

leroy

Chiara Cimini

Christian Krause

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Chapter 1

Leroy Chapter I

1.1 F star

Definition 1 (f^* and f_*). For every continuous Function $f : X \rightarrow Y$ between topological Spaces, there exists a pair of functors (f^*, f_*) .

$$f^* = f^{-1} : O(Y) \rightarrow O(X)$$
$$f_* : O(X) \rightarrow O(Y) := A \mapsto \bigcup_{f^*(v) \leq A} v$$

Lemma 2 ($f^* \dashv f_*$). f^* is the right adjoint to f_*

Proof.

□

Lemma 3 (triangle). (McLane p. 485)

The triangular identities reduce to the following equalities:

$$f^* f_* f^* = f^* \quad \text{and} \quad f_* f^* f_* = f_*$$

Proof. This follows from the triangular identities of the adjunction.

□

1.2 Embedding

Lemma 4 (Embedding). (Leroy Lemme 1) The following arguments are equivalent:

1. f^* is surjective
2. f_* is injective
3. $f^* f_* = 1_{O(X)}$

Proof. This follows from the triangular identities.

□

Definition 5 (Embedding). An embedding is a morphism that satisfies the conditions of lemma 4

1.3 Sublocals

Definition 6 (Nucleus). A nucleus is a map $e : O(E) \rightarrow O(E)$ with the following three properties:

1. e is idempotent
2. $U \leq eU$
3. $e(U \cap V) = e(U) \cap e(V)$

Lemma 7 (Nucleus). (*Leroy Lemme 3*) Let $e : O(E) \rightarrow O(E)$ be monotonic. The following are equivalent:

1. e is a nucleus
2. There is a locale X and a morphism $f : X \rightarrow E$ such that $e = f_*f^*$.
3. Then there is a locale X and an embedding $f : X \rightarrow E$ such that $e = f_*f^*$.

Proof. □

Definition 8 (Sublocal). (*Leroy CH 3*) A sublocal $Y \subset X$ is defined by a nucleus $e_Y : O(X) \rightarrow O(X)$, such that $O(Y) = \text{Im}(e_Y) = \{U \in O(X) \mid e_Y(U) = U\}$. The corresponding embedding is $i_X : O(Y) \rightarrow O(X)$. $i_X^*(V) = e_X(V)$, $(i_X)_*(U) = U$. And every nucleus e on $O(X)$ defines a sublocal Y of X by $O(Y) = \text{Im}(e)$

Definition 9 (Sublocal Inclusion). (*Stimmt das?*)(*Leroy Ch 3*) $X \subset Y$ if $e_Y(u) \leq e_X(u)$ for all u

Lemma 10 (factorisation). (*Leroy Lemme 2*) Let $i : X \rightarrow E$ be an embedding and $f : Y \rightarrow E$ be a morphism of spaces. To have f factor through i , it is necessary and sufficient that $i_*i^*(V) \leq f_*f^*(V)$ for all $V \in O(E)$.

Proof. □

1.3.1 (1.4) Sublocal Union and Intersection

Definition 11 (Union of Sublocals). (*Leroy CH 1.4*) Let $(X_i)_i$ be a family of sublocals of E and $(e_i)_i$ the corresponding nuclei. For all $V \in O(E)$, let $e(V)$ be the union of all $W \in O(E)$ which are contained in all $e_i(V)$.

Lemma 12 (Union of Sublocals). (*Leroy CH 4*) Let X_i be a family of subframes of E and e_i be the corresponding nuclei. For every $V \in O(E)$, let $e(V)$ be the union of all $W \in O(E)$ which are contained in every $(\text{TODO wieso every}) e_i(V)$. Then

1. e is the corresponding nucleus of a subframe X of E
2. a subframe Z of E contains x if and only if it contains all X_i . X is thus called the union of X_i denoted by $\bigcup_i X_i$

Proof. The properties of the nucleus (idempotent, increasing, preserving intersection) can be verified by unfolding the definition of $e(V)$. □

Definition 13 (Intersection of Sublocals). Let $(X_i)_i$ be a family of sublocal of E and $(e_i)_i$ the corresponding nuclei. For all $V \in O(E)$, the intersection $\bigcap X_i$ is the Union of all Nuclei w such that $w \leq x_i$ for all $x_i \in X_i$

1.3.2 (1.5) Direct Images

Definition 14 (Direct Images). Let $f : E \rightarrow F$ be a morphism of Frames. The map $f_* f^* : O(F) \rightarrow O(F)$ is the nucleus of the subframe $Im(f)$ of F . By (lemma 2), $Im(F)$ is the smallest subframe of F through which f can be factored. For any subframe X of E , we define the direct image as

$$f(x) = Im(fi_x)$$

Where i_X is the inclusion of X into E .

Lemma 15 ((4) Direct Images Transitive). (*Leroy Lemme 4*) Given two morphisms $f : E \rightarrow F$ and $g : F \rightarrow G$ and a subspace X of E , we have

$$(gf)(X) = g(f(X))$$

Lemma 16 ((5) Direct Images Families). (*Leroy Lemme 5*) For all morphisms $f : E \rightarrow F$ and a family (X_i) of subspaces of E , the following holds:

$$f(\cup_i X_i) = \cup_i f(X_i)$$

1.3.3 (6) Inverse Images

Definition 17 (Inverse Images). We have a morphism of spaces $f : E \rightarrow F$ and a subspace Y of F . The inverse image $f^{-1}(Y)$ is the biggest subspace X of E such that $f(X) \subset Y$.

More generally for a morphism $h : A \rightarrow E$, the necessary and sufficient condition for h to factor through f^{-1} is that fh factors through Y .

$$Imh \subset f^{-1}(Y) \iff f(Imh) \subset Y \iff Im(fh) \subset Y$$

1.3.4 (7) Open Sublocals

Definition 18 (e_U). Let E be a space with $U, H \in O(E)$. We denote by e_U the largest $W \in O(E)$ such that $W \cap U \subset H$. We verify that e_U is the nucleus of a subspace, which we will temporarily denote by $[U]$.

Lemma 19 (e_U is a nucleus). The map e_U is a nucleus.

Proof. □

Definition 20 (Open sublocal). For any $U \in O(E)$, the sublocal $[U]$ is called an open sublocal of E .

Lemma 21 ((6,7) Open Sublocal Properties). (*Leroy Lemma 6,7*)

1. For all subspaces X of E and any $U \in O(E)$:

$$X \subset [U] \iff e_X(U) = 1_E$$

2. For all $U, V \in O(E)$, we have:

$$[U \cap V] = [U] \cap [V]$$

$$e_{U \cap V} = e_U e_V = e_V e_U$$

$$U \subset V \iff [U] \subset [V]$$

3. For all families V_i of elements of $O(E)$, we have:

$$\cup_i [V_i] = [\cup_i V_i]$$

4.

Proof. □

Definition 22 (Complement). The complement of an open sublocal U of X is the sublocal $X \setminus U$. (Leroy p. 12) (+ Senf brauchen wir das allgemein??)

Lemma 23 (Complement Injective). *The complement is injective.*

Proof. □

Definition 24 (Closed Sublocal). A sublocal X of E is called closed if $X = E \setminus U$ for some open sublocal U of E .

Lemma 25 (Intersection of Closed Sublocals). *For any family X_i of closed sublocals of E , the intersection $\bigcap X_i$ is closed (it can be computed by taking the complement of the union of the complements).*

Proof. □

Lemma 26 ((1.8) Properties of Complements). *For any open sublocal V of E and any sublocal X of E , we have:*

$$V \cup X = E \iff E \setminus V \subset X$$

$$V \cap X = \emptyset \iff X \subset E \setminus V$$

And thereby:

$$(E \setminus U = E \setminus V) \implies U = V$$

Lemma 27 ((1.9) Preimage of complements). *For any morphism of spaces $g : A \rightarrow E$ and any open sublocal V of F , we have:*

$$g^{-1}(E \setminus V) = A \setminus g^{-1}(V)$$

Lemma 28 ((1.8bis) Properties of Complements Part 2). *For any open sublocal V of E and any sublocal X of E , we have:*

$$V \cup (E \setminus V) = E \iff V \subset X$$

$$V \cap (E \setminus V) = \emptyset \iff X \subset V$$

Lemma 29 ((1.10) Intersection of Open and Closed Sublocals). *For any $U \in O(E)$, and sublocal X of E we have:*

$$e_{U \cap X} = e_U e_X$$

And for a closed F

$$e_{X \cap F} = e_X e_F$$

Lemma 30 ((1.11) Distribution of Intersections over Unions). *Let X, Y, L be three sub locals of E . If L is open or closed, we have:*

$$L \cap (X \cap Y) = (L \cap X) \cap (L \cap Y)$$

Definition 31 (Further Topology).

1. $IntX$ is the largest open sublocal contained in X
2. $ExtX$ is the largest open sublocal contained in $E \setminus X$
3. \bar{X} is the smallest closed sublocal containing X
4. $\partial X = \bar{X} \cap (E - IntX)$

Lemma 32 (Properties of Further Topology).

1. $\bar{X} = E \setminus Ext(X)$
2. $\partial X = E \setminus (IntX \cup ExtX)$
3. $IntX \cup \partial X = \bar{X}$
4. $ExtX \cup \partial X = E \setminus IntX$

Proof.

□

Chapter 2

Leroy Chapter II

Definition 33 (Induced boolean algebra). Define $At(U_1, \dots, U_n)$ as the collection of all finite nonempty intersections of opens U_i or their complements.

Define $b(U_1, \dots, U_n)$ as all unions of elements of $At(U_1, \dots, U_n)$

Define $b(X)$ as the union of all $b(U_1, \dots, U_n)$ for finite collections of opens U_i .

Lemma 34 ((2.1) Sublocals and decompositions). *For any open U the natural morphism $p : U \sqcup (E \setminus U)$ induces a bijection of sublocals.*

Lemma 35 ((2.2) Induced morphism of boolean algebras). *For all $H \in B(U_1, \dots, U_n)$ we have $f^{-1}(H) \in b(f^{-1}(U - 1), \dots, f^{-1}(U_n))$ and*

$$H \mapsto f^{-1}(H)$$

Is a morphism of boolean algebras

Proposition 36 (Boolean algebra). *(Leroy Proposition II.1)*

1. $b(X)$ is the generated boolean algebra of the open and closed sublocals of X .
2. For all $H \in b(X)$, we have $f^{-1}(H) \in b(Y)$ and the map $H \rightarrow f^{-1}(H)$ is a homomorphism of boolean algebras $b(X) \rightarrow b(Y)$
3. For two sublocals A, B of X and any $H \in b(X)$, we have:

$$H \cap (A \cup B) = (H \cap A) \cup (H \cap B)$$

Lemma 37 ($b(X)$ generates sublocals). *Every sublocal X is an intersection of elements of $b(X)$.*

Lemma 38 (Union of Intersections). *(Leroy Lemme 2.4) For any family B_i of sublocals of a local E and a sublocal A , we have:*

$$A \cup \left(\bigcap_i B_i \right) = \bigcap_i (A \cup B_i)$$

This implies if A_i and B_j are families of sublocals of X , we have:

$$\left(\bigcap_i A_i \right) \cup \left(\bigcap_j B_j \right) = \bigcap_{ij} (A_i \cup B_j)$$

Theorem 39 (Preimage commutes with unions). *(Leroy resultat principal) For any morphism f of locals, we have:*

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$

Chapter 3

Leroy Chapter III

Definition 40 (Measure on Locals). TODO ggf auspalten mit extra lemma das sagt dass die Open subframes einen frame bilden A measure on a local X is a map $\mu : O(X) \rightarrow [0, \infty)$ such that:

1. $\mu(\emptyset) = 0$
2. $U \subset V \implies \mu(U) \leq \mu(V)$
3. $\mu(U \cup V) = \mu(U) + \mu(V) - \mu(V \cap U)$
4. For any increasingly filtered family V_i of open sublocals of X , we have:

$$\mu(\bigcup_i V_i) = \sup_i \mu(V_i)$$

this means: For all i and j there exists a k such that $V_i \cup V_j \subset V_k$ bzw. $V_i \subset V_k$ and $V_j \subset V_k$.

(Leroy III.1.)

Definition 41 (Caratheodory). For any measure on a local X , the caratheodory extension is:

$$\mu(A) = \inf\{\mu(U) \mid A \subset U \in O(X)\}$$

Lemma 42 (Property 0 (Commutes with sup)). *(Leroy lemme 3.1) The caratheodory extension of a measure on a local commutes with unions of increasing families. (Senf von noa: commutes with filtered colimits)*

Definition 43 (Regular Local). A local is regular, if for all open sublocals U of E , the open sublocals V such that $V \subset U$ recover U .

Definition 44 (Neighborhood).

A neighborhood of a sublocal A of X is an open sublocal V of X such that $A \leq V$.

Lemma 45 (Regularity of Sublocals). *(Leroy lemme 3.2) In a regular local, any sublocal is regular, meaning that it is the intersection of all open neighborhoods.*

Lemma 46 (Property 1). *(Leroy Lemme 3.3) For any open sublocal U of a local X , the caratheodory extension of a measure on X satisfies*

$$\mu(U) + \mu(X \setminus U) = \mu(X)$$

Proposition 47 (Sublocals structure). *Obacht TODO The sublocals of a local E form a complete (distributive vlt.) lattice with the operations of intersection and complement.*

Lemma 48 (Property 2). *(Leroy Lemm 3.4) For any open sublocal U and any sublocal A of a local E , the caratheodory extension of a measure on X satisfies*

$$\mu(A) = \mu(A \cap U) + \mu(A \cap (E \setminus U))$$

Lemma 49 (Property 3). *(Leroy Lemm 3.5) For a increasing family V_α of open sublocals of E and any sublocal A , we have:*

$$\mu(A \cap (\bigcup V_\alpha)) = \sup_\alpha \mu(A \cap V_\alpha)$$

Lemma 50 (Commutates with inf opens). *(Leroy Lemme 3.6) For any measure on a local X and a decreasing family V_i of open sublocals, the caratheodory extension fulfills: $\mu(\inf V_i) = \inf \mu(V_i)$.*

Lemma 51 (Caratheodory Extensions are monotonic). *The caratheodory extension is monotonic i.e.*

$$A \leq B \implies \mu(A) \leq \mu(B)$$

Proof. This is a direct consequence of the definition of the caratheodory extension. \square

Proposition 52 (Elementary Properties of Caratheodory Extensions). *(Leroy lemme 3.3, 3.4, Corollary 3.1, Lemme 3.5) For any measure on a local X , the caratheodory extension satisfies the following properties:*

1. *It is monotonic i.e.*

$$A \leq B \implies \mu(A) \leq \mu(B)$$

2. *Commutates with unions of increasing families*

$$3. \mu(U) + \mu(X \setminus U) = \mu(X)$$

$$4. \mu(A) = \mu(A \cap U) + \mu(A \cap (E \setminus U))$$

5. *For a increasing family V_α of open sublocals of E and any sublocal A , we have:*

$$\mu(A \cap (\bigcup V_\alpha)) = \sup_\alpha \mu(A \cap V_\alpha)$$

6. *For any measure on a local X and a decreasing family V_i of open sublocals, the caratheodory extension fulfills: $\mu(\inf V_i) = \inf \mu(V_i)$.*

In particular, for two open sublocals U and V of X and any sublocal A of X , we have

$$\mu(A \cap (U \cup V)) = \mu(A \cap U) + \mu(A \cap V) - \mu(A \cap U \cap V)$$

Proposition 53 (strictly additive). *(Leroy theorem 3.3.1) For any measure on a local X , the caratheodory extension is strictly additive i.e. $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$*

Proposition 54 (reductive). *(Proposition 3.3.1) For any measure on a local X , the caratheodory extension is reductive i.e. for all $A \leq X$ the set $\{A' \subset A, \mu(A') = \mu(A)\}$ has a minimal element*

Proposition 55 (Commutates with inf). *(Leroy lemme 3.7 et principal) For any measure on a local X , the caratheodory extension is regular $\mu(\inf A_i) = \inf \mu(A_i)$. For decreasing families A_i*

Theorem 56 (Main Theorem (very important)). *For any measure on a local X , the caratheodory extension is*

1. *strictly additive i.e. $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$*
2. *commutes with inf $\mu(\inf A_i) = \inf \mu(A_i)$*
3. *reductive i.e. for all $A \leq X$ the set $\{A' \subset A, \mu(A') = \mu(A)\}$ has a minimal element*

Chapter 4

Leroy Chapter V

Lemma 57 (Regular Top to regular local). *Any regular topological space induces a regular local.*

Lemma 58 (Opens). *(Leroy V.1 Remarque 2) The Open subsets of any good enough topological space correspond precisely to the open sublocals of the corresponding local.*

Lemma 59 (Subset Sublocal). *(Leroy V.1 Remarque 3) Any subset X of a good enough topological space E induces a sublocal $[X]$ of the corresponding local. This is an order preserving embedding.*

Definition 60 (Good enough topological space). blackbox to mathlib????)

Lemma 61 (Subset to sublocal Part 1). *(Leroy Proposition 5.1.1)*

For two subspaces X and Y of E and an open subspaces U of E , we have:

1. $X \subset Y \implies [X] \subset [Y]$
2. $X \subset U \iff [X] \subset [U]$
3. *If E is a good enough topological space, then*

$$X \subset Y \iff [X] \subset [Y]$$

Lemma 62 (Subset to sublocal Part 2). *(Leroy Proposition 5.1.2, 5.1.3) For an open subspace U of E and a subspace X of E , we have:*

$$[U \cap X] = [U] \cap [X]$$

$$F = E \setminus U$$

$$[F] = [E]$$

$$[U]$$

$$X \cap F = [X] \cap [F]$$

Lemma 63 (Part 3). *For any subspaces X of E , we have:*

- 1.

$$Ext[X] = [ExtX]$$

- 2.

$$[\bar{X}] = [\bar{X}]$$

3.

$$[IntX] \subset Int[X]$$

4.

$$\partial[X] \subset [Fr(X)]$$

For a good enough topological space E , we have equality in 3 and 4.

Proposition 64 (Subset to sublocal preserves structure). *For two subspaces X and Y of E and an open subspaces U of E , we have:*

$$1. X \subset Y \implies [X] \subset [Y]$$

$$2. X \subset U \iff [X] \subset [U]$$

3. If E is a good enough topological space, then

$$X \subset Y \iff [X] \subset [Y]$$

4.

$$[U \cap X] = [U] \cap [X]$$

5. ...

Theorem 65 (Measure top to loc). *Any measure on a good enough topological space X induces a measure on the corresponding local. Furthermore, the classical caratheodory extension onto $\mathcal{P}(X)$ agrees with the restriction of the caratheodory extension of the induced measure on the local.*

Theorem 66 (Goal). *One can interpret any classical borel measure as a measure on locals and their life is good :)*