CMPE 548 Monte Carlo Methods
Homework 1

Due on March 2, 2016

1. **(Uniform samples from a circular region)**

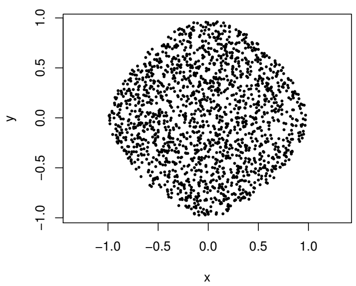
   Let

   $$R \sim p(r) = \begin{cases} 2r & 0 \leq r \leq 1 \\ 0 & otherwise \end{cases}$$
   $$\Theta \sim \mathcal{U}(\theta; 0, 2\pi)$$

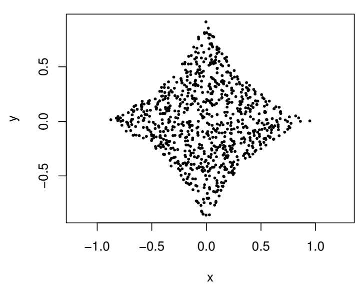
   Generate samples from R and Θ to obtain polar coordinates (r, θ) and plot sample points to visually observe that they are distributed uniformly inside the unit circle. Use **inversion method** to sample from $p(r)$.

2. The region enclosed by the unit circle is called **closed unit ball** corresponding to **2-norm** (Euclidean norm). In general, the closed unit ball corresponding to **p-norm** is the region defined by $(|x|^p + |y|^p)^{\frac{1}{p}} \leq 1$ in 2-dimensional space.

   By using **rejection sampling**, draw samples from the closed unit ball for $p = 1.5$ and $p = 0.7$ using the samples drawn from unit 2-norm ball and report the acceptance rate. Can you improve the efficiency of the sampler for $p = 0.7$ by using a different proposal?



(a) $p = 1.5$



(b) $p = 0.7$

Figure 1: Your results should look like these figures.

Bonus: The sampler you have developed in (1) can be generalized to generate uniform samples from the volume bounded by the unit n-sphere as follows:

In 2D, θ defines the direction between the position vector $\mathbf{r}$ and the x-axis. In higher dimensions the direction cannot be defined by using only one angle. As discussed in the lecture, to obtain a random direction uniformly in n-dimensional space, we can generate a sample from n-variate gaussian distribution with identity covariance matrix, and normalize the resulting vector.

As we have the direction, only remaining coordinate is the distance from the origin, r, in order to obtain a sample. Again, we need to induce a similar bias for larger values of r to satisty the uniformity of the joint distribution. Think about the rate at which the volume of the n-sphere increases with respect to r. As we get further away from the origin, the volume of the shell corresponding to the small neighborhoods of r increases. Therefore the sampler needs to be able to generate more samples for larger values of r.

Implement the sampler described above to generate samples uniformly from the region bounded by the n-sphere.

An alternative way to implement the same sampler would be to use **rejection sampling**. Consider bounding the n-sphere by a corresponding n-cube (In 2D, this corresponds to bounding the unit circle by a square). By definition, we only accept the samples which are inside the n-sphere. In other words, the samples that lie outside the n-sphere get discarded. How would the acceptance rate change as n gets larger?