

Convergence Analysis

November 3, 2022

Notation & Definitions

- t : time index, $t \in \mathbb{Z}^+$.
- z_t : position of the target at time t , $z_t \in \mathbb{R}^d$.
- x_t : position of the agent at time t , $x_t \in \mathbb{R}^d$.
- $\ell_t(x, z)$: stochastic loss as evaluated by the zeroth-order oracle at time t , with the position of agent as x , and the position of target as z for $x, z \in \mathbb{R}^d$.
- $\ell_{\mu,t}(x, z) := \mathbb{E}_u[\ell_t(x + \mu u, z)]$ for $x, z \in \mathbb{R}^d$, $u \sim \mathcal{N}(0, I_d)$ and $\mu \in \mathbb{R}$.
- $\nabla \ell_{\mu,t}(x, z) := \mathbb{E}_u[g_{\mu,t}(x, z)]$ where $g_{\mu,t}(x, z) := \frac{\ell_t(x + \mu u, z) - \ell_t(x, z)}{\mu} u$ for $x, z \in \mathbb{R}^d$, $u \sim \mathcal{N}(0, I_d)$ and $\mu \in \mathbb{R}$.
- $\ell_t(x) := \mathbb{E}_z[\ell_t(x, z)]$ for $x, z \in \mathbb{R}^d$.

Assumptions

Assumption 1. (*Unbiased Stochastic Zeroth-Order Oracle*) For any $t \in \mathbb{Z}^+$ and $x, z \in \mathbb{R}^d$, we have

$$\mathbb{E}_z[\ell_t(x, z)] = \ell_t(x). \quad (1)$$

Assumption 2. (*Unbiased Stochastic First-Order Oracle with Bounded Variance*) For any $t \in \mathbb{Z}^+$ and $x, z \in \mathbb{R}^d$, we have

$$\mathbb{E}_z[\nabla \ell_t(x, z)] = \nabla \ell_t(x) \quad (2)$$

and

$$\mathbb{E}_z[\|\nabla \ell_t(x, z) - \nabla \ell_t(x)\|^2] \leq \sigma^2. \quad (3)$$

Assumption 3. (*L-smoothness*) Each $\ell_t(x, z)$ is continuously differentiable and L -smooth over x on \mathbb{R}^d , that is, there exists an $L \geq 0$ such that for all $x, y, z \in \mathbb{R}^d$ and $t \in \mathbb{Z}^+$, we have

$$\|\nabla \ell_t(x, z) - \nabla \ell_t(y, z)\| \leq L\|x - y\|. \quad (4)$$

We denote this by $\ell_t(x, z) \in C_L^{1,1}(\mathbb{R}^d)$.

Assumption 4. (*Contractive Compression*) The compression function \mathcal{C} is a contraction mapping, that is,

$$\mathbb{E}_{\mathcal{C}} [\|\mathcal{C}(x) - x\|^2 \mid x] \leq (1 - \delta) \|x\|^2 \quad (5)$$

for all $x \in \mathbb{R}^d$ where $0 < \delta \leq 1$, and the expectation is over the randomness generated by compression \mathcal{C} .

Assumption 5. (*Bounded Stochastic Gradients*) For any $t \in \mathbb{Z}^+$ and $x, z \in \mathbb{R}^d$, there exist $\sigma, M > 0$ such that

$$\mathbb{E}_z [\|\nabla \ell_t(x, z)\|^2] \leq \sigma^2 + M \|\nabla \ell_{\mu,t}(x)\|^2. \quad (6)$$

Assumption 6. (*Bounded Drift in Time*) There exists a nonnegative sequence $\{\omega_t\}_{t=1}^T$ such that for all $t \in \mathbb{Z}^+$, $|\ell_t(x, z) - \ell_{t+1}(x, z)| \leq \omega_t$ for any $x, z \in \mathbb{R}^d$. Note that in the case where $\ell_{t+1} = \ell_t$, this assumption holds with $\omega_t = 0$.

Lemmas

Suppose $\ell_t(x, z) \in C_L^{1,1}(\mathbb{R}^d)$ over x . We have the following results:

Lemma 1. $\ell_{\mu,t}(x, z) \in C_{L_\mu}^{1,1}(\mathbb{R}^d)$ over x , where $L_\mu \leq L$.

Lemma 2. $\ell_{\mu,t}(x, z)$ has the following gradient with respect to x :

$$\nabla \ell_{\mu,t}(x, z) = \frac{1}{(2\pi)^{d/2}} \int \frac{\ell_t(x + \mu u, z) - \ell_t(x, z)}{\mu} u e^{(-\frac{1}{2}\|u\|^2)} du, \quad (7)$$

where $u \sim \mathcal{N}(0, I_d)$.

Lemma 3. For any $x, z \in \mathbb{R}^d$, we have

$$|\ell_{\mu,t}(x, z) - \ell_t(x, z)| \leq \frac{\mu^2 L d}{2}. \quad (8)$$

Lemma 4. For any $x, z \in \mathbb{R}^d$, we have

$$\|\nabla \ell_{\mu,t}(x, z) - \nabla \ell_t(x, z)\| \leq \frac{\mu}{2} L(d+3)^{\frac{3}{2}}, \quad (9)$$

where the gradient is with respect to x .

Lemma 5. For any $x, z \in \mathbb{R}^d$, we have

$$\mathbb{E}_u \left[\left\| \frac{\ell_t(x + \mu u, z) - \ell_t(x, z)}{\mu} u \right\|^2 \right] \leq \frac{\mu^2}{2} L^2 (d+6)^3 + 2(d+4) \|\nabla \ell_t(x, z)\|^2, \quad (10)$$

where $u \sim \mathcal{N}(0, I_d)$ and the gradient is with respect to x .

Lemma 6. (*Young's inequality*) For any $x, y \in \mathbb{R}^d$ and $\lambda > 0$, we have

$$\langle x, y \rangle \leq \frac{\|x\|^2}{2\lambda} + \frac{\|y\|^2 \lambda}{2}. \quad (11)$$

EF-ZO-SGD Convergence Analysis

We work with the following algorithm:

Algorithm 1 EF-ZO-SGD

Input: Number of time steps $T \in \mathbb{Z}^+$, smoothing parameter $\mu \in \mathbb{R}$, initial source position $x_0 \in \mathbb{R}^d$, learning rate $\eta \in \mathbb{R}$, sequence of target positions $\{z_t\}_{t=1}^T \subset \mathbb{R}^d$.

Output: Sequence of optimal source positions $\{x_t\}_{t=1}^T \subset \mathbb{R}^d$.

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1:  $e_0 = 0$ 
2: for  $t = 1, \dots, T$  do
3:    $u_t \sim \mathcal{N}(0, I_d)$ 
4:    $g_{\mu,t}(x_t, z_t) = \frac{\ell_t(x_t + \mu u_t, z_t) - \ell_t(x_t, z_t)}{\mu} u_t$ 
5:    $p_t = g_{\mu,t}(x_t, z_t) + e_t$ 
6:    $x_{t+1} = x_t - \eta \mathcal{C}(p_t)$ 
7:    $e_{t+1} = p_t - \mathcal{C}(p_t)$ 
8: end for

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In the analysis, we assume that $z_t \in \mathbb{R}^d$ are *i.i.d.* random variables for all $t \in \mathbb{Z}^+$.

Let \tilde{x}_t be defined as follows (following the analysis in [1]):

$$\tilde{x}_t = x_t - \eta e_t. \quad (12)$$

From algorithm 0, we know that $e_{t+1} = p_t - \mathcal{C}(p_t)$ and $p_t = g_{\mu,t}(x_t, z_t) + e_t$, so we can rewrite \tilde{x}_{t+1} as

$$\begin{aligned}
\tilde{x}_{t+1} &= x_{t+1} - \eta p_t + \eta \mathcal{C}(p_t) \\
&= x_t - \eta \mathcal{C}(p_t) - \eta g_{\mu,t}(x_t, z_t) - \eta e_t + \eta \mathcal{C}(p_t) \\
&= x_t - \eta e_t - \eta g_{\mu,t}(x_t, z_t) \\
&= \tilde{x}_t - \eta g_{\mu,t}(x_t, z_t),
\end{aligned} \quad (13)$$

where $g_{\mu,t}(x_t, z_t) := \frac{\ell_t(x_t + \mu u_t, z_t) - \ell_t(x_t, z_t)}{\mu} u_t$ and $u_t \sim \mathcal{N}(0, I_d)$.

By definition, $\ell_{\mu,t}(x_t, z_t) := \mathbb{E}_{u_t} [\ell_t(x_t + \mu u_t, z_t)]$, so by assumption 3, we can write the following:

$$\ell_{\mu,t}(\tilde{x}_{t+1}, z_{t+1}) \leq \ell_{\mu,t}(\tilde{x}_t, z_t) + \langle \nabla \ell_{\mu,t}(\tilde{x}_t, z_t), \tilde{x}_{t+1} - \tilde{x}_t \rangle + \frac{L}{2} \|\tilde{x}_{t+1} - \tilde{x}_t\|^2. \quad (14)$$

Now by assumption 6, we get:

$$\ell_{\mu,t+1}(\tilde{x}_{t+1}, z_{t+1}) \leq \ell_{\mu,t}(\tilde{x}_t, z_t) - \eta \langle g_{\mu,t}(x_t, z_t), \nabla \ell_{\mu,t}(\tilde{x}_t, z_t) \rangle + \frac{L\eta^2}{2} \|g_{\mu,t}(x_t, z_t)\|^2 + \omega_t \quad (15)$$

Since $\nabla \ell_{\mu,t}(x_t, z_t) = \mathbb{E}_{u_t} [g_{\mu,t}(x_t, z_t)]$, we have the following:

$$\begin{aligned}
\mathbb{E}_{u_t} [\langle g_{\mu,t}(x_t, z_t), \nabla \ell_{\mu,t}(\tilde{x}_t, z_t) \rangle] &= \langle \nabla \ell_{\mu,t}(x_t, z_t), \nabla \ell_{\mu,t}(\tilde{x}_t, z_t) \rangle \\
&= \frac{1}{2} \|\nabla \ell_{\mu,t}(x_t, z_t)\|^2 + \frac{1}{2} \|\nabla \ell_{\mu,t}(\tilde{x}_t, z_t)\|^2 - \frac{1}{2} \|\nabla \ell_{\mu,t}(x_t, z_t) - \nabla \ell_{\mu,t}(\tilde{x}_t, z_t)\|^2.
\end{aligned} \quad (16)$$

In the last step, we use the fact that $2\langle a, b \rangle = \|a\|^2 + \|b\|^2 - \|a - b\|^2$. Plugging this into (15), we get:

$$\begin{aligned} \ell_{\mu,t+1}(\tilde{x}_{t+1}, z_{t+1}) &\leq \ell_{\mu,t}(\tilde{x}_t, z_t) - \frac{\eta}{2} \|\nabla \ell_{\mu,t}(x_t, z_t)\|^2 - \frac{\eta}{2} \|\nabla \ell_{\mu,t}(\tilde{x}_t, z_t)\|^2 \\ &\quad + \frac{L^2\eta}{2} \|x_t - \tilde{x}_t\|^2 + \frac{L\eta^2}{2} \|g_{\mu,t}(x_t, z_t)\|^2 + \omega_t. \end{aligned} \quad (17)$$

Note that $\|\nabla \ell_{\mu,t}(x_t, z_t) - \nabla \ell_{\mu,t}(\tilde{x}_t, z_t)\|^2 \leq L^2 \|x_t - \tilde{x}_t\|^2$ because of lemma 1. Also, we can drop $-\frac{\eta}{2} \|\nabla \ell_{\mu,t}(\tilde{x}_t, z_t)\|^2$ because it is always nonpositive. Using the fact that $\tilde{x}_t - x_t = \eta e_t$, we get:

$$\underbrace{\frac{\eta}{2} \|\nabla \ell_{\mu,t}(x_t, z_t)\|^2}_{\text{Term III}} \leq \underbrace{[\ell_{\mu,t}(\tilde{x}_t, z_t) - \ell_{\mu,t+1}(\tilde{x}_{t+1}, z_{t+1})]}_{\text{Term II}} + \underbrace{\frac{L\eta^2}{2} \|g_{\mu,t}(x_t, z_t)\|^2}_{\text{Term I}} + \underbrace{\frac{\eta^3 L^2}{2} \|e_t\|^2}_{\text{Term IV}} + \omega_t. \quad (18)$$

We will put an upper bound to the terms I, II, IV and a lower bound to term III. Starting with **term I**, by lemma 5, we know that

$$\mathbb{E}_{u_t, z_t} [\|g_{\mu,t}(x_t, z_t)\|^2] \leq 2(d+4) \mathbb{E}_{z_t} [\|\nabla \ell_t(x_t, z_t)\|^2] + \frac{\mu^2 L^2}{2} (d+6)^3, \quad (19)$$

where $\mathbb{E}_{z_t} [\|\nabla \ell_t(x_t, z_t)\|^2] \leq \|\nabla \ell_t(x_t)\|^2 + \sigma^2$ by assumption 2.

We can put the following upper bound to **term II** by means of a telescoping sum and subsequently applying lemma 3:

$$\begin{aligned} \sum_{t=1}^T [\ell_{\mu,t}(\tilde{x}_t, z_t) - \ell_{\mu,t+1}(\tilde{x}_{t+1}, z_{t+1})] &= \ell_{\mu,1}(\tilde{x}_1, z_1) - \ell_{\mu,T+1}(\tilde{x}_{T+1}, z_{T+1}) \\ &\leq \mu L^2 d + \ell_1(\tilde{x}_1, z_1) - \ell_{T+1}(\tilde{x}_{T+1}, z_{T+1}) \\ &= \mu L^2 d + \ell_1(x_1, z_1) - \ell_{T+1}(\tilde{x}_{T+1}, z_{T+1}), \end{aligned} \quad (20)$$

where we use the fact that $\ell(x_1, z_1) = \ell_1(\tilde{x}_1, z_1)$ because $\tilde{x}_1 = x_1$ by definition. If we take the expectation of both sides with respect to $z_{1:T+1} = \{z_1, z_2, \dots, z_{T+1}\}$, owing to the fact that z_t 's are *i.i.d.*, we get

$$\begin{aligned} \ell_{\mu,1}(\tilde{x}_1) - \ell_{\mu,T+1}(\tilde{x}^*) &\leq \mu L^2 d + \ell_1(x_1) - \ell_{T+1}(\tilde{x}_{T+1}) \\ &\leq \mu L^2 d + \ell_1(x_1) - \ell_{T+1}(x_{T+1}^*), \end{aligned} \quad (21)$$

where $x_{T+1}^* = \arg \min_x \ell_{T+1}(x)$.

We can put the following lower bound to **term III** by using lemma 4 and lemma 6:

$$\frac{1}{2} \|\nabla \ell_t(x_t, z_t)\|^2 - \frac{\mu^2 L^2}{4} (d+3)^3 \leq \|\nabla \ell_{\mu,t}(x_t, z_t)\|^2. \quad (22)$$

Lastly, we can put the following upper bound to **term IV** by assumption 4 and lemma 6:

$$\begin{aligned} \mathbb{E}_{u_{1:T}, z_{1:T}, \mathcal{C}_{1:T}} [\|e_{t+1}\|^2] &= \|p_t - \mathcal{C}_t(p_t)\|^2 \leq (1-\delta) \|p_t\|^2 = (1-\delta) \|e_t + g_{\mu,t}(x_t, z_t)\|^2 \\ &\leq (1-\delta)(1+\varphi) \mathbb{E}_{u_{1:T}, z_{1:T}, \mathcal{C}_{1:T}} [\|e_t\|^2] + (1-\delta)(1+\frac{1}{\varphi}) \mathbb{E}_{u_{1:T}, z_{1:T}} [\|g_{\mu,t}(x_t, z_t)\|^2] \\ &= \sum_{i=1}^t [(1-\delta)(1+\varphi)]^{t-i} (1-\delta)(1+\frac{1}{\varphi}) \mathbb{E}_{u_i, z_i} [\|g_{\mu,i}(x_i, z_i)\|^2], \end{aligned} \quad (23)$$

for some $\varphi > 0$, z_t, x_t, \mathcal{C}_t are *i.i.d.*, and $\mathbb{E}_{\mathcal{C}_t}[\cdot]$ denotes the expectation over the randomness at time t due to the compression used. Note that by assumption 5 and using lemma 5,

$$\mathbb{E}_{u_t, z_t} [\|g_{\mu, t}(x_t, z_t)\|^2] \leq A \|\nabla \ell_t(x_t)\|^2 + B, \quad (24)$$

where

$$\begin{aligned} B &= 2\sigma^2(d+4) + \frac{\mu^2 L^2}{2}(d+6)^3 \text{ and} \\ A &= 2M(d+4). \end{aligned} \quad (25)$$

So we can rewrite (23) as follows:

$$\mathbb{E}_{u_{1:T}, z_{1:T}, \mathcal{C}_{1:T}} [\|e_{t+1}\|^2] \leq \sum_{i=1}^t [(1-\delta)(1+\varphi)]^{t-i} (1-\delta)(1+\frac{1}{\varphi}) [A \|\nabla \ell_i(x_i)\|^2 + B]. \quad (26)$$

If we set $\varphi := \frac{\delta}{2(1-\delta)}$, then $1 + \frac{1}{\varphi} \leq \frac{2}{\delta}$ and $(1-\delta)(1+\varphi) = (1-\frac{\delta}{2})$, so we get:

$$\mathbb{E}_{u_{1:T}, z_{1:T}, \mathcal{C}_{1:T}} [\|e_{t+1}\|^2] \leq \sum_{i=1}^t \left(1 - \frac{\delta}{2}\right)^{t-i} [A \|\nabla \ell_i(x_i)\|^2 + B] \frac{2(1-\delta)}{\delta}. \quad (27)$$

If we sum through all $\mathbb{E}_{u_{1:T}, z_{1:T}, \mathcal{C}_{1:T}} [\|e_t\|^2]$, we get:

$$\begin{aligned} \sum_{t=1}^T \mathbb{E}_{u_{1:T}, z_{1:T}, \mathcal{C}_{1:T}} [\|e_t\|^2] &\leq \sum_{t=1}^T \sum_{i=1}^{t-1} \left(1 - \frac{\delta}{2}\right)^{t-i} [A \|\nabla \ell_i(x_i)\|^2 + B] \frac{2(1-\delta)}{\delta} \\ &\leq \sum_{t=1}^T [A \|\nabla \ell_t(x_t)\|^2 + B] \sum_{i=0}^{\infty} \left(1 - \frac{\delta}{2}\right)^i \frac{2(1-\delta)}{\delta} \\ &\leq \sum_{t=1}^T [A \|\nabla \ell_t(x_t)\|^2 + B] C, \end{aligned} \quad (28)$$

where $C = \frac{2(1-\delta)}{\delta} \frac{2}{\delta} \leq \frac{4}{\delta^2}$. If we define $\Delta := \ell_1(x_1) - \ell_{T+1}(x_{T+1}^*)$ and combine the upper bounds derived in (19), (20), (23), and the lower bound derived in (22) and plug them into (18), we get the following:

$$\begin{aligned} &\sum_{t=1}^T \frac{\eta}{4} \mathbb{E}_{z_t} [\|\nabla \ell_t(x_t, z_t)\|^2] - \frac{\eta \mu^2 L^2}{8} (d+3)^3 T \\ &\leq \mu L^2 d + \Delta + \frac{T \mu^2 L^3 \eta^2}{4} (d+6)^3 + \frac{L \eta^2}{2} \sigma^2 T \times 2(d+4) \\ &\quad + \frac{L \eta^2}{2} \times 2M(d+4) \sum_{t=1}^T \mathbb{E}_{z_t} [\|\nabla \ell_t(x_t, z_t)\|^2] + \frac{\eta^3 L^2}{2} \times \frac{4}{\delta^2} T \left[2\sigma^2(d+4) + \frac{\mu^2 L^2}{2} (d+6)^3 \right] \\ &\quad + \frac{\eta^3 L^2}{2} \times \frac{4}{\delta^2} \sum_{t=1}^T 2M(d+4) \mathbb{E}_{z_t} [\|\nabla \ell_t(x_t, z_t)\|^2] + \sum_{t=1}^T \omega_t. \end{aligned} \quad (29)$$

Now, since z_t 's are *i.i.d.* for all $t \in \mathbb{Z}^+$, we have:

$$\begin{aligned} \frac{E}{T} \sum_{t=1}^T \mathbb{E}_{z_t} [\|\nabla \ell_t(x_t, z_t)\|^2] &\leq \frac{\mu L^2 d + \Delta}{T} + \frac{\eta^2 L^3 \mu^2 (d+6)^3}{4} + L\eta^2 \sigma^2 (d+4) + \frac{\eta \mu^2 L^2 (d+3)^3}{8} \\ &\quad + \frac{\eta^3 L^2}{\delta^2} 4\sigma^2 (d+4) + \frac{\eta^3 L^2}{\delta^2} \mu^2 L^2 (d+6)^3 + \frac{1}{T} \sum_{t=1}^T \omega_t, \end{aligned} \quad (30)$$

where

$$\begin{aligned} E &= \frac{\eta}{4} - LM\eta^2(d+4) - \frac{L^2\eta^3}{\delta^2} 4M(d+4) \\ &= \eta \left[\frac{1}{4} - LM\eta(d+4) \left(1 + \frac{4L\eta}{\delta^2} \right) \right]. \end{aligned} \quad (31)$$

If $\eta \leq \frac{1}{4L}$, instead first upper bound will be:

$$1 + \frac{4L\eta}{\delta^2} \leq 1 + \frac{1}{\delta^2} = \frac{\delta^2 + 1}{\delta^2} \leq \frac{2}{\delta^2}. \quad (32)$$

We proceed to find an η such that

$$\frac{2}{\delta^2} LM\eta(d+4) \leq \frac{1}{8}. \quad (33)$$

Then, we get

$$\eta \leq \frac{\delta^2}{16LM(d+4)}, \quad (34)$$

which implies $E \geq \frac{\eta}{8}$. Multiplying all terms in the bound by $\frac{8}{\eta}$,

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^T \mathbb{E}_{z_t} [\|\nabla \ell_t(x_t, z_t)\|^2] &\leq \frac{8\Delta}{(\eta T)} + \frac{8\mu L^2 d}{\eta T} + 2\eta L^3 \mu^2 (d+6)^3 \\ &\quad + 8L\eta \sigma^2 (d+4) + \mu^2 L^2 (d+3)^3 \\ &\quad + \frac{32\eta^2 L^2}{\delta^2} \sigma^2 (d+4) + \frac{8\eta^2 L^4 \mu^2 (d+6)^3}{\delta^2} + \frac{8}{\eta T} \sum_{t=1}^T \omega_t. \end{aligned} \quad (35)$$

Let

$$\eta = \frac{1}{\sigma \sqrt{(d+4)MTL}} \quad \text{and} \quad \mu = \frac{1}{(d+4)\sqrt{T}}. \quad (36)$$

Then, for a numerical constant $C > 0$, we have

$$\frac{1}{CT} \sum_{t=1}^T \mathbb{E}_{z_t} [\|\nabla \ell_t(x_t, z_t)\|^2] \leq \frac{1}{\delta^2} \frac{dL\Delta}{T} + \sigma \sqrt{\frac{d}{T} L\Delta M} + \frac{1}{\eta T} \sum_{t=1}^T \omega_t. \quad (37)$$

Defining $\bar{\omega} := \sum_{t=1}^T \omega_t$, the number of times steps T to obtain a ξ -first order solution is

$$T = \mathcal{O} \left(\frac{d\sigma^2 L\Delta M}{\xi^2} + \frac{dL\Delta}{\delta^2 \xi} + \frac{\bar{\omega} \sigma^2 dML}{\xi^2} \right). \quad (38)$$

Remark: In choosing $\eta = \frac{1}{\sigma\sqrt{(d+4)MTL}}$, we assumed that it satisfies (34). For this to hold, T can be made arbitrarily large as long as it does not exceed the bound we found in (38). (36) and (34) imply that

$$T = \Omega\left(\frac{dLM}{\delta^4\sigma^2}\right). \quad (39)$$

In practice, since $\xi \ll \delta$, this term is smaller than (37). This fact is also demonstrated by our experiments.

Lastly, if $\omega_t = 0$ for all $t \in \mathbb{Z}^+$, i.e., in the case where the loss function is time-invariant, the number of time steps T to obtain a ξ -first order solution is:

$$T = \mathcal{O}\left(\frac{d\sigma^2 L \Delta M}{\xi^2} + \frac{dL \Delta}{\delta^2 \xi}\right). \quad (40)$$

References

- [1] S. P. Karimireddy, Q. Rebjock, S. U. Stich, and M. Jaggi, “Error feedback fixes signsgd and other gradient compression schemes,” 2019.

FL-EF-ZO-SGD Convergence Analysis

We work with the following algorithm:

Algorithm 2 FL-EF-ZO-SGD

Input: Number of time steps $T \in \mathbb{Z}^+$, number of agents $N \in \mathbb{Z}^+$, smoothing parameter $\mu \in \mathbb{R}$, initial agent positions $x_0^{1:N} \in \mathbb{R}^{Nd}$, learning rate $\eta \in \mathbb{R}$, sequence of target positions $\{z^{1:N}\}_{t=1}^T \subset \mathbb{R}^{Nd}$.

Output: Sequence of optimal target positions $\{x^{1:N}\}_{t=1}^T \subset \mathbb{R}^{Nd}$.

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1: for  $i = 1, \dots, N$  do
2:    $e_0^i = 0$ 
3: end for
4: for  $t = 1, \dots, T$  do
   Runs on each agent:
5:   for  $i = 1, \dots, N$  do
6:      $u_t^i \sim \mathcal{N}(0, I_{Nd})$ 
7:      $g_{\mu,t}^i(x_t^{1:N}, z_t^i) = \frac{\ell_t(x_t^{1:N} + \mu u_t^i, z_t^i) - \ell_t(x_t^{1:N}, z_t^i)}{\mu} u_t^i$ 
8:      $p_t^i = g_{\mu,t}^i(x_t^{1:N}, z_t^i) + e_t^i$ 
9:      $\gamma_t^i = \mathcal{C}(p_t^i)$ 
10:     $e_{t+1}^i = p_t^i - \gamma_t^i$ 
11:    transmit  $(\gamma_t^i)$ 
12:   end for
   Runs on the server:
13:    $\mathcal{G}_t = \frac{1}{N} \sum_{i=1}^N \gamma_t^i$ 
14:    $x_{t+1}^{1:N} = x_t^{1:N} - \mathcal{G}_t$ 
15:   transmit  $(x_{t+1}^{1:N})$ 
16: end for
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In the analysis, we assume that $z_t \in \mathbb{R}^d$ are *i.i.d.* random variables for all $t \in \mathbb{Z}^+$.