

CLI – Øvelsestimer

Øvelsestime 1

Problem 1.1 Given the following sets $S = \{1; 2; 3; 6; 7; 9\}$ and $R = \{2; 4; 5; 6; 7\}$ state

a) $S \cup R$
• {1, 2, 3, 4, 5, 6, 7, 9}

b) $S \cap R$
• {2, 6, 7}

c) $S \setminus R$
• {1, 3, 9}

d) $R \setminus S$

• {4, 5}

e) $|S|$

• 6

f) $|R \setminus S|$

• 2

Problem 1.2 State these expression as simple rational numbers or fractions:

a) $\frac{7}{3} - \frac{1}{2}$

$$\begin{aligned}\frac{7 * 2}{3 * 2} - \frac{1 * 3}{2 * 3} &= & -\frac{a}{b} &= \frac{-a}{b} = \frac{a}{-b} \\ \frac{14}{6} - \frac{3}{6} &= & \frac{a}{b} + \frac{c}{d} &= \frac{a \cdot d + c \cdot b}{b \cdot d} \\ \frac{11}{6} && &\end{aligned}$$

$$b) \pi + \frac{3}{5}$$

$$pi = 3,14$$

$$\begin{aligned} \frac{3}{5} &= 0,6 & a &= \frac{a}{1} \\ 3,14 + 0,6 &= \\ 3,74 & & \frac{a}{b} + \frac{c}{d} &= \frac{a \cdot d + c \cdot d}{b \cdot d} \end{aligned}$$

$$c) \frac{3x^2}{a} + 2x$$

$$\begin{aligned} a &= \frac{1}{x} \\ \frac{3x^2}{a} + \frac{2x}{1} &= & a &= \frac{a}{1} \\ \frac{3x^2}{a} + \frac{a * 2x}{a * 1} &= & \frac{a}{b} + \frac{c}{d} &= \frac{a \cdot d + c \cdot d}{b \cdot d} \\ \frac{3x^2}{a} + \frac{a * 2x}{a} &= \\ \frac{3x^2}{a} + \frac{2ax}{a} &= \\ \frac{3x^2 + 2ax}{a} & \end{aligned}$$

$$d) \frac{a}{4x^2} - \frac{x^2}{a}$$

$$\begin{aligned} \frac{a * a}{4x^2 * a} - \frac{4x^2 * x^2}{4x^2 * a} &= & -\frac{a}{b} &= \frac{-a}{b} = \frac{a}{-b} \\ \frac{a^2}{4ax^2} - \frac{4x^4}{4ax^2} &= & \frac{a}{b} + \frac{c}{d} &= \frac{a \cdot d + c \cdot d}{b \cdot d} \\ \frac{a^2 - 4x^4}{4ax^2} & & a^2 - b^2 &= (a + x)(a - x) \end{aligned}$$

$$e) \frac{\frac{1}{a}}{(x^2+1)} - \frac{1}{a}$$

$$\begin{aligned}
& \frac{x^2 + 1}{a} - \frac{-1}{a} = & \frac{1}{\frac{a}{b}} = \frac{b}{a} \\
& \frac{x^2 + 1}{a} + \frac{-1}{a} = & -\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b} \\
& \frac{(x^2 + 1)a - 1 * a}{a * a} = & \frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + c \cdot d}{b \cdot d} \\
& \frac{a * x^2 + a - a}{a^2} = & a(b + c) = ab + ac \\
& \frac{a * x^2}{a^2} = & \frac{x^2}{a}
\end{aligned}$$

$$f) \frac{a}{2b+3a} - \frac{b}{a+1}$$

$$\begin{aligned}
& \frac{a}{2b+3a} + \frac{-b}{a+1} = & -\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b} \\
& \frac{a(a+1) - (2b+3a)b}{(2b+3a)(a+1)} = & \frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + c \cdot d}{b \cdot d}
\end{aligned}$$

Problem 1.3 Using the rules for exponentials simplify the following expressions:

$$a) 10^3 \cdot 5^{-2} \text{ (hint } 10 = 2 \cdot 5\text{)}$$

$$\begin{aligned}
10^3 * 5^{-2} = & \quad 1. \text{ Since } 10 = 2 \cdot 5, \text{ we can substitute } 10^3 \cdot 5^{-2} = (2 \cdot 5)^3 \cdot 5^{-2}. \\
2^3 * 5^3 * \frac{1}{5^2} = & \quad 2. \text{ Using the rule } (a \cdot b)^n = a^n \cdot b^n \text{ we get } (2 \cdot 5)^3 \cdot 5^{-2} = 2^3 \cdot 5^3 \cdot 5^{-2} \\
& \quad 3. \text{ Using the rule } a^n \cdot a^m = a^{n+m} \text{ we get } 2^3 \cdot 5^3 \cdot 5^{-2} = 2^3 \cdot 5^{3-2} = 2^3 \cdot 5
\end{aligned}$$

$$2^3 * 5^3 * \frac{1}{5^2} =$$

$$2^3 * 5 * 1 =$$

$$2^3 * 5 =$$

$$8 * 5 =$$

b) $\frac{3^5}{3^2}$

$$\frac{3^5}{3^2} =$$

1. Using the rule $\frac{a^n}{a^m} = a^{n-m}$ we get $\frac{3^5}{3^2} = 3^{5-2} = 3^3$

$$3^{5-2} =$$

$$3^3$$

c) $5 \cdot 25^8$ (hint $25 = 5^2$)

$$5 * (5^2)^8 =$$

1. Since $25 = 5^2$, we can substitute to get $5 \cdot 25^8 = 5 \cdot (5^2)^8$

$$5 * 5^{2*8} =$$

2. Using the rule $(a^n)^m = a^{n \cdot m}$ we get $5 \cdot (5^2)^8 = 5 \cdot 5^{2 \cdot 8} = 5 \cdot 5^{16}$

$$5 * 5^{16} =$$

3. Using the rule $a^n \cdot a^m = a^{n+m}$ we get $5 \cdot 5^{16} = 5^{17}$

$$5^1 * 5^{16} =$$

$$5^{17}$$

d) $2^3 \sqrt[5]{6^{10}}$ (hint $6 = 3 \cdot 2$)

$$2^3 * 6^{\frac{10}{5}} =$$

1. Using the rule $\sqrt[n]{a^m} = a^{\frac{m}{n}}$ we get $2^3 \sqrt[5]{6^{10}} = 2^3 \cdot 6^{\frac{10}{5}} = 2^3 \cdot 6^2$

$$2^3 * 6^{\frac{10}{5}} =$$

2. Since $6 = 3 \cdot 2$ we can substitute $2^3 \cdot 6^2 = 2^3 \cdot (3 \cdot 2)^2$

$$2^3 * 6^2 =$$

3. Using the rule $(a \cdot b)^n = a^n \cdot b^n$ we get $2^3 \cdot (3 \cdot 2)^2 = 2^3 \cdot 3^2 \cdot 2^2$

$$2^3 * 2^2 * 3^2 =$$

4. Using the rule $a^n \cdot a^m = a^{n+m}$ we get $2^3 \cdot 3^2 \cdot 2^2 = 2^{3+2} \cdot 3^2 = 2^5 \cdot 3^2$

$$2^3 * 6^2 =$$

$$2^3 * 6^2 =$$

$$2^3 * (2 * 3)^2 =$$

$$2^3 * 2^2 * 3^2 =$$

$$2^5 * 3^2$$

e) $\sqrt[3]{7^{-21}}$

$$7^{-\frac{21}{3}} =$$

1. Using the rule $\sqrt[n]{a^m} = a^{\frac{m}{n}}$ we get $\sqrt[3]{7^{-21}} = 7^{-\frac{21}{3}} = 7^{-7}$

$$7^{-\frac{21}{3}} =$$

2. Using the rule $a^{-n} = \frac{1}{a^n}$ we get $7^{-7} = \frac{1}{7^7}$

$$7^{-7} =$$

$$\frac{1}{7^7}$$

f) $\sqrt[4]{(2^2)^5}$

$$\begin{aligned}\sqrt[4]{2^{2 \cdot 5}} &= && \text{1. Using the rule } (a^n)^m = a^{n \cdot m} \text{ we get } \sqrt[4]{(2^2)^5} = \sqrt[4]{2^{10}} \\ \sqrt[4]{2^{10}} &= && \text{2. Using the rule } \sqrt[m]{a^n} = a^{\frac{n}{m}} \text{ we get } \sqrt[4]{2^{10}} = 2^{\frac{10}{4}} = 2^{\frac{5}{2}} \\ 2^{\frac{10}{4}} &= && \\ 2^{\frac{5}{2}} & && \end{aligned}$$

Problem 1.4 Write the following inequalities as intervals of the real line

a) $x \leq 100$

Numbers smaller or equal than 100: $(-\infty; 100]$

b) $x > 0$ and $x < 10$

Numbers between 0 and 10 (not included) $(0; 10)$

c) $x \geq -10$ and $x < 5$

Numbers larger or equal than -10 and smaller than 5: $[-10; 5)$

d) $x \geq -1$

Numbers larger or equal than -1: $[-1, \infty)$

e) $x < -5$ or $x \geq 5$

Numbers smaller than -5 or larger than 5: $(-\infty; -5) \cup [5; \infty)$

f) $x \neq 3$

Numbers different than 3: $(-\infty, \infty) \setminus \{3\}$ or $(-\infty, 3) \cup (3, \infty)$

Problem 1.5 Solve the following inequalities for x:

a) $6x - 5 > 7$

$$6x - 5 + 5 > 7 + 5 =$$

$$6x > 12 =$$

$$\frac{6}{6} * x > \frac{12}{6} =$$

$$x > 2 (2, \infty)$$

b) $-2x \geq 4$

$$-\frac{2x}{-2} < \frac{4}{-2} =$$

$$x > -2(-\infty; -2)$$

c) $5x - 3 < 7 - 3x$

$$5x - 3 + 3 < 7 + 3 - 3x =$$

$$5x < 10 - 3x =$$

$$5x + 3x < 10 - 3x + 3x =$$

$$8x < 10 =$$

$$\frac{8x}{8} < \frac{10}{8} =$$

$$x = \frac{\frac{10}{8}}{2} =$$

$$x = \frac{5}{4}$$

d) $\frac{6-x}{-4} \geq \frac{3x-4}{2}$

$$4 * \frac{6-x}{-4} > 4 * \frac{3x-4}{2} =$$

$$-(6-x) > 2 * (3x-4) =$$

$$-6 + x > 6x - 8 =$$

$$-6 + 6 + x - 6x > -8 + 6 =$$

$$x - 6x > -2 =$$

$$5x > -2 =$$

$$\frac{-5x}{-5} > \frac{-2}{-5} =$$

$$x = \frac{2}{5}$$

Kommenterede [BKI]: Ethvert negativt tal delt med et negativt giver et positivt tal

$$e) \frac{1}{x} > 6$$

$$\begin{aligned}\frac{1}{x} - 6 &> 6 - 6 = \\ \frac{1-6}{x} &> 0 = \\ 1-6x &> 0 = \\ -1+1-6x &> -1 = \\ -6x &> -1 = \\ -\frac{6}{-6} * x &> -\frac{1}{-6} = \\ x &> \frac{1}{6} \quad (0, \frac{1}{6})\end{aligned}$$

$$f) \frac{x+1}{x-1} \geq 2$$

f) $\frac{x+1}{x-1} \geq 2$ we can multiply both sides by $x-1$ but since $x-1$ can be negative we need to account for the change of sign of the inequality.

- $x-1 > 0$, i.e. $x > 1$, leads to $(x-1)\frac{x+1}{x-1} = x+1 \geq 2(x-1) = 2x-2$, and adding $2-x$ on both sides leads to $x+1+2-x = 3 \geq 2x-2+2-x = x$, i.e. $3 \geq x$. Both inequalities must hold simultaneously $x > 1$ and $x \leq 3$, which leads to the solution $1 < x \leq 3$.
- $x-1 < 0$, i.e. $x < 1$, leads to $(x-1)\frac{x+1}{x-1} = x+1 \leq 2(x-1) = 2x-2$ (change of inequality sign), which by adding leads to $2-x$ on both sides leads to $x+1+2-x = 3 \leq 2x-2+2-x = x$, i.e. $x \geq 3$. Both conditions cannot hold simultaneously and therefore $x-1 < 0$ does not lead to a valid solution.

$$g) \frac{x}{2-x} < 0$$

g) $\frac{x}{2-x} < 0$. We can multiply both sides by $2 - x$ but since $2 - x$ can be negative we need to account for the change on the sign of the inequality.

- If $2 - x > 0$ (i.e. $x < 2$) we get $x < 0$ (i.e. no sign change). Since both inequalities ($x < 0$ and $x < 2$) must be fulfilled simultaneously, the numbers which fulfil both are $x < 0$ (the most restrictive).
- If $2 - x < 0$ (i.e. $x > 2$) we get $x > 0$ (i.e. sign change after multiplication). Since both inequalities ($x > 0$ and $x > 2$) must be fulfilled simultaneously, the numbers which fulfil both are $x > 2$ (the most restrictive).

If $2 - x > 0$ we get $x < 0$, and if $2 - x < 0$ we get $x > 0$, therefore the solution is $x < 0$ or $x > 2$, which as interval can be written as $(-\infty, 0) \cup (2, \infty)$

Alternative solution: (Using function notation that we will see later in the course) There is an alternative way of solving inequalities of the form $\frac{f(x)}{g(x)} > 0$, $\frac{f(x)}{g(x)} < 0$, $\frac{f(x)}{g(x)} \geq 0$ and $\frac{f(x)}{g(x)} \leq 0$ (or any you can convert to this form). In this case, since $\frac{x}{2-x} < 0$, that happens when: ($x < 0$ and $2 - x > 0$) or ($x > 0$ and $2 - x < 0$). This leads to the combination of two solutions:

- $x < 0$ and $2 - x > 0$ (i.e. $x < 2$). In this case, the numbers which are both smaller than zero ($x < 0$) and simultaneously smaller than two ($x < 2$) are only those smaller than zero, i.e. $x < 0$ which as an interval is $(-\infty, 0)$
- $x > 0$ and $2 - x < 0$ (i.e. $x > 2$). In this case, the numbers which are bigger than zero ($x > 0$) and simultaneously bigger than two ($x > 2$) are those bigger than two, i.e. $x > 2$, which as an interval is $(2, \infty)$.

Since one or the other condition must hold the result is $(-\infty, 0) \cup (2, \infty)$, which is the same as for the solution above.

$$h) \frac{x^2-1}{x-3} \geq 0$$

h) $\frac{x^2-1}{x-3} \geq 0$. We can multiply both sides by $x - 3$ but since $x - 3$ can be negative we need to account for the change on the sign of the inequality.

- If $x - 3 > 0$ (i.e. $x > 3$) we get $x^2 - 1 \geq 0$ (i.e. no sign change), which leads to $x^2 \geq 1$, and this occurs when $x \geq 1$ or $x \leq -1$. There is no number x bigger than three ($x > 3$) which is simultaneously smaller than negative one ($x \leq -1$), therefore these conditions do not lead to a valid solution. On the other hand, $x \geq 1$ and $x > 3$ have to be fulfilled simultaneously, and that happens for any $x > 3$.

- If $x - 3 < 0$ (i.e. $x < 3$) we get $x^2 - 1 \leq 0$ (i.e. sign change after multiplication), which leads to $x^2 \leq 1$, and this occurs when $-1 \leq x \leq 1$. Since all the numbers between negative one and one are smaller than three, the resulting condition is $-1 \leq x \leq 1$

If $x - 3 > 0$ we get $x > 3$, and if $x - 3 < 0$ we get $-1 \leq x \leq 1$, therefore the solution as interval can be written as $[-1, 1] \cup (3, \infty)$

Alternative solution: Since $\frac{x^2-1}{x-3} \geq 0$, that happens when: $(x^2-1 \geq 0 \text{ and } x-3 > 0) \text{ or } (x^2-1 \leq 0 \text{ and } x-3 < 0)$. Note that the denominator $x-3$ cannot be zero, but the numerator x^2-1 can since we have equality (\geq) sign, so we use equality in the numerator and strict inequality in the denominator. This leads to the combination of two solutions:

- $x^2 - 1 \geq 0$ (i.e. $x^2 \geq 1$) and $x - 3 > 0$ (i.e. $x > 3$). Like in the previous solution we get that these two conditions are fulfilled simultaneously when $x > 3$.
- $x^2 - 1 \leq 0$ (i.e. $x^2 \leq 1$) and $x - 3 < 0$ (i.e. $x < 3$). Like in the previous solution we get that these conditions are fulfilled simultaneously when $-1 \leq x \leq 1$.

Since one or the other condition must hold the result is $[-1, 1] \cup (3, \infty)$, which is the same as for the solution above.

Øvelsestime 2

Problem 1.1 Given the matrices:

$$A = \begin{bmatrix} 9 & -4 & 7 \\ -5 & 2 & -4 \\ 10 & -4 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 10 & 7 & -7 & 4 \\ -5 & -9 & 4 & -7 \\ -5 & 10 & -6 & -5 \end{bmatrix} \quad C = \begin{bmatrix} -3 & -9 & -6 \\ 7 & 1 & 7 \\ -4 & 5 & 1 \\ 3 & -4 & -5 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 10 & 1 \\ 0 & 5 & 0 \end{bmatrix}$$

Obtain the following matrices

a) A^T

$$\begin{bmatrix} 9 & -5 & 10 \\ -5 & 2 & -4 \\ -7 & -4 & 3 \end{bmatrix}$$

b) C^T

$$\begin{bmatrix} -3 & 7 & -4 & 3 \\ -9 & 1 & 5 & -4 \\ -6 & 7 & 1 & -5 \end{bmatrix}$$

c) $A + D$

$$\begin{bmatrix} 12 & 2 & 14 \\ -2 & 12 & -3 \\ 10 & 1 & -3 \end{bmatrix}$$

d) $B + C^T$

$$B + C^T = \begin{bmatrix} -3 & 7 & -4 & 3 \\ -9 & 1 & 5 & -4 \\ -6 & 7 & 1 & -5 \\ 7 & 14 & -11 & 7 \\ -14 & -8 & 9 & -11 \\ 11 & 17 & -5 & -10 \end{bmatrix}$$

e) $D - A$

$$\begin{bmatrix} -6 & 10 & 0 \\ 8 & 8 & 5 \\ -10 & 9 & 3 \end{bmatrix}$$

f) AB

$$A = \begin{bmatrix} 9 & -4 & 7 \\ -5 & 2 & -4 \\ 10 & -4 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 10 & 7 & -7 & 4 \\ -5 & -9 & 4 & -7 \\ -5 & 10 & -6 & -5 \end{bmatrix}$$

$$\begin{aligned} \text{Formula(Only the first number)} &= a_{11} * b_{11} + a_{12} * b_{21} + a_{13} * b_{31} \\ 9 * 10 + (-4) * (-5) + 7 * (-5) &= \\ 90 + 20 - 35 &= \\ 75 \end{aligned}$$

$$\begin{bmatrix} 75 & 169 & -121 & 29 \\ -40 & -93 & 67 & -14 \\ 135 & 76 & -68 & 83 \end{bmatrix}$$

g) BC

$$BC = \begin{bmatrix} 59 & -134 & -38 \\ -85 & 84 & 6 \\ 94 & 45 & 119 \end{bmatrix}$$

h) CB

$$CB = \begin{bmatrix} 45 & 0 & 21 & 81 \\ 30 & 110 & -87 & -14 \\ -70 & -63 & 42 & -56 \\ 75 & 7 & -7 & 65 \end{bmatrix}$$

Problem 1.2 Calculate the determinant of the following matrices

a)

$$\begin{bmatrix} 1 & 2 & 0 \\ -3 & 5 & 0 \\ 7 & 4 & 1 \end{bmatrix}$$

$$\begin{aligned} (1 * 5 * 1) + (2 * 0 * 7) + (-3 * 4 * 0) &= 5 + 0 + 0 = 5 \\ (7 * 5 * 0) - (-3 * 2 * 1) - (4 * 0 * 1) &= 0 - 6 - 0 = -6 \\ 5 - (-6) &= \\ 11 \end{aligned}$$

b)

$$\begin{bmatrix} 5 & 1 & -8 \\ -3 & 7 & 2 \\ -4 & 1 & -1 \end{bmatrix}$$

$$\begin{aligned} (5 * 7 * (-1)) + (1 * 2 * (-4)) + (-3 * 1 * (-8)) &= -35 - 8 + 24 = -19 \\ (4 * 7 * -8) - (-3 * 1 * (-1)) - (1 * 2 * 5) &= -224 - 3 - 10 = -237 \\ -19 - 237 &= \\ -256 \end{aligned}$$

c)

$$\begin{vmatrix} 3 & 9 & -3 \\ 9 & 1 & -1 \\ 6 & -8 & 2 \end{vmatrix} = 3 \times 1 \times 2 + 9 \times (-1) \times 6 + 9 \times (-8) \times (-3) - 6 \times 1 \times (-3) \\ - 9 \times 9 \times 2 - (-8) \times (-1) \times 3 \\ = 6 - 54 + 216 + 18 - 162 - 24 = 0$$

d)

$$\begin{bmatrix} 2 & 0 & 7 & 0 \\ -1 & 3 & 6 & 1 \\ 2 & 0 & 1 & 3 \\ -2 & 4 & 3 & 1 \end{bmatrix}$$
$$+ 2 \begin{bmatrix} 3 & 6 & 1 \\ 0 & 1 & 3 \\ 4 & 3 & 1 \end{bmatrix}$$
$$2((3 * 1 * 1) + (6 * 3 * 4)) + (0 * 3 * 1) = 3 + 72 + 0 = 75$$
$$2((-2 * 0 * 1) - (4 * 1 * 1) - (3 * 3 * 3)) = 0 - 4 - 27 = -31$$
$$2(75 - 31) = 2(44)$$
$$2 * 44 = 88$$
$$+ 7 \begin{bmatrix} -1 & 3 & 1 \\ 2 & 0 & 3 \\ -2 & 4 & 1 \end{bmatrix}$$
$$7((-1 * 0 * 1) + (3 * 3 * (-2)) + (2 * 3 * 1)) = -18 + 8 = -10$$
$$7((-2 * 0 * 1) - (2 * 3 * 1) - (4 * 3 * (-1))) = 6 - (-12) = 6$$
$$7(-10 + 6) = 7(-4)$$
$$7 * (-4) = -28$$

$$88 - 28 = \\ 60$$

e)

$$\begin{vmatrix} 0 & -4 & -4 & 0 \\ -1 & 2 & -3 & -4 \\ 0 & -4 & 1 & 4 \\ 0 & 2 & 3 & 0 \end{vmatrix} = (-1)^{(2+1)}(-1) \begin{vmatrix} -4 & -4 & 0 \\ -4 & 1 & 4 \\ 2 & 3 & 0 \end{vmatrix}$$

$$= (-1)^{(2+3)}4 \begin{vmatrix} -4 & -4 \\ 2 & 3 \end{vmatrix}$$

$$= -4(-12 - (-8)) = 16$$

f)

$$\begin{vmatrix} 0 & -2 & 0 & -5 & 3 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 5 & 0 & -1 & 0 \\ 7 & -1 & -1 & 2 & -9 \\ 1 & 0 & 0 & -10 & 0 \end{vmatrix} = (-1)^{(4+3)}(-1) \begin{vmatrix} 0 & -2 & -5 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 5 & -1 & 0 \\ 1 & 0 & -10 & 0 \end{vmatrix}$$

$$= (-1)^{(4+1)}1 \begin{vmatrix} -2 & -5 & 3 \\ 1 & -2 & -1 \\ 5 & -1 & 0 \end{vmatrix}$$

$$= (-1)(25 - 3 + 30 + 2) = -54$$

Problem 1.3 Given the following vectors \mathbf{u} and \mathbf{v} calculate their norm ($|\mathbf{u}|$ and $|\mathbf{v}|$), sum, difference ($\mathbf{u} - \mathbf{v}$ and $\mathbf{v} - \mathbf{u}$), dot/scalar product and the angle between them.

a) $\mathbf{u} = [1, 3]$ and $\mathbf{v} = [6, -2]$

Addition

$$\begin{matrix} 1 + 6 = 7 \\ 3 + (-2) = 1 \\ [7, 1] \end{matrix}$$

Subtraction

$$\begin{matrix} u - v \\ 1 - 6 = 5 \\ 3 - (-2) = 5 \\ [5, 5] \end{matrix}$$

$$\begin{matrix} v - u \\ 6 - 1 = 5 \\ -2 - 3 = -5 \\ [5, -5] \end{matrix}$$

Norms

$$[1, 3] = \sqrt{1^2 + 3^2} = \sqrt{1 + 9} = \sqrt{10}$$

$$[6, -2] = \sqrt{\frac{|v|}{6^2 + (-2)^2}} = \sqrt{\frac{36 + (-4)}{40}} = 4$$

Dot product

$$\begin{array}{r} u * v \\ 1 * 6 = 6 \\ 3 * -2 = -6 \\ 6 - 6 = 0 \\ 0 \end{array}$$

Angle

$$\begin{array}{r} 1 * 6 + 3 * (-2) \\ \sqrt{1^2 + 3^2} \sqrt{6^2 + (-2)^2} \\ 6 + \sqrt{(-6)} \\ \sqrt{1 + 9} \sqrt{36 + 4} \\ 0 \\ \sqrt{10} \sqrt{40} \end{array}$$

b) $\mathbf{u} = [\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$ and $\mathbf{v} = [0, -1]$

- Addition: $\mathbf{u} + \mathbf{v} = [\frac{\sqrt{2}}{2} + 0, \frac{\sqrt{2}}{2} - 1] = [\frac{\sqrt{2}}{2}, \frac{\sqrt{2}-2}{2}]$
- Subtraction: $\mathbf{u} - \mathbf{v} = [\frac{\sqrt{2}}{2} - 0, \frac{\sqrt{2}}{2} - (-1)] = [\frac{\sqrt{2}}{2}, \frac{\sqrt{2}+2}{2}]$ and $\mathbf{v} - \mathbf{u} = [0 - \frac{\sqrt{2}}{2}, -1 - \frac{\sqrt{2}}{2}] = [-\frac{\sqrt{2}}{2}, -\frac{2+\sqrt{2}}{2}]$
- Norms: $|\mathbf{u}| = \sqrt{\left[\frac{\sqrt{2}}{2}\right]^2 + \left[\frac{\sqrt{2}}{2}\right]^2} = \sqrt{\frac{2}{4} + \frac{2}{4}} = 1$ and $|\mathbf{v}| = \sqrt{0^2 + (-1)^2} = \sqrt{1} = 1$
- Dot product: $\mathbf{u} \cdot \mathbf{v} = \frac{\sqrt{2}}{2} \times 0 + \frac{\sqrt{2}}{2} \times (-1) = -\frac{\sqrt{2}}{2}$
- Angle: $\theta = \arccos \left[\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \right] = \arccos \left[\frac{-\frac{\sqrt{2}}{2}}{1 \times 1} \right] = \arccos \left[-\frac{\sqrt{2}}{2} \right] = \frac{3\pi}{4}$

c) $\mathbf{u} = [-3, 1]$ and $\mathbf{v} = [-2, -1]$

- Addition: $\mathbf{u} + \mathbf{v} = [-5, 0]$
- Subtraction: $\mathbf{u} - \mathbf{v} = [-1, 2]$ and $\mathbf{v} - \mathbf{u} = [1, -2]$
- Norms: $|\mathbf{u}| = \sqrt{(-3)^2 + 1^2} = \sqrt{10}$ and $|\mathbf{v}| = \sqrt{(-2)^2 + (-1)^2} = \sqrt{5}$
- Dot product: $\mathbf{u} \cdot \mathbf{v} = (-3) \times (-2) + 1 \times (-1) = 5$
- Angle: $\theta = \arccos \left[\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \right] = \arccos \left[\frac{5}{\sqrt{10} \sqrt{5}} \right] = \arccos \left[\frac{5}{\sqrt{50}} \right] = \arccos \left[\frac{5}{5\sqrt{2}} \right] = \arccos \left[\frac{\sqrt{2}}{2} \right] = \frac{\pi}{4}$

d) $\mathbf{u} = [\sqrt{5}, \sqrt{3}, \sqrt{2}]$ and $\mathbf{v} = [0, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}]$

- Addition: $\mathbf{u} + \mathbf{v} = [\sqrt{5}, \sqrt{3} - \frac{1}{\sqrt{3}}, \sqrt{2} + \frac{1}{\sqrt{2}}] = [\sqrt{5}, \frac{3-1}{\sqrt{3}}, \frac{2+1}{\sqrt{2}}] = [\sqrt{5}, \frac{2}{\sqrt{3}}, \frac{3}{\sqrt{2}}]$
- Subtraction: $\mathbf{u} - \mathbf{v} = [\sqrt{5}, \sqrt{3} + \frac{1}{\sqrt{3}}, \sqrt{2} - \frac{1}{\sqrt{2}}] = [\sqrt{5}, \frac{3+1}{\sqrt{3}}, \frac{2-1}{\sqrt{2}}] = [\sqrt{5}, \frac{4}{\sqrt{3}}, \frac{1}{\sqrt{2}}]$ and $\mathbf{v} - \mathbf{u} = [-\sqrt{5}, -\frac{1}{\sqrt{3}} - \sqrt{3}, \frac{1}{\sqrt{2}} - \sqrt{2}] = [-\sqrt{5}, \frac{-1-3}{\sqrt{3}}, \frac{1-2}{\sqrt{2}}] = [-\sqrt{5}, -\frac{4}{\sqrt{3}}, -\frac{1}{\sqrt{2}}]$
- Norms: $|\mathbf{u}| = \sqrt{(\sqrt{5})^2 + (\sqrt{3})^2 + (\sqrt{2})^2} = \sqrt{5+3+2} = \sqrt{10}$ and $|\mathbf{v}| = \sqrt{0^2 + \left[\frac{1}{\sqrt{3}}\right]^2 + \left[\frac{1}{\sqrt{2}}\right]^2} = \sqrt{\frac{1}{3} + \frac{1}{2}} = \sqrt{\frac{2+3}{6}} = \sqrt{\frac{5}{6}}$
- Dot product: $\mathbf{u} \cdot \mathbf{v} = \sqrt{5} \times 0 + \sqrt{3} \times (-\frac{1}{\sqrt{3}}) + \sqrt{2} \times \frac{1}{\sqrt{2}}) = 0 - 1 + 1 = 0$
- Angle: $\theta = \arccos \left[\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \right] = \arccos \left[\frac{0}{\sqrt{10} \sqrt{\frac{5}{6}}} \right] = \arccos(0) = \frac{\pi}{2}$

e) $\mathbf{u} = [2, 3, 6]$ and $\mathbf{v} = [-6, -9, -18]$

- Addition: $\mathbf{u} + \mathbf{v} = [-4, -6, -12]$
- Subtraction: $\mathbf{u} - \mathbf{v} = [8, 12, 24]$ and $\mathbf{v} - \mathbf{u} = [-8, -12, -24]$
- Norms: $|\mathbf{u}| = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{4+9+36} = \sqrt{49} = 7$ and $|\mathbf{v}| = \sqrt{(-6)^2 + (-9)^2 + (-18)^2} = \sqrt{36+81+324} = \sqrt{441} = 21$
- Dot product: $\mathbf{u} \cdot \mathbf{v} = 2 \times (-6) + 3 \times (-9) + 6 \times (-18) = -147$
- Angle: $\theta = \arccos \left[\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \right] = \arccos \left[\frac{-147}{7 \times 21} \right] = \arccos(-1) = \pi$

f) $\mathbf{u} = [6, 3\sqrt{2}, 3\sqrt{2}]$ and $\mathbf{v} = [1, -1 + \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2}]$

- Addition: $\mathbf{u} + \mathbf{v} = [7, -1 + \frac{7\sqrt{2}}{2}, 1 + \frac{7\sqrt{2}}{2}]$
- Subtraction: $\mathbf{u} - \mathbf{v} = [5, 1 + \frac{5\sqrt{2}}{2}, -1 + \frac{5\sqrt{2}}{2}]$ and $\mathbf{v} - \mathbf{u} = [-5, -1 - \frac{5\sqrt{2}}{2}, 1 - \frac{5\sqrt{2}}{2}]$
- Norms: $|\mathbf{u}| = \sqrt{6^2 + (3\sqrt{2})^2 + (3\sqrt{2})^2} = \sqrt{36+18+18} = 6\sqrt{2}$ and $|\mathbf{v}| = \sqrt{1^2 + (-1 + \frac{\sqrt{2}}{2})^2 + (1 + \frac{\sqrt{2}}{2})^2} = \sqrt{1+1-\sqrt{2}+\frac{2}{4}+1+\sqrt{2}+\frac{2}{4}} = 2$
- Dot product: $\mathbf{u} \cdot \mathbf{v} = 6 \times 1 + 3\sqrt{2} \times (-1 + \frac{\sqrt{2}}{2}) + 3\sqrt{2} \times (1 + \frac{\sqrt{2}}{2}) = 12$
- Angle: $\theta = \arccos \left[\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \right] = \arccos \left[\frac{12}{6\sqrt{2} \times 2} \right] = \arccos(\frac{\sqrt{2}}{2}) = \frac{\pi}{4}$

g) $\mathbf{u} = [1, -1, 1, 1]$ and $\mathbf{v} = [3, 2, 4, -5]$

- Addition: $\mathbf{u} + \mathbf{v} = [4, 1, 5, -4]$
- Subtraction: $\mathbf{u} - \mathbf{v} = [-2, -3, -3, 6]$ and $\mathbf{v} - \mathbf{u} = [2, 3, 3, -6]$
- Norms: $|\mathbf{u}| = \sqrt{1^2 + (-1)^2 + 1^2 + 1^2} = \sqrt{4} = 2$ and $|\mathbf{v}| = \sqrt{3^2 + 2^2 + 4^2 + (-5)^2} = \sqrt{9 + 4 + 16 + 25} = \sqrt{54} = 3\sqrt{6}$
- Dot product: $\mathbf{u} \cdot \mathbf{v} = 1 \times 3 + (-1) \times 2 + 1 \times 4 + 1 \times (-5) = 0$
- Angle: $\theta = \arccos \left[\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \right] = \arccos \left[\frac{0}{2 \times 3\sqrt{6}} \right] = \arccos(0) = \frac{\pi}{2}$

h) $\mathbf{u} = [2, 3, -1, 6]$ and $\mathbf{v} = [-4, -6, 2, -12]$

- Addition: $\mathbf{u} + \mathbf{v} = [-2, 9, 1, -6]$
- Subtraction: $\mathbf{u} - \mathbf{v} = [6, -3, -3, 18]$ and $\mathbf{v} - \mathbf{u} = [-6, 3, 3, -18]$
- Norms: $|\mathbf{u}| = \sqrt{2^2 + 3^2 + (-1)^2 + 6^2} = \sqrt{4 + 9 + 1 + 36} = \sqrt{50} = 5\sqrt{2}$ and $|\mathbf{v}| = \sqrt{(-4)^2 + (-6)^2 + 2^2 + (-12)^2} = \sqrt{16 + 36 + 4 + 144} = \sqrt{200} = 10\sqrt{2}$
- Dot product: $\mathbf{u} \cdot \mathbf{v} = 2 \times (-4) + 3 \times (-6) + (-1) \times 2 + 6 \times (-12) = -64$
- Angle: $\theta = \arccos \left[\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \right] = \arccos \left[\frac{-64}{(5\sqrt{2}) \times (10\sqrt{2})} \right] = \arccos(-\frac{16}{25}) \approx 2.26529$