Sedež (2.01)

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1. izpit iz Matematike 2, FMF, Aplikativna fizika

12. 6. 2024

Čas pisanja je 120 minut. Veliko uspeha!

Ime in priimek

Vpisna številka

1. naloga (25 točk)

Dana je funkcija $f(x, y, z) = xyz^2$.

a) Izračunajte vrednost smernega odvoda funkcije f v smeri $\vec{v} = (2, 0, 3)$ v točki $(1, 2, 3)$.

b) Poiščite ekstreme funkcije f na območju $x^2 + 4y^2 + z^2 = 16$.

a) $f_x = yz^2, \; f_y = xz^2, \; f_z = 2xyz \qquad \nabla f = (yz^2, xz^2, 2xyz) \qquad \nabla f(x=1, y=2, z=3) = (18, 9, 12)$

$\vec{n} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{(2,0,3)}{\sqrt{13}} \qquad \boxed{\frac{\partial f}{\partial \vec{n}}} = \vec{n} \cdot \nabla f = \frac{2 \cdot 18 + 3 \cdot 12}{\sqrt{13}} = \boxed{\frac{72}{\sqrt{13}}}$

b) $x^2 + 4y^2 + z^2 = 16 \Rightarrow z^2 = 16 - x^2 - 4y^2$

$g(x,y) = xy(16 - x^2 - 4y^2) = 16xy - x^3y - 4y^3x$

$g_x = 16y - 3x^2y - 4y^3 = y(16 - 3x^2 - 4y^2) = 0$

$g_y = 16x - x^3 - 12y^2x = x(16 - x^2 - 12y^2) = 0$

KANDIDATI: $x = 0 \rightarrow y = 0 \quad \boxed{T(0,0,\pm 4) \quad f(0,0,\pm 4) = 0}$

$\searrow 16 - 4y^2 = 0 \Rightarrow y = \pm 2, \; z^2 = 16 - 16 = 0 \quad \boxed{T(0, \pm 2, 0), f(0, \pm 2, 0) = 0}$

$x \neq 0, y \neq 0: \; 16 - 3x^2 - 4y^2 = 0 \; / \cdot 3 \Rightarrow 48 - 9x^2 - 12y^2 = 0$

$16 - x^2 - 12y^2 = 0 \longrightarrow 32 - 8x^2 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

$16 - 12 - 4y^2 = 0 \Rightarrow y = \pm 1, \; z^2 = 16 - 4 - 4 = 8 \Rightarrow z = \pm\sqrt{8}$

KANDIDATI: $T(2, 1, \pm\sqrt{8}), T(-2, -1, \pm\sqrt{8}) \quad f = 16 \leftarrow$ ŠTIRJE MAKSIMUMI

$T(2, -1, \pm\sqrt{8}), T(-2, 1, \pm\sqrt{8}) \quad f = -16 \leftarrow$ ŠTIRJE MINIMUMI

2. NAČIN: LAGRANGE $\quad L(x,y,z,\lambda) = f(x,y,z) - \lambda(16 - x^2 - 4y^2 - z^2)$

2. naloga (25 točk)

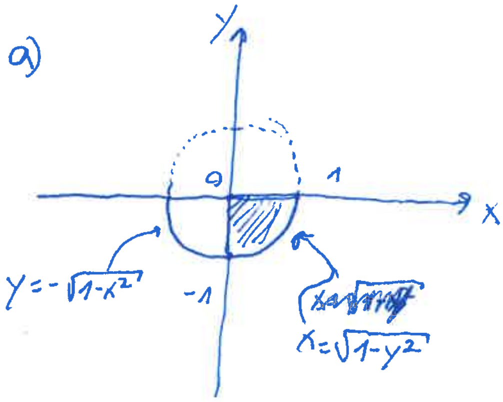
a) Za spodnji integral najprej skicirajte integracijsko območje, nato obrnite vrstni red integracije in integral izračunajte.

$$\int_0^1 \left(\int_{-\sqrt{1-x^2}}^0 2x\cos\left(y - \frac{y^3}{3}\right)\, dy \right) dx.$$

b) Izračunajte integral

$$\int_0^\infty x^5 e^{-x^4}\, dx.$$

a)



$$\int_0^1 dx \int_{-\sqrt{1-x^2}}^0 dy \left(2x\cos\left(y-\frac{y^3}{3}\right)\right) =$$

$$= 2\int_{-1}^0 dy \int_0^{\sqrt{1-y^2}} dx\; x\cos(y - y^3/3) =$$

$$= 2\int_{-1}^0 dy\, \frac{(1-y^2)}{2}\cos\left(y-\frac{y^3}{3}\right) = \qquad u = y - \frac{y^3}{3} \quad du = (1-y^2)\,dy$$

$$= \int_{-\frac{2}{3}}^0 \cos(u)\, du = \sin(u)\Big|_{-\frac{2}{3}}^0 = \boxed{\sin\left(\frac{2}{3}\right)}$$

b) $$\int_0^\infty x^5 e^{-x^4} dx = \frac{1}{4}\int_0^\infty x^2 e^{-u} du = \frac{1}{4}\int_0^\infty \sqrt{u}\, e^{-u} du = \frac{1}{4}\Gamma\left(\frac{3}{2}\right) = \frac{1}{8}\cdot\Gamma\left(\frac{1}{2}\right) = \boxed{\frac{\sqrt{\pi}}{8}}$$

$u = x^4$

$du = 4x^3 dx$

3. naloga (25 točk)

Dano je vektorsko polje $\vec{F}(x,y,z) = (-y, x, xyz)$.

a) Izračunajte $\vec{G} = \text{rot}\,(\vec{F})$.

b) Naj bo S del sfere $x^2+y^2+z^2=25$, ki leži pod ravnino $z=4$. Parametrizirajte ploskev S.

c) S pomočjo Stokesovega izreka izračunajte ploskovni integral

$$\int_S \vec{G}\,d\vec{S},$$

kjer je S orientirana tako, da je normalni vektor v točki $(0,0,-5)$ enak $(0,0,-1)$.

a) $\vec{G}$: $\begin{vmatrix} \partial x & \partial y & \partial z \\ -y & x & xyz \end{vmatrix} = (xz, -yz, 2)$ $\quad \boxed{\vec{G} = (xz, -yz, 2)}$

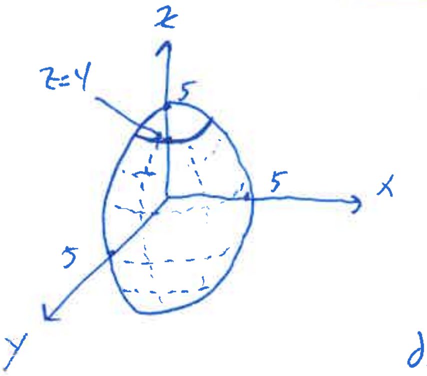
b) SFERIČNE KOORDINATE:

$x = r\cos\theta\cos\varphi$ $\quad x^2+y^2+z^2=25 \Rightarrow r=5 \quad \theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$

$y = r\cos\theta\sin\varphi$ $\quad z\le 4 \Rightarrow 5\sin\theta\le 4 \Rightarrow \theta \le \arcsin(\frac{4}{5})$

$z = r\sin\theta$

$$\boxed{\vec{r}(\varphi,\theta) = (5\cos\theta\cos\varphi,\ 5\cos\theta\sin\varphi,\ 5\sin\theta)} \quad \varphi\in[0,2\pi),\ \theta\in[-\tfrac{\pi}{2}, \arcsin(\tfrac{4}{5})]$$



$$\int_S \vec{G}\,d\vec{S} = \int_S \vec{\nabla}\times\vec{F}\,d\vec{S} \overset{\text{STOKES}}{=} -\int_{\partial S} \vec{F}\cdot d\vec{s}$$

GLEDE NA SPODNJO ORIENTACIJO RABIMO MINUS, DA BOSTA

∂S: $z=4$: $x^2+y^2+16=25 \Rightarrow x^2+y^2=9$, KROŽNICA, PARAMETRIZIRAMO

$\vec{r}(\varphi) = (3\cos\varphi, 3\sin\varphi, 4)$, $\varphi\in[0,2\pi)$

$\dot{\vec{r}}(\varphi) = (-3\sin\varphi, 3\cos\varphi, 0)$

ORIENTACIJI S IN ∂S SKLADNI.

$$\Rightarrow \int_S \vec{G}\,d\vec{S} = -\int_{\partial S}\vec{F}\cdot d\vec{s} = -\int_0^{2\pi} d\varphi\,(-3\sin\varphi, 3\cos\varphi, xyz)\cdot(-3\sin\varphi, 3\cos\varphi, 0) =$$

$$= -9\int_0^{2\pi} d\varphi\,\underbrace{(\sin^2\varphi+\cos^2\varphi)}_{1} = \boxed{-18\pi}$$

4. naloga (25 točk)

Poiščite splošno rešitev diferencialne enačbe

$$y'' + 4y' + 4y = \frac{e^{-2x}}{x^3}$$

HOMOGEN DEL: $y''+4y'+4y=0$

$$y=e^{\lambda x}: \quad \lambda^2+4\lambda+4=0$$
$$(\lambda+2)^2=0$$

$$\Rightarrow \boxed{y_H(x) = Ae^{-2x} + Bxe^{-2x}}$$

PARTIKULARNA REŠITEV:

$$y_P(x) = A(x)e^{-2x} + B(x)\cdot xe^{-2x}$$

$$W = \begin{bmatrix} e^{-2x}, & xe^{-2x} \\ -2e^{-2x}, & e^{-2x}(1-2x) \end{bmatrix} \qquad W\begin{bmatrix} A'(x) \\ B'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{e^{-2x}}{x^3} \end{bmatrix}$$

$$W^{-1} = \frac{1}{e^{-4x}(1-2x+2x)} \begin{bmatrix} e^{-2x}(1-2x) & -xe^{-2x} \\ 2e^{-2x} & e^{-2x} \end{bmatrix} = e^{2x}\begin{bmatrix} (1-2x) & -x \\ 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} A'(x) \\ B'(x) \end{bmatrix} = W^{-1}\begin{bmatrix} 0 \\ \frac{e^{-2x}}{x^3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{x^2} \\ \frac{1}{x^3} \end{bmatrix} \quad \Rightarrow \quad \begin{matrix} A'(x) = -\frac{1}{x^2} \Rightarrow A(x) = \frac{1}{x} \\ B'(x) = \frac{1}{x^3} \Rightarrow B(x) = -\frac{1}{2x^2} \end{matrix}$$

$$\Rightarrow \boxed{y_P} = \frac{e^{-2x}}{x} - \frac{1}{2x}e^{-2x} = \boxed{\frac{1}{2}\frac{e^{-2x}}{x}}$$

SPLOŠNA REŠITEV: $\boxed{y(x) = Ae^{-2x} + Bxe^{-2x} + \frac{e^{-2x}}{2x}}$