Envelopes and their mathematical applications with reflections

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1 Introduction

In this report we will be exploring the nature of Envelopes and their applications in calculating mathematical reflections, know as **Caustics**. These calculations are not completely accurate representation of real life waves as they ignore the wave structure of light, but they give very good approximation for the caustic produced.

To produce the upcoming graphs I used software know as Maple, which plots a changing parameter, **t**, applied the family of lines. This produces multiple lines which are plotted simultaneously to produce the envelope. Some graphs also have lines of certain length, instead of infinite, to make them easier to read.

Also, when using my algorithms I sometimes $\pm$ 0.01 to my equations to avoid any division by zero, this allows me to be allows me to plot accurate approximations of the envelopes.

2 Envelopes

2.1 What is an Envelope of a Curve?

In the context of curves, an envelope of a family of curves is a curve that is tangent to each member of the family at some point. A **family of lines** is a set of lines that have something in common and are of the form:

$$A(t)x + B(t)y + C(t) = 0$$

As can been seen A,B,C are functions of the parameter **t** and it should be noted that $A(t)$ and $B(t)$ are never both equal to zero at the same value of **t**.

In the figure below we can see how a curve is formed as the multiple tangents overlap, this curve being the envelope.

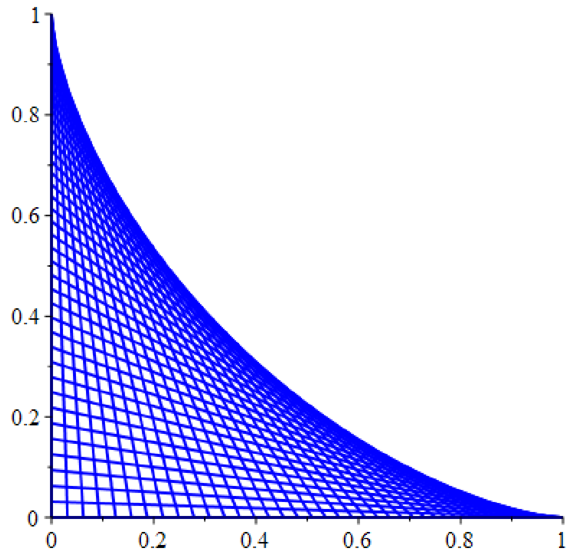


Figure 1: Envelope for family of lines from $(\sin(t), 0)$ to $(0, \cos(t))$

2.2 How do we find the Envelope?

To find the general equation of a envelope we simutaneouly solve the derivative and the equation to find x and y, this will give the general point.

$$y = tx - t^2$$
$$0 = x - 2t$$
$$x = 2t$$
$$y = t^2$$
$$E = (2t, t^2)$$

We can find the equation of the evnvelope by finding y in terms of x

$$y = T^2 = \text{relationship to x} * (2t)$$

$$y = \frac{(2t)^2}{4}$$

To give us the equation of the envelope:

$$y = \frac{x^2}{4}$$

3 Evolute

3.1 What is an Evolute?

The envolute is a type of envelope, specifically it is the evelope of the normals to a given curve.

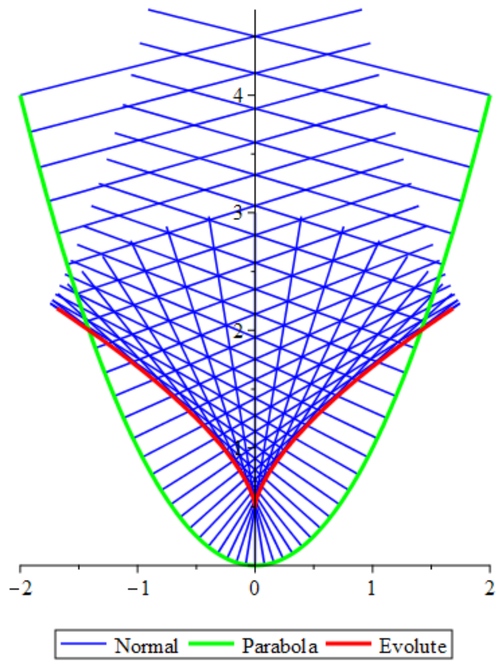
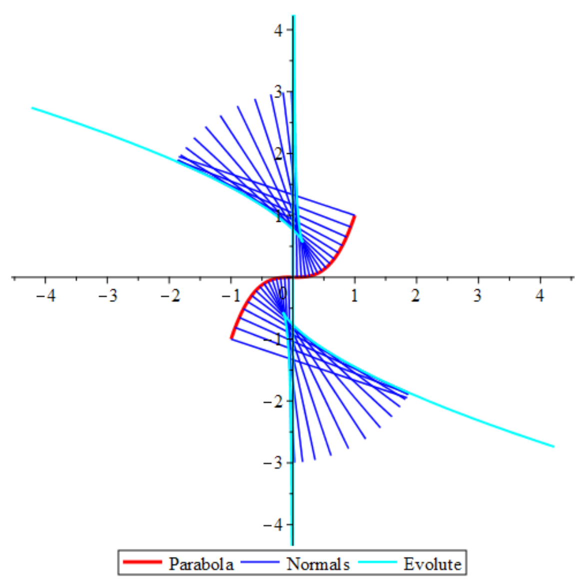


Figure 2: For $y = x^2$

On the evolute, at the point (0,0.5) we can see a unqiue feature which is called a **Cusp**. These cusps are points of maximun curvature,k, on the parabola, in *figure 2* the curvature increase going towards (0,0) and then decreases when it goes past it. Cusps can be caluclated as they are only found when...

$$x' = y' = 0$$

Here is an other example for the curve $y = x^3$ where we can also see two cusps.



3.2 Using Vectors

To find the evolute we have to use vectors methods, specifically unit vectors and the dot product, as we are primarily working with the tangent and normals A **unit vector** is a vector that has a overall distance equal to 1

$$\text{Unit Vector} = \frac{\begin{pmatrix} a \\ b \end{pmatrix}}{\sqrt{a^2 + b^2}}$$

The **Dot Product** of two vector is..

$$A \cdot B = \begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} c \\ b \end{pmatrix} = ac + bd$$

And the **Dot Product Rule**...

$$A \cdot B = |A| \times |B| \times \cos(\theta)$$

Where θ is the angle between the vectors

Specifically, we can use the dot product rule to determine if two vectors are perpendicular to each other, as when vector A and B are perpendicular, the value of θ will be equal to 0. Therefore

$$A \cdot B = 0$$

Lastly, using the dot product rule and unit vectors we can prove the derivative of a vector is it's own perpendicular

$$T \cdot T = 1$$

$$T \cdot T' + T' \cdot T = 0$$

$$2(T \cdot T') = 0$$

$$T \cdot T' = 0$$

3.3 Finding the evolute

We use these vectors methods to find the equation of the line...

Let be $\gamma(t)$ general point on the line

$$\text{Tangent Vector} = \gamma' = (x', y')$$

$$\text{Normal to the vector } \gamma = (-y', x')$$

We need to convert the vectors into unit vectors so that we can manipulate our dot products

$$\text{Speed of vector} = s = \sqrt{x'^2 + y'^2}$$

$$\text{Therefor Unit Tangent} = \mathrm{T} = \frac{\gamma'}{s} = \frac{(x', y')}{\sqrt{x'^2 + y'^2}}$$

$$\text{Unit Normal} = \text{N} = \frac{(-y', x')}{\sqrt{x'^2 + y'^2}}$$

From here we can form an equation between the normal and the tangent

$$T' \text{is a scalar multiple of } N$$

$$T' = skN$$

This equation defines the curvature,k and we also include the speed, s as T' isn't a unit vector.

To find the curvature...

$$\text{Rearranging equation for unit tangent } \gamma' = sT$$

$$\gamma'' = s'T + sT' = s'T + s^2kN$$

$$\gamma'' \cdot N = S'T \cdot N + s^2kN \cdot N$$

$$\text{Substitute } T \cdot N = 0 \text{ and } N \cdot N = 1$$

$$\gamma'' \cdot N = s^2k$$

From here we substitute the x, y values of the vectors and simplify

$$(x'', y'') \cdot \frac{(-y', x')}{s} = s^2k$$

$$\frac{x''(-y') + y''x'}{s^3} = k$$

$$\frac{x'y'' - x''y'}{(x'^2 + y^2)^{\frac{3}{2}}} = k$$

Now that we have found the curvature, k, we can find the general point on the evolute

$$T \cdot N = 0$$

$$T' \cdot N + T \cdot N' = 0$$

$$skN \cdot N + T \cdot N' = 0$$

$$T \cdot N' = -sk$$

$$\text{N' is parallel to T so } N' = \lambda T$$

$$\text{Substituting this back in } \lambda T \cdot T = -sk$$

$$T \cdot T = 1$$

So we get...

$$\lambda = -sk$$

And

$$N' = skT$$

This is known as the third fundamental vector formula for curves.

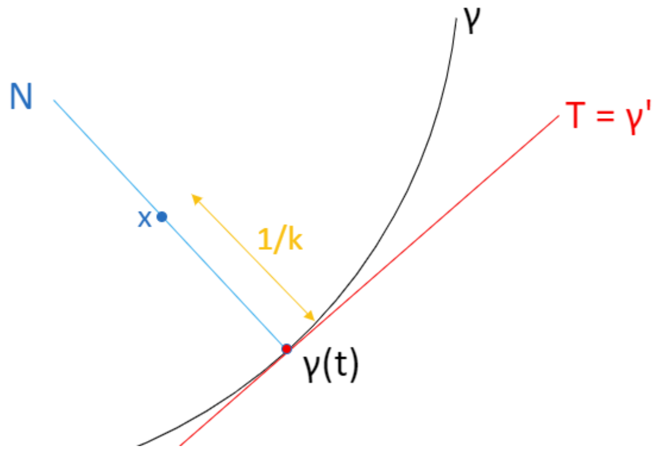


Figure 3: A visual represenation of the vectors we are using

Looking at the graph we can determine that the general point for x on the normal to γ has the equation

$$X = \gamma + \lambda N$$

And as the equation of the normal has the equation:

$$(X - \gamma)$$

$$(X - \gamma) \cdot T = 0$$
$$-\gamma' \cdot T + (x - \gamma) \cdot T' = 0$$
$$-1 + \lambda N \cdot kN = 0$$
$$\lambda = \frac{1}{k}$$

$$X = \gamma + \frac{1}{k} N$$

This gives us the equation of the line for the normals. From here, we can use presviously mentioned methods to find the envelope of this equation to form the evolute.

4 Caustics

A caustic is a type of envelope, similar to an evolute, and is the envelope of reflected light rays. When calculating the caustic of a surface there are two main vairables, the light source and the curvature on the surface. Below are two real life examples, and which are sometimes refered to as "cofffe cup caustics" due to being prevalent in common surfaces, such as coffee cups e.c.t.

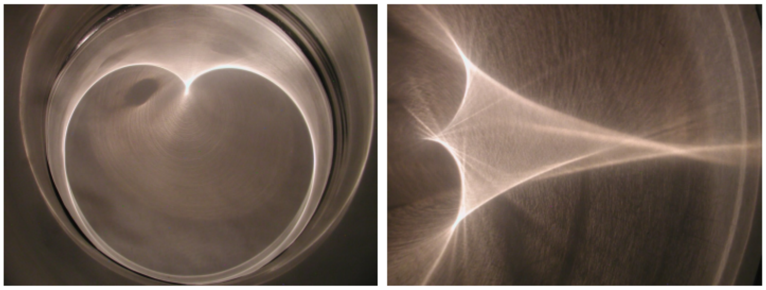


Figure 4: Real life examples

As can be seen in the images the caustic also has cusps, which are concentrated areas of light- this for example is what lets us to burn paper with a magnifying glass, due to the concentrated light rays. On thing that should be noted is that all light rays diffract outwards, so no real light wave will be perdendicular to our axes, but for light coming from incredibly far distances, the gradient is so small that we assume them to be perpendicular.

4.1 Calculating the caustics

4.1.1 For parallel rays

As stated before, although all light rays diffract(i.e will never go in straight line), we can assume at a long distance the graident of the ray from the source is near enough zero that we can model the incoming light ray (known as the **incident ray**) as parallel to the x-axis. This allows us to apply our same equation from when we calculated the light source at a point (a,0).

To find the caustic, we need to find the equation for reflected line,for which we need to manipulate angles for to find m in terms of t.

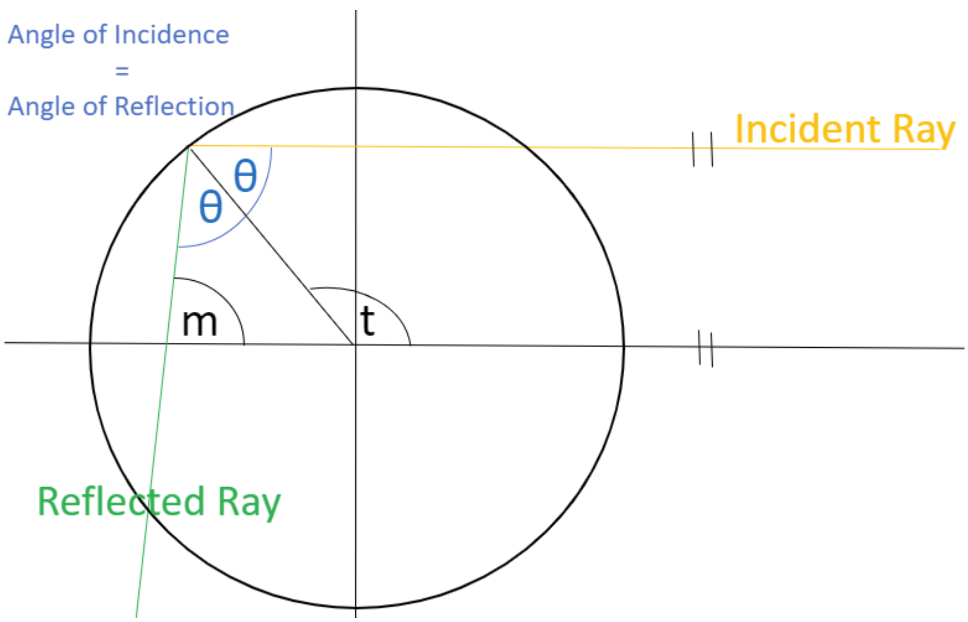


Figure 5: Visual representation of a light rays coming parallel to the x-axis

As can be seen in the figure, the gradient of the reflectred ray is m. We can find m in terms of t by first finding in terms of t using angles rules. We can manipulate angles to find the value of the gradient in terms of parameter t.

$$2\pi = \theta + t + \pi$$

$$\theta = \pi - t$$

We can substitute this into the triangle formed by the angle of reflecction and reflected ray.

$$\pi = m + \theta + (\pi - t)$$

$$\begin{aligned} \pi &= m + (\pi - t) + (\pi - t) \\ \pi &= m + 2\pi - t \\ m &= -\pi + 2t \\ slope = m &= tan(2t - \pi) \\ m &= \frac{sin(\pi - 2t)}{cos(\pi - 2t)} \\ &= \frac{sin(2t)}{-cos(2t)} \\ &= -\tan(2t) \end{aligned}$$

We know the graident and the intercept between the reflected line and the circle, so we can form an equation and find the caustic using previously mentioned methods.

4.1.2 For rays from source (a,0)

For rays from (a,0) we need to employ the use of the Sin and Cos rules, due to not being able to use only angles. To use the Sin and Cos rules, we exploit the use of unit circles as that way we can fill out enough variables to acutally be able to use the equations.

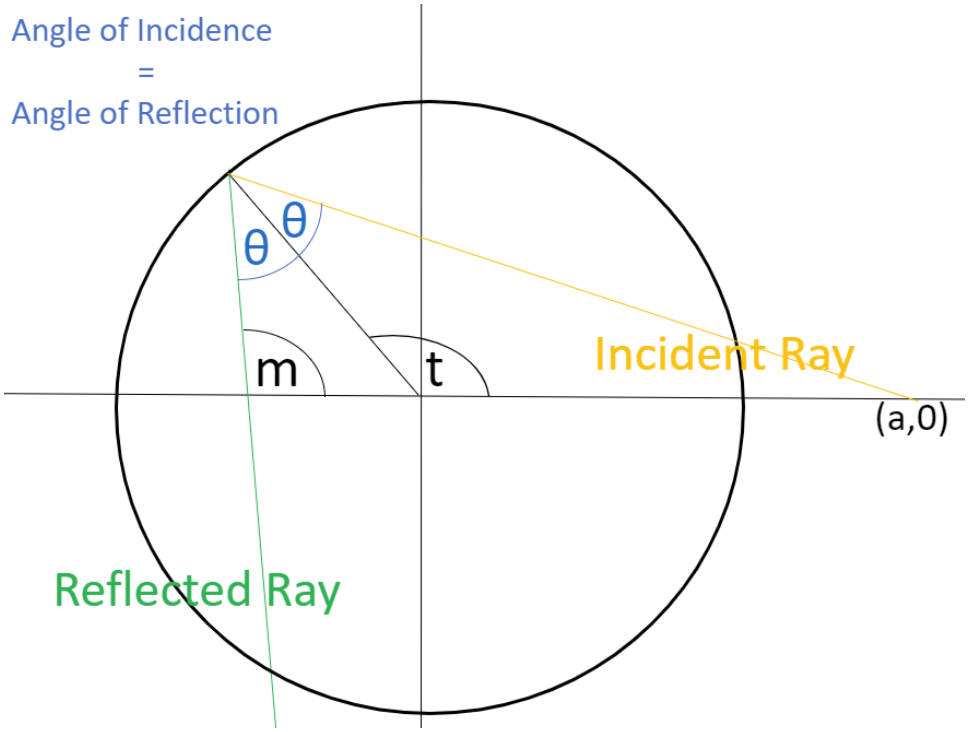


Figure 6: Visual representaiton of a light source from point (a,b) to a circle

We find the angle m in terms of t and θ using ccoordinate geometry and the Cos rule, leaving us with:

$$\tan(t - \theta)$$

Using our trigonometric values for $\tan(x)$...

$$\frac{y - \sin(t)}{x - \cos(t)} = \frac{\sin(t - \theta)}{\cos(t - \theta)}$$

As this equation leaves us with both $\cos(theta)$ and $\sin(theta)$, and as they may take on different values, so we find them in terms of a and t which we used the Sin and Cos rule for, giving us...

$$\sin(\theta) = \frac{a\sin(t)}{sqrt(1 - 2a\cos(t) + a^2)}$$

$$\cos(\theta) = \frac{1 - a\cos(t)}{\sqrt{(1 - 2a\cos(t) + a^2)}}$$

We substitute both these values in when using their corrosponding values to find angle θ in terms of a and t, as we control the value a and use a range of known valuess for t.

4.2 Shapes of the cacustic

There are two factors that will affect the shape of the caustic: the distance of the source from the surface as it changes the angle of incidence; and the shape of the surface as it will change how much reflection there is for a given value of t. These values of a and$t$ are applied to the above equations to form the images below.

4.2.1 Source outside surface

Using these gradients and finding their envelopes gives us...

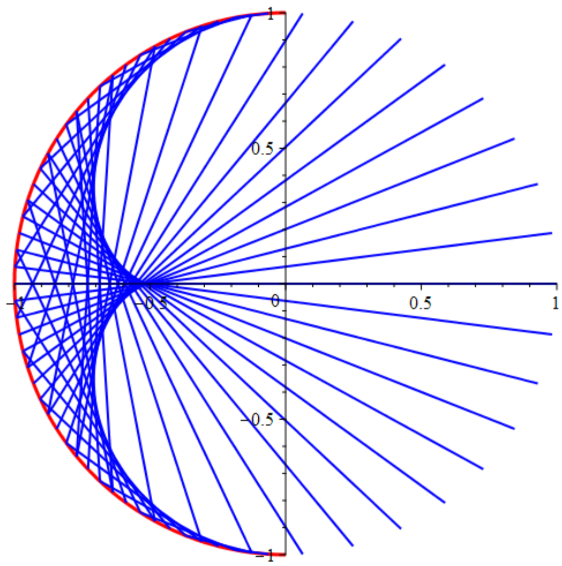


Figure 7: Caustic with source (a,0), where a is a large number

This forms a unique shape called a Nephroid, in which this case is a half-nethroid with only one visible cusp, however, if we calculate the caustic we can see two cusp.

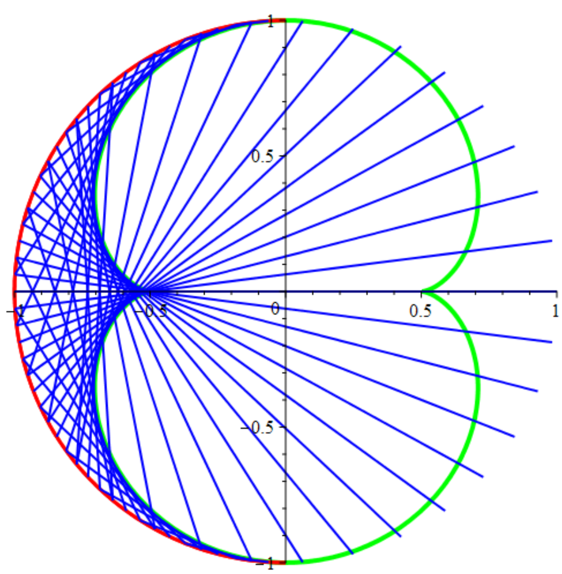


Figure 8: Overlap of reflected lines and the caustic

The forms the shape called a (full) Nephroid. The reason only one side has the reflected light rays is due to the fact light can from from both directions, however to recreate the on the otherside we would have to flip the direction of light(i.e the light source) and also flip the semi-circle into the opposite quadrtant so it can poroperly reflect.

Also, by using the same equation of the caustic we can predict how the shape of the caustic changes as the light source shifts from close to far, i.e from $1 < a < \infty$. And as the graident from point(∞,0) will have such a small gradient, we can also model light coming in parallel to the x-axis using the same equations.

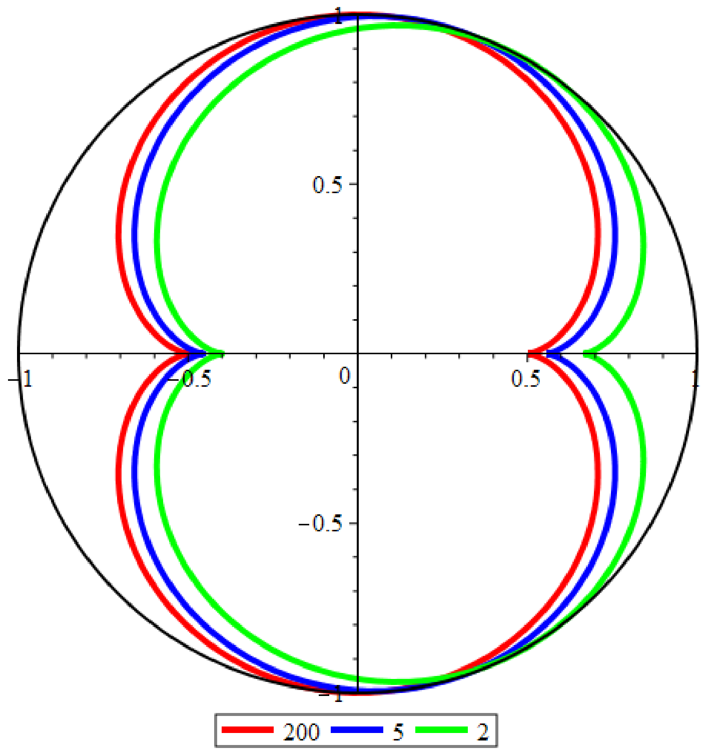


Figure 9: Above are 3 caustic for changing values of a, where a is (0,a)

We can see the caustic shifting exponentially towards the light source for smaller values of a, but after approximetly $a > 10$ the change to the graideint of the incident rays are so small it becomes essentially null.

Also the caustics shift position towards the light source, as a decreases, and they streche in the x-axis and compresse in the y-axis, although only slightly.

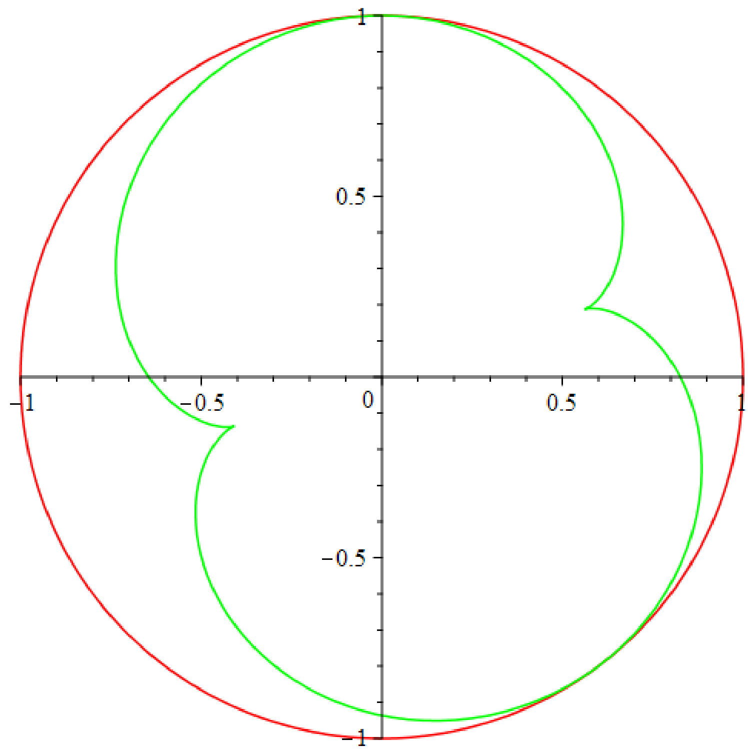


Figure 10: Caustic for unit circle with light source (3,1)

In the above image, as the light is coming in from a different direction, so the caustic rotates and shifts in response.

Using a similar method, but with different equation we can also model the caustic of an elipse from point (a,0). As seen below, their caustics are not too dissimilar to the caustics of the circles.

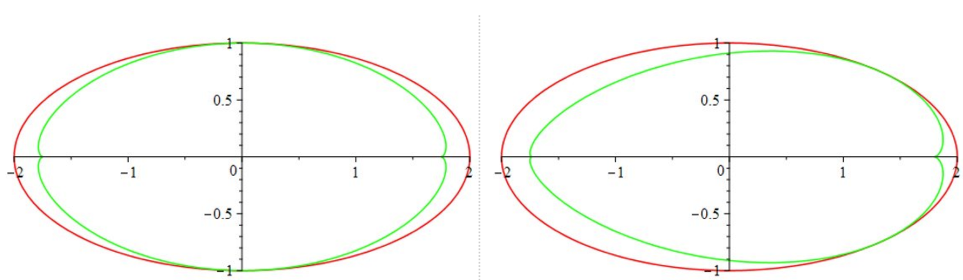


Figure 11: Caustic of elipse as a varies from $100 \rightarrow 3$

The caustic of the elipse, like the circle's, shifts towards the source as it moves nearer to the surface of the shape.

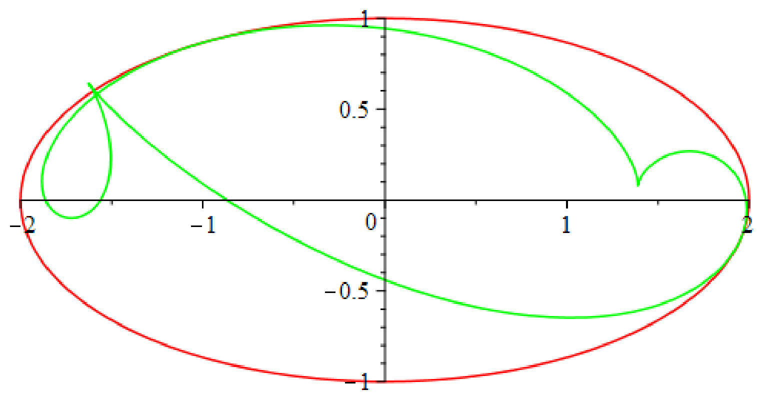


Figure 12: Caustic for elipse with light source (3,2)

4.2.2 Source on the surface of the shape

When the light source in the actaul surface of the shape, we get the following shapes.

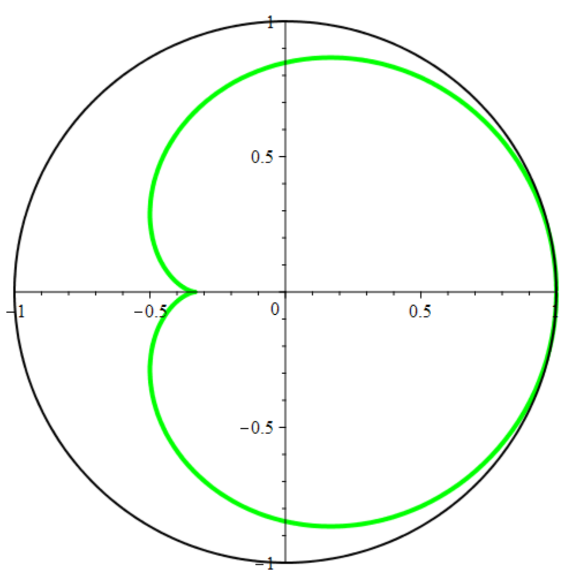


Figure 13: This is identical to one of the real life examples shown before

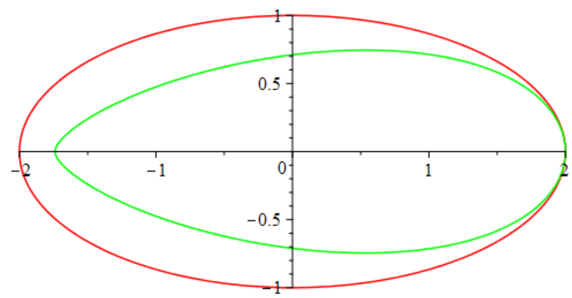


Figure 14: Ellipse with light source on surface

What interesting is that both the caustics intercept their own respective original shapes at the source. This may be a way we can model partial reflections.

Also, both caustic apear to only have one cusp and shift towards the source of light. It apears as if the caustic of the elipse is just an elongated caustic of a circle.

4.3 Exploring the equation of the caustic

Our general equation for the caustic of a circle is...

$$x = \frac{-2a^2cos(t)^3 + 3a^2cos(t) - a}{-3acos(t) + 2a^2 + 1}$$

$$y = -\frac{2a^2sin(t)^3}{3acos(t) - 2a^2 - 1}$$

As we cannot divide by zero, we can use the denominators to find what values of a the equation will not work for.

$$3acos(t) - 2a^2 - 1 = 0$$

We know that $\cos(x) \geq 1$ and ≤ -1, we we rearragne to find cos(t)

$$\cos(t) = \frac{1 + 2a^2}{3a}$$

So applying the rule...

$$-1 \leq \cos(t) = \frac{1 + 2a^2}{3a} \leq 1$$

Where $a > 0$, as a is just a point and it doesn't matter what direction it's coming from (in the context of this equation).

$$-3a \geq 2a^2 + 1 \geq 3a$$

Solving the inequality gives us a range for values of a that cause division by zero...

$$-0.5 \geq a \geq 0.5$$

4.4 Source inside the shape

When the source of light is inside the shape, it forms very different shapes to the previous and also when modelling we have some issues with points on the caustic jumping to infinity.

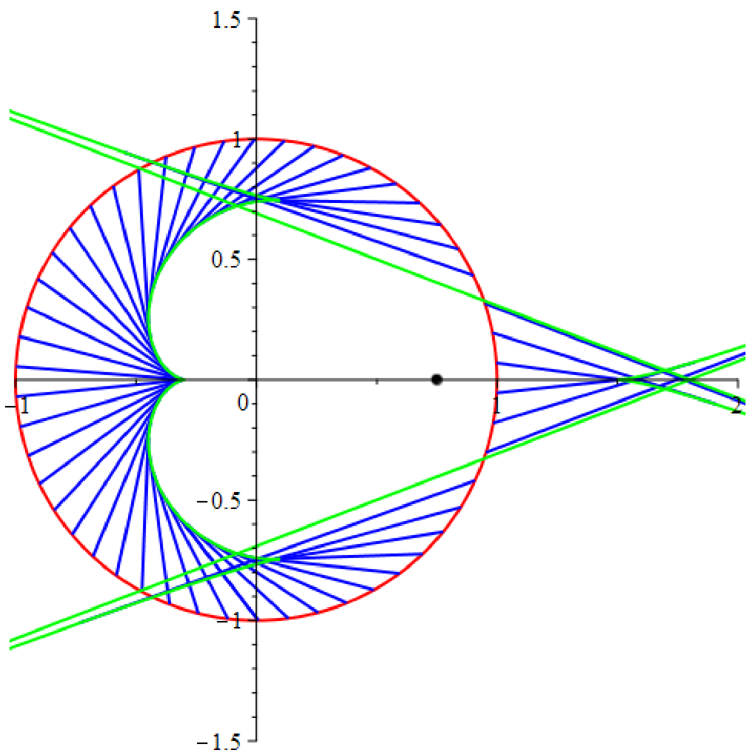


Figure 15: Caustic of a circle from (a,0) where $\frac{1}{2} < a < 1$

As can been seen in the figure about the caustic moves towards infinity, due to the denominator being $= 0$ for these ranges. This is due to the equation making a point at infinty (which the caustic joins up to) and then making another point at infinity on the other side. This causes the erratic sharp lines we see in the image. We can remove these points at infinity by calculating their values and avoiding to displace those values ——————————————DONT UNDERSTAND INFINITY ASPECT

4.4.1 For what values do we get points at infinity?

Similar to the last calculation, we calculate when the denominater is $= 0$ for a given value of a. We need to find what values of t cause this and alter them slightly by ± 0.01 to avoid the devision by zero.

$$cos(S) = \frac{1 + 2a^2}{3a}$$

Where S value of t that gives us the point(s) at ∞

$$S = \arccos(\frac{1+2a^2}{3a})$$

To better represent the caustic, we split it up into 3 intervals of t into the range..

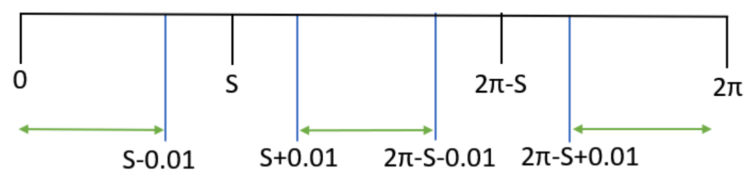


Figure 16: The green liness encompass what we want to plot

We use these intervals to clean up the caustic and produce a more accurate image relating to real life.

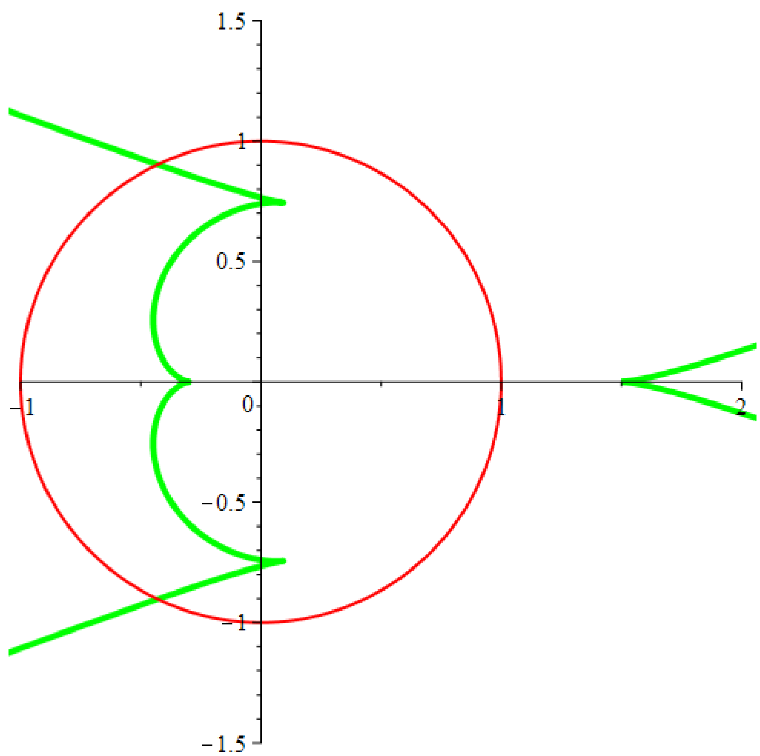


Figure 17: Caustic using the intervals

5 Orthortomics

A caustic is the evolute of a orthotomic. Due to their linked nature, they can give alot of infomation about the caustic, and we can also use the orthotomic to find the caustic. It is for this reason, that the orthotomic is also know as the "secondary caustic" or the "anti-caustic".

5.1 Finding the Orthotomic

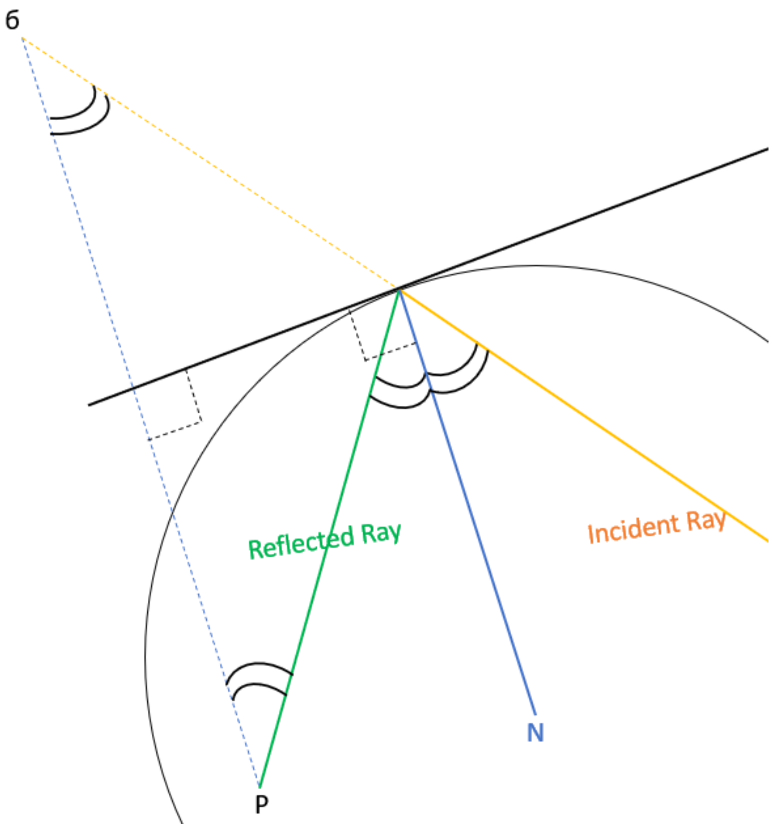


Figure 18: Caustic using the intervals

From the graph above we can figure that...

$$\delta = P + \lambda N$$

However, we do not know what this value of λ so we will calculating it, finding the general equation for an orthotomic. We find this equation by first finding the distance between the light source,P, and the orthotomic, σ and then we multiply with the unit normal so it is going in the right direction. As the two triangles formed by the incident ray and the reflected ray are identical...

$$2(\gamma - p) \cdot N = \text{distance between points}$$

$$\delta = (2(\gamma - P) \cdot N)N$$

This gives the general equation of the orthotomic, which we can use to graph the orthotomic A more specific example below is for the orthotomic of a circle

$$\delta = (a, b) + (2(\cos(t) - a, \sin(t)) \cdot (-\cos(t), -\sin(t))(-\cos(t), -\sin(t))$$

Which simplify's down into...

$$\delta = (a + 2(-1 + a\cos(t)(-\cos(t), b + 2(-1 + a\cos(t))(-\sin(t)))$$

From this we can find the caustic, by using the methods previously when working with the evolute.

5.2 Shapes of orthotomics

Sometimes the envelope formed by the orthotomic is called Pascal's limaçons as it can be defined as a line formed by the path of a fixed point on a circle of equal radias rolling around the outside.

5.2.1 Orthotomics with source outside surface

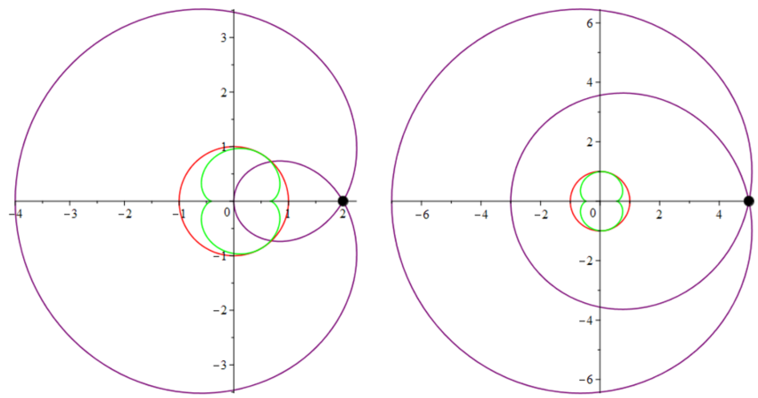


Figure 19: Orthotomic of a unit circle, with light source at point (2,0)

Where the orthotomic overlaps it's self (in this example) is called a *ordinary double point*. As can be seen in the graph, the cusps on the caustic translate to points of maximum curvature on the orthotomic, i.e their points of inflection. As the caustic and orthotomics shapes are linked, we can predict the shape of one by the other- so for example, even as a increases where (a,0) is the source of light, it will still follow the pattern of having one ordinary double point- as a increases the inner loop increases in diameter and nears the outer loop, but they don't cross over.

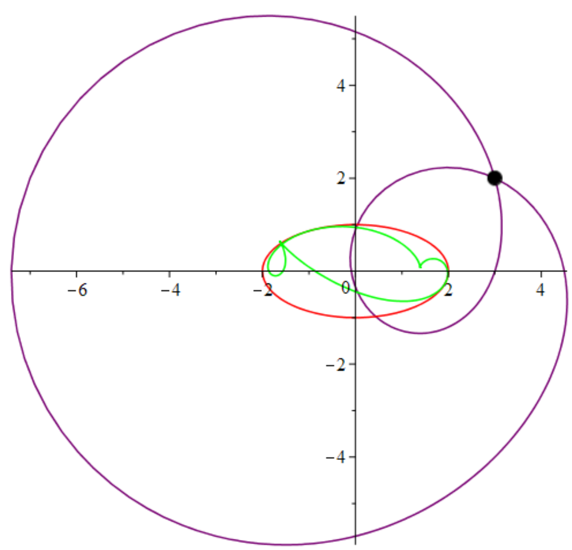


Figure 20: Orthotomic of a elipse, with light (3,2)

As can be seen in the figure above, even with a different surface and co-ordinates off the axis, we still produce a similar shape with it's ordinary double point.

5.2.2 Orthotomics on the surface

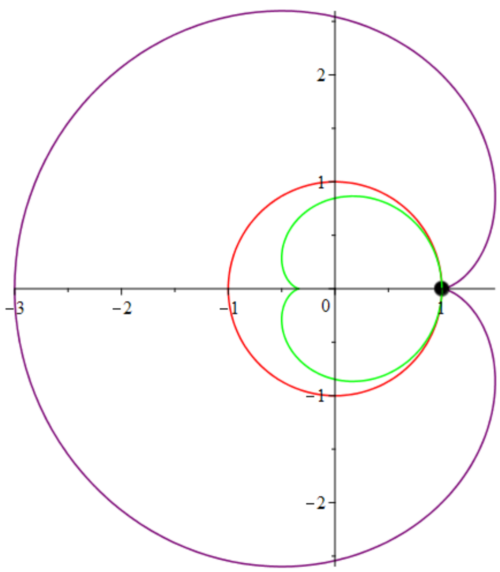


Figure 21: Orthotomic of a circle, with light source at the surface

When the light source is on the surface, for both the example above and below, a cusp forms at the light source.

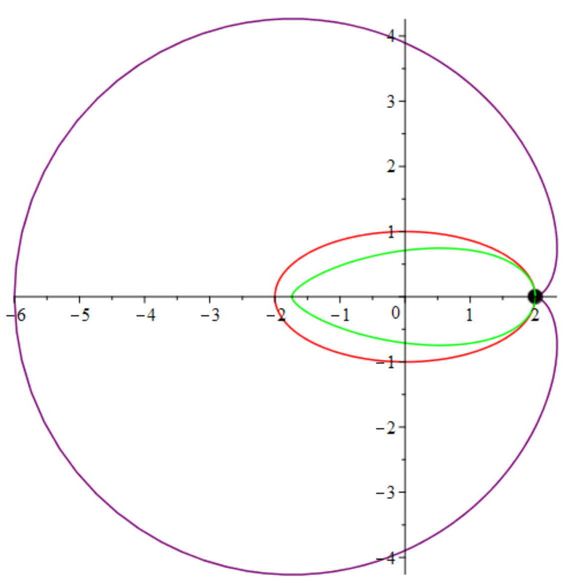


Figure 22: Orthotomic of a elipse, with light sourceat the surface

5.2.3 Orthotomics with source inside surface

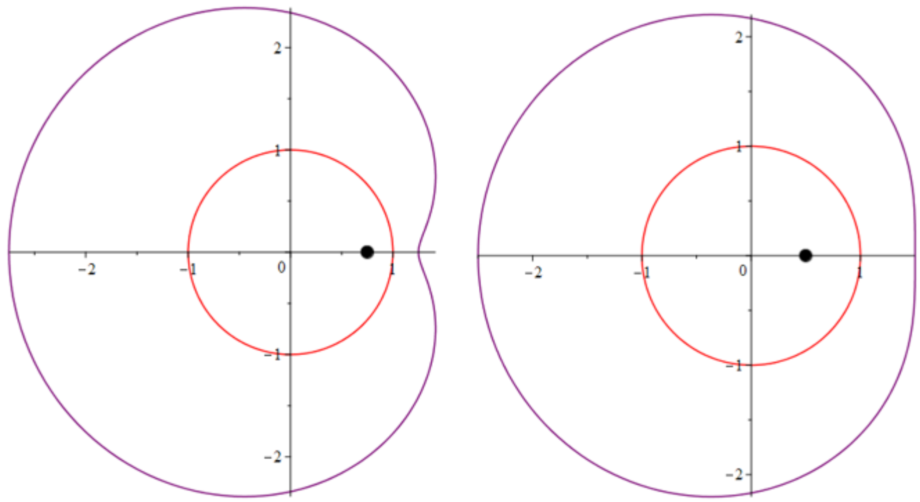


Figure 23: Orthotomic of unit circle, when $\frac{1}{2} < a < 1$

5.2.4 Special Cases

As the light sources nears the centre of the circle, the orthotomic looks more and more circle like.

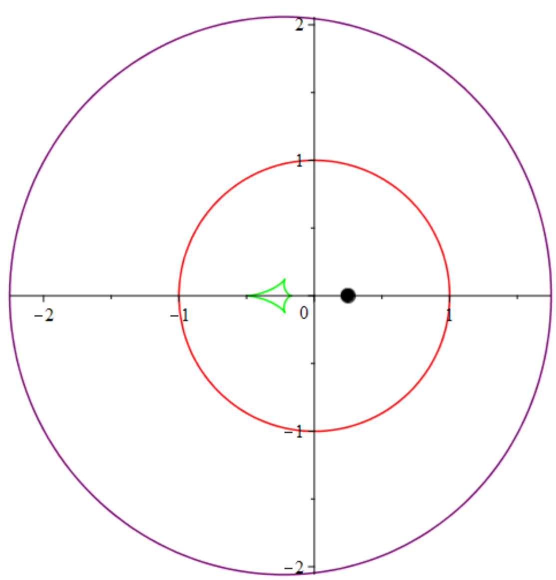


Figure 24: Orthotomic of unit circle, when $a < 0.25$

However, even the example above doesn't give a perfect circle. The is only one value which accomplishes this and that is $a = \pm\sqrt{3}$ applied to an elipse,so that all the normals pass through a centre.

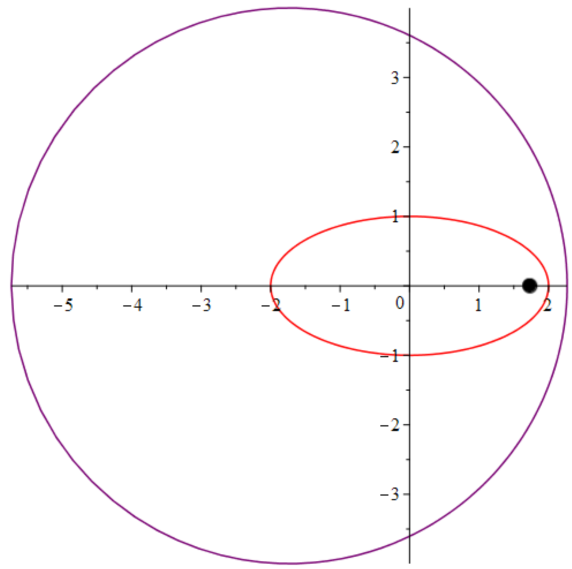


Figure 25: Special cases of Orthotomic when $a = \sqrt{3}$

6 Ackowledgements