İşaret İşleme
Zaman Domaini Analizi-H3CD2

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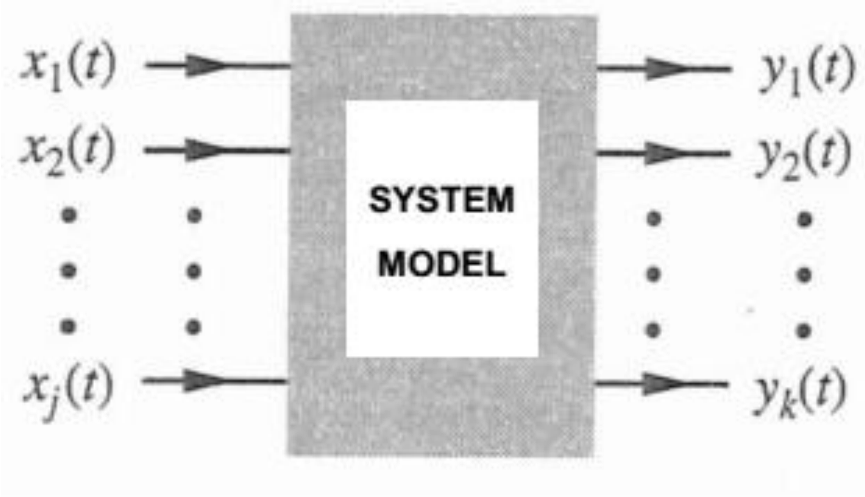
Sürekli zamanlı sistemleri tekrar hatırlayalım



In general, relationship between *x(t)* and *y(t)* in a **linear time-invariant** (**LTI**) differential system is given by (where all coefficients a_i and b_i are constants):

$$\frac{d^N y}{dt^N} + a_1 \frac{d^{N-1} y}{dt^{N-1}} + \cdots + a_{N-1} \frac{dy}{dt} + a_N y(t)$$

$$= b_{N-M} \frac{d^M x}{dt^M} + b_{N-M+1} \frac{d^{M-1} x}{dt^{M-1}} + \cdots + b_{N-1} \frac{dx}{dt} + b_N x(t)$$

Use compact notation **D** for **operator *d/dt***,

$$\frac{dy}{dt} \equiv Dy(t) \text{ and } \frac{d^2 y}{dt^2} \equiv D^2 y(t)$$

We get: $$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t)$$

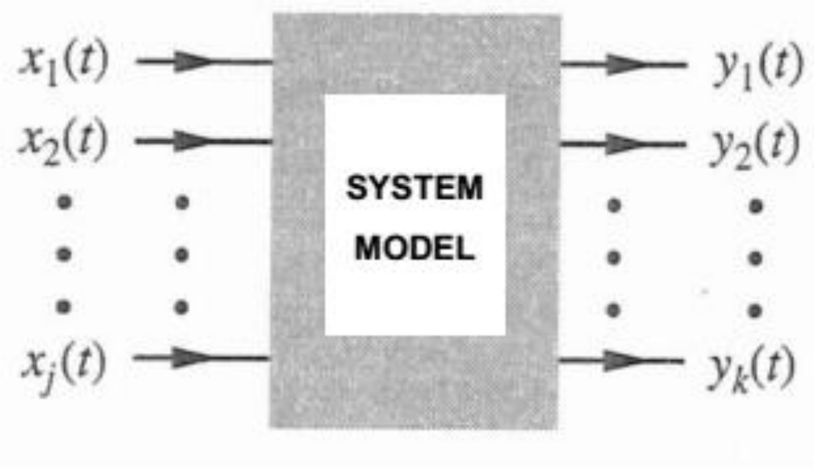
$$= (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) x(t)$$

or $$Q(D) y(t) = P(D) x(t)$$

$$Q(D) = D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N$$

$$P(D) = b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N$$

Bir sistemin toplam tepkisi - (Total response)



Remember that for a Linear System

Total response = zero-input response + zero-state response

Sıfır Giriş Cevabı (SGC)-(Zero input response)

In this lecture, we will focus on a linear system's **zero-input response**, $y_0(t)$, which is the solution of the system equation when input $x(t) = 0$.

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t) = (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) x(t)$$

$$\Rightarrow \quad Q(D) y(t) = P(D) x(t)$$

$$\Rightarrow \quad Q(D) y_0(t) = 0$$

$$\Rightarrow \quad (D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y_0(t) = 0$$

SGC için genel çözüm-1

From maths course on differential equations, we may solve the equation:

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y_0(t) = 0 \quad \ldots\ldots\ldots \quad (3.1)$$

by letting $y_0(t) = ce^{\lambda t}$, where **c and λ are constants**

Then:

$$\left.\begin{aligned} Dy_0(t) &= \frac{dy_0}{dt} = c\lambda e^{\lambda t} \\ D^2 y_0(t) &= \frac{d^2 y_0}{dt^2} = c\lambda^2 e^{\lambda t} \\ &\vdots \\ D^N y_0(t) &= \frac{d^N y_0}{dt^N} = c\lambda^N e^{\lambda t} \end{aligned}\right\}$$

Substitute into (3.1)

SGC için genel çözüm-2

We get:

$$c(\lambda^N + a_1\lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N)e^{\lambda t} = 0 \quad \Rightarrow$$

$$\lambda^N + a_1\lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N = 0$$

This is identical to the polynomial Q(D) **with λ replacing D**, i.e.

$$Q(\lambda) = 0$$

We can now express Q(λ) in **factorized form**:

$$Q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_N) = 0 \quad \text{..........} \quad (3.2)$$

Therefore **λ has N solutions**: λ_1, λ_2, ... , λ_N, assuming that all λ_i are distinct.

Characteristic Polynomial of a system

$Q(\lambda)$ is called the **characteristic polynomial** of the system

$Q(\lambda) = 0$ is the **characteristic equation** of the system

The **roots** to the characteristic equation $Q(\lambda) = 0$, i.e. $\lambda_1, \lambda_2, \ldots, \lambda_N$, are extremely important.

They are called by different names:

- Characteristic values
- **Eigenvalues**
- **Natural frequencies**

The exponentials $e^{\lambda_i t} (i = 1, 2, \ldots, n)$ are the **characteristic modes** (also known as **natural modes**) of the system

Characteristics modes determine the system's behaviour

Q(λ) is called the **characteristic polynomial** of the system

The **roots** to the characteristic equation $Q(\lambda) = 0$, i.e. $\lambda_1, \lambda_2, \ldots, \lambda_N$, are extremely important.

- **NOT:**
  - **Sistemin karakteristik polinomuna ait kökler (λ) 3 farklı durumda olabilir.**
    - **Katsız (tekrar etmeyen) kök**
    - **Katlı (tekrar eden) kök**
    - **Kompleks kök**
  - **Her 3 durum için sistemin sıfır giriş cevabı farklı genel çözümlerle bulunur.**

SGC için genel çözüm-3

Therefore, equation (3.1): $(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y_0(t) = 0$

has **N possible solutions**: $c_1 e^{\lambda_1 t}, c_2 e^{\lambda_2 t}, \ldots, c_N e^{\lambda_N t}$

where $c_1, c_2, \ldots, c_N$ are arbitrary constants.

It can be shown that the **general solution** is the sum of all these terms:

$$y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_N e^{\lambda_N t}$$

Katsız kök olduğunda

In order to determine the N arbitrary constants, we need to have **N constraints** (i.e. initial or boundary or auxiliary conditions).

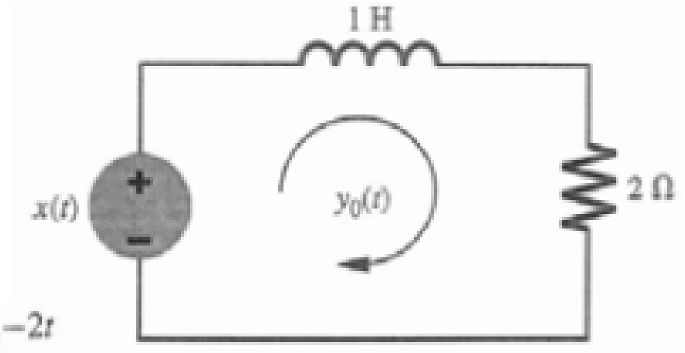
- Sistemin karakteristik polinomuna ait kökler (λ_i) Katsız (tekrar etmeyen) kökler olduğunda sistemin sıfır giriş cevabına ait genel çözüm;

$$y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_N e^{\lambda_N t}$$

Bir Örnek;

- This example demonstrates that any combination of characteristic modes can be sustained by the system with no external input.
- Consider this RL circuit:
- The loop equation is: $(D+2)y(t) = x(t)$
- It has a single characteristic root λ = -2,

  and the characteristic mode is e^{-2t}
- Therefore, the loop current equation is $y(t) = ce^{-2t}$



The loop current is sustained by the RL circuit on its own without any external input.

Sistemin karakteristik polinomuna ait kökler (λ) **katlı** yani **tekrar eden** durumda ise sistemin sıfır giriş cevabı için genel çözüm formu değişecektir.

Sıfır Giriş Cevabı Bulunurken Tekrar Eden Kökler Varsa;

- The discussions so far assume that all characteristic roots are **distinct**. If there are **repeated roots**, the form of the solution is modified.
- The solution of the equation:

$$(D-\lambda)^2 y_0(t) = 0$$

is given by:

$$y_0(t) = (c_1 + c_2 t)e^{\lambda t}$$

- In general, the characteristic modes for the differential equation:

$$(D-\lambda)^r y_0(t) = 0$$

are:

$$e^{\lambda t}, te^{\lambda t}, t^2 e^{\lambda t}, \ldots, t^{r-1} e^{\lambda t}$$

- The solution for $y_0(t)$ is

$$y_0(t) = (c_1 + c_2 t + \cdots + c_r t^{r-1})e^{\lambda t}$$

Tekrar Eden Kökler için Bir Örnek

Find $y_0(t)$, the zero-input component of the response for a LTI system described by the following differential equation: $(D^2 + 6D + 9) = (3D + 5)x(t)$

when the initial conditions are $y_0(0) = 3, \quad \dot{y}_0(0) = -7.$

- The **characteristic polynomial** for this system is:

$$\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2$$

- The **repeated roots** are therefore $\lambda_1 = -3$ and $\lambda_2 = -3$.
- The **zero-input response** is $y_0(t) = (c_1 + c_2 t)e^{-3t}$
- Now, determine the constants using the initial conditions gives $c_1 = 3$ and $c_2 = 2$.
- Therefore: $y_0(t) = (3 + 2t)e^{-3t} \qquad t \geq 0$

L2.2 p156

- Sistemin karakteristik polinomuna ait kökler (λ_i) Katlı (tekrar eden) kökler olduğunda sistemin sıfır giriş cevabına ait genel çözüm;

$$y_0(t) = (c_1 + c_2 t + \cdots + c_r t^{r-1})e^{\lambda t}$$

Sistemin karakteristik polinomuna ait kökler (λ) kompleks ise sistemin sıfır giriş cevabı için genel çözüm formu değişecektir.

Sıfır Giriş Cevabı Bulunurken Kompleks Kökler Varsa;

- Solutions of the characteristic equation may result in **complex roots**.
- For real (i.e. physically realizable) systems, all complex roots must occur in **conjugate pairs**. In other words, the coefficients of the characteristic polynomial $Q(\lambda)$ are real.
- In other words, if $\alpha + j\beta$ is a root, then there **must exists** the root $\alpha - j\beta$.
- The zero-input response corresponding to this pair of conjugate roots is:

$$y_0(t) = c_1 e^{(\alpha+j\beta)t} + c_2 e^{(\alpha-j\beta)t}$$

- For a real system, the response $y_0(t)$ must also be real. This is possible only if c_1 and c_2 are conjugates too.
- Let $c_1 = \frac{c}{2}e^{j\theta}$ and $c_2 = \frac{c}{2}e^{-j\theta}$
- This gives

$$\begin{aligned} y_0(t) &= \frac{c}{2}e^{j\theta}e^{(\alpha+j\beta)t} + \frac{c}{2}e^{-j\theta}e^{(\alpha-j\beta)t} \\ &= \frac{c}{2}e^{\alpha t}\left[e^{j(\beta t+\theta)} + e^{-j(\beta t+\theta)}\right] \\ &= ce^{\alpha t}\cos(\beta t+\theta) \end{aligned}$$

L2.2 p155

Kompleks Kökler için Bir Örnek

Find $y_0(t)$, the zero-input component of the response for a LTI system described by the following differential equation: $(D^2+4D+40) = (D+2)x(t)$

when the initial conditions are $y_0(0)=2, \quad \dot{y}_0(0)=16.78.$

- The characteristic polynomial for this system is:

$$\lambda^2+4\lambda+40 = (\lambda^2+4\lambda+4)+36 = (\lambda+2)^2+(6)^2$$
$$= (\lambda+2-j6)(\lambda+2+j6)$$

- The complex roots are therefore $\lambda_1 = -2+j6$ and $\lambda_2 = -2-j6$

- The zero-input response in real form is (α = -2, β = 6)

$$y_0(t) = ce^{-2t}\cos(6t+\theta) \qquad \text{.... (12.1)}$$

Kompleks Kökler için Bir Örnek-devam

- To find the constants c and θ, we use the initial conditions $y_0(0) = 2, \quad \dot{y}_0(0) = 16.78.$
- Differentiating equation (12.1) $y_0(t) = ce^{-2t}\cos(6t + \theta)$ gives:

$$\dot{y}_0(t) = -2ce^{-2t}\cos(6t + \theta) - 6ce^{-2t}\sin(6t + \theta)$$

- Using the initial conditions, we obtain:

$$2 = c\cos\theta$$

$$16.78 = -2c\cos\theta - 6c\sin\theta$$

- This reduces to:

$$c\cos\theta = 2$$

$$c\sin\theta = -3.463$$

- Hence

$$c^2 = (2)^2 + (-3.464)^2 = 16 \Longrightarrow c = 4 \qquad \theta = \tan^{-1}\left(\frac{-3.463}{2}\right) = -\frac{\pi}{3}$$

- Finally, the solution is

$$y_0(t) = 4e^{-2t}\cos\left(6t - \frac{\pi}{3}\right)$$

L2.2 p157

- **Sistemin karakteristik polinomuna ait kökler (λ_i) Kompleks olduğunda sistemin sıfır giriş cevabına ait genel çözüm;**

$$\begin{aligned} y_0(t) &= \frac{c}{2}e^{j\theta}e^{(\alpha+j\beta)t} + \frac{c}{2}e^{-j\theta}e^{(\alpha-j\beta)t} \\ &= \frac{c}{2}e^{\alpha t}\left[e^{j(\beta t+\theta)} + e^{-j(\beta t+\theta)}\right] \\ &= ce^{\alpha t}\cos(\beta t+\theta) \end{aligned}$$

The Resonance Behaviour

- Any signal consisting of a system's characteristic mode is **sustained by the system** on its own.
- In other words, the system offers NO obstacle to such signals.
- It is like asking an alcoholic to be a whisky taster.
- Driving a system with an input of the form of the characteristic mode will cause **resonance behaviour.**

Tacoma Bridge Disaster

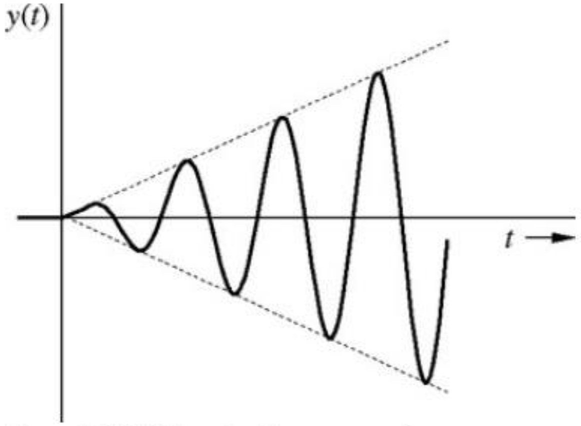
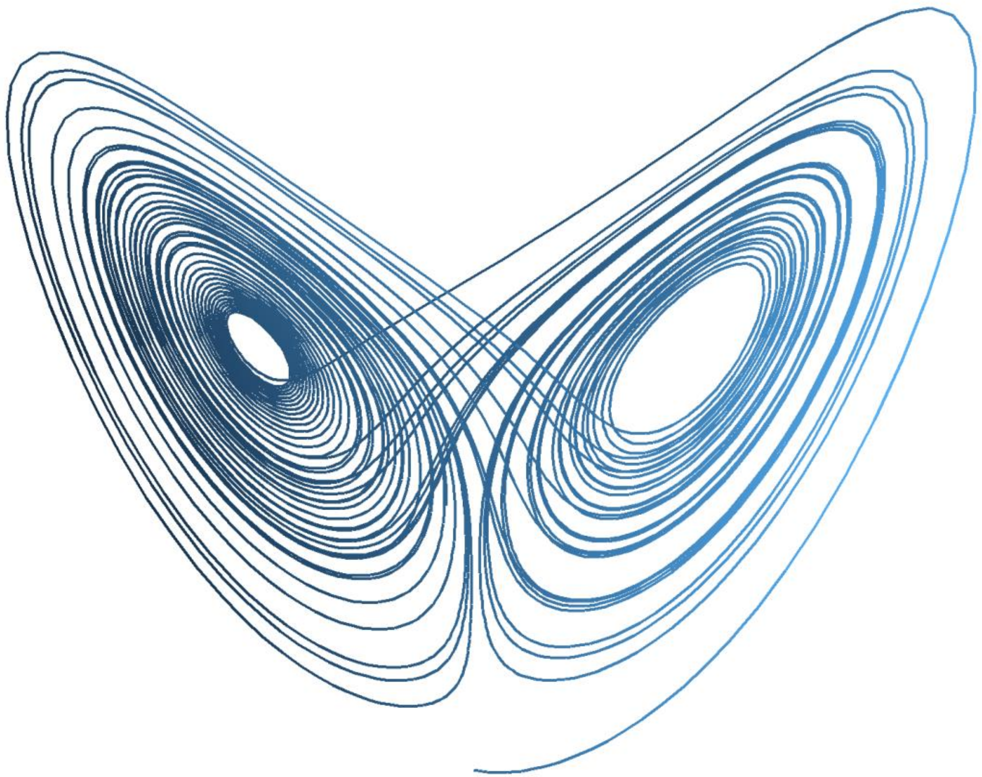


Figure 2.24: Buildup of system response in resonance.

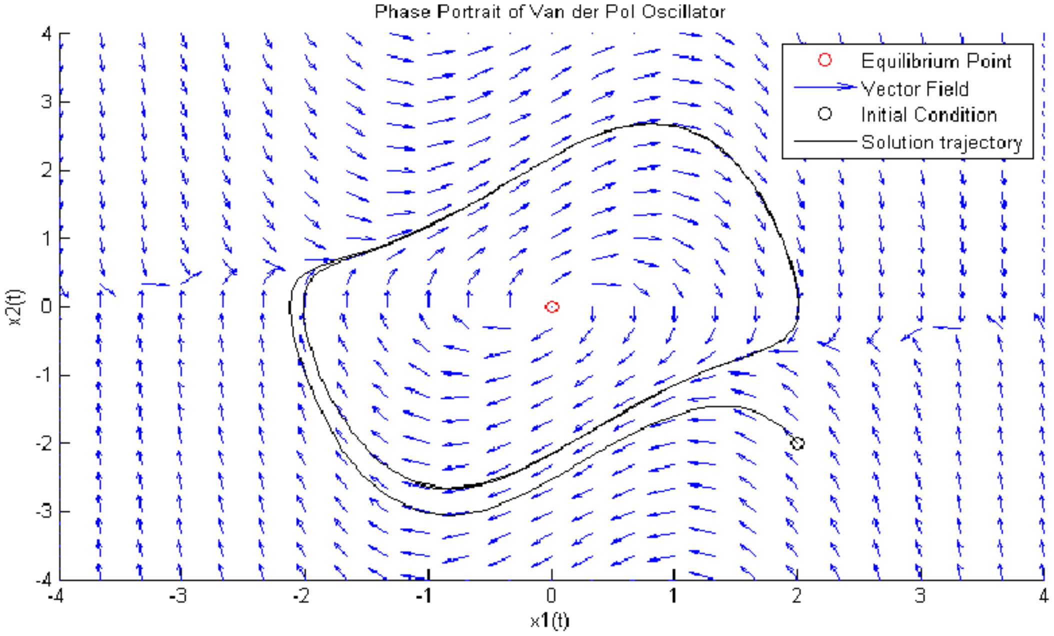
- https://www.youtube.com/watch?v=3mclp9QmCGs

DEMO

- Strange Attractors
- https://www.youtube.com/watch?v=dP3qAq9RNLg



Van der Pol oscillator



Bu ders notu için faydalanılan kaynaklar

Lecture 3

Time-domain analysis:
Zero-input Response
(Lathi 2.1-2.2)

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