CLI – Øvelsestimer

Øvelsestime 1

Problem 1.1 Given the following sets S = {1; 2; 3; 6; 7; 9} and R = {2; 4; 5; 6; 7} state

a) $S \cup R$

- {1, 2, 3, 4, 5, 6, 7, 9}

b) $S \cap R$

- {2, 6, 7}

c) $S \setminus R$

- {1, 3, 9}

d) $R \setminus S$

- {4, 5}

e) $|S|$

- 6

f) $|R \setminus S|$

- 2

Problem 1.2 State these expression as simple rational numbers or fractions:

a) $\frac{7}{3} - \frac{1}{2}$

$$\frac{7*2}{3*2} - \frac{1*3}{2*3} =$$

$$\frac{14}{6} - \frac{3}{6} =$$

$$\frac{11}{6}$$

$$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$$

$$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + c \cdot d}{b \cdot d}$$

$b)\ \pi + \frac{3}{5}$

$$pi = 3{,}14$$

$$\frac{3}{5} = 0{,}6$$

$$3{,}14 + 0{,}6 =$$

$$3{,}74$$

$$a = \frac{a}{1}$$

$$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + c \cdot d}{b \cdot d}$$

$c)\ \frac{3x^2}{a} + 2x$

$$a = \frac{1}{x}$$

$$\frac{3x^2}{a} + \frac{2x}{1} =$$

$$\frac{3x^2}{a} + \frac{a * 2x}{a * 1} =$$

$$\frac{3x^2}{a} + \frac{a * 2x}{a} =$$

$$\frac{3x^2}{a} + \frac{2ax}{a} =$$

$$\frac{3x^2 + 2ax}{a}$$

$$a = \frac{a}{1}$$

$$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + c \cdot d}{b \cdot d}$$

$d)\ \frac{a}{4x^2} - \frac{x^2}{a}$

$$\frac{a * a}{4x^2 * a} - \frac{4x^2 * x^2}{4x^2 * a} =$$

$$\frac{a^2}{4ax^2} - \frac{4x^4}{4ax^2} =$$

$$\frac{a^2 - 4x^4}{4ax^2}$$

$$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$$

$$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + c \cdot d}{b \cdot d}$$

$$a^2 - b^2 = (a + x)(a - x)$$

e) $\frac{1}{\frac{a}{(x^2+1)}} - \frac{1}{a}$

$$\frac{x^2+1}{a} - \frac{-1}{a} =$$

$$\frac{x^2+1}{a} + \frac{-1}{a} =$$

$$\frac{(x^2+1)a - 1 * a}{a * a} =$$

$$\frac{a * x^2 + a - a}{a^2} =$$

$$\frac{a * x^2}{a^2} =$$

$$\frac{x^2}{a}$$

$$\frac{1}{\frac{a}{b}} = \frac{b}{a}$$

$$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$$

$$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + c \cdot d}{b \cdot d}$$

$$a(b+c) = ab + ac$$

f) $\frac{a}{2b+3a} - \frac{b}{a+1}$

$$\frac{a}{2b+3a} + \frac{-b}{a+1} =$$

$$\frac{a(a+1) - (2b+3a)b}{(2b+3a)(a+1)} =$$

$$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$$

$$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + c \cdot d}{b \cdot d}$$

Problem 1.3 Using the rules for exponentials simplify the following expressions:

a) $10^3 \cdot 5^{-2}$ *(hint* $10 = 2 \cdot 5$*)*

$$10^3 * 5^{-2} =$$

$$2^3 * 5^3 * \frac{1}{5^2} =$$

$$2^3 * 5^3 * \frac{1}{5^2} =$$

$$2^3 * 5 * 1 =$$

$$2^3 * 5 =$$

$$8 * 5 =$$

$$40$$

1. Since $10 = 2 \cdot 5$, we can substitute $10^3 \cdot 5^{-2} = (2 \cdot 5)^3 \cdot 5^{-2}$.
2. Using the rule $(a \cdot b)^n = a^n \cdot b^n$ we get $(2 \cdot 5)^3 \cdot 5^{-2} = 2^3 \cdot 5^3 \cdot 5^{-2}$
3. Using the rule $a^n \cdot a^m = a^{n+m}$ we get $2^3 \cdot 5^3 \cdot 5^{-2} = 2^3 \cdot 5^{3-2} = 2^3 \cdot 5$

$b)\ \frac{3^5}{3^2}$

$$\frac{3^5}{3^2} =$$
$$3^{5-2} =$$
$$3^3$$

1. Using the rule $\frac{a^n}{a^m} = a^{n-m}$ we get $\frac{3^5}{3^2} = 3^{5-2} = 3^3$

$c)\ 5 \cdot 25^8\ \ (hint\ 25 = 5^2)$

$$5 * (5^2)^8 =$$
$$5 * 5^{2*8} =$$
$$5 * 5^{16} =$$
$$5^1 * 5^{16} =$$
$$5^{17}$$

1. Since $25 = 5^2$, we can substitute to get $5 \cdot 25^8 = 5 \cdot (5^2)^8$
2. Using the rule $(a^n)^m = a^{n \cdot m}$ we get $5 \cdot (5^2)^8 = 5 \cdot 5^{2 \cdot 8} = 5 \cdot 5^{16}$
3. Using the rule $a^n \cdot a^m = a^{n+m}$ we get $5 \cdot 5^{16} = 5^{17}$

$d)\ 2^3 \sqrt[5]{6^{10}}\ \ (hint\ 6 = 3 \cdot 2)$

$$2^3 * 6^{\frac{10}{5}} =$$
$$2^3 * 6^{\frac{10/5}{5/5}} =$$
$$2^3 * 6^{\frac{2}{1}} =$$
$$2^3 * 6^2 =$$
$$2^3 * (2 * 3)^2 =$$
$$2^3 * 2^2 * 3^2 =$$
$$2^5 * 3^2$$

1. Using the rule $\sqrt[m]{a^n} = a^{\frac{n}{m}}$ we get $2^3 \sqrt[5]{6^{10}} = 2^3 \cdot 6^{\frac{10}{5}} = 2^3 \cdot 6^2$
2. Since $6 = 3 \cdot 2$ we can substitute $2^3 \cdot 6^2 = 2^3 \cdot (3 \cdot 2)^2$
3. Using the rule $(a \cdot b)^n = a^n \cdot b^n$ we get $2^3 \cdot (3 \cdot 2)^2 = 2^3 \cdot 3^2 \cdot 2^2$
4. Using the rule $a^n \cdot a^m = a^{n+m}$ we get $2^3 \cdot 3^2 \cdot 2^2 = 2^{3+2} \cdot 3^2 = 2^5 \cdot 3^2$

e) $\sqrt[3]{7^{-21}}$

$$7^{\frac{-21}{3}} =$$
$$7^{\frac{-7}{1}} =$$
$$7^{-7} =$$
$$\frac{1}{7^7}$$

1. Using the rule $\sqrt[m]{a^n} = a^{\frac{n}{m}}$ we get $\sqrt[3]{7^{-21}} = 7^{-\frac{21}{3}} = 7^{-7}$
2. Using the rule $a^{-n} = \frac{1}{a^n}$ we get $7^{-7} = \frac{1}{7^7}$

f) $\sqrt[4]{(2^2)^5}$

$$\sqrt[4]{2^{2*5}} =$$
$$\sqrt[4]{2^{10}} =$$
$$2^{\frac{10}{4}} =$$
$$2^{\frac{5}{2}}$$

1. Using the rule $(a^n)^m = a^{n \cdot m}$ we get $\sqrt[4]{(2^2)^5} = \sqrt[4]{2^{2 \cdot 5}} = \sqrt[4]{2^{10}}$
2. Using the rule $\sqrt[m]{a^n} = a^{\frac{n}{m}}$ we get $\sqrt[4]{2^{10}} = 2^{\frac{10}{4}} = 2^{\frac{5}{2}}$

Problem 1.4 Write the following inequalities as intervals of the real line

$a)\ x \leq 100$

Numbers smaller or equal than 100: (-∞; 100]

$b)\ x > 0\ and\ x < 10$

Numbers between 0 and 10 (not included) (0; 10)

$c)\ x \geq -10\ and\ x < 5$

Numbers larger or equal than -10 and smaller than 5: [-10; 5)

$d)\ x \geq -1$

Numbers larger or equal than -1: [-1, ∞)

$e)\ x < -5\ or\ x \geq 5$

Numbers smaller than -5 or larger than 5: (-∞; -5) U [5; ∞)

$f)\ x \neq 3$

Numbers different than 3: (-∞,∞) \ {3} or (-∞, 3) U (3; ∞)

Problem 1.5 Solve the following inequalities for x:

$a)\ 6x - 5 > 7$

$$6x - 5 + 5 > 7 + 5 =$$

$$6x > 12 =$$

$$\frac{6}{6} * x > \frac{12}{6} =$$

$$x > 2\ (2, \infty)$$

$b)\ -2x \geq 4$

$$-\frac{2x}{-2} < \frac{4}{-2} =$$

$$x > -2(-\infty; -2)$$

$c)\ 5x - 3 < 7 - 3x$

$$5x - 3 + 3 < 7 + 3 - 3x =$$
$$5x < 10 - 3x =$$
$$5x + 3x < 10 - 3x + 3x\text{-} =$$
$$8x < 10 =$$
$$\frac{8x}{8} < \frac{10}{8} =$$
$$x = \frac{\frac{10}{2}}{\frac{8}{2}} =$$
$$x = \frac{5}{4}$$

$d)\ \frac{6-x}{-4} \geq \frac{3x-4}{2}$

$$4 * \frac{6-x}{-4} > \cancel{4 *} \frac{3x-4}{\cancel{2}} =$$

$$-(6 - x) > 2 * (3x - 4) =$$

$$-6 + x > 6x - 8 =$$

$$-6 + 6 + x - 6x > -8 + 6 =$$

$$x - 6x > -2 =$$

$$5x > -2 =$$

$$\frac{-5x}{-5} > \frac{-2}{-5} =$$

$$x = \frac{2}{5}$$

Kommenterede [BK1]: Ethvert negativt tal delt med et negativt giver et positivt tal *e)* $\frac{1}{x} > 6$

$$\frac{1}{x} - 6 > 6 - 6 =$$
$$\frac{1-6}{x} > 0 =$$
$$1 - 6x > 0 =$$
$$-1 + 1 - 6x > -1 =$$
$$-6x > -1 =$$
$$-\frac{6}{-6} * x > -\frac{1}{-6} =$$
$$x > \frac{1}{6} \quad (0, \frac{1}{6})$$

f) $\frac{x+1}{x-1} \geq 2$

f) $\frac{x+1}{x-1} \geq 2$ we can multiply both sides by $x-1$ but since $x-1$ can be negative we need to account for the change of sign of the inequality.

- $x - 1 > 0$, i.e. $x > 1$, leads to $(x-1)\frac{x+1}{x-1} = x + 1 \geq 2(x-1) = 2x - 2$, and adding $2 - x$ on both sides leads to $x + 1 + 2 - x = 3 \geq 2x - 2 + 2 - x = x$, i.e. $3 \geq x$. Both inequalities must hold simultaneously $x > 1$ and $x \leq 3$, which leads to the solution $1 < x \leq 3$.
- $x - 1 < 0$, i.e. $x < 1$, leads to $(x-1)\frac{x+1}{x-1} = x + 1 \leq 2(x-1) = 2x - 2$ (change of inequality sign), which by adding leads to $2 - x$ on both sides leads to $x + 1 + 2 - x = 3 \leq 2x - 2 + 2 - x = x$, i.e. $x \geq 3$. Both conditions cannot hold simultaneously and therefore $x - 1 < 0$ does not lead to a valid solution.

g) $\frac{x}{2-x} < 0$

g) $\frac{x}{2-x} < 0$. We can multiply both sides by $2-x$ but since $2-x$ can be negative we need to account for the change on the sign of the inequality.

- If $2-x>0$ (i.e. $x<2$) we get $x<0$ (i.e. no sign change). Since both inequalities ($x<0$ and $x<2$) must be fulfilled simultaneously, the numbers which fulfil both are $x<0$ (the most restrictive).
- If $2-x<0$ (i.e. $x>2$) we get $x>0$ (i.e. sign change after multiplication). Since both inequalities ($x>0$ and $x>2$) must be fulfilled simultaneously, the numbers which fulfil both are $x>2$ (the most restrictive).

If $2-x>0$ we get $x<0$, and if $2-x<0$ we get $x>0$, therefore the solution is $x<0$ or $x>2$, which as interval can be written as $(-\infty,0)\cup(2,\infty)$

Alternative solution: (Using function notation that we will see later in the course) There is an alternative way of solving inequalities of the form $\frac{f(x)}{g(x)}>0$, $\frac{f(x)}{g(x)}<0$, $\frac{f(x)}{g(x)}\geq 0$ and $\frac{f(x)}{g(x)}\leq 0$ (or any you can convert to this form). In this case, since $\frac{x}{2-x}<0$, that happens when: ($x<0$ **and** $2-x>0$) **or** ($x>0$ **and** $2-x<0$). This leads to the combination of two solutions:

- $x<0$ **and** $2-x>0$ (i.e. $x<2$). In this case, the numbers which are both smaller than zero ($x<0$) and simultaneously smaller than two ($x<2$) are only those smaller than zero, i.e. $x<0$ which as an interval is $(-\infty,0)$
- $x>0$ **and** $2-x<0$ (i.e. $x>2$). In this case, the numbers which are bigger than zero ($x>0$) and simultaneously bigger than two ($x>2$) are tjose bigger than two, i.e. $x>2$, which as an interval is $(2,\infty)$.

Since one **or** the other condition must hold the result is $(-\infty,0)\cup(2,\infty)$, which is the same as for the solution above.

h) $\frac{x^2-1}{x-3} \geq 0$

h) $\frac{x^2-1}{x-3}\geq 0$. We can multiply both sides by $x-3$ but since $x-3$ can be negative we need to account for the change on the sign of the inequality.

- If $x-3>0$ (i.e. $x>3$) we get $x^2-1\geq 0$ (i.e. no sign change), which leads to $x^2\geq 1$, and this occurs when $x\geq 1$ or $x\leq -1$. There is no number x bigger than three ($x>3$) which is simultaneously smaller than negative one ($x\leq -1$), therefore these conditios do not lead to a valid solution. On the other hand, $x\geq 1$ and $x>3$ have to be fulfilled simultaneously, and that happens for any $x>3$.

- If $x-3<0$ (i.e. $x<3$) we get $x^2-1 \leq 0$ (i.e. sign change after multiplication), which leads to $x^2 \leq 1$, and this occurs when $-1 \leq x \leq 1$. Since all the numbers between negative one and one are smaller than three, the resulting condition is $-1 \leq x \leq 1$

If $x-3>0$ we get $x>3$, and if $x-3<0$ we get $-1 \leq x \leq 1$, therefore the solution as interval can be written as $[-1,1] \cup (3,\infty)$

Alternative solution: Since $\frac{x^2-1}{x-3} \geq 0$, that happens when: ($x^2-1 \geq 0$ **and** $x-3>0$) **or** ($x^2-1 \leq 0$ **and** $x-3<0$). Note that the denominator $x-3$ cannot be zero, but the numerator x^2-1 can since we have equality ($\geq$) sign, so we use equality in the numerator and strict inequality in the denominator. This leads to the combination of two solutions:

- $x^2-1 \geq 0$ (i.e. $x^2 \geq 1$) **and** $x-3>0$ (i.e. $x>3$). Like in the previous solution we get that these two conditios are fulfilled simultaneously when $x>3$.
- $x^2-1 \leq 0$ (i.e. $x^2 \leq 1$) **and** $x-3<0$ (i.e. $x<3$). Like in the previous solution we get that these conditions are fulfilled simultaneously when $-1 \leq x \leq 1$.

Since one **or** the other condition must hold the result is $[-1,1] \cup (3,\infty)$, which is the same as for the solution above.