PROVE THAT UP TO EQUIVALENCE, A_5 HAS ONLY ONE FAITHFUL, TRANSITIVE ACTION ON 6 POINTS.

$\rho : A_5 \rightsquigarrow \Omega$, $|\Omega| = 6$ TRANSITIVE $\Rightarrow$ EQUIVALENT TO $A_5 \rightsquigarrow A_5/H$ COSET ACTION, WHERE $H \leq A_5$

$|A_5/H| = 6 \Rightarrow |H| = \frac{|A_5|}{6} = 10$

THIS WILL BE FAITHFUL $\Leftrightarrow$ $\mathrm{CORE}_{A_5}(H) = 1$ $\Leftrightarrow$ ALWAYS, BECAUSE A_5 IS SIMPLE.

(CORE WOULD BE A NORMAL SUBGROUP OF A_5)

SO WE ARE LOOKING FOR $|H| = 10$, $H < A_5$. AS $|H| = 5 \cdot 2$ SYLOW THEOREMS TELL US THAT A_5 IS A SEMIDIRECT PRODUCT OF A 5-CYCLE AND A 2-CYCLE.

SO $H = \langle (12345), (12)(34) \rangle$, WHICH WE KNOW IS REALLY A GROUP.

NOW WE NEED TO SHOW ANY SUBGROUPS OF THIS FORM ARE CONJUGATE.

$H_1 = \langle (abcde)(fg)(hi) \rangle$, $H_2 = \langle (a'b'c'd'e'), (f'g')(h'i') \rangle$

$|A_5| = 60 = 5 \cdot 3 \cdot 2^2 \Rightarrow$ 5 CYCLES ARE SYLOW 5-GROUPS OF A_5 SO THEY ARE ALL CONJUGATE

AS FOR THE 2-CYCLES $(fg)^{(gg'*)} = (fg')$ SOME ELEMENT NOT IN $\{f, g, g'\}$

SOME OF THEM HAVE TO BE EQUAL (PIGEONHOLE)

OTHER CASES OF 2-CYCLES FOLLOW SIMILARLY. ALL SUBGROUPS OF ORDER 10 ARE THEREFORE CONJUGATE IN A_5 AND THE CLAIM FOLLOWS. $\square$

LET G ACT TRANSITEVLY ON Ω. IF $\Delta \subseteq \Omega$ AND $\alpha \in \Omega$ PROVE THAT

$$\phi = \bigcap_{g \in G, \alpha \in \Delta^g} \Delta^g \quad \text{IS A BLOCK FOR } G$$

DEFINE FOR $w_1, w_2 \in \Omega$ $\quad w_1 \sim w_2 \Leftrightarrow \forall g \in G \; (w_1 \in \Delta^g \Leftrightarrow w_2 \in \Delta^g)$

OBVIOUSLY SYMMETRIC, REFLEXIVE.

TRANSITIVE

$$w_1 \sim w_2 \,,\; w_2 \sim w_3$$

$$\Rightarrow \forall g \in G \; (w_1 \in \Delta^g \Leftrightarrow w_2 \in \Delta^g \Leftrightarrow w_3 \in \Delta^g) \Rightarrow w_1 \sim w_3$$

$\sim$ IS AN EQUIV. RELATION.

$\sim$ IS A G-CONGRUENCE

$$w_1 \sim w_2 \Leftrightarrow w_1^g \sim w_2^g$$

$(\Rightarrow)$ $w_1 \sim w_2$ $\quad \forall g' \in G \; (w_1 \in \Delta^{g'} \Leftrightarrow w_2 \in \Delta^{g'})$

$\forall g' \in G \; (w_1^g \in \Delta^{g'g} \Leftrightarrow w_2^g \in \Delta^{g'g})$ $\quad$ REINDEX $h = g'g$

$\forall h \in G \; (w_1^g \in \Delta^h \Leftrightarrow w_2^g \in \Delta^h)$

$\square$

$(\Leftarrow)$ SIMILAR

BLOCKS ARE EQUIVALENCE CLASSES.

$$[\alpha] = \{ w \in \Omega \,;\; \forall g \in G \; (w \in \Delta^g \Leftrightarrow \alpha \in \Delta^g) \} = \phi \quad \square$$

~~KO NONABOM~~ K NONABELIAN AND $G = K \times K$. CONSIDER $G \curvearrowright K$ $u^{(x,y)} = x^{-1}uy$.

a) IS IT TRANSITIVE? FAITHFUL? WHAT ARE ITS STABILISERS?

b) PROVE THAT THE ACTION IS PRIMITIVE $\Leftrightarrow$ K SIMPLE GROUP

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a) TRANSITIVE $k_1, k_2 \in K$ $\quad k_1^{(k_1, k_2)} = k_1^{-1} k_1 k_2 = k_2$ $\square$

FAITHFUL $\quad \forall k \in K : k = x^{-1} k y$. TAKE $k = id \Rightarrow x = y$

$\Rightarrow \forall k \in K \quad k = x^{-1} k x \Rightarrow xk = kx$ $\quad \boxed{\ker(\beta) = \{(x,x) ; x \in Z(K)\}}$ NOT IN GENERAL.

STABILISERS

$$G_1 = \{(x,y) \in G \mid 1 = x^{-1}y\} = \{(x,x) \in G \mid x \in K\} = D$$

AS THE ACTION IS TRANSITIVE, ALL STABILISERS ARE ISOMORPHIC TO D.

b) $(\Rightarrow)$ ASSUME $N \lhd K$ NONTRIVIAL. THEN N IS A BLOCK: $N^{(x,y)} = x^{-1} N y = \overbrace{x^{-1} N x}^{N} x^{-1} y = N x^{-1} y$

THIS IS EITHER EQUAL OR DISJOINT FROM $N \Rightarrow$ BLOCK $\rightarrow\!\leftarrow$

$(\Leftarrow)$ ASSUME WE HAVE A BLOCK $N \subseteq K$. WLOG (TRANSLATE IF NEEDED) ASSUME $id \in N$.

THEN $N^{(x,x)} = x^{-1} N x = N \quad \forall x \in K \Rightarrow N \lhd K$

CAN'T BE DISJOINT BECAUSE $id \in N$, $id \in x^{-1} N x$. $\rightarrow\!\leftarrow$

a) PROVE THAT THE AFFINE GROUP $AGL(n,\mathbb{F})$ IS 2-TRANSITIVE ON $\mathbb{F}^n$

b) PROVE THAT $AGL(n,2)$ IS 3-TRANSITIVE ON $\mathbb{F}_2^n$

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a) $AGL(n,\mathbb{F}) \curvearrowright \mathbb{F}^n$ BY $\vec{x}^{(A,B)} = \vec{x}A + \vec{b}$ CLEARLY TRANSITIVE: $\vec{x}^{(I_n,\vec{y}-\vec{x})} = \vec{y} \quad \forall \vec{x},\vec{y}$

$AGL(n,\mathbb{F})$ 2-TRANS. $\Leftrightarrow$ STABILISER TRANSITIVE

$AGL(n,\mathbb{F})_0 = \{(A,b) \mid \vec{0} = \vec{0}A + b\} = \{(A,\vec{0}) \mid A \in GL(n,\mathbb{F})\}$,

BUT $GL(n,\mathbb{F})$ IS TRANSITIVE ON $\mathbb{F}^n \setminus \{\vec{0}\}$. $\square$

b) AS BEFORE, WE NEED TO PROVE $GL(n,\mathbb{F})$ IS 2-TRANSITIVE ON $\mathbb{F}^n \setminus \{\vec{0}\}$

BUT $GL(n,\mathbb{F})$ TAKES ANY PAIR OF LIN. INDEP. VECTORS TO ANY OTHER PAIR OF LIN. INDEP. VECTORS.

SINCE $\mathbb{F} = GF(2)$, ANY DISTINCT NONZERO VECTORS ARE LIN. INDEP. $\square$