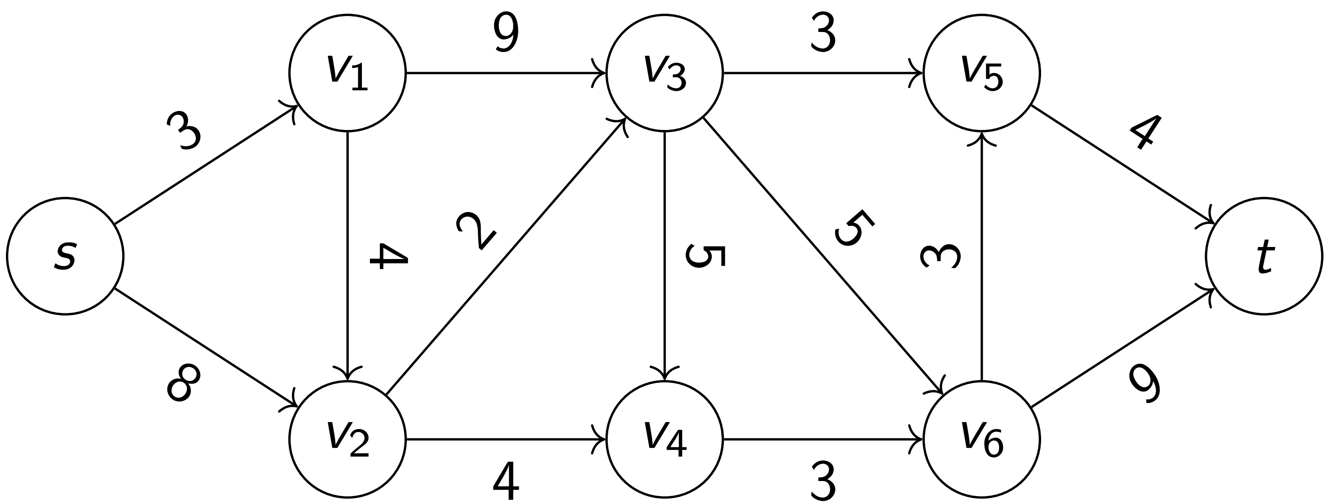
# Contents Lecture 7

- The maximum flow problem
- The Ford-Fulkerson algorithm
- Maximum flows and minimum cuts
- The preflow-push maximum flow algorithm

# A flow network

- A directed (or undirected) graph $G(V, E)$
- Each edge $e \in E$ has a nonnegative capacity $c(e)$
- A source node $s \in V$ with no predecessor
- A sink node $t \in V$ with no successor
- An example:




- A previous version of Lab 6 was about being an CCCP party member and solving a problem for railway transportations passing Minsk, using capacities estimated by US spies — hence book cover

# Terminology

- An $st - cut$ is a partition $(A, B)$ with $s \in A$ and $t \in B$. Also called simply a *cut*
- The **capacity** of a cut is

$$cap(A, B) = \sum_{e \text{ out from } A} c(e)$$

- For the previous graph, $cap(\{s\}, V - \{s\}) = 3 + 8$
- The **min-cut problem** is to find a cut of minimum capacity
- Useful information when bombing enemy railroads for instance
- Honest and respectful diplomacy towards a happy world is preferable

# A flow

- A **flow** is a function $f$ which says how much flows on each edge
- Often we want to use the edges to maximize the total flow from $s$ to $t$
- The algorithm design techniques we have studied so far are insufficient to solve this problem
- The **capacity constraint** says: for each $e \in E$, $0 \leq f(e) \leq c(e)$
- For undirected graphs, we need to specify the direction of the flow
- One way to do that is to fix the order of the nodes and use:
  - flow from $u$ to $v$ is positive
  - flow from $v$ to $u$ is negative
  - $u$ and $v$ need to agree on what is meant by positive flow

# Flow conservation constraint

- The flow coming in to a vertex $v$ must equal the flow going out from $v$
- This **flow conservation constraint** does not apply to the source $s$ and the sink $t$

$$v \in V - \{s, t\} : \sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out from } v} f(e)$$

- A water hose (vattenslang) cannot store any water
- Water systems are a good mental model for network flow

# The maximum flow problem

- The value of a flow $f$ is $\sum_{e \text{ out from } s} f(e) = \sum_{e \text{ in to } t} f(e)$
- The **maximum flow problem** is to find a flow $f$ with maximum value

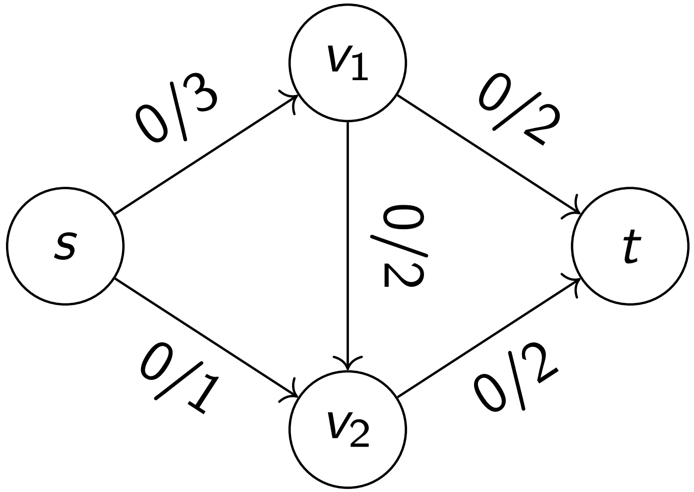
# The Ford-Fulkerson algorithm: overview

- The basic idea is very simple
  1. Start with a flow $f(e) = 0$ for every $e \in E$
  2. Look for a simple path $p$ from $s$ to $t$ such that on every edge $(u, v)$ in $p$ we can increase the flow in the direction from $u$ to $v$
  3. If we could not find any such path, we have the maximum flow
  4. Let each edge $e = (u, v)$ on $p$ have a value $\delta(e)$, which means room for improvement, or how much we can increase the flow on that edge
  5. Let $\Delta$ be the minimum of all $\delta(e)$ on $p$
  6. Increase the flow by $\Delta$ along the path $p$
  7. `goto` 2
- The risky part is number 3: how can we be sure of that?
- We will prove it is correct

# Some more details

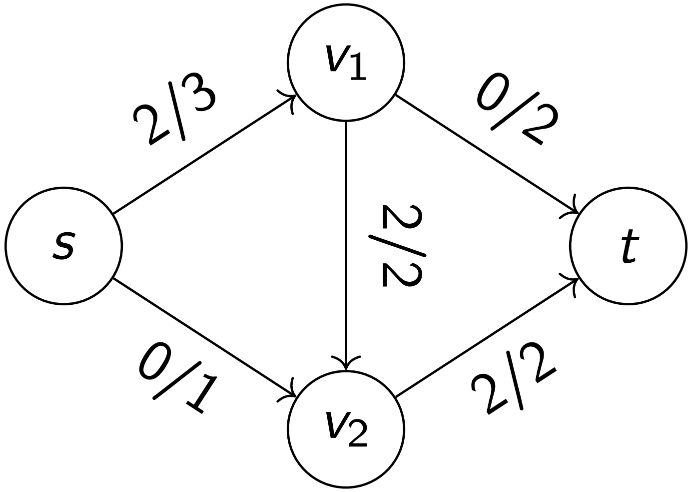
- What does it mean that $\delta(u, v) > 0$ ?
- Answer: $f(u, v) < c(u, v)$
- It is clear that if we find such a path $p$ we can increase the flow on each edge of that path $p$ by $\Delta$
- From what we have so far, we cannot decrease the flow of any edge, so we still easily can get stuck
- But consider an edge $e = (u, v)$ with a flow $f(e)$
- To decrease this flow, we can instead increase the flow of a new edge $(v, u)$ by up to the amount $f(e)$
- We thus need additional edges and therefore create a new graph $G_f$ for that

# An example



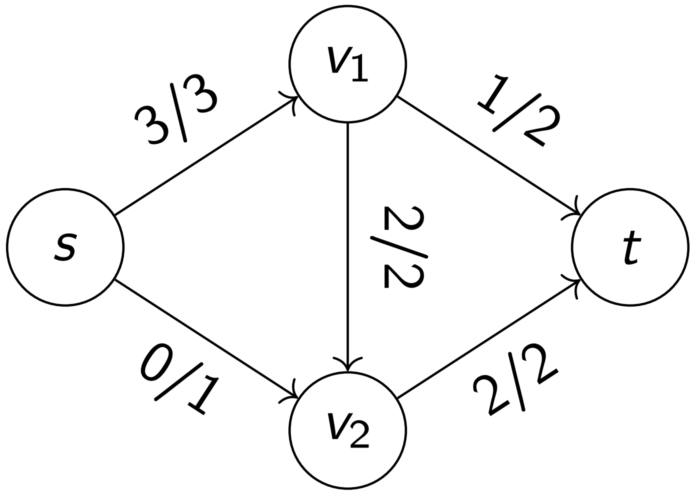

- Using BFS to find an $s - t$ path is a good idea
- Let the first BFS find the path: $p_1 = (s, v_1, v_2, t)$ with $\delta = 2$.

# An example



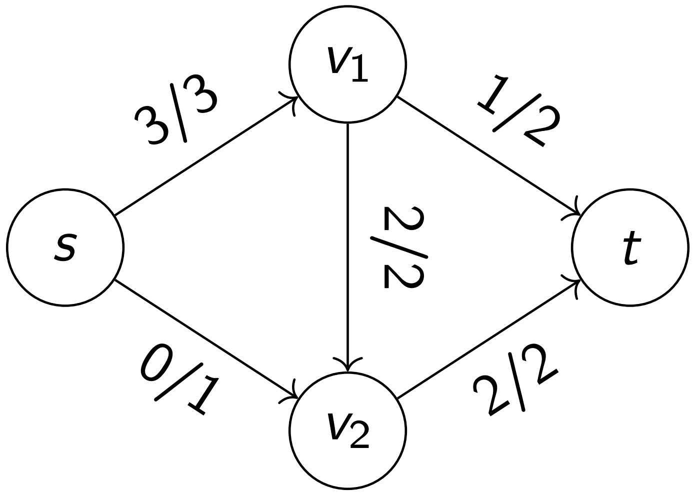

- In the second BFS there is "no edge" between $v_2$ and $t$ since its flow cannot be increased
- So BFS cannot reach $t$ going through $v_2$
- The path $p_2 = (s, v_1, t)$ with $\delta = 1$ is found.

# An example



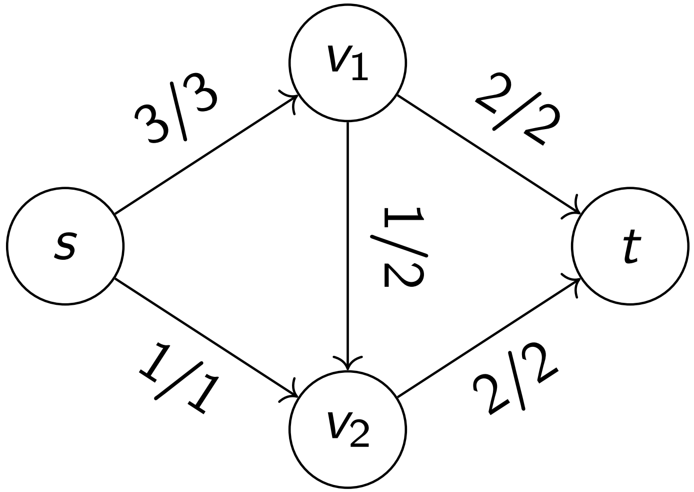

- Now no path can be found!
- What to do?

# An example



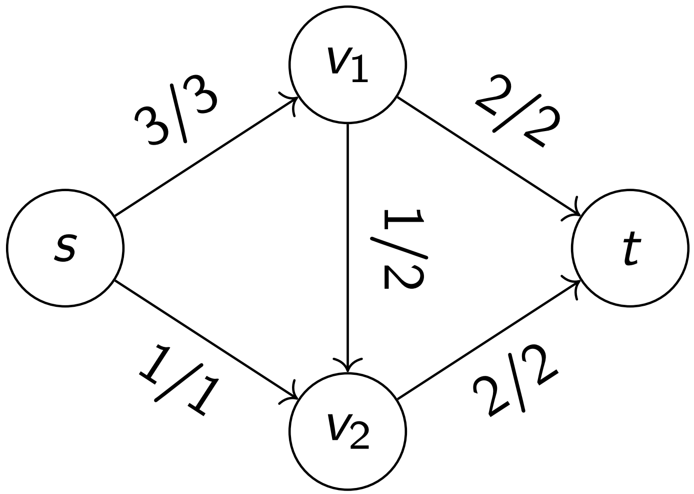

- Can we somehow send flow using $p_3 = (s, v_2, v_1, t)$?

# An example




- We have in some sense reduced the flow from $v_1$ to $v_2$
- Note we only changed the flow along $p_3 = (s, v_2, v_1, t)$?

# An example




- This is the maximum flow
- We need a simple and systematic approach for this
- The "residual graph" has the same nodes but edges correspond to where we can increase or decrease flow.
- In this graph an original edge is called a forward edge
- To reduce flow a backward edge is created

# The residual graph

- We create a **residual graph** $G_f$ with the same nodes as $G$
- An edge in $G$ becomes either one or two edges in $G_f$ (one of them with reversed direction)
- Edges in $G_f$ have the capacities:

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E \quad \text{a forward edge} \\ f(v,u) & \text{if } (v,u) \in E \quad \text{a backward edge} \\ 0 & \text{otherwise} \end{cases}$$

- If $c(u,v) = a$, $f(u,v) = b$, and $a > b$, two edges are created in $G_f$:
  - One forward edge $(u,v)$ with capacity $a - b$ since this is how much we can increase the flow
  - One backward edge $(v,u)$ with capacity $b$ since this is how much we can decrease the flow
- If $f(u,v) = c(u,v)$ in $G$ then only a backward edge is created in $G_f$
- With $f(u,v) = c(u,v)$ in $G$ we can only decrease the flow on $(u,v)$

# The Ford-Fulkerson algorithm

**procedure** $\mathit{ford\_fulkerson}(G, s, t, c)$
    **for** each $e \in E$ **do** $f(e) \leftarrow 0$
    $G_f \leftarrow$ create initial residual graph
    **while** $(p \leftarrow \mathit{find\_path}(G_f)) \neq \mathit{null}$ **do**
        update $G_f$ according to previous slide

# Ford-Fulkerson algorithm or method?

- Ford and Fulkerson did not specify how the path should be found
- Different options result in different time complexity and therefore it is sometimes called a **method** and not an algorithm — such as in Cormen, Leiserson, Rivest and Stein *Introduction to Algorithms* — i.e. CLRS (about 1300 pages)
- If breadth first search is used, it is called the Edmond-Karp algorithm
- We will use the name Ford-Fulkerson algorithm

# Correctness of the Ford-Fulkerson algorithm

- We need to show that after updating $G_f$ it still satisfies the two constraints for being a network flow, the capacity and conservation constraints
- We also need to prove that it actually terminates — maybe it does not?

## Termination of the Ford-Fulkerson algorithm

- Will it eventually terminate?
- It depends. If we use infinite precision of the representation of the capacities and flows, and the capacities are carefully selected irrational numbers, it will not terminate
- Showing this is beyond the scope of the course
- In practise this is not a problem because real numbers are represented as floating point numbers which means they really are rational numbers
- If the capacities are integers, then all flows will also be integers and the algorithm clearly will terminate since it improves the flow at least by one each iteration (exactly by $\Delta$)
- The sum $C$ of capacities out from $s$ is an upper bound on the maximum flow so it will terminate after at most $C$ iterations

# Running time of the Ford-Fulkerson algorithm

- As usual, $n$ is the number of nodes and $m$ the number of edges in $G$
- Assume all capacities are integers
- Let the sum of capacities out from $s$ be $C$
- We assume $m \geq n$ to make our analysis simpler

**Lemma**

*The Ford-Fulkerson algorithm can be implemented to run in $O(Cm)$ time*

**Proof.**

At most $C$ iterations to find a path are needed. Finding a path using e.g. breadth-first search and an adjacency list representation, can be done in $O(n+m)$ and by our assumption this is equal to $O(m)$. Updating $G$ and $G_f$ using the path also needs $O(m)$ time $\square$

# Cuts

- Recall a partitioning of $V$ into $A$ and $B$ means
  - $V = A \cup B$, and
  - $A \cap B = \emptyset$
- A cut is a partitioning $(A, B)$ such that $s \in A$ and $t \in B$
- How are cuts and flows related?

# Flows and cuts

- The value of a flow is denoted by $v(f)$
- Consider any cut $(A, B)$ with $s \in A$ and $t \in B$
- $f^{\text{in}}(s) = 0$
- $v(f) = f^{\text{out}}(s)$
- So $v(f) = f^{\text{out}}(s) - f^{\text{in}}(s)$
- For all nodes $u \in V - \{s, t\}$ we have $f^{\text{out}}(u) = f^{\text{in}}(u)$
- Thus for all nodes $u \in A - \{s\}$ we have $f^{\text{out}}(u) - f^{\text{in}}(u) = 0$
- Therefore we can write: $v(f) = f^{\text{out}}(s) = \sum_{u \in A} f^{\text{out}}(u) - f^{\text{in}}(u)$
- See next slide

# Edges, flows, and cuts

- Again $v(f) = \sum_{u \in A} f^{\text{out}}(u) - f^{\text{in}}(u)$
- Consider any edge $e \in E$. We have four cases:
  1. No end in $A$: The edge does not affect the flow in $A$
  2. From $B$ to $A$: the flow will be counted only as $-f^{\text{in}}(u)$
  3. From $A$ to $B$: the flow will be counted only as $f^{\text{out}}(u)$
  4. Both ends in $A$: the flow will be counted both as $f^{\text{out}}(u)$ and as $f^{\text{in}}(u)$ above and thus cancels (by different terms in the sum)
- Thus: $v(f) = \sum_{u \in A} f^{\text{out}}(u) - f^{\text{in}}(u) = \sum_{e \text{ out of } A} f^{\text{out}}(e) - \sum_{e \text{ in to } A} f^{\text{in}}(e)$
- We have just shown:

**Lemma**

*Let $f$ be any $s - t$ flow and $(A, B)$ any $s - t$ cut. Then $v(f) = f^{out}(A) - f^{in}(A)$*

# Viewing the flow from $t$

- The value of a flow $f$ can also be written $v(f) = f^{\text{in}}(t)$
- This is clear but we can also see it follows from what we just saw
- Since the edges out of $A$ are the edges in to $B$, we have $f^{\text{out}}(A) = f^{\text{in}}(B)$
- And since the edges out of $B$ are the edges in to $A$ we have $f^{\text{out}}(B) = f^{\text{in}}(A)$
- Therefore $v(f) = f^{\text{in}}(B) - f^{\text{out}}(B)$
- Since $f^{\text{out}}(t) = 0$ we have with $B = \{t\}$ the expected $v(f) = f^{\text{in}}(t)$

# Capacities, flows, and cuts

- The capacity of a cut $(A, B)$ is $\sum_{e \text{ out of } A} c(e)$ and it is denoted $c(A, B)$
- We have:

$$\begin{aligned} v(f) &= f^{\text{out}}(A) - f^{\text{in}}(A) \\ &\leq f^{\text{out}}(A) \\ &= \sum_{e \text{ out of } A} f(e) \\ &\leq \sum_{e \text{ out of } A} c(e) \\ &= c(A, B) \end{aligned}$$

- Therefore:

**Lemma**

*The value of any flow is limited by the capacity of any cut:* $v(f) \leq c(A, B)$

# Exploiting $v(f) \leq c(A, B)$

- If we can show that the flow $f$ found by the Ford-Fulkerson algorithm is equal to the capacity of any cut $(A, B)$ then we know the algorithm finds the maximum flow since the flow must pass every cut

**Lemma**

*If there is an $s - t$ flow $f$ in $G$ such that there is no $s - t$ path in $G_f$ then $f$ has the maximum flow in $G$*

- See the next slides for the proof

# No s-t path in $G_f$ means $f(e) = c(e)$ for $e$ crossing cut

## Proof.

- Let $A$ be the set of nodes reachable from $s$ in $G_f$ and $B = V - A$
- Since $s$ is reachable from itself, $s \in A$ and therefore $A$ is not empty
- By assumption, there is no $s - t$ path in $G_f$ and therefore $t \in B$ and $B$ is not empty
- Thus $(A, B)$ is both a partition and a cut
- For any edge $e = (u, v)$ such that $u \in A$ and $v \in B$ we will next see that $f(e) = c(e)$
- Assume in contradiction that $f(e) < c(e)$. Since $c(e) - f(e) > 0$ there exists a forward edge $e$ in $G_f$ with $c_f(e) > 0$. Since $u \in A$ there is a path from $s$ to $v$ in $G_f$. Since this is a contradiction, $f(e) = c(e)$

$\square$

# No s-t path in $G_f$ means no flow back across cut

**Proof.**

- For any edge $e = (v, u)$ such that $v \in B$ and $u \in A$ we will next see that $f(e) = 0$
- Assume in contradiction that $f(e) > 0$. Since $f(e) > 0$ there exists a backward edge $e' = (u, v)$ with $c(e') > 0$ in $G_f$. But $e'$ makes $v$ reachable from $s$ in $G_f$ which is a contradiction, and therefore $f(e) = 0$
- We have showed that all edges $e$ out from $A$ have $f(e) = c(e)$ and all edges $e$ in to $A$ have $f(e) = 0$

$\square$

# Proving optimality of the Ford-Fulkerson algorithm

**Proof.**

$$\begin{aligned} v(f) &= f^{\text{out}}(A) - f^{\text{in}}(A) \\ &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to of } A} f(e) \\ &= \sum_{e \text{ out of } A} c(e) - 0 \\ &= c(A, B) \end{aligned}$$

□

- We have shown that the flow computed by the Ford-Fulkerson algorithm is equal to a cut, and this means it is optimal

# The max-flow min-cut theorem

- Recall: *the value of any flow is limited by the capacity of any cut:* $v(f) \leq c(A, B)$

**Theorem**

*The maximum flow is equal to the minimum cut*

**Proof.**

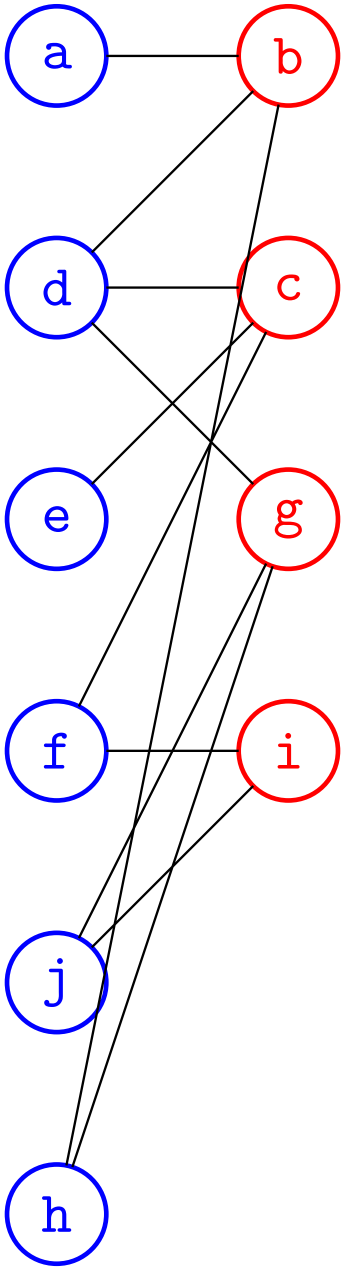
- Consider any flow $f$ and cut $(A, B)$ such that $v(f) = c(A, B)$
- Assume in contradiction there exists a flow $v'(f) > v(f)$
- This would contradict $v(f') \leq c(A, B)$, and therefore $f$ is maximum
- Also assume in contradiction there exists a cut $(A', B')$ with $c(A', B') < c(A, B)$
- Again this would contradict $v(f) \leq c'(A, B)$, and therefore $c$ is minimum

$\square$

# Improving the running time

- Recall that the flow $f$ is incremented by the smallest value of $c_f(u, v)$ on the $s - t$ path in the residual graph $G_f$ (which we called $\Delta$)
- Therefore it is useful to find a path with a high $\Delta$
- If $C = \sum_{e \text{ out from } s} c(e)$ is a huge number this is particularly important
- In this and many other situations it is not worthwhile to find the optimal value of a parameter (here $\Delta$) used to speed up an algorithm
- We can look for paths with $\Delta \geq C/2^i$ for $i = 0, 1, 2, \ldots$
- Another idea is to search for paths with the fewest number of edges
- We will instead soon look at a completely different approach: preflow-push

# Bipartite graph matching




- In a bipartite graph the nodes can be partitioned in two sets
- No edge between nodes in the same set
- We seek a matching of blue and red nodes
- Blue can be employees and red can be tasks
- An edge says somebody can perform a task
- We want to find a maximal matching
- We want as many tasks performed as possible
- Quiz: how can we solve this with Ford-Fulkerson?

# The preflow-push maximum network flow algorithm

- Variants of the preflow-push maximum network flow algorithm, we will study next, are the fastest algorithms for finding the maximum network flow
- For brevity we simply call it the preflow-push algorithm
- The preflow-push algorithm also uses of the residual graph
- Instead of maintaining a valid flow which satisfies both the conservation constraint and the capacity constraint, it uses a weaker type of flow which only satisfies the capacity constraint
- The weaker flow is called a **preflow**
- At algorithm termination, the preflow will have become a valid flow
- In addition, it uses a **height** function for each node

# The preflow

- For each edge $e \in E$ we have $0 \leq f(e) \leq c(e)$
- Thus the capacity constraint is always satisfied
- Instead of the conservation constraint, a node $u \neq s$ is allowed to have more incoming flow than outgoing
- Thus for each node $u \in V - \{s\}$ we have

$$\sum_{e \text{ into } u} f(e) \geq \sum_{e \text{ out from } u} f(e)$$

- The **excess preflow** of a node $u$ is

$$e_f(u) = \sum_{e \text{ into } u} f(e) - \sum_{e \text{ out from } u} f(e)$$

- Only $s$ has a negative excess preflow

# The height function

- There is a height function $h : V \to \mathbb{N}$
- $h(s) = n$
- $h(t) = 0$
- For $s$ and $t$ the heights cannot change and for other nodes they start at 0 and can increase
- The preflow on an edge $(u, v)$ can only increase if $h(u) = h(v) + 1$
- As we will see, $0 \leq h(u) \leq 2n - 1$ for $u \neq s$

# Compatible $h$ and $f$

- Recall: $(v, w) \in E_f$ if the flow on $(v, w)$ can be increased
- The height function $h$ and a preflow $f$ are **compatible** if the following conditions are satisfied:
  1. $h(s) = n$ and $h(t) = 0$
  2. For all edges $(v, w) \in E_f$ we have $h(v) \leq h(w) + 1$, or $h(w) \geq h(v) - 1$
- In $G_f$ a simple path $p = v_1, v_2, \ldots, v_k$ we have $v_i$ at most one higher than $v_{i+1}$
- Consider a simple path $v_0, v_1, v_2, \ldots, v_k$ in $G_f$ with $v_0 = s$
- $h(v_0) = n$, $h(v_1) \geq n - 1$, $h(v_2) \geq n - 2$, ..., $h(v_k) \geq n - k$.
- In Ford-Fulkerson we look for an $s - t$ path
- Quiz: can there be an $s - t$ path in $G_f$ here?

# Preflow paths in $G_f$

**Lemma**

*There can be no $s - t$ path in $G_f$ for a preflow $f$ compatible with $h$*

**Proof.**

- Assume in contradiction there is a simple $s - t$ path $p$ in $G_f$
- Let $p = v_0, v_1, v_2, \ldots, v_k$, i.e. $s = v_0$ and $t = v_k$
- Then $h(t) \geq n - k$ and since $h(t) = 0$ it must be the case that $k = n$, and that the length of $p$ is $n$.
- This path cannot be simple. A contradiction.

$\square$

# Finding a maximum flow using $h$

## Lemma

*If an $s-t$ flow $f$ is compatible with a height function $h$, then $f$ is a maximum flow.*

## Proof.

- Recall: if there is an $s-t$ flow $f$ in $G$ such that there is no $s-t$ path in $G_f$ then $f$ has the a maximum flow in $G$
- Since a flow $f$ also satisfies the conservation constraint, $f$ is a preflow.
- Therefore for a flow $f$ compatible with a height function $h$, there cannot be an $s-t$ path in $G_f$ (from previous slide)
- And no s-t path in $G_f$ means $f$ is maximal

$\square$

- If we can transform a preflow to a flow compatible with a height function $h$, we have found a maximal flow

# Overview

- We start with a preflow $f$ which, as we will see, is not a flow since it violates the conservation constraint
- The preflow $f$ is compatible with the height function $h$ and thus there is no $s - t$ path in $G_f$
- We will maintain the preflow so it remains compatible with an $h$
- The preflow will be modified until it becomes a flow $f$ which then will be a maximum flow
- Instead of maintaining valid but suboptimal flows which are improved, we will work towards a valid optimal flow
- The height of a node $u \in V - \{s, t\}$ can be at most $2n - 1$

# Initial preflow and height function

- Each edge $(s, u)$ is assigned the initial preflow $f(s, u) = c(s, u)$
- For all other edges $f(u, v) = 0$
- $h(s) = n$ and $h(u) = 0$ for every node $u \neq s$

# push

- Three conditions must be satisfied for a push:
  1. $e_f(v) > 0$
  2. $h(v) > h(w)$
  3. $(v, w) \in G_f$

**procedure** *push* $(f, h, v, w)$
  **assert** $e_f(v) > 0$ and $h(v) > h(w)$ and $(v, w) \in G_f$
  $e \leftarrow (v, w)$
  **if** $e$ is a forward edge **then**
    $\delta \leftarrow \min(e_f(v), c(e) - f(e))$
    increase $f(e)$ by $\delta$
  **else**
    $e \leftarrow (w, v)$
    $\delta \leftarrow \min(e_f(v), f(e))$
    decrease $f(e)$ by $\delta$

# relabel

- The purpose of a relabel is to increase the height of a node
- It is done when the node has excess flow but nowhere to push it due to neighbors have too high height

**procedure** *relabel*$(f, h, v)$
  **assert** $e_f(v) > 0$ and for all edges $(v, w) \in E_f$ we have $h(w) \geq h(v)$
  $h(v) \leftarrow h(v) + 1$

# The preflow push algorithm

**function** $\mathit{preflow\_push}(G, s, t)$

  $h(s) \leftarrow n$

  **for** each node $u \neq s$ **do** $h(u) \leftarrow 0$

  **for** each edge $(s, v)$ **do** $f(s, v) \leftarrow c(s, v)$

  **for** each edge $(u, v)$ such that $u \neq s$ **do** $f(u, v) \leftarrow 0$

  **while** there is a node $v \neq t$ with $e_f(v) > 0$ **do**

    **if** there is a node $w$ such that $h(v) > h(w)$ and $(v, w) \in G_f$ **then**

      $\mathit{push}(h, f, v, w)$

    **else**

      $\mathit{relabel}(h, f, v)$

  **return** $f$

# Correctness of the preflow-push algorithm

- Initially the preflow $f$ and height function $h$ are compatible
- Each push satisfies the capacity constraints due to how the $\delta$ is calculated
- Each relabel increases the height of a node $v$ by one.
- This could violate the compatibility of $f$ and $h$
- The relevant condition for compatibility is:

$$\textit{For all edges } (v, w) \in E_f \textit{ we have } h(v) \leq h(w) + 1$$

- If it is the case $h(v) > h(w)$ then a push and not a relabel is performed
- In the other case, $h(v) \leq h(w)$ the height of $v$ is incremented by one, and this still satisfies the condition
- Therefore after a relabel, $f$ and $h$ remain compatible

# Correctness of the preflow-push algorithm

- The algorithm terminates when only $e_f(t) > 0$
- When this happens the preflow is a flow and as proved earlier, this is a maximum flow

# Paths to $s$ in $G_f$

## Lemma

*A node $v$ with $e_f(v) > 0$ has a path in $G_f$ to $s$*

## Proof.

- Let $A$ be the set of nodes with a path to $s$ in $G_f$, and $B = V - A$.
- $s \in A$
- An edge $(v, w)$ with $v \in A$ and $w \in B$, $(v, w)$ cannot have flow since that would create a backward edge $(w, v)$ in $G_f$ so that then $w \in A$, which contradicts the assumption that $w \in B$
- The sum of excess flow of nodes in $B$ is nonnegative (since only $s \in A$ has negative excess flow) and can be written:

$$0 \leq \sum_{w \in B} e_f(w) = \sum_{w \in B} f^{\text{in}}(w) - \sum_{w \in B} f^{\text{out}}(w)$$

$\square$

# Paths to $s$ in $G_f$

## Proof.

- From the previous slide

$$0 \leq \sum_{w \in B} e_f(w) = \sum_{w \in B} f^{\text{in}}(w) - \sum_{w \in B} f^{\text{out}}(w)$$

- Considering edges which contribute to the above sums we have different cases.
- For an edge $(u, v)$ with $u, v \in B$ these cancel.
- For an edge $(u, v)$ with $u \in A$ and $v \in B$ its flow is 0 as shown on the previous slide.
- Only edges $(u, v)$ with $u \in B$ and $v \in A$ remain

$$0 \leq \sum_{w \in B} e_f(w) = -\sum_{w \in B} f^{\text{out}}(w)$$

$\square$

# Paths to $s$ in $G_f$

**Proof.**

- From the previous slide

$$0 \le \sum_{w \in B} e_f(w) = -\sum_{w \in B} f^{\text{out}}(w)$$

- But flows are nonnegative which implies they are all zero.
- Therefore, all nodes with excess are in the set $A$ and the claim follows.

$\square$

# Maximum height of a node and relabel operations

**Lemma**

$h(u) \leq 2n - 1$

**Proof.**

- A height is increased by a relabel operation, which is applicable to nodes other than $s$ and $t$
- As was proved by the previous lemma, a node $u$ with $e_f(u) > 0$ has a simple path $p$ to $s$ in $G_f$
- The length of this path is at most $n - 1$.
- For a compatible $h$ and $f$ the heights on this path decrease at most by the length of the path, i.e. at most $n - 1$
- Since $h(s) = n$ we have $h(u) - h(s) \leq n - 1$ i.e. $h(u) \leq 2n - 1$
- Since each node can have height at most $2n - 1$ and there are $n$ nodes, the number of relabel operations is less than $2n^2$

□

# Push operations

- A push operation increases the flow along an edge $(v, w)$
- As much excess flow $e_f(v)$ as possible is added to $f(v, w)$
- There are two limits:
  1. At most $e_f(v)$ can be used since excess flow can never be negative
  2. The capacity of the edge cannot be exceeded
- A push at an edge $(v, w)$ is **saturating** if the only limit was edge capacity:
  1. $(v, w)$ is a forward edge and $\delta = c(v, w) - f(v, w)$, and
  2. $(v, w)$ is a backward edge and $\delta = f(v, w)$.
- Note *only*: we assume $v$ still has excess flow after a saturating push
- All other push operations are **nonsaturating** and were limited by the amount of excess flow for $v$
- After a nonsaturating push, $v$ no longer has any excess flow: $e_f(v) = 0$

# Saturating push operations

## Lemma

*The number of saturating push operations is less than $2nm$.*

## Proof.

- Consider any two nodes $v$ and $w$ such that they have an edge $(v, w)$
- At a saturating push at the edge $(v, w)$ we have $h(v) = h(w) + 1$
- Before a new push at the same edge, the height of $w$ must be increased by 2. Since the height of any node always is less than $2n - 1$, any node can increase by 2 at most $n - 1$ times.
- Counting both $v$ and $w$ the number of saturating pushes between them is less than $2n$.
- Since there are $m$ edges the total number of saturating pushes is less than $2nm$

□

# Nonsaturating push operations

## Lemma

*The number of nonsaturating push operations is at most $4n^2m$.*

## Proof.

- This lemma is proved using the **potential function** method.
- For a given preflow $f$ and height function $h$ we define

$$\Phi(f, h) = \sum_{v : e_f(v) > 0} h(v)$$

- Initially $\Phi(f, h) = 0$ since $h(s) > 0$ but $e_f(s) < 0$
- $\Phi(f, h) \geq 0$ since no negative heights

$\square$

# Approach to counting nonsaturating push operations

- Both relabel and saturating push increase $\Phi$
- We can find the max value of $\Phi$
- Nonsaturating push decrease $\Phi$
- If we find the minimal decrease for each nonsaturating push, we can calculate an upper bound on their number (max $\Phi$ / minimal decrease)

# Effects on $\Phi(f, h)$ of different operations

**Proof.**

- A relabel increases $\Phi(f, h)$ by one and $2n^2$ relabels so at most $+2n^2$
- A saturating push $(v, w)$ may increase $e_f(w)$ from 0 and therefore increase $\Phi$ by at most $2n - 1$, and with at most $2nm$ saturating push operations, at most $+4n^2m$
- After a nonsaturating push $\Phi$ is reduced by $h(v)$ since $v$ no longer has any excess flow
- After that $\Phi$ is incremented by $h(w)$ if $e_f(w) = 0$ before the push and not incremented if $w$ already had excess flow
- So at least $-1$ by each nonsaturating push but possibly reduced more
- $\Phi(f, h) \geq 0$ so at most $4n^2m$ nonsaturating pushes

$\square$

# Maximum flow running times

| | |
|---|---|
| Ford-Fulkerson | $O(Cm)$ |
| Preflow-push | $O(4mn^2)$ |

- Both relabel and push take constant time
- The theoretical limitation of preflow-push is the number of nonsaturating push operations
- The preflow-push algorithm has $O(mn^2)$ nonsaturating push operations
- In a dense graph this is $O(n^4)$
- It can be shown that if we always take the node $v$ with $e_f(v) > 0$ and maximum height $h(v)$ the number of nonsaturating push operations is at most $4n^3$

# Parallel preflow push

- EDAN26 multicore programming in LP1
- IBM POWER8 computer with 80 hardware threads
- Lots of synchronizations
- Two phases
- I. Henckel and D. Söderberg: if the sum of capacities in to $t$ is less than the sum out from $s$ and the graph is undirected, it can be useful to let $s$ and $t$ switch roles.
- Nils Ceberg: the algorithm can terminate when $-e_f(s) = e_f(t)$ which is especially useful in a distributed implementation (such as with Scala/Akka) so no need to maintain a set of nodes with $e_f(v) > 0$ and check that it is empty as in the sequential algorithm
- Called Ceberg preflow-push termination (since 24/4 2023)