z)$$

3 Question three

In the case that the semiring S is *commutative*, the binary product operation on **Set** extends to a bifunctor $\times : \mathbf{Set}_V \times \mathbf{Set}_V \to \mathbf{Set}_V$ whose action on objects (sets) maps X, Y to the product set $X \times Y$. This bifunctor is part of a symmetric monoidal structore on the Kleisli category $\mathbf{Set}_V$.

Define the morphism action of the bifunctor $\times : \mathbf{Set}_V \times \mathbf{Set}_V \to \mathbf{Set}_V$*.*

A morphism in $\mathbf{Set}_V \times \mathbf{Set}_V$ from $X \times Y$ to $V(X') \times V(Y')$ is given by a pair of maps (f, g):

$$\begin{aligned} f &: X \to V(X') \\ g &: Y \to V(Y') \end{aligned}$$

The bifunctor $\times$ maps it to a map:

$$f \times g : X \times Y \to V(X' \times Y'),$$

that given $x \in X, y \in Y$ (or equivalently a pair $(x, y) \in X \times Y$) produces a function:

$$(x', y') \mapsto f(x)(x') \cdot g(y)(y').$$

4 Question four

Give an example showing that the definition you gave in Question 3 does not give rise to a bifunctor in the case of a non-commutative semiring S.

I've decided to show this the long way (sorry) by showing that in the case of a non-commutative S one of the two functor laws (the composition one) does not necessarily hold. I provide a specific example at the end.

Let's take:

$$\begin{aligned} f &: X \to V(X') \\ g &: Y \to V(Y') \\ f' &: X' \to V(X'') \\ g' &: Y' \to V(Y'') \end{aligned}$$

If we perform the composition in the product category first we get a map:

$$\begin{aligned} &f' \circ f : X \to V(X'') \\ &g' \circ g : Y \to V(Y'') \\ &(f' \circ f, g' \circ g) : X \times Y \to V(X'') \times V(Y'') \end{aligned}$$

For this we need (just repeating the already mentioned)

$$\begin{aligned} (f' \circ f)(x) = x'' &\to \sum_{x' \in X'} f(x)(x') \cdot f'(x')(x'') \\ (g' \circ g)(y) = y'' &\to \sum_{y' \in Y'} g(y)(y') \cdot g'(y')(y'') \end{aligned}$$

Applying the bifunctor $\times$ to these gives a map that for each $(x, y) \in X \times Y$ gives:

$$(x'', y'') \mapsto \sum_{x' \in X'} f(x)(x') \cdot f'(x')(x'') \cdot \sum_{y' \in Y'} g(y)(y') \cdot g'(y')(y'')$$

which (distributivity again) is equal to:

$$(x'', y'') \mapsto \sum_{x' \in X', y' \in Y'} f(x)(x') \cdot f'(x')(x'') \cdot g(y)(y') \cdot g'(y')(y'')$$

On the other hand, if we first apply the bifunctor $\times$ to the maps (f, g) and (f', g') we get (I hope by now the domains/codomains are clear from the notation):

$$(f \times g)(x, y) = (x', y') \mapsto f(x)(x') \cdot g(y)(y')$$
$$(f' \times g')(x', y') = (x'', y'') \mapsto f'(x')(x'') \cdot g'(y')(y'')$$

By composing these in $\mathbf{Set}_V$ we get:

$$(f' \times g') \circ (f \times g)(x, y) = (x'', y'') \mapsto \sum_{x' \in X', y' \in Y'} f(x)(x') \cdot g(y)(y') \cdot f'(x')(x'') \cdot g'(y')(y'')$$

Let us for clarity repeat what we have just reached. If we first compose the maps and then apply the bifunctor we get a map:

$$(x, y) \mapsto \left((x'', y'') \mapsto \sum_{x' \in X', y' \in Y'} f(x)(x') \cdot f'(x')(x'') \cdot g(y)(y') \cdot g'(y')(y'') \right)$$

While if we first apply the bifunctor and only then compose we get:

$$(x, y) \mapsto \left((x'', y'') \mapsto \sum_{x' \in X', y' \in Y'} f(x)(x') \cdot g(y)(y') \cdot f'(x')(x'') \cdot g'(y')(y'') \right)$$

These two should be equal. If the ring S is commutative, they are equal and there are no problems. But in the case of non-commutative ring there can be problems if it happens that $f'(x')(x'') \cdot g(y)(y') \neq g(y)(y') \cdot f'(x')(x'')$.

A simple concrete example:

Let's take the sets X, Y, X', Y', X'', Y'' to all be equal to the terminal object 1 in **Set** (singleton set). Take f and g' to map to the identity function (so $f(x) = x' \mapsto 1$) (they have finite support since the domain is the singleton set). Take f' to be defined as $f'(x') = x'' \mapsto s_1$, where s_1 is a chosen element from the semi-ring. Take g to be defined as $g(y') = y' \mapsto s_2$, where s_2 is a chosen element from the semi-ring. Then the functor laws will not hold if $s_1 \cdot s_2 \neq s_2 \cdot s_1$. It's easy to find an example of such a ring S and elements s_1 and s_2. For example, take a ring of quaternions and take $s_1 = i$ and $s_2 = j$. Another example would be if ring S was the ring of 2×2 matrices over the reals and the elements s_1 and s_2 two matrices that do not commute such as $s_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $s_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

5 Question five

Let $F\colon \mathbf{Set} \to \mathbf{Set}_V$ be the left adjoint in the adjunction associated with the Kleisli category. For any set X define $\mathsf{Y}_X \in \mathbf{Set}_V(X, X \times X)$ and $\mathsf{o}_X \in \mathbf{Set}_V(X, 1)$ (where 1 is a chosen terminal object in **Set**) by:

$$\mathsf{Y}_X \ := \ F(\Delta_X) \qquad\qquad \mathsf{o}_X \ := \ F(!_X) \ ,$$

where $\Delta_X = x \mapsto (x,x) : X \to X \times X$ and $!_X\colon X \to 1$ is the unique map. We can write these maps as boxes for use in *string diagrams* based on the monoidal structure on $\mathbf{Set}_V$. (For an introduction to string diagrams, read the paper *"A survey of graphical languages for monoidal categories"*, by Peter Selinger, as far as the send of Section 3.1. There is a link to the paper from the course webpage.)

Question 5 *Explicitly characterise those Kleisli maps $g \in \mathbf{Set}_V(X,Y)$ for which the following string-diagram equation holds.*

Let's first state what the left Kleisli adjunction does. This is straightforward to see from the Kleisli category. On an object X it simply returns the object $F(X) = X$. On a morphism $f : X \to Y$ it gives us a map $X \to V(Y)$ given by $\eta_Y \circ f$ or explicitly it gives us the morphism $x \mapsto 1_{f(x)}$.

Using this we can get explicit form of the map Y_X:

$$Y_X(x) = 1_{(x,x)}$$

and the map $\circ_X$:

$$\sigma_X(x) = 1$$

(The latter maps all elements of X to a constant function 1. Actually all maps to the terminal object here are just constants.)

The string diagram equation says that for each $x \in X$ the following must hold:

$$\circ_X(x) = (\circ_Y \circ g)(x)$$

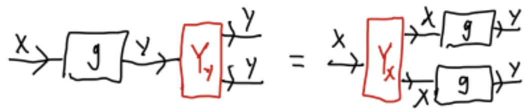
Writing out the composition operation and the definition of $\circ_X$ and $\circ_Y$ we get:

$$1 = \sum_{y \in Y} g(x)(y)$$

This equation provides the required characterisation of the Kleisli maps g for which the given string diagram equation holds - the appropriate $g \in \mathbf{Set}_V(X,Y)$ are such that the above equation holds for all $x \in X$.

6 Question six

Question 6 *Explicitly characterise those Kleisli maps $g \in \mathbf{Set}_V(X, Y)$ for which the following string-diagram equation holds.*



Can you simplify your characterisation in the case in which the semiring S satisfies the property: $x \cdot y = 0$ only if (at least) one of x or y is 0?

The "left hand side of string diagram equation" is a map:

$$(Y_y \circ g)(x) = (y_1, y_2) \mapsto \sum_{y \in Y} g(x)(y) \cdot 1_{(y,y)}(y_1, y_2)$$

This is a map that returns 0 if $y_1 \neq y_2$ and $g(x)(y)$ if $(y_1, y_2) = (y, y)$

The "right hand side of string diagram equation" is a map:

$$((g \times g) \circ Y_X)(x) = (y_1, y_2) \mapsto \sum_{x_1 \in X, x_2 \in X} 1_{(x,x)}(x_1, x_2) \cdot g(x_1)(y_1) \cdot g(x_2)(y_2)$$

which is equal to

$$(y_1, y_2) \mapsto g(x)(y_1) \cdot g(x)(y_2)$$

The conditions that the string diagram equality holds is then:

If $y_1 \neq y_2$ it must be that $g(x)(y_1) \cdot g(x)(y_2) = 0$.

if $y_1 = y_2 = y$ it must hold that $g(x)(y) \cdot g(x)(y) = g(x)(y)$

We can use this to characterise the suitable functions g.

For each $x \in X$, the image of $g(x)$ must be some subset of the idenpotent elements of S. In addition, the product of each two different elements of the image must be zero. Also, if $g(x)$ maps two elements y_1, y_2 to the same element of S, that element must be zero.

If S is an integral domain we can say more. For each x $g(x)$ maps one element of Y to an arbitrary idenpotent element and all other to zero.