leroy

Chiara Cimino Christian Krause

March 30, 2025

Chapter 1

Localic Caratheodory Extension

1.1 Basics of Locales

Definition 1 (f^* and f_*)**.** For every continuous Function $f : X \to Y$ between topological Spaces, there exists a pair of functors (f^*, f_*).

$$f* = f^{-1} : O(Y) \to O(X)$$
$$f_* : O(X) \to O(Y) := A \mapsto \bigcup_{f^*(v) \leq A} v$$

Lemma 2 ($f^* \dashv f_*$)**.** *f^* is the right adjoint to f_**

Proof. □

1.2 Sublocals

Definition 3 (Nucleus)**.** A nucleus is a map $e : O(E) \to O(E)$ with the following three properties:

1. e is idempotent
2. $U \leq eU$
3. $e(U \cap V) = e(U) \cap e(V)$

Lemma 4 (Nucleus)**.** *(Leroy Lemme 3) TODO Let $e : O(E) \to O(E)$ be monotonic. The following are equivalent:*

1. *e is a nucleus*
2. *There is a locale X and a morphism $f : X \to E$ such that $e = f_* f^*$.*
3. *Then there is a locale X and a embedding $f : X \to E$ such that $e = f_* f^*$.*

Proof. □

Definition 5 (Nucleus Partial Order)**.** For two nuclei e and f on $O(E)$, we say that $e \leq f$ if $e(U) \leq f(U)$ for all $U \in O(E)$. This relation is a partial order.

Lemma 6 (Nucleus Intersection)**.** *TODO Quelle StoneSpaces S.51 For a set S of nuclei, the intersection $\bigcap S$ can be computed by $\bigcap S(a) = \bigcap\{j(a) | j \in S\}$. This function satisfies the properties of a nucleus and of an infimum.*

Proof. □

Definition 7 (Sublocal)**.** (Leroy CH 3) A sublocal $Y \subset X$ is defined by a nucleus $e_Y : O(X) \to O(X)$, such that $O(Y) = Im(e_Y) = \{U \in O(X) | e_Y(U) = U\}$. The corresponding embedding is $i_X : O(Y) \to O(X)$. $i_X^*(V) = e_X(V)$, $(i_X)_*(U) = U$ And every nucleus e on $O(X)$ defines a sublocal Y of X by $O(Y) = Im(e)$

Definition 8 (Sublocal Inclusion)**.** (Stimmt das?)(Leroy Ch 3) $X \subset Y$ if $e_Y(u) \leq e_X(u)$ for all u. This means that the Sublocals are a dual order to the nuclei.

1.2.1 (1.4) Sublocal Union and Intersection

Definition 9 (Union of Sublocals)**.** (Leroy CH 1.4) Let $(X_i)_i$ be a family of sublocals of E and $(e_i)_i$ the corresponding nuclei. For all $V \in O(E)$, let $e(V)$ be the union of all $W \in O(E)$ which are contained in all $e_i(V)$.

Lemma 10 (Union of Sublocals)**.** *(Leroy CH 4) Let X_i be a family of subframes of E and e_i be the corresponding nuclei. For every $V \in O(E)$, let $e(V)$ be the union of all $W \in O(E)$ which are contained in every $e_i(V)$. Then*

1. *e is the corresponding nucleus of a sublocale X of E*
2. *a sublocale Z of E contains x if and only if it contains all X_i. X is thus called the union of X_i denoted by $\bigcup_i X_i$*

Proof. The properties of the nucleus (idempotent, increasing, preserving intersection) can be verified by unfolding the definition of $e(V)$. □

Lemma 11 (Sublocal Union equals Nucleus Intersection)**.** *For a family of sublocals X_i of E, the union $\bigcup X_i$ is the intersection of the corresponding nuclei. TODO lean link*

Proof. The infimum of the Nuclei is a Supremum of the sublocals, because the Nuclei are a dual order to the sublocals.) This means that it suffices to show that suprema are unique.
TODO Quelle `https://proofwiki.org/wiki/Infimum_is_Unique`
Suppose there are two different suprema c and c' of a set S. Because of the definition of a supremum, we that they are both upper bounds of S. But we also know that the supremum is smaller than any other upper bound, so we get $c \leq c'$ and $c' \leq c$. This means that $c = c'$.

□

Definition 12 (Intersection of Sublocals)**.** Let $(X_i)_i$ be a family of sublocal of E and $(e_i)_i$ the corresponding nuclei. For all $V \in O(E)$, the intersection $\bigcap X_i$ is the Union of all Nuclei w such that $w \leq x_i$ for all $x_i \in X_i$

Lemma 13 (Nucleus Complete Lattice)**.** *The Nuclei (and therefore the sublocals) form a complete lattice.*

Proof. One can prove that the Nuclei are closed under arbitrary intersections by unfolding the definition of the intersection. The supremum is defined as the infimum of the upper Bound.

□

Proposition 14 (Complete Heyting Algebra)**.** *A complete Lattice is a Frame if and only if it as a Heyting Algebra.*

Proof. (Source Johnstone:) The Heyting implication is right adjoint to the infimum. This means that the infimum preserves Suprema, since it is a left adjoint.

□

Lemma 15 (Nucleus Heyting Algebra)**.** *The Nuclei form a Heyting Algebra.*

Proof. Quelle Johnstone □

Lemma 16 (Nucleus Frame)**.** *The Nuclei form a frame.*

Proof. □

1.2.2 (7) Open Sublocals

Definition 17 (e_U)**.** Let E be a space with $U, H \in O(E)$. We donote by e_U the largest $W \in O(E)$ such that $W \cap U \subset H$. We verify that e_U is the nucleus of a subspace, which we will temporarily denote by $[U]$.

Lemma 18 (e_U is a nucleus)**.** *The map e_U is a nucleus.*

Proof. □

Definition 19 (Open sublocal)**.** For any $U \in O(E)$, the sublocal $[U]$ is called an open sublocal of E.

Lemma 20 ((6,7) Open Sublocal Properties)**.** *(Leroy Lemma 6,7)*

1. *For all subspaces X of E and any $U \in O(E)$:*

$$X \subset [U] \iff e_X(U) = 1_E$$

2. *For all $U, V \in O(E)$, we have:*

$$[U \cap V] = [U] \cap [V]$$

$$e_{U \cap V} = e_U e_V = e_V e_U$$

$$U \subset V \iff [U] \subset [V]$$

3. *For all families V_i of elements of $O(E)$, we have:*

$$\cup_i [V_i] = [\cup_i V_i]$$

4.

Proof. □

Definition 21 (Complement)**.** The complement of an open sublocal U of X is the sublocal $X \setminus U$. (Leroy p. 12) (+ Senf brauchen wir das allgemein??)

Lemma 22 (Complement Injective)**.** *The complement is injective.*

Proof. □

Definition 23 (Closed Sublocal)**.** A sublocal X of E is called closed if $X = E \setminus U$ for some open sublocal U of E.

Lemma 24 (Intersection of Closed Sublocals)**.** *For any family X_i of closed sublocals of E, the intersection $\bigcap X_i$ is closed (it can be computed by taking the complment of the union of the complements).*

Proof. □

Lemma 25 ((1.8) Properties of Complements)**.** *For any open sublocal V of E and any sublocal X of E, we have:*

$$V \cup X = E \iff E \setminus V subset X$$

$$V \cap X = \emptyset \iff X \subset E \setminus V$$

And thereby:

$$(E - U = E - V) \implies U = V$$

Proof. □

Lemma 26 ((1.8bis) Properties of Complements Part 2)**.** *For any open sublocal V of E and any sublocal X of E, we have:*

$$V \cup (E - V) = E \iff V \subset X$$

$$V \cap (E - V) = \emptyset X \subset V$$

Proof. □

Definition 27 (Further Topology)**.**

1. $IntX$ is the largest open sublocal contained in X
2. $ExtX$ is the largest open sublocal contained in $E \setminus X$
3. $\bar{X}$ is the smallest closed sublocal containing X
4. $\partial X = \bar{X} \cap (E - IntX)$

Lemma 28 (Properties of Further Topology)**.**

1. $\bar{X} = E \setminus Ext(X)$
2. $\partial X = E \setminus (IntX \cup ExtX)$
3. $IntX \cup \partial X = \bar{X}$
4. $ExtX \cup \partial X = E \setminus IntX$

Proof. □

1.3 Caratheodory Extnesion on Locales

Definition 29 (Measure on Locales)**.** A measure on a local X is a map $\mu : O(X) \to [0, \infty)$ such that:

1. $\mu(\emptyset) = 0$
2. $U \subset V \implies \mu(U) \leq \mu(V)$
3. $\mu(U \cup V) = \mu(U) + \mu(V) - \mu(V \cap V)$
4. For any increasingly filtered family V_i of open sublocals of X, we have:
$$\mu(\bigcup V_i) = \sup_i \mu(V_i)$$
this means: For all i and j there exists a k such that $V_i \cup V_j \subset V_k$ bzw. $V_i \subset V_k$ and $V_j \subset V_k$.

(Leroy III.1.)

Definition 30 (Caratheodory)**.** For any measure on a local X, the caratheodory extension is:

$$\mu(A) = \inf\{\mu(U) | A \subset U \in O(X)\}$$

Lemma 31 (Proptery 0 (Commutes with sup))**.** *(Leroy lemme 3.1) The caratheodory extension of a measure on a local commutes with unions of increasing families.*

Proof. □

Lemma 32 (Caratheodory Extensions are monotonic)**.** *The caratheodory extension is monotonic i.e.*
$$A \leq B \implies \mu(A) \leq \mu(B)$$

Proof. This is a direct consequence of the definition of the caratheodory extension. □

Lemma 33 (Subadditivity)**.** *The Caratheodory extension is subaddtive:*

$$\mu(A \cup B) \leq \mu(A) + \mu(b)$$

Proof. □

Definition 34 (Regular Local)**.** A local is regular, if for all open sublocals U of E, the open sublocals V such that $V \bar{\subset} U$ recover U.

Definition 35 (Neighborhood)**.**
A neighborhood of a sublocal A of X is an open sublocal V of X such that $A \leq V$.

Lemma 36 (Regularity of Sublocals)**.** *(Leroy lemme 3.2) In a regular local, any sublocal is regular, meaning that it is the intersection of all open neighborhoods.*

Proof. □

Lemma 37 (Measure add compl eq top)**.** *(Leroy Lemme 3.3) For any open sublocal U of a local X, the caratheodory extension of a measure on X satisfies*

$$\mu(U) + \mu(X \setminus U) = \mu(X)$$

Proof. Siehe Leroy □

Lemma 38 (Restriction)**.** *The Restriction of a Measure to any open Sublocal is a Measure.*

Proof. □

Lemma 39 (Property 2)**.** *(Leroy Lemm 3.4) For any open sublocal U and any sublocal A of a local E, the caratheodory extension of a measure on X satisfies*

$$\mu(A) = \mu(A \cap U) + \mu(A \cap (E \setminus U))$$

Proof. Siehe Leroy □

Lemma 40 (Property 3)**.** *(Leroy Lemm 3.5) For a increasing family V_α of open sublocals of E and any sublocal A, we have:*

$$\mu(A \cap (\bigcup V_\alpha)) = \sup_\alpha \mu(A \cap V_\alpha)$$

Proof. □

Lemma 41 (Restriction to a Sublocale)**.** *Let A be a sublocale of E with the embedding $i : A \to E$. The restriction of a measure μ on E to A is a measure on A:*

$$V \mapsto \mu(i(V)) : Open(A) \to \mathbb{R}$$

Proof. □

Proposition 42 (strictly additve)**.** *(Leroy theorem 3.3.1) For any measure on a local X, the caratheodory extension is strictly additive i.e. $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$*

Proof. □

Proposition 43 (reductive)**.** *(Proposition 3.3.1) For any measure on a local X, the caratheodory extension is reductive i.e. for all $A \leq X$ the set $\{A' \subset A, \mu(A') = \mu(A)\}$ has a minimal element*

Proof. □

Lemma 44 (Commutes with inf opens)**.** *(Leroy Lemme 3.6) For any measure on a local X and a decreasing family V_i of open sublocals, the caratheodory extension fulfills: $\mu(\inf V_i) = \inf \mu(V_i)$.*

Proof. □

Proposition 45 (Commutes with inf)**.** *(Leroy lemme 3.7 et principal) For any measure on a local X, the caratheodory extension is regular $\mu(\inf A_i) = \inf \mu(A_i)$. For decreasing families A_i*

Proof. □

Theorem 46 (Main Theorem (very important))**.** *For any measure on a local X, the caratheodory extension is*

1. *strictly additive i.e. $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$*
2. *commutes with inf $\mu(\inf A_i) = \inf \mu(A_i)$*
3. *reductive i.e. for all $A \leq X$ the set $\{A' \subset A, \mu(A') = \mu(A)\}$ has a minimal element*

Proof. □

Chapter 2

Locales correspond to Topology

2.1 Leroy Chapter V

Lemma 47 ((1.10) Intersection of Open and Closed Sublocals)**.** *For any $U \in O(E)$, and sublocal X of E we have:*

$$e_{U \cap X} = e_U e_X$$

And for a closed F

$$e_{X \cap F} = e_X e_F$$

Lemma 48 (Regular Top to regular local)**.** *Any regular topological space induces a regular local.*

Lemma 49 (Opens)**.** *(Leroy V.1 Remarque 2) The Open subsets of any good enough topological space correspond precisely to the open sublocals of the corresponding local.*

Lemma 50 (Subset Sublocal)**.** *(leroy V.1 Remarque 3) Any subset X of a good enough topological space E induces a sublocal $[X]$ of the corresponding local. This is an order preserving embedding.*

Definition 51 (Good enough topological space)**.** blackbox to mathlib????

Lemma 52 (Subset to sublocal Part 1)**.** *(Leroy Proposition 5.1.1)*

For two subspaces X and Y of E and an open subspaces U of E, we have:

1. $X \subset Y \implies [X] \subset [Y]$
2. $X \subset U \iff [X] \subset [U]$
3. *If E is a good enough topological space, then*

$$X \subset Y \iff [X] \subset [Y]$$

Lemma 53 (Subset to sublocal Part 2)**.** *(Leroy Proposition 5.1.2, 5.1.3) For an open subspace U of E and a subspace X of E, we have:*

$$[U \cap X] = [U] \cap [X]$$
$$F = E \setminus U$$
$$[F] = [E]$$
$$[U]$$
$$[X \cap F] = [X] \cap [F]$$

Lemma 54 (Part 3). *For any subspaces X of E, we have:*

1.
$$Ext[X] = [ExtX]$$

2.
$$[\bar{X}] = [\bar{X}]$$

3.
$$[IntX] \subset Int[X]$$

4.
$$\partial[X] \subset [Fr(X)]$$

For a good enough topological space E, we have equality in 3 and 4.

Proposition 55 (Subset to sublocal preserves structure). *For two subspaces X and Y of E and an open subspaces U of E, we have:*

1. $X \subset Y \implies [X] \subset [Y]$
2. $X \subset U \iff [X] \subset [U]$
3. *If E is a good enough topological space, then*
$$X \subset Y \iff [X] \subset [Y]$$
4.
$$[U \cap X] = [U] \cap [X]$$
5. ...

Theorem 56 (Measure top to loc). *Any measure on a good enough topological space X induces a measure on the corresponding local. Furthermore, the classical caratheodory extension onto $\mathcal{P}(X)$ agrees with the restriction of the caratheodory extension of the induced measure on the local.*