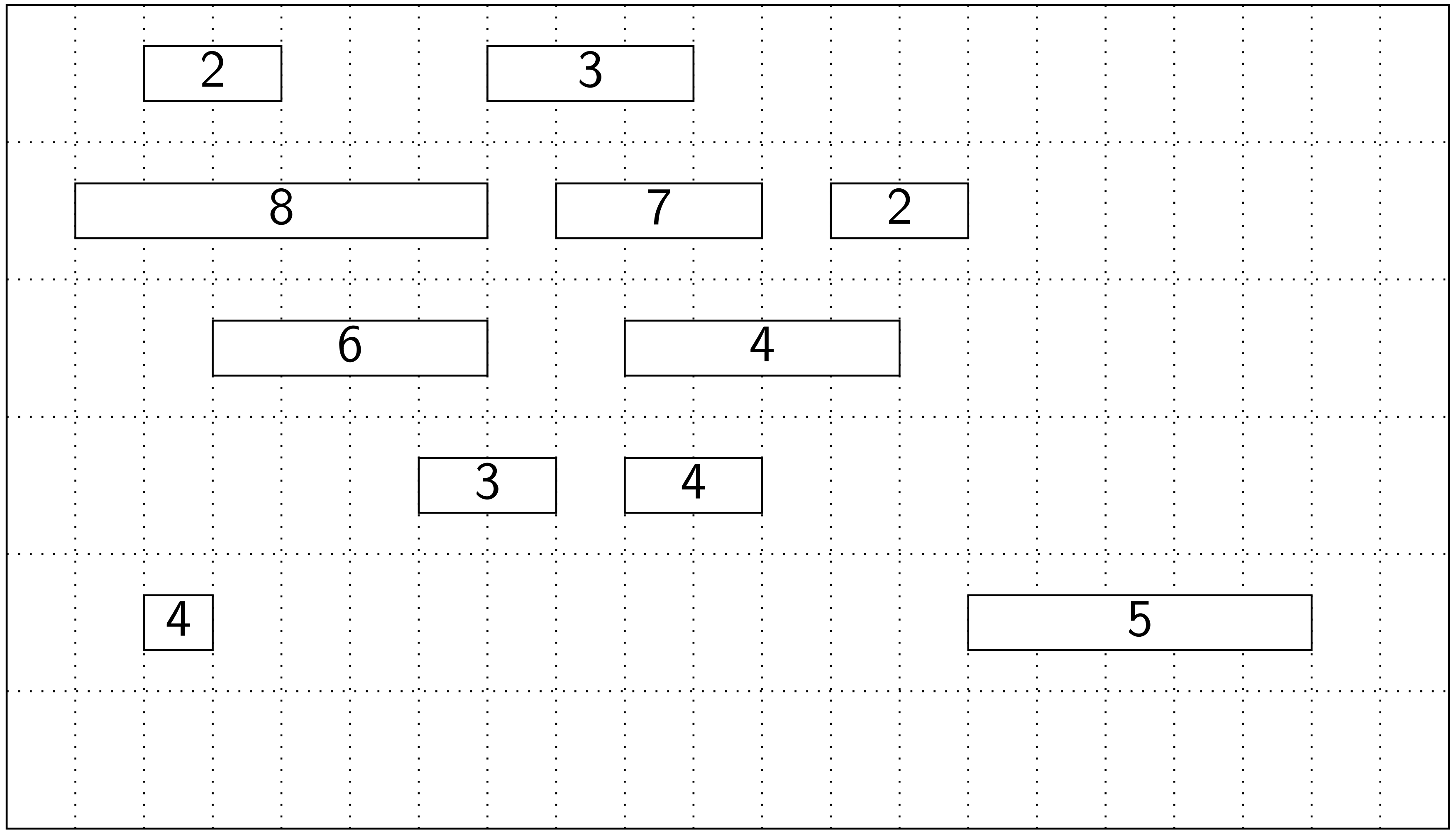
Contents Lecture 6

- Weighted interval scheduling
- Recursion or iteration
- Subset sums and knapsacks
- RNA: pairs of molecules: in $(1, n)$, find a t which splits $(1, n)$ optimally
- DNA: sequence alignment (lab 5)
- Shortest paths in directed graph with negative edge costs: Bellman-Ford

Weighted interval scheduling

- Recall interval scheduling:
    - Input is a set of requests r_i with start s_i and finish f_i times
    - Two requests conflict if their intervals overlap
    - We want to select the maximum number of nonconflicting requests
    - We have seen this can be done with a greedy algorithm which selects as next request the request with earliest finish time and which does not conflict with any already selected request
- In the weighted interval scheduling problem each request has a value v_i
- Now we want to maximize the sum of values v_i of the selected requests
- We will take an approach which at first may seem to be extremely slow

An example set R with values



- Values of requests are shown
- Later we will also need the following $p(k) =$ index of rightmost request that request k does not overlap with

Naming requests

- Assume the requests are named such that $f(r_1) \leq f(r_2) \leq \ldots \leq f(r_n)$
- We write instead: $f(1) \leq f(2) \leq \ldots \leq f(n)$
- Each request has a value $v(i)$
- In interval scheduling we started with r_1
- Now we will instead consider each request starting with the last, r_n
- Let T be an optimal schedule and $OPT(n)$ be the sum of the selected $v(i)$ from requests $r_1, r_2, \ldots, r_n$.
- We will make our own optimal schedule S, also with value $OPT(n)$

Our algorithm

- $p(n)$ is the request with maximum $f(i)$ which does not conflict with r_n
- We need to decide if r_n should be selected or not, so we have two cases:
  1. r_n is selected: in this case $OPT(n) = v(n) + OPT(p(n))$
  2. r_n is not selected: in this case $OPT(n) = OPT(n-1)$
- To decide which case to use, we evaluate both and see which is best.

```
**function** $OPT(n)$
    **if** $n = 0$ **then**
        **return** 0
    $a \leftarrow v(n) + OPT(p(n))$
    $b \leftarrow OPT(n-1)$
    **return** $\max(a, b)$
```

- This algorithm will recompute $OPT(i)$ a huge number of times
- If for all i $p(i) = i - 1$ then OPT will be called 2^n times
- Quiz: how can we make this practical instead of hopelessly slow?

Answer: remember already computed values

- Note that the value of $OPT(i)$ never changes
- So when we have computed $OPT(i)$ we can remember the value
- We save it in an array and use it next time $OPT(i)$ is needed
- Let $m[1], m[2], \ldots, m[n] = -\infty$ initially

```
function $OPT(n)$
    if $n = 0$ then
        return 0
    else if $m[n] = -\infty$ then
        $a \leftarrow v(n) + OPT(p(n))$
        $b \leftarrow OPT(n-1)$
        $m[n] \leftarrow \max(a, b)$
    return $m[n]$
```

- Remembering values like this is called **memoization**

Avoiding recursion

- Recursion can simplify life but function calls and returns take time
- We can just produce the array *m* directly

```
procedure $make\_table(n)$
    $m[0] \leftarrow 0$
    $i \leftarrow 1$
    while $i \leq n$
        $a \leftarrow v(i) + m[p(i)]$
        $b \leftarrow m[i-1]$
        $m[i] \leftarrow \max(a, b)$
        $i \leftarrow i + 1$

function $OPT(n)$
    return $m[n]$
```

Dynamic programming

- What we just saw is an example of **dynamic programming**
- We express a solution in terms of solutions to smaller problems
- This aspect is similar to divide and conquer
- There is a big difference: with dynamic programming we come back to the same problem multiple times — called **overlapping subproblems**
- With divide and conquer we solve **independent** smaller problems
- The power of dynamic programming comes from avoiding recomputing already solved subproblems
- We find an optimal solution by combining optimal solutions to smaller problems — called **optimal substructure**
- What is nice with dynamic programming is that it usually is trivial to prove optimality since we check all solutions.

Origin of dynamic programming

- We will see several examples how we should think when using dynamic programming
- This technique was invented in the 1950's by Richard Bellman
- In this context programming is not ”computer programming” but instead finding an optimal solution, or, program, to achieve typically a military scheduling problem (as linear programming in mathematics)
- Bellman wanted a fancy name so he could continue working on this with funding from the US department of defence

The subset sum problem

- How to bring as much hand luggage as possible on a flight
- You are allowed to bring at most W kilograms of hand luggage
- You have n items, and an item i has weight w_i
- Select a subset S of these items so that
    - $T = \sum_{i \in S} w_i \leq W$
    - T is as large as possible
- No greedy algorithm is known for this problem
- How can we use dynamic programming here?
- We need to consider both weights w_i and W
- If we select item i with weight w_i we have $W - w_i$ left...

Dynamic programming approach

- $T = \sum_{i \in S} w_i \leq W$, maximize T
- Consider an optimal solution which can choose from n items for an allowable weight W
- Either item n is included or it is not. Excluding n may be due to $w_n > W$ or because it is simply better to skip it
- For instance if the items have weights $\{3, 7, 8\}$ and $W = 10$ it is better to skip the 8 kg item
- If we then select the 7 kg item, we clearly have $W - 7$ kg left

$$OPT(n, W) = \begin{cases} 0, & n = 0 \\ OPT(n-1, W), & w_n > W \\ \max(OPT(n-1, W), & \\ \quad w_n + OPT(n-1, W - w_n)), & \textit{otherwise} \end{cases}$$

Running time

- This is not polynomial time
- The running time is dependent on the value of W
- This is called pseudo-polynomial time
- The time complexity is $O(nW)$ which is bad for large nW

The knapsack problem

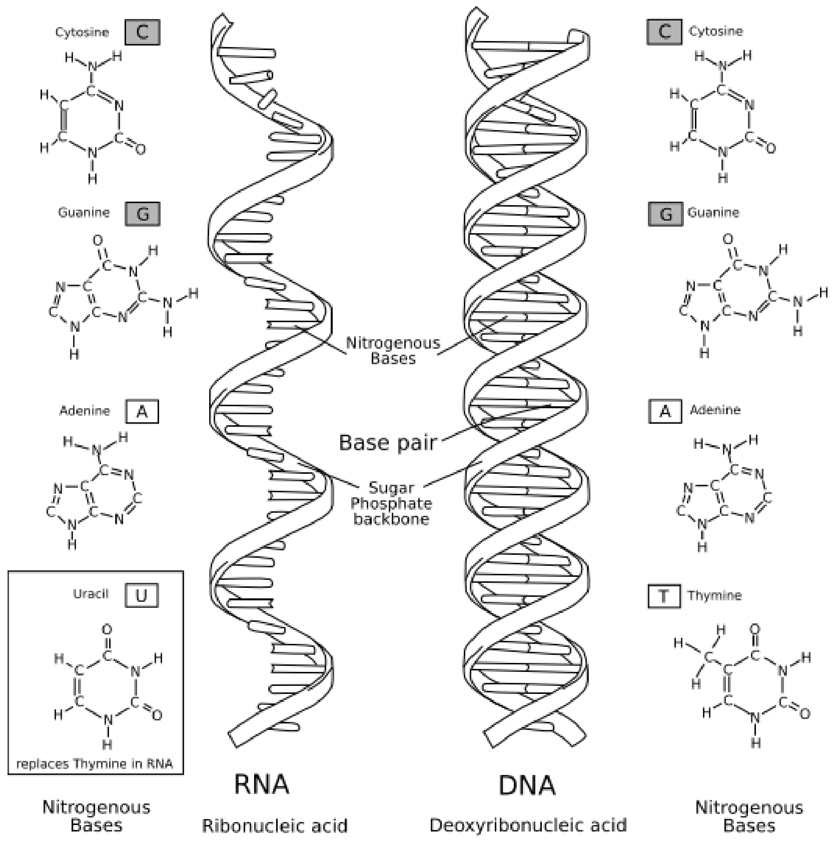
- Similar to the subset sum problem
- Now each item has both a weight w_i and a value v_i
- Select a subset S of n items so that
    - $\sum w_i \leq W$
    - $\max \sum v_i$
- The solution is very similar to that of subset sum. Just add the values instead:

$$OPT(n, W) = \begin{cases} 0, & n = 0 \\ OPT(n-1, W), & w_n > W \\ \max(OPT(n-1, W), & \\ \quad v_n + OPT(n-1, W - w_n)), & \textit{otherwise} \end{cases}$$

Solving subset-sum and knapsack problems

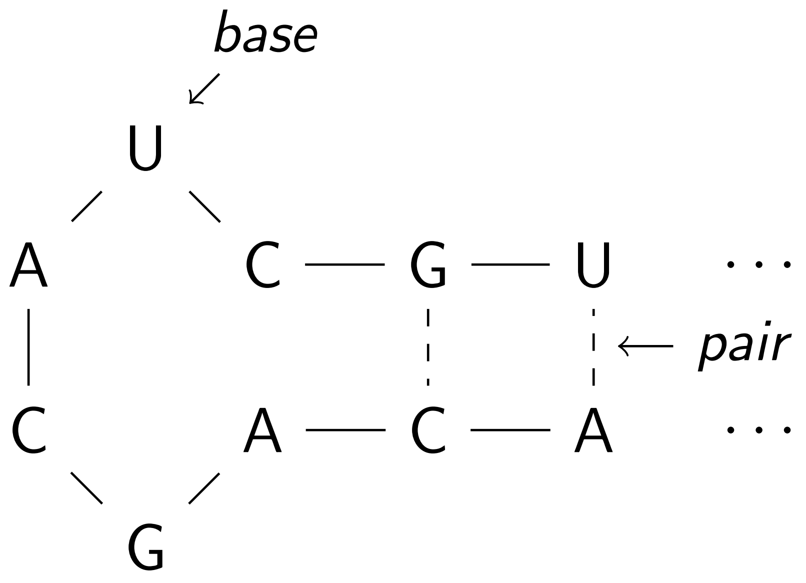
- These are real world problems
- Variants include cutting paper in a clever way to reduce waste
- They are examples of so called NP-complete problems
- Practical approaches include using
  - dynamic programming if $W \cdot n$ is sufficiently small
  - branch-and-bound — see last lecture
  - integer linear programming — see last lecture

RNA



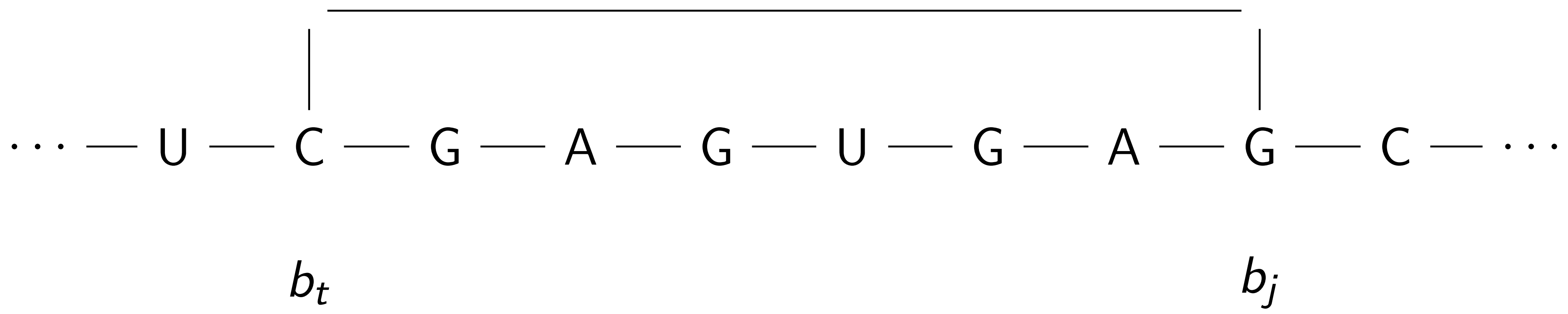
- RNA is a string $B = b_1 b_2 \dots b_n$ over the alphabet $\{C, G, A, U\}$
- Compared with DNA it is single stranded and due to this there are secondary structures when it connects to itself according to certain rules

Secondary structures



- A secondary structure is a matching $S = \{(b_i, b_j)\}$
- A pair is shown as two molecules connected with a dashed edge
- A molecule can only pair with at most one other molecule
- A and U can pair, and C and G can pair
- Pairing molecules cannot be too close: $(b_i, b_j) \in S \Rightarrow i < j - 4$
- No crossing pairs: if $i < j < k < l$ then (b_i, b_k) and (b_j, b_l) cannot both be in S
- The problem is to find an S with a maximal number of pairs

An $OPT(i, j)$ function



- Initially called with $OPT(1, n)$
- Then for some arbitrary call we have $OPT(i, j)$
- b_j is our rightmost symbol, or molecule.
- When b_j pairs with some b_t the noncrossing condition splits up our remaining interval in two halves:
    - $b_i \dots b_{t-1}$
    - $b_{t+1} \dots b_{j-1}$

An $OPT(i,j)$ function

- Case 1: $i \geq j-4$: $OPT(i,j) = 0$
- Case 2: There is no available molecule to create a pair for b_j: $OPT(i,j) = OPT(i,j-1)$
- Case 3: Taking rules used in Cases 1 and 2 into account, a t is selected which maximizes: $OPT(i,j) = 1 + \max_t\{OPT(i,t-1) + OPT(t+1,j-1)\}$
- $\max_t$ means select the t which maximizes the expression
- The time complexity is $O(n^3)$, since there are $O(n^2)$ intervals and selecting t is $O(n)$

String alignment: how similar are two strings?

- Comparing `"abcd"` and `"abd"` we can say that there is a `'c'` missing
- Comparing `"abcd"` and `"abed"` we may say:
  - the `'c'` and `'e'` should have been the same but where not, or
  - the right string has a missing `'c'` and the left a missing `'e'`
- We can put a value on these differences:
  - For a mismatch: there is a cost of α_{pq} with p and q being Unicode characters or members of some other alphabet such as symbols in DNA strings
  - If there is a missing character: δ
- For instance, α_{qw} may be 1 since 'q' and 'w' are close on a keyboard and $\alpha_{qk} = 3$ since they are more distant
- For a missing character, we may give it a cost $\delta = 2$ for instance
- To say how similar two strings are, we want to find the smallest cost of "fixing" the strings so they become identical.

An example of using α_{pq}

- Assume $\alpha_{cd} = 3$
- Of course $\alpha_{pp} = 0$ for every character p
- We can compare `"abc"` and `"abd"`
- Starting from the end we simply note the cost $\alpha_{cd} = 3$ and move on to the next pair of characters
- `"ab"` and `"ab"` remain with no cost

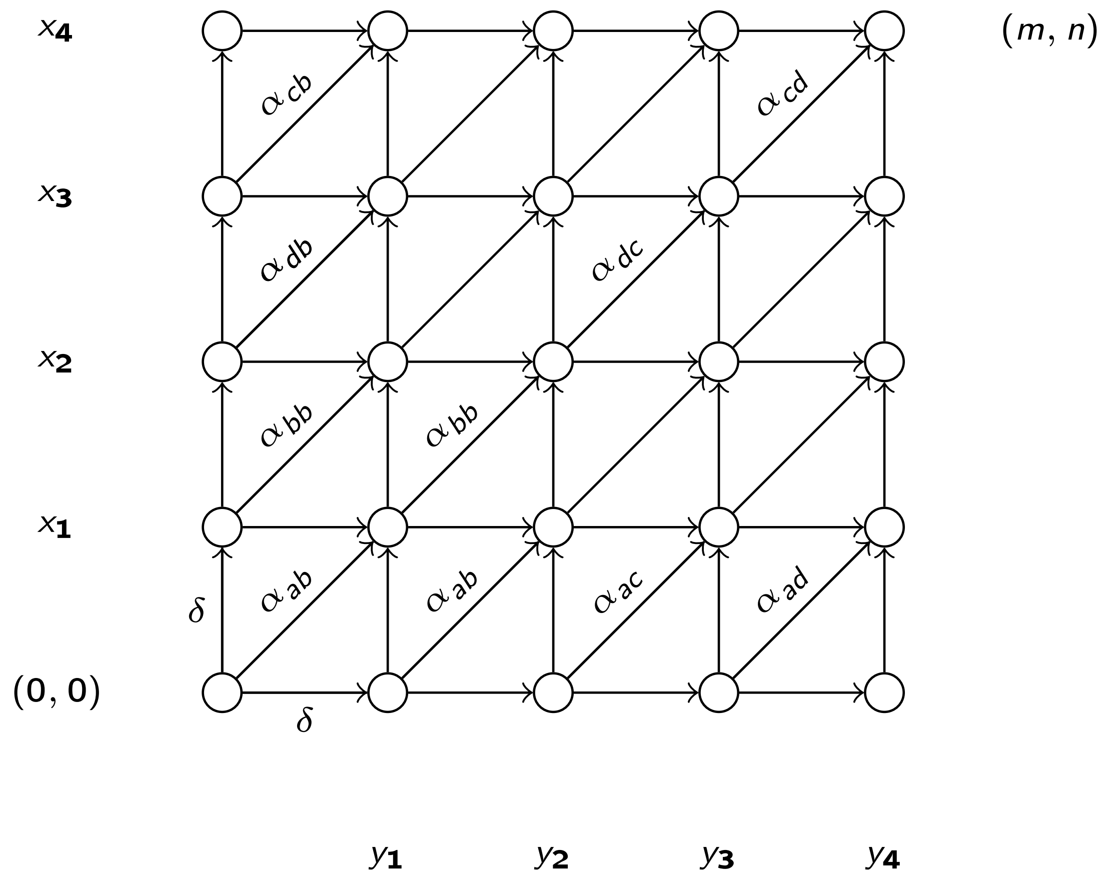
An example of using δ

- We again compare `"abc"` and `"abd"`
- Starting from the end we either see this as
  - the left string misses a `'d'`, or
  - the right string misses a `'c'`
- Let us use the first case. It means we "insert" the `'-'` and get: `"abc-"` i.e. there is a gap in the left string
- We don't actually insert any `'-'` in the algorithms we will see soon, but the dashes are used when printing the output
- The gap in the left string is removed together with the `'d'` in the right string
- We then have `"abc"` and `"ab"`

X, Y, and $OPT(i,j)$

- $X = "abc"$ and $Y = "abd"$
- $X = x_1x_2x_3$ and $Y = y_1y_2y_3$
- The cost of an optimal alignment of $X = x_1x_2 \dots x_i$ and $Y = y_1y_2y \dots y_j$ is denoted $OPT(i,j)$
- $OPT(i,0) = i\delta$ since it ignores Y and aligns a string of i symbols with an empty string — which must be done with $i\delta$
- $OPT(0,j) = j\delta$ for similar reason
- $OPT(1,1)$ is the minimum of $\alpha_{x_1y_1}$ and 2δ
- It is clear what happens when we use $\alpha_{x_1y_1}$ — we are "charged" with the mismatch cost of $\alpha_{x_1y_1}$, which in our example is $\alpha_{aa} = 0$
- In the other case, we use one δ to skip either x_1 or y_1 and then another δ to skip the other of x_1 and y_1

We can view the alignment as a graph



- Another example: $X = "abdc"$ and $Y = "bbcd"$
- A vertical δ eats one symbol from X and leaves Y unchanged
- A horizontal δ eats one symbol from Y and leaves X unchanged
- $OPT(i, j)$ is equivalent to finding a shortest path from $(0, 0)$ to (i, j) in this graph, called G_{XY}

An $OPT(i,j)$ function

- It is probably clear now how we can write an optimal function to find the set of α and δ operations with minimum cost
- We have two strings $X = x_1x_2\dots x_m$ and $Y = y_1y_2\dots y_n$
- Using dynamic programming we write:
  - Case 1: $OPT(i,j) = \alpha_{x_i,y_j} + OPT(i-1,j-1)$
  - Case 2: $OPT(i,j) = \delta + OPT(i,j-1)$
  - Case 3: $OPT(i,j) = \delta + OPT(i-1,j)$
- As usual, we evaluate all cases and select the minimum
- We compute a table $A[0..m][0..n]$ using the recurrence for OPT
- A is initialized with $A[i][0] \leftarrow i\delta$ for each i, and
- A is initialized with $A[0][j] \leftarrow j\delta$ for each j.

Bellman-Ford shortest path algorithm

- Consider a directed graph (V, E) with n nodes and m edges.
- Edge costs c_{vw} are allowed to be negative in this algorithm
- The sum of costs on the edges in a cycle must be positive (otherwise no shortest path)
- The problem is to find the minimum cost path from s to t
- Let $OPT(i, v)$ be the minimum cost of a path from v to t which uses at most i edges
- The initial problem is $OPT(n-1, s)$ which can be solved by:

$$OPT(i, v) = \begin{cases} 0, & v = t \\ \infty, & i = 0 \\ \min\{OPT(i-1, v), OPT(i-1, w) + c_{vw}\} & i \geq 1 \end{cases}$$

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- We can create the table M with $O(n^2)$ space from $OPT(i, v)$
- M can be created in time $O(n^3)$ for a dense graph

Creating M

```
**int** $M[n][n]$ // $M[i][j]$ is distance from $j$ to $t$ using $i$ edges
**procedure** $make\_table(G, s, t)$
    $n \leftarrow |V|$
    $M[0][t] \leftarrow 0$
    $M[0][v] \leftarrow \infty$ for $v \in V - \{t\}$
    $i \leftarrow 1$
    **while** $i \leq n - 1$ **do**
        **for** $v \in V$ **do**
            $M[i][v] \leftarrow \min\{M[i-1][v], M[i-1, w] + c_{vw}\}$
```

- This is a direct translation from the $OPT(i, v)$ recurrence
- $M[i][v]$ is the shortest path from v to t with at most i edges
- The M table can be used to compute a shortest path from s to t
- The Bellman-Ford algorithm is better than this, as we will see next

The Bellman-Ford algorithm

- Consider the for-loop again:

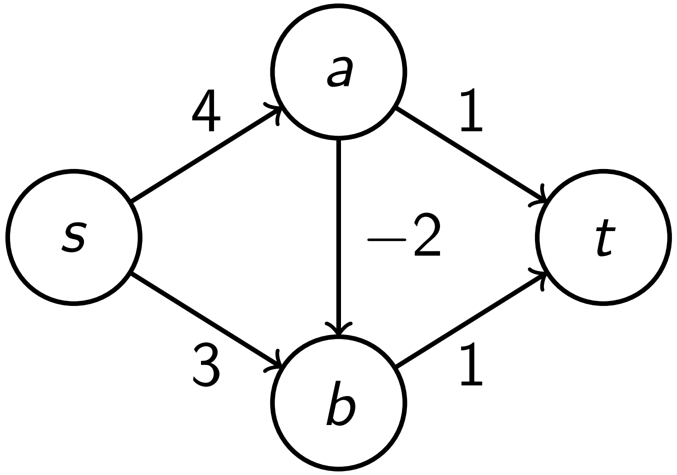
```
**for** $v \in V$ **do**
    $M[i][v] \leftarrow \min\{M[i-1][v], M[i-1, w] + c_{vw}\}$
```

- It checks each edge (v, w) to discover a shorter path from v
- We do not need a two-dimensional matrix
- Each vertex can have two attributes: `distance` and `succ`

```
**for** $e = (v, w) \in E$ **do**
    **if** $distance(v) > c_{vw} + distance(w)$ **then**
    **begin**
        $distance(v) \leftarrow c_{vw} + distance(w)$
        $succ(v) \leftarrow w$
    **end**
```

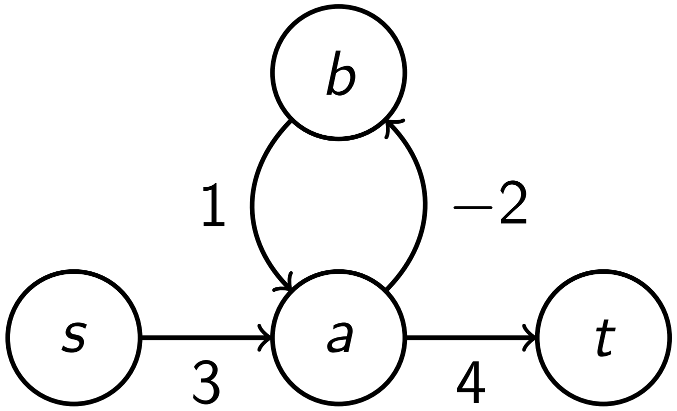
- This gives a running time $O(mn)$ — still $O(n^3)$ in a dense graph

An example



	s	a	b	t
0	∞	∞	∞	0
1	∞	1	1	0
2	4	1	-1	0
3	3	1	-1	0

Another example



	s	a	b	t
0	∞	∞	∞	0
1	∞	4	∞	0
2	7	4	5	0
3	7	5	5	0

```
for $e = (v, w) \in E$ do
    if $distance(v) > c_{vw} + distance(w)$ then
        print negative cycle detected
```