Assignment #4

Linear Algebra, Fall 2020

Due: December 12, 2022
(15 questions in 3 pages)

1. Suppose S is spanned by the vectors $(1, 2, 2, 3)$ and $(1, 3, 3, 2)$. Find two vectors that span $S^{\perp}$. This is the same as solving $\mathbf{Ax} = \mathbf{0}$ for which $\mathbf{A}$?

ex. $(-5, 0, 1, 1), (0, 1, -1, 0)$ span $S^{\perp}$ = null space of $A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix}$

2. Fill in the blanks: Suppose V is the whole space $\mathbb{R}^4$. Then, $V^{\perp}$ contains only the vector zero vector. Then $(V^{\perp})^{\perp}$ is V. So $(V^{\perp})^{\perp}$ is the same as $\mathbb{R}^4$.

3. Project the vector $\mathbf{b}$ onto the line through $\mathbf{a}$. Check that $\mathbf{e} = \mathbf{b} - \mathbf{p}$ is perpendicular to $\mathbf{a}$. Find the projection matrix $\mathbf{P} = \mathbf{aa}^{\mathrm{T}}/\mathbf{a}^{\mathrm{T}}\mathbf{a}$ onto the line through the vector $\mathbf{a}$. Verify that $\mathbf{P}^2 = \mathbf{P}$. Multiply $\mathbf{Pb}$ to compute the projection $\mathbf{p}$.

Solution is on below

(a) $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $\mathbf{a} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$.

(b) $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

4. Project $\mathbf{b}$ onto the column space of $\mathbf{A}$ by solving $\mathbf{A}^{\mathrm{T}}\mathbf{A}\hat{\mathbf{x}} = \mathbf{A}^{\mathrm{T}}\mathbf{b}$ and $\mathbf{p} = \mathbf{A}\hat{\mathbf{x}}$. Find $\mathbf{e} = \mathbf{b} - \mathbf{p}$. Find the projection matrix $\mathbf{P}$ onto the column space of $\mathbf{A}$.

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

(b) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$.

5. Suppose $\mathbf{A}$ is the 4 by 4 identity matrix with its last column removed. $\mathbf{A}$ is 4 by 3. Project $\mathbf{b} = (1, 2, 3, 4)$ onto the column space of $\mathbf{A}$. What shape is the projection matrix $\mathbf{P}$ and what is $\mathbf{P}$?

6. To find the projection matrix onto the plane $x - y - 2z = 0$, choose two vectors in that plane and make them the columns of $\mathbf{A}$. The plane should be the column space. Then, compute $\mathbf{P} = \mathbf{A}(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}$.

7. Write down three equations for the line $b = C + Dt$ to go through $b = 7$ at $t = -1$, $b = 7$ at $t = 1$, and $b = 21$ at $t = 2$. Find the least squares solution $\hat{\mathbf{x}} = (C, D)$ and draw the closest line.

8. Find the best line $C + Dt$ to fit $b = 4, 3, 1, 0$ at times $t = -2, -1, 0, 2$.

9. Find orthogonal vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ by Gram-Schmidt from $\mathbf{a}, \mathbf{b}, \mathbf{c}$:
$$\mathbf{a} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

10. Find an orthonormal basis for the column space of $\mathbf{A}$:
$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -4 \\ -3 \\ 3 \\ 0 \end{bmatrix}.$$
Then compute the projection of $\mathbf{b}$ onto that column space.

11. Find $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ (orthonormal) as combinations of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ (independent columns). Then write $\mathbf{A}$ as $\mathbf{QR}$:
$$\mathbf{A} = \begin{bmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

12. If a 3 by 3 matrix $\mathbf{A}$ has $\det \mathbf{A} = -1$, compute the followings:

    (a) $\det(\frac{1}{2}\mathbf{A})$

    (b) $\det(-\mathbf{A})$

    (c) $\det(\mathbf{A}^2)$

    (d) $\det(\mathbf{A}^{-1})$

13. By applying row operations to produce an upper triangular $\mathbf{U}$, compute
$$\det \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix} \quad \text{and} \quad \det \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

14. We say that $\mathbf{K}$ is skew-symmetric if $\mathbf{K}^{\mathrm{T}} = -\mathbf{K}$. Show that $\det \mathbf{K} = 0$ if $\mathbf{K}$ is a 3×3 skew-symmetric matrix.

15. Compute the determinants of $\mathbf{A}$ and $\mathbf{B}$ from the big formula introduced in Section 5.2. Are their columns independent?

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

3.

a) $a=\begin{bmatrix}2\\-1\\2\end{bmatrix}$ $b=\begin{bmatrix}1\\3\\1\end{bmatrix}$

$\Rightarrow p=\dfrac{a^Tb}{a^Ta}=\dfrac{1}{9} \Rightarrow P_a=p=\dfrac{1}{9}a$

$\rightsquigarrow e=b-p=\begin{bmatrix}1\\3\\1\end{bmatrix}-\begin{bmatrix}\frac{2}{9}\\-\frac{1}{9}\\\frac{2}{9}\end{bmatrix}$

$=\begin{bmatrix}\frac{7}{9}\\\frac{28}{9}\\\frac{7}{9}\end{bmatrix}$

$\rightsquigarrow e\cdot a=\begin{bmatrix}\frac{7}{9}\\\frac{28}{9}\\\frac{7}{9}\end{bmatrix}\begin{bmatrix}2\\-1\\2\end{bmatrix}=0$

$\therefore e\perp a.$

b) $a=\begin{bmatrix}1\\2\\3\end{bmatrix}$ $b=\begin{bmatrix}1\\2\\2\end{bmatrix}$

$\Rightarrow P=\dfrac{a^Tb}{a^Ta}=\dfrac{11}{14} \Rightarrow P_a=p=\dfrac{11}{14}a$

$\rightsquigarrow e=b-p=\begin{bmatrix}1\\2\\2\end{bmatrix}-\begin{bmatrix}\frac{11}{14}\\\frac{11}{7}\\\frac{33}{14}\end{bmatrix}$

$=\begin{bmatrix}\frac{3}{14}\\\frac{3}{7}\\-\frac{5}{14}\end{bmatrix}$

$\rightsquigarrow e\cdot a=\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}\frac{3}{14}\\\frac{3}{7}\\-\frac{5}{14}\end{bmatrix}$

$=\dfrac{3}{14}\times 5-3\times\dfrac{5}{14}=0$

$\therefore e\perp a.$

4. a) $A=\begin{bmatrix}1&2\\1&3\\0&0\end{bmatrix}$ $b=\begin{bmatrix}4\\5\\6\end{bmatrix}$

$A^TA=\begin{bmatrix}2&5\\5&13\end{bmatrix}$ $A^Tb=\begin{bmatrix}9\\23\end{bmatrix}$

$\begin{bmatrix}2&5\\5&13\end{bmatrix}\hat{x}=\begin{bmatrix}9\\23\end{bmatrix}\Rightarrow \hat{x}=\begin{bmatrix}2\\1\end{bmatrix}$

$p=2\begin{bmatrix}1\\1\\0\end{bmatrix}+1\begin{bmatrix}2\\3\\0\end{bmatrix}=\begin{bmatrix}4\\5\\0\end{bmatrix}$

$e=b-p=\begin{bmatrix}0\\0\\-6\end{bmatrix}$

$(0,0,-6)$ is perpendicular to $(1,1,0)$ and $(2,3,0)$

$\therefore P=A(A^TA)^{-1}A^T=\begin{bmatrix}74&105&0\\105&149&0\\0&0&0\end{bmatrix}$

b) $A=\begin{bmatrix}1&1\\1&-1\\-2&4\end{bmatrix}$ $b=\begin{bmatrix}1\\2\\7\end{bmatrix}$

$A^TA=\begin{bmatrix}6&-8\\-8&18\end{bmatrix}$ $A^Tb=\begin{bmatrix}-11\\27\end{bmatrix}$

$\begin{bmatrix}6&-8\\-8&18\end{bmatrix}\hat{x}=\begin{bmatrix}-11\\27\end{bmatrix}$ $\hat{x}=\begin{pmatrix}\frac{9}{22}\\\frac{37}{22}\end{pmatrix}$

$p=A\hat{x}=\begin{bmatrix}\frac{23}{11}\\-\frac{14}{11}\\\frac{65}{11}\end{bmatrix}$ $e=b-p=\begin{bmatrix}-\frac{12}{11}\\\frac{36}{11}\\\frac{12}{11}\end{bmatrix}$

$(-\frac{12}{11},\frac{36}{11},\frac{12}{11})$ is perpendicular to $(1,1,-2)$ and $(1,-1,4)$

$P=A(A^TA)^{-1}A^T=\begin{bmatrix}8&-12&44\\-12&40&-132\\44&-132&440\end{bmatrix}$

5. $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\rightsquigarrow$ Column space.

$AA^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $Ab = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \hat{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \hat{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$p = A^T \hat{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} \Rightarrow e = b - p = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix}$

$P = A^T (AA^T)^{-1} A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\rightsquigarrow$ P is square matrix.

6. $x - y - 2z = 0 \Rightarrow (1,1,0)\ (2,0,1)$

$\rightsquigarrow P = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{6} & \frac{5}{6} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$

7. $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$

$\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix}$

$\rightsquigarrow \hat{x} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$

8. $b = C + Dt$. // $b = 4, 3, 1, 0$ $t = -2\ -1\ 0\ 2$

$Ax = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix}$

$A^T A \hat{x} = A^T b$.

$\rightsquigarrow \begin{bmatrix} 4 & -1 \\ -1 & 9 \end{bmatrix} \hat{x} = \begin{bmatrix} 8 \\ -11 \end{bmatrix}$

$\rightsquigarrow C = \frac{61}{35}$ $D = \frac{-36}{35}$.

9. $a = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ $b = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ $c = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$

Set $A = a$. $A^T A = 2$.

Compute $B = b - \frac{A^T b}{A^T A} A = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ 0 \end{bmatrix}$

$C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B$ $\to -1$

$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ -1 \end{bmatrix}$ $\frac{16+2}{9} + 1$

$\rightsquigarrow q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ $q_2 = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ 0 \end{bmatrix}$

$q_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{4}{3} \\ -1 \end{bmatrix}$

10. $A=\begin{bmatrix}1 & -2\\1 & 0\\1 & 1\\1 & 3\end{bmatrix}$ $b=\begin{bmatrix}-4\\-3\\3\\0\end{bmatrix}$

$q_1=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

$B=\begin{bmatrix}-2\\0\\1\\3\end{bmatrix}-\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}-5\\-1\\1\\5\end{bmatrix}$

$\Rightarrow q_2=\frac{1}{2\sqrt{13}}\begin{bmatrix}-5\\-1\\1\\5\end{bmatrix}$

$\Rightarrow b=\begin{bmatrix}-4\\-3\\3\\0\end{bmatrix} \rightsquigarrow p=\frac{1}{2}\begin{bmatrix}-7\\-3\\-1\\3\end{bmatrix}$

$e=b-p=\frac{1}{2}\begin{bmatrix}-1\\-3\\7\\-3\end{bmatrix}$

e is orthogonal to q_1 and q_2

11. $A=\begin{bmatrix}a & b & c\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\1 & 1 & 0\\1 & 1 & 1\end{bmatrix}$

$q_1=\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$

$\begin{bmatrix}0\\1\\1\end{bmatrix}-\frac{2}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}-\frac{2}{3}\\\frac{1}{3}\\\frac{1}{3}\end{bmatrix}$

$\Rightarrow q_2=\frac{1}{2}\begin{bmatrix}-2\\1\\1\end{bmatrix}$

$\begin{bmatrix}0\\0\\1\end{bmatrix}-\frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}-\frac{1}{2}\begin{bmatrix}-\frac{2}{3}\\\frac{1}{3}\\\frac{1}{3}\end{bmatrix}$

$=\begin{bmatrix}0\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix} \rightsquigarrow q_3=\frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\end{bmatrix}$

$Q=\begin{bmatrix}\frac{1}{\sqrt{3}} & -1 & 0\\\frac{1}{\sqrt{3}} & \frac{1}{2} & -\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{3}} & \frac{1}{2} & \frac{1}{\sqrt{2}}\end{bmatrix}$

$R=\begin{bmatrix}\sqrt{3} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\0 & \frac{2}{3} & \frac{2}{3}\\0 & 0 & \frac{1}{2}\end{bmatrix}$

12. 1) $\det(\frac{1}{2}A)=\frac{1}{8}\det A$

$=-\frac{1}{8}$

2) $\det(-A)=(-1)^3\det A$

$=1$

3) $\det(A^2)=1.$

4) $\det(A^{-1})=-1$

13. $\det\begin{bmatrix}1 & 2 & 3 & 0\\2 & 6 & 6 & 1\\-1 & 0 & 0 & 3\\0 & 2 & 0 & 7\end{bmatrix}$

$=\begin{bmatrix}1 & 2 & 3 & 0\\0 & 2 & 0 & 1\\0 & 0 & 3 & 2\\0 & 0 & 0 & 6\end{bmatrix}$

$=6\begin{bmatrix}1 & 2 & 3 & 0\\0 & 2 & 0 & 1\\0 & 0 & 3 & 2\\0 & 0 & 0 & 1\end{bmatrix}$

$=36$

$$\det \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$= \det \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}$$

$$= 5.$$

14. $K^T = -K \xrightarrow{\text{show}} \det K = 0.$

Let $K = \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix}$ $(a, b, c \in \mathbb{R})$

$$\det K = \det \begin{bmatrix} a & 0 & -c \\ 0 & -a & -b \\ b & c & 0 \end{bmatrix}$$

$$= \det \begin{bmatrix} a & 0 & -c \\ 0 & -a & -b \\ 0 & 0 & \frac{bc}{a} - \frac{cb}{a} \end{bmatrix}$$

$$= 0$$

$\therefore$ For every skew-symmetric matrix K,

$$\det K = 0$$

15.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{matrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{matrix}$$

$$\det A = 1(0-1) + 1(0-1) + 0(1-0)$$

$$= -2$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\det B = (45-48) + 2(42-36) + 3(32-35)$$

$$= -3 + 12 - 9$$

$$= 0$$