In the following, K is a field of characteristic 0 and G an algebraic group, that is a group whose set is an affine variety, and whose multiplcation and inversion are morphisms of affine varieties.

Definition 0.1
We denote by m the group multiplication of the group G. We want to view the pullback of m as a map $m^* : K[G] \longrightarrow K[G] \otimes K[G]$, which makes sense, because m and $\otimes$ are associative. The strict pullback, which I will call $\hat{m}$, should be a map of the type $K[G] \longrightarrow K[G \times G]$, where $f \mapsto f \circ m$. If we want to give the variables names, we can equivalently say it is a map $K[Z]|_G \longrightarrow K[X, Y]|_{G \times G}$, where $Z = \{Z_1, \dots, Z_k\}$, X and Y analogously (here, m canonically takes its left input via X and its right input via Y). Define

$$\begin{aligned} t\colon\ K\left[X,Y\right]|_{G\times G} &\longrightarrow K\left[Z\right]|_G \otimes K\left[Z\right]|_G \\ \sum_i \lambda_i \prod_j X_j^{d_{i,j}} \prod_j Y_j^{e_{i,j}} &\mapsto \sum_i \lambda_i \prod_j Z_j^{d_{i,j}} \otimes \prod_j Z_j^{e_{i,j}} \end{aligned} \tag{1}$$

This is independant of the choice of the representatives and therefore well-defined. It is even an isomorphism. Now, finally, define $m^* := t \circ \hat{m} : K[G] \longrightarrow K[G] \otimes K[G]$.

One might ask, why we want to look at these objects $m^*(f)$ instead of $\hat{m}(f)$. Really, these objects are hardly different, but it helps to formalize performing operations only on the “left part” or the “right part”, as we will soon see. This is an approach that [**?**] follows, but other literature such as [**?**] (and probably also Cayley) rather consider $\hat{m}(f)$ written as $\hat{m}(f) = f(XY)$. To give a very simple example: If $f \in K[Z]|_G$, we will write $\mathrm{id} \otimes \frac{\partial}{\partial Z_i}(m^* f)$ as in [**?**], whereas [**?**] would write $\frac{\partial}{\partial Y_i} f(XY)$.

Definition 0.2: rational representation
Let V be a vector space (not necessarily finite dimensional), and $\mu : G \times V \longrightarrow V$ an action. We call μ a **rational representation** iff there exists a linear map $\mu^*\colon V \longrightarrow K[G] \otimes V$ such that $\mu(\sigma, v) = ((\epsilon_\sigma \otimes \mathrm{id}) \circ \mu^*)(v)$.

Definition 0.3: Regular Action
Let G be a linear algebraic group, X an affine variety. We call an action $G \times X \longrightarrow X$ a **regular action**, iff it is a morphism of affine varieties. We say G **acts regularly on** X.

Definition 0.4: Rational Representation
Let G be a linear algebraic group. A representation V of G is called a **rational representation**, iff its corresponding action $G \times V \longrightarrow V$ is a regular action.

Claim: If V is finite dimensional, the notions of the definitions coincide.
Proof: First, let V be a rational representation of G with basis $\{v_1, \dots, v_N\}$ be a basis of V. By our assumption, we have a rational representation, therefore there exist $p_{i,j} \in K[G]$ such that $\mu(\sigma, v_j) = \Sigma_{i=1}^N p_{i,j}(\sigma) \cdot v_i$. Define $\mu^*(v_j) :=$

$\Sigma_{i=1}^{N} p_{i,j} \otimes v_i$. Now we easily see:

$$\begin{aligned} \mu(\sigma, v) &= \mu\left(\sigma, \Sigma_{j=1}^{N} \lambda_j v_j\right) \\ &= \sum_{j=1}^{N} \lambda_j \sum_{i=1}^{N} p_{i,j}(\sigma) \cdot v_i \\ &= \sum_{j=1}^{N} \lambda_j \left(\left(\epsilon_\sigma \otimes \mathrm{id}\right) \circ \mu^*\right)(v_j) \quad = \left(\left(\epsilon_\sigma \otimes \mathrm{id}\right) \circ \mu^*\right)(v) \end{aligned} \tag{2}$$

which was to show.

Remark 0.4.1
A rational representation $G \longrightarrow \mathrm{GL}(V)$ is of the following form:
If $a_{i,j} : G \longrightarrow K$ is the function of the (i, j)-entry, then $a_{i,j} \in K[G]$.
In fact, it is equivalent to define a representation as rational, iff its map $G \longrightarrow \mathrm{GL}(V)$ is a map of affine varieties.

Definition 0.5: Invariants
Let G act on X regularly.

$$X^G := \{\, x \in X \mid \forall g \in G : g.x = x \,\} \tag{3}$$

It defines a linear subspace. This given action induces an action $G \times K[X] \longrightarrow K[X]$, where $K[X]$ is the coordinate ring, as follows:

$$(g, f) \longmapsto g \cdot f := \left(x \mapsto f\left(\sigma^{-1}.x\right)\right) \tag{4}$$

The **invariant ring** of the representation is defined as

$$K[X]^G := \{\, f \in K[X] \mid \forall g \in G : g \cdot f = f \,\} \tag{5}$$

As the name implies, $K[X]^G$ defines a subalgebra of $K[X]^G$.

The general theme of my work revolves around the question of whether the invariant ring $K[X]^G$ is finitely generated.
Hilbert's finiteness theorem states that if the group G is linearly reductive, $K[V]^G$ is finitely generated. The strict definition of "linearly reductive" is quite tricky, but we will give an alternate equivalent definition shortly.