Homology, Persistence and Magnitude

Homework

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Exercise 1

Problem

Let $Z(G)$ denote the center of the group G. Does there exist a functor $F : \mathbf{Grp} \to \mathbf{Grp}$ such that $F(G) = Z(G)$?

Solution

The answer is no. Assume that such a functor would exist. Consider $C_2 = \{id, x\}$, the group with 2 elements. It is abelian so we have $F(C_2) = Z(C_2) = C_2$. Consider also the permutation group S_3. We can check that $F(S_3) = Z(S_3) = 1$, where 1 is the trivial group. Now consider a homomorphism (actually a monomorphism) $f : C_2 \to S_3$ that map x to the transposition $(1, 2)$. The map $Ff : C_2 \to 1$ is the trivial map. Consider also a homorphism (epismorphism, actually) $g : S_3 \to C_2$ given by a sign of the permutation (i. e., assigning id to even permutation and x to odd ones). $Fg : 1 \to C_2$ is the identity map. Their composition $gof : C_2 \to C_2$ is the identity. Thus by the properties of functors also $F(gof) : C_2 \to C_2$ is the identity. Hower for covariant functors we also have $F(gof) = F(g)oF(f)$, which is the trivial map. Therefore, such a functor F cannot exist.

Exercise 4

Problem

According to Schanuel's paper *What is the length of a potato?* the length of a ball is twice the diameter. Work out the details.

Solution

Schanuel in the paper writes Steiner's formula for the area of a given compact, convex set as something like:

$$\text{Total\_Area} = 1 * \text{Area}(T) + (2R\text{in})\text{Length}(T) + (\pi R^2\text{in}^2)\text{Number}(T).$$

He says the formula also holds for higher dimensional volumes, where the coefficiens on the right hand side are the volumes of $0, 1, 2$ balls of radius R. For our cases, T will be a ball of radius S and we will use this formula to calculate the volume of a ball (denoted T') of radius $R + S$. We have:

$$\text{Vol}(T') = 1 * \text{Vol}(T) + (2R\text{in})\text{Area}(T) + (\pi R^2\text{in}^2)\text{Length}(T) + (4/3\pi R^3\text{in}^3)\text{Number}(T).$$

We know:

- $\text{Vol}(T') = 4/3\pi(R + S)^3\text{in}^3$, the volume of a ball of radius $R + S$
- $\text{Vol}(T) = 4/3\pi S^3\text{in}^3$
- $\text{Area}(T) = 1/2 * 4\pi S^2$. This is analogous to the case in the paper - half the surface of a sphere (only one half of the surface is "exposed").
- $\text{Number}(T) = 1$, the Euler characteristic of a ball.

Plugging in and simplifying we have:

$$4/3\pi(R + S)^3 = 4/3\pi S^3 + 4R\pi S^2 + (\pi R^2\text{in}^{-1})\text{Length}(T) + 4/3\pi R^3$$

$$4\pi R^2 S + 4\pi S^2 R = +4R\pi S^2 + (\pi R^2\text{in}^{-1})\text{Length}(T)$$

$$4S = (\text{in}^{-1})\text{Length}(T)$$

And we get $\text{Length}(T) = 4S\text{in}$, the desired result.

Exercise 6

Problem

Show that there is an infinite set of points S in $\mathbb{R}^k$ such that each $(k+1)$-element subset of S is affinely independent. Use this to prove that any finite simplicial complex of dimension n has a geometric realization in $\mathbb{R}^{2n+1}$.

Solution

Take an infinite subset of real numbers A (without a zero), for example $A = [1/4, 1/2]$ Define the set S as $S = \{v_x | x \in A\} := \{(1, x, x^2, \ldots, x^k) | x \in A\}$. Let's say we have picked $k+1$ distinct $v_{x_0}, \ldots v_{x_k}$ They are affinely independent iff $v_{x_k} - v_{x_0}, \ldots, v_{x_1} - v_{x_0}$. Are linearly independent. Which is true iff a $k \times k$ matrix V, with these vectors as rows is nonsingular. A typical vector $v_{x_i} - v_{x_0}$ looks like

Exercise 7

Problem

Prove that, up to homotopy, every finite simplicial complex can be realized as a clique complex $Cl(G)$ for some finite graph G.

Solution

Pick a finite simplicial complex K and assume it is an abstract simplicial complex (otherwise, pass to its abstract version as in the lectures). Therefore, our simplicial complex K consists of a vertex set X and a family of subsets $\mathcal{K}$.

Define a graph G as follows: The graph vertices are elements of X. For each $\sigma \in \mathcal{K}$, all the vertices in σ form a clique - that is, for each two elements $x, y \in \sigma$ there is an edge between them.

By construction, the clique complex $Cl(G)$ is exactly the abstract simplicial complex K and as two geometric realizations of a single abstract simplicial complex are homotopy equivalent, we have proved the claim.

Exercise 9

Problem

Let $A_k = \{1, 2, \ldots, k\}$ be a set of k points, viewed as a CW complex. Given a non-empty CW complex X:

- Determine $\tilde{H}_*(A_k * X)$ in terms of $\tilde{H}_*(X)$.
- Determine $\tilde{H}_*(S^1 * X)$ in terms of $\tilde{H}_*(X)$.

In both cases, $Y * X$ is the join of Y and X.

Solution

We will use Lemma 2.15 from https://people.math.wisc.edu/ lmaxim/vanishing.pdf

$$\tilde{H}_{r+1}(Y * X) \cong \bigoplus_{i+j=r} \left(\tilde{H}_i(Y) \otimes \tilde{H}_j(X)\right) \oplus \bigoplus_{i+j=r-1} \mathrm{Tor}(\tilde{H}_i(Y), \tilde{H}_j(X))$$

Let us first take $Y = A_k$. If $k = 1$, $A_1 * X = CX$ is the Cone of X, which is contractible, so we all the reduced homology groups of $A_1 * X$ are trivial.

Assume now $k > 1$. First of all, since it is a discrete space, the only nonzero homology group is $\tilde{H}_0(A_k) = \mathbb{Z}^{k-1}$. Additionally the number of components in $A_k * X$ is the same as in X so we immediately know that $\tilde{H}_0(A_k * X) = \tilde{H}_0(X)$. As the homology groups of A_k are torsion-free, the Tor term in the formula is zero. We are left with, for $r > 0$: $\tilde{H}_r(A_k * X) \cong \bigoplus_{i+j=r-1} \left(\tilde{H}_i(A_k) \otimes \tilde{H}_j(X)\right) \cong \mathbb{Z}^{k-1} \otimes \tilde{H}_{r-1}(X)$.

Now take $Y = S^1$. We know that the only nontrivial reduced homology group is now $\tilde{H}_1(S^1) = \mathbb{Z}$. We also know that $S^1 * X$ is connected, so we have $\tilde{H}_0(S^1 * X) = 0$. For higher homology groups we once again use the formula: $\tilde{H}_r(S^1 * X) \cong \mathbb{Z} \otimes \tilde{H}_{r-2}(X) \cong \tilde{H}_{r-2}(X)$.

Exercise 10

Problem

Prove that the hexasphere doesn't really exist. You can assume it is given as a regular CW complex structure on S^2, whose 2-cells are hexagons, and their closures interesect in at most one edge.

Solution

First, we know the homology groups of S^2. Those are $H_i(S^2) = \mathbb{Z}$ for $i = 0, 2$ and trivial otherwise. The "vertices"of the hexagon are our 0-cells. The edges of the hexagons are attached to the 0-skeleton in such a way that we get the desired hexagonal pattern. Finally, the hexagons (2-cells) are attached as well. The resulting body is our hexasphere.

Let's say we have h hexagons. Each of these hexagons has 6 edges, and each edge is shared by two so we have $3h$ edges. We also have $2h$ vertices.

Using cellular homology, we know that $H_i(X^i, X^{i-1}) \cong \mathbb{Z}(e_i^1, \ldots, e_i^k)$, the free group on set of generators that correspond to the i-cells. The chain complex we have is:

$$\ldots \to 0 \to \mathbb{Z}^h \xrightarrow{d_2} \mathbb{Z}^{3h} \xrightarrow{d_1} \mathbb{Z}^{2h} \xrightarrow{d_0} 0$$

Clearly, $\mathrm{rk}(\ker(d_0)) = 2h$. As we know that the H_0 homology group should have rank 1, this implies $\mathrm{rk}(\mathrm{im}(d_1)) = 2h - 1$. Rank-nullity theorem gives $\mathrm{rk}(\ker(d_1)) = h + 1$, which as the H_1 group is trivial should equal $\mathrm{rk}(\mathrm{im}(d_2)) = h + 1$, which is a contradiction.

Exercise 12

Problem

Suppose $f : A \to B$ and $g : B \to C$ are homomorphisms of abelian groups. Show that there is an exact sequence:

$$0 \to \ker f \to \ker gf \to \ker g \to \mathrm{coker} f \to \mathrm{coker} gf \to \mathrm{coker} g \to 0$$

Solution

First, let's add some notations for these maps:

$$0 \xrightarrow{f_1} \ker f \xrightarrow{f_2} \ker gf \xrightarrow{f_3} \ker g \xrightarrow{f_4} \mathrm{coker} f \xrightarrow{f_5} \mathrm{coker} gf \xrightarrow{f_6} \mathrm{coker} g \xrightarrow{f_7} 0$$

- f_1 is obviously just a zero map.
- For the sequence to be exact at $\ker f$, f_2 must then be a monomorphism. Since $\ker(f) \leq \ker(gf)$ we can take f_2 to be the inclusion $\ker(f) \hookrightarrow \ker(gf)$.
- Exactness at $\ker(gf)$ demands us to take f_3 such that $\ker(f_3) = \ker(f)$. The natural choice is $f_3 = f$, with the domain restricted to $\ker(gf)$.
- Exactness at $\ker(g)$ demands that $\ker(f_4) = \mathrm{im}(f_3)$. This will be satisfied if we take f_4 to be the quotient projection $\ker(g) \to \ker(g)/\mathrm{im}(f) \subset \mathrm{coker}(f)$.

- Now since $\mathrm{im}(f_4) = \ker(g)/\mathrm{im}(f)$, this will be exact at $\mathrm{coker}(gf)$ if we take f_5 to be the induced map of g on the quotient $B/\mathrm{im}(f)$, mapping $x + \mathrm{im}(f)$ to $g(x) + \mathrm{im}(gf)$.
- $\mathrm{im}(f_5) = \mathrm{im}(g)/\mathrm{im}(gf)$. We get $\ker(f_6) = \mathrm{im}(f_5)$ if we define $f_6(x + \mathrm{im}(gf)) = x + \mathrm{im}(g)$.
- f_6 is clearly surjective, so the zero map f_7 finishes the exact sequence.

Exercise 13

Problem

Suppose (E^r, d^r) is a spectral sequence that converges to $(H_n)_n$.

- If $E^2_{p,q} = 0$ for all $p \neq 0, 1$ show that there are short exact sequences.

$$0 \to E^2_{0,n} \to H_n \to E^2_{1,n-1} \to 0$$

- If $E^2_{p,q} = 0$ for all $q \neq 0, 1$ show that there is a long exact sequence

$$\cdots \to H_{n+1} \to E^2_{n+1,0} \to E^2_{n-1,1} \to H_n \to E^2_{n,0} \to E^2_{n-2,1} \to H_{n-1} \to \ldots$$

Solution

For the first case, the second page of a spectral sequence looks like this:

$$\begin{array}{cccccc}
\cdots & 0 & E^2_{0,3} & E^2_{1,3} & 0 & \cdots \\
\\
\cdots & 0 & E^2_{0,2} & E^2_{1,2} & 0 & \cdots \\
\\
\cdots & 0 & E^2_{0,1} & E^2_{1,1} & 0 & \cdots \\
\\
\cdots & 0 & E^2_{0,0} & E^2_{1,0} & 0 & \cdots
\end{array}$$

All the differential maps on the second page are zero maps: For example $d^2_{0,q}$ maps from $E^r_{0,q}$ to $E^2_{-2,q+1} = 0$. Similar conclusion can be made for $d^r_{1,q}$. Likewise, the map of which codomain is, for example, $E^r_{0,q}$ is also a zero map, as it is $d^2_{2,q-1} : E^2_{2,q-1} \to E^2_{0,q}$, and the domain is 0. From this it holds that $E^3_{p,q} \cong H_{p,q}(E^2_{p,q}) = E^2_{p,q}/0 \cong E^2_{p,q}$. A similar argument shows that also $E^3_{p,q} = E^4_{p,q}$ and, actually, the spectral sequence stabilises already at the second page: $E^\infty_{p,q} = E^2_{p,q}$. Since by assumption, the spectral sequence converges to H_n we have: $E^2_{p,q} \cong F_pH_n/F_{p-1}H_n$ for $n = p + q$, where the filtration is: $0 = F_{-1}H_n \subseteq \cdots \subseteq F_pH_n \subseteq \cdots \subseteq F_nH_n = H_n$. So we have: $E^2_{0,n} \cong F_0H_n/F_{-1}H_n \cong F_0H_n$ $E^2_{1,n-1} \cong F_1H_n/F_0H_n \cong F_1H_n/E^2_{0,n}$ and $0 \cong F_pH_n/F_{p-1}H_n$ for $p > 1$. This implies $H_n = F_nH_n \cong F_{n-1}H_n \cong \ldots.. \cong F_1H_n$. So we actually have $E^2_{1,n-1} \cong H_n/E^2_{0,n}$. We can thus build a short exact sequence:

$$0 \to E^2_{0,n} \to H_n \to E^2_{1,n-1} \to 0,$$

where the first arrow is the zero map, the second arrow is inclusion of $E^2_{0,n}$ into H_n, the third arrow the projection $H_n \to H_n/E^2_{0,n}$ and the final arrow the zero map. It is easy to see that this sequence is exact.

Now let's see what happens if we instead have $E^2_{p,q} = 0$ for $q \neq 0, 1$. In this case the second page looks like:

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 \\
E^2_{0,1} & E^2_{1,1} & E^2_{2,1} & E^2_{3,1} \\
E^2_{0,0} & E^2_{1,0} & E^2_{2,0} & E^2_{3,0} \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots
\end{array}$$

This time we have some non-zero differential maps. Namely the maps of the form $d^2_{p,0} : E^2_{p,0} \to E^2 p-2, 1$. This implies the third page of our spectral sequence looks like this:

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 \\
E^3_{0,1} & E^3_{1,1} & E^3_{2,1} & E^3_{3,1} \\
E^3_{0,0} & E^3_{1,0} & E^3_{2,0} & E^3_{3,0} \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots
\end{array}$$

The zeros from the 2nd page remain zeros, as the maps involved are the zero maps (similar argument to the first part of this exercise), while for $E^3_{p,0}$ we have: $E^3_{p,0} \cong H_{p,0}(E^2_{p,0}) = \ker(d^2_{p,0})/0 \cong \ker(d^2_{p,0})$ and $E^3_{p,1} \cong H_{p,1}(E^2_{p,1}) = E^2_{p,1}/\mathrm{im}(d^2_{p+2,0})$. The maps $d^3_{p,q} : E^3_{p,q} \to E^3_{p-3,q+2}$ are all zero, similarly as in the first part of the exercise. So the spectral sequence stabilises and we have $E^\infty_{p,q} = E^3_{p,q}$. Now similarly as in first part of the exercise we have: $E^3_{n,0} = E^\infty_{n,0} \cong F_n H_n / F_{n-1} H_n \cong H_n / F_{n-1} H_n$ and $E^3_{n-1,1} \cong F_{n-1} H_n / F_{n-2} H_n$ and $E^3_{n-k,k} = 0 \cong F_{n-k} H_n / F_{n-k-1} H_n$ for $k > 1$. Therefore $F_{n-2} H_n \cong F_{n-3} H_n \cong ... \cong F_{-1} H_n = 0$ and $E^3_{n-1,1} \cong F_{n-1} H_n$. So we have $E^3_{n,0} \cong H_n / E^3_{n-1,1}$ and from this we get, like in the first part of the exercise, the short exact sequence

$$0 \to E^3_{n-1,1} \to H_n \to E^3_{n,0} \to 0$$

Also, from $E^3_{p,0} \cong \ker(d^2_{p,0})$ and $E^3_{p,1} \cong E^2_{p,1}/\mathrm{im}(d^2_{p+2,0})$ we have the exact sequence

$$0 \to E^3_{p,0} \to E^2_{p,0} \to E^2_{p-2,1} \to E^3_{p-2,1} \to 0.$$

The first map is, as usual, the zero map. The second arrow is the inclusion of $\ker(d^2_{p,0}) \hookrightarrow E^2_{p,0}$. The third arrow is the differential map $d^2_{p,0}$, the fourth the quotient projection $E^2_{p-2,1} \to E^2_{p-2,1}/\mathrm{im}(d^2_{p,0})$. It is easy to check that this sequence really is exact.

Let us write down again, the long exact sequence we are trying to derive:

$$\cdots \to H_{n+1} \xrightarrow{f_1} E^2_{n+1,0} \xrightarrow{f_2} E^2_{n-1,1} \xrightarrow{f_3} H_n \xrightarrow{f_4} E^2_{n,0} \xrightarrow{f_5} E^2_{n-2,1} \xrightarrow{f_6} H_{n-1} \to \dots$$

We are going to construct the maps f_i by applying both of the exact sequences we have just written down.

- f_1 is the composition of the surjective map $H_{n+1} \to E^3_{n+1,0}$ and the injective map $E^3_{n+1,0} \to E^2_{n+1,0}$. It's image is the copy of $E^3_{n+1,0}$ inside $E^2_{n+1,0}$.
- f_2 is the map $E^2_{n+1,0} \to E^2_{n-1,1}$ from the second exact sequence (the derivative). Note that it's kernel is $E^3_{n+1,0}$, so the sequence is exact here. It's image is $\mathrm{im}(d^2_{n+1,0})$.
- f_3 is the composition of the surjective map $E^2_{n-1,1} \to E^3_{n-1,1}$ and the injective map $E^3_{n-1,1} \to H_n$. It's kernel is equal to the kernel of the first map, which by construction is $\mathrm{im}(d^2_{n+1,0})$. It's image is the copy of $E^3_{n-1,1}$ in H_n.
- f_5 is similar to f_1 by replacing $n+1$ with n. Its kernel is the kernel of the map $H_n \to E^3_{n,0} \cong H_n/E^3_{n-1,1}$, so the sequence is exact here as well.
- Maps f_6 (and other not shown) fall into one of the classes already described. The long sequence thus really is exact.

Exercise 15

Problem

Prove that the Euler characteristic of $\mathrm{Inj}(n)$ is equal to $\chi(\mathrm{Inj}(n)) = 1 + (-1)^{n-1} d_n$, where d_n is the number of derangements in S_n.

Solution

The plan is to compute the Euler characteristic from the Betti numbers (ranks of homology groups). In lecture notes, we have proved that the complex of injective words $X = \mathrm{Inj}(n)$ has the homotopy type: $X \simeq \bigvee_{d_n} S^{n-1}$. Now let's compute the homology groups. We know that $H_0(X) = \mathbb{Z}$ as X is connected. For the higher homology groups, we can use the fact that $H_i(A \vee B) = H_i(A) \oplus H_i(B)$, $i > 0$. Additionally, we know that $H_i(S^n) = \mathbb{Z}$ if $i = n$ and trivial for other $i > 0$. From this it holds that the only non-trivial homology groups are: $H_0(X) = \mathbb{Z}$ and $H_{n-1}(X) = \mathbb{Z}^{d_n}$. Now using the formula for the Euler characteristic: $\chi = \sum_{i=0}^{\infty} (-1)^i b_i = \sum_{i=0}^{\infty} (-1)^i \mathrm{rk}(H_i(X)) = 1 + (-1)^{n-1} d_n$, which shows that $\chi(\mathrm{Inj}(n)) = 1 + (-1)^{n-1} d_n$

Exercise 16

Problem

Given CW-complexes X and Y, suppose that $f : X \to Y$ is an n-equivalence. Prove that the induced homomorphism $f_* : H_q(X) \to H_q(Y)$ is an isomorphism for $q < n$ and an epimorphism for $q = n$.

Solution

This is the well known theorem, proved for example in Hatcher. Nvm ni v hatcherju lol luzer

Exercise 17

Problem

Determine the Vietoris-Rips persistence barcode (in all homological degrees) for the 9-cycle graph $G = C_9$, equipped with the graph metric.

Solution

Remember in the Vietoris-Rips complex at scale r, two vertices of G x, y are in the same simplex iff $d(x, y) \leq r$. It is easy to see that at the scale $r < 1$ there are no simplices except for each of the vertices:

$$X_0 = \{\{i\} | i = 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

On the other hand for $r \geq 5$ (graph diameter) we have all the vertices:

$$X_9 = \{\{i\} | i = 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

Other changes happen at integer values of r between 1 and 5. With some drawing we can see that we have the following:

This gives us a filtration:

Applying H_0 we get...

Exercise 18

Problem

Calculate the magnitude of the 3-cube graph $G = Q_3$ (with vertices given as $V(G) = \{0, 1\}^3$ and edges given by those pairs that differ in exactly one componenet). Using results from the provided paper, deduce the magnitude homology of G.

Solution

The cube graph $G = Q_3$ can be expressed as the cartesian product: $G = K_2 \square K_2 \square K_2$, where K_2 is the complete graph on two vertices (so, having two nodes $0, 1$ and an edge between them). 1.2.2. in the provided paper shows that we can express magnitude of G as $\#G = \#(K_2 \square K_2 \square K_2) = \#K_2 \cdot \#K_2 \cdot \#K_2$. To proceed, let us recall the general formula for the graph magnitude:

$$\#G = \sum_{l \geq 0} \left(\sum_{k \geq 0} (-1)^k \left| \{(x_0, \ldots, x_k) : x_i \in V(G), x_i \neq x_{i+1} \sum_{i=0}^{k-1} d(x_i, x_{i+1}) = l\} \right| \right) q^l$$

Let us calculate $\#K_2$. The constant term is 2, the coefficient of q is -2, coefficient of q^2 is again 2 and so on. We have: $\#K_2 = 2 - 2q + 2q^2 - 2q^3 + \cdots = 2(1 - q + q^2 - q^3 + \ldots) = \frac{2}{1+q}$, where we have used geometric series to write the sum in a closed form. It follows $\#G = (\frac{2}{1+q})^3 = 8(\frac{1}{1+q})^3 = 4 \sum_{l=2}^{\infty} (-1)^l (l-1) l q^{l-2}$, where we used $\frac{1}{(1-x)^3} = \sum_{n=2}^{\infty} \frac{(n-1)n}{2} x^{n-2}$. So, to repeat the main result, we found

$$\#G = 4 \sum_{l=2}^{\infty} (-1)^l (l-1) l q^{l-2}$$

The magnitude homology of G can be obtained in a similar manner. Example 5 in the paper shows that $MH_{k,l}(K_2) = 0$ for $k \neq l$ and $MH_{l,l}(K_2) = \mathbb{Z}^2, l \geq 0$. Now using Künneth theorem for magnitude homology we know there is a short exact sequence:

$$0 \to MH_{*,*}(G) \otimes MH_{*,*}(H) \to MH_{*,*}(G \square H) \to \operatorname{Tor}(MH_{*-1,*}(G), MH_{*,*}(H)) \to 0$$

Picking $G = H = K_2$ we have $\operatorname{Tor}(MH_{*-1,*}(G), MH_{*,*}(H)) = 0$, as the magnitude homologies of K_2 are torsion-free. So we are left with an isomorphism: $MH_{*,*}(K_2 \square K_2) \cong MH_{*,*}(K_2) \otimes MH_{*,*}(K_2)$. We get $MH_{k,l}(K_2 \square K_2)$ equals 0 if $k \neq l$ or $\mathbb{Z}^2 \otimes \mathbb{Z}^2 \cong \mathbb{Z}^4$ otherwise.

Since the resulting homology groups are still torsion-free we can use the same formula to finally compute: $MH_{k,l}(G) = 0$ if $k \neq l$ and $MH_{k,l}(G) = \mathbb{Z}^8$ otherwise.