1. IZRAČUNAJ LIMITE

a) $\lim\limits_{(x,y)\to(5,1)} \frac{xy}{x+y} = \frac{5}{6}$ ZA ZVEZNO FUNKCIJO KAR VSTAVIMO x,y.

b) $\lim\limits_{(x,y)\to(1,1)} \frac{2x^2-xy-y^2}{x^2-y^2} = \lim\limits_{(x,y)\to(1,1)} \frac{(2x+y)(x-y)}{(x+y)(x-y)} = \lim\limits_{(x,y)\to(1,1)} \frac{2x+y}{x+y} = \frac{3}{2}$

2. ZA NASLEDNJE $f(x,y)$ IZRAČUNAJ $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)$, $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$

a) $f(x,y) = \frac{x^2-y^2+2x^3+3y^3}{x^2+y^2}$

$\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y) = \lim\limits_{x\to 0} \frac{x^2+2x^3}{x^2} = 1$, $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y) = \lim\limits_{y\to 0} \frac{-y^2+3y^3}{y^2} = -1$

$\lim\limits_{(x,y)\to(0,0)} f(x,y)$ NE OBSTAJA (ČE BI OBSTAJALA, BI BILA VREDNOST NEODVISNA OD POTI DO (0,0))

b) $f(x,y) = \frac{x^2 y}{(x^2+y)^2}$

$\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y) = 0$ $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y) = 0$

$\lim\limits_{(x,y)\to(0,0)} f(x,y) = ?$ NIČ SE NE POKRAJŠA, POLARNE KOORDINATE NE DELUJEJO... DAJMO PROBAT POKAZAT, DA NE OBSTAJA, S TEM DA NAJDEMO POT DO (0,0) KI DA DRUGAČEN REZULTAT.

(0,0) SE PRIBLIŽUJEMO PO POTI $y=x$: $\lim\limits_{x\to 0} f(x,x) = \lim\limits_{x\to 0} \frac{x^3}{(x^2+x)^2} = 0$

$y=x^2$:

$\lim\limits_{x\to 0} f(x,x^2) = \lim\limits_{x\to 0} \frac{x^4}{4x^4} = \frac{1}{4} \Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y)$ NE OBSTAJA

1. KOLOKVIJ 2020 1. NALOGA

$$f(x,y) = 1 + e^{\sqrt{1-x^2-y^2}}$$

a) DOLOČI $D_f$, $Z_f$

$D_f$: $1-x^2-y^2 \geq 0 \Rightarrow x^2+y^2 \leq 1$

$D_f = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$

$Z_f$: V $D_f$ $\sqrt{1-x^2-y^2}$ ZAVZAME VREDNOSTI IZ $[0,1]$

$\sqrt{1-x^2-y^2} = 0 \Rightarrow f(x,y) = 2$

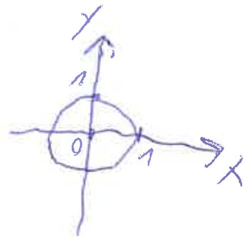
$\sqrt{1-x^2-y^2} = 1 \Rightarrow f(x,y) = 1+e$

$Z_f = [2, 1+e]$

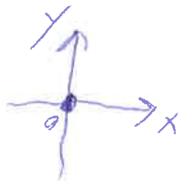
b) NARIŠI NIVOJNICE $f$ ZA VREDNOSTI $2, 1+e, 1+\sqrt{e}$

NIVOJNICA: $f(x,y) = c$

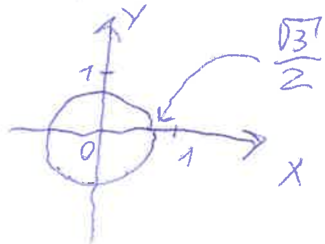
$c=2$: $2 = f(x,y) = 1 + e^{\sqrt{1-x^2-y^2}} \Rightarrow \sqrt{1-x^2-y^2} = 0 \Rightarrow x^2+y^2 = 1$




$c=1+e$: $1+e = f(x,y) \Rightarrow \sqrt{1-x^2-y^2} = 1 \Rightarrow (x,y) = (0,0)$

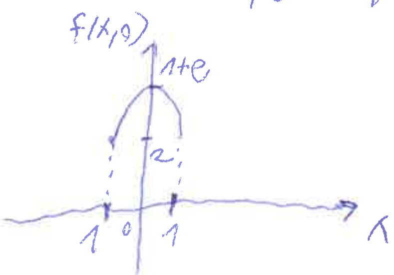


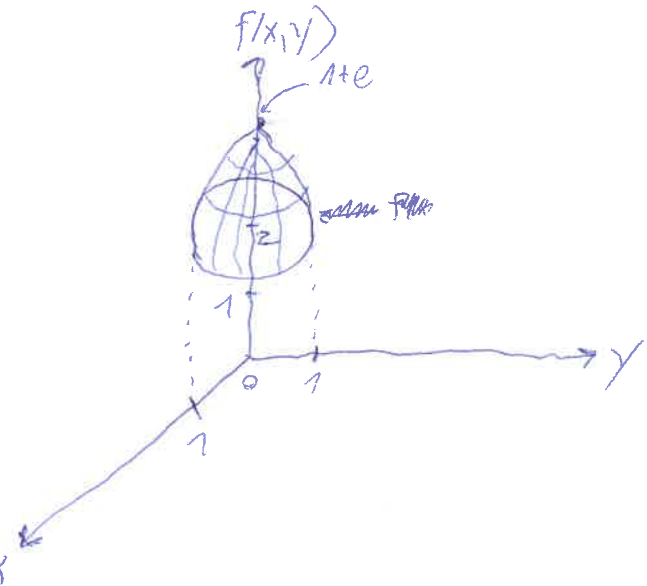

$c = 1+\sqrt{e}$: $1+\sqrt{e} = f(x,y) \Rightarrow \sqrt{1-x^2-y^2} = \frac{1}{2} \Rightarrow x^2+y^2 = \frac{3}{4}$




c) SKICIRAJ GRAF $f$

PREREZ $f(x,0) = 1 + e^{\sqrt{1-x^2}}$



d) $z = f(x,y)$, $x = s$, $y = st^2$. IZRAČUNAJTE $\frac{\partial z}{\partial t}$, $\frac{\partial z}{\partial s}$ ZA $t = 1$, $s = \frac{1}{2}$

$$z = f(x(s,t), y(s,t)) \qquad \frac{\partial z}{\partial t} = \underbrace{\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}}_{0} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} =$$

$$= e^{\sqrt{1-x^2-y^2}} \cdot \frac{(-y)}{\sqrt{1-x^2-y^2}} \cdot 2st = e^{\sqrt{1/2}} \cdot \frac{(-1/2)}{\sqrt{1/2}} \cdot 1 =$$

$t = 1, s = \frac{1}{2} \Rightarrow x = \frac{1}{2}, y = \frac{1}{2}$

$$= -\frac{\sqrt{2}}{2} e^{\sqrt{1/2}} = \boxed{-\frac{\sqrt{2}}{2} e^{\sqrt{2}/2}}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = -\frac{e^{\sqrt{1-x^2-y^2}}}{\sqrt{1-x^2-y^2}}\left(x \cdot 1 + y \cdot t^2\right) = -e^{\sqrt{2}/2}\sqrt{2}\left(\frac{1}{2} + \frac{1}{2}\right)$$

$$= \boxed{-\sqrt{2}\, e^{\sqrt{2}/2}}$$

3. NALOGA $f(x,y) = y^{14} e^x + \sqrt{x} \sin(\sqrt{x}\, y)$

a) DOLOČI $D_f$ $\qquad D_f = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0\}$

b) RAZVIJ $f(x,y)$ V TAYLORJEVO VRSTO OKOLI $(x,y) = (0,0)$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad \sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$\Rightarrow f(x,y) = y^{14} \sum_{k=0}^{\infty} \left[\frac{x^k}{k!}\right] + \sqrt{x} \sum_{k=0}^{\infty} \left[(-1)^k \frac{(\sqrt{x}\, y)^{2k+1}}{(2k+1)!}\right] =$$

$$= \sum_{k=0}^{\infty} \left( \frac{x^k y^{14}}{k!} + (-1)^k \frac{x^{k+1} y^{2k+1}}{(2k+1)!} \right)$$

c) IZRAČUNAJTE $\frac{\partial^{14} f}{\partial x^5 \partial y^9}(0,0)$ IN $\frac{\partial^{14} f}{\partial x^7 \partial y^7}(0,0)$

$$\frac{\partial^{14} f}{\partial x^5 \partial y^9}(0,0) = 5! \cdot 9! \cdot (\text{KOEF. PRED } x^5 y^9) = 5! \cdot 9! \cdot \frac{1}{9!} = 5!$$

$$\frac{\partial^{14} f}{\partial x^7 \partial y^7}(0,0) = 0 \text{, KER ČLENA } x^7 y^7 \text{ NI V RAZVOJU}$$

d) S POMOČJO LINEARNEGA PRIBLIŽKA OCENI VREDNOST $f(-0.5, 0.1)$

LINEARNIH ČLENOV V RAZVOJU NI, ZATO JE REZULTAT 0.

1\. KOLOKVIJ 2019 1. NALOGA

$$f(x,y) = \sqrt{x}\,\sqrt[5]{1+y^2}$$

a) RAZVIJ f V TAYLORJEVO VRSTO OKOLI (1,0).

$$f(x,y) = (1+x-1)^{1/2}\cdot(1+y^2)^{1/5} = \left[\sum_{k=0}^{\infty}\binom{1/2}{k}(x-1)^k\right]\cdot\left[\sum_{k=0}^{\infty}\binom{1/5}{k}y^{2k}\right]$$

b) S POMOČJO TAYLORJEVEGA POLINOMA DRUGE STOPNJE PRIBLIŽNO IZRAČUNAJ f(1.01, −0.5)

$$f(x,y) = \left(1+\frac{1}{2}(x-1)+\left(-\frac{1}{8}\right)(x-1)^2 + \text{VIŠJI ČLENI}\right)\cdot\left(1+\frac{1}{5}y^2+\text{VIŠJI ČL.}\right)$$

$$\binom{1/2}{2} = \frac{1/2\cdot(1/2-1)}{2!} = -\frac{1}{8}$$

$$= 1+\frac{y^2}{5}+\frac{x-1}{2}-\frac{(x-1)^2}{8} + \text{VIŠJI ČLENI} =$$

$$= 1+\underbrace{\frac{25\cdot 10^{-2}}{5}}_{0.05}+\underbrace{\frac{0.01}{2}}_{0.005}-\underbrace{\frac{10^{-4}}{8}}_{0.0000125} + \text{VIŠJI ČLENI} =$$

$$= 1.055 - 0.0000125 + \text{VIŠJI} = \boxed{= 1.0548875 + \text{VIŠJI ČL.}}$$

c) IZRAČUNAJ $\dfrac{\partial^{2020} f}{\partial x^{2019}\partial y}(1,0)$ IN $\dfrac{\partial^{2020} F}{\partial x^{2018}\partial y^2}(1,0)$

$$\frac{\partial^{2020} f}{\partial x^{2019}\partial y}(1,0) = 0 \quad \text{KER ČLENA } (x-1)^{2019}y \text{ NI V RAZVOJU}$$

$$\frac{\partial^{2020} f}{\partial x^{2018}\partial y^2}(1,0) = 2018!\cdot 2!\cdot\binom{1/2}{2018}\cdot\frac{1}{5} = \boxed{\frac{2}{5}\cdot\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)\cdot\ldots\left(\frac{1}{2}-2017\right)}$$

③ DOLOČI STACIONARNE TOČKE IN LOKALNE EKSTREME

$f(x,y,z) = (3x^2+2y^2+z^2-2xy-2yz)e^{-x}$

STAC. TOČKE $\nabla f = 0$

$f_x = (6x-2y)e^{-x} - f(x,y,z) = 0 \rightarrow 4xe^{-x} - 2x^2e^{-x} = 0 \Rightarrow 2x(2-x) = 0$

$f_y = (4y-2x-2z)e^{-x} = 0 \rightarrow x=y=z$

$f_z = (2z-2y)e^{-x} = 0 \Rightarrow x=y$

$x=0$, $x=2$

$\boxed{\text{STAC. TOČKI } (0,0,0),\ (2,2,2)}$

$f_{xx} = 6e^{-x} - (6x-2y)e^{-x} - f_x(x,y,z)$, $f_{xy} = -2e^{-x} - f_y(x,y,z)$

$f_{yy} = 4e^{-x}$

$f_{xz} = -f_z(x,y,z)$

$f_{zz} = 2e^{-x}$

$f_{yz} = -2e^{-x}$

$$H(0,0,0) = \begin{bmatrix} 6 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

$D_1 = 6 > 0$

$D_2 = 24-4 > 0$

$D_3 = 6\begin{vmatrix} 4 & -2 \\ -2 & 2 \end{vmatrix} + 2\begin{vmatrix} -2 & -2 \\ 0 & 2 \end{vmatrix} = 6\cdot(8-4) - 2\cdot 4 = 24-8 > 0$

$\Rightarrow (0,0,0)$ LOKALNI MINIMUM

$$H(2,2,2) = \begin{bmatrix} \overbrace{e^{-2}(6-8)}^{-2e^{-2}} & -2e^{-2} & 0 \\ -2e^{-2} & 4e^{-2} & -2e^{-2} \\ 0 & -2e^{-2} & 2e^{-2} \end{bmatrix}$$

$D_1 = -2e^{-2} < 0$

$D_2 = -8e^{-4} - 4e^{-4} < 0$

$D_3 = -2e^{-2}\begin{vmatrix} 4e^{-2} & -2e^{-2} \\ -2e^{-2} & 2e^{-2} \end{vmatrix} + 2e^{-2}\begin{vmatrix} -2e^{-2} & -2e^{-2} \\ 0 & 2e^{-2} \end{vmatrix} =$

$= -2e^{-2}(8e^{-4} - 4e^{-4}) + 2e^{-2}(-4e^{-4}) =$

$= e^{-6}(-8-8) < 0 \Rightarrow$ SEDLO $(2,2,2)$

# INTEGRALI S PARAMETROM

1) IZRAČUNAJ $\int_0^\infty \frac{\sin(x)}{x}\,dx$

NAMIG: • OPAZUJ $F(a) = \int_0^\infty e^{-ax}\frac{\sin(x)}{x}\,dx$

• PRIVZAMI, DA LAHKO ODVAJAŠ SKOZI INTEGRAL IN, DA JE $F(a)$ ZVEZNA ZA $a \geq 0$

PO NAMIGU: $\int_0^\infty \frac{\sin(x)}{x}\,dx = F(0)$

ZANIMIVOST: TAKŠNO UMETNO DODAJANJE PARAMETROV ZA LAŽJI IZRAČUN INTEGRALA JE ZNANO KOT „FEYNMAN'S TRICK"

IZRAČUNAJMO $F'(a) = -\int_0^\infty e^{-ax}\sin(x)\,dx$ ZA $a > 0$

TAKE INTEGRALE SE RAČUNA Z DVAKRATNIM PER PARTESOM

$$F'(a) = -\int_0^\infty e^{-ax}\sin(x)\,dx = e^{-ax}\cos(x)\Big|_0^\infty + a\int_0^\infty e^{-ax}\cos(x)\,dx = -1 + a\left[e^{-ax}\sin(x)\Big|_0^\infty + a\int_0^\infty e^{-ax}\sin(x)\,dx\right]$$

$u = e^{-ax}$ $\quad du = -a\,u\,dx$

$dv = \sin(x)dx$ $\quad v = -\cos(x)$

$u = e^{-ax}$ $\quad du = -a\,u\,dx$

$dv = \cos(x)$ $\quad v = \sin(x)$

$$= -1 + a^2\int_0^\infty e^{-ax}\sin(x)\,dx = -1 - a^2F'(a)$$

$$\Rightarrow F'(a) = -1 - a^2F'(a) \qquad F'(a) = -\frac{1}{1+a^2}$$

$$F(a) = \int\frac{-da}{1+a^2} + C = -\arctan(a) + C$$

$$F(\infty) = 0 = -\frac{\pi}{2} + C \Rightarrow C = \frac{\pi}{2} \qquad \Rightarrow F(a) = -\arctan(a) + \frac{\pi}{2}\,, \quad a > 0$$

KER $F(a)$ ZVEZNA ZA $a \geq 0$, JE $\boxed{\int_0^\infty \frac{\sin(x)}{x}\,dx} = \cancel{\lim_{a\to 0}F(a)} = F(0) =$

$$= \lim_{a\to 0}F(a) = \boxed{\frac{\pi}{2}}$$

ALTERNATIVNA POT ZA $-\int_0^\infty e^{-ax}\sin(x)dx$

$\sin(x)=\mathrm{Im}(e^{ix})$, $\boxed{-\int_0^\infty e^{-ax}\sin(x)dx} = -\mathrm{Im}\left(\int_0^\infty e^{-ax+ix}dx\right) = -\mathrm{Im}\left(\int_0^\infty e^{(i-a)x}dx\right) =$

REALNOST $e^{-ax}$
+
LINEARNOST INTEGRALA

$= -\mathrm{Im}\left(\frac{1}{i-a}\cdot(-1)\right) = \mathrm{Im}\left(\frac{1}{i-a}\right) = \mathrm{Im}\left(\frac{i+a}{-1-a^2}\right) = \boxed{-\frac{1}{1+a^2}}$ ISTO!

2) $F(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\cdot e^{-\frac{x^2}{2\sigma^2}}$ $(\sigma>0)$

ZANIMIVOST: f(x) JE DOBRO ZNANA GOSTOTA GAUSSOVE PORAZDELITVE

IZRAČUNAJ $\int_{-\infty}^{\infty} f(x)dx$

$\boxed{\int_{-\infty}^{\infty} f(x)dx} = 2\cdot\int_0^\infty f(x)dx = \sqrt{\frac{2}{\pi}}\cdot\frac{1}{\sigma}\int_0^\infty e^{-\frac{x^2}{2\sigma^2}}dx = \sqrt{\frac{2}{\pi}}\,\sigma\cdot\int_0^{+\infty} x^{-1}e^{-u}du =$

f(x) SODA FUNKCIJA

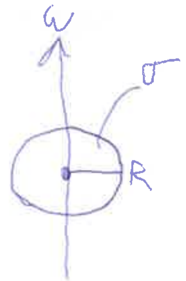
$u = +\frac{x^2}{2\sigma^2} \Rightarrow x = \sqrt{2}\,\sigma\sqrt{u}$

$du = +x\frac{1}{\sigma^2}dx$

$= \sqrt{\frac{2}{\pi}}\ \cancel{\ldots} = \sqrt{\frac{2}{\pi}}\cdot\sigma\,\frac{1}{\sqrt{2}\cdot\sigma}\int_0^\infty u^{-\frac{1}{2}}e^{-u}du = \frac{\Gamma(\frac{1}{2})}{\sqrt{\pi}} = \boxed{1}$

# VEČKRATNI INTEGRALI

1) IZRAČUNAJ VZTRAJNOSTNI MOMENT HOMOGENEGA DISKA RADIJA R




$dm = \sigma\, dS$

$$J = \int r^2 dm = \sigma \iint_D r^2 dS = \sigma \int_0^{2\pi} d\varphi \int_0^R r^3 dr =$$

$$= 2\pi\sigma \frac{R^4}{4} = \frac{1}{2}\sigma(\pi R^2)R^2 = \boxed{\frac{m}{2}R^2}$$

2) IZRAČUNAJ VZTR. MOMENT HOMOGENE KROGLE

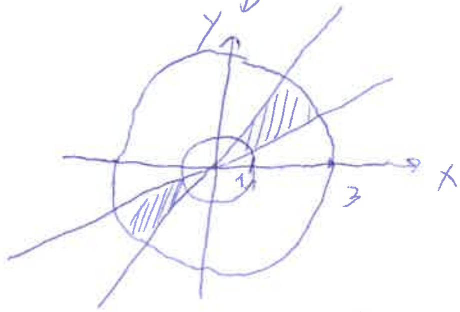
$$J = \rho \iiint_K r_\perp^2 dV = \rho \int_{-\pi/2}^{\pi/2} d\theta \int_0^{2\pi} d\varphi \int_0^R r^2\cos^2\theta \cdot r^2 \cos\theta\, dr =$$

$$= 2\pi\rho \int_{-\pi/2}^{\pi/2} \cos^3\theta\, d\theta \frac{R^5}{5} = \frac{2\pi\rho R^5}{5} \int_{-\pi/2}^{\pi/2} (1-\sin^2\theta)\cos\theta\, d\theta = \frac{2}{5}\pi\rho R^5 \int_{-1}^{1} (1-u^2)\,du =$$

$u = \sin\theta$

$$= \frac{2}{5}\pi\rho R^5\left(2 - \frac{2}{3}\right) = \frac{2}{5}\rho R^2 \cdot \frac{4}{3}\pi R^3 = \boxed{\frac{2}{5}mR^2}$$

3) IZRAČUNAJ $S = \iint_D \arctan\left(\frac{y}{x}\right) dx\,dy$, KJER D OMEJUJEJO $x^2+y^2=1$, $x^2+y^2=9$, $y = \frac{\sqrt{3}}{3}x$ IN $y = \sqrt{3}\,x$




$r$ IZ 1 DO 3

$\varphi$ IZ $\frac{\pi}{6}$ DO $\frac{\pi}{3}$

$tg\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$

$tg\left(\frac{\pi}{3}\right) = \sqrt{3}$

$\frac{y}{x} = tg(\varphi)$

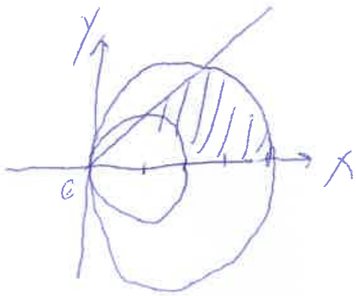
$$S = 2 \int_{\pi/6}^{\pi/3} d\varphi \int_1^3 r\,dr \cdot \varphi = 2 \cdot \int_{\pi/6}^{\pi/3} \varphi\, d\varphi \cdot \frac{1}{2}(9-1) =$$

DVE OBMOČJI ISTE PLOŠČINE

$$= \frac{8}{2}\left(\frac{\pi^2}{9} - \frac{\pi^2}{36}\right) = 4 \cdot \frac{3\pi^2}{36} = \boxed{\frac{\pi^2}{3}}$$

4) IZRAČUNAJ PLOŠČINO LIKA, OMEJENEGA Z $x^2+y^2=2x$, $x^2+y^2=4x$, $y=x$, $y=0$

$(x-1)^2+y^2=1 \qquad (x-2)^2+y^2=4$




$\varphi$ OD 0 DO $\frac{\pi}{4}$

$r$ OD $r_1$ DO $r_2$

$x^2+y^2=2x \Rightarrow r^2 = 2r\cos\varphi$

$r_1 = 2\cos\varphi$

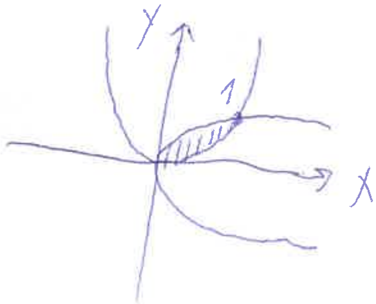
$x^2+y^2=4x \Rightarrow r_2 = 4\cos\varphi$

$$S = \int_0^{\pi/4} d\varphi \int_{2\cos\varphi}^{4\cos\varphi} r\,dr = \frac{1}{2}\int_0^{\pi/4} d\varphi\,(16\cos^2\varphi - 4\cos^2\varphi) =$$

$$= 2\int_0^{\pi/4} d\varphi\,(3\cos^2\varphi\, d\varphi) = 6\int_0^{\pi/4}\left(\frac{1+\cos(2\varphi)}{2}\right)d\varphi = \boxed{3\cdot\left(\frac{\pi}{4}+\frac{1}{2}\right)}$$

⑤ ZA PLOŠČO $D$, OMEJENO Z $y=x^2$ IN $x=y^2$ IN GOSTOTO $\sigma=\frac{x^2}{y}$

IZRAČUNAJ MASO IN TEŽIŠČE




$$M=\iint_D \sigma\, dS = \int_0^1 dx \int_{x^2}^{\sqrt{x}} \frac{x^2}{y}\, dy = \int_0^1 dx\; x^2 \cdot \ln\left(\frac{\sqrt{x}}{x^2}\right) =$$

$$= -\frac{3}{2}\int_0^1 dx\; x^2 \ln(x) = -\frac{3}{2}\left[\frac{x^3}{3}\ln x\Big|_0^1 - \int_0^1 \frac{x^2}{3}\,dx\right] = +\frac{3}{2}\cdot\frac{1}{9} = \boxed{\frac{1}{6}}$$

$u=\ln x$

$dv = x^2\,dx$

ALTERNATIVA: $$M=\int_0^1 dy \int_{y^2}^{\sqrt{y}} \frac{x^2}{y}\,dx = \int_0 \frac{dy}{y}\cdot\frac{1}{3}\left(y^{3/2}-y^6\right) =$$

$$= \frac{1}{3}\int_0^1 dy\left(y^{1/2}-y^5\right) = \frac{1}{3}\left(\frac{2}{3}-\frac{1}{6}\right) = \boxed{\frac{1}{6}}$$

$$X_T = \frac{1}{M}\iint_D x\sigma\, dS = \frac{1}{M}\int_0^1 dx \int_{x^2}^{\sqrt{x}} \frac{x^3}{y}\,dy = \frac{1}{M}\int_0^1 dx\; x^3 \ln x^{-3/2} =$$

$$= \frac{1}{M}\left(-\frac{3}{2}\right)\int_0^1 dx\; x^3 \ln x = -\frac{3}{2}\,\frac{1}{M}\left[-\int_0^1 dx\,\frac{x^3}{4}\right] = \frac{3}{2}\cdot\frac{1}{16}\cdot\frac{1}{M} = \boxed{\frac{9}{16}}$$

$$Y_T = \frac{1}{M}\iint_D y\sigma\, dS = 6\cdot\int_0^1 dy \int_{y^2}^{\sqrt{y}} x^2\,dx = \frac{6}{3}\cdot\int_0 dy\left(y^{3/2}-y^6\right) =$$

$$= 2\cdot\left[\frac{2}{5}-\frac{1}{7}\right] = 2\cdot\frac{9}{35} = \boxed{\frac{18}{35}}$$

$$\boxed{\vec{r}_T = \left(\frac{9}{16}, \frac{18}{35}\right)}$$