2006 Final

Problem 1

Proof (by ind)

I.H. $P(n) ::= \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ ✓

B.C. $P(0)$ $0=0$ ✓

I.S. $\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{n(n+1)(2n+1) + 6(n+1)^2}{6}$

$= \frac{(n+1)(n(2n+1) + 6(n+1))}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$ ✓

$\square$

Problem 2

(a) Proof. by ind

I.H. $P(k) ::= \forall\, 0 \leq \cancel{j}^{\,l} \leq k,\ \forall\, l < j \leq n\ \ P_l \rightarrow P_j$ ✓

B.C. $P(0)$. Since it has 0 wins, everybody beats them. ✓

I.S. Since $P(k)$, ~~k~~ nobody up until k beats k+1. ✓

$\square$

(b) for any fixed permutation, $\left(\frac{1}{2}\right)^{\binom{n}{2}} \to n!\left(\frac{1}{2}\right)^{\binom{n}{2}}$

(c) $n!\,\frac{1}{2}^{\binom{n}{2}} \le n^n\,\frac{1}{2}^{\binom{n}{2}} = 2^{n\log n}\,\frac{1}{2}^{\binom{n}{2}} = \frac{1}{2}^{\binom{n}{2} - n\log n} \le \frac{1}{2}^{cn^2}$

$\binom{n}{2} - n\log n = \Omega(n^2)$

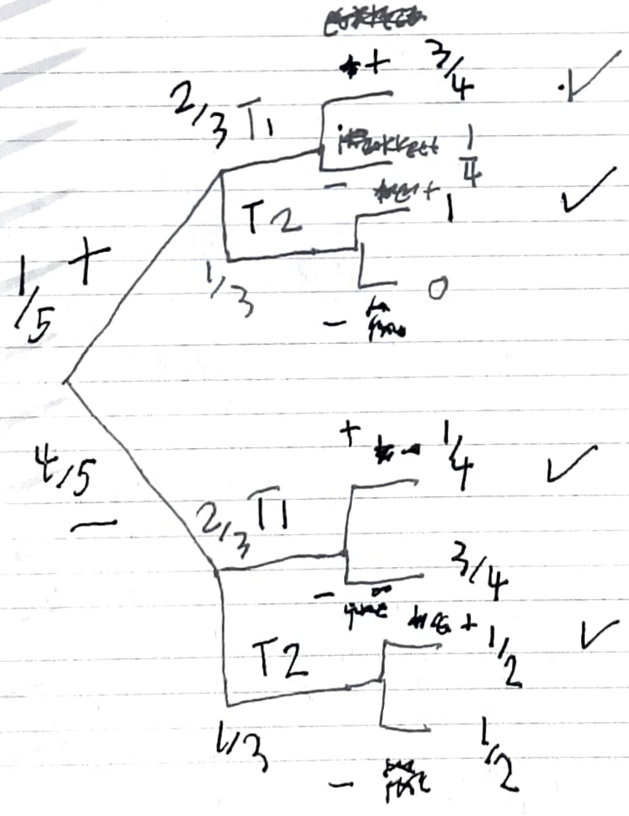
Problem 3

$$P_n = \frac{1}{2}(P_{n-3}) + \frac{1}{2}(P_{n-4})$$

$P_0 = 1 \qquad P_1 = 0$

$P_3 = \frac{1}{2} \qquad P_2 = 0$

Problem 4



$$\frac{\frac{1}{5}\cdot\frac{3}{4}\cdot\frac{2}{3} + \frac{1}{3}\cdot\frac{1}{5}}{\frac{1}{5}\cdot\frac{3}{4}\cdot\frac{2}{3} + \frac{1}{3}\cdot\frac{1}{5} + \frac{1}{4}\cdot\frac{2}{3}\cdot\frac{4}{5} + \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{4}{5}}$$

$$= \frac{5}{13}$$

Problem 5

(a) $\dfrac{104 \cdot 102 \cdot 100 \cdot 98}{\binom{104}{5}}$ $\quad\binom{52}{5} 2^5$

(b) $\dfrac{104 \cdot 1 \cdot 102 \cdot 100}{\binom{104}{5}}$ $\quad 52 \binom{52}{3} 2^3$

Problem 6

(a) $\cancel{\frac{128 \cdot 129}{2 \cdot 128}}$ # $\cancel{\frac{149}{2}}$ $\cancel{1494}$ $\dfrac{(50+98)}{49} \dfrac{49}{2} = 74$

(b) $E[\cancel{S}] = E[S]$

$S = \frac{1}{128}(S_1 + \dots + S_{128}) \to \frac{74 \cdot 128}{128} = Ex(S)$

(c) $Pr[S \geq 88] \leq \frac{74}{88} = \frac{37}{44}$ by markov

(d) $Pr(S \geq 88] \leq \frac{74-50}{88-50} = \frac{24}{38} = \frac{12}{19}$

(e) $Ex(S^2) - Ex^2(S) = \frac{48 \cdot 49 \cdot 97}{49 \cdot 6} - 24^2 = 200$

(f) $\frac{200 \cdot 128}{128^2} = \frac{200}{128}$

(g) $= \frac{5}{4}$

(h) $Pr[S \leq 69] \leq Pr[|S - 74| \leq 5] = \frac{4 \cdot 6}{1} \quad \frac{1}{16}$

Problem 7

(a) $T_1 + \cdots + T_{1000} = \frac{1000}{4} = 250$

(b) $\sum_{i=1}^{1000} \frac{1}{4} S_i = \frac{1}{4} 400 = 100$

(c) $Pr(X \geq c \underbrace{Ex(X)}_{2 \cdot 100}) \leq e^{-(2\ln 2 - 2 + 1)100}$

(d) $Pr[X] \leq 4 \cdot e^{-(2\ln 2 - 1)100}$