3. kolokvij iz Matematike 2, FMF, Aplikativna fizika

18. 4. 2024

Čas pisanja je 120 minut. Veliko uspeha!

Sedež (2.01)

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Ime in priimek

Vpisna številka

1. naloga (25 točk)

Dana je funkcija $f(x) = x \sin x$.

a) Funkcijo f razvijte v Fourierovo vrsto na intervalu $[-\pi, \pi]$. Poleg a_0 posebej izračunajte še a_1, saj ni enake splošne oblike kot a_n za $n \geq 2$.

b) S pomočjo dobljenega Fourierovega razvoja določite

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1}.$$

Pomoč: $\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha+\beta) + \sin(\alpha-\beta))$, $\sin\alpha\sin\beta = -\frac{1}{2}(\cos(\alpha+\beta) - \cos(\alpha-\beta))$

a) f SODA $\Rightarrow b_n = 0 \quad \forall n \in \mathbb{N}$

0 INT. COSX PO PERIODI

$$\boxed{a_0} = \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{u}{x}\,\underbrace{\sin x\, dx}_{dv} = \frac{1}{\pi}\left(-x\cos x\Big|_{-\pi}^{\pi} + \cancel{\int_{-\pi}^{\pi}\cos x\, dx}^{\,0}\right) = \frac{1}{\pi}(\pi+\pi) = \boxed{2}$$

$$\boxed{a_1} = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos x\, dx \overset{\text{Pomoč}}{=} \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} x\sin(2x)\,dx + \cancel{\int_{-\pi}^{\pi} x\sin(0)\,dx}^{\,0}\right) =$$

$$= \frac{1}{2\pi}\left(-\frac{x\cos(2x)}{2}\Big|_{-\pi}^{\pi} + \cancel{\int_{-\pi}^{\pi}\frac{\cos(2x)}{2}}^{\,0}\right) = \frac{1}{2\pi}\left(-\frac{2\pi}{2}\right) = \boxed{-\frac{1}{2}}$$

$$\boxed{a_n} = \frac{1}{\pi}\left(\int_{-\pi}^{\pi} x\sin(x)\cos(nx)\,dx\right) = \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} x\left(\sin((n+1)x) + \sin((n-1)x)\right)dx\right) =$$

$$= \frac{1}{2\pi}\left(-\frac{x\cos((n+1)x)}{(n+1)}\Big|_{-\pi}^{\pi} + \cancel{\int_{-\pi}^{\pi}\frac{\cos((n+1)x)}{n+1}dx}^{\,0} + \frac{x\cos((n-1)x)}{n-1}\Big|_{-\pi}^{\pi} - \cancel{\int_{-\pi}^{\pi}\frac{\cos((n-1)x)}{n-1}dx}^{\,0}\right)$$

$$= \frac{1}{2\pi}\left((-1)^n\frac{2\pi}{n+1} + (-1)^{n+1}\frac{2\pi}{n-1}\right) = (-1)^n\left(\frac{1}{n+1} - \frac{1}{n-1}\right) = \boxed{(-1)^n\frac{-2}{n^2-1}}$$

$\cos((n+1)\pi) = (-1)^{n+1}$

$$\boxed{\tilde{f}(x) = 1 - \frac{1}{2}\cos(x) + \sum_{n\geq 2}\frac{2(-1)^{n+1}}{n^2-1}\cos(nx)}$$

b) $f(0) = 0 = \tilde{f}(0) = 1 - \frac{1}{2} - 2\sum_{n\geq 2}\frac{(-1)^n}{n^2-1} \Rightarrow \boxed{\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1} = \frac{1}{4}}$

UJEMANJE

$\tilde{f}$ IN f V 0

2. naloga (30 točk)

Naj bo K krivulja v $\mathbb{R}^3$, ki jo dobimo kot presek ploskev $S_1: x^2+y^2=1$ in $S_2: z=x^2+y^2-2y+2$.

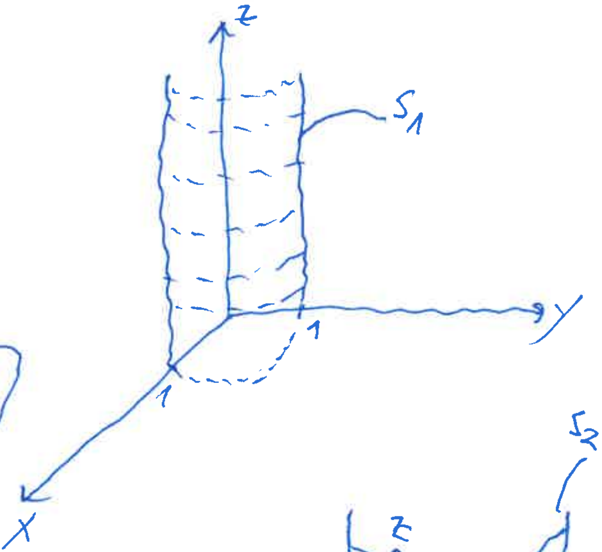
a) Skicirajte obe ploskvi in parametrizirajte krivuljo K.

b) Krivuljo K orientiramo tako, da je projekcija na ravnino $z=0$ orientirana pozitivno. Izračunajte krivuljni integral vektorskega polja $\vec{F}(x,y,z)=(z,y,-x)$ po krivulji K.

c) S premaknjenimi cilindričnimi koordinatami izračunajte površino ploskve S_2 pod ravnino $z=5$.

a) S_1 - VALJ

S_2 - $x^2+y^2-2y+2 = x^2+(y-1)^2+1$ PARABOLOID

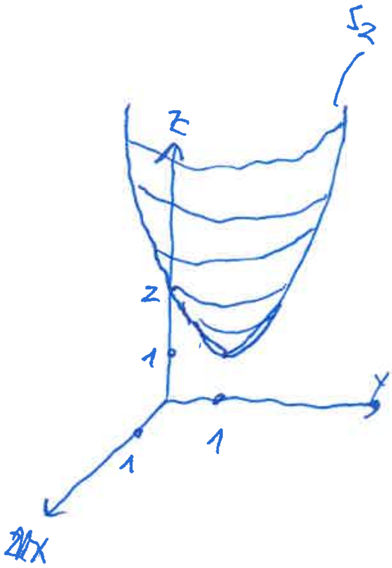


CILIND. KOORD.: $x = r\cos\varphi$, $y = r\sin\varphi$, $z = z$

$x^2+y^2=1 \Rightarrow \boxed{r=1}$

$z = x^2+y^2-2y+2 = 3-2\sin\varphi$

$$\boxed{\vec{r}(\varphi) = (\cos\varphi, \sin\varphi, 3-2\sin\varphi) \quad \varphi\in[0,2\pi)}$$



b) JE ŽE ORIENTIRANA V POZ. SMERI.

$\dot{\vec{r}}(\varphi) = (-\sin\varphi, \cos\varphi, -2\cos\varphi)$

$$\int_K \vec{F}\cdot d\vec{s} = \int_0^{2\pi} (3-2\sin\varphi, \sin\varphi, -\cos\varphi)\cdot(-\sin\varphi, \cos\varphi, -2\cos\varphi)\,d\varphi =$$

$$= \int_0^{2\pi} (\underbrace{-3\sin\varphi}_{0} + 2\sin^2\varphi + \underbrace{\sin\varphi\cos\varphi}_{0} + 2\cos^2\varphi)\,d\varphi = \boxed{4\pi}$$

(INT $\sin\varphi$ PO CELI PERIODI)

(INT $\sin(2\varphi)$ PO CELI PERIODI)

c) $x = r\cos\varphi$

$y = r\sin\varphi + 1$

$z = z = x^2+(y-1)^2+1 = r^2+1$

$\Rightarrow \vec{r}(r,\varphi) = (r\cos\varphi, r\sin\varphi+1, r^2+1)$ $\quad \varphi\in[0,2\pi)$

$z\in[1,5] \Rightarrow r\in[0,2]$

$\vec{r}_r = (\cos\varphi, \sin\varphi, 2r)$

$\vec{r}_\varphi = (-r\sin\varphi, r\cos\varphi, 0)$

$\vec{r}_r\times\vec{r}_\varphi = (-2r^2\cos\varphi, -2r^2\sin\varphi, r)$ $\quad \|\vec{r}_r\times\vec{r}_\varphi\| = \sqrt{4r^4+r^2} = r\sqrt{4r^2+1}$

$$S = \iint_{S_2} 1\,dS = \int_0^{2\pi} d\varphi \int_0^2 dr\, r\sqrt{\underbrace{4r^2+1}_{u}} = \frac{2\pi}{8}\int_1^{17}\sqrt{u}\,du = \frac{\pi}{4}\cdot\frac{u^{3/2}}{3/2}\Big|_1^{17} = \boxed{\frac{\pi}{6}\left(17\sqrt{17}-1\right)}$$

3. naloga (25 točk)

Dana je krivulja s parametrizacijo:

$$x(t) = t\cos^2(\pi t),$$
$$y(t) = t\sin^2(\pi t),$$
$$z(t) = t,$$

kjer je $t \in \mathbb{R}$.

a) Izračunajte spremljajoči trieder v točki $(1, 0, 1)$.

b) Izračunajte fleksijsko in torzijsko ukrivljenost v točki $(1, 0, 1)$.

c) Kakšna je torzijska ukrivljenost v poljubni točki na krivulji in kaj nam to pove o krivulji? Pri izračunu si pomagajte z dejstvom, da je dano parametrizacijo možno zapisati kot $\vec{r}(t) = (x(t), t - x(t), t)$.

a) $(1,0,1)$ USTREZA $t=1$ $\qquad \vec{r}(t) = (t\cos^2(\pi t), t\sin^2(t), t)$

$$\dot{\vec{r}} = (\cos^2(\pi t) - 2\pi t\cos(\pi t)\sin(\pi t),\ \sin^2(\pi t) + 2\pi t\sin(\pi t)\cos(\pi t),\ 1)$$

$$\dot{\vec{r}} = (\cos^2(\pi t) - \pi t\sin(2\pi t),\ \sin^2(\pi t) + \pi t\sin(2\pi t),\ 1)$$

$$\dot{\vec{r}}(1) = (1, 0, 1) \Rightarrow \boxed{\vec{T} = \frac{(1,0,1)}{\sqrt{2}}}$$

$$\ddot{\vec{r}}(t) = (-2\pi\cos(\pi t)\sin(\pi t) - \pi(\sin(2\pi t) + 2\pi t\cos(2\pi t)),\ 2\pi\sin(\pi t)\cos(\pi t) + \pi(\sin(2\pi t) + 2\pi t\cos(2\pi t)),\ 0)$$

$$\ddot{\vec{r}}(1) = (-2\pi^2, 2\pi^2, 0)$$

$$\dot{\vec{r}}(1)\times\ddot{\vec{r}}(1) = (-2\pi^2, -2\pi^2, 2\pi^2) = 2\pi^2(-1,-1,1)$$

$$\Rightarrow \boxed{\vec{B} = \frac{(-1,-1,1)}{\sqrt{3}}}$$

b) $$\boxed{\kappa} = \frac{\|\dot{\vec{r}}(1)\times\ddot{\vec{r}}(1)\|}{\|\dot{\vec{r}}(1)\|^3} = \frac{2\pi^2\sqrt{3}}{2\sqrt{2}} = \boxed{\pi^2\sqrt{\frac{3}{2}}}$$

$$\vec{N} = \vec{B}\times\vec{T} = \boxed{\frac{1}{\sqrt{6}}(-1,2,1)}$$

$$\ddot{\vec{r}}(t) = (-2\pi\sin(2\pi t) - 2\pi^2 t\cos(2\pi t),\ 2\pi\sin(2\pi t) + 2\pi^2 t\cos(2\pi t),\ 0)$$

$$\dddot{\vec{r}}(t) = (-4\pi^2\cos(2\pi t) - 2\pi^2(\cos(2\pi t) - 2\pi t\sin(2\pi t)),\ 4\pi^2\cos(2\pi t) + 2\pi^2(\cos(2\pi t) - 2\pi t\sin(2\pi t)),\ 0)$$

$$\dddot{\vec{r}}(1) = (-6\pi^2, 6\pi^2, 0) \Rightarrow \boxed{\omega} = \frac{[\dot{\vec{r}}, \ddot{\vec{r}}, \dddot{\vec{r}}]}{\|\dot{\vec{r}}\times\ddot{\vec{r}}\|^2} \overset{t=1}{=} \frac{12\pi^4\cdot 0}{\dots} = \boxed{0}$$

c) $$\vec{r}(t) = (x, t-x, t)$$
$$\dot{\vec{r}} = (\dot{x}, 1-\dot{x}, 1)$$
$$\ddot{\vec{r}} = (\ddot{x}, -\ddot{x}, 0)$$
$$\dddot{\vec{r}} = (\dddot{x}, -\dddot{x}, 0)$$

$$\dot{\vec{r}}\times\ddot{\vec{r}} = (\ddot{x}, \ddot{x}, -\dot{x}\ddot{x} + (\dot{x}-1)\ddot{x}) = (\ddot{x}, \ddot{x}, -\ddot{x})$$

$$[\dot{\vec{r}}, \ddot{\vec{r}}, \dddot{\vec{r}}] = (\ddot{x}, \ddot{x}, -\ddot{x})(\dddot{x}, -\dddot{x}, 0) = 0 \Rightarrow \omega = 0 \text{ ZA } \forall t \in \mathbb{R} \Rightarrow$$ KRIVULJA JE RAVNINSKA

4. naloga (25 točk)

$\partial x \quad \partial y \quad \partial z$

Dano je vektorsko polje $\vec{F}(x,y,z) = (Ax^2y^2z, 2x^3yz, x^3y^n)$.

a) Določi naravno število n in parameter $A \in \mathbb{R}$ tako, da bo polje $\vec{F}$ potencialno.

b) Izračunaj integral polja $\vec{F}$ vzdolž poljubne krivulje med točkama $(1,1,3)$ in $(1,2,-1)$. Odgovor utemelji!

c) Izračunaj pretok polja $\vec{F}$ skozi ploskev $P = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 < 1,\ x,y > 0 \text{ in } z = 0\}$, kjer je normala orientirana navzdol.

a) NA $\mathbb{R}^3$ $\vec{F}$ POTENCIALNO $\iff \nabla \times \vec{F} = \vec{0}$?

$$\nabla \times \vec{F} = (n y^{n-1}x^3 - 2x^3y,\ Ax^2y^2 - 3x^2y^n,\ 6x^2yz - 2Ax^2yz) = \vec{0}$$

$$\Rightarrow \boxed{n=2,\ A=3} \Rightarrow \vec{F} = (3x^2y^2z, 2x^3yz, x^3y^2)$$

b) $\vec{F} = \nabla u$

$u_x = 3x^2y^2z$

$u_y = 2x^3yz$

$u_z = x^3y^2 \Rightarrow u = x^3y^2z + C(x,y)$

$u_y = 2x^3yz + \frac{\partial C(x,y)}{\partial y} = 2x^3yz \Rightarrow C(x,y) = D(x)$

$u_x = 3x^2y^2z + D'(x) = 3x^2y^2z \Rightarrow D(x) = E$

$\Rightarrow \boxed{u = x^3y^2z + E}$

$$\int_{(1,1,3)}^{(1,2,-1)} \vec{F}\,d\vec{s} \overset{\text{POTENCIALNO}}{=} -4 - 3 = \boxed{-7}$$

c) CIL. KOORD.

$x = r\cos\varphi$, $y = r\sin\varphi$, $z = 0$, $r \in (0,1)$, $\varphi \in (0, \frac{\pi}{2})$

$\vec{r} = (r\cos\varphi, r\sin\varphi, 0)$

$\vec{r}_r = (\cos\varphi, \sin\varphi, 0)$

$\vec{r}_\varphi = (-r\sin\varphi, r\cos\varphi, 0)$

$\vec{r}_r \times \vec{r}_\varphi = (0, 0, r)$ ← NORMALA KAŽE GOR

SMER NORMALE

$$\iint_P \vec{F}\,d\vec{s} = -\int_0^1 dr \int_0^{\frac{\pi}{2}} d\varphi\ (0, 0, r^5\cos^3\varphi\sin^2\varphi)\cdot(0,0,r) =$$

$$= -\int_0^1 dr \int_0^{\frac{\pi}{2}} d\varphi\ r^6\cos^3\varphi\sin^2\varphi = -\frac{1}{7}\int_0^{\frac{\pi}{2}} \cos^3\varphi\sin^2\varphi\,d\varphi = -\frac{1}{7}\cdot\frac{1}{2}B\left(2,\frac{3}{2}\right) = -\frac{1}{14}\frac{\Gamma(2)\Gamma(\frac{3}{2})}{\Gamma(\frac{7}{2})}$$

LAHKO BI TUDI Z $u = \sin\varphi$

$$= -\frac{1}{14}\frac{1}{\frac{5}{2}\cdot\frac{3}{2}} = -\frac{2}{7\cdot 15} = \boxed{-\frac{2}{105}}$$