leroy

Chiara Cimino Christian Krause

January 1, 2025

Chapter 1

Leroy Chapter I

1.1 F star

Definition 1 (f^* and f_*)**.** For every continuous Function $f : X \to Y$ between topological Spaces, there exists a pair of functors (f^*, f_*).

$$f* = f^{-1} : O(Y) \to O(X)$$
$$f_* : O(X) \to O(Y) := A \mapsto \bigcup_{f^*(v) \leq A} v$$

Lemma 2 ($f^* \dashv f_*$)**.** *f^* is the right adjoint to f_**

Proof. □

Lemma 3 (triangle)**.** *(Mclane p. 485)*

The triangular identities reduce to the following equalities:

$$f^* f_* f^* = f^* \quad and \quad f_* f^* f_* = f_*$$

Proof. This follows from the triangular identities of the adjunction. □

1.2 Embedding

Lemma 4 (Embedding)**.** *(Leroy Lemme 1) The following arguments are equivalent:*

1. *f^* is surjective*
2. *f_* is injective*
3. *$f^* f_* = 1_{O(X)})$*

Proof. This follows from the triangular identities. □

Definition 5 (Embedding)**.** An embedding is a morphism that satisfies the conditions of lemma 4

1.3 Sublocals

Definition 6 (Nucleus)**.** A nucleus is a map $e : O(E) \to O(E)$ with the following three properties:

1. e is idempotent
2. $U \leq eU$
3. $e(U \cap V) = e(U) \cap e(V)$

Lemma 7 (Nucleus)**.** *(Leroy Lemme 3) Let $e : O(E) \to O(E)$ be monotonic. The following are equivalent:*

1. *e is a nucleus*
2. *There is a locale X and a morphism $f : X \to E$ such that $e = f_* f^*$.*
3. *Then there is a locale X and a embedding $f : X \to E$ such that $e = f_* f^*$.*

Proof. □

Definition 8 (Nucleus Partial Order)**.** For two nuclei e and f on $O(E)$, we say that $e \leq f$ if $e(U) \leq f(U)$ for all $U \in O(E)$. This relation is a partial order.

Lemma 9 (Nucleus Intersection)**.** *TODO Quelle StoneSpaces S.51 For a set S of nuclei, the intersection $\bigcap S$ can be computed by $\bigcap S(a) = \bigcap \{j(a) | j \in S\}$. This function satisfies the properties of a nucleus and of an infimum.*

Proof. □

Definition 10 (Sublocal)**.** (Leroy CH 3) A sublocal $Y \subset X$ is defined by a nucleus $e_Y : O(X) \to O(X)$, such that $O(Y) = Im(e_Y) = \{U \in O(X) | e_Y(U) = U\}$. The corresponding embedding is $i_X : O(Y) \to O(X)$. $i_X^*(V) = e_X(V)$, $(i_X)_*(U) = U$ And every nucleus e on $O(X)$ defines a sublocal Y of X by $O(Y) = Im(e)$

Definition 11 (Sublocal Inclusion)**.** (Stimmt das?)(Leroy Ch 3) $X \subset Y$ if $e_Y(u) \leq e_X(u)$ for all u. This means that the Sublocals are a dual order to the nuclei.

Lemma 12 (factorisation)**.** *(Leroy Lemme 2) Let $i : X \to E$ be an embedding and $f : Y \to E$ be a morphism of spaces. To have f factor through i, it is necessary and sufficient that $i_* i^*(V) \leq f_* f^*(V) forall V \in O(E)$.*

Proof. □

1.3.1 (1.4) Sublocal Union and Intersection

Definition 13 (Union of Sublocals)**.** (Leroy CH 1.4) Let $(X_i)_i$ be a family of sublocals of E and $(e_i)_i$ the corresponding nuclei. For all $V \in O(E)$, let $e(V)$ be the union of all $W \in O(E)$ which are contained in all $e_i(V)$.

Lemma 14 (Union of Sublocals)**.** *(Leroy CH 4) Let X_i be a family of subframes of E and e_i be the corresponding nuclei. For every $V \in O(E)$, let $e(V)$ be the union of all $W \in O(E)$ which are contained in every (TODO wieso every) $e_i(V)$. Then*