Numerical Integration

1

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{j=1}^{n} f\left(x_0 + \left(\frac{b-a}{n}\right) j\right)$$

$$= \lim_{n\to\infty} \frac{b-a}{n} \sum_{j=1}^{n} f(x_j)$$

$$x_j = a + hj \quad , \quad x_0 = a$$
$$x_n = b$$

$j=1 \Rightarrow \quad x_1 = x_0 + h$

$j=2 \Rightarrow \quad x_2 = x_0 + 2h$

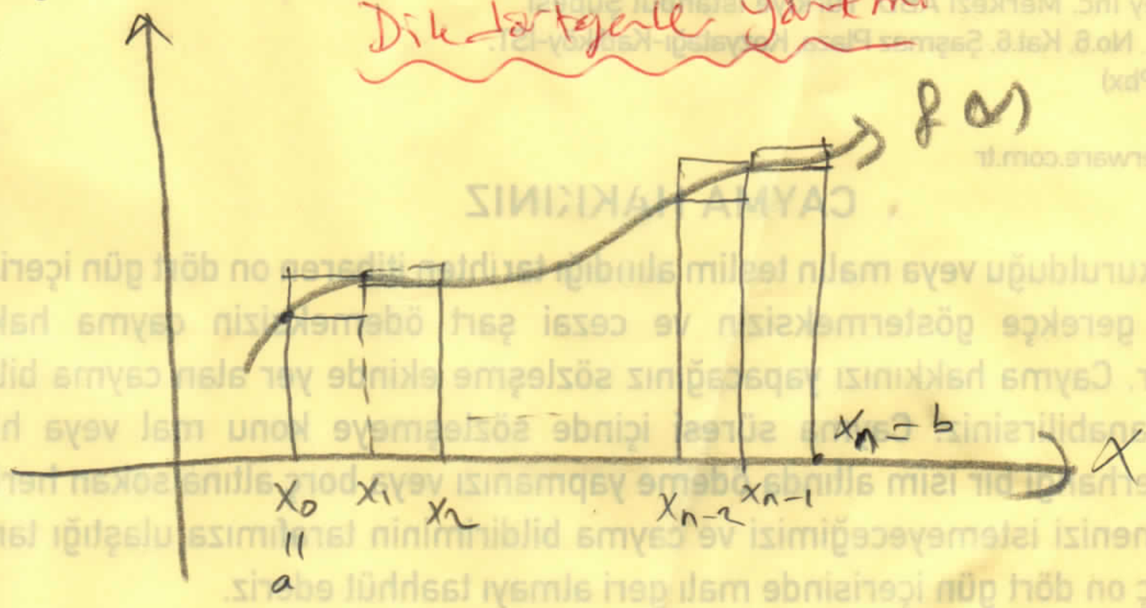
$\vdots$

$j=n \Rightarrow \quad x_n = x_0 + nh.$

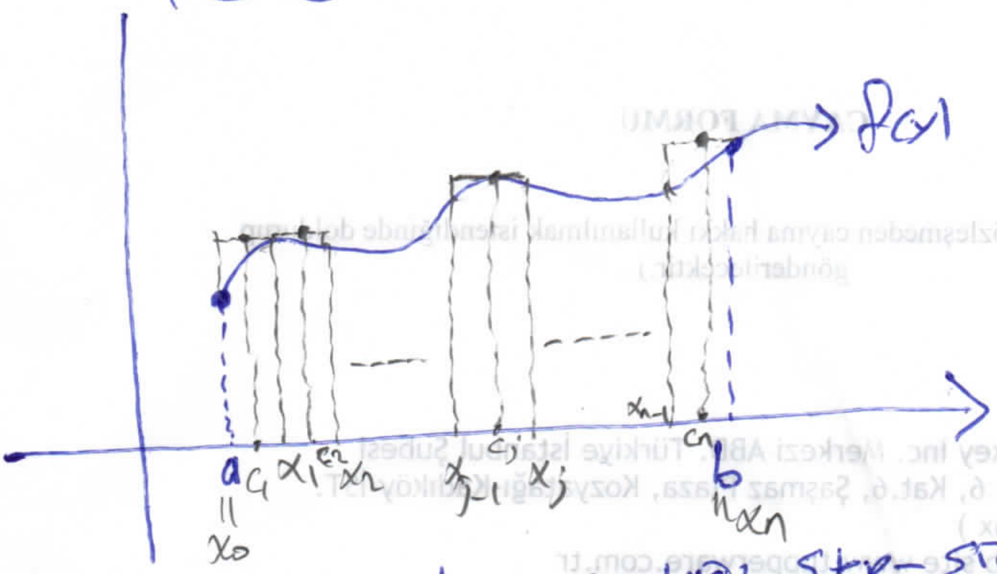
$$\int_a^b f(x)\,dx \simeq h\left[f_0 + f_1 + f_2 + \cdots + f_{n-1}\right] = h\sum_{j=0}^{n-1} f_j$$

$$\int_a^b f(x)\,dx \simeq h\left[f_1 + f_2 + f_3 + \cdots + f_n\right] = h\sum_{j=1}^{n} f_j$$

Dikdörtgenle yaklaşımı



Mid-point Rule



$$\Delta x = h = \frac{b-a}{n} \Rightarrow$$ Adım uzunluğu, step-size, step-length

$$\int_a^b f(x)\,dx \cong w_j \sum_{j=}^{n} f_{j+\frac{1}{2}} \qquad w_j = 1 \cdot h$$

$$\simeq h\left[f_{\frac{1}{2}} + f_{3/2} + f_{5/2} + \cdots + f_{n-\frac{1}{2}} \right]$$

$$x_j = x_0 + \left(j - \frac{1}{2}\right) h$$

$$j = \Rightarrow x_1 = x_0 + \frac{h}{2}$$

$$j = 2 \Rightarrow x_2 = x_0 + \frac{3}{2} h$$

$$\vdots$$

$$j = n-1 \Rightarrow x_{n-1} = x_0 + \left(n - 1 - \frac{1}{2}\right) h = x_0 + \left(\frac{2n-}{2}\right) h$$

$$j = n \Rightarrow x_n = x_0 + \left(n - \frac{1}{2}\right) h = x_0 + \left(\frac{2n-1}{2}\right) h$$

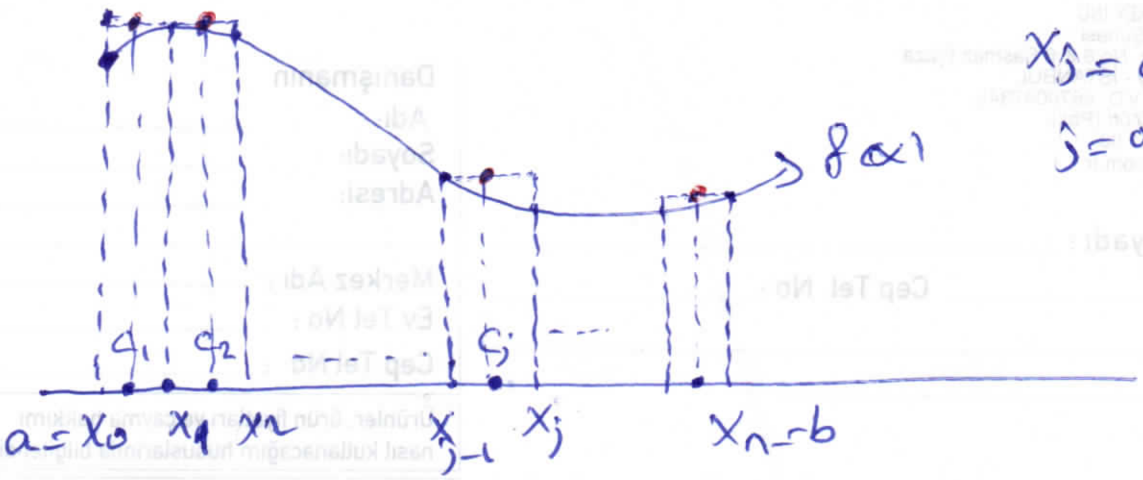
3

mid-point - hata sınırı

$\Delta x = h = \frac{b-a}{n}$

$x_j = a + j h$

$j = 0, 1, \ldots, n$



$[x_{j-1}, x_j]$ aralığının orta noktası $\Rightarrow c_j = a + (j - \frac{1}{2})h$

dikdörtgenin Alanı $= f(c_j) \cdot \Delta x$

$$\int_a^b f(x)\,dx \simeq \underbrace{h[f(c_1) + f(c_2) + \cdots + f(c_n)]}_{M_n}$$

$$\text{Error}(M_n) = \left| \int_a^b f(x)\,dx - M_n \right|$$

Teorem:

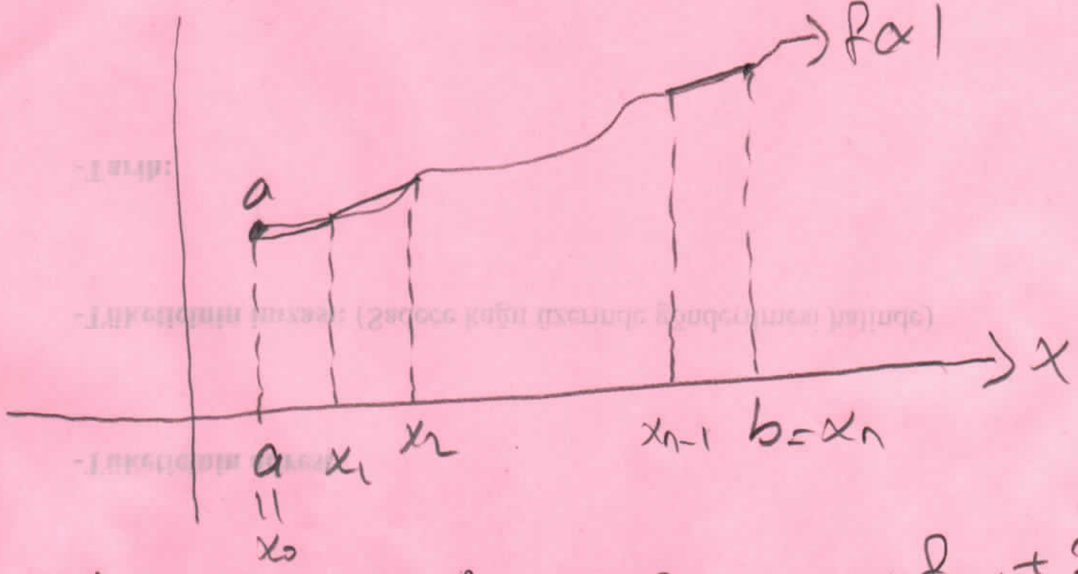
$M_n = \Delta x [f(c_1) + f(c_2) + \cdots + f(c_n)]$

Error bounds for M_n, Let K be a number such that $|f''(x)| \leq K$ for all $x \in [a,b]$. Then

$$\text{Error}(M_n) \leq \frac{K(b-a)^3}{24 n^2}.$$

Trapezler (Yamuklar yöntemi)

4



$$\int_a^b f(x)\,dx \cong h\left[\frac{f_0}{2}+f_1+f_2+\dots+f_{n-1}+\frac{f_n}{2}\right]$$

$$\int_a^b f(x)\,dx \Rightarrow$$

$$\int_{x_0}^{x_1} f(x)\,dx = (x_1-x_0)\cdot\frac{[f(x_0)+f(x_1)]}{2}$$

$$\int_{x_1}^{x_2} f(x)\,dx = (x_2-x_1)\frac{[f(x_2)+f(x_1)]}{2}$$

$\vdots$

$$\int_{x_{n-1}}^{x_n} f(x)\,dx = (x_n-x_{n-1})\frac{[f(x_n)+f(x_{n-1})]}{2}$$

$$\int_{x_0}^{x_n} f(x)\,dx \cong \Delta x\left[\frac{f_0}{2}+\frac{f_1}{2}+\frac{f_1}{2}+\frac{f_2}{2}+\frac{f_2}{2}+\dots+\frac{f_{n-1}}{2}+\frac{f_{n-1}}{2}+\frac{f_n}{2}\right]$$

$$= h\left[\frac{f_0}{2}+f_1+f_2+\dots+f_{n-1}+\frac{f_n}{2}\right]$$

Derivation of Trapezoidal Rule

$$\int_a^b f(x)\,dx \cong \int_a^b \left[\frac{x-b}{a-b} f(a) + \frac{x-a}{b-a} f(b) \right] dx$$

$$= \frac{f(a)}{a-b} \int_a^b (x-b)\,dx + \frac{f(b)}{b-a} \int_a^b (x-a)\,dx$$

$$= \frac{f(a)}{a-b} \left. \frac{(x-b)^2}{2} \right|_{x=a}^{b} + \frac{f(b)}{b-a} \left. \frac{(x-a)^2}{2} \right|_{x=a}^{b}$$

$$= \frac{f(a)}{a-b} \left[0 - \frac{(a-b)^2}{2} \right] + \frac{f(b)}{b-a} \left[\frac{(b-a)^2}{2} - 0 \right]$$

$$= -\frac{f(a)}{a-b} \frac{(a-b)^2}{2} + \frac{f(b)}{b-a} \frac{(b-a)^2}{2}$$

$$= \frac{f(a)}{2} \frac{(b-a)^2}{b-a} + \frac{f(b)}{2} \frac{(b-a)^2}{b-a}$$

$$= \frac{(b-a)}{2} \left[f(a) + f(b) \right]$$

$$= (b-a) \left[\frac{f(a)}{2} + \frac{f(b)}{2} \right]$$ bulunur.

$$\int_0^1 \cos(x^2)\,dx \ , \ \text{mid-point} \ , \ n=8$$ 6

Error bound $\Rightarrow$ $$|E_M| \le \frac{k\cdot(b-a)^3}{24\cdot n^2}$$

$$\le \frac{3.8449(1-0)^3}{24(8)^2}$$

$$\le 0.002503$$

$$\int_0^1 \cos(x^2)\,dx \ , \ \text{Trap. Rule} \quad n=8$$

$$|E_T| \le \frac{k(b-a)^3}{12n^2}$$ k bir sayı

$|f''(x)| \le k$

$a \le x \le b$ için

$$\le \frac{3.8449(1-0)^3}{12(8)^2}$$

$$\le 0.005006$$

$f(x) = \cos(x^2)$ d

$f'(x) = -2x\sin(x^2)$

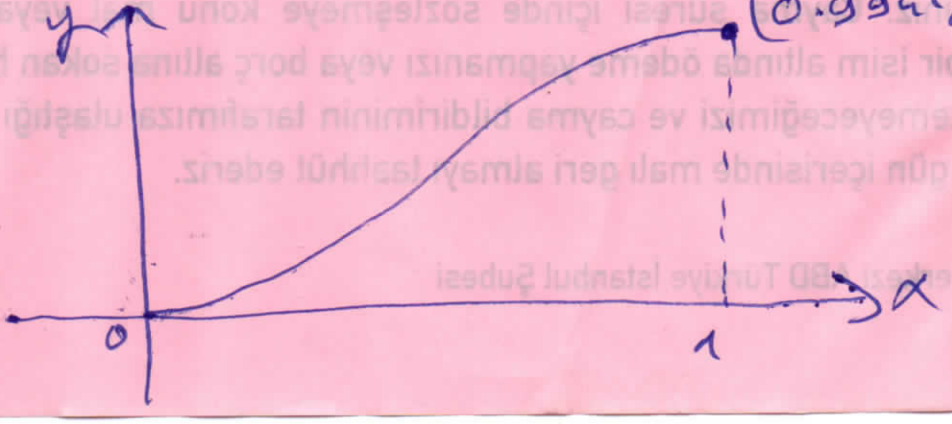
$f''(x) = -2x\cdot\cos(x^2)\cdot 2x + 2\sin(x^2)$

$= -4x^2\cos(x^2) + 2\sin(x^2)$

$y = f(x) = |-4x^2\cos(x^2) + 2\sin(x^2)|$

x , y

$(0.941, 3.8449)$



Simpson $\frac{1}{3}$ Rule

$$\int_{x_0}^{x_n} f(x)\,dx \simeq \frac{h}{3}\left[f_0 + 4f_1 + 2f_2 + \cdots + 4f_{n-1} + f_n \right]$$ (8)

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx, \quad x_0 < x_1 < x_2$$

$(a, f(a)), \ (c, f(c)), \ (b, f(b))$

$$\int_{x_0}^{x_2} f(x)\,dx \simeq \int_{x_0}^{x_2} \left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \right] dx$$

$$= \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2) \right] \ d.i.s.$$

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

$$= \frac{h}{3}\left[f_0 + 4f_1 + f_2 \right] + \frac{h}{3}\left[f_2 + 4f_3 + f_4 \right] + \cdots + \frac{h}{3}\left[f_{n-2} + 4f_{n-1} + f_n \right]$$

$$= \frac{h}{3}\left[f_1 + 4f_1 + 2f_2 + 4f_3 + \cdots + 4f_{n-1} + f_n \right]$$

Simpson $\frac{3}{8}$ kuralı

9

$$\int_a^b f(x)\,dx \simeq \frac{3h}{8}\left[f_0 + 3f_1 + 3f_2 + 2f_3 + 3f_4 + 3f_5 + 2f_6 + \dots + 3f_{n-1} + f_n\right]$$

$$\int_{x_0}^{x_3} f(x)\,dx \simeq \frac{3h}{8}\left[f_0 + 3f_1 + 3f_2 + f_3\right]$$

$$\int_{x_3}^{x_6} f(x)\,dx \simeq \frac{3h}{8}\left[f_3 + 3f_4 + 3f_5 + f_6\right]$$

x_3

$\vdots$

x_n

$$\int_{x_{n-3}}^{x_n} f(x)\,dx \simeq \frac{3h}{8}\left[f_{n-3} + 3f_{n-2} + 3f_{n-1} + f_n\right]$$

Simpson's Rule

(18)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}\left[f(x_0)+4f(x_1)+2f(x_2)+\cdots+4f(x_{n-1})+f(x_n)\right]$$

$$\int_1^2 \frac{dx}{x} = ? \qquad a) \quad n = 10$$

$$\int_1^2 \frac{dx}{x} = \frac{2-1}{3(10)}\left[\frac{1}{1}+\frac{4}{1.1}+\frac{2}{1.2}+\cdots+\frac{4}{1.9}+\frac{1}{2}\right]$$

$$= \frac{1}{30}(20.794507) \approx 693150$$

$$n = 16$$

$$\int_0^2 e^{x^2}\,dx \approx \frac{b-a}{3n}\left[\right.$$

$$= \frac{2-0}{3\cdot 16}\left[f(x_0)+4f(x_1)+\cdots+f(x_{16})\right)$$

$x_0 = 0$

$x_1 = \frac{1}{8}$

$x_2 = \frac{2}{8}$

$\cdots$

$x_{14} = \frac{14}{8}$

$x_{15} = \frac{15}{8}$

$x_{16} = \frac{16}{8}$

$$= \frac{1}{3}\cdot\frac{1}{8}\left[e\right.$$

$$= \frac{1}{24}(395.0118)$$

$$\approx 16.4588$$

Time in Hours, t	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1	$\frac{7}{6}$
Velocity in miles per Hour, $v(t)$	45	55	52	60	64	58	47

$n = 6$, $a = t_0 = \frac{1}{6}$, $b = t_6 = \frac{7}{6}$

By trapozoidal rule:

$$\int_{\frac{1}{6}}^{7/6} v(t)\,dt \approx \frac{\frac{7}{6} - \frac{1}{6}}{6}\left[\frac{1}{2}\cdot 45 + 55 + 52 + 60 + 64 + 58 + \frac{1}{2}\cdot 47\right]$$

$$\approx 55.83$$

By Simpson rule:

$$\int_{\frac{1}{6}}^{7/6} v(t)\,dt \approx \frac{\frac{7}{6} - \frac{1}{6}}{3\cdot 6}\left[45 + 4(55) + 2(52) + 4(60) + 2(64) + 4(58) + 47\right]$$

$$\approx 55.44$$

$$\int_1^5 \frac{1}{x+1}\,dx = \ln(x+1)\Big|_1^5$$

(12)

$$= \ln(6) - \ln(2)$$

$$= \ln\left(\frac{6}{2}\right) = \ln(3)$$

$$= 1.09861$$

$$\left.\begin{aligned} &= 1.79175 \\ &= 0.69314 \end{aligned}\right] 1.09861$$

$h = \frac{b-a}{n}$, $n = 4$

$$\int_1^5 \frac{dx}{x+1} \approx \frac{5-1}{4}\left[\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + f(x_3) + \frac{1}{2}f(x_4)\right]$$

$$= \left[\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{2}\cdot\frac{1}{6}\right]$$

$$= \left[\frac{1}{4} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{12}\right]$$

$$= \left[\frac{1}{2} + \frac{8}{15} + \frac{1}{12}\right]$$

$$= [0.5 + 0.53333 + 0.08333 = 1.11663$$

$$\int_1^5 \frac{dx}{x+1} \simeq \frac{5-1}{4\cdot 3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$$

$$= \frac{1}{3}\left[\frac{1}{2} + 4\frac{1}{3} + 2\frac{1}{4} + 4\frac{1}{5} + \frac{1}{6}\right]$$

$$= \frac{1}{3}\left[\frac{1}{2} + \frac{4}{3} + \frac{1}{2} + \frac{4}{5} + \frac{1}{6}\right]$$

$$= \frac{1}{3}\left[1 + 1.333 + 0.8 + 0.16666\right] = 1.09997$$

$$\int_1^3 e^x \, dx$$


```
for m=1:2
a=1;
b=3;
n= m*600;
h=(b-a)/n;
tplm = 0;
for k=1:n
x= a+k*h; % sağdan dikdörtgen
% x= a+(k-1)*h; % soldan "
% x= a+(k-0.5)*h; ortanokta kuralı
tplm= tplm + exp(x);
end
grck = exp(3)-exp(1);
ytp= h*tplm;
hata= abs(ytp-grck);
fprintf('%12.5f %15.5f %10.5e\n', grck, ytp, hata)
end
```

$$\int_1^3 e^{x^2}\,dx$$

```
a = 1;
b = 3;
n = 600;
tplm = 0;
for k = 1:n
% x = a + k*h;  % sağdan dikdörtgen
% x = a + (k-1)*h; % soldan dikdörtgen
  x = a + (k-0.5)h; % ortanokta kuralı
tplm = tplm + exp(x^2);
end
ytp = h*tplm;
fprintf('snc = %12.5f\n', ytp)
```

```
%% Simpson 1/3 yöntemi (Belirli integral için)
clear all
n= input('n sayisini giriniz = ');
if mod(n,2) != 0
disp('n sayisini çift sayi girmelisiniz!')
break
end
a= 0.6
b= 0.8
h= (b-a)/n
tplm= f(a)+f(b)
for k= 1:(n-1)
    x= a+k*h;
    cf= 3+(-1)^(k+1);
    tplm= tplm + cf*f(x);
end
ytplm= h*tplm/3;
fprintf('%12.5f\n', ytplm)
```

```
function yt= f(x)
yt= exp(-x*x);
```

```
function yt= f(x)
yt = sin(x)/x;
```

In[157]:=

```
(*
 Mathematica ile  Numerik integrasyon
    Trepezium Rule
*)
Clear[f,x,a,b,h,n]
f[x_] = Sin[x]^2 * Exp[-2*x];
a=0;
b=2;
n=10;
h=(b-a)/n;
TrapI=N[h*(0.5*f[a] +  Sum[f[a+i*h],{i,1,n-1}] + 0.5*f[b])  ]
Int = Integrate[f[x], {x,0,2}]//N
TrapI-Int
```

Out[164]=

```
0.120484
```

Out[165]=

```
0.120657
```

Out[166]=

```
-0.00017346
```

```
(*
 Mathematica ile  Numerik integrasyon
    Simpson 1/3  Rule
*)
Clear[f,x,a,b,h,n]
f[x_] = Exp[x^2] * Exp[-2*x];
a=0;
b=1;
n=4;
h=(b-a)/n;
SimpI=N[h*(f[a] +  Sum[(3+(-1)^(i+1))*f[a+i*h],{i,1,n-1}] + f[b])/3 ]
Int = Integrate[f[x], {x,0,1}]//N
Abs[SimpI-Int]
```

Out[146]=

```
0.538469
```

Out[147]=

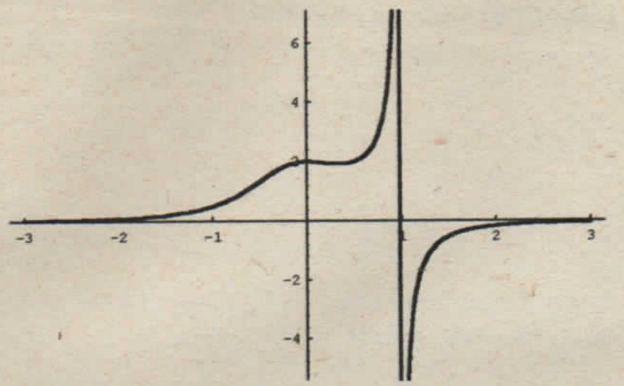
```
0.53808
```

Out[148]=

```
0.00038959
```

In[5]:=

```
Plot[-1/(Cos[x]+x^3 -1.5), {x,-3,3}]
```



Out[5]=

```
-Graphics-
```

In[149]:=

```
(*
 Mathematica ile  Numerik integrasyon
    Rectangle Rule
*)
f[x_]=x^2;
a=1;
b=3;
n=100;
h=(b-a)/n;
N[Sum[h*f[a+i*h],{i,0,99}],4]
N[Integrate[f[x],{x,1,3}],4]
```

Out[155]=

```
8.587
```

Out[156]=

```
8.667
```

10'dan küçük olanları sayı tara