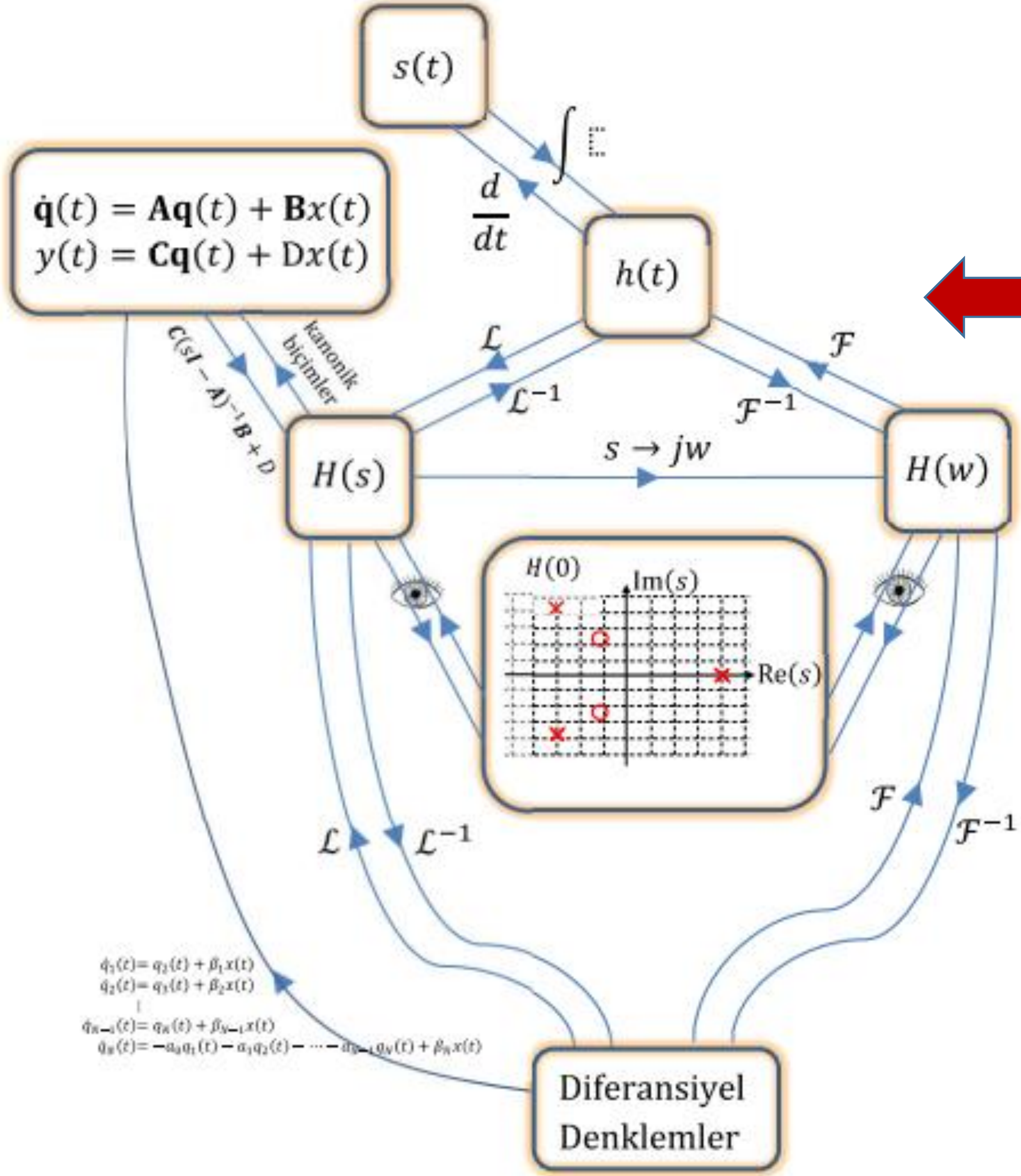
İşaret İşleme

Fourier Serileri ve Fourier Dönüşümü-H10CD2

Dr. Meriç Çetin

versiyon291020

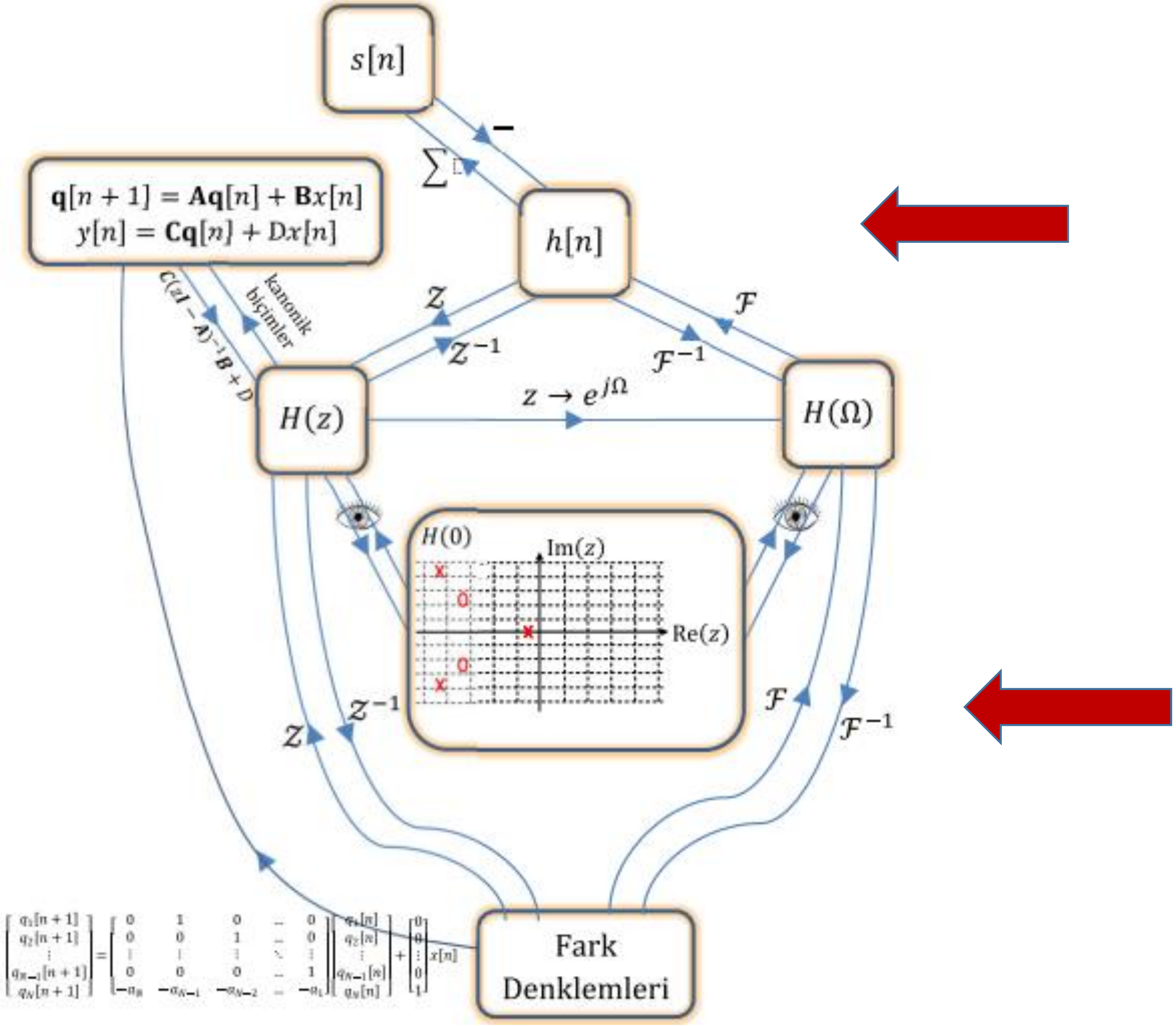
Sürekli-zamanlı sistemlerde giriş-çıkış ilişkileri



EEN343 Sinyaller ve Sistemler

Ders Notları

Prof. Dr. Serdar İplikçi



EEN343 Sinyaller ve Sistemler

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Giriş

- Önceki bölümlerde, zaman domenindeki sürekli ve ayrık-zamanlı sinyallerin Laplace ve z dönüşümleriyle **s ve z domenine** dönüştürerek daha kolay analiz ve işlem yapılabileceği görülmüştü.
- Bunun yanısıra, bu dönüşümler bir çok sinyal ve sistemin özelliklerinin daha iyi kavranmasını sağlar.
- Bu bölümde ilk olarak Fourier serileri görülecek, ardından da sürekli-zaman domenindeki sinyallerin frekans domenine dönüştürülmesi ve bu domende analiz edilmesi görülecektir.
- Fourier serileri, sinyalleri sinüzoidler cinsinden temsil eder. Bu gösterim sistemlerin yeni bir temsille (filtre gibi) gösterilmesini sağlar.

Fourier Serileri

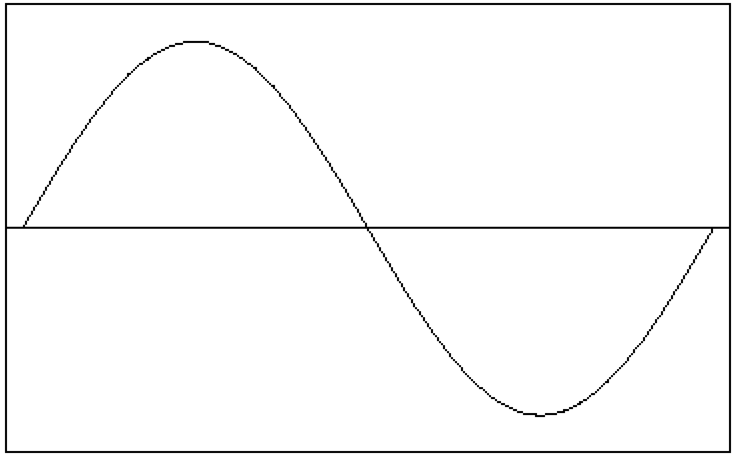
Jean-Baptiste Joseph Fourier (1768-1830)

Fourier Serisi $= \quad f(t) = A_0 + C_1.\cos(\omega t - \phi_1) + \sum_{1}^{\infty} C_k.\cos(k.\omega t - \phi_k)$

DC Bileşen

Temel Bileşen

Harmonikler

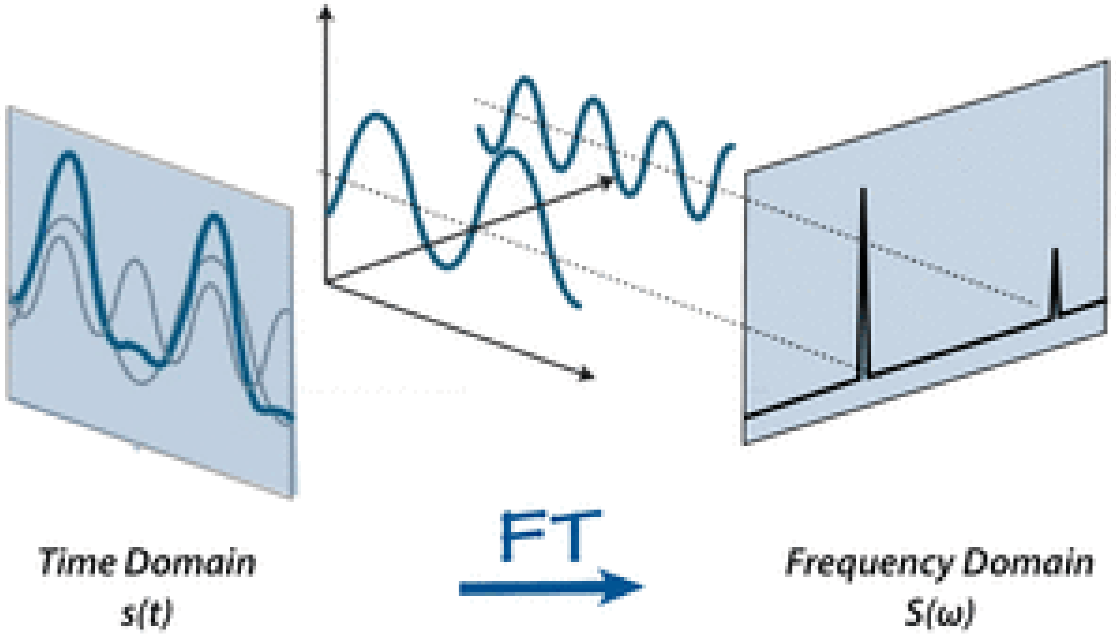


- Fourier showed that any periodic signal *s*(*t*) can be written as a sum of sine waves with various amplitudies, frequencies and phases

$$s(t) = a_0 + a_1 \sin(\omega t + \phi_1) + a_2 \sin(2\omega t + \phi_2) + a_3 \sin(3\omega t + \phi_3) + \cdots$$

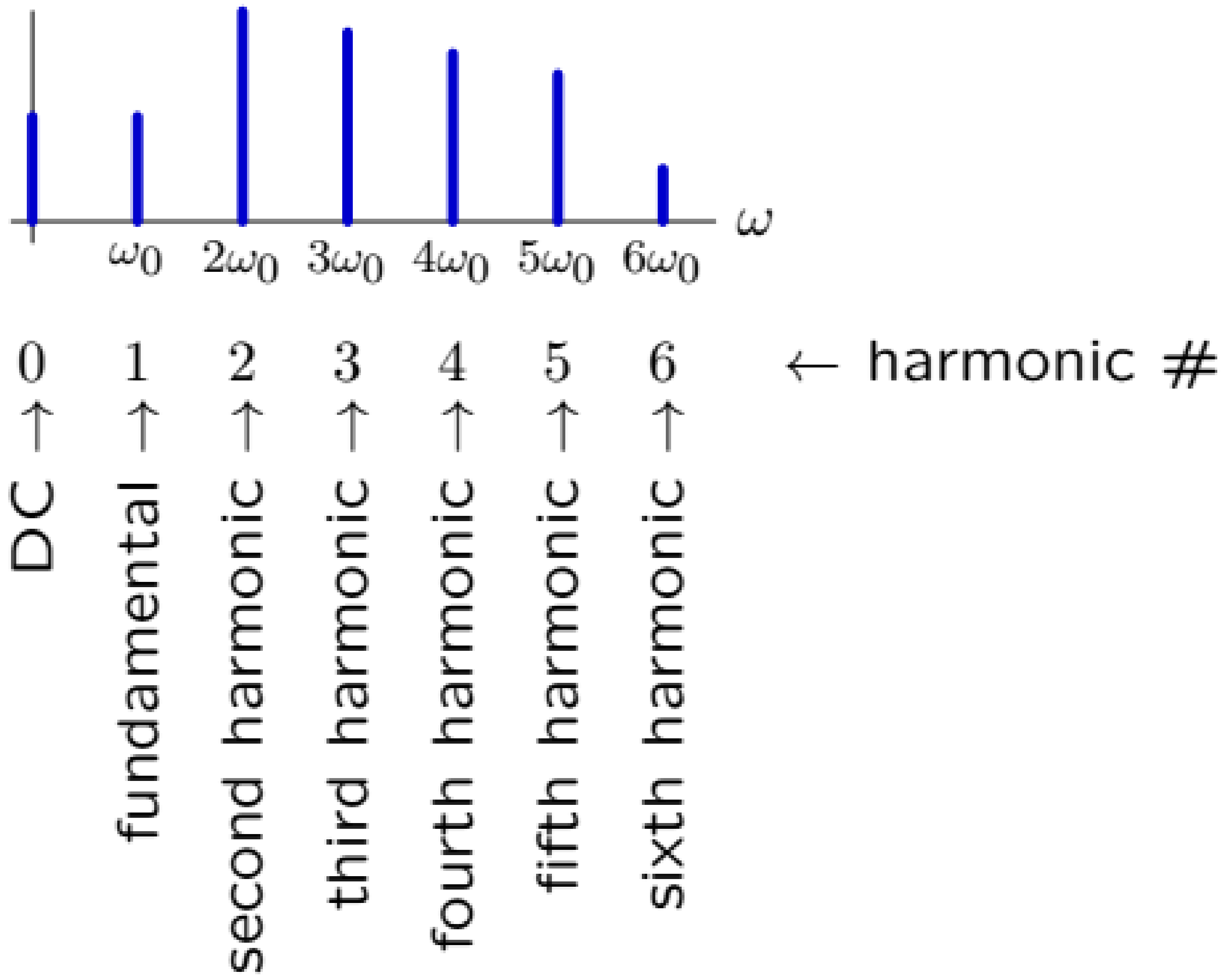
- where ***ai***'s are amplitudes, ***φi***'s are phase shifts, and ***ω*** is the ***fundamental frequency***. The higher order frequencies *2ω*, *3ω*, etc. are called ***harmonics***.

- The ***time domain*** signal of the square wave, s(t), is shown on the left. The so-called ***frequency domain*** representation, *S*(*ω*), is shown on the right. *S*(*ω*) is called the ***Fourier transform*** of s(t)

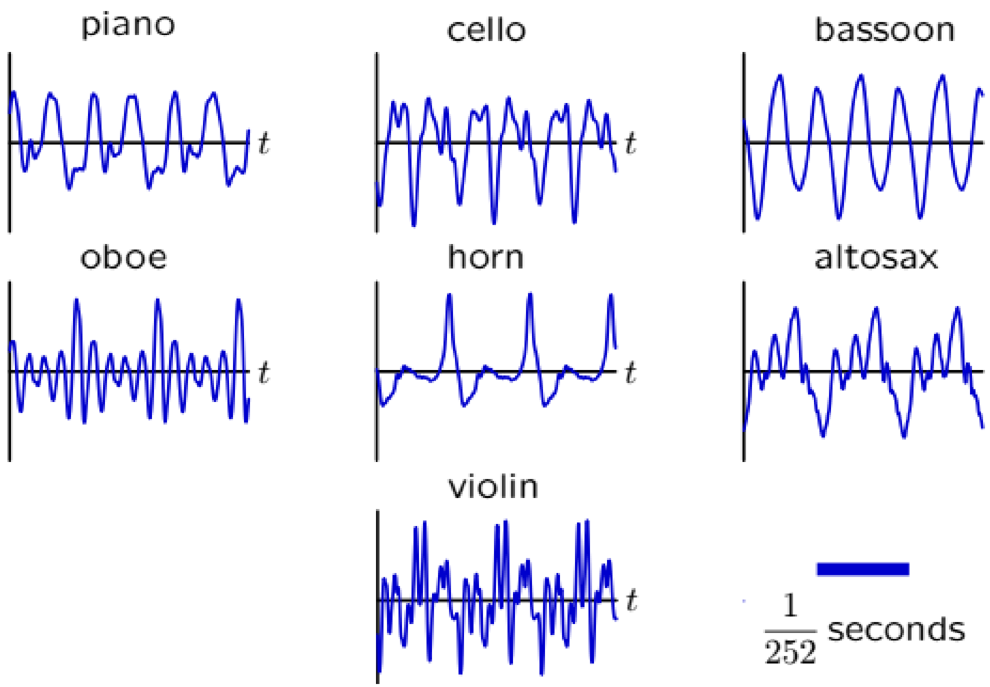


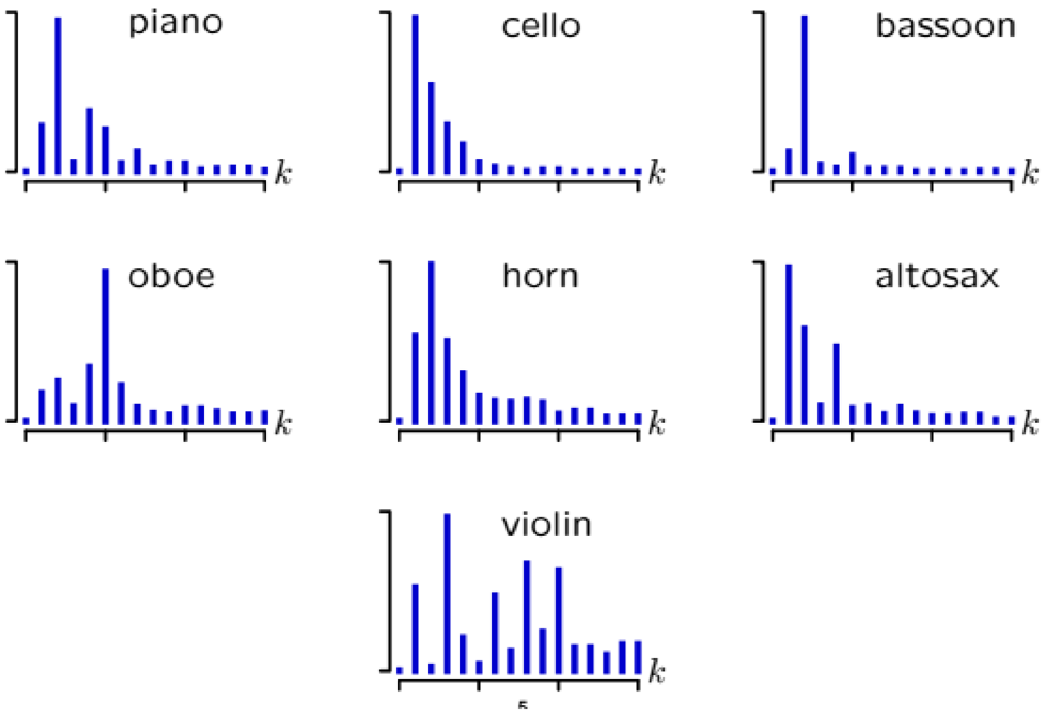
Fourier Series

Representing signals by their harmonic components.



Musical Instruments



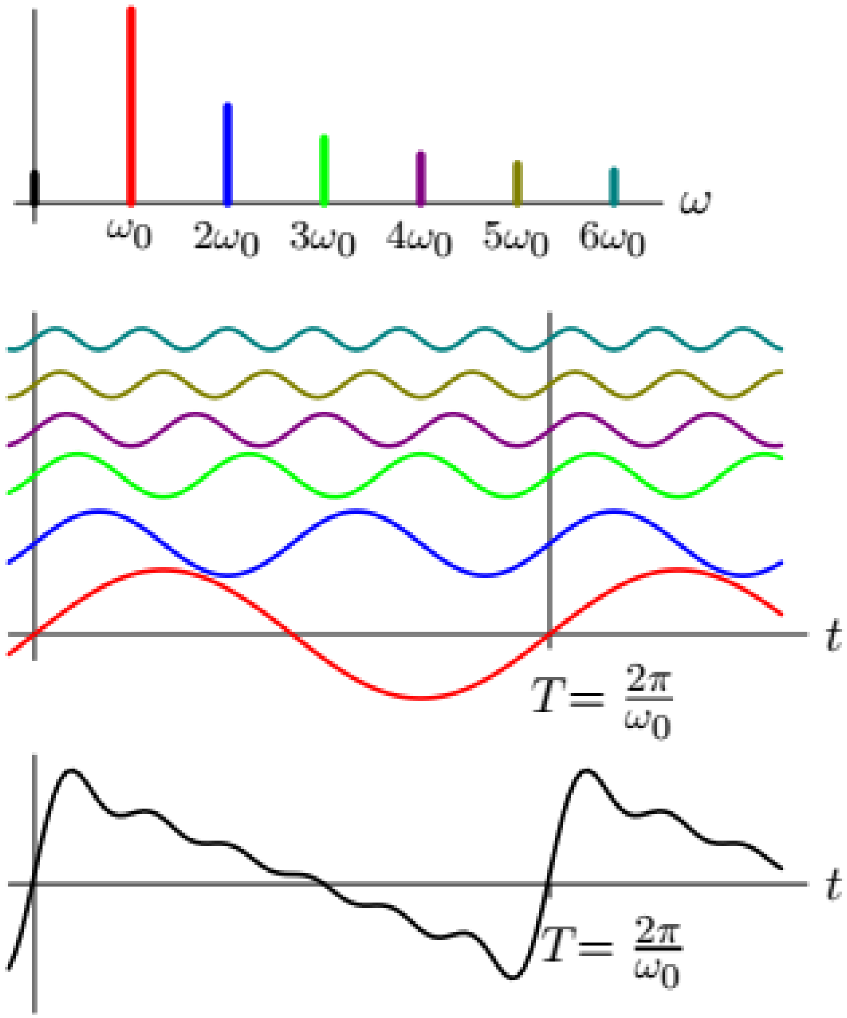


Ex: musical instruments (http://theremin.music.uiowa.edu/MIS.html)

Harmonic Representations

6.003 Signals and Systems
Fall 2011

What signals can be represented by sums of harmonic components?



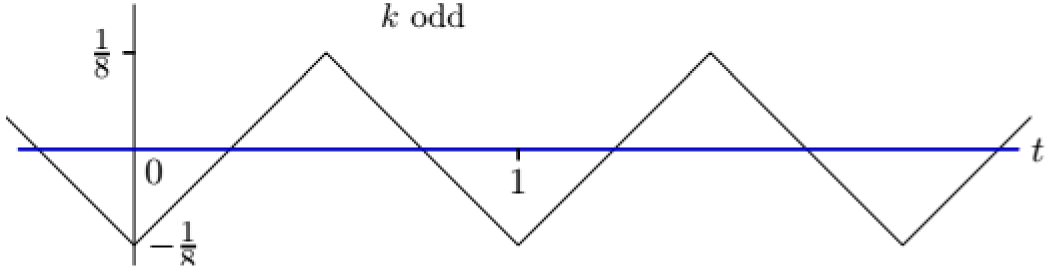
Only periodic signals: all harmonics of ω_0 are periodic in $T = 2\pi/\omega_0$.

Fourier Series

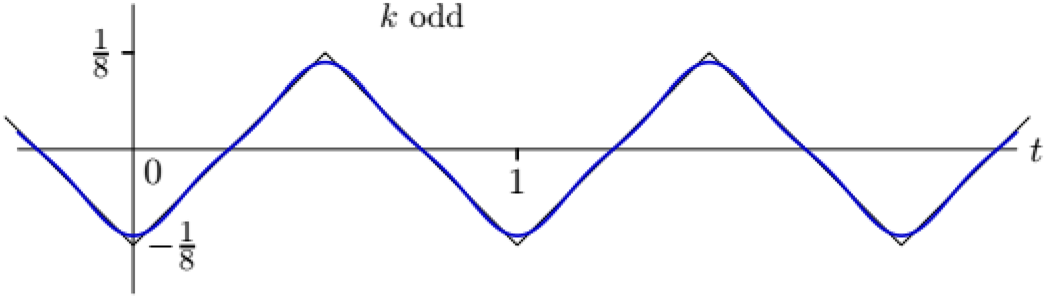
One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

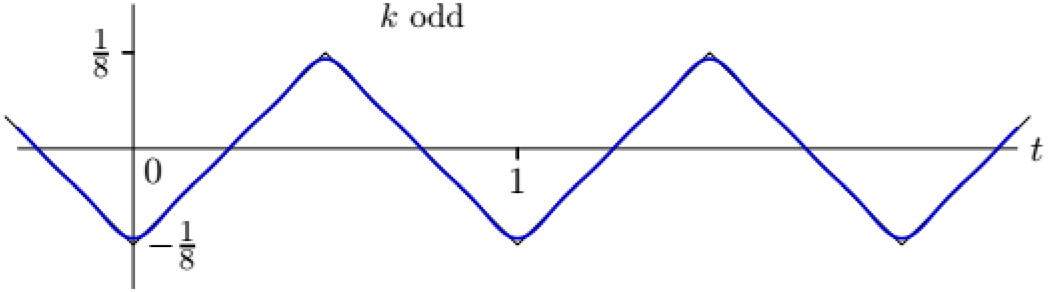
$$\sum_{\substack{k=-0\\ k\ \text{odd}}}^{0} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$



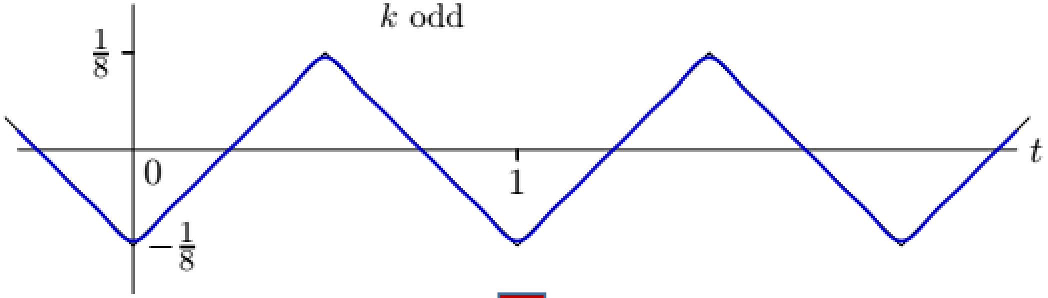
$$\sum_{\substack{k=-3\\ k\ \text{odd}}}^{3} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$



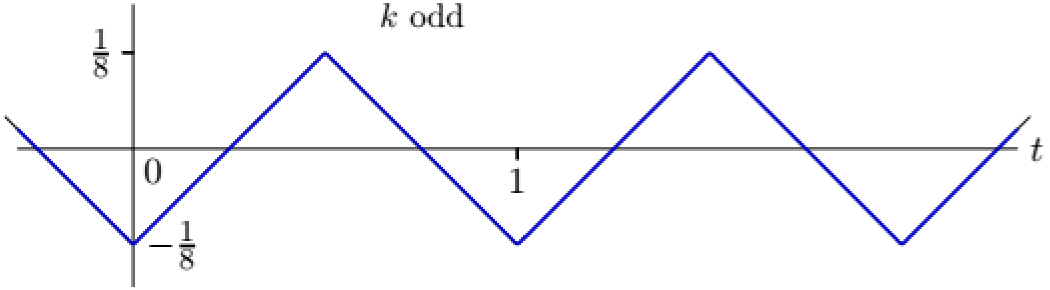
$$\sum_{\substack{k=-5\\ k\ \text{odd}}}^{5} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$



$$\sum_{\substack{k=-7\\ k\ \text{odd}}}^{7} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$



$$\sum_{\substack{k=-39\\ k\ \text{odd}}}^{39} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

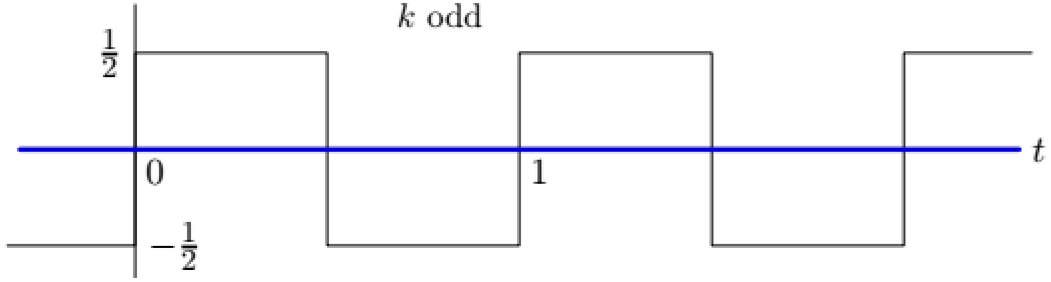


Fourier Series

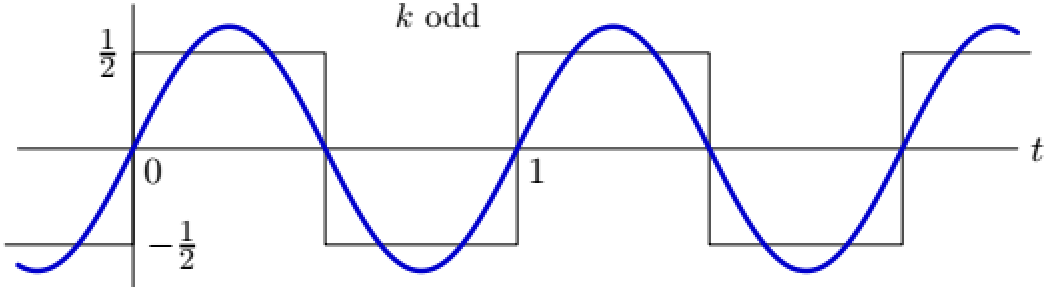
One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

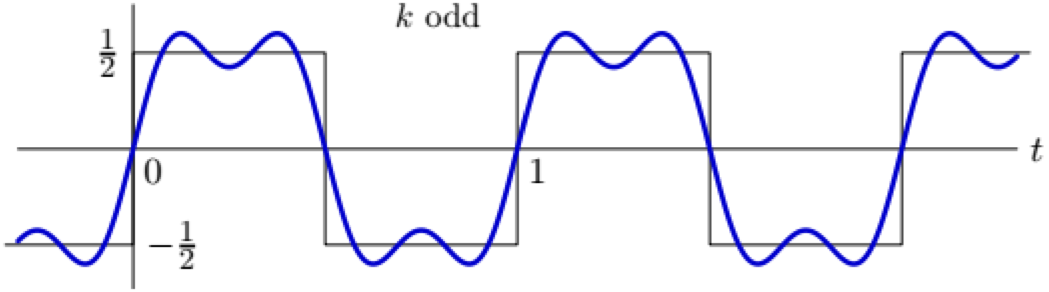
$$\sum_{\substack{k=-0\\ k\text{ odd}}}^{0} \frac{1}{jk\pi} e^{j2\pi kt}$$



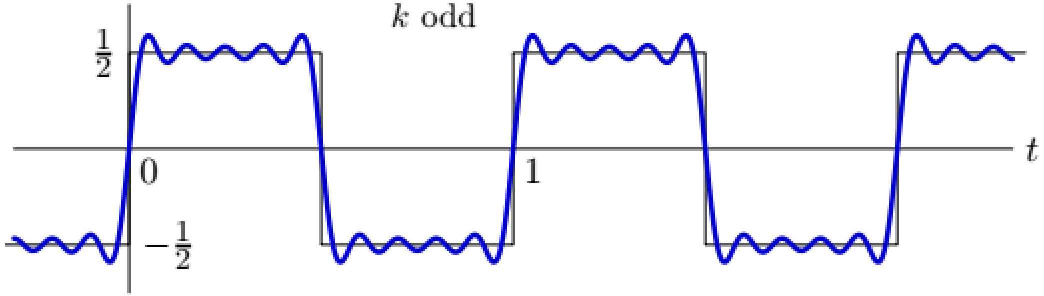
$$\sum_{\substack{k=-1\\ k\text{ odd}}}^{1} \frac{1}{jk\pi} e^{j2\pi kt}$$



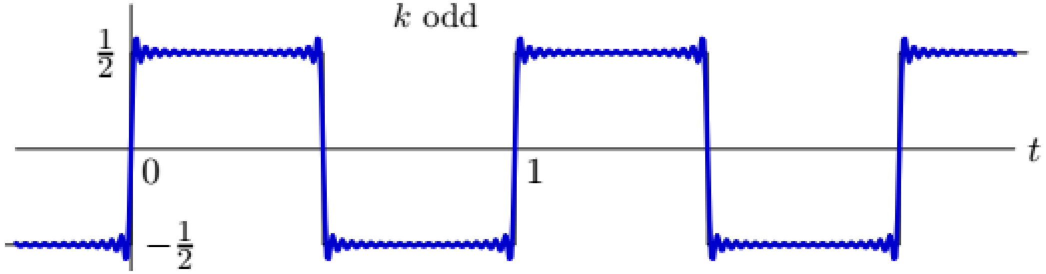
$$\sum_{\substack{k=-3\\ k\text{ odd}}}^{3} \frac{1}{jk\pi} e^{j2\pi kt}$$



$$\sum_{\substack{k=-9\\ k\text{ odd}}}^{9} \frac{1}{jk\pi} e^{j2\pi kt}$$



$$\sum_{\substack{k=-39\\ k\text{ odd}}}^{39} \frac{1}{jk\pi} e^{j2\pi kt}$$



Periyodik bir sinyalin genlik ve faz spektrumu

- **Temel periyodu To olan bir x(t) periyodik sinyalinin karmaşık üstel Fourier serisi gösterilimi:**

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jkw_0 t} \qquad w_0 = \frac{2\pi}{T_0}$$

buradaki c_k katsayılar

$$c_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jkw_0 t} dt$$

Periyodik bir sinyalin genlik ve faz spektrumu

Bu c_k katsayıları karmaşık sayılar olup

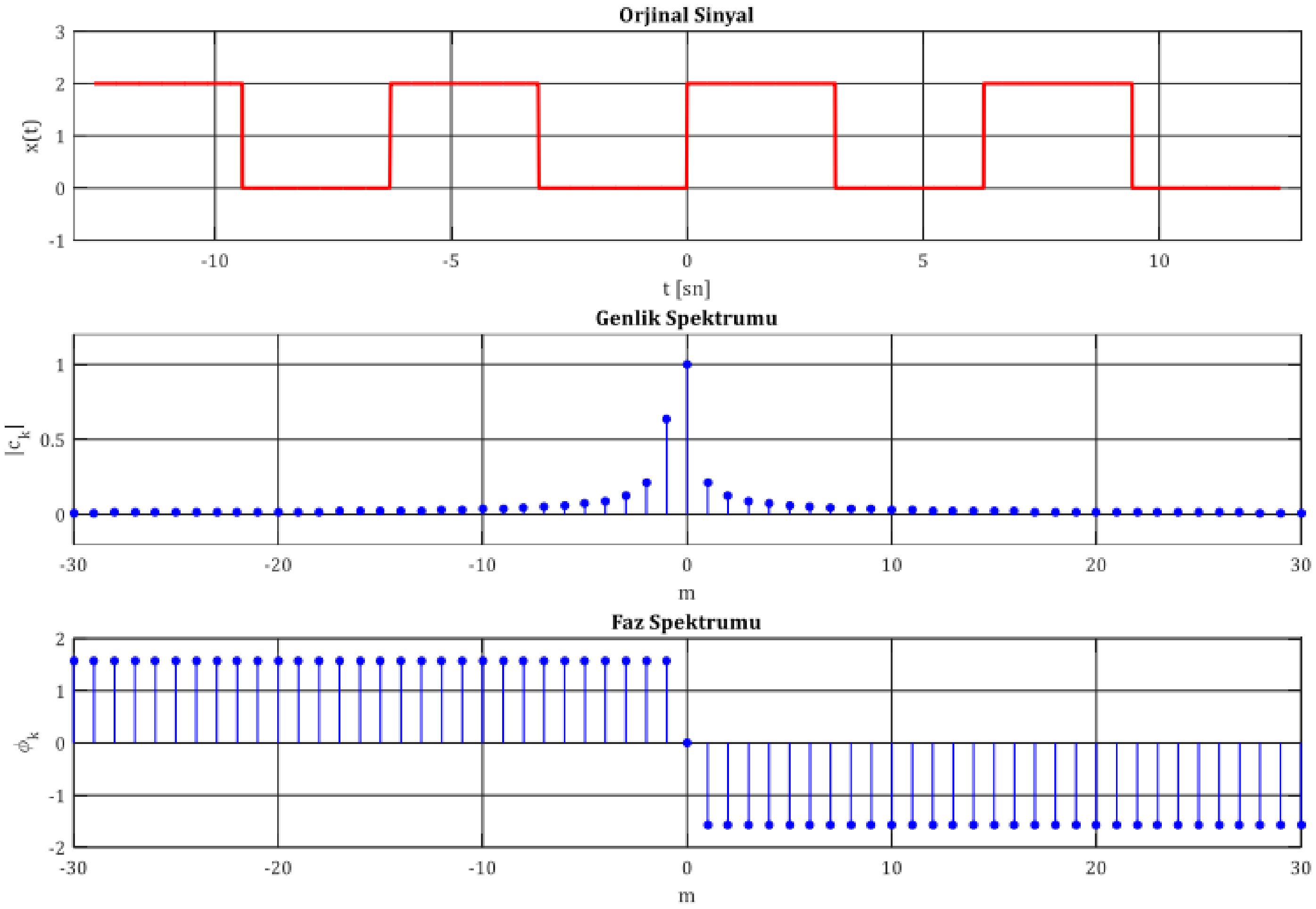
$$c_k = |c_k| e^{j\phi_k}$$

şeklinde ifade edilebilir, burada $|c_k|$ genlik, ϕ_k ise fazdır. Açısal frekans w'a karşı $|c_k|$'nın grafiğine $x(t)$ periyodik sinyalinin *genlik spektrumu*, açısal frekans w'a karşı ϕ_k'nın grafiğine $x(t)$ periyodik sinyalinin *faz spektrumu* denir. k indisi yalnızaca tamsayı değerler aldığı için bu spektrumlar sürekli eğriler olmayıp yalnızca kw_0 ayrık frekans değerlerinde ortaya çıkarlar. Bu nedenle de bunlara *ayrık frekans spektrumları* ya da *çizgi spektrumları* denir. Şekil 5.3'te, karmaşık üstel Fourier serisi katsayıları

Periyodik bir sinyalin genlik ve faz spektrumu

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jkw_0 t}$$

$$c_k = |c_k| e^{j\phi_k}$$



Şekil 5.3 Periyodik bir Sinyalin Genlik ve Faz Spektrumları

Fourier Transformu

Fourier Dönüşümü

Sürekli-zamanlı bir $x(t)$ işaretinin Fourier dönüşümü $X(w) = \mathcal{F}\{x(t)\}$ ile gösterilir

$$x(t) \leftrightarrow X(w) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

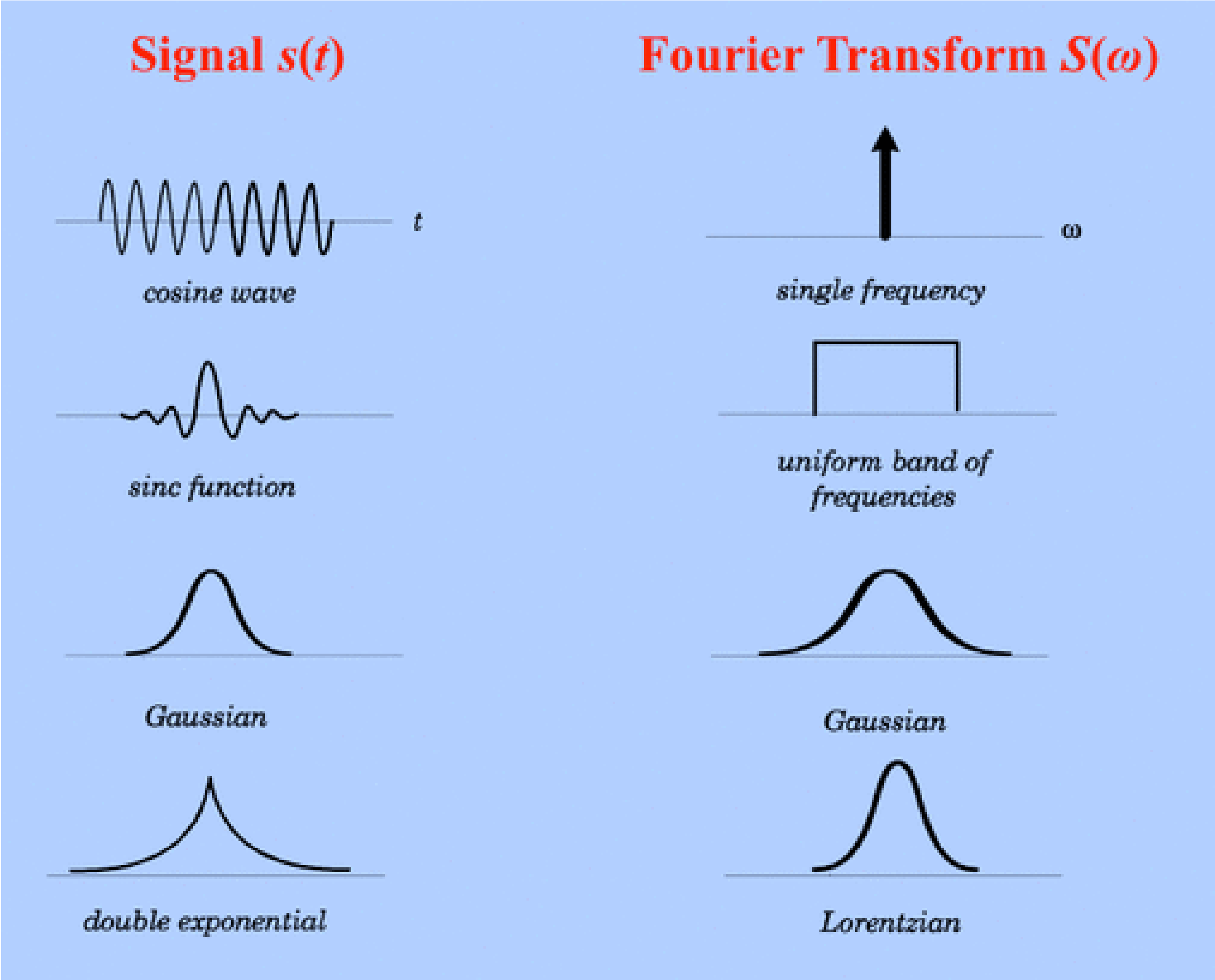
buradaki w değişkeni açısal frekansı temsil eden bağımsız bir değişkendir. Fourier dönüşümünün bulunduğu ortama *frekans-domeni* adı verilmektedir. Benzer şekilde, Fourier dönüşümü $X(w)$ olan sürekli-zamanlı bir $x(t)$ işareti, aşağıdaki gibi verilen ters Fourier dönüşümü ile elde edilir:

$$X(w) \leftrightarrow x(t) = \mathcal{F}^{-1}\{X(w)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(w)e^{jwt}dw.$$

- ***Fourier transformation*** is the mathematical procedure connecting *s(t)* and $S(\omega)$. If *s(t)* is specified, $S(\omega)$ may be computed, and vice versa. The equations require some knowledge of complex numbers and calculus to make sense. Here I will simply provide the defining equations for completeness:

$$S(\omega) = \int_{-\infty}^{\infty} s(t) e^{-i\omega t} dt$$

$$s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{i\omega t} \, d\omega$$



Örnek olarak

- birim darbe fonksiyonunun Fourier dönüşümünü bulalım.

$$x(t) = \delta(t) \leftrightarrow X(w) = \int_{-\infty}^{+\infty} \delta(t) e^{-jwt} dt \quad = 1$$

- $x(t) = e^{-at}u(t)$ sinyalinin $a > 0$ olmak üzere Fourier dönüşümü

$$x(t) = e^{-at}u(t) \leftrightarrow X(w) = \int_{-\infty}^{+\infty} e^{-at}u(t)e^{-jwt} dt$$

$$= \int_{0}^{+\infty} e^{-at}e^{-jwt} dt$$

$$= \int_{0}^{+\infty} e^{-(jw+a)t} dt$$

$$= \frac{-1}{jw+a} e^{-(jw+a)t} \Big|_{0}^{+\infty}$$

$$= \frac{-1}{jw+a} e^{-(jw+a)\infty} - \frac{-1}{jw+a} e^{-(jw+a)0}$$

$$= \frac{-1}{jw+a} e^{-(jw+a)\infty} + \frac{1}{jw+a}$$

$$= \frac{1}{jw+a}$$

Genel Karmaşık Üstel Sinyaller

$$x(t) = e^{(\sigma + j\omega_0)t}.$$

Euler bağıntısı

Genel Karmaşık Üstel Sinyaller

$$x(t) = e^{(\sigma + j\omega_0)t}.$$

Euler bağıntısı

$$x(t) = e^{(\sigma + j\omega_0)t} = e^{\sigma t}\cos\omega_0 t + je^{\sigma t}\sin\omega_0 t.$$

Şimdi de $x(t) = \cos(w_0 t)$ sinyalinin Fourier dönüşümünü bulalım. Bunun için $\cos(w_0 t) = \frac{1}{2}e^{jw_0 t} + \frac{1}{2}e^{-jw_0 t}$ şeklindeki Euler formülünden yararlanacağız. Ancak, öncelikle $e^{\mp jw_0 t}$ sinyalinin Fourier dönüşümünü bulalım. $1 \leftrightarrow 2\pi\delta(w)$ dönüşümüne frekans-domeninde öteleme özelliği olan $e^{jw_0 t}x(t) \leftrightarrow X(w - w_0)$ eşitliğini uygularsak,

$$e^{\mp jw_0 t} \leftrightarrow 2\pi\delta(w \pm w_0)$$

elde edilir. Böylece,

$$\begin{aligned} x(t) = \cos(w_0 t) \leftrightarrow X(w) &= \mathcal{F}\left\{\frac{1}{2}e^{jw_0 t} + \frac{1}{2}e^{-jw_0 t}\right\} \\ &= \frac{1}{2}2\pi\delta(w - w_0) + \frac{1}{2}2\pi\delta(w + w_0) \\ &= \pi\delta(w - w_0) + \pi\delta(w + w_0) \end{aligned}$$

Tablo 5.1 Fourier Dönüşümünün Özellikleri

Özellik	$x(t)$	$X(w)$
	$x(t)$ $x_1(t)$ $x_2(t)$	$X(w)$ $X_1(w)$ $X_2(w)$
Doğrusallık	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(w)+a_2X_2(w)$
Zamanda Öteleme	$x(t-t_0)$	$e^{-jwt_0}X(w)$
Frekans -domeninde Öteleme	$e^{jw_0t}x(t)$	$X(w-w_0)$
Zamanda Ölçekleme	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{w}{a}\right)$
Zamanda Geri Dönüş	$x(-t)$	$X(-w)$
Zamanda Türev	$\frac{d}{dt}x(t)$	$jwX(w)$
Frekans-domeninde Türev	$-jtx(t)$	$\frac{d}{dw}X(w)$
Çifteşlik	$X(t)$	$2\pi x(-w)$
Konvolüsyon	$x_1(t)*x_2(t)$	$X_1(w)X_2(w)$
Çarpma	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(w)*X_2(w)$
Parseval Bağıntısı	$\int_{-\infty}^{+\infty}\lvert x(t)\rvert^2dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\lvert X(w)\rvert^2dw$	

Tablo 5.2 Bazı Fourier Dönüşüm Çiftleri

$x(t)$	$X(w)$	$X(s)$
$\delta(t)$	1	1
$u(t)$	$\pi\delta(w)+\frac{1}{jw}$	$\frac{1}{s}$
$-u(-t)$	$\pi\delta(w)-\frac{1}{jw}$	$\frac{1}{s}$
1	$2\pi\delta(w)$	
$\mathrm{sgn}(t)$	$\frac{2}{jw}$	
$tu(t)$		$\frac{1}{s^2}$
$e^{-at}u(t)$	$\frac{1}{jw+a}$	$\frac{1}{s+a}$
$-e^{-at}u(-t)$	$\frac{1}{jw+a}$	$\frac{1}{s+a}$
$te^{-at}u(t)$	$\frac{1}{(jw+a)^2}$	$\frac{1}{(s+a)^2}$
$-te^{-at}u(-t)$	$\frac{1}{(jw+a)^2}$	$\frac{1}{(s+a)^2}$

$x(t)$	$X(w)$	$X(s)$
$e^{-at}\cos(w_0t)u(t)$	$\frac{jw+a}{(jw+a)^2+w_0^2}$	$\frac{s+a}{(s+a)^2+w_0^2}$
$e^{-at}\sin(w_0t)u(t)$	$\frac{w_0}{(jw+a)^2+w_0^2}$	$\frac{w_0}{(s+a)^2+w_0^2}$
$e^{\mp jw_0t}$	$2\pi\delta(w\pm w_0)$	
$\cos(w_0t)$	$\pi\delta(w-w_0)+\pi\delta(w+w_0)$	
$\sin(w_0t)$	$-j\pi\delta(w-w_0)+j\pi\delta(w+w_0)$	
$e^{-a\|t\|}$	$\frac{2a}{a^2+w^2}$	
$\frac{1}{a^2+t^2}$	$\frac{\pi}{a}e^{-a\|w\|}$	
$P_a(t)$	$2\frac{\sin(aw)}{w}$	
$\frac{\sin(at)}{\pi t}$	$P_a(w)$	
e^{-at^2}	$\sqrt{\frac{\pi}{a}}e^{-\frac{w^2}{4a}}$	
$\sum_{k=-\infty}^{\infty}\delta(t-kT_0)$	$w_0\sum_{k=-\infty}^{\infty}\delta(w-kw_0)$	

Fourier Dönüşümü ile Laplace Dönüşümü Arasındaki İlişki

Bilindiği gibi, sürekli-zamanlı bir $x(t)$ işaretinin Fourier dönüşümü

$$x(t) \leftrightarrow X(w) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

eşitliği ile bulunurken Laplace dönüşümü de

$$x(t) \leftrightarrow X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)e^{-st}dt$$

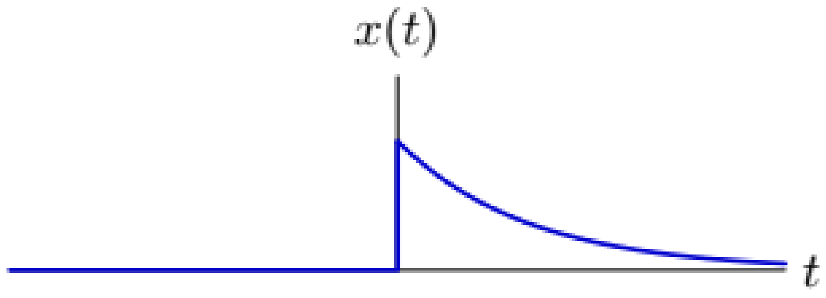
eşitliği ile bulunmaktadır. Bu iki eşitlik karşılaştırıldığında aralarında

$$\mathcal{F}\{x(t)\} = X(w) = X(s)|_{s \to jw}$$

Relation between Fourier and Laplace Transforms

There are also important differences.

Compare Fourier and Laplace transforms of $x(t) = e^{-t}u(t)$.



Laplace transform

$$X(s) = \int_{-\infty}^{\infty} e^{-t}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-(s+1)t}dt = \frac{1}{1+s} \ ; \ \mathrm{Re}(s) > -1$$

a complex-valued function of **complex** domain.

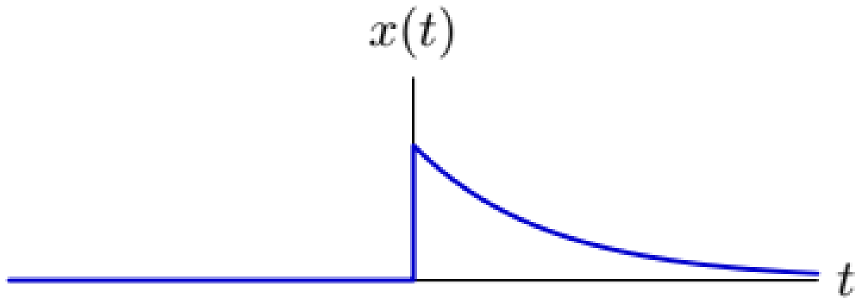
Fourier transform

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-t}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(j\omega+1)t}dt = \frac{1}{1+j\omega}$$

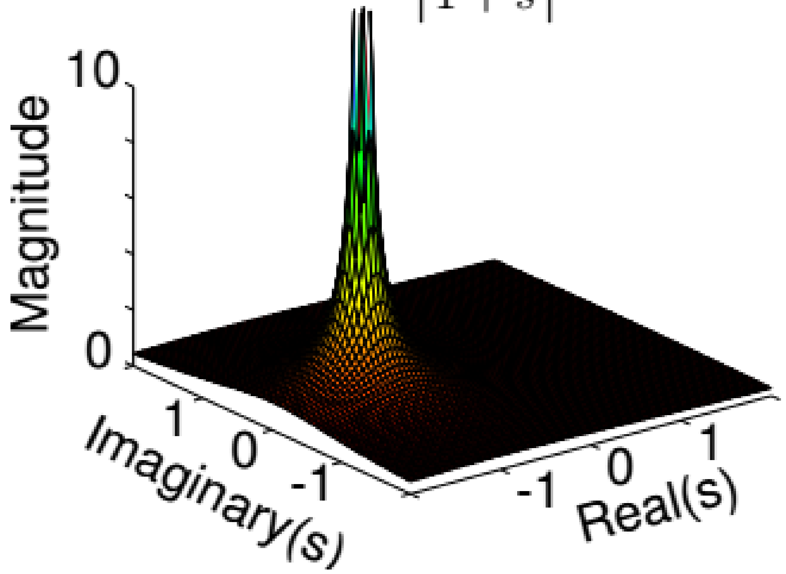
a complex-valued function of **real** domain.

Laplace Transform

The Laplace transform maps a function of time t to a complex-valued function of complex-valued domain s.

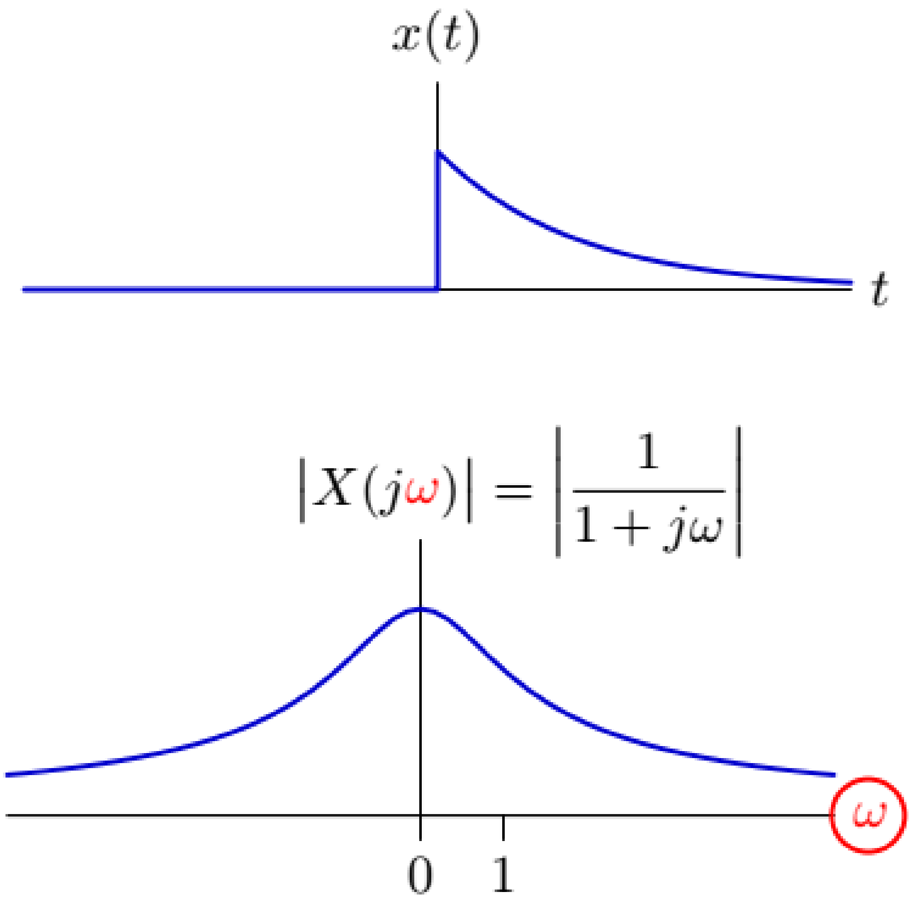


$$|X(s)| = \left|\frac{1}{1+s}\right|$$



Fourier Transform

The Fourier transform maps a function of time t to a complex-valued function of real-valued domain ω.



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Özellik	$x(t)$	$X(w)$
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Doğrusallık	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(w) + a_2X_2(w)$
Zamanda Öteleme	$x(t - t_0)$	$e^{-jwt_0}X(w)$
Frekans -domeninde Öteleme	$e^{jw_0t}x(t)$	$X(w - w_0)$
Zamanda Ölçekleme	$x(at)$	$\frac{1}{\lvert a \rvert} X\left(\frac{w}{a}\right)$
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Konvolüsyon	$x_1(t) * x_2(t)$	$X_1(w)X_2(w)$
Çarpma	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(w) * X_2(w)$
Parseval Bağıntısı	$\int_{-\infty}^{+\infty} \lvert x(t) \rvert^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \lvert X(w) \rvert^2 dw$	

Tablo 5.2 Bazı Fourier Dönüşüm Çiftleri

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$te^{-at}u(t)$	$\frac{1}{(jw+a)^2}$	$\frac{1}{(s+a)^2}$
$-te^{-at}u(-t)$	$\frac{1}{(jw+a)^2}$	$\frac{1}{(s+a)^2}$

$x(t)$	$X(w)$	$X(s)$
$e^{-at}\cos(w_0t)u(t)$	$\frac{jw+a}{(jw+a)^2+w_0^2}$	$\frac{s+a}{(s+a)^2+w_0^2}$
$e^{-at}\sin(w_0t)u(t)$	$\frac{w_0}{(jw+a)^2+w_0^2}$	$\frac{w_0}{(s+a)^2+w_0^2}$
$e^{\mp jw_0t}$	$2\pi\delta(w\pm w_0)$	
$\cos(w_0t)$	$\pi\delta(w-w_0)+\pi\delta(w+w_0)$	
$\sin(w_0t)$	$-j\pi\delta(w-w_0)+j\pi\delta(w+w_0)$	
$e^{-a\|t\|}$	$\frac{2a}{a^2+w^2}$	
$\frac{1}{a^2+t^2}$	$\frac{\pi}{a}e^{-a\|w\|}$	
$P_a(t)$	$2\frac{\sin(aw)}{w}$	
$\frac{\sin(at)}{\pi t}$	$P_a(w)$	
e^{-at^2}	$\sqrt{\frac{\pi}{a}}e^{-\frac{w^2}{4a}}$	
$\sum_{k=-\infty}^{\infty}\delta(t-kT_0)$	$w_0\sum_{k=-\infty}^{\infty}\delta(w-kw_0)$	

Bu ders notu için faydalanılan kaynaklar

EEEN343 Sinyaller ve Sistemler

Ders Notları

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