

Dynamic Bayesian Neural Networks

Lorenzo Rimella and Nick Whiteley

School of Mathematics, University of Bristol, and the Alan Turing Institute

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Abstract

We define an evolving in time Bayesian neural network called a Hidden Markov neural network. The weights of a feed-forward neural network are modelled with the hidden states of a Hidden Markov model, whose observed process is given by the available data. A filtering algorithm is used to learn a variational approximation to the evolving in time posterior over the weights. Training is pursued through a sequential version of Bayes by Backprop [Blundell et al., 2015], which is enriched with a stronger regularization technique called variational DropConnect. The experiments test variational DropConnect on MNIST and display the performance of Hidden Markov neural networks on time series.

1 Introduction

Hidden Markov models (HMMs) have appeared as an efficient statistical tool to identify patterns in dynamic dataset, with applications ranging from speech recognition [Rabiner and Juang, 1986] to computational biology [Krogh et al., 2001]. Neural networks (NNs) are nowadays the most popular models in Machine learning and Artificial intelligence, and they have shown outstanding performances in several fields. Multiple attempts have been accomplished in the literature to combine HMM and NN. In Franzini et al. [1990] an NN is trained to approximate the emission distribution of an HMM. Bengio et al. [1990] and Bengio et al. [1991] preprocess the data with an NN and then use the output as the observed process of a discrete HMM. Krogh and Riis [1999] propose *Hidden neural networks* where NNs are used to parameterize Class HMM, an HMM with a distribution over classes assigned to each state. In neuroscience, Aitchison et al. [2014] explores the idea of updating measures of uncertainty over the weights in a mathematical model of a neuronal network as part of a "Bayesian Plasticity" hypothesis of how synapses take uncertainty into account during learning. However, they did not focus on artificial neural networks and the computational challenges of using them for data analysis when network weights are statistically modelled as being time-varying. In this paper, we propose a novel hybrid model between Factorial hidden Markov model [Ghahramani and Jordan, 1997] and neural network called: *Hidden Markov neural networks* (HMNNs).

Intuitively, we need to perform Bayesian inference on a time-evolving NN. However, even computing the posterior over the weights of a single NN is a complex task and it is generally intractable. Monte Carlo sampling techniques, such as particle filtering, provide a way to approximate an evolving posterior, however, they suffer from the curse of dimensionality [Rebeschini et al., 2015] and high computational cost [Rimella and Whiteley, 2019]. The rich literature on variational Bayes [Blei et al., 2017] and its success on Bayesian inference for NN [Graves, 2011, Kingma and Welling, 2013, Blundell et al., 2015] have motivated us to use this technique in HMNNs. In particular, the resulting procedure ends up being a sequential counterpart of the algorithm Bayes by Backprop proposed by Blundell et al. [2015]. As for Blundell et al. [2015] a pivotal role is played by the reparameterization trick [Kingma and Welling, 2013], which generates unbiased estimates of the considered gradient. Online learning makes HMNNs appealing, indeed the posterior over the weights at time t can be easily updated once new observations are available without retraining it from scratch. This can find applications not only on time series analysis and forecasting but also on streaming data.

Contributions We introduce a novel evolving in time Bayesian neural network where the posterior over the weights is estimated sequentially through variational Bayes with an evolving prior. We use a

variational approximation that induces a regularization that resembles DropConnect [Wan et al., 2013] and variational DropOut [Kingma et al., 2015] and we reformulate the reparameterization trick according to it. Our method is tested on the MNIST dataset, dynamic classification and time series forecasting, and it appears to perform better than multiple baselines.

1.1 Related Work

Bayesian DropConnect & DropOut DropConnect [Wan et al., 2013] and DropOut [Srivastava et al., 2014] are well-known techniques to prevent NN from overfitting. Kingma et al. [2015] proposes variational DropOut where they combined fully factorized Gaussian variational approximation with the local reparameterization trick to re-interpret DropOut with continuous noise as a variational method. Gal and Ghahramani [2016] extensively treat the connections between DropOut and Gaussian processes, and they show how to train NNs with DropOut (or DropConnect [Mobiny et al., 2019]) through a variational Bayes setting. Our version of variational DropConnect has several common aspects with the cited works, however, the whole regularization is induced by the variational approximation’s choice and the corresponding reformulation of the reparameterization trick, which is, to the authors’ knowledge, a novel approach.

Bayesian filtering In the literature, there are multiple examples of NN training through Bayesian filtering [Puskorius and Feldkamp, 1991, 1994, 2001, Shah et al., 1992, Feldkamp et al., 2003, Ollivier et al., 2018]. In particular, the recent work of Aitchison [2018] proposed AdaBayes and AdaBayes-SS where updates resembling the Kalman filter are employed to model the conditional posterior over a weight of an NN given the states of all the other weights. The main difference with HMNN lies on the time dimension and so the dynamical evolution of the underlined NN.

Continual learning There are multiple similarities between our work and continual learning methods, hence we do a quick overview of the most popular ones. Elastic Weight Consolidation [Kirkpatrick et al., 2017] uses an L2-regularization that guarantees the weights of the NN for the new task being in the proximity of the ones from the old task. Variational continual learning [Nguyen et al., 2017] learns a posterior over the weights of an NN by approximating sequentially the true posterior with the last variational approximation. Online Laplace approximation [Ritter et al., 2018] proposes a recursive update for the parameters of a Gaussian variational approximation which involves Hessian of the newest negative log-likelihood. However, continual learning does not build dynamic models so the time evolution of HMNN supports the novelty also in this field.

2 Hidden Markov Neural Networks

For a time horizon $T \in \mathbb{N}$, a Hidden Markov model is a bivariate process composed by an unobserved Markov chain $(W_t)_{t=0,\dots,T}$, called the hidden process, and a collection of observations $(\mathcal{D}_t)_{t=1,\dots,T}$, called the observed process, where the single observation at t is conditionally independent of all the others given the hidden process at the corresponding time step. We consider the case where the latent state-space is \mathbb{R}^V , with V finite set, and \mathcal{D}_t is valued in \mathbb{D} whose form is model specific (discrete, \mathbb{R}^d with $d \in \mathbb{N}$, etc.). To describe the evolution of an HMM three quantities are needed: the initial distribution, the transition kernel and the emission distribution. We use $\lambda_0(\cdot)$ for the probability density function of W_0 (initial distribution). We write $p(w_{t-1}, \cdot)$ for the conditional probability density function of W_t given $W_{t-1} = w_{t-1}$ (transition kernel of the Markov chain). We call $g(w_t, \mathcal{D}_t)$ the conditional probability mass or density function of \mathcal{D}_t given $W_t = w_t$ (emission distribution).

In this paper, we introduce a novel HMM called a “Hidden Markov Neural Network” (HMNN) where the hidden process outlines the evolution of the weights of a neural network. Here the finite set V collects the location of each weight and $v \in V$ can be thought of as a triplet (l, i, j) saying that the weight W_t^v is a weight of the NN at time t , and precisely related to the connection of the hidden unit i (or input feature i if $l - 1 = 0$) in the layer $l - 1$ with the hidden unit j in the layer l (which might be the output layer). In such a model we also assume that the weights evolve independently from each other, meaning that the transition kernel factorizes as follows:

$$p(w, \tilde{w}) = \prod_{v \in V} p^v(w^v, \tilde{w}^v), \quad w, \tilde{w} \in \mathbb{R}^V. \quad (1)$$

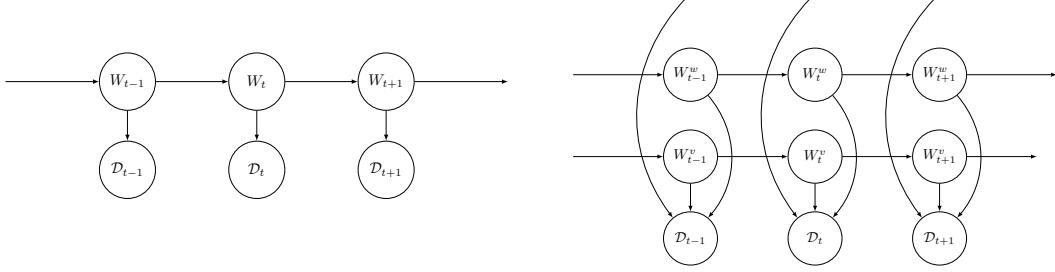


Figure 1: On the left: the conditional independence structure of an HMM. On the right: the conditional independence structure of an FHMM

Under this assumption, an HMNN is a well-known class of HMM called Factorial Hidden Markov model (FHMM), introduced by Ghahramani and Jordan [1997].

There is no restriction on the form of the neural network and the data \mathcal{D}_t , however, we focus on feed-forward neural network and on a supervised learning scenario where the observed process is composed by an input x_t and an output y_t , such that the neural network associated to the weights W_t maps the input into a probability to each possible output, which represents the emission distribution $g(w_t, \mathcal{D}_t)$ for $W_t = w_t$.

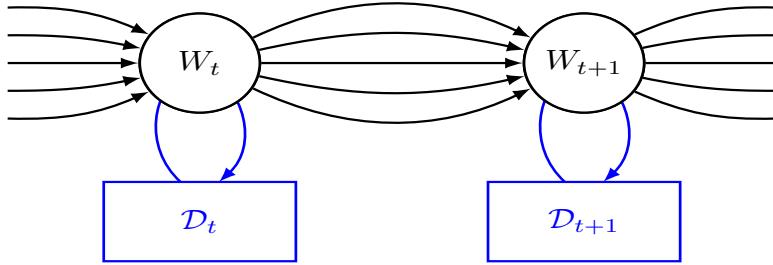


Figure 2: A graphical representation of an HMNN. Black lines show the evolution in time of the weights. Blue lines represent the action of the neural network on the data.

2.1 Filtering algorithm for HMNN

The filtering problem aims to compute the conditional distributions of W_t given $\mathcal{D}_1, \dots, \mathcal{D}_t$, this is called filtering distribution and we denote it with π_t . Ideally, the filtering problem can be solved with a forward step through the data:

$$\begin{cases} \pi_0 := \lambda_0 \\ \pi_t := C_t P \pi_{t-1} \end{cases} \quad t \in \{1, \dots, T\}, \quad (2)$$

where P , C_t are called prediction operator and correction operator and they are defined as follows:

$$P\rho(A) := \int \mathbb{I}_A(w)p(w', w)\rho(w')dw'dw \quad \text{and} \quad C_t\rho(A) := \frac{\int \mathbb{I}_A(w)g(w, \mathcal{D}_t)\rho(w)dw}{\int g(w, \mathcal{D}_t)\rho(w)dw}, \quad (3)$$

with ρ probability density function. Throughout the paper, we refer to the posterior over the weights of the neural network at time t as the filtering distribution π_t , which can be thought as our target time-evolving posterior.

Unfortunately, recursion (2) is intractable for any chosen architecture of the underlying neural network. Variational inference can be used to approximate sequentially the target distribution π_t with a variational approximation q_{θ_t} belonging to a pre-specified class of distributions \mathcal{Q} . The approximate distribution q_{θ_t} is uniquely identified inside the class by a vector of parameters θ_t , which is picked up by minimizing a Kullback-Leibler (**KL**) divergence criteria:

$$q_{\theta_t} = \arg \min_{q_{\theta} \in \mathcal{Q}} \mathbf{KL}(q_{\theta} || \pi_t) = \arg \min_{q_{\theta} \in \mathcal{Q}} \mathbf{KL}(q_{\theta} || C_t P \pi_{t-1}), \quad (4)$$

where the Kullback-Leibler divergence can be rewritten as:

$$\mathbf{KL}(q_\theta || \mathbf{C}_t \mathbf{P} \pi_{t-1}) = \text{constant} + \mathbf{KL}(q_\theta || \mathbf{P} \pi_{t-1}) - \mathbb{E}_{q_\theta(w)} [\log(g(w, \mathcal{D}_t))]. \quad (5)$$

Sequential training using (4) is again intractable because it requires the filtering distribution at time $t-1$, i.e. π_{t-1} . Although under a proper minimization procedure we could consider $q_{\theta_t} \approx \pi_t$ when π_{t-1} is known, similarly $q_{\theta_{t-1}} \approx \pi_{t-1}$ when π_{t-2} is known, and so on. Given that π_0 is our prior knowledge λ_0 on the weights before training, we can find a q_{θ_1} that approximates π_1 using (4) and then we can propagate forward our approximation by following the previous logic. In this way an HMNN is trained sequentially on the same flavour of (4), by substituting the optimal filtering with the last variational approximation. As for the optimal procedure, we define an approximated filtering recursion, where $\tilde{\pi}_t$ stands for the sequential variational approximation of π_t :

$$\begin{cases} \tilde{\pi}_0 := \lambda_0 \\ \tilde{\pi}_t := \mathbf{V}_{\mathcal{Q}} \mathbf{C}_t \mathbf{P} \tilde{\pi}_{t-1} \end{cases} \quad t = 1, \dots, T, \quad (6)$$

where the operators \mathbf{P}, \mathbf{C}_t are as in recursion (2) and the operator $\mathbf{V}_{\mathcal{Q}}$ is defined as follows:

$$\mathbf{V}_{\mathcal{Q}}(\rho) := \arg \min_{q_\theta \in \mathcal{Q}} \mathbf{KL}(q_\theta || \rho), \quad (7)$$

with ρ being a probability distribution and \mathcal{Q} being a class of probability distribution.

2.2 Sequential reparameterization trick for HMNN

The minimization procedure exploited in recursion (6) cannot be solved in a closed form and we propose to find a suboptimal solution through gradient descent. This requires an estimate of the gradient of $\mathbf{KL}(\tilde{\pi}_t || \mathbf{C}_t \mathbf{P} \tilde{\pi}_{t-1})$. As explained in Blundell et al. [2015], if the variational approximation $q_\theta(w)$ can be rewritten as a probability distribution $\nu(\epsilon)$ through a deterministic transformation $w = t(\theta, \epsilon)$ then:

$$\frac{\partial}{\partial \theta} \mathbf{KL}(q_\theta || \mathbf{C}_t \mathbf{P} \tilde{\pi}_{t-1}) = \mathbb{E}_{\nu(\epsilon)} \left[\frac{\partial \log \left(\frac{q_\theta(w)}{\mathbf{C}_t \mathbf{P} \tilde{\pi}_{t-1}(w)} \right)}{\partial w} \frac{\partial w}{\partial \theta} + \frac{\partial \log \left(\frac{q_\theta(w)}{\mathbf{C}_t \mathbf{P} \tilde{\pi}_{t-1}(w)} \right)}{\partial \theta} \right], \quad (8)$$

with $w = t(\theta, \epsilon)$. This result is a reformulation of the Gaussian reparameterization trick [Opper and Archambeau, 2009, Kingma and Welling, 2013, Rezende et al., 2014, Blundell et al., 2015].

Algorithm 1 Approximate filtering recursion for HMNN

Input: $\lambda_0; (p^v(\cdot, \cdot))_{v \in V}; (\mathcal{D}_t)_{t=1, \dots, T}; (\theta_t^{(0)})_{t=1, \dots, T}; \gamma$

Set: $\tilde{\pi}_0 = \lambda_0$

for $t = 1, \dots, T$ **do**

Set the initial condition: $\theta_t = \theta_t^{(0)}$

repeat

$\epsilon^{(i)} \sim \nu, \quad i = 1, \dots, N$

$$\nabla = \sum_{i=1}^N \left[\frac{\partial}{\partial w} \log \left(\frac{q_\theta(w)}{\mathbf{C}_t \mathbf{P} \tilde{\pi}_{t-1}(w)} \right) \frac{\partial w}{\partial \theta} + \frac{\partial}{\partial \theta} \log(q_\theta(w)) \right]_{\theta=\theta_t, w=t(\theta_t, \epsilon^{(i)})}$$

Update the parameters: $\theta_t = \theta_t + \gamma \nabla$

until Maximum number of iterations

Set: $\tilde{\pi}_t = q_{\theta_t}$

end for

Return: $(\theta_t)_{t=1, \dots, T}$

Given (8) we can estimate the expectation $\mathbb{E}_{\nu(\epsilon)}$ via straightforward Monte Carlo sampling. Given the Monte Carlo estimate of the gradient, we can then update the parameters θ_t , related to the variational

approximation at time t , according to any gradient descent technique. Algorithm 1 displays this procedure and for the sake of simplicity, we write the algorithm with an update that follows a vanilla gradient descent.

As suggested multiple times in the literature [Graves, 2011, Blundell et al., 2015], the cost function in (5) is suitable to minibatches optimization. This might be useful when at each time step the datapoint \mathcal{D}_t is made of multiple data and so a full computation of the gradient is computationally expensive.

2.3 Gaussian kernel and Gaussian variational approximation

A fully Gaussian model, i.e. a model where both the transition kernel and the variational approximation are Gaussian distributions, is not only convenient because the form of $t(\theta, \epsilon)$ is trivial, but also because there exists a closed-form solution for $P\tilde{\pi}_{t-1}(w)$. Another appealing aspect of the Gaussian choice is that similar results hold for the scale mixture of Gaussians, which allows us to use a more complex variational approximation and a transition kernel of the same form as the prior in Blundell et al. [2015].

Start by considering the variational approximation. We choose $q_\theta := \bigotimes_{v \in V} q_\theta^v$ where q_θ^v is a scale mixture of Gaussian with parameters $\theta^v = (m^v, s^v)$ and p^v hyperparameter. Precisely, for a given weight w^v of the feed-forward neural network:

$$q_\theta^v(w^v) := p^v \mathcal{N}(w^v | m^v, (s^v)^2) + (1 - p^v) \mathcal{N}(w^v | 0, (s^v)^2), \quad (9)$$

where $p^v \in (0, 1]$, $m^v \in \mathbb{R}$, $(s^v)^2 \in \mathbb{R}_+$. We refer to this technique as variational DropConnect because it can be interpreted as setting around zero with probability $1 - p^v$ the weight in position v of the neural network and so it plays a role of regularization similar to Wan et al. [2013]. Under variational DropConnect the deterministic transformation $t(\theta, \epsilon)$ is still straightforward. Indeed, given that q_θ factorises, then $t(\theta, \epsilon) = (t^v(\theta^v, \epsilon^v))_{v \in V}$ (each w^v depends only on θ^v) and w^v is distributed as (9) which is equivalent to consider:

$$w^v = \eta m^v + \xi \sigma^v, \quad \text{with } \eta \sim \text{Be}(\cdot | p^v), \xi \sim \mathcal{N}(\cdot | 0, 1). \quad (10)$$

Hence $t^v(\theta^v, \epsilon^v) = \eta m^v + \xi s^v$, where $\theta^v = (m^v, s^v)$ and $\epsilon^v = (\eta, \xi)$ with η Bernoulli with parameter p^v and ξ standard Gaussian, meaning that in Algorithm 1 we just need to sample from a Bernoulli and a Gaussian distribution independently. Remark that p must be considered as a fixed hyperparameter and cannot be learnt during training, because from (8) we need the distribution of ϵ to be independent of the learnable parameters.

The kernel is chosen to be a scale mixture of Gaussian with parameters $\pi, \alpha, \sigma, c, \mu$:

$$p(w', w) := \pi \mathcal{N}(w | \mu + \alpha(w' - \mu), \sigma^2 \mathbf{I}_V) + (1 - \pi) \mathcal{N}(w | \mu + \alpha(w' - \mu), (\sigma^2/c^2) \mathbf{I}_V), \quad (11)$$

where $\pi \in (0, 1)$, $\mu \in \mathbb{R}^V$, $\alpha \in (0, 1)$, $\sigma \in \mathbb{R}_+$, \mathbf{I}_V is the identity matrix on $\mathbb{R}^{V,V}$, $c \in \mathbb{R}_+$ and $c > 1$. Intuitively, the transition kernel tells us how we are expecting the weights to be in the next time step given the states of the weights at the current time. We can interpret it as playing the role of an evolving prior which constraints the new posterior in regions that are determined from the previous training step similarly to Variational continual learning [Nguyen et al., 2017]. The choice of the transition kernel is crucial. A too conservative kernel would constrain too much the evolution and the algorithm would not be able to learn patterns in new data. On the contrary, a too flexible kernel could just forget what learnt before and adapt to the new data only.

The term $P\tilde{\pi}_{t-1}(w)$ has a closed form solution when transition kernel and the variational approximation are as in (9) and (12). Consider a general weight $v \in V$ and call $(P\tilde{\pi}_{t-1})^v$ the marginal density of $P\tilde{\pi}_{t-1}$ on the component v . If m_{t-1}^v, s_{t-1}^v are the estimates of m^v, s^v at time $t-1$ then:

$$\begin{aligned} (P\tilde{\pi}_{t-1})^v(w^v) &= p^v \pi \mathcal{N}(w^v | \mu^v - \alpha(\mu^v - m_{t-1}^s), \sigma^2 + \alpha^2 s_{t-1}^2) \\ &\quad + (1 - p^v) \pi \mathcal{N}(w^v | \mu^v - \alpha \mu_k^v, \sigma^2 + \alpha^2 s_{t-1}^2) \\ &\quad + p^v (1 - \pi) \mathcal{N}(w^v | \mu_k^v - \alpha(\mu^v - m_{t-1}^s), \sigma^2/c^2 + \alpha^2 s_{t-1}^2) \\ &\quad + (1 - p^v)(1 - \pi) \mathcal{N}(w^v | \mu^v - \alpha \mu^v, \sigma^2/c^2 + \alpha^2 s_{t-1}^2). \end{aligned} \quad (12)$$

We can observe that (12) is again a scale mixture of Gaussians, where all the variances are influenced by the variances at the previous time step according to α^2 . On the one hand, the DropConnect rate p^v tells how to scale the mean of the Gaussians according to the previous estimates m_{t-1}^v . On the other hand, π controls the entity of the jumps by allowing the weights to stay in place with a small variance σ^2/c^2 and permitting big-jumps with σ^2 if necessary. As in Blundell et al. [2015] we do not use the variance in practice, but a transformation \tilde{s}_t such that $s_t = \log(1 + \exp(\tilde{s}_t))$.

3 Experiments

In this section, we show three experiments. Firstly, we present a static comparison on MNIST [LeCun et al., 1998] between Bayes by Backprop and Bayes by Backprop with variational DropConnect. Secondly, we test HMNN on an artificially modified MNIST to learn an evolving in time classifier. We conclude the section with a time series prediction in the dynamic video texture of a waving flag [Chan and Vasconcelos, 2007, Boots et al., 2008, Basharat and Shah, 2009]. Additional details on the experiments can be found in the appendices. Datasets and codes are available at <https://github.com/LorenzoRimella/HiddenMarkovNeuralNetwork>. As for Blundell et al. [2015] we focus our studies on simple feed-forward neural networks.

The experiments were run on three different clusters: BlueCrystal Phase 4 (University of Bristol), Cirrus (one of the EPSRC Tier-2 National HPC Facilities) and The Cambridge Service for Data Driven Discovery (CSD3) (University of Cambridge). In our experiments, we experience a computational cost per time step that is comparable to the one of Bayes by Backprop.

3.1 Variational DropConnect

The experiment aims to understand if using a scale mixture as variational approximation can help in improving Bayes by Backprop. We train on the MNIST dataset with the same setup of Blundell et al. [2015]. We consider a small architecture with the vectorized image as input, two hidden layers with 400 rectified linear units [Nair and Hinton, 2010, Glorot et al., 2011] and a softmax layer on 10 classes as output. We consider a fully Gaussian HMNN, as in Subsection 2.3, with $T = 1$, $\alpha = 0$ and $\mu = \mathbf{0}$, with $\mathbf{0}$ the zero vector. Note that such HMNN coincides with a single Bayesian neural network, meaning that we are simply training with Bayes by Backprop with the addition of variational DropConnect. We train on about 50 combination of the remaining parameters (p, π, σ, c) and the learning rate, which are randomly extracted from pre-specified grids. For each value of p we report in Figure 3 the performance on the validation set of the three best models.

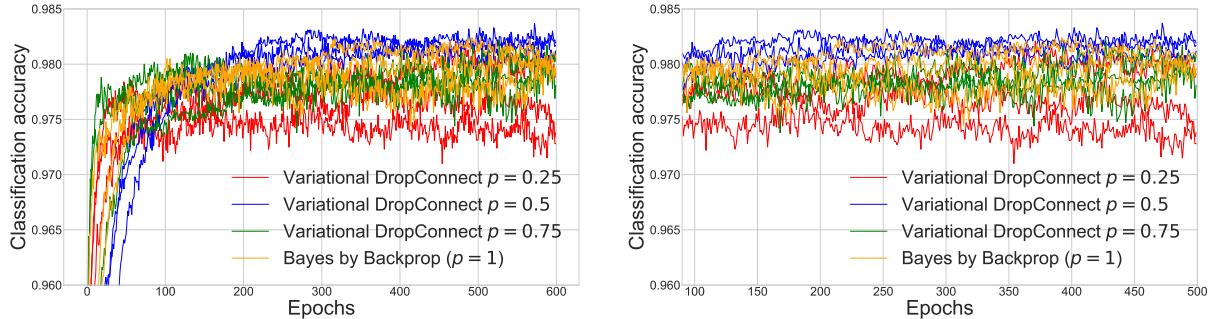


Figure 3: Performance on a validation set of a Bayes by Backprop with and without Variational DropConnect. The plot on the right is a zoom-in of the plot on the left.

We do model selection according to validation on each possible value of p . Test performances are reported in Table 1. We find out that small values of p lead to better performance. In particular, we observe that the performance of $p = 0.25$ moves from the worst on the validation set to the best on the test set, while $p = 1$ drops at test time.

Table 1: Performances on MNIST’s test set (bigger is better). $p = 1$ refers to the case of Standard Bayes by Backprop.

p	0.25	0.5	0.75	1
Accuracy	0.9838	0.9827	0.9825	0.9814

3.2 Evolving classifier on MNIST

In this section, we want to learn an evolving in time classifier from an artificially generated dataset which is built from MNIST.

1. We define two labellers: \mathcal{C}_1 , naming each digit with its label in MNIST; \mathcal{C}_2 , labelling each digits with its MNIST's label shifted by one unit, i.e. 0 is classified as 1, 1 is classified as 2, ..., 9 is classified as 0.
2. We consider 19 time steps where each time step t is associated with a probability $f_t \in [0, 1]$ and a portion of the MNIST's dataset \mathcal{D}_t .
3. At each time step t we randomly label each digit in \mathcal{D}_t with either \mathcal{C}_1 or \mathcal{C}_2 according to the probabilities $f_t, 1 - f_t$.

The resulting $(\mathcal{D}_t)_{t=1,\dots,19}$ is a collection of images where the labels evolves in t by switching randomly from \mathcal{C}_1 to \mathcal{C}_2 and vice-versa. Validation and test sets are built similarly. In such a scenario, we would ideally want to be able to predict the correct labels by sequentially learn a classifier that is capable of inferring part of the information from the previous time step. Remark that when $f_t = 0.5$ the best we can do is a classification accuracy of 0.5 because \mathcal{C}_1 and \mathcal{C}_2 are indistinguishable.

We consider a fully Gaussian HMNN with $\mu = m_{t-1}$, to encourage a strong memory on the previous posterior. The evolving in time NN is composed by the vectorize image as input, two hidden layers with 100 rectified linear units and a softmax layer on 10 classes as output. The parameters α, p are selected through the validation set while the other parameters are fixed before training. Along with the previous HMNN, we train sequentially four additional models for comparison: Variational continual learning; Elastic weight consolidation, with tuning parameter chosen with the validation set; Bayes by Backprop trained sequentially on the dataset; Bayes by Backprop on the full dataset. Selected graphical performances on the validation sets are displayed in Figure 4.

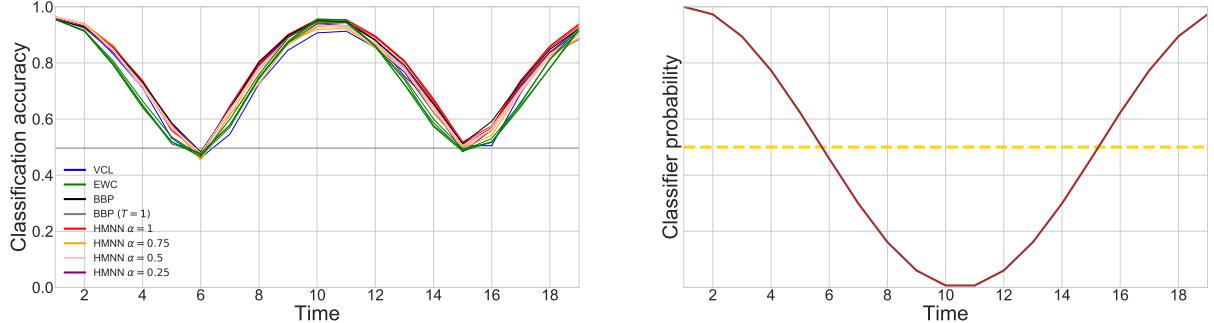


Figure 4: On the left, performances on the validation set of time of evolving classifiers obtained with different algorithms. BBP refers to Bayes by Backprop trained sequentially, while BBP ($T=1$) refers to a training of Bayes by Backprop on the whole dataset. On the right, in brown evolution in time of the probability f_t of choosing the labeller \mathcal{C}_1 ; in yellow the value 0.5.

To test the method we report the mean over time of the classification accuracy, which can be found in Table 2. For HMNN, Bayes by Backprop, EWC and VCL we choose the parameters that perform the best on validation. We find that HMNN and a sequential training of Bayes by Backprop perform the best. It is not surprising that continual learning methods fall behind. Indeed, EWC and VCL are built to preserve knowledge on the previous tasks, which might mix up \mathcal{C}_1 and \mathcal{C}_2 and confuse the network.

Table 2: Performances for the evolving classifier (bigger is better). BBP refers to Bayes by Backprop trained sequentially, while BBP ($T=1$) refers to a training of Bayes by Backprop on the whole dataset.

Method	BBP ($T=1$)	BBP	EWC	VCL	HMNN
Accuracy	0.496	0.774	0.738	0.741	0.777

3.3 One-step ahead prediction for flag waving

We consider now a sequence of images extracted from a video of a waving flag [Basharat and Shah, 2009, Venkatraman et al., 2015]. The idea is to create an HMNN where the neural network at time t can predict the next frame, i.e. the NN maps frame t in frame $t + 1$. To measure the performance we use the metric suggested in Venkatraman et al. [2015] which is a standardized version of the RMSE on a chosen test trajectory:

$$\mathcal{M}(y_{1:T}, \hat{y}_{1:T}) := \sqrt{\frac{\sum_{t=1}^T \|y_t - \hat{y}_t\|_2^2}{\|y_t\|_2^2}}, \quad (13)$$

where $y_{1:T}$ is the ground truth on frames $1, \dots, T$ and $\hat{y}_{1:T}$ are the predicted frames. Unless specified differently, $\hat{y}_{1:T}$ is a sequence of one step ahead predictions. To have a proper learning procedure we need multiple frames per time step, otherwise, the neural network would just learn the current frame. To overcome this problem we create a sliding window with 36 frames, meaning that at time step t we train on predicting frames $t - 35, \dots, t$ from frames $t - 36, \dots, t - 1$, with $t > 36$. The choice of the length for the sliding window was empirical, we tried multiple lengths and stop at the first one that was not overfitting on the data inside the window.

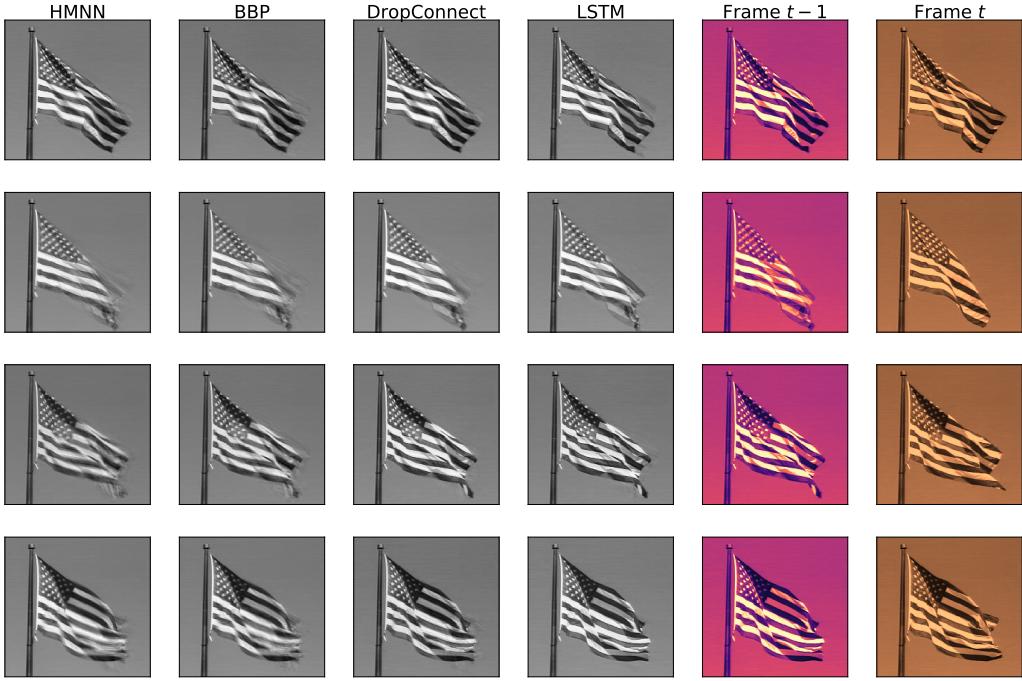


Figure 5: The grey columns show the prediction for different algorithms. The orange column shows the target frame for prediction and the purple column shows the frame at the previous time step. The rows display different frames, i.e. different time steps.

Per each time step, we use a simple architecture of three layers with 500, 20, 500 rectified linear units, the vectorized previous frame as input and the vectorized current frame as output (the dimension is reduced with PCA). We consider fully Gaussian HMNN, with $\mu = m_{t-1}$, to encourage a strong memory of the previous time step, and all the other parameters selected through random grid search and validation. Figure 5 compares HMNN predictions with different baselines: Bayes by Backprop trained sequentially on the sliding windows (column named BBP), DropConnect trained sequentially on the sliding windows (column named DropConnect), LSTM trained with the same sliding window size (column named LSTM), a trivial predictor that uses the previous frame as forecasting for the current frame (column named: frame $t - 1$). We notice that LSTM is prone to overfit on the sliding window and so to predict frame t with the last frame seen without any uncertainty. Similar issues appear in sequential DropConnect. This unwanted behaviour is probably due to the absence of uncertainty quantification and so overconfidence on the considered predictions. HMNN and sequential BBP are less certain about prediction and they create blurred regions where they expect the image to change. This phenomenon is particularly evident in the last row of Figure 5. Table 3 summarizes the performances using metric (13). HMNN performs

overall better than the baselines and it is directly followed by the sequential BBP.

Table 3: Performance on the test set accessed with metric (13) (smaller is better).

Method	Trivial predictor	LSTM	DropConnect	BBP	HMNN
Accuracy	0.2162	0.2080	0.2063	0.1932	0.1891

A Reader's guide to the appendices

The supplementary materials are organized as follows:

- section B includes some mathematical derivations for HMNN;
- section C is a complete description of the experiments' settings.

B Supplementary information about Hidden Markov Neural Network

The full derivation of the **KL**-divergence between the variational approximation and the evolving prior for a general time step t of an HMNN, equation (5), is:

$$\begin{aligned} \mathbf{KL}(q_\theta || \mathcal{C}_t \mathbb{P}\pi_{t-1}) &:= \mathbb{E}_{q_\theta(w)} [\log(q_\theta(w)) - \log(\mathcal{C}_t \mathbb{P}\pi_{t-1}(w))] \\ &= \text{constant} + \mathbb{E}_{q_\theta(w)} [\log(q_\theta(w)) - \log(g(w, \mathcal{D}_t)) - \log(\mathbb{P}\pi_{t-1}(w))] \\ &= \text{constant} + \mathbf{KL}(q_\theta || \mathbb{P}\pi_{t-1}) - \mathbb{E}_{q_\theta(w)} [\log(g(w, \mathcal{D}_t))], \end{aligned} \quad (14)$$

where:

$$\begin{aligned} \log \mathcal{C}_t \mathbb{P}\pi_{t-1}(w) &= \log \left(\frac{g(w, \mathcal{D}_t) \mathbb{P}\pi_{t-1}(w)}{\int g(w, \mathcal{D}_t) \mathbb{P}\pi_{t-1}(w) dw} \right) \\ &= \log(g(w, \mathcal{D}_t)) + \log(\mathbb{P}\pi_{t-1}(w)) - \log \left(\int g(w, \mathcal{D}_t) \mathbb{P}\pi_{t-1}(w) dw \right). \end{aligned} \quad (15)$$

The close form solution of $\mathbb{P}\tilde{\pi}_{t-1}$, equation (12) , for a fully Gaussian HMNN is derived from the following integral:

$$\begin{aligned} (\mathbb{P}\tilde{\pi}_{t-1})^v(w^v) &= \int p(\tilde{w}^v, w^v) (\tilde{\pi}_{t-1})^v(\tilde{w}^v) d\tilde{w}^v \\ &= p^v \pi \int \mathcal{N}(w^v | \mu^v + \alpha(\tilde{w}^v - \mu^v), \sigma^2) \mathcal{N}(\tilde{w}^v | m_{t-1}^v, (s_{t-1}^v)^2) d\tilde{w}^v \\ &\quad + (1 - p^v) \pi \int \mathcal{N}(w^v | \mu^v + \alpha(\tilde{w}^v - \mu^v), \sigma^2) \mathcal{N}(\tilde{w}^v | 0, (s_{t-1}^v)^2) d\tilde{w}^v \\ &\quad + p^v (1 - \pi) \int \mathcal{N}(w^v | \mu^v + \alpha(\tilde{w}^v - \mu^v), \sigma^2/c^2) \mathcal{N}(\tilde{w}^v | m_{t-1}^v, (s_{t-1}^v)^2) d\tilde{w}^v \\ &\quad + (1 - p^v) (1 - \pi) \int \mathcal{N}(w^v | \mu^v + \alpha(\tilde{w}^v - \mu^v), \sigma^2/c^2) \mathcal{N}(\tilde{w}^v | 0, (s_{t-1}^v)^2) d\tilde{w}^v, \end{aligned} \quad (16)$$

the formulation in (12) can be achieved by applying the following lemma to each element of the sum in (16).

Lemma 1. Consider a Gaussian random variable $W \sim \mathcal{N}(\cdot | \mu_1, \sigma_1^2)$ and let:

$$\tilde{W} = \mu_2 - \alpha(\mu_2 - W) + \sigma_2^2 \xi, \quad (17)$$

with $\xi \sim \mathcal{N}(\cdot | 0, 1)$. Then the distribution of \tilde{W} is again Gaussian:

$$\tilde{W} \sim \mathcal{N}(\cdot | \mu_2 - \alpha(\mu_2 - \mu_1), \sigma_2^2 + \alpha^2 \sigma_1^2). \quad (18)$$

Proof. Note that the distribution $p(\tilde{w}|w)$ of $\tilde{W}|W$ is a Gaussian distribution, so $p(\tilde{w}) = \int p(\tilde{w}|w)p(w)dw$ is an integral of the same form of the ones in (16). The distribution of \tilde{W} can be directly computed by noting that \tilde{W} is a linear combination of two Gaussians and a scalar, and consequently it is Gaussian itself. The lemma is proved by computing the straightforward mean and variance from formulation (17). \square

C Experiments

This section presents a subsection per each experiment:

- subsection C.1 treats the experiment on Variational DropConnect;
- subsection C.2 describes the application to the evolving classifier;
- subsection C.3 considers the video texture of a waving flag.

Unless we specify differently, we train using vanilla gradient descent with learning rate γ .

C.1 Variational DropConnect: supplementary information

We train on the MNIST [LeCun et al., 1998] dataset consisting of 60000 images of handwritten digits with size 28 by 28. The images are preprocessed by dividing each pixel by 126. We use 50000 images for training and 10000 for validation. The test set is composed by 10000 images. Both training and test can be downloaded from “<http://yann.lecun.com/exdb/mnist/>” .

As for Blundell et al. [2015] we focus on an ordinary feed-forward neural network without any convolutional layers. We consider a small architecture with the vectorized image as input, two hidden layers with 400 rectified linear units [Nair and Hinton, 2010, Glorot et al., 2011] and a softmax layer on 10 classes as output. We consider a cross-entropy loss and a fully Gaussian HMNN with a single time step. The Variational Dropconnect technique is applied only to the internal linear layers of the network, i.e. we are excluding the initial layer and the final one as in LeCun et al. [1998]. Observe that a single time step HMNN with $p = 1, \alpha = 0$ and $\mu = \mathbf{0}$ (vector of zeros) is equivalent to Bayes by Backprop. For this reason, we can simply use our implementation of HMNN to include Bayes by Backprop. We set $T = 1, \alpha = 0, \mu = \mathbf{0}$ and we consider $p \in \{0.25, 0.5, 0.75, 1\}, \pi \in \{0.25, 0.5, 0.75\}, -\log(\sigma) \in \{0, 1, 2\}, -\log(c) \in \{6, 7, 8\}$, learning rate $\gamma \in \{10^{-5}, 10^{-4}, 10^{-3}\}$.

We generate more than 50 random combinations of the parameters $(p, \pi, \sigma, c, \gamma)$, for each combination we also randomly set the number of monte carlo simulations $N \in \{1, 2, 5\}$. We train each combination for 600 epochs and we consider a minibatch size of 128. We then choose the best three models per each possible value of p and we report the performance on the validation set in Figure 3. Table 1 reports the best performances registered on the test set.

C.2 Evolving classifier on MNIST: supplementary information

The main feature of HMNN is the time dimension, hence we need an example where the data evolves in time. We decide to build this example from the MNIST dataset by simply changing the way in which we assign the labels.

1. We preprocess the data by dividing each pixel by 126. We then define two labellers: \mathcal{C}_1 , naming each digit with its label in MNIST; \mathcal{C}_2 , labelling each digit with its MNIST’s label shifted by one unit, i.e. 0 is classified as 1, 1 is classified as 2, ..., 9 is classified as 0.
2. We consider 19 time steps where each time step t is associated with a probability $f_t \in [0, 1]$ and a portion of the MNIST’s dataset \mathcal{D}_t . The probability of choosing \mathcal{C}_1 evolves as follows:

$$f_t = \frac{1}{2} \sin \left(\frac{\pi}{8} \left(\frac{4t}{5} + \frac{16}{5} \right) \right) + \frac{1}{2}, \quad t = 1, \dots, 19.$$

3. At each time step t we randomly label each digit in \mathcal{D}_t with either \mathcal{C}_1 or \mathcal{C}_2 according to the probabilities $f_t, 1 - f_t$.

The above procedure is used for both training, validation and test sets. The sample size of each time step is 10000 for training and 5000 for validation and test (we resample from the training set of MNIST to reach the desired sample size). To validate and test the models we consider the mean classification accuracy over time:

$$\mathcal{A}(\mathcal{D}_{1:T}, \hat{\mathcal{D}}_{1:T}) := \frac{1}{T} \sum_{t=1}^T \frac{1}{|\mathcal{D}_t|} \sum_{x,y \in \mathcal{D}_t} \mathbb{I}_y(\hat{y}(x)), \quad (19)$$

where \mathcal{D}_t is the generated dataset of images x and labels y , $\hat{\mathcal{D}}_t$ is the collection of images x and predictions $\hat{y}(x)$ on the images x using the considered model, $|\mathcal{D}_t|$ is the number of elements in \mathcal{D}_t , i.e. the total number of labels or images.

We consider a fully Gaussian HMNN with $\mu = m_{t-1}$, to encourage a strong memory on the previous posterior. The evolving in time NN is composed by the vectorize image as input, two hidden layers with 100 rectified linear units and a softmax layer on 10 classes as output, and we consider a cross entropy loss function. The Variational DropConnect technique is again applied to the internal linear layers of the network only. The parameters $\alpha \in \{0.25, 0.5, 0.75, 1\}$ and $p \in \{0.25, 0.5, 0.75, 1\}$ are selected through the validation set (using the metric (19)) while the other parameters are: $\pi = 0.5, -\log(\sigma) = 2, -\log(c) = 4, \gamma = 10^{-3}, N = 1$. We also tried other values for $\pi, \sigma, c, \gamma, N$ but we do not experience significant changes in terms of performances. We try all the possible combinations of α, p and we train on the generated training set for 19 time steps and 100 epochs per each \mathcal{D}_t . Remark that a single \mathcal{D}_t is a collection of images and labels and it can be seen as a whole dataset itself.

We compare our method with four algorithms. The architecture of the NN is the same as for HMNN.

- Sequential Bayes by Backprop. At each time step t we train for 100 epochs on \mathcal{D}_t . The parameters are $\pi = 0.5, -\log(\sigma) = 2, -\log(c) = 4, \gamma = 10^{-3}, N = 1$. The Bayesian NN at time t is initialized with the previous estimates m_{t-1}, s_{t-1} .
- Bayes by Backprop on the whole dataset. We train for 100 epochs on the whole dataset (no time dimension, the sample size is 190000) a Bayesian NN with Bayes by Backprop. The parameters are $\pi = 0.5, -\log(\sigma) = 2, -\log(c) = 4, \gamma = 10^{-3}, N = 1$.
- Elastic Weight Consolidation. At each time step t we train for 100 epochs on \mathcal{D}_t . The tuning parameter is chosen from the grid $\{10, 100, 1000, 10000\}$ through the validation set and the metric (19). We find out that ADAM works better hence we train with it. Remark that this method is not Bayesian, but it is a well-known baseline for continual learning.
- Variational Continual Learning. At each time step t we train for 100 epochs on \mathcal{D}_t . We choose the learning rate from the grid $\{10^{-3}, 10^{-4}, 10^{-5}\}$ through validation and the metric (19). The training is pursued without the use of a coresnet, because we are not comparing rehearsal methods.

Sequential classification accuracies on the validation set are showed in Figure 4 in the main paper. Test performances using (19) are reported in Table 2 in the main paper.

C.3 One-step ahead prediction for flag waving: supplementary information

A video has an intrinsic dynamic given by the sequence of frames, which makes HMNN suitable to one-step ahead prediction video's frame. Here we consider the video of a waving flag.

The preprocessing phase is similar to MNIST: the video is converted in a sequence of frames in grayscale that are additionally divided by 126 and put in a vector form. We then reduce the dimension with PCA (130 principal components). We use 300 frames in total, validate on the frames from 100 to 150 and test on the last 150 frames. Note that we have a single video available, hence we need to perform validation and test online, meaning that validation and test sets are also part of the training, but they are not seen in advance. Precisely, during validation we train on the full path from 1 to 150, we make predictions on the frames from 100 to 150 using HMNN from time 99 to 149 and then we compute (13) on the considered path. Once the validation score is available, we make model-selection and we continue the training on the next frames to get the performance on the test set equivalently. As explained in the main paper, we train sequentially on a sliding window that includes 36 frames.

Consider a fully Gaussian HMNN with a simple architecture of three layers with 500, 20, 500 rectified linear units, the vectorized previous frame as input and the vectorized current frame as output, and an MSE loss. We apply Variational DropConnect to all the linear layers. The parameters $\alpha \in \{0.25, 0.5, 0.75\}$, $p \in \{0.3, 0.8, 1\}$, $\pi = \{0.25, 0.5, 0.75\}$ are chosen according to the validation set. For the other parameters, we find that setting $\log(\sigma) = 2, \log(c) = 8, \gamma = 10^{-4}, N = 1$ performs the best. We train for 150 epochs on each sliding window

We compare with four models. The architectures for sequential Bayes by Backprop and sequential DropConnect are the same of HMNN. For implementation purposes, we consider an architecture of three layers with 500, 500, 500 rectified linear units for the LSTM.

- Sequential Bayes by Backprop. At each time step t we trained on the current sliding window for 150 epochs. The parameter $\pi = \{0.25, 0.5, 0.75\}$ is chosen with grid search and $\log(\sigma) = 2, \log(c) =$

$8, \gamma = 10^{-4}, N = 1$. We find that other choices of σ, c, γ, N do not improve the performance. The Bayesian neural network at time t is initialized with the estimates at time $t - 1$.

- Sequential DropConnect. At each time step t we trained on the current sliding window for 150 epochs. The learning rate $\gamma = \{10^{-3}, 10^{-4}, 10^{-5}\}$ is chosen with grid search using the validation score. The neural network at time t is initialized with the estimates at time $t - 1$. This is not a Bayesian method.
- LSTM. We choose a window size of 36 and we choose the learning rate $\gamma = \{10^{-3}, 10^{-4}, 10^{-5}\}$ with grid search using the validation score. This is not a Bayesian method.
- Trivial predictor. We predict frame t with frame $t - 1$. We decide to include this trivial baseline because it is an indicator of overfitting on the current window.

Table 3 summarizes the performances using metric (13). Given that we extract 30 frames per second, successive frames look almost equal, this makes the comparison with the trivial predictor unfair. Testing on a subpath makes the metric \mathcal{M} more sensitive to big changes and it gives us a clearer measure of the learned patterns. Hence, we test on the subpath 150, 152, 154, … 300 (we skip all the odd frames from 150 to 300). As already explained this is just a subset of the full test set and no algorithm is discarding any frames during training.

References

- Laurence Aitchison. Bayesian filtering unifies adaptive and non-adaptive neural network optimization methods. *arXiv preprint arXiv:1807.07540*, 2018.
- Laurence Aitchison, Alex Pouget, and Peter E Latham. Probabilistic synapses. *arXiv preprint arXiv:1410.1029*, 2014.
- Arslan Basharat and Mubarak Shah. Time series prediction by chaotic modeling of nonlinear dynamical systems. In *2009 IEEE 12th international conference on computer vision*, pages 1941–1948. IEEE, 2009.
- Yoshua Bengio, Régis Cardin, Renato De Mori, and Yves Normandin. A hybrid coder for hidden markov models using a recurrent neural networks. In *International Conference on Acoustics, Speech, and Signal Processing*, pages 537–540. IEEE, 1990.
- Yoshua Bengio, Renato De Mori, Giovanni Flammia, and Ralf Kompe. Global optimization of a neural network-hidden markov model hybrid. In *IJCNN-91-Seattle International Joint Conference on Neural Networks*, volume 2, pages 789–794. IEEE, 1991.
- David M Blei, Alp Kucukelbir, and Jon D McAuliffe. Variational inference: A review for statisticians. *Journal of the American statistical Association*, 112(518):859–877, 2017.
- Charles Blundell, Julien Cornebise, Koray Kavukcuoglu, and Daan Wierstra. Weight uncertainty in neural networks. *arXiv preprint arXiv:1505.05424*, 2015.
- Byron Boots, Geoffrey J Gordon, and Sajid M Siddiqi. A constraint generation approach to learning stable linear dynamical systems. In *Advances in neural information processing systems*, pages 1329–1336, 2008.
- Antoni B Chan and Nuno Vasconcelos. Classifying video with kernel dynamic textures. In *2007 IEEE Conference on Computer Vision and Pattern Recognition*, pages 1–6. IEEE, 2007.
- Lee A Feldkamp, Danil V Prokhorov, and Timothy M Feldkamp. Simple and conditioned adaptive behavior from kalman filter trained recurrent networks. *Neural Networks*, 16(5-6):683–689, 2003.
- Michael Franzini, K-F Lee, and Alex Waibel. Connectionist viterbi training: a new hybrid method for continuous speech recognition. In *International Conference on Acoustics, Speech, and Signal Processing*, pages 425–428. IEEE, 1990.
- Yarin Gal and Zoubin Ghahramani. Dropout as a bayesian approximation: Representing model uncertainty in deep learning. In *international conference on machine learning*, pages 1050–1059, 2016.

- Zoubin Ghahramani and Michael I. Jordan. Factorial hidden markov models. *Machine Learning*, 29(2):245–273, Nov 1997. ISSN 1573-0565. doi: 10.1023/A:1007425814087. URL <https://doi.org/10.1023/A:1007425814087>.
- Xavier Glorot, Antoine Bordes, and Yoshua Bengio. Deep sparse rectifier neural networks. In *Proceedings of the fourteenth international conference on artificial intelligence and statistics*, pages 315–323, 2011.
- Alex Graves. Practical variational inference for neural networks. In *Advances in neural information processing systems*, pages 2348–2356, 2011.
- Diederik P Kingma and Max Welling. Auto-encoding variational bayes, 2013. URL <https://arxiv.org/abs/1312.6114>, 2013.
- Durk P Kingma, Tim Salimans, and Max Welling. Variational dropout and the local reparameterization trick. In *Advances in neural information processing systems*, pages 2575–2583, 2015.
- James Kirkpatrick, Razvan Pascanu, Neil Rabinowitz, Joel Veness, Guillaume Desjardins, Andrei A Rusu, Kieran Milan, John Quan, Tiago Ramalho, Agnieszka Grabska-Barwinska, et al. Overcoming catastrophic forgetting in neural networks. *Proceedings of the national academy of sciences*, 114(13):3521–3526, 2017.
- Anders Krogh and Soren Kamaric Riis. Hidden neural networks. *Neural Computation*, 11(2):541–563, 1999.
- Anders Krogh, BjoÈrn Larsson, Gunnar Von Heijne, and Erik LL Sonnhammer. Predicting transmembrane protein topology with a hidden markov model: application to complete genomes. *Journal of molecular biology*, 305(3):567–580, 2001.
- Yann LeCun, Léon Bottou, Yoshua Bengio, Patrick Haffner, et al. Gradient-based learning applied to document recognition. *Proceedings of the IEEE*, 86(11):2278–2324, 1998.
- Aryan Mobiny, Hien V Nguyen, Supratik Moulik, Naveen Garg, and Carol C Wu. Dropconnect is effective in modeling uncertainty of bayesian deep networks. *arXiv preprint arXiv:1906.04569*, 2019.
- Vinod Nair and Geoffrey E Hinton. Rectified linear units improve restricted boltzmann machines. In *Proceedings of the 27th international conference on machine learning (ICML-10)*, pages 807–814, 2010.
- Cuong V Nguyen, Yingzhen Li, Thang D Bui, and Richard E Turner. Variational continual learning. *arXiv preprint arXiv:1710.10628*, 2017.
- Yann Ollivier et al. Online natural gradient as a kalman filter. *Electronic Journal of Statistics*, 12(2):2930–2961, 2018.
- Manfred Opper and Cédric Archambeau. The variational gaussian approximation revisited. *Neural computation*, 21(3):786–792, 2009.
- Gintaras V Puskorius and Lee A Feldkamp. Decoupled extended kalman filter training of feedforward layered networks. In *IJCNN-91-Seattle International Joint Conference on Neural Networks*, volume 1, pages 771–777. IEEE, 1991.
- Gintaras V Puskorius and Lee A Feldkamp. Neurocontrol of nonlinear dynamical systems with kalman filter trained recurrent networks. *IEEE Transactions on neural networks*, 5(2):279–297, 1994.
- Gintaras V Puskorius and Lee A Feldkamp. Parameter-based kalman filter training: Theory and implementation. *Kalman filtering and neural networks*, page 23, 2001.
- Lawrence R Rabiner and Biing-Hwang Juang. An introduction to hidden markov models. *ieee assp magazine*, 3(1):4–16, 1986.
- Patrick Rebeschini, Ramon Van Handel, et al. Can local particle filters beat the curse of dimensionality? *The Annals of Applied Probability*, 25(5):2809–2866, 2015.
- Danilo Jimenez Rezende, Shakir Mohamed, and Daan Wierstra. Stochastic backpropagation and approximate inference in deep generative models. *arXiv preprint arXiv:1401.4082*, 2014.

Lorenzo Rimella and Nick Whiteley. Exploiting locality in high-dimensional factorial hidden markov models. *arXiv preprint arXiv:1902.01639*, 2019.

Hippolyt Ritter, Aleksandar Botev, and David Barber. Online structured laplace approximations for overcoming catastrophic forgetting. In *Advances in Neural Information Processing Systems*, pages 3738–3748, 2018.

Samir Shah, Francesco Palmieri, and Michael Datum. Optimal filtering algorithms for fast learning in feedforward neural networks. *Neural networks*, 5(5):779–787, 1992.

Nitish Srivastava, Geoffrey Hinton, Alex Krizhevsky, Ilya Sutskever, and Ruslan Salakhutdinov. Dropout: a simple way to prevent neural networks from overfitting. *The journal of machine learning research*, 15(1):1929–1958, 2014.

Arun Venkatraman, Martial Hebert, and J Andrew Bagnell. Improving multi-step prediction of learned time series models. In *Twenty-Ninth AAAI Conference on Artificial Intelligence*, 2015.

Li Wan, Matthew Zeiler, Sixin Zhang, Yann Le Cun, and Rob Fergus. Regularization of neural networks using dropconnect. In *International conference on machine learning*, pages 1058–1066, 2013.