1. kolokvij iz Matematike II, FMF, Aplikativna matematika

29. 1. 2025

Čas pisanja je 120 minut. Veliko uspeha!

Ime in priimek

Sedež (2.01)

Vpisna številka

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1. naloga (25 točk)

a) Dan je dvakratni integral

$$\int_{-6}^{2}\left(\int_{\frac{x^2}{4}-1}^{2-x} f(x,y)\,dy\right)dx.$$

Skicirajte integracijsko območje in obrnite vrstni red integracije.

b) Za $a > 0$ in $n \in \mathbb{N}$ izračunajte

$$\int_0^\infty e^{-ax}x^{n-1}\,dx.$$

c) Dokažite, da za $a, b, c > 0$ velja

$$B(a,b)B(a+b,c) = B(b,c)B(a,b+c),$$

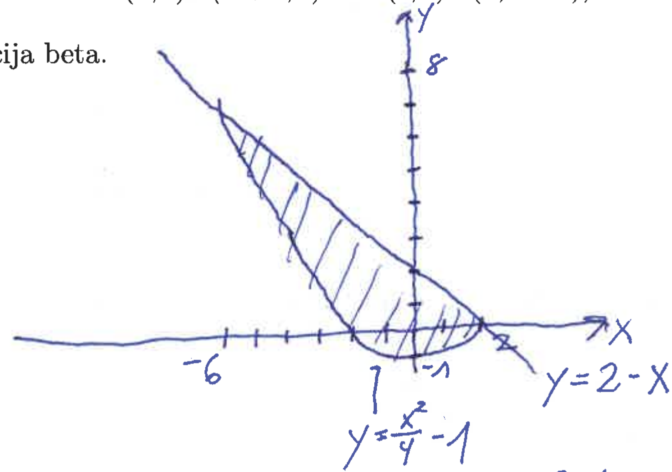
kjer $B(x,y)$ funkcija beta.

a)

$$2 - x = \frac{x^2}{4} - 1$$

$$0 = \frac{x^2}{4} + x - 3$$

$$x_{1,2} = \frac{-1 \pm \sqrt{4}}{1/2} = -2 \pm 4$$



$$\int_{-6}^{2}\left(\int_{\frac{x^2}{4}-1}^{2-x} f(x,y)\,dy\right)dx = \int_{-1}^{0}\left(\int_{-2\sqrt{y+1}}^{2\sqrt{y+1}} f(x,y)\,dx\right)dy + \int_{0}^{8}\left(\int_{-2\sqrt{y+1}}^{2-y} f(x,y)\,dx\right)dy$$

$$y = \frac{x^2}{4} - 1 \Rightarrow x = \pm\sqrt{4(y+1)}$$

$$x = \pm 2\sqrt{y+1}$$

b)

$$\int_0^\infty e^{-ax}x^{n-1}\,dx \underset{u=ax}{=} \frac{1}{a}\int_0^\infty e^{-u}\left(\frac{u}{a}\right)^{n-1}du = \frac{1}{a^n}\int_0^\infty u^{n-1}e^{-u}\,du = \frac{\Gamma(n)}{a^n} = \frac{(n-1)!}{a^n}$$

c)

$$B(a,b)B(a+b,c) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,\frac{\Gamma(a+b)\Gamma(c)}{\Gamma(a+b+c)} = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)}$$

ENAKO

$$B(b,c)B(a,b+c) = \frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)}\,\frac{\Gamma(a)\Gamma(b+c)}{\Gamma(a+b+c)} = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)}$$

2. naloga (25 točk)

Dana je

$$F(\alpha) = \int_0^{\frac{\pi}{2}} \ln(\sin^2 x + \alpha^2 \cos^2 x)\, dx,$$

kjer $\alpha > 0$. ZVEZNO ODVEDLJIVA, LAHKO ODVAJAMO $F(\alpha)$

a) S pomočjo substitucije $t = \tan x$ in trigonometrične zveze $\cos^2 x = \frac{1}{1+\tan^2 x}$ dokažite $F'(\alpha) = \frac{\pi}{\alpha+1}$ za $\alpha \neq 1$.

b) Določite $F(\alpha)$.

a) $\boxed{F'(\alpha)} = \int_0^{\frac{\pi}{2}} \frac{2\alpha \cos^2 x \, dx}{\sin^2 x + \alpha^2 \cos^2 x} = 2\alpha \int_0^{\frac{\pi}{2}} \frac{dx}{\tan^2 x + \alpha^2} = 2\alpha \int_0^{\infty} \frac{dt}{(1+t^2)(\alpha^2+t^2)} = (*)$

$t = \tan x$

$dt = \frac{1}{\cos^2 x} dx = (1+t^2)\,dx$

$$\frac{1}{(1+t^2)(\alpha^2+t^2)} = \frac{At+B}{t^2+1} + \frac{Ct+D}{t^2+\alpha^2} \Rightarrow 1 = (At+B)(t^2+\alpha^2) + (Ct+D)(t^2+1)$$

$t^3: A + C = 0 \Rightarrow A = -C$

$t^2: B + D = 0 \Rightarrow B = -D$

$t: A\alpha^2 + C = A(\alpha^2-1) = 0 \overset{\alpha \neq 1}{\Rightarrow} A = 0$

$1: B\alpha^2 + D = B(\alpha^2-1) = 1 \Rightarrow B = \frac{1}{\alpha^2-1}$

$$(*) = \frac{2\alpha}{\alpha^2-1} \int_0^{\infty} \left(\frac{1}{t^2+1} - \frac{1}{t^2+\alpha^2}\right) dt = \frac{2\alpha}{\alpha^2-1}\left(\operatorname{arctg}(t)\Big|_0^{\infty} - \frac{1}{\alpha} \operatorname{arctg}\left(\frac{t}{\alpha}\right)\Big|_0^{\infty} \right) =$$

$$= \frac{2\alpha}{\alpha^2-1} \cdot \frac{\pi}{2}\left(1 - \frac{1}{\alpha}\right) = \frac{\pi\alpha}{\alpha^2-1} \cdot \frac{\alpha-1}{\alpha} = \boxed{\frac{\pi}{\alpha+1}}$$

b) $F(\alpha) = \int F'(\alpha)\,d\alpha + C = \pi \ln(\alpha+1) + C$

$$F(1) = \int_0^{\frac{\pi}{2}} \ln(\underbrace{\sin^2 x + \cos^2 x}_{1})\, dx = 0 = \pi \ln(2) + C \Rightarrow C = -\pi \ln(2)$$

$F(\alpha)$ ZVEZNA, DEF. TUDI ZA $\alpha = 1$

$$\Rightarrow \boxed{F(\alpha)} = \pi(\ln(\alpha+1) - \ln 2) = \boxed{\pi \ln\left(\frac{\alpha+1}{2}\right)}$$

3. naloga (25 točk)

Naj bo T telo v prvem oktantu ($x, y, z \geq 0$), ki leži nad ploskvijo $z = \sqrt{3x^2 + 3y^2}$ in pod ploskvijo $x^2 + y^2 + z^2 = 9$. ← SFERA

↑ STOŽEC

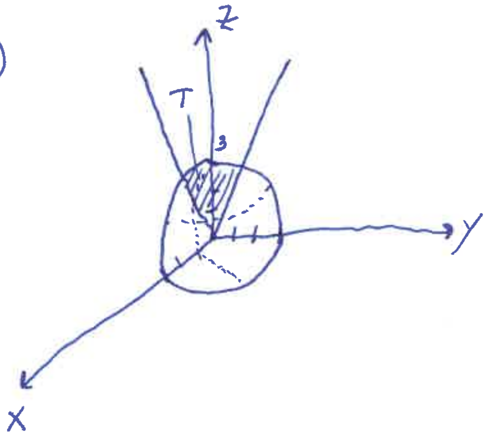
a) Skicirajte telo T.

b) S pomočjo cilindričnih koordinat z integralom izrazite volumen telesa T.

c) S pomočjo sferičnih koordinat z integralom izrazite volumen telesa T.

d) Na poljuben način izračunajte volumen telesa T.

a)



b) $\sqrt{3}\,r \leq z \leq \sqrt{9-r^2}$

$\varphi \in [0, \frac{\pi}{2}]$ $(x, y \geq 0)$

$r \in [0, r_{MAX}]$

PRESEČIŠČE: $\sqrt{3}\,r = \sqrt{9-r^2} \Rightarrow 3r^2 = 9 - r^2$

$4r^2 = 9 \Rightarrow r = \frac{3}{2}$

$$V = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{3}{2}} r \cdot dr \int_{\sqrt{3}r}^{\sqrt{9-r^2}} dz$$

c) $\varphi \in [0, \frac{\pi}{2}]$, $r \in [0, 3]$, $\theta \in [\theta_{MIN}, \frac{\pi}{2}]$ θ_{MIN} - ROB STOŽCA $k = \sqrt{3} = tg\,\theta_{MIN} \Rightarrow \theta_{MIN} = \frac{\pi}{3}$

$$V = \int_0^{\frac{\pi}{2}} d\varphi \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos\theta\, d\theta \int_0^3 r^2\, dr$$

d) Iz b): $V = \frac{\pi}{2} \int_0^{3/2} dr\, r\left(\sqrt{9-r^2} - \sqrt{3}\,r\right) = \frac{\pi}{2} \int_0^{3/2} \left(r\sqrt{9-r^2} - \sqrt{3}\,r^2\right) dr = \frac{\pi}{2}\left(-\frac{1}{2} \frac{(9-r^2)^{3/2}}{3/2}\Big|_0^{3/2} - \sqrt{3}\frac{r^3}{3}\Big|_0^{3/2}\right)$

$= \frac{\pi}{2}\left(-\frac{1}{3}\left((9-\frac{9}{4})^{3/2} - 3^3\right) - \frac{\sqrt{3}}{3}\,\frac{3^3}{2^3}\right) = \frac{\pi}{2}\left(-\frac{1}{3}\left(\frac{27}{4}\right)^{3/2} + 9 - \frac{9\sqrt{3}}{8}\right) =$

$= \frac{\pi}{2}\left(-\frac{1}{3}\,\frac{3^3 \cdot 3\sqrt{3}}{8} + 9 - \frac{9\sqrt{3}}{8}\right) = \frac{\pi}{2}\left(-\frac{27\sqrt{3}}{8} - \frac{9\sqrt{3}}{8} + 9\right) = \frac{\pi}{2}\left(9 - \frac{9}{2}\sqrt{3}\right) =$

$= \frac{9\pi}{2}\left(1 - \frac{\sqrt{3}}{2}\right)$

Iz c) $V = \frac{\pi}{2} \cdot \sin\theta\Big|_{\pi/3}^{\pi/2} \cdot \frac{3^3}{3} = \frac{9\pi}{2}\left(1 - \frac{\sqrt{3}}{2}\right)$

4. naloga (25 točk)

Z uvedbo novih spremenljivk izračunajte integral

$$\iint_D (x-y)e^{x^2-y^2}\,\mathrm{d}x\mathrm{d}y,$$

kjer je D območje, omejeno z $x+y=1$, $x+y=3$, $x^2-y^2=-1$ in $x^2-y^2=1$.

1. NAČIN

$$u = x-y \qquad x^2-y^2 = uv \Rightarrow -1 \le uv \le 1 \Rightarrow -\frac{1}{v} \le u \le \frac{1}{v}$$

$$v = x+y \Rightarrow 1 \le v \le 3$$

$$x = \frac{u+v}{2} \qquad y = \frac{v-u}{2} \qquad J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

$$\iint_D (x-y)e^{x^2-y^2}\,dxdy = \frac{1}{2}\int_1^3 dv \int_{-1/v}^{1/v} du \, \underbrace{u e^{uv}}_{\text{PER PARTES}} = \frac{1}{2}\int_1^3 dv \left(\frac{u e^{uv}}{v}\Big|_{-1/v}^{+1/v} - \frac{1}{v^2} e^{uv}\Big|_{-1/v}^{1/v} \right) =$$

$$= \frac{1}{2}\int_1^3 dv \left(\frac{e}{v^2} + \frac{e^{-1}}{v^2} - \frac{e}{v^2} + \frac{e^{-1}}{v^2} \right) = \frac{1}{e}\int_1^3 dv \frac{1}{v^2} = \frac{1}{e}\left(1-\frac{1}{3}\right) = \boxed{\frac{2}{3e}}$$

2. NAČIN

$$u = x+y \quad 1 \le u \le 3 \qquad x-y = \frac{v}{u} \qquad x = \frac{u+\frac{v}{u}}{2}$$

$$v = x^2-y^2 \quad -1 \le v \le 1 \qquad y = \frac{u-\frac{v}{u}}{2}$$

$$J = \begin{vmatrix} \frac{1-\frac{v}{u^2}}{2} & \frac{1}{2u} \\ \frac{1+\frac{v}{u^2}}{2} & -\frac{1}{2u} \end{vmatrix} = \frac{\frac{v}{u^2}-1}{4u} - \frac{\frac{v}{u^2}+1}{4u} = -\frac{1}{2u}$$

$$\iint_D (x-y)e^{x^2-y^2}\,dxdy = \int_1^3 du \int_{-1}^1 dv \frac{v}{u} e^{v} \cdot \left(\frac{1}{2u}\right) = \frac{1}{2}\int_1^3 \frac{du}{u^2} \int_{-1}^1 v e^{v}\,dv =$$

$$= \frac{1}{2}\left(1-\frac{1}{3}\right)\left(v e^{v}\Big|_{-1}^{1} - e^{v}\Big|_{-1}^{1} \right) = \frac{1}{3}\left(e + e^{-1} - e + e^{-1}\right) = \boxed{\frac{2}{3e}}$$