Московский авиационный институт
(национальный исследовательский университет)

Факультет информационных технологий и прикладной математики

Кафедра вычислительной математики
и программирования

Курсовой проект по курсу «Уравнения математической физики»

Студент: Т. А. Бердикин
Преподаватель: С. А. Колесник
Группа: М8О-307Б-18
Вариант: 8
Дата:
Оценка:
Подпись:

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СОДЕРЖАНИЕ

Курсовая работа по УМФ
Вариант №8

Сформулировать и решить задачи:

1. О нагреве конечного стержня $x \in [0;l]$ с начальным распределением $T_0 = 300$ и источником тепла $f(x) = 300(1+x)$, когда на левом конце задана температура $\mu_1 = 300(1+e^{-t})$, а правый конец теплоизолирован $(a^2 = 10^{-6})$. Исследовать ортогональность и нормировку собственных функций, построить графики $u(x,t)$, $l = 0{,}1$м.

2. О свободных колебаниях конечного стержня $x \in [0;\pi]$, $a^2 = 10^6$ с начальным отклонением $\varphi_1 = x$. $\varphi_2 = \pi - x$, когда левый конец движется по закону $\mu_1 = t$, а на правом задано усилие $\mu_2 = -t$. Результаты вывести графически.

3. Задачу Неймана для уравнения Лапласа в круге с радиусом $r_0 = 1$, когда на границе задан поток $\dfrac{\partial u(r_0,\varphi)}{\partial r} = 2\sin\varphi$, $\varphi \in [0;2\pi]$. Построить графики $u(r,\varphi_i)$, $\varphi_i = 0, \pi/2,\ \pi,\ \dfrac{3}{2}\pi,$ 2π.

№1. $T_0 = 300$, $a^2 = 10^{-6}$, $f(x) = 300(1+x)$

$\mu_1 = 300(1+e^{-t})$, $l = 0{,}1$ м

$$\begin{cases} u_t = a^2 \cdot u_{xx} + 300(1+x) \\ u(x,0) = 300 \\ u(0,t) = \mu_1 = 300(1+e^{-t}) \\ u_x(l,t) = 0 \end{cases}$$

$$\begin{cases} u = u_1 + u_2 \\ u_{1t} + u_{2t} = a^2 u_{1xx} + a^2 u_{2xx} + 300(1+x) \\ u_1(x,0) + u_2(x,0) = 300 \\ u_1(0,t) + u_2(0,t) = 300(1+e^{-t}) \\ u_{1x}(l,t) + u_{2x}(l,t) = 0 \end{cases}$$

Решение относ. u_1:

$$\begin{cases} u_{1xx} = 0 \\ u_1(0,t) = 300(1+e^{-t}) \\ u_{1x}(l,t) = 0 \end{cases}$$

$u_{1x} = C_1$, $u_1 = C_1 \cdot x + C_2$

$u_{1x}(l,t) = C_1 = 0$

$u_1(0,t) = C_2 = 300(1+e^{-t})$

$u_1(x,t) = 300(1+e^{-t})$, $u_{2t} = -300e^{-t}$

Решение относ. u_2:

$$\begin{cases} u_{2t} = a^2 u_{2xx} + 300(1+x) + 300e^{-t} \\ u_2(x,0) = 300 - u_1(x,0) = 300 - 300(1+1) = -300 \\ u_2(0,t) = 0 \\ u_{2x}(l,t) = 0 \end{cases}$$

$u_{2t} = a^2 \cdot u_{2xx}$

$u_2 = X(x)T(t)$

$$X(x)T'(t)=a^2\cdot X''(x)T(t)$$

$$\frac{T'(t)}{a^2T(t)}=\frac{X''(x)}{X(x)}=-\lambda$$

$$X''(x)+\lambda X(x)=0$$

$$\left.\begin{aligned}u_2(0,t)&=X(0)T(t)=0\\ u_{2x}(l,t)&=X'(l)T(t)=0\end{aligned}\right\}\Rightarrow X(0)=X'(l)=0$$

$$\begin{cases}X''(x)+\lambda X(x)=0\\ X(0)=X'(l)=0\end{cases}$$

$$k^2+\lambda=0\Rightarrow k^2=-\lambda$$

1) $\lambda<0,\ k_{1,2}=\pm\sqrt{-\lambda}$

$$X(x)=c_1\cdot e^{-\sqrt{-\lambda}x}+c_2\cdot e^{\sqrt{-\lambda}\cdot x}$$

$$X(0)=c_1+c_2=0$$

$$X'(l)=-\sqrt{-\lambda}c_1\cdot e^{-\sqrt{-\lambda}\cdot l}+\sqrt{-\lambda}\cdot c_2\cdot e^{\sqrt{-\lambda}\cdot l}=0$$

$$c_1=-c_2$$

$$\sqrt{-\lambda}c_2e^{-\sqrt{-\lambda}l}+\sqrt{-\lambda}c_2e^{\sqrt{-\lambda}l}=0$$

$$c_1=0,\ c_2=0$$

Тривиальное решение.

2) $\lambda=0,\ k_{1,2}=0$

$$X(x)=c_1x+c_2$$

$$X(0)=c_2=0$$

$$X'(l)=c_1=0$$

Тривиальное решение.

3) $\lambda>0,\ k_{1,2}=\pm\sqrt{\lambda}\cdot i$

$$X(x)=c_1\cdot\cos(\sqrt{\lambda}\cdot x)+c_2\sin(\sqrt{\lambda}x)$$

$$X(0)=c_1=0$$

$$X'(l)=-\sqrt{\lambda}c_1\cdot\sin(\sqrt{\lambda}l)+\sqrt{\lambda}c_2\cdot\cos(\sqrt{\lambda}l)=0$$

$$\sqrt{\lambda}c_2\cos(\sqrt{\lambda}l)=0$$

$$\cos(\sqrt{\lambda}l)=0$$

$$\sqrt{\lambda_n}\cdot l=\frac{\pi}{2}+\pi n=\frac{\pi(2n+1)}{2}$$

$$\lambda_n = \left(\frac{\pi(2n+1)}{2l}\right)^2$$

$$X_n(x) = \sin\left(\frac{\pi(2n+1)}{2l}\cdot x\right)$$

Квадрат нормы собств. ф-ций:

$$\|X_n(x)\|^2 = \left(X_n(x), X_n(x)\right) = \int_0^l \sin\left(\frac{\pi(2n+1)}{2l}\cdot x\right)\sin\left(\frac{\pi(2n+1)}{2l}\cdot x\right)dx =$$

$$= \int_0^l \sin^2\left(\frac{\pi(2n+1)}{2l}\cdot x\right)dx = \frac{1}{2}\int_0^l\left(1-\cos\left(\frac{\pi(2n+1)}{l}\cdot x\right)\right)dx =$$

$$= \frac{1}{2}\cdot\left(x - \frac{l}{\pi(2n+1)}\cdot\sin\left(\frac{\pi(2n+1)}{l}x\right)\right)\Big|_0^l = \frac{1}{2}\cdot\left(l - \frac{l}{\pi(2n+1)}\cdot\right.$$

$$\left.\cdot\sin\left(\pi(2n+1)\right)\right) = \frac{l}{2}$$

Ортогональность собств. ф-ций:

$$\left(X_n(x), X_m(x)\right) = \int_0^l \sin\left(\frac{\pi(2n+1)}{2l}\cdot x\right)\sin\left(\frac{\pi(2m+1)}{2l}\cdot x\right)dx =$$

$$= \frac{1}{2}\int_0^l\left(\cos\left(\frac{\pi(n-m)x}{l}\right) - \cos\left(\frac{\pi(n+m)}{l}\cdot x\right)\right)dx = |n\neq m| =$$

$$= \frac{1}{2}\cdot\frac{l}{\pi}\cdot\left(\frac{1}{n-m}\cdot\sin\left(\frac{\pi(n-m)\cdot x}{l}\right) - \frac{1}{n+m}\cdot\sin\left(\frac{\pi(n+m)\cdot x}{l}\right)\right)\Big|_0^l =$$

$$= \frac{l}{2\pi}\cdot\left(\frac{1}{n-m}\cdot\sin(\pi(n-m)) - \frac{1}{n+m}\cdot\sin(\pi(n+m))\right) = 0$$

при $n=m$: $\displaystyle\int_0^l \sin\left(\frac{\pi(2n+1)}{2l}x\right)\cdot\sin\left(\frac{\pi(2n+1)}{2l}\cdot x\right)dx = \|X_n\|^2 = \frac{l}{2}$

$$\int_0^l \sin\left(\frac{\pi(2n+1)}{2l}\cdot x\right)\sin\left(\frac{\pi(2m+1)}{2l}\cdot x\right)dx = \begin{cases} 0, & \text{при } n\neq m \\ \|X_n\|^2 = \frac{l}{2}, & \text{при } n=m\end{cases}$$

Решение неоднор. задачи ищется в виде:

$$u_2(x,t) = \sum_{n=0}^{\infty} u_{2n}(t)\cdot\sin\left(\frac{\pi(2n+1)}{2l}\cdot x\right)$$

$$f(x,t) = 300\left(1+x+e^{-t}\right) = \sum_{n=0}^{\infty} f_n(t)\cdot\sin\left(\frac{\pi(2n+1)}{2l}\cdot x\right)dx$$

$$f_n(t) = \frac{2}{l}\int_0^l 300\left(1+x+e^{-t}\right)\cdot\sin\left(\frac{\pi(2n+1)}{2l}x\right)dx =$$

$$= 300\cdot\left(\frac{2}{l}\int_0^l(1+x)\sin\left(\frac{\pi(2n+1)}{2l}\cdot x\right)dx\right) =$$

$$= \left[\begin{matrix} u = 1 + x, \quad du = dx \\ dv = \sin\left(\frac{\pi(2n+1)}{2l} x\right) dx, \quad v = -\frac{2l}{\pi(2n+1)} \cos\left(\frac{\pi(2n+1)}{2l} \cdot x\right) \end{matrix} \right] =$$

$$= 300 \cdot \frac{2}{l} \cdot \frac{-2l}{\pi(2n+1)} \cdot \left((1+x) \cos\left(\frac{\pi(2n+1)}{2l} x\right) \Big|_0^l - \int_0^l \cos\left(\frac{\pi(2n+1)}{2l} x\right) dx + \right.$$

$$\left. + e^{-t} \cos\left(\frac{\pi(2n+1)}{2l} x\right) \Big|_0^l \right) = 300 \cdot \frac{-4}{\pi(2n+1)} \cdot \left((1+l) \cos\left(\frac{\pi(2n+1)}{2}\right) - 1 - \right.$$

$$\left. - \frac{2l}{\pi(2n+1)} \cdot \sin\left(\frac{\pi(2n+1)}{2l} x\right) \Big|_0^l + e^{-t} \left(\cos\left(\frac{\pi(2n+1)}{2}\right) - 1 \right) \right) =$$

$$= 300 \cdot \frac{-4}{\pi(2n+1)} \cdot \left(-1 - \frac{2l}{\pi(2n+1)} \cdot \sin\left(\frac{\pi(2n+1)}{2}\right) - e^{-t} \right) =$$

$$= \underbrace{\frac{1200}{\pi(2n+1)} \cdot \left(1 + \frac{2l \cdot (-1)^n}{\pi(2n+1)} \right)}_{= a_n} + \underbrace{\frac{1200}{\pi(2n+1)}}_{= b_n} \cdot e^{-t} = a_n + b_n \cdot e^{-t}$$

$$u_2(x,0) = -300 = \sum_{n=0}^{\infty} \varphi_n \cdot \sin\left(\frac{\pi(2n+1)}{2l} \cdot x\right)$$

$$\varphi_n = \frac{2}{l} \int_0^l (-300) \cdot \sin\left(\frac{\pi(2n+1)}{2l} x\right) dx = \frac{1200}{\pi(2n+1)} \cos\left(\frac{\pi(2n+1)}{2l} x\right) \Big|_0^l =$$

$$= \frac{1200}{\pi(2n+1)} \cdot \left(\cos\left(\frac{\pi(2n+1)}{2}\right) - 1 \right) = -\frac{1200}{\pi(2n+1)}$$

$$\sum_{n=0}^{\infty} u_{2n}'(t) \sin\left(\frac{\pi(2n+1)}{2l} x\right) = -a^2 \sum_{n=0}^{\infty} u_{2n}(t) \cdot \left(\frac{\pi(2n+1)}{2l}\right)^2 \cdot \sin\left(\frac{\pi(2n+1)}{2l} x\right) +$$

$$+ \sum_{n=0}^{\infty} (a_n + b_n \cdot e^{-t}) \sin\left(\frac{\pi(2n+1)}{2l} x\right)$$

$$u_2(x,0) = \sum_{n=0}^{\infty} u_{2n}(0) \sin\left(\frac{\pi(2n+1)}{2l} x\right) = \sum_{n=0}^{\infty} \varphi_n \cdot \sin\left(\frac{\pi(2n+1)}{2l} x\right)$$

$$\begin{cases} u_{2n}'(t) + a^2 \cdot \left(\frac{\pi(2n+1)}{2l}\right)^2 \cdot u_{2n}(t) = a_n + b_n \cdot e^{-t} \\ u_{2n}(0) = \varphi_n \end{cases}$$

$$u_{2n}(t) = C_n \cdot e^{-a^2 \left(\frac{\pi(2n+1)}{2l}\right)^2} + u_r$$

$$u_r = A + B \cdot e^{-t}$$

$$-Be^{-t} + \left(\frac{a\pi(2n+1)}{2l}\right)^2 \cdot (A + B \cdot e^{-t}) = a_n + b_n \cdot e^{-t}$$

$$A = \frac{a_n \cdot 4l^2}{(a\pi(2n+1))^2} \; ; \; B = \frac{b_n}{\left(\frac{a\pi(2n+1)}{2l}\right)^2 - 1} = \frac{4l^2 b_n}{(a\pi(2n+1))^2 - 4l^2}$$

$$u_{2n}(t) = C_n \cdot e^{-\left(\frac{a\pi(2n+1)}{2l}\right)^2} + A + Be^{-t}$$

$$u_{2n}(0) = C_n + A + B = \varphi_n \; ; \; C_n = \varphi_n - A - B$$

$$u_{2n}(t) = (\varphi_n - A - B) \cdot e^{-\left(\frac{a\pi(2n+1)}{2l}\right)^2} + A + Be^{-t}$$

$$u(x,t) = 300(1+e^{-t}) + \sum_{n=0}^{\infty}\left(-\frac{1200}{\pi(2n+1)} - \frac{a_n \cdot 4l^2}{(a\pi(2n+1))^2} - \frac{4l^2 \cdot b_n}{(a\pi(2n+1))^2 - 4l^2}\right) \cdot$$

$$\cdot e^{-\left(\frac{a\pi(2n+1)}{2l}\right)^2 t} + \frac{a_n \cdot 4l^2}{(a\pi(2n+1))^2} + \frac{4l^2 \cdot b_n}{(a\pi(2n+1))^2 - 4l^2} \cdot e^{-t}\Bigg) \cdot \sin\left(\frac{\pi(2n+1)}{2l} x\right),$$

где $a_n = \frac{1200}{\pi(2n+1)} \cdot \left(1 + \frac{2l \cdot (-1)^n}{\pi(2n+1)}\right)$; $b_n = \frac{1200}{\pi(2n+1)}$

Проверка.

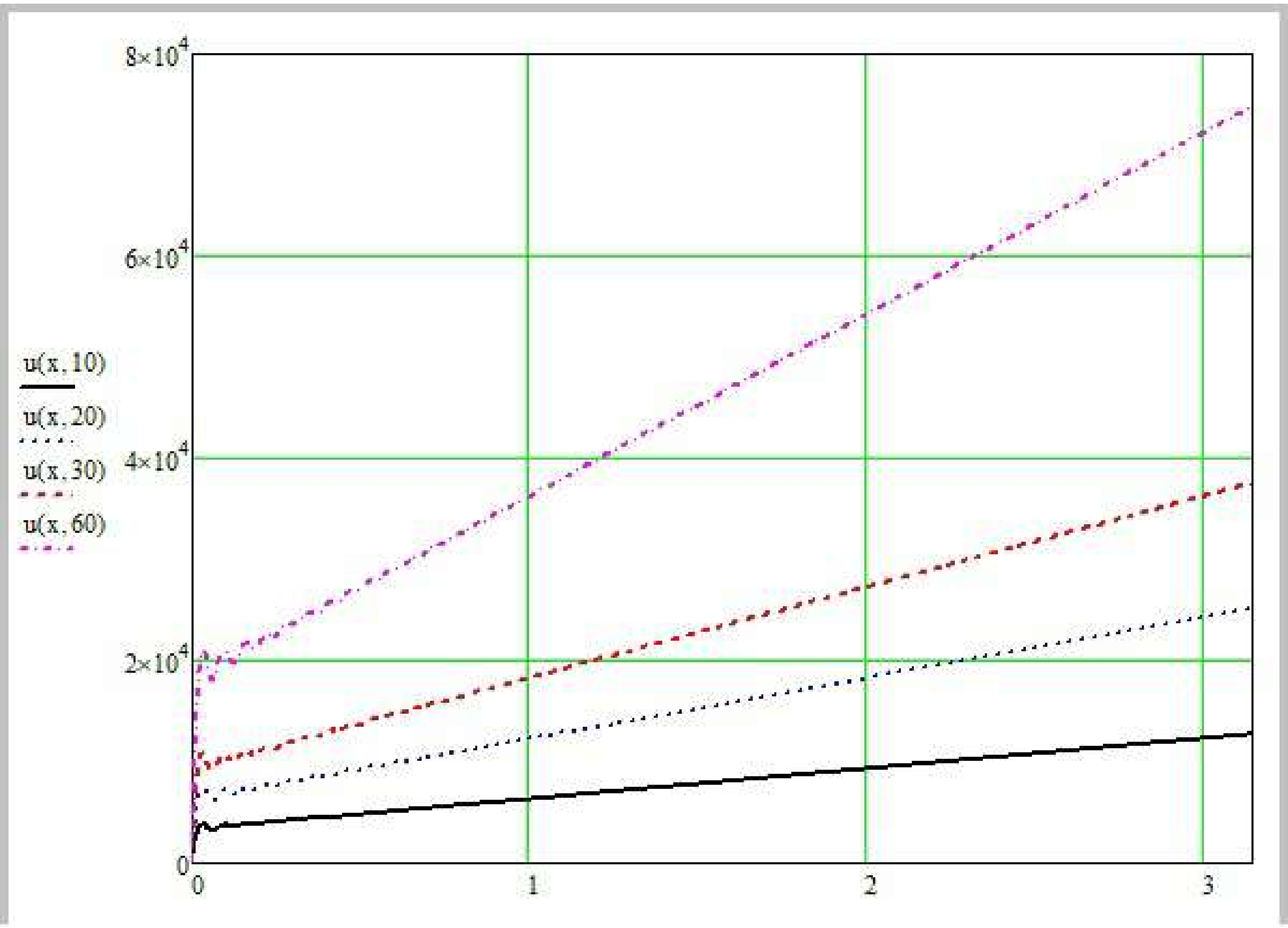
$$u(x,0) = 300 \cdot 2 + \sum_{n=0}^{\infty}\left(-\frac{1200}{\pi(2n+1)} - \frac{a_n \cdot 4l^2}{(a\pi(2n+1))^2} - \frac{4l^2 \cdot b_n}{(a\pi(2n+1))^2 - 4l^2} + \right.$$

$$\left. + \frac{a_n \cdot 4l^2}{(a\pi(2n+1))^2} + \frac{4l^2 \cdot b_n}{(a\pi(2n+1))^2 - 4l^2}\right) \sin\left(\frac{\pi(2n+1)}{2l} x\right) =$$

$$= 600 + \sum_{n=0}^{\infty} \frac{-1200}{\pi(2n+1)} \cdot \sin\left(\frac{\pi(2n+1)}{2l} x\right) = 600 - 300 \quad \text{Верно.}$$

$u(0,t) = 300 \cdot (1+e^{-t})$. Верно.

$$u_x(l,t) = \sum_{n=0}^{\infty}(\ldots) \cdot \frac{\pi(2n+1)}{2l} \underbrace{\cos\left(\frac{\pi(2n+1)}{2}\right)}_{=0} = 0 \quad \text{Верно.}$$



u(x, 10)
u(x, 20)
u(x, 30)
u(x, 60)
8×10^4
6×10^4
4×10^4
2×10^4
0
0
1
2
3

№2.

$a^2=10^6,\ \varphi_1=x,\ \varphi_2=\pi-x,\ \mu_1=t,\ \mu_2=-t$

$$\begin{cases} u_{tt}=a^2\cdot u_{xx},\ x\in[0;\pi] \\ u(x,0)=x \\ u_t(x,0)=\pi-x \\ u(0,t)=t \\ u_x(\pi,t)=-t \end{cases}$$

$u=u_1+u_2$

$$\begin{cases} u_{1tt}+u_{2tt}=a^2(u_{1xx}+u_{2xx}) \\ u_1(x,0)+u_2(x,0)=x \\ u_{1t}(x,0)+u_{2t}(x,0)=\pi-2 \\ u_1(0,t)+u_2(0,t)=t \\ u_{1x}(\pi,t)+u_{2x}(\pi,t)=-t \end{cases}$$

Решение относ. u_1:

$$\begin{cases} u_{1xx}=0 \\ u_1(0,t)=t \\ u_{1x}(\pi,t)=-t \end{cases}$$

$u_{1x}=C_1,\ u_1=C_1\cdot x+C_2$

$u_1(0,t)=C_2=t$

$u_{1x}(\pi,t)=C_1=-t$

$u_1(x,t)=-xt+t=t\cdot(1-x)$

Решение относит. u_2:

$$\begin{cases} u_{2tt}=a^2\cdot u_{2xx} \\ u_2(x,0)=x-u_1(x,0)=x \\ u_{2t}(x,0)=\pi-2-u_{1t}(x,0)= \\ =\pi-2-(1-x)=x+\pi-3 \\ u_2(0,t)=0,\ u_{2x}(\pi,t)=0 \end{cases}$$

$u_2(x,t)=X(x)\cdot T(t)$

$X(x)T''(t)=a^2X''(x)T(t)$

$\dfrac{T''(t)}{a^2T(t)}=\dfrac{X''(x)}{X(x)}=-\lambda$

$T''(t)+a^2\lambda T(t)=0$

$X''(x)+\lambda X(x)=0$

$u_2(0,t)=X(0)T(t)=0$

$u_{2x}(\pi,t)=X'(\pi)T(t)=0$

$k^2+\lambda=0$

$k^2=-\lambda$

a) $\lambda<0,\ k_{1,2}=\pm\sqrt{-\lambda}$

$X(x)=C_1\cdot e^{-\sqrt{-\lambda}x}+C_2\cdot e^{\sqrt{-\lambda}x}$

$X(0)=C_1+C_2=0 \Rightarrow C_1=-C_2$

$X'(\pi)=-\sqrt{-\lambda}C_1e^{-\sqrt{-\lambda}\pi}+$

$+\sqrt{-\lambda}\cdot C_2e^{\sqrt{-\lambda}\pi}=0$

$-\sqrt{-\lambda}C_1e^{-\sqrt{-\lambda}\pi}-\sqrt{-\lambda}C_1e^{\sqrt{-\lambda}\pi}=0$

$C_1=0,\ C_2=0$

Тривиальное решение.

б) $\lambda = 0$, $X_0''(x) = 0$

$X_0'(x) = A_0$ Тривиальное решение

$X(x) = A_0 \cdot x + B_0$

$X(0) = B_0 = 0$

$X'(\pi) = A_0 = 0$

в) $\lambda > 0$, $k_{1,2} = \pm\sqrt{\lambda} \cdot i$

$X(x) = C_1 \cdot \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x)$

$X(0) = C_1 = 0$

$X'(\pi) = -\sqrt{\lambda} C_1 \sin(\sqrt{\lambda}\pi) + \sqrt{\lambda} C_2 \cdot \cos(\sqrt{\lambda}\pi) = 0$

$\sqrt{\lambda} C_2 \cos(\sqrt{\lambda}\pi) = 0$

$\cos(\sqrt{\lambda} \cdot \pi) = 0$

$\sqrt{\lambda_n} \cdot \pi = \frac{\pi}{2} + \pi n = \frac{\pi(2n+1)}{2}$

$\lambda_n = \left(\frac{2n+1}{2}\right)^2$

$X_n(x) = \sin\left(\frac{2n+1}{2} \cdot x\right)$

при $\lambda = \left(\frac{2n+1}{2}\right)^2$: $T_n''(t) + a^2 \cdot \left(\frac{2n+1}{2}\right)^2 \cdot T_n(t) = 0$

$k^2 + a^2 \cdot \left(\frac{2n+1}{2}\right)^2 = 0$, $k_{1,2} = \pm \frac{a \cdot (2n+1)}{2} \cdot i$

$T_n(t) = A_n \cos\left(\frac{a(2n+1)}{2} \cdot t\right) + B_n \cdot \sin\left(\frac{a \cdot (2n+1)}{2} \cdot t\right)$

$u_2(x,t) = \sum\limits_{n=0}^{\infty} T_n(t) X_n(x) = \sum\limits_{n=0}^{\infty} \left(A_n \cdot \cos \frac{a(2n+1)}{2} \cdot t\right) + B_n \cdot$

$\cdot \sin \frac{a(2n+1)}{2} \cdot t\Big)\Big) \cdot \sin\left(\frac{2n+1}{2} \cdot x\right)$

$u_2(x,0) = \sum\limits_{n=0}^{\infty} A_n \sin\left(\frac{2n+1}{2} \cdot x\right) = x$

$A_n = \frac{2}{\pi} \int\limits_0^{\pi} x \sin\left(\frac{2n+1}{2} \cdot x\right) dx = \left[\begin{array}{ll} u = x & du = dx \quad v = -\frac{2}{2n+1} \cdot \cos\left(\frac{2n+1}{2} \cdot x\right) \\ dv = \sin\left(\frac{2n+1}{2} \cdot x\right) dx & \end{array}\right]$

$= \frac{2}{\pi} \cdot \frac{-2}{2n+1} \left(x \cos\left(\frac{2n+1}{2} \cdot x\right)\Big|_0^{\pi} - \int\limits_0^{\pi} \cos\left(\frac{2n+1}{2} x\right) dx\right) =$

$$= -\frac{4}{\pi(2n+1)} \cdot \left(\overbrace{\pi \cos\left(\frac{2n+1}{2} \cdot \pi\right)}^{=0} - \frac{2}{2n+1} \cdot \sin\left(\frac{2n+1}{2} \cdot x\right) \Big|_0^{\pi} \right) =$$

$$= -\frac{4}{\pi(2n+1)} \cdot -\frac{2}{2n+1} \sin\left(\frac{2n+1}{2} \cdot \pi\right) = \frac{8}{\pi(2n+1)^2} \cdot (-1)^n$$

$$u_{2t}(x,0) = \sum_{n=0}^{\infty} \frac{a(2n+1)}{2} \cdot B_n \cdot \sin\left(\frac{2n+1}{2} \cdot x\right) = x + \pi - 3$$

$$\frac{a(2n+1)}{2} \cdot B_n = \frac{2}{\pi} \int_0^{\pi} (x+\pi-3) \cdot \sin\left(\frac{2n+1}{2} \cdot x\right) dx =$$

$$= \left[\begin{array}{l} u = x+\pi-3 \,; \; du = dx \\ dv = \sin\left(\frac{2n+1}{2} x\right) dx, \; v = -\frac{2}{2n+1} \cos\left(\frac{2n+1}{2} \cdot x\right) \end{array} \right] =$$

$$= \frac{2}{\pi} \cdot -\frac{2}{2n+1} \cdot \left((x+\pi-3) \cos\left(\frac{2n+1}{2} \cdot x\right) \Big|_0^{\pi} - \int_0^{\pi} \cos\left(\frac{2n+1}{2} \cdot x\right) dx \right) =$$

$$= -\frac{4}{\pi(2n+1)} \cdot \left((\pi+\pi-3) \cdot \cos\left(\frac{2n+1}{2} \cdot \pi\right) - (\pi-3) - \frac{2}{2n+1} \sin\left(\frac{2n+1}{2} \cdot x\right) \Big|_0^{\pi} \right) =$$

$$= -\frac{4}{\pi(2n+1)} \cdot \left(3 - \pi - \frac{2}{2n+1} \cdot \sin \frac{2n+1}{2} \cdot \pi \right) = -\frac{4}{\pi(2n+1)} \cdot \left(3 - \pi - \right.$$

$$\left. - \frac{2}{2n+1} \cdot (-1)^n \right) = \frac{4}{\pi(2n+1)} \cdot \left(\pi - 3 + \frac{2 \cdot (-1)^n}{2n+1} \right)$$

$$B_n = \frac{8}{a\pi(2n+1)^2} \cdot \left(\pi - 3 + \frac{2 \cdot (-1)^n}{2n+1} \right)$$

$$u_2 = \sum_{n=0}^{\infty} \left(\frac{8}{\pi(2n+1)^2} \cdot (-1)^n \cdot \cos\left(\frac{a(2n+1)}{2} \cdot t\right) + \frac{8}{a\pi \cdot (2n+1)^2} \cdot \right.$$

$$\left. \cdot \left(\pi - 3 + \frac{2 \cdot (-1)^n}{2n+1} \right) \cdot \sin\left(\frac{a(2n+1)}{2} \cdot t\right) \right) \cdot \sin\left(\frac{2n+1}{2} \cdot x\right)$$

$$u = u_1 + u_2$$

$$u = t \cdot (1-x) + \sum_{n=0}^{\infty} \left(\frac{8}{\pi(2n+1)^2} \cdot (-1)^n \cdot \cos\left(\frac{a(2n+1)}{2} \cdot t\right) + \right.$$

$$\left. + \frac{8}{a\pi(2n+1)^2} \cdot \left(\pi - 3 + \frac{2 \cdot (-1)^n}{2n+1} \right) \cdot \sin\left(\frac{a(2n+1)}{2} \cdot t\right) \right) \cdot \sin\left(\frac{2n+1}{2} \cdot x\right)$$

Проверка решения:

$$u(0,t) = t + \sum_{n=0}^{\infty} (\dots) \cdot \sin(0) = t$$

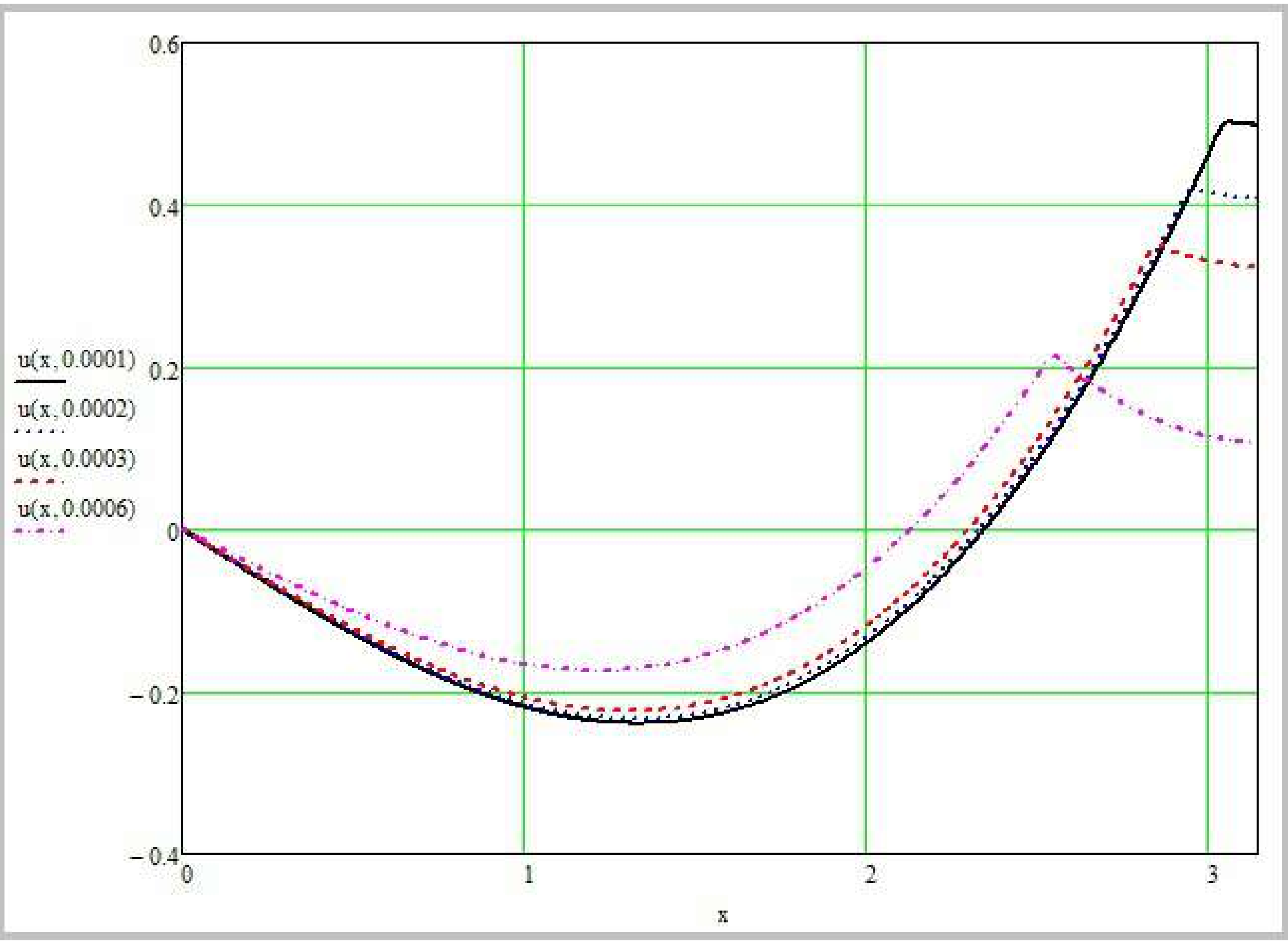
$$u_x(\pi, t) = -t + \sum_{n=0}^{\infty} (\dots) \cdot \frac{2n+1}{2} \overset{=0}{\cos\left(\frac{2n+1}{2} \cdot \pi\right)} = -t$$

$$u(x,0) = \sum_{n=0}^{\infty} \frac{8}{\pi(2n+1)^2} \cdot (-1)^n \cdot \sin\left(\frac{2n+1}{2} \cdot x\right) = x$$

$$u_t(x,0) = 1 - x + \sum_{n=0}^{\infty} \frac{4}{\pi(2n+1)} \cdot \left(\pi - 3 + \frac{2 \cdot (-1)^n}{2n+1}\right) \cdot \sin\left(\frac{2n+1}{2} \cdot x\right) =$$

$$= 1 - x + x + \pi - 3 = \pi - 2$$

Верно.


0.6
0.4
0.2
0
− 0.2
− 0.4
0
1
2
3
x
u(x, 0.0001)
u(x, 0.0002)
u(x, 0.0003)
u(x, 0.0006)

№3.

$$\begin{cases} \Delta u = 0, \ 0<r<1, \ r_0 = 1, \ \varphi \in [0, 2\pi] \\ \dfrac{\partial u(r_0, \varphi)}{\partial r} = 2\sin\varphi \end{cases}$$

$$\Delta u = \frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2} \cdot \frac{\partial^2 u}{\partial \varphi^2} = 0$$

$$u(r, \varphi) = R(r) \cdot \Phi(\varphi)$$

$$\frac{1}{r} \cdot (rR'(r))' \cdot \Phi(\varphi) + \frac{1}{r^2} \cdot R(r) \cdot \Phi''(\varphi) = 0$$

$$\frac{r(rR'(r))'}{R(r)} + \frac{\Phi''(\varphi)}{\Phi(\varphi)} = 0$$

$$\frac{r(r \cdot R'(r))'}{R(r)} = -\frac{\Phi''(\varphi)}{\Phi(\varphi)} = \lambda$$

$$\begin{cases} \Phi''(\varphi) + \lambda \Phi(\varphi) = 0 \\ \Phi(\varphi) = \Phi(\varphi + 2\pi) \end{cases} \qquad \Phi(\varphi) = C_1 \cdot \cos(\sqrt{\lambda}\varphi) + C_2 \sin(\sqrt{\lambda}\varphi)$$

$$\sqrt{\lambda} = n, \ \lambda_n = n^2$$

$$\Phi_n(\varphi) = C_{1n} \cdot \cos(n\varphi) + C_{2n} \cdot \sin(n\varphi)$$

при $n = 0, \lambda_0 = 0, \ \Phi_0''(\varphi) = 0$

$$\Phi_0(\varphi) = A \cdot \varphi + B$$

$$A(\varphi + 2\pi) + B = A\varphi + B \Rightarrow A = 0$$

$$\lambda_0 = 0, \ \Phi_0(\varphi) = 1$$

$$r \cdot (rR'(r))' = \lambda R(r)$$

$$r^2 \cdot R''(r) + r \cdot R'(r) - n^2 \cdot R(r) = 0$$

$$R(r) = r^{\alpha}$$

$$R'(r) = \alpha r^{\alpha-1} \qquad R''(r) = \alpha(\alpha - 1) \cdot r^{\alpha-2}$$

$$r^2 \cdot (\alpha \cdot (\alpha - 1)) r^{\alpha-2} + r \cdot \alpha \cdot r^{\alpha-1} - n^2 \cdot r^{\alpha} = 0$$

$$\alpha^2 - \alpha + \alpha - n^2 = 0$$

$$\alpha^2 - n^2 = 0$$

$$(\alpha - n)(\alpha + n) = 0$$

$$\alpha_{1,2} = \pm n$$

$R_n(r) = A_n \cdot r^n + B_n \cdot r^{-n}$

при $\lambda_0 = 0$: $(r \cdot R_0'(r))' = 0$

$$r \cdot R_0'(r) = A_0$$

$$R_0'(r) = \frac{A_0}{r}$$

$$R_0(r) = A_0 \cdot \ln(r) + B_0$$

$u(r,\varphi) = R_0(r) + \Phi_0(\varphi) + \sum_{n=1}^{\infty} R_n(r) \cdot \Phi_n(\varphi) = A_0 \cdot \ln(r) + B_0 +$

$+ \sum_{n=1}^{\infty} (A_n \cdot r^n + B_n \cdot r^{-n}) \cdot (C_{1n} \cdot \cos(n\varphi) + C_{2n} \sin(n\varphi))$

Реш-е должно быть огранич. при $r=0 \Rightarrow A_0 = B_n = 0$

Пусть $B_0 = \frac{a_0}{2}$, $A_n \cdot C_{1n} = a_n$, $A_n \cdot C_{2n} = b_n$

т.е. решение ищется в виде

$u(r,\varphi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \cdot (a_n \cdot \cos(n\varphi) + b_n \cdot \sin(n\varphi))$

$\frac{\partial u}{\partial r} = \sum_{n=1}^{\infty} n \cdot r^{n-1} (a_n \cdot \cos(n\varphi) + b_n \cdot \sin(n\varphi))$

$\frac{\partial u(1,\varphi)}{\partial r} = \sum_{n=1}^{\infty} n \cdot (a_n \cdot \cos(n\varphi) + b_n \cdot \sin(n\varphi)) = 2\sin\varphi$

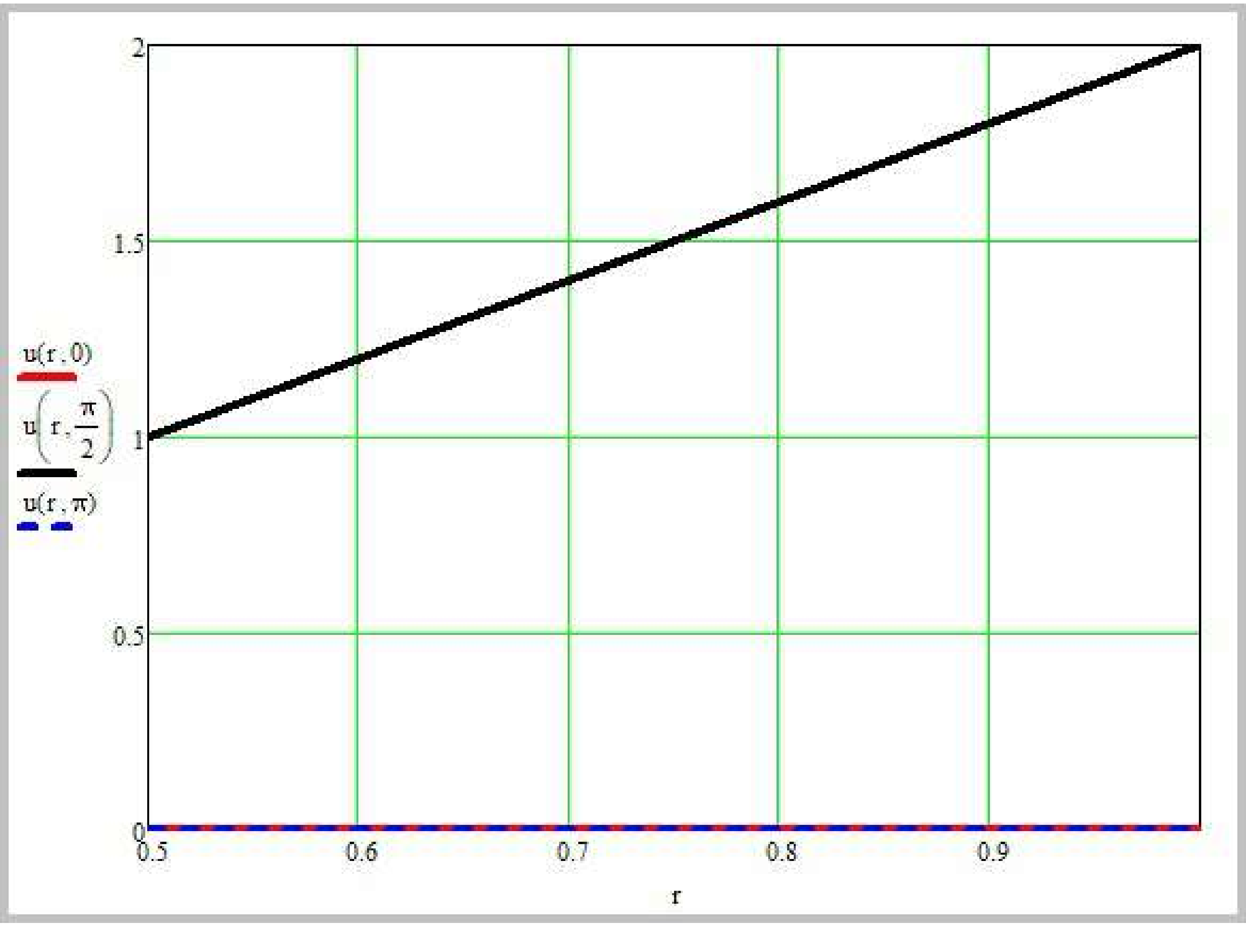
$a_n = 0$

$b_1 = 2$

$b_n = 0$ при $n \neq 1$

При постр. графиков будем считать $a_0 = 0$

$u(r,\varphi) = 2r\sin\varphi + \frac{a_0}{2}$


2
1.5
1
0.5
0
0.5
0.6
0.7
0.8
0.9
r
u(r, 0)
u(r, π/2)
u(r, π)

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[3] Боголюбов А. Н, Кравцов В. В. *Задачи по математической физике* — М.: Издательство Московского университета, 1998.