Some properties of RBF-FD differential operator approximations

Doctoral dissertation proposal

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Numerical treatment of PDEs - Finite Difference Method

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- Step 4 - Invert the system to obtain the values $u(x_i)$.

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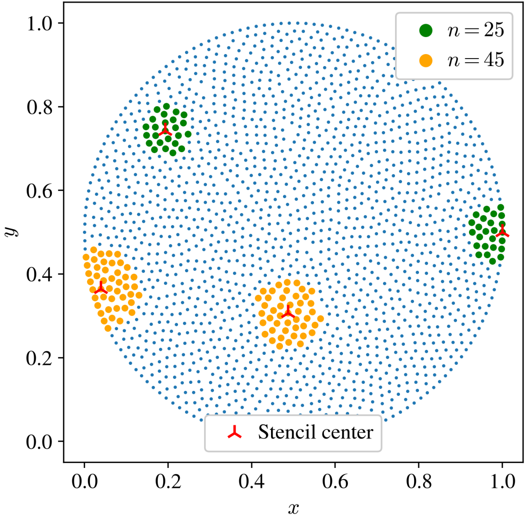
$$\begin{pmatrix} A & P \\ P^T & 0 \end{pmatrix} \begin{pmatrix} w \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathcal{L}(\phi) \\ \mathcal{L}(p) \end{pmatrix}$$

$$A = \begin{pmatrix} \phi(||x_1 - x_1||) & \dots & \phi(||x_1 - x_n||) \\ \vdots & \ddots & \vdots \\ \phi(||x_n - x_1||) & \dots & \phi(||x_n - x_n||) \end{pmatrix}$$

$$P = \begin{pmatrix} p_1(x_1) & \dots & p_M(x_1) \\ \vdots & \ddots & \vdots \\ p_1(x_n) & \dots & p_M(x_n) \end{pmatrix}$$

Example meshless discretisation



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- Polyharmonic Spline (PHS) RBF: $\phi(r) = r^3$ - no shape parameter, improved stability.
- Monomial augmentation of order m - Polynomial reproduction up to order m.

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- Smoothness of the RBF-FD approximation.

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- Simple problem (circular domain, $\nabla^2 u(x, y) = f(x, y)$, Dirichlet BC)
- Choose a function $u(x, y)$ and solve the corresponding PDE with RBF-FD
- Study the error behaviour and the approximation properties.

Oscillatory behaviour of RBF-FD error under increasing stencil size

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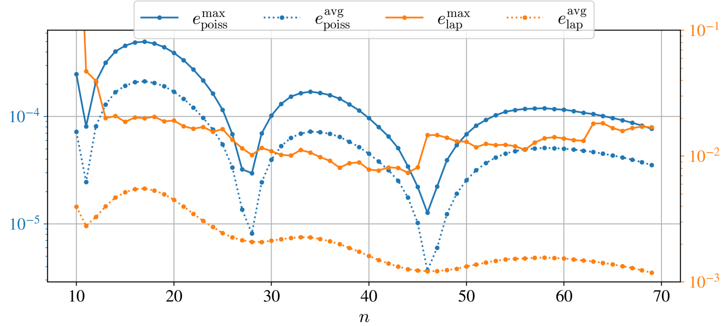
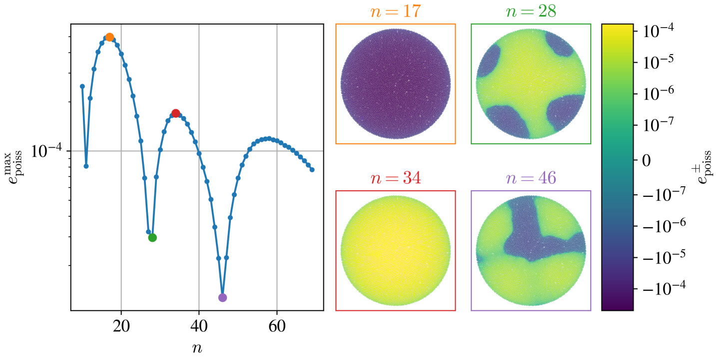
- $u(x, y) = \sin(\pi x)\sin(\pi y)$.
- Fix monomial augmentation $m = 3$, $h = 0.01$, $\phi(r) = r^3$, increase the stencil size and observe that the solution error oscillates.



Oscillatory behaviour of RBF-FD error under increasing stencil size

- Oscillatory error can be connected to the pointwise error behaviour.



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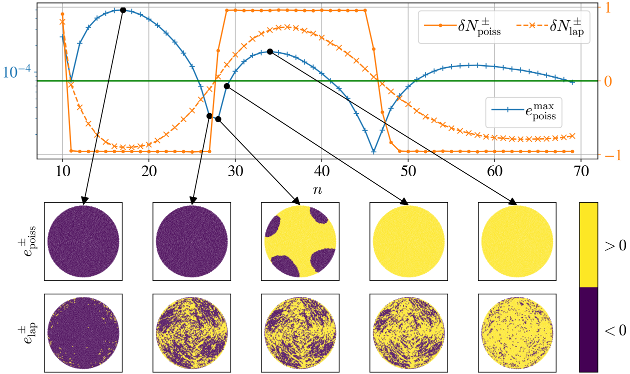
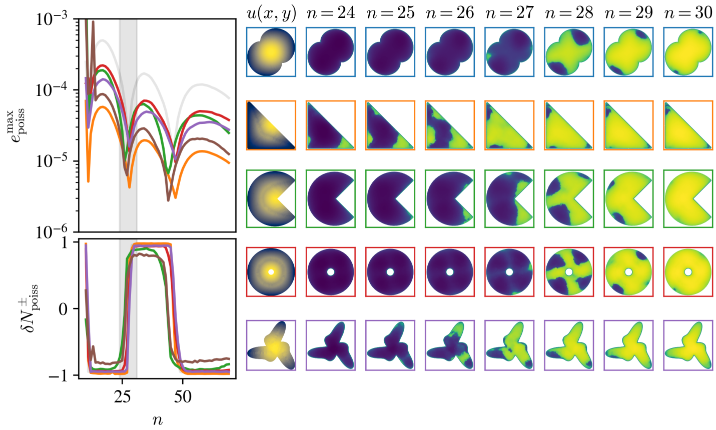
- Quantity $\delta N^{\pm}_{\text{poiss}}$ ("mean sign of pointwise error") - minima indicator?



Oscillatory behaviour of RBF-FD error under increasing stencil size

- The behaviour is robust under various different changes of the problem setup and method parameters.

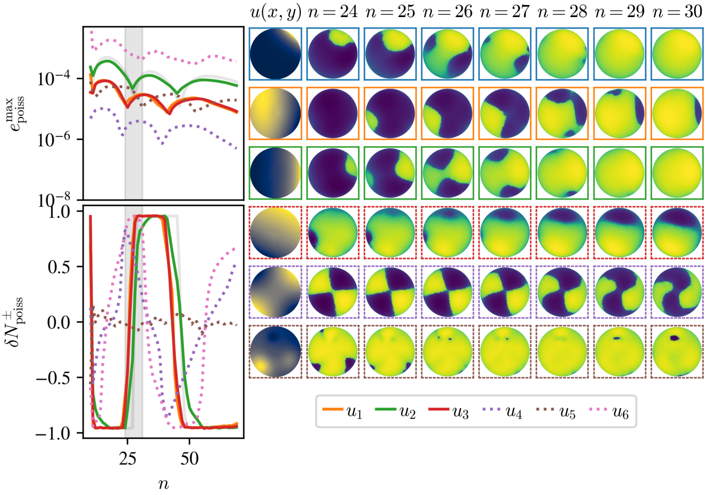


Oscillatory behaviour of RBF-FD error under increasing stencil size

- ... But there are exceptions.

Label	$u(x, y)$
u_1	$x^4 y^5$
u_2	$1 + \sin(4x) + \cos(3x) + \sin(2y)$
u_3	$\exp(x^2)$
u_4	$\mathrm{arsinh}(x + 2y)$
u_5	$\cos(\pi x)\cos(\pi y)$
u_6	$\mathrm{franke}(x, y)$

Oscillatory behaviour of RBF-FD error under increasing stencil size



Superconvergence of the RBF-FD method

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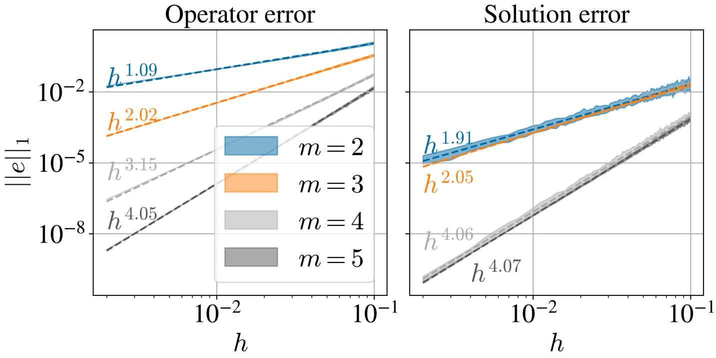

Superconvergence of the RBF-FD method

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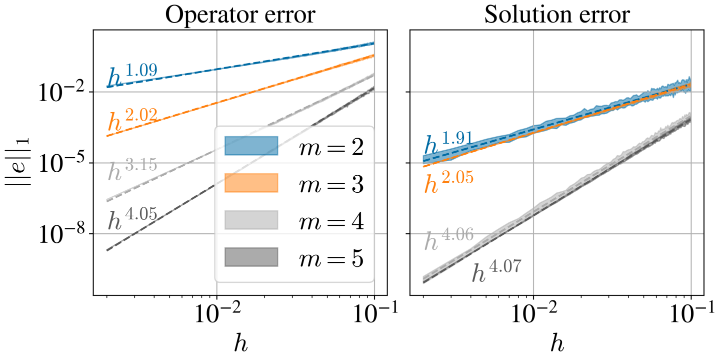
- Operator approximation order matches theory - h^{m-1} (h^{m+1-k}).
- Solution order is h^{m-1} or h^m



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- The form of e_{op} is actually known[1]:
- $\nabla^2 \hat{u}(x_i, y_i) - \nabla^2 u(x_i, y_i) = \sum_{k=s+1}^{\infty} L_k[u(x_i, y_i)](\mathbf{p}_k^T \cdot \mathbf{w})$.

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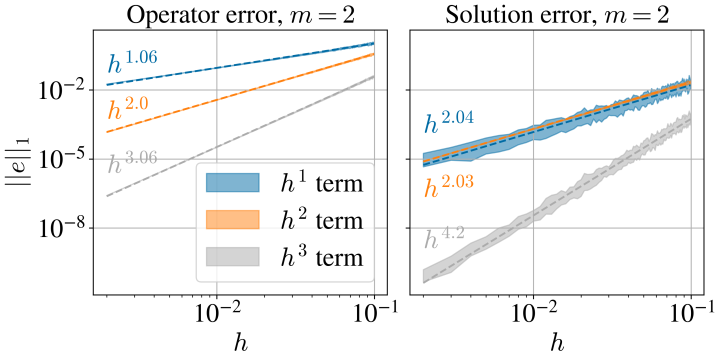
- Study the error behaviour term by term in powers of h.



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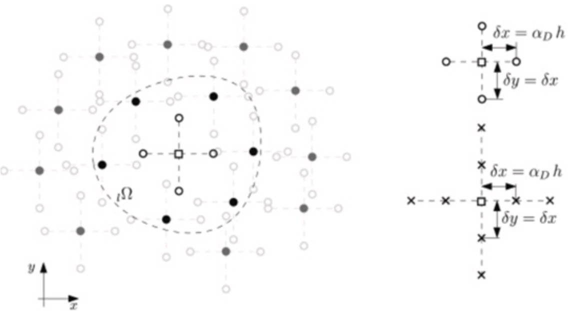

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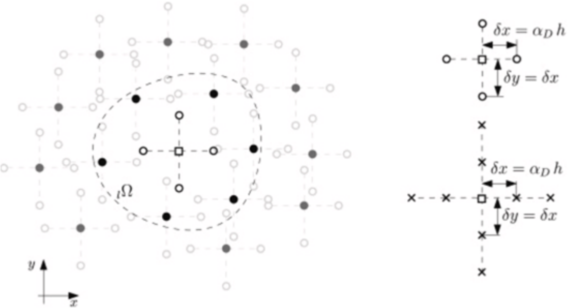
- New parameters - shape and spacing of the "virtual" FDM stencil.



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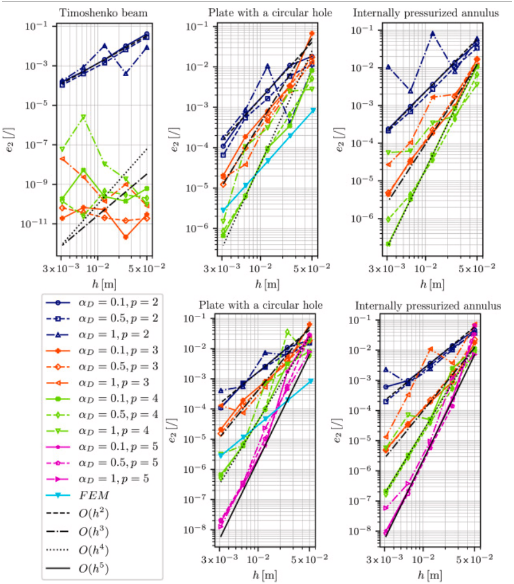
RBF-FDM applications



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- Sometimes a grid may also be desired, i.e. Yee's algorithm.

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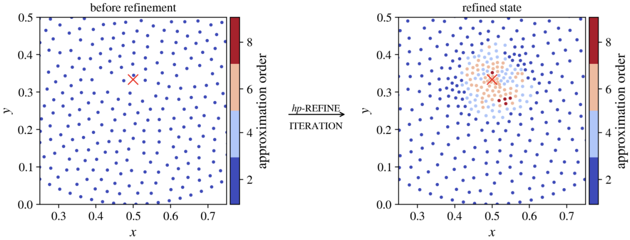
- Taylor series remainder - $\propto f^{(p+1)}(\xi)h^{p+1}$
- For a badly resolved function, increasing the method order is undesirable!
- Main question - when should the order be increased?

Smoothness of the approximation - Application and Motivation

- hp refinement - iterative spatial modification of nodal density (h-) and method order (p-)[3].



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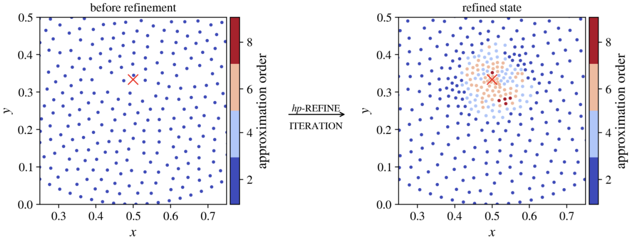

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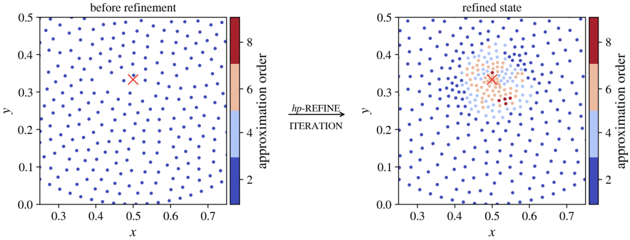
- Connection with RBFs.



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- We aim to analyse the two observed behaviours - oscillating error and order elevation.
- We plan to systematically research the RBF FDM.
- We wish to incorporate solution smoothness information in the solution procedure.