$\varsigma: G \rightsquigarrow \Omega_1$ TRANSITEVLY $|\Omega_1|=|\Omega_2|=n$

$\sigma: G \rightsquigarrow \Omega_2$ TRANSITEVLY DEFINE $G \rightsquigarrow \Omega_1 \times \Omega_2$ BY $(w_1,w_2)^g = (w_1^{\overset{\varsigma}{g}}, w_2^{\overset{\sigma}{g}})$

SHOW THAT ς, σ ARE EQUIVALENT $\iff$ $G \rightsquigarrow \Omega_1 \times \Omega_2$ HAS AN ORBIT OF SIZE n

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($\Rightarrow$) ς, σ ARE EQ. SO $\exists\, \varphi: \Omega_1 \to \Omega_2$ BIJECTION AND $\varphi(\omega_1^g) = \varphi(\omega_1)^g$ $\forall\, \omega_1 \in \Omega_1$

ALSO $|\omega_1 G| = |\Omega_1| = n$ SINCE TRANSITIVE. PICK $w_1 \in \Omega_1$, DEFINE $w_2 = \varphi(w_1)$, $|w_2^G| = |\Omega_2| = n$

THEN $(w_1, w_2)^G = \{(w_1^g, w_2^g) \mid g \in G\} = \{(w_1^g, \varphi(w_1)^g) \mid g \in G\}$ IS AN ORBIT OF SIZE n □

($\Leftarrow$) WE NEED TO FIND A BIJECTION S.T. $\varphi(w_1)^g = \varphi(w^g)$ $\forall\, w, g$

$\varphi: \Omega_1 \to \Omega_2$

~~DEFINE~~ LIST THE ORBIT ELEMENTS AS $(w_{11}, w_{21})^G = \{(w_{11}, w_{21}), \underset{(w_{11},w_{21})^{g_2}}{(w_{12}, w_{22})}, \dots, \underset{(w_{11},w_{21})^{g_n}}{(w_{1n}, w_{2n})}\}$

DEFINE φ: $\varphi(w_{1i}) = w_{2i}$ ~~THIS IS INJECTIVE: ASSUME $w_{1i} \neq w_{1j}$~~ ~~$w_{21}^{g_i} \neq w_{21}^{g_j}$~~

THIS IS WELL-DEFINED:

$\forall\, w \in \Omega_1$ BY TRANSITIVITY EQUALS $w_{11}^{g_i}$ FOR SOME g_i.

IT IMEDIATELLY FOLLOWS THAT ALL w_{1i} ARE DISTINCT (THERE ARE $|\Omega_1|$ OF THEM)

$\Rightarrow \varphi$ INJECTIVE $\Rightarrow \varphi$ BIJECTIVE

SETS ARE FINITE

$\varphi(w)^g = \varphi(w^g)$

WLOG: $w = w_{1i} = w_{11}^{g_i}$ $\varphi(w_{11}^{g_i g}) = w_{2i}^{g} = w_{21}^{g_i g}$, HOLDS BY DEFINITION □

LET $A \in \left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right] \in PGL(2,5)$

a) SHOW $A \in PSL(2,5)$

b) FIND A SYLOW $\underset{\overset{\shortparallel}{P}}{\text{PODGROUP}}$ OF $\underset{\overset{\shortparallel}{G}}{PSL(2,5)}$ THAT CONTAINS A

c) SHOW THAT $N_G(P) \neq P$

e) PROVE $PSL(2,5) \cong A_5$

a) $A = \left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right] = \left[\begin{pmatrix}2 & 0\\ 0 & -2\end{pmatrix}\right]$ HAS DETERMINANT $-4 = 1$ ✓

b) $|PGL(2,5)| = \dfrac{(5^2-1)(5^2-5)}{4} = \dfrac{24 \cdot 20}{4} = 120$, $|PSL(2,5)| = \dfrac{120}{2} = 60 = 5 \cdot 2^2 \cdot 3$

$(1^2 = 1, 4^2 = 1)$

A IS IN A SYLOW 2-SUBGROUP P, $|P| = 4$

$P = \left\{ \left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right], \left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right], \left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right], \left[\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right] \right\}$ ROUTINE TO CHECK P IS A GROUP.

c) $\left[\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}\right] \in PSL(2,5)$ $\left[\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}\right]^{-1} = \left[\begin{pmatrix}3 & 0\\ 0 & 2\end{pmatrix}\right]$

$$\left[\begin{pmatrix}3 & 0\\ 0 & 2\end{pmatrix}\right]\left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right]\left[\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}\right] = \left[\begin{pmatrix}3 & 0\\ 0 & -2\end{pmatrix}\right]\left[\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}\right] = \left[\begin{pmatrix}6 & 0\\ 0 & -6\end{pmatrix}\right] = \left[\begin{pmatrix}1 & 0\\ 0 & 4\end{pmatrix}\right] \checkmark$$

$$\left[\begin{pmatrix}3 & 0\\ 0 & 2\end{pmatrix}\right]\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right]\left[\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}\right] = \left[\begin{pmatrix}0 & 3\\ 2 & 0\end{pmatrix}\right]\left[\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}\right] = \left[\begin{pmatrix}0 & 4\\ 4 & 0\end{pmatrix}\right] = \left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right] \checkmark$$

...FIND AN ELEMENT $\notin P$, BUT $\in N_G(P)$ $\left[\begin{pmatrix}2 & 2\\ 1 & 4\end{pmatrix}\right]^{-1} = \left[\begin{pmatrix}4 & 3\\ 4 & 2\end{pmatrix}\right]$ $\left[\begin{pmatrix}4 & 3\\ 4 & 2\end{pmatrix}\right]\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right]\left[\begin{pmatrix}2 & 2\\ 1 & 4\end{pmatrix}\right] =$

$= \left[\begin{pmatrix}3 & 4\\ 2 & 4\end{pmatrix}\right]\left[\begin{pmatrix}2 & 2\\ 1 & 4\end{pmatrix}\right] = \left[\begin{pmatrix}0 & 2\\ 3 & 0\end{pmatrix}\right] \notin P$ ✓

d) FIND THE NUMBER OF SYLOW 2-GROUPS IN G

$n_2 = 1 \bmod 2$ $n_2 \mid 15$ $n_2 = 1, 3, 5, 15$

$n_2 = 1$ NOT POSSIBLE ($PSL(2,5)$ IS SIMPLE)

$n_2 = 3$ NOT POSSIBLE $G \to S_3$ FAITHFUL (G SIMPLE) BUT $|G| \nmid 3! = 6$

$n_2 = 15$ NOT POSSIBLE NUMBER OF SYL SUBGROUPS IS $[G : N_G(P)] < 15$

$\boxed{n_2 = 5}$

e) G ACTS ON $Syl_2(5)$ SO $G \to Sym(5)$ IS MONO;

BUT $|G| = 60$ SO ALSO THE IMAGE HAS SIZE 60. $\Rightarrow$ IMAGE IS A_5, $G \to A_5$ ISO. $\square$

$G \curvearrowright \Omega$, $|\Omega| = n$ SHARPLY 2-TRANSITIVE.

a) SHOW G HAS $n(n-2)$ ELEMENTS THAT FIX EXACTLY ONE POINT AND $n-1$ ELEMENTS THAT FIX NO POINTS.

b) SHOW THAT THE CENTRALISER OF ONE OF $n-1$ POINTS CONTAINS ONLY ~~ONE~~ OF ~~AMONG~~ $n-1$ POINTS AND ITS ~~THE~~ CONJUGACY CLASS AT LEAST $n-1$ POINTS.

c) SHOW THAT ID ~~& POINTS~~ AND ELEMENTS THAT FIX NO POINTS FORM A NORMAL SUBGROUP

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a) BY SHARP 2-TRANS: $G_{\omega_1,\omega_2} = 1$ $\forall \omega_1,\omega_2 \in \Omega$ AND $|G| = n\cdot(n-1)$ (ORBIT-STABILIZER)

$\omega \in \Omega$, $|G| = |G_\omega|\,|\omega^G| \Rightarrow |G_\omega| = \frac{|G|}{\underbrace{|\omega^G|}_{\text{TRANS.}}} = \frac{n(n-1)}{n} = n-1$. EACH G_ω FIXES ONE POINT AND HAS

$n-2$ NON-IDENTITY ELEMENTS. $\omega_1 \neq \omega_2 \Rightarrow G_{\omega_1} \cap G_{\omega_2} = 1$ (ELSE THE INTERSECTION WOULD FIX 2 ELEMENTS)

SO WE GET $\underbrace{n}_{|\Omega|} \cdot \underbrace{(n-2)}_{|G_\omega \setminus \{id\}|}$ ELEMENTS. THE REMAINING $\underbrace{n(n-1)}_{|G|} - n(n-2) - \underbrace{1}_{id} = n-1$ FIX NO POINTS ✓

b) x FIXES NO POINTS. TAKE $y \in C_G(x)$, $y \neq id$: ASSUME y FIXES ω: $\omega y = \omega$ THEN y FIXES $\omega^x \neq \omega$:

$\omega^{xy} = \omega^{yx} = \omega^x$

y FIXES TWO POINTS $\Rightarrow y = id$ ↯

SO $C_G(x)$ HAS ONLY $n-1$ ~~POINTS~~ ELEMENTS THAT FIX NO POINTS + id (OR MAYBE LESS) $\Rightarrow |C_G(x)| \leq n$

~~KNOWING~~ CONJ. CLASS SIZE $= [G : C_G(x)] = \frac{|G|}{|C_G(x)|} \geq \frac{n(n-1)}{n} = n-1$

c) BY CLASS FORMULA: $|G| = \sum_{g \in G} [G : C_G(g)] = \underbrace{1}_{\text{IDENTITY}} + \underbrace{n(n-2)}_{\text{STABILISERS ARE CONJUGATE}} + \sum_{\substack{x \in G \\ \text{FIXES} \\ \text{NO POINTS}}} [G : C_G(x)]$

$$n(n-1) = 1 + n(n-2) + \sum_{\substack{x \in G \\ \text{NO FIX}}} [G : C_G(x)] \Rightarrow 0 = 1 - n + \sum_{\substack{x \in G \\ \text{NO FIX}}} \underbrace{[G : C_G(x)]}_{\geq\, n-1}$$

THE ONLY WAY THIS HOLDS IS IF $[G : C_G(x)] = n-1$ AND $\Rightarrow |C_G(x)| = n$ SUBGROUP THAT WE ARE LOOKING FOR. NORMAL BECAUSE ALL ELEMENTS ARE IN A SAME CONJUGACY CLASS.

4) CONSIDER W_{12}, A $S(5,6,12)$ STEINER SYSTEM WITH $\Omega = AG_2(3) \cup \{\alpha,\beta,\gamma\}$ AND BLOCKS GIVEN BY $\mathcal{F} = \bigcup_i \mathcal{F}_i$, WHERE

$$\mathcal{F}_1 = \{ \{\alpha,\beta,\gamma\} \cup L \ ; \ L \text{ LINE OF } AG_2(3)\}$$
$$\mathcal{F}_2 = \{ \{\alpha,\gamma\} \cup Q \ ; \ Q \in D_1\}$$
$$\mathcal{F}_3 = \{ \{\beta,\gamma\} \cup Q \ ; \ Q \in D_2\}$$
$$\mathcal{F}_4 = \{ Q \cup \{\delta\} \cup \{\gamma\} \ ; \ Q \in D_3,\ \delta \text{ DIAG. PT. OF } Q\}$$
$$\mathcal{F}_5 = \{ \{\alpha,\beta\} \cup Q \ ; \ Q \in D_3\}$$
$$\mathcal{F}_6 = \{ (Q \cup \{\delta\}) \cup \{\beta\} \ ; \ Q \in D_1,\ \delta \text{ DIAG. PT.}\}$$
$$\mathcal{F}_7 = \{ (Q \cup \{\delta\}) \cup \{\alpha\} \ ; \ Q \in D_2,\ -\text{''}- \}$$
$$\mathcal{F}_8 = \{ L \cup L' \ ; \ L, L' \text{ PARALLEL LINES OF } AG_2(3)\}$$

WHERE D_1, D_2, D_3 ARE ONE OF THE FAMILIES OF QUADRANGLES IN $AG_2(3)$ S.T. EVERY TRIANGLE IS CONTAINED IN A UNIQUE QUADRANGLE FROM D_i.

SHOW THAT W_{12} CANNOT BE EXTENDED TO A $S(6,7,13)$ STEINER SYSTEM

LET'S CALCULATE THE STEINER SYSTEM PARAMETERS $\lambda_i = \dfrac{\binom{N-i}{t-i}}{\binom{k-i}{t-i}}$

$$\cancel{b = \lambda_0 = \frac{\binom{13}{6}}{\binom{8}{6}} = \frac{13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} = 13\cdot 11\cdot 2\cdot 3\cdot 2 = 13\cdot 11\cdot 3\cdot 2^2}$$

$$\cancel{bk = Nr \Rightarrow r = \frac{bk}{N} = \frac{13\cdot 11\cdot 3\cdot 2^2\cdot 7}{13} = 11\cdot 3\cdot 2^2\cdot 7}$$

$$\cancel{\lambda_2 = \frac{\binom{11}{4}}{\binom{5}{4}} = \frac{11\cdot 10\cdot 9\cdot 8}{5\cdot 4\cdot 3\cdot 2}}$$

$$b = \lambda_0 = \frac{\binom{13}{6}}{\binom{7}{6}} = \frac{13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8}{7!} = \frac{13\cdot \overset{2}{\cancel{12}}\cdot 11\cdot \cancel{10}\cdot \overset{3}{\cancel{9}}\cdot \overset{2}{\cancel{8}}}{7\cdot \cancel{6}\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}} = \frac{13\cdot 11\cdot 3\cdot 2^2}{7}$$ NON INTEGER

$\Rightarrow S(6,7,13)$ CANNOT EXIST $\square$.