2006 Final

Problem 1

Proof (by ind)

I.H. $P(n) ::= \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ ✓

B.C. $P(0)$ $0=0$ ✓

I.S. $\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{n(n+1)(2n+1) + 6(n+1)^2}{6}$

$= \frac{(n+1)(n(2n+1) + 6(n+1))}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$ ✓

Problem 2

□

(a) Proof. by ind

I.H. $P(\cancel{m}^{k}) ::= \forall 0 \le \cancel{j}^{l} \le k,\ \forall l < j \le n\ P_l \to P_j$ ✓

B.C. $P(0)$. Since it has 0 wins, everybody beats them. ✓

I.S. Since $P(k)$, ~~k+1~~ nobody until up k beats k+1. ✓

□

(b) for any fixed permutation, $\left(\frac{1}{2}\right)^{\binom{n}{2}} \to n!\left(\frac{1}{2}\right)^{\binom{n}{2}}$

(c) $n!\,\frac{1}{2}^{\binom{n}{2}} \le n^n\,\frac{1}{2}^{\binom{n}{2}} = 2^{n\log n}\,\frac{1}{2}^{\binom{n}{2}} = \frac{1}{2}^{\binom{n}{2}-n\log n} \le \frac{1}{2}^{cn^2}$

$\binom{n}{2} - n\log n = \Omega(n^2)$

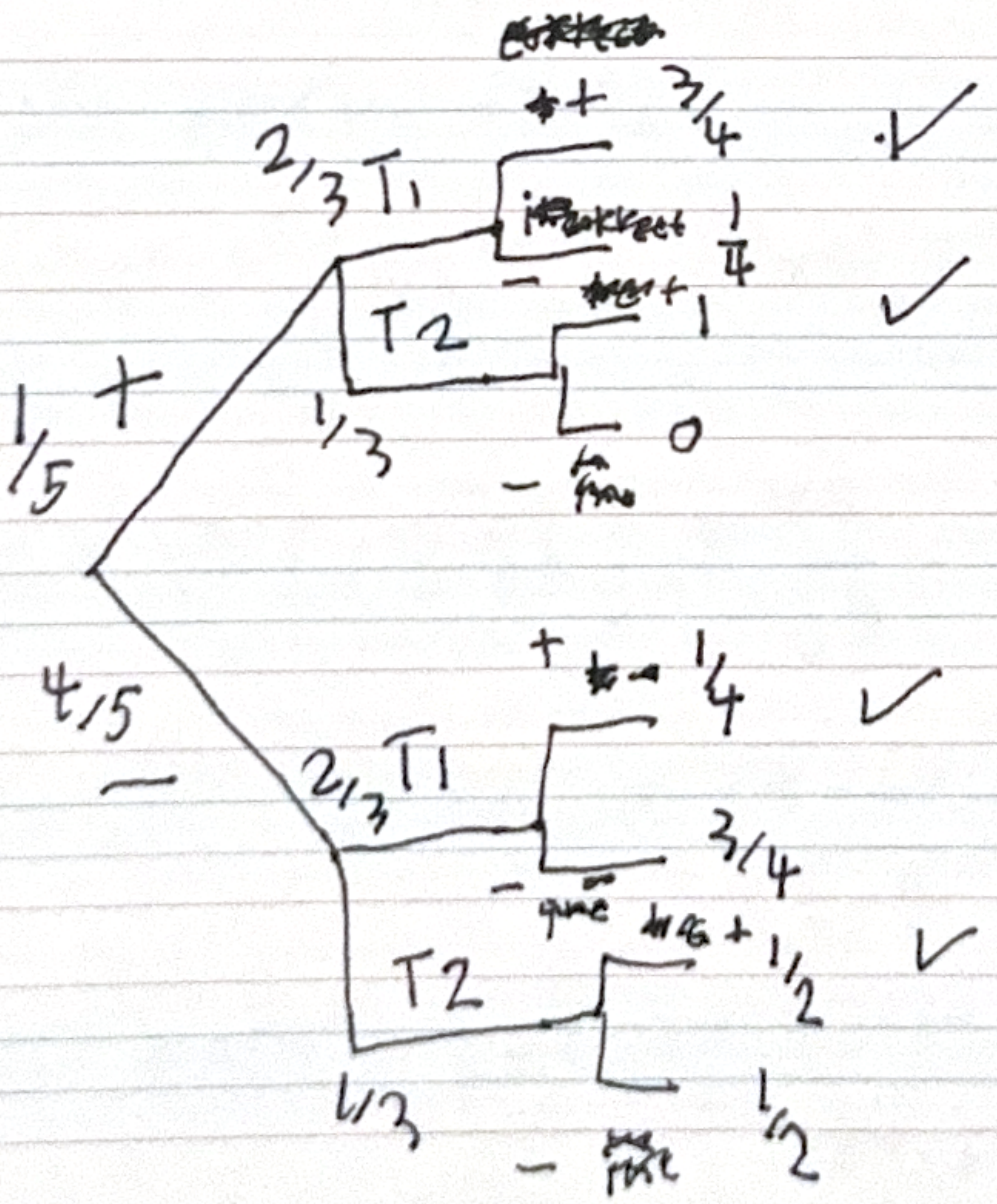
Problem 3

$$P_n = \frac{1}{2}(P_{n-3}) + \frac{1}{2}(P_{n-4})$$

$P_0 = 1 \qquad P_1 = 0$

$P_3 = \frac{1}{2} \qquad P_2 = 0$

Problem 4



$$\frac{\frac{1}{5}\cdot\frac{3}{4}\cdot\frac{2}{3} + \frac{1}{3}\cdot\frac{1}{5}}{\frac{1}{5}\cdot\frac{3}{4}\cdot\frac{2}{3} + \frac{1}{3}\cdot\frac{1}{5} + \frac{1}{4}\cdot\frac{2}{3}\cdot\frac{4}{5} + \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{4}{5}}$$

$$= \frac{5}{13}$$

Problem 5

(a) $\dfrac{104\cdot102\cdot100\cdot98}{\binom{104}{5}}$ $\qquad\binom{52}{5}2^5$

(b) $\dfrac{104\cdot1\cdot102\cdot\overset{100}{\cancel{\ \ }}}{\binom{104}{5}}$ $\qquad 52\binom{52}{3}2^3$

Problem 6

(a) ~~[illegible]~~ $\dfrac{(50+98)}{49}\dfrac{49}{2} = 74$

(b) $E\cancel{x}p = E[S]$

$S = \frac{1}{128}(S_1+\dots+S_{128}) \to \frac{74\cdot128}{128} = Ex[S]$

(c) $\Pr[S \geq 88] \leq \frac{74}{88} = \frac{37}{44}$ by markov

(d) $\Pr[S \geq 88] \leq \frac{74-50}{88-50} = \frac{24}{38} = \frac{12}{19}$

(e) $Ex(S^2) - Ex^2(S) = \frac{48\cdot49\cdot97}{49\cdot6} - 24^2 = 200$

(f) $\frac{200 \cdot 128}{128^2} = \frac{200}{128}$

(g) $= \frac{5}{4}$

(h) $Pr[S \leq 69] \leq Pr[|S-74| \leq \overset{4.6}{5}] = \frac{1}{16}$

Problem 7

(a) $T_1 + \cdots + T_{1000} = \frac{1000}{4} = 250$

(b) $\sum_{i=1}^{1000} \frac{1}{4} S_i = \frac{1}{4} 400 = 100$

(c) $Pr(X \geq \underbrace{c\,Ex(X)}_{2 \cdot 100}) \leq e^{-(2\ln 2 - 2 + 1)100}$

(d) $Pr[X] \leq 4 \cdot e^{-(2\ln 2 - 1)100}$