Midterm Exam 1

BLM 1541: Probability and Statistics — Fall 2016

Print family (or last) name: ____________________

Print given (or first) name: ____________________

Print student number: ____________________ Group: ______ Order: ______

I have read and understand all of the instructions below, and I will obey the Academic Integrity Code of Yıldız Technical University.

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Signature and Date

- This exam has 4 pages in total, numbered 1 to 4. Make sure your exam has all the pages.
- Note the number written on the upper right-hand corner of the first page. On the sign-up sheet being passed around, sign your name next to this number.
- This exam will be 1 hour and 25 minutes in length.
- This is a closed-book and closed-note exam. Electronic devices (e.g., cellphone, smart watch) are not allowed. Single page A4 note sheet is allowed. *Don't forget to attach your A4 note sheet into your solution; otherwise your solution will not be graded.*
- For all problems, follow these instructions:
    - Give only your answers in the spaces provided. I will only grade what you put in the answer space, and I will take off points for any scratch work in the answer space. Use the scratch-work area or the backs of the sheets to work out your answers before filling in the answer space.
    - PMF stands for probability mass function; CDF stands for cumulative distribution function; $var(X)$ stands for the variance of the random variable X; $cov(X, Y)$ stands for the covariance between the random variables X and Y.
    - For any proofs, be sure to provide a step-by-step argument, with justifications for every step.

Problem	1	2	3	4	5	6	Total
Points	20	15	20	15	15	15	100

1. **[20 points]** Let us given two events A and B, then answer the following questions

(a) If events A and B are not independent and the probabilities $P(A \cup B) = 0.6$, $P(\bar{A} \cup \bar{B}) = 0.7$ and $P(A \cap \bar{B}) = 0.2$ are given, then find the probability of event A, i.e., $P(A)$?

Hints: You can start from $A = (A \cap B) \cup (A \cap B)$.

Herein write your answer.

Most efficient way:

$$P(A) = P(A \cap B) + P(A \cap \bar{B})$$
$$P(A \cap B) = 1 - P(\bar{A} \cup \bar{B})$$

Therefore we have $P(A) = 0.3 + 0.2 = 0.5$

Other way:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$0.6 = P(A) + P(B) - 0.3 \text{ so } P(A) + P(B) = 0.9$$

$P(A \cup B) = P(B) + P(A \cap \bar{B})$ so we have $P(B) = P(A \cup B) - P(A \cap \bar{B}) = 0.6 - 0.2 = 0.4$

At the end we have $P(A) + P(B) = 0.9$ then $P(A) = 0.9 - P(B) = 0.9 - 0.4 = 0.5$

(b) If events A and B are independent, then show that $P(A \cap \bar{B}) = P(A)P(\bar{B})$.

Herein write your answer.

We can write $P(A) = P(A \cap B) + P(A \cap \bar{B})$, then we have $P(A \cap \bar{B}) = P(A) - P(A \cap B)$. Under the condition of that A and B are independent, then we have $P(A \cap B) = P(A)P(B)$. Accordingly, $P(A \cap \bar{B})$ can be written as

$$P(A \cap \bar{B}) = P(A) - P(A)P(B)$$
$$= P(A)\big(1 - P(B)\big)$$

where substituting $P(\bar{B}) = 1 - P(B)$ results in $P(A \cap \bar{B}) = P(A)P(\bar{B})$, which proves the equality.

2. **[15 points]** A computer manufacturer uses chips from three sources. Chips from sources A, B, and C are defective with probabilities 0.005, 0.001, and 0.010, respectively. If a randomly selected chip is found to be defective, find the probability that the manufacturer was A. Assume that the proportions of chips from A, B, and C are 0.5, 0.1, and 0.4, respectively.

Herein write your answer.

$$\begin{aligned} P(\text{chip defective}) &= P(\text{def.}|A)P(A) + P(\text{def.}|B)P(B) + P(\text{def.}|C)P(C) \\ &= 5\times10^{-3}P(A) + 10^{-3}P(B) + 10^{-2}P(C) \\ &= 5\times10^{-3}\times0.5 + 10^{-3}\times0.1 + 10^{-2}\times0.4 = 6.6\times10^{-3} \end{aligned}$$

By means of using Bayes theorem, we have

$$P(A|\text{chip defective}) = \frac{P(\text{def.}|A)P(A)}{P(\text{def.})} = \frac{5\times10^{-3}\times0.5}{6.6\times10^{-3}} = 0.3788$$

3. **[20 points]** If X and Y have a covariance of $cov(X,Y)$, we can transform them to a new pair of random variables whose covariance is zero. To do so we let

$$W = X,$$

$$Z = cX + Y,$$

where c is a constant real number, then answer the following questions. Express $cov(W,Z)$ in terms of c, $var(X)$ and $cov(X,Y)$.

Herein write your answer.

Let say $\mu_X = E[X]$ and $\mu_Y = E[Y]$. Using the equation of covariance, we can write

$$\begin{aligned} Cov(W,Z) &= E[(W - E[W])(Z - E[Z])] \\ &= E[(X - E[X])(cX + Y - E[cX + Y])] \\ &= E[(X - \mu_X)(cX + Y - c\mu_X - \mu_Y)] \\ &= E[cX^2 + XY - 2cX\mu_X - Y\mu_X + c\mu_X^2 - X\mu_Y + \mu_X\mu_Y] \\ &= E[cX^2] + E[XY] - 2c\mu_X^2 - \mu_Y\mu_X + c\mu_X^2 - \mu_X\mu_Y + \mu_X\mu_Y \\ &= E[cX^2] - c\mu_X^2 + \underbrace{E[XY] - \mu_Y\mu_X}_{Cov(X,Y)} \\ &= c\,\underbrace{(E[X^2] - \mu_X^2)}_{var(X)} + \underbrace{E[XY] - \mu_Y\mu_X}_{Cov(X,Y)} \\ &= c\,var(X) + cov(X,Y) \end{aligned}$$

4. **[15 points]** Assume that X and Y are two discrete random variables whose joint PMF is given as

$$P_{X,Y}(x,y) = \begin{cases} \frac{1}{6}, & x = 0, y = 0 \\ \frac{1}{3}, & x = 0, y = 1 \\ \frac{1}{3}, & x = 1, y = 0 \\ \frac{1}{6}, & x = 1, y = 1 \end{cases}$$

Then, find the PMF of random variable X, i.e., $P_X(x) =$?

Herein write your answer.

We can find the probabilities of X by using Bayes theorem as follows

$$P(X = 0) = P(X = 0|Y = 0) + P(X = 0|Y = 1) = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$$

$$P(X = 1) = P(X = 1|Y = 0) + P(X = 1|Y = 1) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

Then we have the PMF of X as

$$\begin{aligned} P_X(x) &= \begin{cases} P(X = 0), & x = 0, \\ P(X = 1), & x = 1 \end{cases} \\ &= \begin{cases} 1/2, & x = 0, \\ 1/2, & x = 1 \end{cases} \end{aligned}$$

5. **[15 points]** A television storeowner figures that 70 percent of the customers entering his store buy a television. The rest is just browsing. 5 customers enter the store that day. What is the probability that our storeowner sells three or more televisions on that day?

Herein write your answer.

Let say X be a random variable representing the Number of televisions sold that day. $P(X \geq 3) = ?$

Each customer is a Bernoulli trial with definitions:

Success→Customer buys a TV.

Failure→Customer does not buy a TV

Then the random variable X is a binomial random variable, i.e., $X \sim Binomial(n = 5, p = 0.7)$. Then, we can compute the probability as

$$\begin{aligned} P(X \geq 3)) &= 1 - P(X \leq 2) \\ &= 1 - F(2) \\ &= 1 - P(0) - P(1) - P(2) \\ &= 1 - 0.163 = 0.837 \end{aligned}$$

6. **[15 points]** Let X be a random variable whose PDF is given as

$$f(x) = \begin{cases} c(1 - x^2), & -1 < x < 1 \\ 0, & otherwise \end{cases}$$

a) What is the value of the constant c?

Herein write your answer.

$$\int_{-1}^{1} f(x)dx = 1 \text{ therefore } \int_{-1}^{1} c(1 - x^2)dx$$

$$= c\left(x - \frac{x^3}{3}\Big|_{-1}^{1}\right) = c\left(1 - \frac{1}{3} - 1 + \frac{1}{3}\right)$$

$$= c\left(2 - \frac{2}{3}\right) = c\frac{4}{3} = 1 \text{ then we find that } c = \frac{3}{4}$$

b) What is the CDF of X? In other words $F_X(x) = P(X < x) = ?$

Herein write your answer.

$$F_X(x) = P(X < x) = \int_{-\infty}^{x} f(u)du$$

$$= \int_{-1}^{x} \frac{3}{4}(1 - u^2)du$$

$$= \frac{3}{4}\left(u - \frac{u^3}{3}\Big|_{-1}^{x}\right) = \frac{3}{4}\left(x - \frac{x^3}{3}\right) - \frac{3}{4}\left(-1 + \frac{1}{3}\right)$$

$$= -\frac{x^3}{4} + \frac{3}{4}x + \frac{1}{2}$$