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Simple Harmonic Motions

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Abstract:

In this report, the Earth's gravitational constant was measured by investigating the period of a simple pendulum when the pendulum was doing simple harmonic motion (SHM) with small oscillations in degrees less than 10□with respect to the horizontal axis of the pendulum using the period formula for SHM. The same experiment was conducted three times with different methodologies. Earth's gravitational constant (G) was determined from the first methodology as 11.6 □ 0.81 × ms^{-2} was also determined as 8.81 $\pm$ 0.94 × ms^{-2} , and finally, the result from the third methodology was $9.74 \pm 0.02\, ms^{-2}$. These results were determined by using the relation between G with the periodic formula for SHM. However, for the third result only, additionally, it was determined by taking the gradient of the graph.

Introduction:

The primary aim of this experiment was to ascertain Earth's gravitational constant. The investigation hinged on the connection between the formula for the period of simple harmonic motion (SHM) in a pendulum with small oscillations (1) and Earth's gravitational constant. SHM involves the oscillation of a load from its origin at an angle with the horizontal plane. However, the experiment was confined to small oscillations, as angles exceeding 10° would invalidate the equation (1).

$$T = 2\pi\sqrt{\frac{L}{g}} \tag{1}$$

Here, T represents the period in seconds which is a complete cycle of a load, L is the length of the pendulum in meters from the oscillation point to the centre of mass of the load at the end of the pendulum. G is the acceleration due to Earth's gravity(g) in meters per second squared or ms^{-2}.Equation (1) comes from solving the second-order differential equation of Hooke's Law with respect to the time equation (2). Hooke's Law explains force relation with extension and spring constant in springs.

$$F = -kx \tag{2}$$

Where F represents force in Newtons, which is causing stress on the elastic object, x is the extension caused by the force in meters, and k is the spring constant of the material used in newtons per meter.

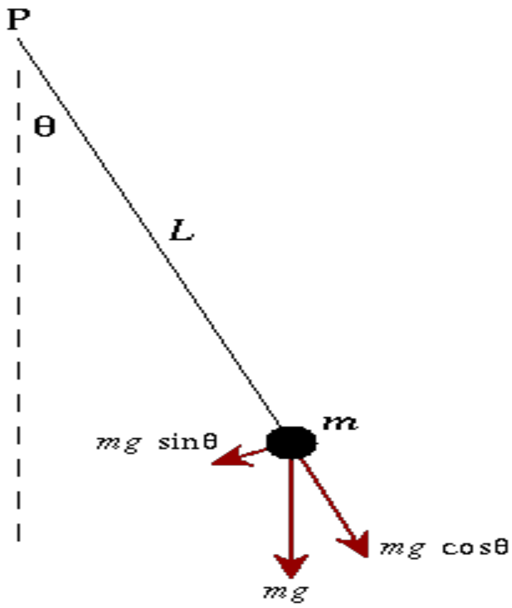


Figure 1: diagram showing a simple pendulum with the forces acting on an object. (Russel, https://www.acs.psu.edu/drussell/Demos/Pendulum/Pendulum.html, 2018)

To get equation (1), using Hooke's Law, the angle θ in Figure 1 must be replaced by its value of arc(theta), which is the load-displacement divided by the Length of the cable L. It is known that this calculation can only be done mathematically if the angle θ is smaller than 10□ That explains the reason for using small oscillations in this experiment.

The occurrence of harmonic motion in practical situations can be exemplified by the vibrations of the eardrum. Upon the arrival of a sound wave at the pinna, it traverses through the ear canal and reaches the eardrum. The transfer of energy from the sound wave to the eardrum causes it to vibrate, initiating a harmonic motion on the eardrum. The parts of the ear can be seen in the figure 3.

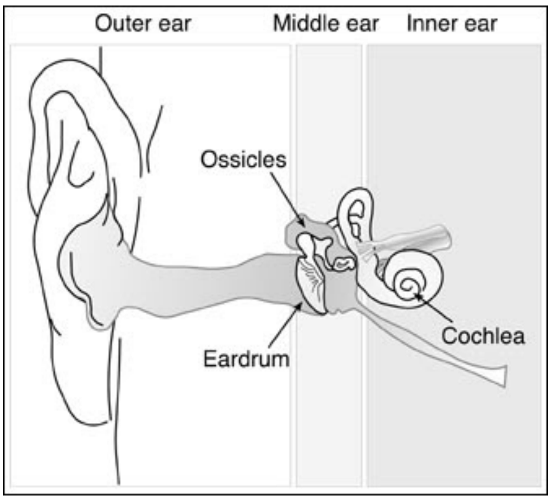


Figure 3: A diagram showing the part of the ear. (Pharmacy, https://pharmacyimages.blogspot.com/2020/09/simpleeardiagrams-eardiagramwithlabels-Innereardiagram.html, 2020)

For this investigation, the period formula for pendulum (1) is assumed to be correct, so the c constant in the line equation will be zero because the line crosses through the origin.

The literature value of g was taken as 9.81 ms^{-2}.

Methods:

In this investigation, three different methods were used to calculate g.

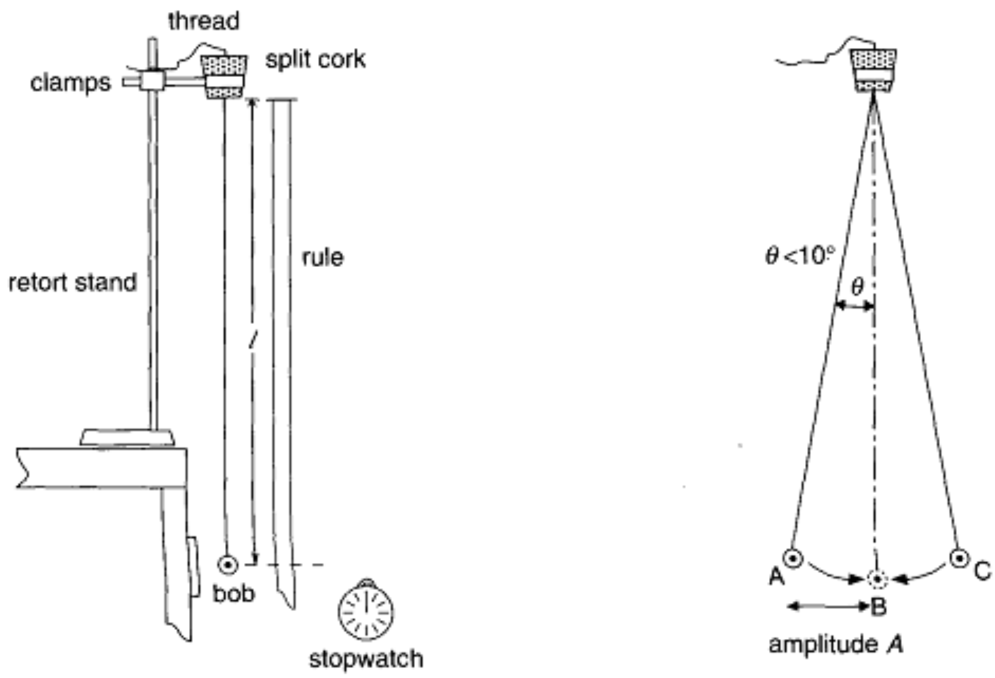


Figure 2: shows the pendulum setup and highlights the critical angle for simple harmonic motion. (Minseo, https://kimminseodimension.wordpress.com/2014/04/17/practicals-4-pendulum-predictions, 2014)

The first method employed in this study aimed to determine the acceleration due to gravity (g) by conducting a single small oscillation in a pendulum. The pendulum has consisted of a load attached to the bottom end with a fixed length of 15 cm, which was measured using a ruler with an uncertainty of 0.05cm, which, when combined in quadrature, resulted in a total uncertainty of 0.07cm. However, parallax errors were raised when taking such measurements, leading to a misleading measurement of the pendulum's length. In this experiment, however, such errors were deemed negligible compared to other sources of error. It should be noted that the pendulum length was measured from the point of oscillation to the point where the masses were attached, whereas the measurement locations for the rest of the experiment have remained constant. The time taken for one oscillation was measured using a stopwatch on a phone. However, taking a reading with a stopwatch has introduced a random error due to the delay in reaction time between the stopwatch and the observer. To minimise such errors, multiple readings were taken, and the average period and length, along with their error, were calculated, and used to calculate g.

The second method aimed to determine the acceleration due to gravity by conducting multiple oscillations in a pendulum. The pendulum's length was fixed at 15cm, and the period was calculated by measuring the timing length of ten oscillations and dividing it by ten. This process was repeated ten times, and the standard deviation and average of the period data were calculated for statistical analysis. The average uncertainty was also determined, and the length and its error, along with the period and its error, were used to calculate g.

The third method focused on calculating g with varying lengths of the pendulum. The pendulum length was set to 17cm, 19cm, 21cm, 23cm, and 25cm, and ten oscillations were conducted for each length. The period T was calculated by dividing the time reading by ten and repeating the process ten times. The average uncertainty of the period was calculated using the average uncertainty formula for large data sets. Additionally, a graph was plotted in Excel with the y-axis representing the period T and the x-axis representing the $2\pi\sqrt{L}$; L(Length). The line of best fit was plotted using both the informal method and the least squares method, and the gradient and its error were measured. The line equation was written in the form of $y = mx + c$, where the gradient m was used to determine the value of g, assuming equation [1] was correct.

As a precaution, a glove should have been worn to prevent contact with the metal of the pendulum.

Results:

For investigation one:

The result of the investigation one can be seen in the tables below. Table 1 shows the measurements taken of period, length and their errors. This first investigation used a statistical formula valid for small data sets to calculate errors. For small data sets, the uncertainty of the average equation (3) was used.

$$\Delta\bar{X} = \frac{\Delta x}{\sqrt{N}} \qquad (3)$$

Where $\Delta\bar{X}$ stands for average uncertainty Δx represents the uncertainty of the set. N denotes the number of repeatings in the experiment, the formulas used in the first investigation.

Table 1: A table showing measurements and calculations for the case where a pendulum was doing small oscillations with a length of 15cm.

	Period of Pendulum(s)	Length of Pendulum(cm)
Uncertainty of the set	0.5	0
Error on the Average	0.14	0.07
Result	0.14±0.02	15±0.07

For period Δx was calculated as $\frac{0.75-0.65s}{2}$, and the lt was 0.5s; when this result was substituted into equation (3), $\Delta\bar{x}$, it became $\pm$0.02s. Moreover, the average period was calculated as 0.14. Thus, the final result for the period is 0.14$\pm$0.02.

When the results are combined with equation (1) to calculate g, the length units must be converted to meters. Then, the g was calculated as 11.6 ms^{-2}. The quadrature formula will be used to calculate the total error on the g equation[4], and it was calculated as $\pm 0.81\ ms^{-2}$. Thus, the final result of g became $g = 11.6 \pm 0.81\ ms^{-2}$.

$$\Delta t = \left|\frac{T}{L}\right| \times \sqrt{\left(\frac{\Delta T}{T}\right)^2 + \left(\frac{\Delta L}{L}\right)^2} \tag{4}$$

Where T is the period, L is the length of the pendulum, and Δ in the right-hand side terms are errors of these measurements.Δt is the total error of the calculation.

For investigation two:

The result of the second investigation can be seen in Table 2, which shows the measurements taken of the period, length and their errors. A statistical formula valid for large data sets will be used to calculate errors in this second investigation. Equation (5) was used for large data sets by taking the standard derivation of the data set and substituting it into the equation.

$$\Delta\bar{x} = \frac{2\sigma}{\sqrt{N}} \tag{5}$$

Where $\Delta\bar{x}$ stands for average uncertainty, σ denotes the standard derivation of the data set. It can be calculated in equation (6), and N represents the number of repeatings in the experiment.

$$\sqrt{\frac{1}{N}\left(\sum_{i=1}^{N}(xi)^2\right) - \bar{x}^2} \tag{6}$$

N in this equation represents the number of repeating (xi), which denotes each value in the data set, and $\bar{x}$ shows the average of the data and can be calculated by adding all values in a data set and dividing by a number of terms.

Table 2: A table showing measurements and calculations for the case where a pendulum was doing small oscillations with a length of 15cm.

	Period of Pendulum(s)	Length of Pendulum(cm)
Uncertainty of the set(σ)	0.66	0
Error on the average	0.04	0.07
Result	0.82$\pm$0.04	15$\pm$0.07

For this experiment, ten oscillations time for a pendulum was measured and divided by ten. T is calculated for each run, and then the average of the values was calculated as T = 0.82. The standard derivation of the investigation was calculated as 0.66s using (6). After that, using equation (5), the average error was calculated as ± 0.04 s. So, the result for T became $\mathrm{T} = 0.82 \pm 0.04$ s.

When the results are combined with equation (1) to calculate g, the length units must be converted into meters. Then, the g was calculated as 8.81 ms^{-2}. The quadrature formula will be used to calculate the total error on g (4), and it was calculated as $\pm 0.94\, ms^{-2}$. Thus, the result of g becomes $g = 8.81 \pm 0.94\, ms^{-2}$. Notice that the error ranges, which were calculated as a standard deviation, are both the %68 and %95 confidence levels.

For investigation three:

The result of the third investigation can be seen in Table 3, which shows the measurements taken of the period, with different length and their errors.

Table 3: A table showing measurements and calculations for the case where a pendulum was doing small oscillations with a length of 17,19,21,23 and 25 cm. Which length has an uncertainty of 0.07cm

Length of Pendulum(cm)	Average Period of Pendulum(s)	Uncertainty of the set of period(s)	Error on the average of period(s)
17	0.84	$8.3 \times (10^{-3})$	5.2×10^{-3}
19	0.89	5.1×10^{-3}	3.2×10^{-3}
21	0.93	5.1×10^{-3}	3.2×10^{-3}
23	0.96	6.3×10^{-3}	3.9×10^{-3}
25	1.00	4.3×10^{-3}	2.7×10^{-3}

To calculate g from a graph, a graph was drawn with the x-axis values. Data for the graph can be seen below.

Table 4: Shows the variables for the x and y axes values with their errors for the graph. Assuming equation (1) is correct, the values can start from the origin (0,0).

$2\pi\sqrt{Lenght}$ (m) [x-axis]	Period T (s) [y-axis]
2.6 $\pm 0.7 \times 10^{-2}$	0.84 $\pm$ 5.2 $\times 10^{-3}$
2.7 $\pm 0.7 \times 10^{-2}$	0.89 $\pm$ 3.2 $\times 10^{-3}$
2.9 $\pm 0.7 \times 10^{-2}$	0.93 $\pm$ 3.2 $\times 10^{-3}$
3.0 $\pm 0.7 \times 10^{-2}$	0.96 $\pm$ 3.9 $\times 10^{-3}$
3.1 $\pm 0.7 \times 10^{-2}$	1.00 $\pm$ 2.7 $\times 10^{-3}$

To calculate the line of best fit for the graph using the informal method first line was drawn with lower error bars on the left-hand side, and for the right-hand side, upper error bars must be used; this is the opposite for drawing the second line.

So, the first line equation will come from points from the upper error bound for the last term of the period in Table 4; 1.00+2.7 $\times 10^{-3} s$ to the lower bound is in Table 4; 0.84 $-$ 5.2 $\times\ 10^{-3} s$. The second line will come from a lower error bound for the last term of the period in Table 4, 1.00 $-$2.7$\times 10^{-3}$ and the first term with an upper bound, which is 0.84 $+$ 5.2 $\times\ 10^{-3}$. Therefore, the third line gradient must be between the gradients of these two lines, which are almost equal. So, the gradient from these two points will give m(gradient) $\approx$ 0.32 with an error of + 2.7 $\times 10^{-3} s$ and $-$ 5.2 $\times\ 10^{-3} s$.So, the result of the gradient would be 0.32 + 2.7 $\times 10^{-3}$ $-$ 5.2 $\times\ 10^{-3}$ $(\frac{1}{\sqrt{ms^2}})$.

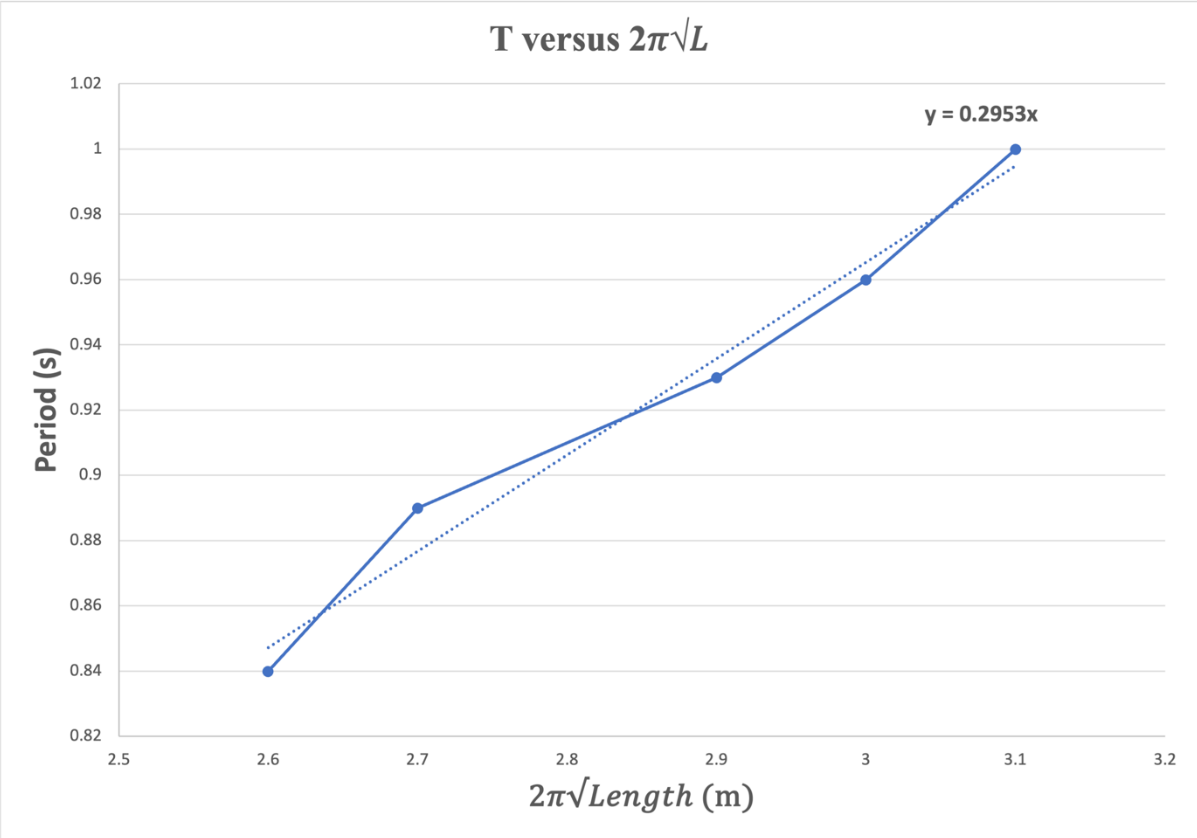
Moreover, to calculate the line of best fit by using the least squares method, the sum of data on the x-axis and on the y-axis, the sum of the squares of data on the x-axis, and the sum of the products of the data in both the x and y axes are needed.

It's because a line is desired to plot into the graph; the equation must be in the form of $y = mx + c$. The variables mentioned above will be substituted in equation (6) to calculate m in our equation, which is the gradient in the line equation.

$$m = \frac{n\sum_{i=1}^{n} xy - \sum_{i=1}^{n} x \times \sum_{i=1}^{n} y}{n\sum_{i=1}^{n} x^2 - (\sum_{i=1}^{n} x^2)} \tag{6}$$

Where m is the gradient and $\sum figure$ denotes the sum.

Using equation (6) and the LINEST function in Excel, m (gradient) was calculated as 0.29 $\pm$0.02.



Graph 1: A graph results from Table 4. A best-fit line and its equation are included and calculated using the weighted least squares method. The error on both the x and y axes was too small to be represented.

Notice that the gradient of the equation of the line of best fit from the least squares method. gives $\frac{1}{\sqrt{g}}$, which comes from equation (1). Thus, g was calculated as $9.74 \pm 0.02\ ms^{-2}$. The literature value of g = 9.81 ms^{-2} , and the result from the experiment, calculated in the first methodology, is g = 11.6 □ 0.81 $\times\ ms^{-2}$, so the error is %18. Furthermore, the second investigation g = 8.81 $\pm$ 0.94 ms^{-2}, so the error is %10. Finally,
The result from the third investigation is 9.74 $\pm$ 0.02 ms^{-2} that the error is %0.71.

Discussion:

The experiment encountered a primary source of error in measuring the period due to random factors such as parallax and the conductor's reaction time. Repeated measurements could not mitigate these errors in the initial two parts of the investigation. However, the third part, involving varying pendulum lengths, was yielded a more accurate outcome with an error rate of 0.71%. This section has demonstrated that repeated measurements were effective in addressing the issue. Importantly, it should be noted that this error cannot be easily quantified.

The second experiment had a smaller error compared to the first one because no repeated measurements were taken initially. However, repeated measurements were employed in the second part, resulting in a smaller error. When compared to the literature value, the error in the results from both parts was 18% for the first part and 10% for the second part. This stark difference underscores the importance of conducting repeated measurements.

Moreover, the third experiment had a smaller error compared to the second part. In the second part, 'g' was determined through calculations, whereas in the third part, it was calculated using the line of the best fit method, resulting in a more accurate outcome than the second part. Consequently, the error in both the second and third parts of the investigation, when compared to the literature value of 'g,' was 10% and 0.71%, respectively, highlighting a substantial difference between the two results, as previously mentioned.

It was also observed that a parallax error occurred during length measurements using a ruler, which led to misleading measurements. Nevertheless, this error wasn't significantly impacted the results due to its relative insignificance compared to the uncertainty of the ruler, making it negligible. To address this issue more effectively, it would have been advantageous to connect a digital stopwatch mechanism to both ends of the pendulum. This would ensure that the pendulum automatically stops at the end of each cycle, significantly reducing errors related to human reaction time, as the speed of electrons far exceeds that of a human. Additionally, increasing the number of repeated measurements could further decrease error, as it would distribute the error across a larger dataset.

One limitation of this experiment was the inability to determine the length at which the period changes precisely. Although a difference was observed, future investigations could address this limitation by using more distinct length differences in part three to detect variations readily.

In conclusion, the experiment was encountered various sources of error, primarily associated with the measurement of the period, influenced by parallax and the conductor's reaction time. Nonetheless, repeated measurements, varying pendulum lengths, and the least square method were effectively reduced error and were yielded accurate outcomes.

Conclusions:

This study aimed to validate Hooke's Law, equation (2), and the period formula for a pendulum equation (1). The results demonstrated that the aforementioned equations were broadly validated. The best result for the acceleration due to gravity (g) was determined to be $9.74 \pm 0.02\ ms^{-2}$. In addition, the period formula (1) was validated in the third part of the study, as increasing the length resulted in an increase in the period. The least squares method was found to be more accurate than calculating by using raw data, as evidenced by the calculation of g in the third part of the experiment, which had an error of only %0.71 compared to the literature value.

Further investigation may focus on the different types of loads because, in theory, the pendulum has a load of particle mass; what would have been different if a much smaller object, similar to the theory, was used in the experiment?

In summation, the most minor error was founded on measurement from the experiment is g = $9.74 \pm 0.02\ ms^{-2}\ with\ an\ error\ \%0.71$, which indicates the Earth's gravitational constant was determined successfully from this experiment. Improvements to the method and to the other areas have been suggested.

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