

leroy

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Chapter 1

Leroy Chapter I

1.1 F star

Definition 1 (f^* and f_*). For every continuous Function $f : X \rightarrow Y$ between topological Spaces, there exists a pair of functors (f^*, f_*) .

$$f^* = f^{-1} : O(Y) \rightarrow O(X)$$
$$f_* : O(X) \rightarrow O(Y) := A \mapsto \bigcup_{f^*(v) \leq A} v$$

Lemma 2 ($f^* \dashv f_*$). f^* is the right adjoint to f_*

Proof.

□

Lemma 3 (triangle). (McLane p. 485)

The triangular identities reduce to the following equalities:

$$f^* f_* f^* = f^* \quad \text{and} \quad f_* f^* f_* = f_*$$

Proof. This follows from the triangular identities of the adjunction.

□

1.2 Embedding

Lemma 4 (Embedding). (Leroy Lemme 1) The following arguments are equivalent:

1. f^* is surjective
2. f_* is injective
3. $f^* f_* = 1_{O(X)}$

Proof. This follows from the triangular identities.

□

Definition 5 (Embedding). An embedding is a morphism that satisfies the conditions of lemma 4

1.3 Sublocals

Definition 6 (Nucleus). A nucleus is a map $e : O(E) \rightarrow O(E)$ with the following three properties:

1. e is idempotent
2. $U \leq eU$
3. $e(U \cap V) = e(U) \cap e(V)$

Lemma 7 (Nucleus). (*Leroy Lemme 3*) Let $e : O(E) \rightarrow O(E)$ be monotonic. The following are equivalent:

1. e is a nucleus
2. There is a locale X and a morphism $f : X \rightarrow E$ such that $e = f_*f^*$.
3. Then there is a locale X and an embedding $f : X \rightarrow E$ such that $e = f_*f^*$.

Proof. □

Definition 8 (Nucleus Partial Order). For two nuclei e and f on $O(E)$, we say that $e \leq f$ if $e(U) \leq f(U)$ for all $U \in O(E)$. This relation is a partial order.

Lemma 9 (Nucleus Intersection). *TODO Quelle StoneSpaces S.51* For a set S of nuclei, the intersection $\bigcap S$ can be computed by $\bigcap S(a) = \bigcap \{j(a) \mid j \in S\}$. This function satisfies the properties of a nucleus and of an infimum.

Proof. □

Definition 10 (Sublocal). (*Leroy CH 3*) A sublocal $Y \subset X$ is defined by a nucleus $e_Y : O(X) \rightarrow O(X)$, such that $O(Y) = \text{Im}(e_Y) = \{U \in O(X) \mid e_Y(U) = U\}$. The corresponding embedding is $i_X : O(Y) \rightarrow O(X)$. $i_X^*(V) = e_X(V)$, $(i_X)_*(U) = U$ And every nucleus e on $O(X)$ defines a sublocal Y of X by $O(Y) = \text{Im}(e)$

Definition 11 (Sublocal Inclusion). (*Stimmt das?*)(*Leroy Ch 3*) $X \subset Y$ if $e_Y(u) \leq e_X(u)$ for all u . This means that the Sublocals are a dual order to the nuclei.

Lemma 12 (factorisation). (*Leroy Lemme 2*) Let $i : X \rightarrow E$ be an embedding and $f : Y \rightarrow E$ be a morphism of spaces. To have f factor through i , it is necessary and sufficient that $i_*i^*(V) \leq f_*f^*(V)$ for all $V \in O(E)$.

Proof. □

1.3.1 (1.4) Sublocal Union and Intersection

Definition 13 (Union of Sublocals). (*Leroy CH 1.4*) Let $(X_i)_i$ be a family of sublocals of E and $(e_i)_i$ the corresponding nuclei. For all $V \in O(E)$, let $e(V)$ be the union of all $W \in O(E)$ which are contained in all $e_i(V)$.

Lemma 14 (Union of Sublocals). (*Leroy CH 4*) Let X_i be a family of subframes of E and e_i be the corresponding nuclei. For every $V \in O(E)$, let $e(V)$ be the union of all $W \in O(E)$ which are contained in every $e_i(V)$. Then

1. e is the corresponding nucleus of a sublocale X of E
2. a sublocale Z of E contains x if and only if it contains all X_i . X is thus called the union of X_i denoted by $\bigcup_i X_i$

Proof. The properties of the nucleus (idempotent, increasing, preserving intersection) can be verified by unfolding the definition of $e(V)$. \square

Lemma 15 (Sublocal Union equals Nucleus Intersection). *For a family of sublocales X_i of E , the union $\bigcup X_i$ is the intersection of the corresponding nuclei. TODO lean link*

Proof. The infimum of the Nuclei is a Supremum of the sublocales, because the Nuclei are a dual order to the sublocales.) This means that it suffices to show that suprema are unique.

TODO Quelle https://proofwiki.org/wiki/Infimum_is_Unique

Suppose there are two different suprema c and c' of a set S . Because of the definition of a supremum, we have that they are both upper bounds of S . But we also know that the supremum is smaller than any other upper bound, so we get $c \leq c'$ and $c' \leq c$. This means that $c = c'$. \square

Definition 16 (Intersection of Sublocales). Let $(X_i)_i$ be a family of sublocal of E and $(e_i)_i$ the corresponding nuclei. For all $V \in O(E)$, the intersection $\bigcap X_i$ is the Union of all Nuclei w such that $w \leq x_i$ for all $x_i \in X_i$

Lemma 17 (Nucleus Complete Lattice). *The Nuclei (and therefore the sublocales) form a complete lattice.*

Proof. One can prove that the Nuclei are closed under arbitrary intersections by unfolding the definition of the intersection. The supremum is defined as the infimum of the upper Bound. \square

Proposition 18 (Complete Heyting Algebra). *A complete Lattice is a Frame if and only if it is a Heyting Algebra.*

Proof. (Source Johnstone:) The Heyting implication is right adjoint to the infimum. This means that the infimum preserves Suprema, since it is a left adjoint. \square

Lemma 19 (Nucleus Heyting Algebra). *The Nuclei form a Heyting Algebra.*

Proof. Quelle Johnstone \square

Lemma 20 (Nucleus Frame). *The Nuclei form a frame.*

Proof. \square

1.3.2 (7) Open Sublocales

Definition 21 (e_U). Let E be a space with $U, H \in O(E)$. We denote by e_U the largest $W \in O(E)$ such that $W \cap U \subset H$. We verify that e_U is the nucleus of a subspace, which we will temporarily denote by $[U]$.

Lemma 22 (e_U is a nucleus). *The map e_U is a nucleus.*

Proof. \square

Definition 23 (Open sublocal). For any $U \in O(E)$, the sublocal $[U]$ is called an open sublocal of E .

Lemma 24 ((6,7) Open Sublocal Properties). (*Leroy Lemma 6,7*)

1. For all subspaces X of E and any $U \in O(E)$:

$$X \subset [U] \iff e_X(U) = 1_E$$

2. For all $U, V \in O(E)$, we have:

$$[U \cap V] = [U] \cap [V]$$

$$e_{U \cap V} = e_U e_V = e_V e_U$$

$$U \subset V \iff [U] \subset [V]$$

3. For all families V_i of elements of $O(E)$, we have:

$$\cup_i [V_i] = [\cup_i V_i]$$

4.

Proof. □

Definition 25 (Complement). The complement of an open sublocal U of X is the sublocal $X \setminus U$. (Leroy p. 12) (+ Senf brauchen wir das allgemein??)

Lemma 26 (Complement Injective). *The complement is injective.*

Proof. □

Definition 27 (Closed Sublocal). A sublocal X of E is called closed if $X = E \setminus U$ for some open sublocal U of E .

Lemma 28 (Intersection of Closed Sublocals). *For any family X_i of closed sublocals of E , the intersection $\bigcap X_i$ is closed (it can be computed by taking the complement of the union of the complements).*

Proof. □

Lemma 29 ((1.8) Properties of Complements). *For any open sublocal V of E and any sublocal X of E , we have:*

$$V \cup X = E \iff E \setminus V \subset X$$

$$V \cap X = \emptyset \iff X \subset E \setminus V$$

And thereby:

$$(E - U = E - V) \implies U = V$$

Lemma 30 ((1.9) Preimage of complements). *For any morphism of spaces $g : A \rightarrow E$ and any open sublocal V of E , we have:*

$$g^{-1}(E - V) = A - g^{-1}(V)$$

Lemma 31 ((1.8bis) Properties of Complements Part 2). *For any open sublocal V of E and any sublocal X of E , we have:*

$$V \cup (E - V) = E \iff V \subset X$$

$$V \cap (E - V) = \emptyset \subset V$$

Lemma 32 ((1.10) Intersection of Open and Closed Sublocals). *For any $U \in O(E)$, and sublocal X of E we have:*

$$e_{U \cap X} = e_U e_X$$

And for a closed F

$$e_{X \cap F} = e_X e_F$$

Definition 33 (Further Topology).

1. $IntX$ is the largest open sublocal contained in X
2. $ExtX$ is the largest open sublocal contained in $E \setminus X$
3. \bar{X} is the smallest closed sublocal containing X
4. $\partial X = \bar{X} \cap (E - IntX)$

Lemma 34 (Properties of Further Topology).

1. $\bar{X} = E \setminus Ext(X)$
2. $\partial X = E \setminus (IntX \cup ExtX)$
3. $IntX \cup \partial X = \bar{X}$
4. $ExtX \cup \partial X = E \setminus IntX$

Proof.

□

Chapter 2

Leroy Chapter III

Definition 35 (Measure on Locals). A measure on a local X is a map $\mu : O(X) \rightarrow [0, \infty)$ such that:

1. $\mu(\emptyset) = 0$
2. $U \subset V \implies \mu(U) \leq \mu(V)$
3. $\mu(U \cup V) = \mu(U) + \mu(V) - \mu(V \cap U)$
4. For any increasingly filtered family V_i of open sublocals of X , we have:

$$\mu\left(\bigcup_i V_i\right) = \sup_i \mu(V_i)$$

this means: For all i and j there exists a k such that $V_i \cup V_j \subset V_k$ bzw. $V_i \subset V_k$ and $V_j \subset V_k$.

(Leroy III.1.)

Definition 36 (Caratheodory). For any measure on a local X , the caratheodory extension is:

$$\mu(A) = \inf\{\mu(U) \mid A \subset U \in O(X)\}$$

Lemma 37 (Property 0 (Commutates with sup)). *(Leroy lemme 3.1) The caratheodory extension of a measure on a local commutes with unions of increasing families. (Sens von noa: commutes with filtered colimits)*

Lemma 38 (Caratheodory Extensions are monotonic). *The caratheodory extension is monotonic i.e.*

$$A \leq B \implies \mu(A) \leq \mu(B)$$

Proof. This is a direct consequence of the definition of the caratheodory extension. □

Lemma 39 (Subadditivity). *The Caratheodory extension is subadditive:*

$$\mu(A \cup B) \leq \mu(A) + \mu(B)$$

Proof. □

Definition 40 (Regular Local). A local is regular, if for all open sublocals U of E , the open sublocals V such that $V \subset U$ recover U .

Definition 41 (Neighborhood).

A neighborhood of a sublocal A of X is an open sublocal V of X such that $A \leq V$.

Lemma 42 (Regularity of Sublocals). (*Leroy lemme 3.2*) In a regular local, any sublocal is regular, meaning that it is the intersection of all open neighborhoods.

Lemma 43 (Measure add compl eq top). (*Leroy Lemme 3.3*) For any open sublocal U of a local X , the caratheodory extension of a measure on X satisfies

$$\mu(U) + \mu(X \setminus U) = \mu(X)$$

Proof. Siehe Leroy □

Lemma 44 (Restriction). *The Restriction of a Measure to any open Sublocal is a Measure.*

Proof. □

Lemma 45 (Property 2). (*Leroy Lemm 3.4*) For any open sublocal U and any sublocal A of a local E , the caratheodory extension of a measure on X satisfies

$$\mu(A) = \mu(A \cap U) + \mu(A \cap (E \setminus U))$$

Proof. Siehe Leroy □

Lemma 46 (Property 3). (*Leroy Lemm 3.5*) For a increasing family V_α of open sublocals of E and any sublocal A , we have:

$$\mu(A \cap (\bigcup_\alpha V_\alpha)) = \sup_\alpha \mu(A \cap V_\alpha)$$

Lemma 47 (Restriction to a Sublocale). *Let A be a sublocale of E with the embedding $i : A \rightarrow E$. The restriction of a measure μ on E to A is a measure on A :*

$$V \mapsto \mu(i(V)) : \text{Open}(A) \rightarrow \mathbb{R}$$

Proposition 48 (strictly additive). (*Leroy theorem 3.3.1*) For any measure on a local X , the caratheodory extension is strictly additive i.e. $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$

Proposition 49 (reductive). (*Proposition 3.3.1*) For any measure on a local X , the caratheodory extension is reductive i.e. for all $A \leq X$ the set $\{A' \subset A, \mu(A') = \mu(A)\}$ has a minimal element

Lemma 50 (Commutates with inf opens). (*Leroy Lemme 3.6*) For any measure on a local X and a decreasing family V_i of open sublocals, the caratheodory extension fulfills: $\mu(\inf V_i) = \inf \mu(V_i)$.

Proof. □

Proposition 51 (Commutates with inf). (*Leroy lemme 3.7 et principal*) For any measure on a local X , the caratheodory extension is regular $\mu(\inf A_i) = \inf \mu(A_i)$. For decreasing families A_i

Theorem 52 (Main Theorem (very important)). For any measure on a local X , the caratheodory extension is

1. strictly additive i.e. $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$
2. commutes with inf $\mu(\inf A_i) = \inf \mu(A_i)$
3. reductive i.e. for all $A \leq X$ the set $\{A' \subset A, \mu(A') = \mu(A)\}$ has a minimal element

Chapter 3

Leroy Chapter V

Lemma 53 (Regular Top to regular local). *Any regular topological space induces a regular local.*

Lemma 54 (Opens). *(Leroy V.1 Remarque 2) The Open subsets of any good enough topological space correspond precisely to the open sublocals of the corresponding local.*

Lemma 55 (Subset Sublocal). *(Leroy V.1 Remarque 3) Any subset X of a good enough topological space E induces a sublocal $[X]$ of the corresponding local. This is an order preserving embedding.*

Definition 56 (Good enough topological space). blackbox to mathlib????)

Lemma 57 (Subset to sublocal Part 1). *(Leroy Proposition 5.1.1)*

For two subspaces X and Y of E and an open subspaces U of E , we have:

1. $X \subset Y \implies [X] \subset [Y]$
2. $X \subset U \iff [X] \subset [U]$
3. *If E is a good enough topological space, then*

$$X \subset Y \iff [X] \subset [Y]$$

Lemma 58 (Subset to sublocal Part 2). *(Leroy Proposition 5.1.2, 5.1.3) For an open subspace U of E and a subspace X of E , we have:*

$$[U \cap X] = [U] \cap [X]$$

$$F = E \setminus U$$

$$[F] = [E]$$

$$[U]$$

$$X \cap F = [X] \cap [F]$$

Lemma 59 (Part 3). *For any subspaces X of E , we have:*

- 1.

$$Ext[X] = [ExtX]$$

- 2.

$$[\bar{X}] = [\bar{X}]$$

3.

$$[IntX] \subset Int[X]$$

4.

$$\partial[X] \subset [Fr(X)]$$

For a good enough topological space E , we have equality in 3 and 4.

Proposition 60 (Subset to sublocal preserves structure). *For two subspaces X and Y of E and an open subspaces U of E , we have:*

$$1. X \subset Y \implies [X] \subset [Y]$$

$$2. X \subset U \iff [X] \subset [U]$$

3. If E is a good enough topological space, then

$$X \subset Y \iff [X] \subset [Y]$$

4.

$$[U \cap X] = [U] \cap [X]$$

5. ...

Theorem 61 (Measure top to loc). *Any measure on a good enough topological space X induces a measure on the corresponding local. Furthermore, the classical caratheodory extension onto $\mathcal{P}(X)$ agrees with the restriction of the caratheodory extension of the induced measure on the local.*