leroy

Chiara Cimino Christian Krause

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Definition 1 (Frame)**.** (Already in Mathlib)A Frame can be viewed as a Category. There exists a morphism between A and B iff $A \leq B$.

Definition 2 (Top -> Frame)**.** (Brauchen wir das überhaupt?) There exists a contravariant Functor from a topological Space to the corresponding Frame Category with open Sets as objects. $f : X \to O(X)$

Definition 3 (f^* and f_*)**.** For every continuous Function $f : X \to Y$ between topological Spaces, there exists a pair of functors (f^*, f_*).

$$f* = f^{-1} : O(Y) \to O(X)$$
$$f_* : O(X) \to O(Y) := A \mapsto \bigcup_{f^*(v) \leq A} v$$

Lemma 4 (f^* commutes)**.**
f^* commutes with finite meets and arbitrary joins. This is the same as saying that the Frame Category of open Sets has all small coproducts and all finite Limits. nlab

Lemma 5 (Homsets)**.** *(Already in Mathlib)*

$$Hom_{O(Y)}(f * (V), A) = Hom_{O(x)}(v, f_*(A))$$

Lemma 6 ($f^* \dashv f_*$)**.** *f^* is the right adjoint to f_**

Proof. □

Lemma 7 (triangle)**.** *(Mclane p. 485)*
The triangular identities reduce to the following equalities:

$$f^* f_* f^* = f^* \quad \textit{and} \quad f_* f^* f_* = f_*$$

Proof.
This follows from the triangular identities of the adjunction. □

Lemma 8 (Embedding)**.** *(Leroy Lemme 1) The following arguments are equivalent:*

1. *f^* is surjective*
2. *f_* is injective*
3. *$f^* f_* = 1_{O(X)}$)*

Proof. This follows from the triangular identities. □

Definition 9 (Embedding)**.** An embedding is a morphism that satisfies the conditions of lemma 8

Definition 10 (Nucleus)**.** A nucleus is a map $e : O(E) \to O(E)$ with the following three properties:

1. e is idempotent
2. $U \leq eU$

3. $e(U \cap V) = e(U) \cap e(V)$

Lemma 11 (Nucleus)**.** *(Leroy Lemme 3) Let $e : O(E) \to O(E)$ be a nucleus. Then there is a space X and a morphism $f : X \to E$ such that $e = f_* f^*$. (The same holds for embeddings ??)*

Definition 12 (Subframe)**.** (Leroy CH 3) A subspace $Y \subset X$ is defined by a nucleus e_Y : $O(X) \to O(X)$, such that $O(Y) = Im(e_Y) = \{U \in O(X) | e_Y(U) = U\}$. The corresponding embedding (???) is $i_X : O(Y) \to O(X)$. $i_X^*(V) = e_X(V)$, $(i_X)_*(U) = U$

Definition 13 (Subframe Inclusion)**.** (Stimmt das?)(Leroy Ch 3) $X \subset Y$ if $e_Y \leq e_X$

Lemma 14 (factorisation)**.** *(Leroy Lemme 2) Let $i : X \to E$ be an embedding and $f : Y \to E$ be a morphism of spaces. To have f factor through i, it is necessary and sufficient that $i_* i^*(V) \leq f_* f^*(V) forall V \in O(E)$.*

Lemma 15 (Familiy of subspaces)**.** *(Leroy CH 4) Let X_i be a family of subspaces of E and e_i be the corresponding nuclei. For every $V \in O(E)$, let $e(V)$ be the onion of all $W \in O(E)$ which are contained in every $e_i(V)$. Then*

1. *e is the corresponding nuclus of a subspace X of E*
2. *a subspace Z of E conatains x if and only if it containd all X_i x is thus called the union of X_i denoted by $\bigcup_i X_i$*