Derivative Pricing, Vasicek Model, and Heston Model

A Mathematical Approach to Option Valuation and Stochastic Processes

Back to Black Scholes

Plan For Today

1. Black Scholes for Call and Put Options
2. Code explanations in Python
3. Heston SDE
4. Vasicek SDE
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Pricing Derivatives using Expected Value and PDFs

The price of a derivative is computed as the expected value of a function of the underlying price.

This expectation is weighted by the probability density function (PDF) of a normal distribution and discounted.

$$\text{Price of a derivative} = e^{-r(T-t)}\,\mathbb{E}^{\mathbb{Q}}\left[f\left(S_t e^{\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma x\sqrt{T-t}}\right)\right]\cdot\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}dx.$$

Simplification using Standard Normal Distribution

Now, since the PDF corresponds to a standard normal distribution $N(0,1)$ (i.e., $x \sim N(0,1)$), we simplify:

$$\text{Price of a derivative} = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[f\left(S_t e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma x\sqrt{T-t}}\right)\right]\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\,dx.$$

The function on the underlying asset is typically given by:

$$f(S_T) = \max\{S_T - K, 0\}.$$

Now detailed derivations

For $S_T \geq K$, we have:

$$S_t e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma x\sqrt{T-t}} \geq K.$$

Dividing by K:

$$\frac{S_t}{K} e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma x\sqrt{T-t}} \geq 1.$$

Taking the natural logarithm:

$$\ln\left(\frac{S_t}{K}\right) + (r - \frac{\sigma^2}{2})(T-t) + \sigma x\sqrt{T-t} \geq 0.$$

$$x \geq -\frac{\ln\left(\frac{S_t}{K}\right) + (r - \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}}.$$

Since x follows a standard normal distribution

We need the probability that the random variable x falls above this point:

$$x \geq -\frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}}.$$

Which is often called - d2 (the point to the right). Lets draw it

And using symmetry of normal distribution

We want the probability of random variable x to fall below d2.

Remember in normal distribution -d2 and d2 are the mirrors of each other.

This is, as we know from statistics, the definition of CDF. Why?

So:

We are going to be paying K with probability of N(d2).

The payment occurs at expiry. T

So we need to discount it back.

- **K.N(d2) exp(-r(T-t))**

This is half the “equation”

We are going to pay K with that probability.

And we then receive S_T. **So, we need to also calculate S_T**

Calculating S_T

- K.N(d2) exp(-r(T-t))

S0 . N(d1)

d2 = d1 - sigma * sqrt(T-t)

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}}.$$

Complete Black Scholes Equation

$$\text{Price of a European Call Option} = S_t N(d_1) - Ke^{-r(T-t)} N(d_2),$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + (r + \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}},$$

$$d_2 = d_1 - \sigma\sqrt{T - t}.$$

Here, $N(d)$ represents the cumulative distribution function (CDF) of the standard normal distribution:

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{x^2}{2}} \, dx.$$

The term $S_t N(d_1)$ represents the expected stock price at time T in a risk-neutral world,
but only counting cases where the stock price is above K (otherwise, it's zero).
The term $Ke^{-r(T-t)} N(d_2)$ represents the present value of the strike price payment,
which only occurs if the option is exercised.
Since the probability of the option being in the money is $N(d_2)$, the expected payout of K is weighted by that probability.

Explanation of Black Scholes in Python

Explanation of Black Scholes Monte Carlo

Black Scholes for Put

I will leave derivations to you.

$$\text{Price of a European Put Option} = Ke^{-r(T-t)}N(-d_2) - S_tN(-d_1),$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Derivation

I will leave that to you but:

1. Use the price dynamics
2. Use this formula: Max {K - S_T, 0} or use Put-Call Parity
3. Very very similar to our call derivations

Put Call Parity

$$C - P = S_t - Ke^{-r(T-t)}$$

Call Put Parity

A **call (C)** gives you the right to buy S_T at K.

A **put (P)** gives someone else the right to sell S_T to you at K.

These two together minic the stock price because:

1. If $S_T > K$ and the Call is exercised => You buy stock at K
2. Otherwise $S_K < K$. The Put is exercised => Someone else sells you S_T at K (so you buy the stock at K)
3. Own the stock and borrow present value of K to replicate it

Limitation of Black Scholes

1. Assumes constant volatility
    a. Volatility smile
        i. => Stochastic Volatility SDE (Heston)
2. Assumes Log-normality
    a. => Jump diffusion SDE (Merton)
3. Assumes Constant risk free rate:
    a. => Vasicek SDE
4. European Options only (exercise at maturity):
    a. => Numerical Techniques
5. Assumes continuous distribution

Vasicek Interest Rate Model

$$dr = \theta(\lambda - r)dt + \sigma x\sqrt{T - t}$$

where $x \sim N(0, 1)$.

Here we have:

1. θ speed of mean reversion . How quickly r moves back to long-term equilibrium. A higher θ means faster reversion.
2. λ is the long-term mean level of the interest rate
3. σ is the volatility of interest rate changes, representing the magnitude of random fluctuations.

Given an initial interest rate (r_0), the evolution follows:

[$r_1 = r_0 + dr_1$]

[$r_2 = r_1 + dr_2$]

[$r_n = r_{n-1} + dr_n$]

where each (dr_t) is determined by the Vasicek process:

$$dr = \theta(\lambda - r)dt + \sigma x \sqrt{T - t}$$

where $x \sim N(0, 1)$.

Heston Stochastic Volatility Model

SDE for the Asset Price:

$$dS_t = \mu S_t dt + \sqrt{\nu_t} S_t dZ_1$$

SDE for Variance Evolution:

$$d\nu_t = \theta(\lambda - \nu_t)dt + \sigma dZ_2 \sqrt{\nu_t}$$

Where dZ1 and dZ2 are the same as dW1 and dW2 seen earlier in BS.

These are stochastic processes with correlation ϱ, essentially linking the asset price process and the variance process.

In Equation 1 (SDE for asset), under a risk-neutral setting, we'll have $\mu = r$.

In Equation 2 (SDE for variance):

- θ, rate of mean-reversion.
- where λ is the long run average variance.
- σ, volatility of volatility.

Next Session

1. Derivation of Heston SDE
2. Merton Jump Diffusion Model
3. Option Greeks
4. Fourier Transform
5. Inverse Fourier Transform
6. Pricing derivatives with fourier methods
7. Characteristic functions

Brain Teasers