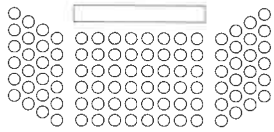
1. kolokvij iz Matematike 2, FMF, Praktična matematika

29. 11. 2022

Čas pisanja je 120 minut. Veliko uspeha!



Sedež (2.01)

Ime in priimek

Vpisna številka

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1. naloga (20 točk)

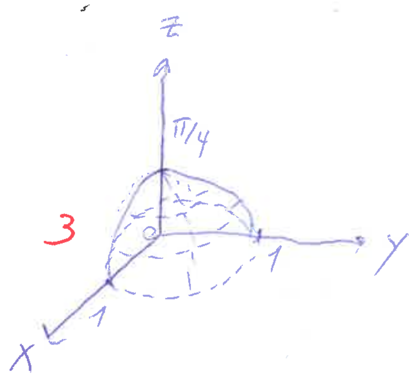
a) Določite definicijsko območje in zalogo vrednosti funkcije $f(x,y) = \arctan(\sqrt{1-x^2-y^2})$. Skicirajte tudi graf funkcije f.

b) Določite definicijsko območje funkcije $g(x,y) = \ln(x^2+y^2) + \sqrt{1-x^2} + \sqrt{1-y^2}$. Zapišite množici notranjih in robnih točk definicijskega območja in ugotovite ali je definicijsko območje funkcije g zaprta množica? (Svoj odgovor dobro utemeljite!)

a) $f(x,y) = \arctan(\sqrt{\underbrace{1-x^2-y^2}_{\geq 0}})$ 1 $\Rightarrow 1-x^2-y^2 \geq 0 \Rightarrow D_f = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 \leq 1\}$ 1

ROBNE VREDNOSTI $\arctan(0) = 0$

$\arctan(1) = \frac{\pi}{4} \Rightarrow Z_f = [0, \frac{\pi}{4}]$ 3



b) $g(x,y) = \ln(x^2+y^2) + \sqrt{1-x^2} + \sqrt{1-y^2}$

$x^2+y^2 > 0 \Rightarrow (x,y) \neq (0,0)$ 1

$1-x^2 \geq 0 \Rightarrow x^2 \leq 1 \Rightarrow |x| \leq 1$ 1

$1-y^2 \geq 0 \Rightarrow |y| \leq 1$ 1

$D_g = \{(x,y)\in\mathbb{R}^2 \mid (x,y)\neq(0,0), |x|\leq 1, |y|\leq 1\}$ 3

$\operatorname{int} D_g = \{(x,y)\in\mathbb{R}^2 \mid (x,y)\neq(0,0), |x|<1, |y|<1\}$ 2

$\partial D_g = \{(x,y)\in\mathbb{R}^2 \mid (|x|=1 \wedge |y|\leq 1) \vee (|y|=1 \wedge |x|\leq 1)\} \cup \{(0,0)\}$ 2

NI ZAPRTA, KER NE VSEBUJE CELOTNEGA ROBA 2

2. naloga (20 točk)

a) Dokaži, da ima funkcija $f(x, y) = x^2y^3 - 4xy$ v točki $(0, 0)$ sedlo. Svoj odgovor dobro utemeljite!

b) Naj bo $g(u, v)$ diferenciabilna funkcija, kjer je $u = 2x + y$ in $v = x - 2y$. Določite konstanto α tako, da bo veljalo

$$\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 = \alpha\left(\left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2\right).$$

a) $f_x = 2xy^3 - 4y$ 1 $f_x(0,0) = 0$ 1

$f_y = 3x^2y^2 - 4x$ 1 $f_y(0,0) = 0$ 1

$f_{xx} = 2y^3$ 1

$f_{yy} = 6x^2y$ 1

$f_{xy} = 6xy^2 - 4$ 1

$H = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix}$ 1 $\det H = -16$ 1

$\Rightarrow$ KER $\det H < 0$ IMAMO SEDLO. 1

b) $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial g}{\partial u} 2 + \frac{\partial g}{\partial v}$ 4

$\frac{\partial g}{\partial y} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial g}{\partial u} - \frac{\partial g}{\partial v}\cdot 2$ 4

$\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 = 4\left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2 + 4\frac{\partial g}{\partial u}\frac{\partial g}{\partial v} + \left(\frac{\partial g}{\partial u}\right)^2 + 4\left(\frac{\partial g}{\partial v}\right)^2 - 4\frac{\partial g}{\partial u}\frac{\partial g}{\partial v} =$

$= 5\left(\left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2\right)$ 1

$\Rightarrow \alpha = 5$ 1

3. naloga (20 točk)

Dana je funkcija f s predpisom

$$f(x,y) = \begin{cases} y^2 \sin \frac{x}{\sqrt{x^2+y^2}} & ;(x,y) \neq (0,0) \\ a & ;(x,y) = (0,0). \end{cases}$$

a) Določite a tako, da bo f zvezna v $(0,0)$.

b) Določite oba prva parcialna odvoda funkcije f

c) Ali je f diferenciabilna v točki $(0,0)$?

$x = r\cos\varphi$
$y = r\sin\varphi$

a) $$\lim_{(x,y)\to(0,0)} y^2 \sin\frac{x}{\sqrt{x^2+y^2}} = \lim_{r\to 0} r^2 \sin^2\varphi \sin\left(\frac{r\cos\varphi}{r}\right) = \lim_{r\to 0} r^2 \sin^2\varphi \underbrace{\sin(\cos\varphi)}_{\text{OMEJENO}} = 0 \quad 4$$

$\Rightarrow a = 0$ 2

b) $(x,y) \neq (0,0)$

$$f_x = y^2 \cos\left(\frac{x}{\sqrt{x^2+y^2}}\right) \frac{\sqrt{x^2+y^2} - \frac{x^2}{\sqrt{x^2+y^2}}}{x^2+y^2} = \cos\left(\frac{x}{\sqrt{x^2+y^2}}\right) \frac{y^4}{(x^2+y^2)^{3/2}} \quad 2$$

$$f_y = 2y \sin\left(\frac{x}{\sqrt{x^2+y^2}}\right) + y^2 \cos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\cdot(-1) \frac{xy}{(x^2+y^2)^{3/2}} \quad 2$$

$(x,y) = (0,0)$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0 \quad 2, \quad f_y(0,0) = \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0 \quad 2$$

$$f_x = \begin{cases} \cos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\frac{y^4}{(x^2+y^2)^{3/2}} ; & (x,y)\neq(0,0) \\ 0 ; & (x,y) = (0,0) \end{cases}$$

$$f_y = \begin{cases} 2y\sin\left(\frac{x}{\sqrt{x^2+y^2}}\right) - \cos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\frac{xy^3}{(x^2+y^2)^{3/2}} ; & (x,y)\neq(0,0) \\ 0 ; & (x,y)=(0,0) \end{cases}$$

c) $$f(h,k) = \overset{0}{f(0,0)} + \overset{0}{f_x(0,0)}h + \overset{0}{f_y(0,0)}k + R(h,k) \quad 2 \quad \Rightarrow R(h,k) = f(h,k)$$

$$\lim_{(h,k)\to(0,0)} \frac{R(h,k)}{\sqrt{h^2+k^2}} \; 1 \; = \lim_{(h,k)\to(0,0)} \frac{k^2 \sin\left(\frac{h}{\sqrt{h^2+k^2}}\right)}{\sqrt{h^2+k^2}} = \lim_{r\to 0}\frac{r^2\sin^2\varphi \sin(\cos\varphi)}{r} =$$

$h = r\cos\varphi$
$k = r\sin\varphi$

$$= \lim_{r\to 0} r \underbrace{\sin^2\varphi \sin(\cos\varphi)}_{\text{OMEJENO}} = 0 \quad 2$$

$\Rightarrow$ JE DIFERENCIABILNA V $(0,0)$ 1

4. naloga (20 točk)

Dana je funkcija f s predpisom $f(x,y) = x + xy + x^2 + y\sin(xy^2 - y^2)$.

a) Izračunajte vse parcialne odvode prvega in drugega reda funkcije f v točki $(1,0)$.

b) Zapišite Taylorjev polinom 2. stopnje in z njim izračunajte približno vrednost funkcije f v točki $(0.8, 0.1)$

c) S pomočjo Taylorjevega razvoja funkcije $\sin x$, razvijte f v Taylorjevo vrsto okoli točke $(1,0)$ in določite vrednosti mešanih odvodov $\frac{\partial^{32} f}{\partial x^{11}\partial y^{21}}(1,0)$ in $\frac{\partial^{22} f}{\partial x^{7}\partial y^{15}}(1,0)$.

a) $f_x = 1+y+2x+y^3\cos(xy^2-y^2)$ 1

$f_y = x+\sin(y^2(x-1)) + y\cos(xy^2-y^2)\cdot(2yx-2y) = x+\sin(y^2(x-1)) + \cos(y^2(x-1))(2y^2x-2y^2)$ 1

$f_{xx} = 2 - \sin(xy^2-y^2)y^5$ 1

$f_{yy} = \cos(y^2(x-1))\cdot 2y(x-1) - \sin(y^2(x-1))\cdot 2y(x-1)\cdot(2y^2x-2y^2)+\cos(y^2(x-1))(4yx-4y)$ 1

$f_{xy} = 1+3y^2\cos(xy^2-y^2) - \sin(xy^2-y^2)y^3(2xy-2y)$ 1

$f_x(1,0)=3$ 1 $\quad f_{xx}(1,0)=2$ 1 $\quad f_{xy}(1,0)=1$ 1

$f_y(1,0)=1$ 1 $\quad f_{yy}(1,0)=0$ 1

b) $T(x,y) = f(1,0)+f_x(1,0)\cdot(x-1)+f_y(1,0)\cdot y+\frac{1}{2}f_{xx}(1,0)\cdot(x-1)^2+\frac{1}{2}f_{yy}(1,0)y^2+f_{xy}(1,0)(x-1)y=$ 1 1

$= 2+3(x-1)+y+(x-1)^2+(x-1)y$ 1

$f(0.8,0.1)\approx 2+3\cdot(-0.2)+0.1+0.2^2+(-0.2)\cdot 0.1 = 2-0.6+0.1+0.04-0.02=$ 1

$= 1.52$ 1

c) $y\sin(y^2(x-1)) = y\sum_{k=0}^{\infty}(-1)^k\frac{(y^2(x-1))^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty}(-1)^k\frac{(x-1)^{2k+1}y^{4k+3}}{(2k+1)!}$ 1

SKUPAJ Z b) DOBIMO RAZVOJ $f(x,y)=2+3(x-1)+y+(x-1)^2+(x-1)y+\sum_{k=0}^{\infty}(-1)^k\frac{(x-1)^{2k+1}y^{4k+3}}{(2k+1)!}$ 1

$\frac{\partial^{32} f}{\partial x^{11}\partial y^{21}}(1,0) = 0$ 1 (KER $x^{11}y^{21}$ NE NASTOPA V RAZVOJU)

$\frac{\partial^{22} f}{\partial x^{7}\partial y^{15}}(1,0) \overset{(k=3)}{=} 7!\cdot 15!\cdot(-1)\frac{1}{7!} = -15!$ 2

5. naloga (20 točk)

Dana je funkcija f s predpisom $f(x,y) = x^2 + 10y + y^2$.

a) Določite globalne ekstreme funkcije f na območju

$$D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + 4y^2 \leq 16, y \leq 0\}.$$

Območje tudi skicirajte!

b) Poiščite točko na nivojnici $f(x,y) = -16$, ki je najbližje koordinatnemu izhodišču.

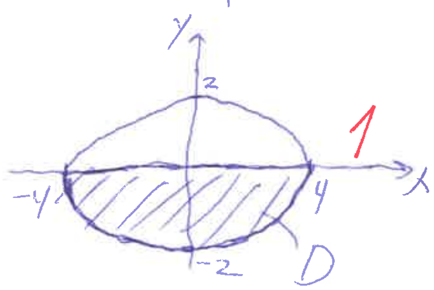
a) BREZ OMEJITEV:

$f_x = 2x = 0$ 1

$f_y = 10 + 2y = 0$ 1 $\Rightarrow$ $x=0$, $y=-5$ 1 ZUNAJ OBMOČJA D 1

$x^2 + 4y^2 = 16$

$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$



ROB $y=0$: $f(x,0) = x^2$ 1 KANDIDAT $T_1(0,0)$ 1

$f'(x,0) = 2x = 0$

ROB ROBA: $(\pm 4, 0)$ 1

ELIPSA:

$g(x,y,\lambda) = x^2 + 10y + y^2 - \lambda(16 - x^2 - 4y^2)$ 1

1 $g_x = 2x + 2\lambda x = 0$

1 $g_y = 10 + 2y + 8y\lambda = 0$

1 $g_\lambda = -16 + x^2 + 4y^2 = 0$

$x = 0$ ALI $\lambda = -1$ 1

$x=0$: $-16 + 4y^2 = 0$, $y = \pm 2$ 1

KANDIDAT $T_2(0,-2)$ ($+2$ ZUNAJ D)

$\lambda=-1$: $10 + 2y - 8y = 0$, $10 = 6y$, $y = \frac{10}{6}$ ZUNAJ D 1

KANDIDATI

$T_1(0,0)$ $f = 0$

$T_2(4,0)$ $f = 16$

$T_3(-4,0)$ $f = 16$ > MAKSIMUMA 1

$T_4(0,-2)$ $f = -16$ - MINIMUM

b)

$x^2 + 10y + y^2 = -16$ 1

MIN $\sqrt{x^2+y^2} \Leftrightarrow$ MIN x^2+y^2

$g(x,y,\lambda) = x^2 + y^2 - \lambda(16 + x^2 + 10y + y^2)$ 1

$g_x = 2x - 2\lambda x = 0$

$g_y = 2y - 10\lambda - 2\lambda y = 0$

$g_\lambda = -16 - x^2 - 10y - y^2 = 0$ 1

$x = 0$: $y^2 + 10y + 16 = 0$, $y = \frac{-10 \pm \sqrt{100-64}}{2} = \frac{-10 \pm 6}{2}$

$\lambda = 1$: $2y - 10 - 2y = 0$ //

KANDIDATA $(0,-2)$ ← NAJBLIŽJI 1

$(0,-8)$ ← NAJBOLJ ODDALJEN

5.b)

ALTERNATIVA:

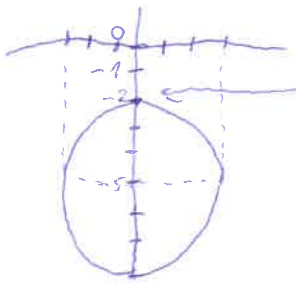
$$f(x,y) = -16$$

$$x^2 + 10y + y^2 = -16$$

$$x^2 + (y+5)^2 - 25 = -16$$

$$x^2 + (y+5)^2 = 3^2$$

KROŽNICA RADIJA 3 S SREDIŠČEM (0,-5)



NAJBLIŽJE IZHODIŠČU JE (0,-2)

4