

# CATEGORY THEORY HW 1

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1) LET US DEFINE THE CATEGORY OF PARTIAL MAPS  $\mathcal{C}_{PAR}$ .  
 $|\mathcal{C}_{PAR}| = |\mathcal{C}|$   $\mathcal{C}_{PAR}(X, Y) =$  PARTIAL MAPS FROM  $X$  TO  $Y$ .

LET'S DEFINE THE COMPOSITION OF MORPHISMS.

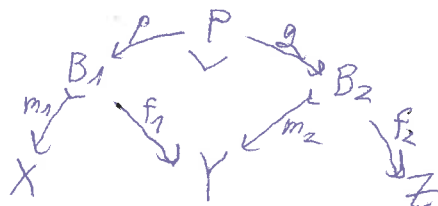
NOTE: I WILL ALWAYS WORK WITH A SPECIFIC REPRESENTATIVE OF THE EQUIV CLASS.

$$(m_1, f_1): X \rightarrow Y \quad (m_1, f_1) \in \mathcal{C}_{PAR}(X, Y)$$

$$(m_2, f_2): Y \rightarrow Z$$

$$(m_2, f_2) \circ (m_1, f_1) = ?$$

$$= (m_1 \circ p, f_2 \circ q)$$



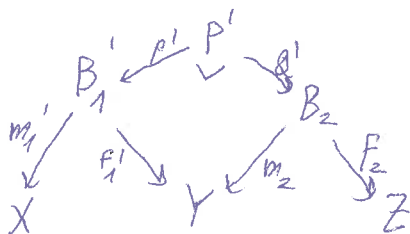
$\mathcal{C}$  HAS PULLBACKS SO WE COMPOSE MAPS BY FORMING A PULLBACK  $P$  OF THE COSPAN

$$B_1 \xrightarrow{f_1} Y \xleftarrow{f_2} B_2$$

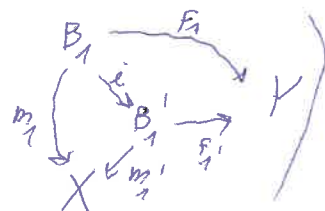
$P$  IS MONO BECAUSE  $m_2$  IS MONO AND PULLBACKS PRESERVE MONOS.

$(m_1, f_1)$  IS THEN ALSO MONO  $\Rightarrow (m_1 \circ p, f_2 \circ q): X \rightarrow Z$  IS INDEED A PARTIAL MAP REPRESENTATIVE.

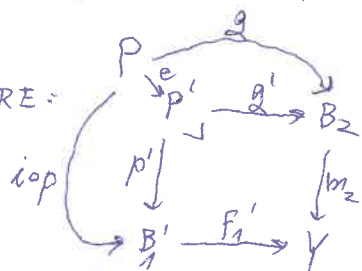
IS THIS WELL DEFINED? LET'S CHOOSE ANOTHER REPRESENTATIVE  $(m'_1, f'_1) \equiv (m_1, f_1)$



WE KNOW THIS COMMUTES:  
 WHERE  $i: B_1 \rightarrow B'_1$  IS  
 ISO SINCE  $(m_1, f_1) \equiv (m'_1, f'_1)$



LOOK AT THE PULLBACK SQUARE:



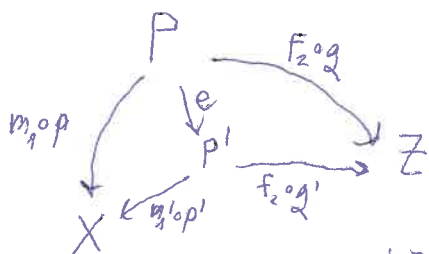
$$m_2 \circ q = f'_1 \circ p = (f'_1 \circ i) \circ (p) = f'_1 \circ (i \circ p)$$

$$\Rightarrow \exists ! e: P \rightarrow P' \text{ s.t. } q = q' \circ e$$

$$i \circ p = p' \circ e$$

$$(m_2 \circ f_2) \circ (m'_1, f'_1) = (m_1 \circ p, f_2 \circ q) = (m'_1 \circ p', f_2 \circ q')$$

WE HAVE:



THIS COMMUTES:  $f_2 \circ q' \circ e = f_2 \circ q$

$$m'_1 \circ p' \circ e = m'_1 \circ i \circ p = m_1 \circ i^{-1} \circ i \circ p = m_1 \circ p$$

THEREFORE

$$(m_1, p, f_2, q) \sqsubset (m'_1, p', f_2, q')$$

WE CAN REPEAT THIS ARGUMENT SWITCHING THE

ROLES OF  $P$  AND  $P'$  TO GET  $(m'_1, p', f_2, q') \sqsubset (m_1, p, f_2, q)$

AND THEREFORE  $(m'_1, p', f_2, q') \equiv (m_1, p, f_2, q)$

IF WE INSTEAD REPLACE  $(m_2, f_2)$  BY  $(m_2', f_2') \equiv (m_2, f_2)$  THE ARGUMENT IS ANALOGOUS.  
COMPOSITION IS WELL DEFINED ✓

IDENTITY MORPHISM IS OBVIOUSLY  $(1_X, 1_X) \in \mathcal{P}_{\text{PAR}}(X, X)$   $\left( \begin{array}{c} X \\ \downarrow 1 \\ X \end{array} \right)$

$$(1, 1) \circ (m, f) = (m, f)$$



(AND ALSO  $(m, f) \circ (1, 1) = (m, f)$  OF COURSE)

HERE WE USED THAT  $B \xrightarrow{1} B$  IS A PULLBACK  $\begin{array}{ccc} B & \xrightarrow{1} & B \\ \downarrow m & & \downarrow m \\ X & \xrightarrow{1} & X \end{array}$

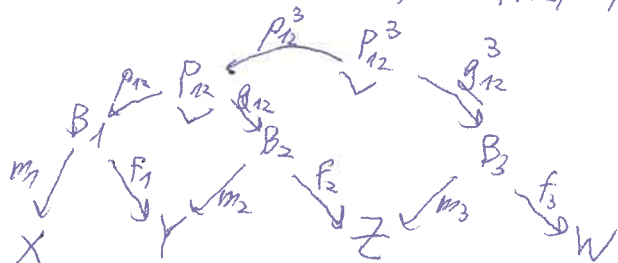
LET'S CHECK ASSOCIATIVITY

$$(m_1, f_1): X \rightarrow Y$$

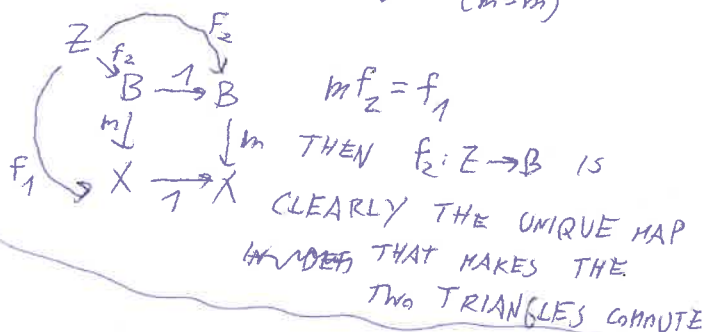
$$(m_2, f_2): Y \rightarrow Z$$

$$(m_3, f_3): Z \rightarrow W$$

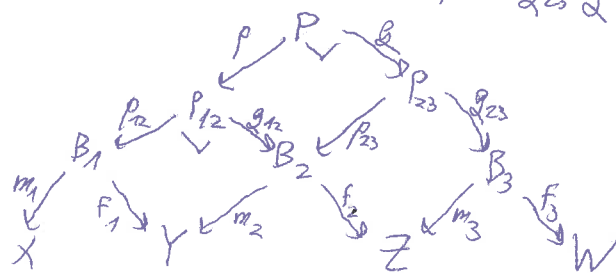
$$(m_3, f_3) \circ ((m_2, f_2) \circ (m_1, f_1)) = (m_1 p_{12} p_{12}^3, f_3 q_{12}^3)$$



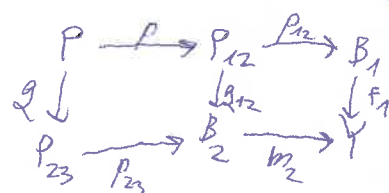
PROOF: ~~COMMIT~~ COMMUTES ✓ ( $m=m$ )



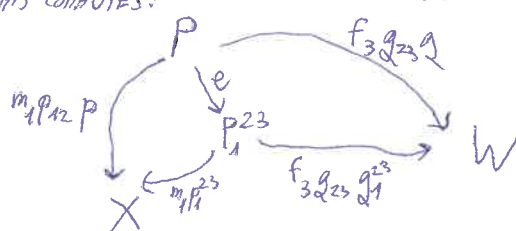
I WILL SHOW THAT BOTH OF RESULTS ON THE LEFT ARE EQUIVALENT TO THE FOLLOWING PARTIAL MAP REPR.  $(m_1 p_{12} p_{12}^3, f_3 q_{12}^3)$



THIS IS TRUE BECAUSE BY PULLBACK LEMMA THE FOLLOWING RECTANGLE IS A PULLBACK



PULLBACKS ARE UNIQUE SO THERE EXISTS AN ISO  $e: P \rightarrow P_{12}^{23}$ . THIS COMMUTES:



THIS SHOWS  $(m_1 p_{12} p_1, f_3 g_{23} g_1) \equiv (m_1 p_{11}^{23}, f_3 g_{23} g_1^{23}) \equiv (m_1 p_{12} p_{12}^3, f_3 g_{12}^3)$

SAME ARGUMENT WITH OTHER 2 PULLBACKS ( $P$  AND  $P_{12}^3$ ) SHOWS THIS

SO ASSOCIATIVITY IS SATISFIED.  $\mathcal{C}_{\text{PAR}}$  IS A CATEGORY

②  $X \in \mathcal{C}$

$I X = X$

$I(X \xrightarrow{f} Y) = \begin{array}{ccc} X & & \\ \downarrow 1 & \searrow f & \\ X & & Y \end{array} = (1, f)$

$I$  IS A FUNCTOR:  $I(X \xrightarrow{1} X) = \begin{array}{ccc} X & & \\ \downarrow 1 & \searrow 1 & \\ X & & X \end{array} = (1, 1) \checkmark$

~~$f: Y \rightarrow Z$~~   $I(X \xrightarrow{g \circ f} Z) = (1, g \circ f)$

$g: Y \rightarrow Z$   
 $f: X \rightarrow Y$

$I(Y \xrightarrow{g} Z) \circ I(X \xrightarrow{f} Y) = (1, g) \circ (1, f) =$

$\begin{array}{ccccc} & & X & & \\ & \swarrow 1 & & \searrow f & \\ X & & & & Y \\ \downarrow 1 & \searrow f & & \swarrow 1 & \searrow g \\ X & & Y & & Z \end{array}$

$= (1, g \circ f) \checkmark$

$I$  IS FAITHFUL:

$f, g: X \rightarrow Y, f \neq g$  THEN  $I(f) = (1, f)$   
 $I(g) = (1, g)$

ASSUME  $(1, f) \equiv (1, g)$ :

$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow i & \\ & X & \xrightarrow{g} & Y \\ \downarrow 1 & \swarrow 1 & \\ X & & \end{array}$

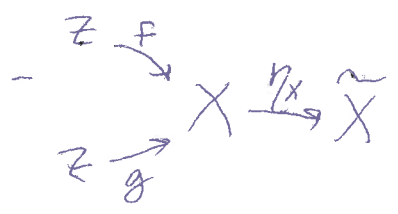
$1 \circ i = 1 \Rightarrow i = 1$

$f = g \circ i = g \circ 1 = g$

CONTRADICTION!

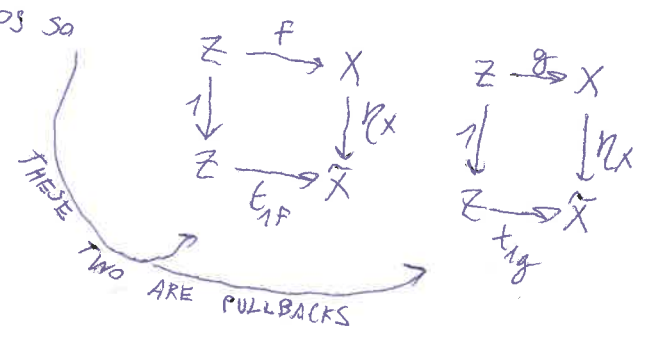
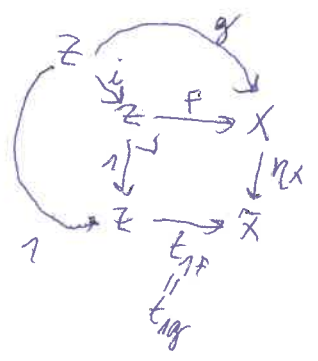
$I$  IS INDEED FAITHFUL.  $\square$

③ FIRST, LET US SHOW THAT  $\eta_X$  IS MONO ( $\forall X \in \mathcal{C}$ )



ASSUME  $\eta_X \circ f = \eta_X \circ g$ , WE NEED TO SHOW  $f = g$

$(1, f)$  ARE PARTIAL MAPS SO  
AND  $(1, g)$



THESE TWO ARE PULLBACKS  
 $t_{1f} = \eta_X \circ f = \eta_X \circ g = t_{1g}$

SO  $i=1$  AND  $g = f \circ i = f \Rightarrow \eta_X$  IS MONO  $\square$

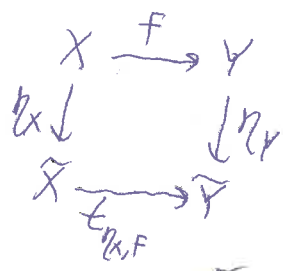
NOW  
LET'S DEFINE THE FUNCTOR  $(\tilde{-}) : \mathcal{C} \rightarrow \mathcal{C}$

$(\tilde{-})X = \tilde{X}$  (THE OBJECT IN PART. MAP CLASSIFIER)

$(\tilde{-})(X \xrightarrow{F} Y) \in \mathcal{C}(\tilde{X}, \tilde{Y})$

SO LET'S DEFINE

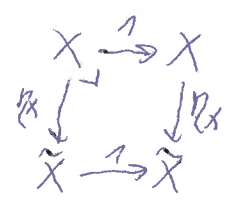
$(\tilde{-})(X \xrightarrow{F} Y) = t_{\eta_X, F}$



$\eta_X$  IS MONO SO  
 $(\eta_X, F) \in \mathcal{C}_{PAR}(\tilde{X}, \tilde{Y})$   
 $\exists ! t_{\eta_X, F} \in \mathcal{C}(\tilde{X}, \tilde{Y})$   
S.T. THIS IS A PULLBACK

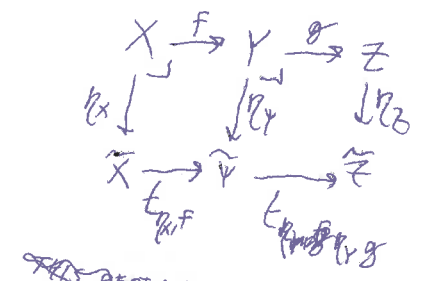
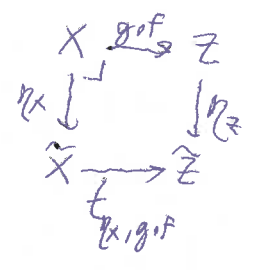
IT'S A FUNCTOR:

$(\tilde{-})(X \xrightarrow{1} X) = 1$

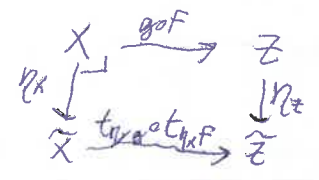


$f: X \rightarrow Y$   
 $g: Y \rightarrow Z$

$(\tilde{-})(g \circ f)$



PULLBACK LEMMA



By UNIQUENESS OF  $t_{m,f}$  WE HAVE  $t_{\eta_X, g \circ f} = t_{\eta_Y g} \circ t_{\eta_X f} \Rightarrow (\sim)$  IS A FUNCTOR  $\vee$

$(\sim)$  IS FAITHFUL:

$$f = g \Rightarrow f, g: \tilde{X} \rightarrow \tilde{Y}, f = g$$

$$f = (\sim) f'$$

$$g = (\sim) g'$$

$$\begin{array}{ccc} X & \xrightarrow{f'} & Y \\ \eta_X \downarrow & & \downarrow \eta_Y \\ \tilde{X} & \xrightarrow{f} & \tilde{Y} \end{array}$$

$$\begin{array}{ccc} X & \xrightarrow{g'} & Y \\ \eta_X \downarrow & & \downarrow \eta_Y \\ \tilde{X} & \xrightarrow{g} & \tilde{Y} \end{array}$$

$$f = g \Rightarrow f \circ \eta_X = g \circ \eta_X \Rightarrow \eta_Y \circ f' = \eta_Y \circ g' \Rightarrow f' = g' \vee$$

DIAGRAM COMMUTATIVITY

$\eta_Y$  MONO

$\Rightarrow (\sim)$  IS FAITHFUL  $\square$

$$X \in \mathcal{C}$$

$$f \in \mathcal{C}(X, Y)$$

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \tilde{X} \\ f \downarrow & & \downarrow t_{\eta_X, f} \\ Y & \xrightarrow{\eta_Y} & \tilde{Y} \end{array}$$

NATURALITY CONDITION IS AUTOMATICALLY SATISFIED BY OUR DEFINITION OF  $(\sim)$

$$(t_{\eta_X, f} \circ \eta_X = \eta_Y \circ f)$$

④  $\Rightarrow$

$$(m, f) \equiv (m', f')$$

$$\begin{array}{ccc} B & \xrightarrow{f} & Y \\ m \downarrow & & \downarrow \eta_Y \\ X & \xrightarrow{t_{m,f}} & \tilde{Y} \end{array}$$

$$\begin{array}{ccc} B & \xrightarrow{f} & Y \\ m \downarrow & & \downarrow \eta_Y \\ X & \xrightarrow{t_{m,f}} & \tilde{Y} \end{array}$$

$$\begin{array}{ccc} B' & \xrightarrow{f'} & Y \\ m' \downarrow & & \downarrow \eta_Y \\ X & \xrightarrow{t_{m',f'}} & \tilde{Y} \end{array}$$

$$t_{m,f} \circ m = \eta_Y \circ f$$

$$t_{m,f} \circ m \circ i = \eta_Y \circ f \circ i$$

$$t_{m,f} \circ m' = \eta_Y \circ f'$$

SO  $B' \xrightarrow{f'} Y$  IS ALSO PULLBACK.

$$\begin{array}{ccc} B' & \xrightarrow{f'} & Y \\ m' \downarrow & & \downarrow \eta_Y \\ X & \xrightarrow{t_{m',f'}} & \tilde{Y} \end{array}$$

BY UNIQUENESS

$$\text{OF } t_{m,f} \Rightarrow t_{m,f} = t_{m',f'} \square$$

SWITCH ROLES OF  $B'$  AND  $B$  AND WE ALSO GET  $(m, f) \sqsubseteq (m', f')$  SO  $(m, f) \equiv (m', f') \square$

$$⑤ \quad \mathcal{C}(-, Z) : \mathcal{C}^{op} \rightarrow \underline{\text{Set}}$$

$$\mathcal{C}(-, Z)X = \mathcal{C}(X, Z)$$

$$\mathcal{C}(X \xleftarrow{F} Y) : \mathcal{C}(X, Z) \mapsto \mathcal{C}(Y, Z) : g \mapsto g \circ F$$

$$F = \mathcal{C}_{\text{PAR}}(-, Y) \circ I^{op} : \mathcal{C}^{op} \rightarrow \underline{\text{Set}}$$

$$FX = \mathcal{C}_{\text{PAR}}(X, Y)$$

$$F(X \xleftarrow{F} Z) = \mathcal{C}_{\text{PAR}}(X, Y) \rightarrow \mathcal{C}_{\text{PAR}}(Z, Y) : (m_1, F_1) \mapsto (m_1, F_1) \circ (1, F)$$

$(\Rightarrow)$  ~~we will show~~ WE WILL SHOW THAT  $F$  IS NAT. ISO. TO  $\mathcal{C}(-, \tilde{Y})$   
 $X \in \mathcal{C}^{op} \quad F : X \leftarrow Z$

$$\begin{array}{ccc} \mathcal{C}_{\text{PAR}}(X, Y) & \xrightarrow{t} & \mathcal{C}(X, \tilde{Y}) \\ \downarrow (m_1, F_1) \mapsto (m_1, F_1) \circ (1, F) & & \downarrow g \mapsto g \circ F \\ \mathcal{C}_{\text{PAR}}(Z, Y) & \xrightarrow{t} & \mathcal{C}(Z, \tilde{Y}) \end{array}$$

$t$  IS DEFINED AS FOLLOWS: FOR  $(m_1, F_1) \in \mathcal{C}_{\text{PAR}}(X, Y)$  USE CLASSIFIER

$$\begin{array}{ccc} B_1 & \xrightarrow{F_1} & Y \\ m_1 \downarrow & & \downarrow \eta_Y \\ X & \xrightarrow{t_{m_1, F_1}} & \tilde{Y} \end{array} \quad \text{AND SET } t((m_1, F_1)) = t_{m_1, F_1}$$

NATURALITY CONDITION:

$$\begin{array}{ccccc} P & \xrightarrow{q} & B_1 & \xrightarrow{F_1} & Y \\ p \downarrow & & m_1 \downarrow & & \downarrow \eta_Y \\ Z & \xrightarrow{F} & X & \xrightarrow{t_{m_1, F_1}} & \tilde{Y} \end{array}$$

$$t_{p, (F \circ q)} = t((m_1, F_1) \circ (1, F)) = t_{m_1, F_1} \circ F$$

$t$  IS ISO IN Set  $\Rightarrow t$  IS BIJECTIVE

$t$  IS INJECTIVE BY EXERCISE ④

$t$  IS SURJECTIVE:  $g \in \mathcal{C}(X, \tilde{Y})$

$$\begin{array}{ccc} P & \xrightarrow{q} & Y \\ p \downarrow & & \downarrow \eta_Y \\ X & \xrightarrow{g} & \tilde{Y} \end{array} \quad t((p, q)) = g \quad \checkmark$$

$\Rightarrow t$  IS NATURAL ISOMORPHISM  $\square$

UNIQUENESS OF  $t$  BECAUSE IT'S A PART. MAP CLASSIFIER.



( $\Leftarrow$ ) AGAIN LET'S WRITE  $\tilde{Y}$  INSTEAD OF  $Z$ .

WE HAVE NAT. ISO.  $t$

$$\begin{array}{ccc} \mathcal{C}_{\text{PAR}}(X, Y) & \xrightarrow{t^X} & \mathcal{C}(X, \tilde{Y}) \\ \downarrow (m, f) \rightarrow (m, f) \circ (1, f) & & \downarrow g \rightarrow g \circ f \\ \mathcal{C}_{\text{PAR}}(Z, Y) & \xrightarrow{t^Z} & \mathcal{C}(Z, \tilde{Y}) \end{array}$$

DEFINE  $\eta_Y = t^Y(1) = t_{11}$

~~TAKE~~

TAKE  $(m, f) \in \mathcal{C}_{\text{PAR}}(X, Y)$

LET'S SHOW THIS COMMUTES:

$$\begin{array}{ccc} B & \xrightarrow{f} & Y \\ m \downarrow & & \downarrow t_{11}^Y \\ X & \xrightarrow{t_{mf}^X} & \tilde{Y} \end{array}$$

BY NAT. CONDITION:

$$t_{11}^Y \circ f = t_{((1,1) \circ (1,f))}^B = t_{(1,f)}^B$$

$$t_{mf}^X \circ m = t_{((m,f) \circ (1,m))}^B = t_{(1,f)}^B$$

$$\begin{array}{ccccc} & B & & B & \\ & \swarrow & & \searrow & \\ B & & B & \xrightarrow{f} & Y \\ & \searrow & & \downarrow m & \\ & X & & & \end{array}$$

COMMUTES  $\checkmark$

THE PULLBACK OF COSPAN

$$X \xrightarrow{t_{mf}^X} \tilde{Y} \xleftarrow{t_{11}^Y} Y \text{ OBVIOUSLY ALSO COMMUTES}$$

$$\begin{array}{ccc} P & \xrightarrow{g} & Y \\ P \downarrow & & \downarrow t_{11}^Y \\ X & \xrightarrow{t_{mf}^X} & \tilde{Y} \end{array}$$

SINCE  $t$  IS ISO IT FOLLOWS  $(m, f) \equiv (p, g)$

WHICH MEANS

$$\begin{array}{ccc} B & \xrightarrow{f} & Y \\ m \downarrow & & \downarrow t_{11}^Y \\ X & \xrightarrow{t_{mf}^X} & \tilde{Y} \end{array}$$

IS ALSO A PULLBACK SQUARE.

UNIQUENESS OF  $t_{mf}^X$  IS CLEAR FROM THE FACT THAT  $t^X$  IS ISO.

$\Rightarrow Y$  HAS A PART. MAP CLASSIFIER GIVEN BY  $\tilde{Y}$  AND  $\eta_Y = t_{11}^Y \quad \square$

An additional note:

I realise that at several points in the homework I have been using the same name (eg. function name f) for different things, sometimes in the same exercise (but for different, hopefully clearly separated parts of the exercise). Since the instructions mention that “quality of exposition is an important concern” I’d just like to point out I am aware that this could be an issue in an actual paper and that I would of course not be so lazy in that case.

I hope my solution procedure is clear enough and my handwriting readable. Have a nice day!