1	
2	
3	
4	
Σ	

Sedež (2.05)

Vpisna številka

2. kolokvij iz Analize 4, FMF, Finančna Matematika

15. 1. 2025

Čas pisanja je 120 minut. Veliko uspeha!

Ime in priimek

1. naloga

Dan je sistem enačb:

$$\dot{x} = 3x + y$$
$$\dot{y} = -x + y$$

a) Izračunaj matriko Wronskega.

b) Skiciraj fazni portret v bližini izhodišča.

c) Poišči splošno rešitev sistema:

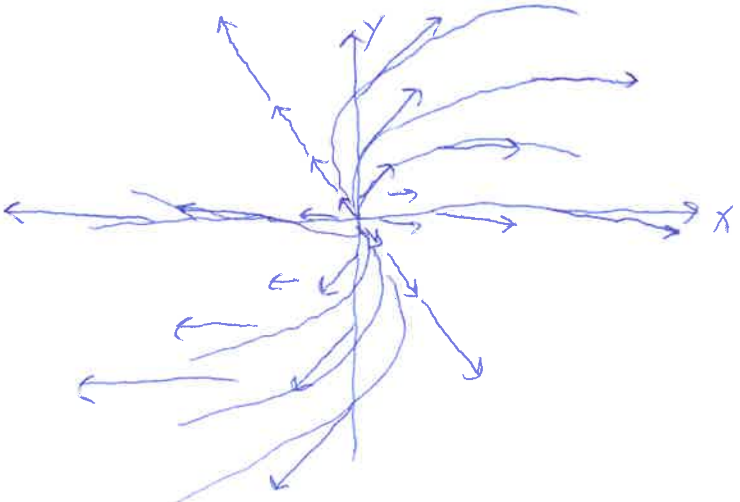
$$\dot{x} = 3x + y$$
$$\dot{y} = -(x - 1) + y$$

a) $A=\begin{bmatrix}3&1\\-1&1\end{bmatrix}$ $|A-\lambda I| = (3-\lambda)(1-\lambda)+1 = \lambda^2-4\lambda+4 = (\lambda-2)^2=0 \Rightarrow \lambda_{1,2}=2$, $A-2I=\begin{bmatrix}1&1\\-1&-1\end{bmatrix}$, $v_1=\begin{bmatrix}1\\-1\end{bmatrix}$

$\begin{bmatrix}1&1\\-1&-1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix}$ $v_2=\begin{bmatrix}1\\0\end{bmatrix}$

$P=\begin{bmatrix}1&1\\-1&0\end{bmatrix}$, $J=\begin{bmatrix}2&1\\0&2\end{bmatrix}$ $\boxed{W(t)}=Pe^{Jt}=\begin{bmatrix}1&1\\-1&0\end{bmatrix}\begin{bmatrix}1&t\\0&1\end{bmatrix}e^{2t}=\boxed{e^{2t}\begin{bmatrix}1&1+t\\-1&-t\end{bmatrix}}$

b)



$A\begin{bmatrix}\alpha\\0\end{bmatrix}=\begin{bmatrix}3\alpha\\-\alpha\end{bmatrix}$ $A\begin{bmatrix}0\\\alpha\end{bmatrix}=\begin{bmatrix}\alpha\\\alpha\end{bmatrix}$

$A\begin{bmatrix}\alpha\\\alpha\end{bmatrix}=\begin{bmatrix}4\alpha\\0\end{bmatrix}$

c) $\boxed{\vec{y}(t)}=W(t)\left(\vec{c}+\int W(t)^{-1}\vec{b}\,dt\right)$ $W^{-1}=\frac{1}{e^{4t}}\begin{bmatrix}-t&-1-t\\1&1\end{bmatrix}e^{2t}$, $\vec{b}=\begin{bmatrix}0\\1\end{bmatrix}$

$=W(t)\left(\vec{c}+\int e^{-2t}\begin{bmatrix}-t&-1-t\\1&1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}dt\right)=W(t)\left(\vec{c}+\int\begin{bmatrix}-e^{-2t}(t+1)\\e^{-2t}\end{bmatrix}dt\right)=W(t)\left(\begin{bmatrix}A\\B\end{bmatrix}+\begin{bmatrix}\frac{t}{2}e^{-2t}+\overbrace{\frac14 e^{-2t}+\frac12 e^{-2t}}^{\frac34 e^{-2t}}\\-\frac12 e^{-2t}\end{bmatrix}\right)$

$\int te^{-2t}dt=-\frac12 te^{-2t}+\frac12\int e^{-2t}dt=-\frac{t}{2}e^{-2t}-\frac14 e^{-2t}$

$=\begin{bmatrix}1&1+t\\-1&-t\end{bmatrix}e^{2t}\left(\begin{bmatrix}A+\frac{t}{2}e^{-2t}+\frac34 e^{-2t}\\B-\frac12 e^{-2t}\end{bmatrix}\right)=e^{2t}\begin{bmatrix}A+\frac{t}{2}e^{-2t}+\frac34 e^{-2t}+B(1+t)-(1+t)\frac12 e^{-2t}\\-A-\frac{t}{2}e^{-2t}-\frac34 e^{-2t}-Bt+\frac12 te^{-2t}\end{bmatrix}=$

$=\boxed{e^{2t}\begin{bmatrix}A+B(1+t)+\frac14 e^{-2t}\\-A-Bt-\frac34 e^{-2t}\end{bmatrix}}$

2. naloga

Dan je sistem

$$\begin{aligned}\dot{x} &= -x\\ \dot{y} &= z + x^2\\ \dot{z} &= y + x^2\end{aligned}$$

a) Poišči in klasificiraj stacionarne točke.

b) Reši sistem.

c) Za sedla določi stabilno in nestabilno mnogoterost.

a) $T(0,0,0)$ JE STAC. $\quad DF = \begin{bmatrix} -1 & 0 & 0 \\ 2x & 0 & 1 \\ 2x & 1 & 0 \end{bmatrix} \quad DF(0,0,0) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$(-1-\lambda)(\lambda^2-1) = 0 \Rightarrow \lambda_{1,2} = -1, \lambda_3 = 1$ SEDLO

b) $\boxed{x(t) = x_0 e^{-t}}$ $\quad u = y - z \Rightarrow \dot{u} = -u \Rightarrow u(t) = Ae^{-t} \quad A = y_0 - z_0$

$y = u + z \Rightarrow \quad \dot{z} = Ae^{-t} + z + x_0^2 e^{-2t}$

H: $z(t) = Be^{t}$

P: $z^P = c_1 e^{-t} + c_2 e^{-2t}$

$-c_1 e^{-t} - 2c_2 e^{-2t} = Ae^{-t} + c_1 e^{-t} + c_2 e^{-2t} + x_0^2 e^{-2t}$

$\Rightarrow c_1 = -\frac{A}{2}, \; c_2 = -\frac{x_0^2}{3}$

$\Rightarrow z(t) = Be^{t} - \frac{A}{2} e^{-t} - \frac{x_0^2}{3} e^{-2t}$

$z_0 = B - \frac{A}{2} - \frac{x_0^2}{3} \Rightarrow B = z_0 + \frac{y_0 - z_0}{2} + \frac{x_0^2}{3} = \frac{y_0 + z_0}{2} + \frac{x_0^2}{3}$

$$\Rightarrow \boxed{z(t) = \left(\frac{y_0 + z_0}{2} + \frac{x_0^2}{3}\right)e^{t} + \frac{z_0 - y_0}{2} e^{-t} - \frac{x_0^2}{3} e^{-2t}}$$

$$\boxed{y(t) = u + z = \left(\frac{y_0 + z_0}{2} + \frac{x_0^2}{3}\right)e^{t} + \frac{y_0 - z_0}{2} e^{-t} - \frac{x_0^2}{3} e^{-2t}}$$

c) $W^S = \{(x,y,z) \mid \lim_{t \to \infty} \varphi_t(x,y,z) = (0,0,0)\} = \{(x,y,z) \in \mathbb{R}^3 \mid \frac{y+z}{2} + \frac{x^2}{3} = 0\}$

$W^N = \{(x,y,z) \mid \lim_{t \to -\infty} \varphi_t(x,y,z) = (0,0,0)\} = \{(0,y,y) \mid y \in \mathbb{R}\}$

3. naloga

Poišči ekstremalo funkcionala:

$$F(y) = \int_0^1 y'^2 \mathrm{d}x,$$

ki zadošča $y(0) = y(1) = 0$ in $\int_0^1 xy\mathrm{d}x = 1$.

$$\tilde{F}(y) = \int_0^1 (y'^2 - \lambda xy)\,dx$$

$$\text{E-L:}\quad \frac{d}{dx}(2y') + \lambda x = 0 \Rightarrow 2y'' = -\lambda x \Rightarrow y'' = -\frac{\lambda x}{2}$$

$$y' = -\frac{\lambda x^2}{4} + A \qquad y = -\frac{\lambda x^3}{12} + Ax + B$$

$$y(0) = 0 \Rightarrow B = 0$$

$$y(1) = 0 \Rightarrow -\frac{\lambda}{12} + A = 0 \Rightarrow A = \frac{\lambda}{12}$$

$$\int_0^1 xy\,dx = \frac{\lambda}{12}\int_0^1 (-x^4 + x^2)\,dx = \frac{\lambda}{12}\left(\frac{1}{3} - \frac{1}{5}\right) = 1 \Rightarrow \frac{\lambda}{12}\left(\frac{2}{15}\right) = 1 \Rightarrow \lambda = 90$$

$$y(x) = \frac{90}{12}(x - x^3) = \boxed{\frac{15}{2}(x - x^3)}$$

4. naloga

Imamo enačbo

$$x^2u_x + xyu_y = u^2$$

in sledeči pogoj: $u = 1$ na krivulji, definirani z $x = y^2, y > 0$. $\Gamma(s) = (s^2, s, 1),\ s > 0$

a) Preveri, da obstaja enolična rešitev enačbe v okolici dane krivulje.

b) Reši enačbo (torej, poišči predpis $u(x, y)$).

a) • x^2, xy, u^2 GLADKE

• (s^2, s) BREZ SAMOPRESEČIŠČ

• TP: $\begin{vmatrix} 2s & s^4 \\ 1 & s^3 \end{vmatrix} = 2s^4 - s^4 = s^4 \neq 0 \Rightarrow \exists$ ENOLIČNA REŠITEV

b) $\dot{x} = x^2 \Rightarrow \frac{dx}{x^2} = dt \Rightarrow -\frac{1}{x} = t + A(s) \Rightarrow x = -\frac{1}{A+t},\ x(t=0) = s^2 \Rightarrow A = -\frac{1}{s^2}$

$\Rightarrow x = \frac{1}{\frac{1}{s^2} - t} = \boxed{\frac{s^2}{1-s^2t}}$

$\dot{y} = xy \Rightarrow \frac{dy}{y} = \frac{s^2}{1-s^2t}dt \Rightarrow \ln y = -\ln(1-s^2t) + \ln B(s) \Rightarrow y = \frac{B(s)}{1-s^2t} \Rightarrow B(s) = s$

$\Rightarrow \boxed{y = \frac{s}{1-s^2t}}$

$\dot{u} = u^2 \Rightarrow u = -\frac{1}{C+t} \Rightarrow C = -1 \Rightarrow \boxed{u = \frac{1}{1-t}}$

$s = \frac{x}{y}$ $\qquad y = \frac{s}{1-s^2t} \Rightarrow y(1-s^2t) = s \Rightarrow 1 - s^2t = \frac{s}{y} \Rightarrow t = \frac{1}{s^2} - \frac{1}{sy} =$

$= \frac{y^2}{x^2} - \frac{1}{x} = \frac{y^2 - x}{x^2}$

$\Rightarrow \boxed{u} = \frac{1}{1 - \frac{y^2 - x}{x^2}} = \boxed{\frac{x^2}{x^2 - y^2 + x}}$