Yamukler Yönteminin Hata Sınırı

$$T_N = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n) \right]$$

$$T_N = h\left[\frac{f_0}{2} + f_1 + f_2 + \cdots + f_{n-1} + \frac{f_n}{2} \right]$$

$$\text{Error}(T_N) = \left| \int_a^b f(x)\,dx - T_N \right|$$

<u>Teorem:</u> Error bound for T_N. Let K be a number such that $|f''(x)| \le K$ for all $x \in [a,b]$.

Then $\text{Error}(T_N) \le \dfrac{K(b-a)^3}{12n^2}$.

Trapezoidal rula

$$\int_a^b f(x)dx \simeq \frac{b-a}{n}\left[\frac{1}{2}f(x_0)+f(x_1)+\dots+f(x_{n-1})+\frac{1}{2}f(x_n)\right]$$

$$\int_1^2 \frac{dx}{x} = ?$$ a) $n=5$

b) $n=10$

a) $$= \frac{b-a}{n}\left[\frac{1}{2}f(x_0)+f(x_1)+f(x_2)+f(x_3)+f(x_4)+\frac{1}{2}f(x_5)\right]$$

$x_0=1$, $x_1=1.2$, $x_2=1.4$, $x_3=1.6$ $x_4=1.8$ $x_5=2$

$$= \frac{2-1}{5}\left[\frac{1}{2}1+\frac{1}{1.2}+\dots+\frac{1}{2}\cdot\frac{1}{2}\right] = 0.6956$$

b) $\frac{n=10}{/}$ $x_0=1$, $x_1=1.1$, $x_2=1.2$, ... $x_9=1.9$. $x_{10}=2$

$$= \frac{2-1}{10}\left[\frac{1}{2}\cdot 1+\frac{1}{1.1}+\dots+\frac{1}{2}\cdot\frac{1}{10}\right]$$

$= 0.6938$

$$\int_1^2 \frac{dx}{x} = \ln x\Big|_1^2 = \ln 2 - \cancel{\ln(1)}^{0} = \ln 2 = 0.6931$$

THEOREM 4

Error estimates for the Trapezoid and Midpoint Rules

If f has a continuous second derivative on $[a, b]$ and satisfies $|f''(x)| \le K$ there, then

$$\left|\int_a^b f(x)\,dx - T_n\right| \le \frac{K(b-a)}{12}h^2 = \frac{K(b-a)^3}{12n^2},$$

$$\left|\int_a^b f(x)\,dx - M_n\right| \le \frac{K(b-a)}{24}h^2 = \frac{K(b-a)^3}{24n^2},$$

where $h = (b-a)/n$. Note that these error bounds decrease like the square of the subinterval length as n increases.

PROOF We will prove only the Trapezoid Rule error estimate here. (The one for the Midpoint Rule is a little easier to prove; the method is suggested in Exercise 14 below.) The straight line approximating $y = f(x)$ in the first subinterval $[x_0, x_1] = [a, a+h]$ passes through the two points (x_0, y_0) and (x_1, y_1). Its equation is $y = A + B(x - x_0)$, where

$$A = y_0 \qquad \text{and} \qquad B = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1 - y_0}{h}.$$

Let the function $g(x)$ be the vertical distance between the graph of f and this line:

$$g(x) = f(x) - A - B(x - x_0).$$

Since the integral of $A + B(x - x_0)$ over $[x_0, x_1]$ is the area of the first trapezoid, which is $h(y_0 + y_1)/2$ (see Figure 6.20), the integral of $g(x)$ over $[x_0, x_1]$ is the error in the approximation of $\int_{x_0}^{x_1} f(x)\,dx$ by the area of the trapezoid:

$$\int_{x_0}^{x_1} f(x)\,dx - h\frac{y_0 + y_1}{2} = \int_{x_0}^{x_1} g(x)\,dx.$$

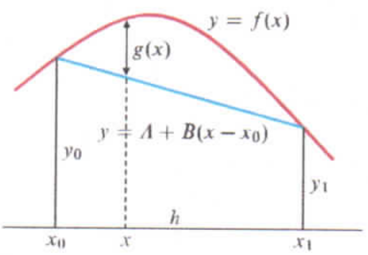

Figure 6.20 The error in approximating the area under the curve by that of the trapezoid is $\int_{x_0}^{x_1} g(x)\,dx$

Now g is twice differentiable, and $g''(x) = f''(x)$. Also $g(x_0) = g(x_1) = 0$. Two integrations by parts (see Exercise 36 of Section 6.1) show that

$$\int_{x_0}^{x_1} (x - x_0)(x_1 - x)\,f''(x)\,dx = \int_{x_0}^{x_1} (x - x_0)(x_1 - x)\,g''(x)\,dx$$
$$= -2\int_{x_0}^{x_1} g(x)\,dx.$$

By the triangle inequality for definite integrals (Theorem 3(f) of Section 5.4),

$$\begin{aligned}\left|\int_{x_0}^{x_1} f(x)\,dx - h\frac{y_0 + y_1}{2}\right| &\le \frac{1}{2}\int_{x_0}^{x_1} (x - x_0)(x_1 - x)\,|f''(x)|\,dx \\ &\le \frac{K}{2}\int_{x_0}^{x_1} \left(-x^2 + (x_0 + x_1)x - x_0x_1\right)dx \\ &= \frac{K}{12}(x_1 - x_0)^3 = \frac{K}{12}h^3.\end{aligned}$$

A similar estimate holds on each subinterval $[x_{j-1}, x_j]$ $(1 \le j \le n)$. Therefore,

$$\begin{aligned}\left|\int_a^b f(x)\,dx - T_n\right| &= \left|\sum_{j=1}^{n}\left(\int_{x_{j-1}}^{x_j} f(x)\,dx - h\frac{y_{j-1} + y_j}{2}\right)\right| \\ &\le \sum_{j=1}^{n}\left|\int_{x_{j-1}}^{x_j} f(x)\,dx - h\frac{y_{j-1} + y_j}{2}\right| \\ &= \sum_{j=1}^{n}\frac{K}{12}h^3 = \frac{K}{12}nh^3 = \frac{K(b-a)}{12}h^2,\end{aligned}$$

since $nh = b - a$.

We illustrate this error estimate for the approximations of Examples 1 and 2 above.

EXAMPLE 3 Obtain bounds for the errors for T_4, T_8, T_{16}, M_4, and M_8 for $I = \int_1^2 \frac{1}{x}\,dx$.

Solution If $f(x) = 1/x$, then $f'(x) = -1/x^2$ and $f''(x) = 2/x^3$. On $[1, 2]$ we have $|f''(x)| \le 2$, so we may take $K = 2$ in the estimate. Thus,

$$|I - T_4| \le \frac{2(2-1)}{12}\left(\frac{1}{4}\right)^2 = 0.0104\ldots,$$

$$|I - M_4| \le \frac{2(2-1)}{24}\left(\frac{1}{4}\right)^2 = 0.0052\ldots,$$

$$|I - T_8| \le \frac{2(2-1)}{12}\left(\frac{1}{8}\right)^2 = 0.0026\ldots,$$

$$|I - M_8| \le \frac{2(2-1)}{24}\left(\frac{1}{8}\right)^2 = 0.0013\ldots,$$

$$|I - T_{16}| \le \frac{2(2-1)}{12}\left(\frac{1}{16}\right)^2 = 0.00065\ldots.$$

The actual errors calculated earlier are considerably smaller than these bounds, because $|f''(x)|$ is rather smaller than $K = 2$ over most of the interval $[1, 2]$.

Remark Error bounds are not usually as easily obtained as they are in Example 3. In particular, if an exact formula for $f(x)$ is not known (as is usually the case if the values of f are obtained from experimental data), then we have no method of calculating $f''(x)$, so we can't determine K. Theorem 4 is of more theoretical than practical importance. It shows us that, for a "well-behaved" function f, the Midpoint Rule error is typically about half as large as the Trapezoid Rule error and that both the Trapezoid Rule and Midpoint Rule errors can be expected to decrease like $1/n^2$ as n increases; in terms of big-O notation,

$$I = T_n + O\left(\frac{1}{n^2}\right) \quad \text{and} \quad I = M_n + O\left(\frac{1}{n^2}\right) \qquad \text{as } n \to \infty.$$

Of course, actual errors are not equal to the error bounds, so they won't always be cut to exactly a quarter of their size when we double n.

EXERCISES 6.6

In Exercises 1–4, calculate the approximations T_4, M_4, T_8, M_8, and T_{16} for the given integrals. (Use a scientific calculator or computer spreadsheet program.) Also calculate the exact value of each integral, and so determine the exact error in each approximation. Compare these exact errors with the bounds for the size of the error supplied by Theorem 4.

1. $I = \int_0^2 (1 + x^2)\,dx$

2. $I = \int_0^1 e^{-x}\,dx$

3. $I = \int_0^{\pi/2} \sin x\,dx$

4. $I = \int_0^1 \frac{dx}{1 + x^2}$

5. Figure 6.21 shows the graph of a function f over the interval $[1, 9]$. Using values from the graph, find the Trapezoid Rule estimates T_4 and T_8 for $\int_1^9 f(x)\,dx$.

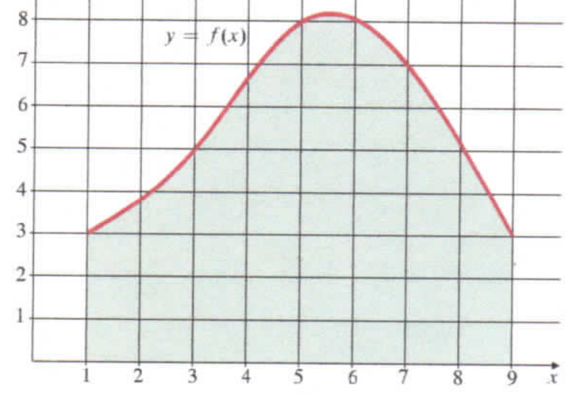

Figure 6.21

$$k = \sum_{n=1}^{2006} n^{2005};$$

```
Mod[k, 2005]
```

```
1
```

$$\int a^x \, dx$$

$$\frac{a^x}{\text{Log}[a]}$$

$$\int 3^x \, dx$$

$$\frac{3^x}{\text{Log}[3]}$$

$$\int_2^3 3^x \, dx$$

$$\frac{18}{\text{Log}[3]}$$

$$\int x^x \, dx$$

$$\int x^x \, dx$$

$$\int_2^3 x^x \, dx$$

$$\int_2^3 x^x \, dx$$

$$\int_2^3 x^x \, dx \; // \; N$$

```
11.6747
```

```
Clear[g, x, a, b, exact, approx, hata, h, n]
g[x_] = x^x;
a = 2;
b = 3;
n = 8;
h = (b - a) / n;
trap = N[ h * ( g[a]  + g[b] + Sum[ (3 + (-1) ^ (i - 1)) * g[a + h * i], {i, 1, n - 1}]) / 3 ]
```

```
11.675
```

```
Clear[f, x, a, b, exact, approx, hata]
f[x_] = x^2;
a = 1;
b = 3;
n = 10;
h = (b - a) / n;
approx = N[Sum[h * f[a + h * i], {i, 0, n - 1}]]
exact = N[Integrate[x^2, {x, 1, 3}]]
hata = exact - approx
```

```
7.88
```

```
8.66667
```

```
0.786667
```

```
Clear[g, x, a, b, exact, approx, hata, h, n]
g[x_] = x * Exp[-x];
a = 0;
b = 3;
n = 20;
h = (b - a) / n;
approx = N[Sum[h g[a + h * i], {i, 0, n - 1}]]
exact = N[Integrate[x * Exp[-x], {x, 0, 3}]]
hata = exact - approx
```

```
0.78759
```

```
0.800852
```

```
0.0132617
```

```
Clear[g, x, a, b, exact, approx, hata, h, n]
g[x_] = Sin[x]^2 * Exp[-2 * x];
a = 0;
b = 2;
n = 10;
h = (b - a) / n;
trap = N[ h * ( 0.5 * g[a]  + 0.5 * g[b] + Sum[ g[a + h * i], {i, 1, n - 1}]) ]
int = N[Integrate[g[x], {x, 0, 2}]]
hata = exact - approx
```

```
0.120484
```

```
0.120657
```

```
-approx + exact
```

```
Clear[g, x, a, b, exact, approx, hata, h, n]
g[x_] = Exp[-x * x];
a = 0;
b = 1;
n = 4;
h = (b - a) / n;
trap = N[h * ( g[a]  + g[b] + Sum[ (3 + (-1) ^ (i - 1)) * g[a + h * i], {i, 1, n - 1}]) / 3 ]
int = N[Integrate[g[x], {x, 0, 2}]]
hata = int - trap
```

0.746855

0.882081

0.135226

```
DSolve[{y'[t] == (t + y[t]) / t, y[2] == 2}, y[t], t]
```

{{y[t] → t - t Log[2] + t Log[t]}}

```
(*Ali Bey'den*)
```

$\int_2^4 \mathrm{Exp}[-x] * x^2 \, dx$

$\frac{2\,(-13 + 5\,e^2)}{e^4}$

$N\left[\frac{2\,(-13 + 5\,e^2)}{e^4}, 10\right]$

0.8771462213

$\frac{18}{\mathrm{Log}[3]}$ // N

16.3843

```
f[x_] := Cos[x^2];
a = 1; b = 3; m = 4;
h = (b - a) / m;
numrect = Sum[h * f[a + i * h], {i, 1, m}] // N;
nummid = Sum[h * f[a + (i - 0.5) * h], {i, 1, m}] // N;
numtrap = N[h * (0.5 * f[a] + Sum[f[a + i * h], {i, 1, m - 1}] + 0.5 * f[b])];
numSimp3h = N[(h / 3) * (f[a] + Sum[ (3 + (-1) ^ (i + 1)) * f[a + i * h], {i, 1, m - 1}] + f[b])]
exa = Integrate[f[x], {x, a, b}] // N
err1 = Abs[numrect - exa];
err2 = Abs[nummid - exa];
err3 = Abs[numtrap - exa];
err4 = Abs[numSimp3h - exa]
```

-0.0321687

-0.201661

0.169492

```
2 / 180 * 3 / 4 * 1 / 3^4 // N
```

0.000102881

$$N\left[\int_0^2 \left(\frac{2}{t^2+4}\right) dt,\ 15\right]$$

0.785398163397448

```
N[0.785398163397448 - 0.78539794523401, 15]
```

2.18163×10^{-7}

```
Clear[t]
f[t_] := 2/(t^2 + 4);
a = 0; b = 2; n = 6;
h = (b - a) / n;
numSimp3h =
 N[(h / 3) * (f[a] + Sum[(3 + (-1) ^ (j + 1)) * f[a + j * h], {j, 1, n - 1}] + f[b]), 8]
exa = Integrate[f[t], {t, a, b}] // N
err = Abs[numSimp3h - exa]
```

0.78539795

0.785398

2.18163×10^{-7}