

$$\sinh(x+y) = \frac{e^{x+y} - e^{-x-y}}{2} = \frac{1}{4} (2e^{x+y} - 2e^{-x-y})$$

$$= \frac{1}{4} (e^{x+y} - e^{-x-y} + e^{y-x} - e^{x-y} + e^{x+y} - e^{-x-y} - e^{y-x} + e^{x-y})$$

$$= \frac{1}{4} (e^{x+y} - e^{-x-y} + e^{y-x} - e^{x-y}) + \frac{1}{4} (e^x - e^{-x})(e^y + e^{-y})$$

$$= \cosh x \sinh y + \sinh x \cosh y$$

$$\begin{aligned} \cosh(x+y) &= \frac{1}{4} (e^{x+y} + e^{-x-y} + e^{y-x} + e^{x-y}) \\ &\quad + \frac{1}{4} (e^{x+y} + e^{-x-y} - e^{y-x} - e^{x-y}) \\ &= \cosh(x) \cosh(y) + \sinh(x) \sinh(y) \end{aligned}$$