McGill University

ECSE 506

Final Project

A review by Feras Al Taha of

Witsenhausen's counterexample

Feras Al Taha 260741500

May 1st, 2020

Abstract

This review covers the Witsenhausen counterexample, a toy example in decentralized systems which demonstrates that linear policies are not always optimal given linear dynamics, quadratic cost and Gaussian noise. Indeed, the information structure of the decentralized system can result in non-linear policies performing better than the most optimal affine solution. The main results and proofs of this problem are presented along with further work that was developed following it.

1 Introduction

The covered paper [1] describes the Witsenhausen counterexample, an optimal control problem which demonstrates that even with linear dynamics, quadratic cost and Gaussian disturbance, non-linear controllers can perform better than the optimal affine controller if given a decentralized system. The next section provides some background on the standard linear-quadratic-Gaussian problem. Section 3 presents the problem formulation while Section 4 gives the main results from the original paper. Section 5 shows further research that followed as well as attempts at finding the optimal solution to the problem. Finally, the review ends with some concluding remarks in Section 6.

2 Background

The linear-quadratic-Gaussian (LQG) problem is a standard and fundamental optimal control problem. It regards systems with linear dynamics driven by additive Gaussian noise for which an output feedback law has to be determined subject to a quadratic cost whose expectation is to be minimized.

Given the state $x_t \in \mathbb{R}^n$, action $u_t \in \mathbb{R}^m$ and observation $y_t \in \mathbb{R}^{\ell}$, the linear noisy dynamics are

$$x_{t+1} = A_t x_t + B_t u_t + w_t$$

The noisy observations are given with

$$y_t = C_t x_t + v_t$$

The random variables x_0, $w_{0:T}$ and $v_{0:T}$ are independent and Gaussian with zero mean. The per-step cost is

$$c_t(x_t, u_t) = x_t^T Q_t x_t + u_t^T R_t u_t$$

The optimal control problem is to choose $u_t = g_t(y_{1:t}, u_{1:t-1})$ such that the performance $J(g)$ is minimized.

$$J(g) = \mathbb{E}^g \sum_{t=1}^{T-1} c_t(x_t, u_t)$$

In this case, the control variables are unconstrained and the information pattern is classical since there is a single controller with perfect memory. In general, classical information pattern is when all actions taken at a given time are based on the same data and any data available at time t is still available at any later time $t' > t$.

For such a setup, it is found that the optimal solution is an affine controller, both for discrete and continuous time systems [2]. Indeed, the separation principle can be used to determine a control law which combines a Kalman filter with a linear-quadratic regulator (LQR) to form the unique LQG controller which will be optimal [3].

A natural conjecture following the LQG result was that this should generalize to the different information patterns found in decentralized systems. However, affine policies were not always optimal and this was demonstrated with the Witsenhausen counterexample [1].

3 Problem Formulation

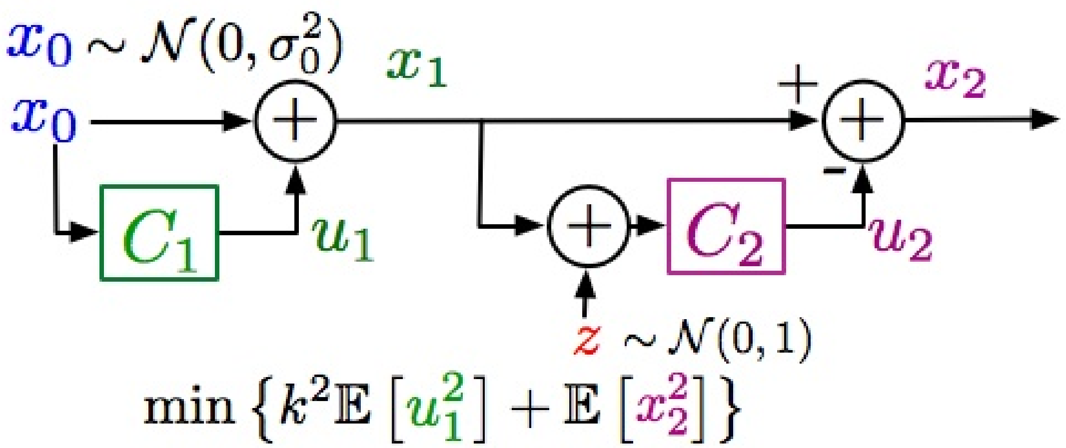

Figure 1: Witsenhausen's counterexample

Let the initial state x_0 and noise process v be independent random variables such that $x_0 \sim N(0, \sigma^2)$ and $v \sim N(0, 1)$. Consider a 2-stage stochastic control problem as depicted in Figure 1. One possible application where this kind of setup could be found is in transmission over a noisy medium. Indeed, one can interpret the first stage to be the encoder sending data and the second stage being a decoder receiving the data with some noise [4].

The linear state equations, which involve no noise, are

$$\begin{aligned} x_1 &= x_0 + u_1 \\ x_2 &= x_1 - u_2 \end{aligned}$$

The observations are such that at the first stage, the state is observed perfectly but at the second stage, it is observed with noise. Accordingly, the output equations are

$$\begin{aligned} y_0 &= x_0 \\ y_1 &= x_1 + v \end{aligned}$$

The quadratic per-step cost is defined as

$$c(x, u) = k^2 u_1^2 + x_2^2 \qquad k^2 > 0$$

There are two separate controllers such that the first acts on the first set of observations and the second acts on the second set of noisy observations. Moreover, the information pattern is non classical since the second control stage does not have access to the value of observation y_0 from the first stage.

$$\begin{aligned} u_1 &= \gamma_1(y_0) \\ u_2 &= \gamma_2(y_1) \end{aligned}$$

where (γ_1, γ_2) are the chosen policies taken from the set of Borel functions Γ.

The objective is to find the optimal policies that will minimize the expectation of the cost, using the data available.

$$\min_{\gamma_1, \gamma_2 \in \Gamma} \mathbb{E}\{k^2 u_1^2 + x_2^2\}$$

4 Main Results

4.1 Reformulation

The problem can be reformulated with a change of variable, to facilitate the analysis. Let $x = x_0$, $f(x) = x + \gamma_1(x)$ and $g(y) = \gamma_2(y)$. Then, the optimization problem (which will be named OCP) is to minimize the performance function $J(f, g)$.

$$J^* = \inf_{f,g\in\Gamma} J(f,g) = \inf_{f,g\in\Gamma} \mathbb{E}\{k^2(x - f(x))^2 + (f(x) - g(f(x) + v))^2\}$$

4.2 Existence of an optimal solution

A few lemmas giving bounds on the performance function can be derived to help characterize the optimal solution.

Lemma 1 *J^* is bounded such that*

$$0 \leq J^* \leq \min(1, k^2\sigma^2)$$

Proof.

Since the quadratic cost is positive, its expectation will necessarily be positive and so $J^* \geq 0$. An upper bound can be obtained by considering the two following policies, which give the highest possible expectations of the cost.

For $(f, g) = (0, 0)$, the performance function gives

$$J(f,g)|_{(f,g)=(0,0)} = \mathbb{E}\{k^2(x - 0)^2 + (0 - 0)^2\} = k^2\mathbb{E}\{x^2\} = k^2\sigma^2$$

For $(f(x), g(y)) = (x, y)$, the performance function gives

$$J(f,g)|_{(f,g)=(x,y)} = \mathbb{E}\{k^2(x - x)^2 + (x - (x + v))^2\} = \mathbb{E}\{v^2\} = 1$$

As such, J^* is upper bounded by $\min(1, k^2\sigma^2)$.

Lemma 2 *Given $(f, g) \in \Gamma$, there exists $(f^*, g^*) \in \Gamma$ such that*

- $\mathbb{E}\{f^*(x)\} = 0$
- $\mathbb{E}\{(x - f^*(x))^2\} \leq \sigma^2$
- $J(f^*, g^*) \leq J(f, g)$
- $\mathbb{E}\{f^*(x)^2\} \leq 4\sigma^2$

Proof.

Consider the case where $\mathbb{E}\{(x - f(x))^2\} > \sigma^2$. Then,

$$\begin{aligned} J(f,g) &= \mathbb{E}\{k^2(x - f(x))^2 + (f(x) - g(f(x) + v))^2\} \\ &\geq k^2\mathbb{E}\{(x - f(x))^2\} \\ &\geq k^2\sigma^2 \end{aligned}$$

By letting $(f^*, g^*) = (0, 0)$, all the stated properties are satisfied. Namely,

- $\mathbb{E}\{f^*(x)\} = 0$ since $f^*(x) = 0$
- $\mathbb{E}\{(x - f^*(x))^2\} = \mathbb{E}\{(x)^2\} = \sigma^2$
- $J(f^*, g^*) = 0 \leq k^2\sigma^2 \leq J(f, g)$
- $\mathbb{E}\{f^*(x)^2\} = 0 \leq 4\sigma^2$

Now, consider the case where $\mathbb{E}\{(x-f(x))^2\} \leq \sigma^2$. Then, $\mathbb{E}\{f^2(x)\} \leq 4\sigma^2$ such that $\mathbb{E}\{f(x)\} = m$ for some nonzero and finite m. Let $(f^*, g^*) = (f(x)-m, g(y+m)-m)$.

- $\mathbb{E}\{f^*(x)\} = \mathbb{E}\{f(x) - m\} = 0$
- $\mathbb{E}\{(x - f^*(x))^2\} = \mathbb{E}\{(x - f(x) - m)^2\} \leq \mathbb{E}\{(x - f(x))^2\} - m^2 \leq \sigma^2$
- When considering $J(f^*, g^*)$,

$$\begin{aligned} J(f^*, g^*) &= \mathbb{E}\{k^2(x - f(x) - m)^2 + (f(x) - m - g(f(x) - m + v) + m)^2\} \\ &= \mathbb{E}\{k^2((x - f(x))^2 - 2m(x - f(x)) + m^2) + (f(x) - g(f(x) + v))^2\} \\ &= \mathbb{E}\{k^2(x - f(x))^2 - 2k^2m^2 + k^2m^2 + (f(x) - g(f(x) + v))^2\} \\ &= J(f, g) - k^2m^2 \\ &\leq J(f, g) \end{aligned}$$

- $\mathbb{E}\{f^*(x)^2\} \leq 4\sigma^2$ since $\mathbb{E}\{(x - f^*(x))^2\} \leq \sigma^2$

From these lemmas and using further results, Witsenhausen could prove the following theorem [1].

Theorem 1 *Given the optimal control problem (OCP) described in Section 3, for any $k^2 > 0$ and any distribution for x_0, the optimization problem has a solution that exists.*

Wu and Verdu also demonstrated this result later on with an alternative proof using Optimal Transportation theory [5].

4.3 Optimization over affine policies

Consider the optimal control problem (OCP) described in Section 3 but instead of minimizing J over all possible policies, take a subset $\Gamma_a \subset \Gamma$ which consists of the affine policies only. These have the following form:

$$\begin{aligned} f(x) &= \lambda x \\ g(x) &= \mu y \end{aligned}$$

Theorem 2 *If only affine strategies are considered for the first stage in the problem setup, i.e. $f(x) = \lambda x$, then the best response in the second stage is also an affine strategy $g(y) = \mu y$ where*

$$\mu = \frac{\sigma^2 \lambda^2}{1 + \sigma^2 \lambda^2}$$

The corresponding performance for this policy is

$$J(\lambda) = k^2 \sigma^2 (1 - \lambda)^2 + \frac{\sigma^2 \lambda^2}{1 + \sigma^2 \lambda^2}$$

Proof.

If $f(x) = \lambda x$, then $y = f(x) + v = \lambda x + v$. The performance to minimize is then

$$\begin{aligned} J(f, g) &= \mathbb{E}\{k^2 (x - f(x))^2 + (f(x) - g(f(x) + v))^2\} \\ &= \mathbb{E}\{k^2 (x - \lambda x)^2 + (\lambda x - g(\lambda x + v))^2\} \end{aligned}$$

The first term $k^2(x-\lambda x)^2$ being fixed, only $\mathbb{E}\{(\lambda x - g(\lambda x + v))^2\}$ has to be considered in the minimization, for finding the best $g(y)$. This expectation term can be interpreted as the estimation error by choosing the estimate $\hat{X} = g(Y)$ for the Gaussian random variable $X = \lambda x = \lambda x_0$ given the Gaussian random variable $Y = X + v$. Since all the involved distributions are Gaussian, the best estimator is affine:

$$\begin{aligned}
\hat{X} &= g(Y) = \mathbb{E}\{X|Y\} \\
&= \frac{\mathbb{E}\{XY\}}{\mathbb{E}\{Y^2\}}Y + \mathbb{E}\{X\} - \frac{\mathbb{E}\{XY\}}{\mathbb{E}\{Y^2\}}\mathbb{E}\{Y\} \\
&= \frac{\mathbb{E}\{X(X+v)\}}{\mathbb{E}\{(X+V)(X+V)\}}Y + 0 - 0 \\
&= \frac{\sigma^2\lambda^2 + 0}{\sigma^2\lambda^2 + 0 + 0 + 1}Y \\
&= \frac{\sigma^2\lambda^2}{\sigma^2\lambda^2 + 1}Y
\end{aligned}$$

Its corresponding performance function is

$$\begin{aligned}
J(f,g) &= \mathbb{E}\{k^2(x-\lambda x)^2 + (\lambda x - g(\lambda x + v))^2\} \\
&= k^2(1-\lambda)^2\mathbb{E}\{x^2\} + \mathbb{E}\left\{\left(\lambda x - \frac{\sigma^2\lambda^2}{\sigma^2\lambda^2+1}(\lambda x + v)\right)^2\right\} \\
&= k^2(1-\lambda)^2\sigma^2 + \mathbb{E}\left\{\left(\frac{1}{\sigma^2\lambda^2+1}\lambda x - \frac{\sigma^2\lambda^2}{\sigma^2\lambda^2+1}v\right)^2\right\} \\
&= k^2(1-\lambda)^2\sigma^2 + \left(\frac{1}{\sigma^2\lambda^2+1}\right)^2\lambda^2\sigma^2 + \left(\frac{\sigma^2\lambda^2}{\sigma^2\lambda^2+1}\right)^2 \\
&= k^2(1-\lambda)^2\sigma^2 + \frac{\sigma^2\lambda^2}{\sigma^2\lambda^2+1}
\end{aligned}$$

Furthermore, the best affine strategy can be solved for by determining λ's value.

Theorem 3 *For the optimal affine strategies described in Theorem 2, the best affine policy is determined by $\lambda = \frac{t}{\sigma}$ where t satisfies the following equation*

$$(t-\sigma)(1+t^2)^2 + \frac{1}{k^2}t = 0$$

Proof.

The optimal affine solution will minimize the performance function $J(\lambda)$ described in Theorem 2. Since $J(\lambda)$ is convex in λ, the minimizing parameter can be determined by setting the first derivative equal to zero.

$$\begin{aligned}\frac{\partial J(\lambda)}{\partial \lambda} &= 0\\ -2k^2\sigma^2(1-\lambda) + \frac{2\sigma^2\lambda(1+\sigma^2\lambda^2) - \sigma^2\lambda^2 \cdot 2\sigma^2\lambda}{(1+\sigma^2\lambda^2)^2} &= 0\\ 2k^2\sigma^2(\lambda-1) + \frac{2\sigma^2\lambda}{(1+\sigma^2\lambda^2)^2} &= 0\\ (\lambda-1)(1+\sigma^2\lambda^2)^2 + \frac{\lambda}{k^2} &= 0\end{aligned}$$

By substituting $\lambda = \frac{t}{\mu}$ and multiplying through by σ, the following equation is obtained $(t-\sigma)(1+t^2)^2 + \frac{t}{k^2} = 0$.

4.4 Non-linear policies

Theorem 4 *Given the problem OCP, there exists values for the parameters k, σ such that the optimal performance J^* is less than the minimal cost $J(\lambda)$ obtained by affine controllers, meaning that the optimal policy is non-linear.*

Witsenhausen used the following policies as counterexample to prove this result [1].

$$\begin{aligned} f(x) &= \sigma sgn(x)\\ g(y) &= \sigma tanh(\sigma y)\end{aligned}$$

When evaluating the first term of the performance function, we obtain

$$\begin{aligned} k^2\mathbb{E}\{(x-\sigma sgn(x))^2\} &= k^2\mathbb{E}\{x^2 - 2x\sigma sgn(x) + \sigma^2\}\\ &= k^2(\sigma^2 - 2\mathbb{E}\{x\sigma sgn(x)\} + \sigma^2)\\ &= 2k^2(\sigma^2 - \sigma\mathbb{E}\{xsgn(x)\})\\ &= 2k^2\sigma^2\left(1 - \frac{1}{\sigma}\mathbb{E}\{|x|\}\right)\\ &= 2k^2\sigma^2\left(1 - \sqrt{\frac{2}{\pi}}\right)\end{aligned}$$

For $g(y) = \sigma tanh(\sigma y)$, the second term can be shown to be the expression

$$\sigma^2 \int_{-\infty}^{\infty} \frac{(1 - tanh(\sigma^2 + \sigma v))^2}{\sqrt{2\pi}} e^{\frac{v^2}{2}} dv$$

Accordingly, the performance for this solution can be found for small k by taking the limit $k \to 0$ and obtaining that $J(f, g) \to 0.404$ while the same limit is taken on the optimal affine performance, we get $J(\lambda) \to 1$.

Mitter and Sahai have proposed a slightly simpler policy which uses the same first stage policy but a different second stage control law [6].

$$\begin{aligned} f(x) &= \sigma sgn(x) \\ g(y) &= \sigma sgn(y) \end{aligned}$$

This new controller pair can be thought of as a 1-bit quantizer which encodes the observed output, followed by a simple decoder. The first term of the associated performance function is still the same whereas the second is as follows:

$$\begin{aligned} \mathbb{E}\{(f(x) - g(f(x) + v))^2\} &= \mathbb{E}\{(\sigma sgn(x) - \sigma sgn(\sigma sgn(x) + v))^2\} \\ &= \sigma^2 \mathbb{E}\{(sgn(x) - sgn(\sigma sgn(x) + v))^2\} \\ &= 4\sigma^2 \mathbb{P}(v > \sigma) \end{aligned}$$

As was done previously, the performance for small k can be obtained by taking the limit $k \to 0$ and again, the performance is about 0.404, which is better than the best affine policy. Mitter and Sahai also further demonstrate that for certain parameters, a non-linear controller pair can arbitrarily outperform the best affine controller pair [6].

5 Further research

Among the multiple works that were derived or inspired from the Witsenhausen counterexample, some were further characterizations of the optimal policies depending on the problem parameters while others were attempts at beating the current best performing strategy.

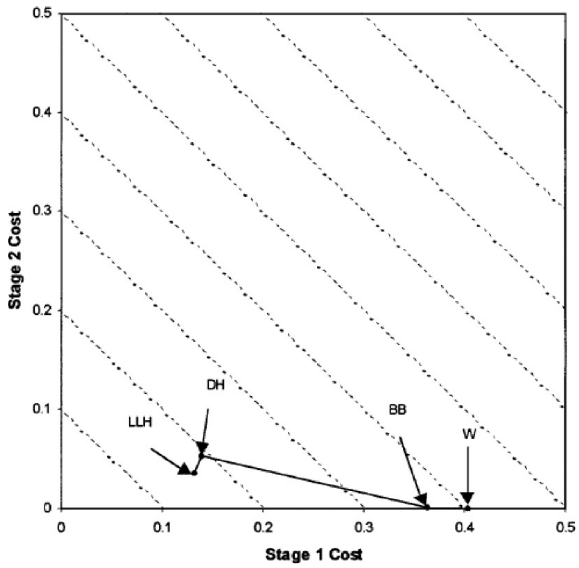

Figure 2: Some attempts at obtaining the optimal solution for the $k = 0.2$ and $\sigma = 5$ benchmark, from [7]

Bansal and Basar demonstrated that for different quadratic cost criterion, the optimality of affine policies could vary [4]. Indeed, for certain regions in the parameter space of a generalized version of the problem, the optimal solution is a linear function of the observation. The optimization problem is generalized by considering the following performance function:

$$J = \mathbb{E}\{k_0 u_0^2 + k_1 u_1^2 + k_{01} u_0 u_1 + s_{01} u_0 x + s_{11} u_1 x\}$$

The Witsenhausen counterexample is then a special case of this problem. If $k_{01} \neq 0$, Bansal and Basar showed that the optimal strategy is necessarily non-linear. Otherwise, without the decision variable cross-term, the optimal strategy is affine.

5.1 Beating the benchmark

Ever since the formulation of the problem by Witsenhausen in 1968 [1], there have been several attempts at finding the optimal non-linear control law. However, it remains an open problem. Figure 2 shows some of the attempts at beating the benchmark performance for the parameters $k = 0.2$ and $\sigma = 5$ [7]. The original solution from Witsenhausen is depicted with W. Then, there are Bansal and Basar (BB) [4], Deng and Ho (DH) [8] with a sampling-selection approach and Lee, Lau and Ho (LLC) [9] with a hierarchical search approach. An interesting detail to note is how better solutions (DH and LLC) were obtained when compromising the second stage cost to allow reducing the first stage cost. Indeed, it is believed that the implicit communication of the controllers is essential for achieving the optimal strategy [6] and that each controller can't be acting on its own, optimizing its own cost.

As of the writing of this report, the best non-linear strategy with a performance of $J = 0.166897$ for the benchmark parameters was found by Tseng and Tang [10]. They used a numerical approach to obtain the solution via a local search algorithm which alternatively fixes x_1 and u_1 to search for a local minimizer. Interestingly, they applied their method to other types of problems, such as the inventory control problem, to demonstrate how it could be generalized.

6 Conclusion

Despite being a toy example meant to disprove the conjecture of optimal affine policies for all types of information patterns, the Witsenhausen counterexample provided important insight into how to deal with decentralized systems. Even the attempts at reaching better performance on the optimal control law have provided some numerical techniques for sub-optimal policy search that can be generalized to other types of problems. Perhaps one day, an optimal strategy can be analytically found and demonstrated.

References

[1] H. S. Witsenhausen, "A counterexample in stochastic optimum control," *SIAM Journal on Control*, vol. 6, no. 1, pp. 131–147, 1968.

[2] C. Striebel, "Sufficient statistics in the optimum control of stochastic systems," *Journal of Mathematical Analysis and Applications*, vol. 12, no. 3, pp. 576 – 592, 1965.

[3] K. J. Åström, *Introduction to stochastic control theory.* Courier Corporation, 2012.

[4] R. Bansal and T. Basar, "Stochastic teams with nonclassical information revisited: When is an affine law optimal?," in *1986 American Control Conference*, pp. 45–50, 1986.

[5] Y. Wu and S. Verdú, "Witsenhausen's counterexample: A view from optimal transport theory," in *2011 50th IEEE Conference on Decision and Control and European Control Conference*, pp. 5732–5737, 2011.

[6] S. Mitter and A. Sahai, "Information and control: Witsenhausen revisited," in *Learning, control and hybrid systems* (Y. Yamamoto and S. Hara, eds.), (London), pp. 281–293, Springer London, 1999.

[7] Y. Ho, "Review of the witsenhausen problem," in *2008 47th IEEE Conference on Decision and Control*, pp. 1611–1613, 2008.

[8] Y.-C. Ho *et al.*, "An ordinal optimization approach to optimal control problems," *Automatica*, vol. 35, no. 2, pp. 331–338, 1999.

[9] J. T. Lee, E. Lau, and Y.-C. Ho, "The witsenhausen counterexample: A hierarchical search approach for nonconvex optimization problems," *IEEE Transactions on Automatic Control*, vol. 46, no. 3, pp. 382–397, 2001.

[10] S. Tseng and A. Tang, "A local search algorithm for the witsenhausen's counterexample," in *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, pp. 5014–5019, 2017.