Tutorial 4

The core exercises are considered part of the course material, and you are advised to at least finish these. For additional practice and interesting applications, advanced exercises are available as well – feel free to pick and choose from these.

Unless otherwise specified, we expect you to create your own algorithms for the exercises in each tutorial. This means that you should not use libraries (e.g. SciPy) for the methods that are part of the course material.

Core exercises

1. **Integration of Functions**

   (a) Write a function that implements the simple trapezoid rule, and use it to calculate the following integrals:

   $$\int_1^5 x^2 \mathrm{d}x, \tag{1}$$

   $$\int_0^\pi \sin(x) \mathrm{d}x. \tag{2}$$

   (b) Now implement a function that uses Simpson's method to calculate the integrals. First, call the trapezoid method twice - once using N points, and once with $2N$ points, then combine them to get Simpson's method as in the lecture.

   (c) Now extend this routine into a Romberg integration routine by following the algorithm showed in the lecture. Compare the results given by the three methods, and see which one gives the most accurate estimation. Use $N = 8$ and a 6th order Romberg method (that is, 6 initial approximations).

   (d) Calculate the value of

   $$\int_0^{10} \left[3\exp(-2x) + \left(\frac{x}{10}\right)^4 \right] \mathrm{d}x \tag{3}$$

   for N equally spaced points in the given range for x. Use the trapezoid and Simpson's rule for $N = 10$, 100, 1000 and 10000, as well as the Romberg method with $N = 10$ and $m = 2$, 4 and 6. Calculate the exact value of the integral, or a highly precise estimation using e.g. WolframAlpha, and compare the results. Which method gave the best estimation and for which value of N? What is the dependence of N with the error of calculation? (error = Exact - Estimated value)

2. **Improper Integrals**

   (a) Consider the following integral:

   $$\int_0^1 \frac{\mathrm{d}x}{1-x}. \tag{4}$$

   What kind of integral is this, and can you use your functions from the previous exercise to solve it, or do you need to change anything in your algorithms?

   (b) The Gamma function has various uses in astronomy to define or estimate properties of objects. One way to define the complete Gamma function is by using the following convergent improper integral:

   $$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} \mathrm{d}x. \tag{5}$$

   Estimate $\Gamma(6)$. (Hint: Divide the integral in parts with convenient limits first.)

   (c) The complete Gamma function is equal to $\Gamma(z) = (z-1)!$ for integer z. Calculate $\Gamma(6)$ using this formula and compare with your estimation.

(d) The Incomplete Gamma Function is another variety of Gamma function and has multiple uses in astronomy too. One way of defining it is as follows:

$$\Gamma(z,x) = \int_x^\infty t^{z-1} e^{-t} \mathrm{d}t. \tag{6}$$

Estimate $\Gamma(6, 2.5)$.

Advanced exercises

3. **Linear Driven Oscillator**

Consider a linear driven oscillator with damping, which is described by the following differential equation:

$$\ddot{x} + 2\beta\dot{x} + \omega_r^2 x = F(t). \tag{7}$$

In general there are several methods in which a differential equation of second order can be solved. In our case we will use a more advanced analytical technique that uses Green's functions. A Green's function is defined as the response to a driving force $F(t) = \delta(t - t')$. In the case the linear driven oscillator the Green's function can be calculated analytically as:

$$G(t;t') = \begin{cases} \frac{1}{\omega_r m} e^{-\beta(t-t')} \sin(\omega_r(t-t')) & \text{if } t > t'; \\ 0 & \text{if } t < t'. \end{cases} \tag{8}$$

(a) Give the expression for the position $x(t)$ if $F(t)$ is known.

(b) In this exercise we will consider an driving force of the following form:

$$F(t) = \begin{cases} 0 & \text{if } t < -\tau; \\ A\left(1 - 3\left(\frac{t}{\tau}\right)^2\right) & \text{if } -\tau < t < \tau; \\ 0 & \text{if } t > \tau. \end{cases} \tag{9}$$

Plot the driving force. For the moment consider the following variable values in your problem: $\tau = 0.8$, $\beta = 0.05$ and $\omega_r = 6$.

(c) Write down the formal expression for $x(t)$ using the Green's function technique.

(d) Write a computer program to solve for $x(t)$ using the Green's function technique. Use your previously written functions for the Trapezoid and Simpson's rules to calculate this integral, also compare with a `scipy.integrate` function of your choice.

(e) Take the limit $\tau\omega_r \gg 1$. Explain your result qualitatively.

(f) Take the limit $\tau\omega_r \ll 1$. Explain your result qualitatively.