

İşaret İşleme

Frekans Cevabı-H11CD1

Dr. Meriç Çetin
versiyon231120

Tablo 5.1 Fourier Dönüşümünün Özellikleri

Özellik	$x(t)$	$X(w)$
	$x(t)$ $x_1(t)$ $x_2(t)$	$X(w)$ $X_1(w)$ $X_2(w)$
Doğrusallık	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(w) + a_2X_2(w)$
Zamanda Öteleme	$x(t - t_0)$	$e^{-j\omega t_0}X(w)$
Frekans -domeninde Öteleme	$e^{j\omega_0 t}x(t)$	$X(w - \omega_0)$
Zamanda Ölçekleme	$x(at)$	$\frac{1}{ a }X\left(\frac{w}{a}\right)$
Zamanda Geri Dönüş	$x(-t)$	$X(-w)$
Zamanda Türev	$\frac{d}{dt}x(t)$	$jwX(w)$
Frekans-domeninde Türev	$-jtx(t)$	$\frac{d}{dw}X(w)$
Çiftleşlik	$X(t)$	$2\pi x(-w)$
Konvolüsyon	$x_1(t) * x_2(t)$	$X_1(w)X_2(w)$
Çarpma	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(w) * X_2(w)$
Parseval Bağıntısı	$\int_{-\infty}^{+\infty} x(t) ^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(w) ^2 dw$	

Tablo 5.2 Bazı Fourier Dönüşüm Çiftleri

$x(t)$	$X(w)$	$X(s)$
$\delta(t)$	1	1
$u(t)$	$\pi\delta(w) + \frac{1}{jw}$	$\frac{1}{s}$
$-u(-t)$	$\pi\delta(w) - \frac{1}{jw}$	$\frac{1}{s}$
1	$2\pi\delta(w)$	
$\text{sgn}(t)$	$\frac{2}{jw}$	
$tu(t)$		$\frac{1}{s^2}$
$e^{-at}u(t)$	$\frac{1}{jw + a}$	$\frac{1}{s + a}$
$-e^{-at}u(-t)$	$\frac{1}{jw + a}$	$\frac{1}{s + a}$
$te^{-at}u(t)$	$\frac{1}{(jw + a)^2}$	$\frac{1}{(s + a)^2}$
$-te^{-at}u(-t)$	$\frac{1}{(jw + a)^2}$	$\frac{1}{(s + a)^2}$

$x(t)$	$X(w)$	$X(s)$
$e^{-at}\cos(w_0t)u(t)$	$\frac{jw + a}{(jw + a)^2 + w_0^2}$	$\frac{s + a}{(s + a)^2 + w_0^2}$
$e^{-at}\sin(w_0t)u(t)$	$\frac{w_0}{(jw + a)^2 + w_0^2}$	$\frac{w_0}{(s + a)^2 + w_0^2}$
$e^{\mp jw_0t}$	$2\pi\delta(w \pm w_0)$	
$\cos(w_0t)$	$\pi\delta(w - w_0) + \pi\delta(w + w_0)$	
$\sin(w_0t)$	$-j\pi\delta(w - w_0) + j\pi\delta(w + w_0)$	
$e^{-a t }$	$\frac{2a}{a^2 + w^2}$	
$\frac{1}{a^2 + t^2}$	$\frac{\pi}{a}e^{-a w }$	
$P_a(t)$	$2\frac{\sin(aw)}{w}$	
$\frac{\sin(at)}{\pi t}$	$P_a(w)$	
e^{-at^2}	$\sqrt{\frac{\pi}{a}}e^{-\frac{w^2}{4a}}$	
$\sum_{k=-\infty}^{\infty} \delta(t - kT_0)$	$w_0 \sum_{k=-\infty}^{\infty} \delta(w - kw_0)$	

Frekans Cevabı

$h(t)$ darbe cevabı bilinen bir sistemin girişine belli bir $x(t)$ sinyali uygulandığında $y(t)$ çıkışının

$$y(t) = x(t) * h(t)$$

konvolüsyonu ile bulunabileceği ve bu nedenle de $h(t)$ darbe cevabının sistemin giriş-çıkış ilişkisini ifade etmede kullanılabileceği ifade edilmişti. Fourier dönüşümünün özelliklerinden olan

$$x_1(t) * x_2(t) \leftrightarrow X_1(w)X_2(w) \quad \text{şeklindeki konvolüsyon özelliği kullanılırsa,}$$

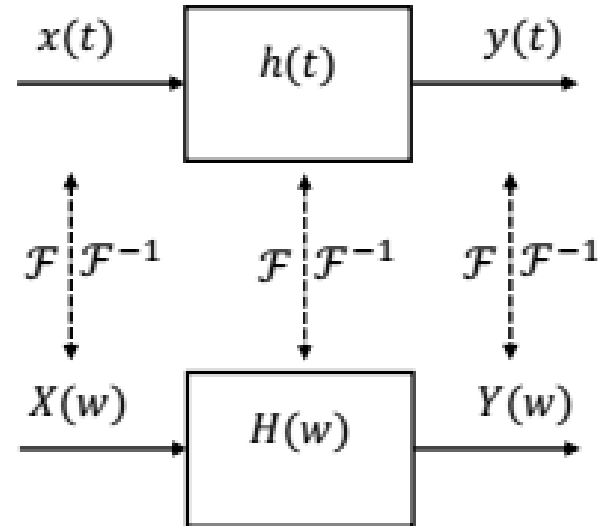
$$y(t) = x(t) * h(t) \leftrightarrow X(w)H(w) = Y(w)$$

ifadesi elde edilir. Burada $X(w)$, $H(w)$ ve $Y(w)$ sırasıyla $x(t)$, $h(t)$ ve $y(t)$ 'nin Fourier dönüşümleridir. Aynı eşitlik

$$H(w) = \frac{Y(w)}{X(w)}$$

Frekans Cevabı - devam

ile ifade edilebilir. $h(t)$ 'nin Fourier dönüşümü $H(w)$ 'ye *frekans cevabı* denir. Artık, darbe cevabı bilinen bir sisteme belli bir sinyal uygulandığında çıkış sinyali frekans cevabı yardımıyla bulunabilir. Bu nedenle de DZD bir sistemin frekans cevabı sistemin giriş-çıkış ilişkisini ifade etmede kullanılabilir. Bunu aşağıdaki şekille görmek mümkündür.



Frekans Cevabı - devam

Frekans cevabı $H(w)$, bağımsız frekans değişkenine bağlı bir fonksiyon olup karmaşık değerler alabilmektedir. Bu nedenle $H(w)$ kutupsal olarak

$$H(w) = |H(w)|e^{j\angle H(w)}$$

$$\begin{aligned} Y(w) &= X(w)H(w) \\ |Y(w)|e^{j\angle Y(w)} &= |X(w)|e^{j\angle X(w)}|H(w)|e^{j\angle H(w)} \end{aligned}$$

şeklinde yazılabileceği için çıkış sinyalinin genliği

$$|Y(w)| = |X(w)||H(w)|$$

fazı da

$$\angle Y(w) = \angle X(w) + \angle H(w)$$

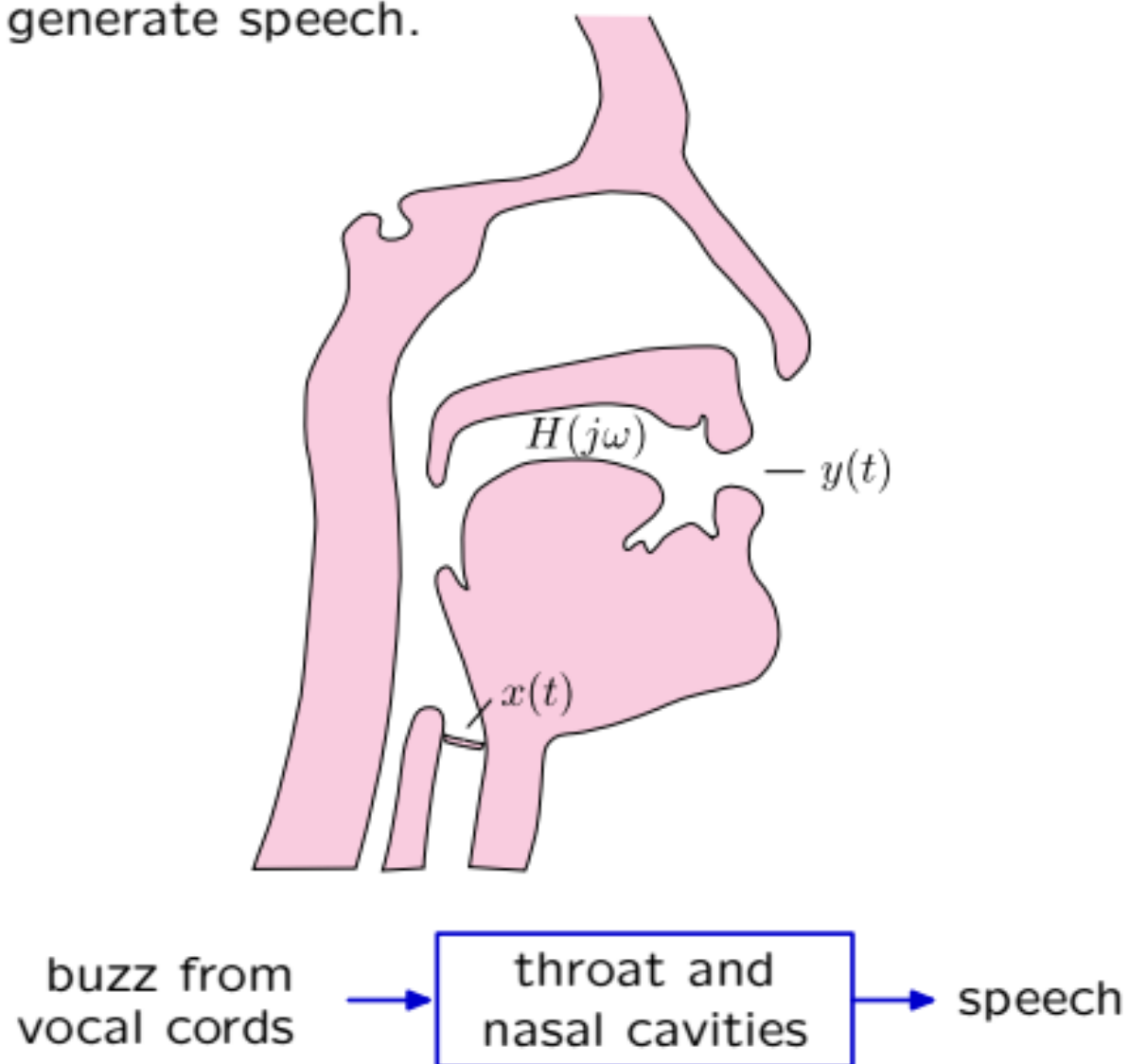
biçiminde ifade edilebilir.

Filtre Kavramı

Önceki altkısımda da görüldüğü gibi DZD bir sistemin girişinden periyodik bir sinyal uygulandığında, sistem bu giriş sinyalinin frekans bileşenlerinden bazılarını geçirmekte bazılarını da süzmektedir. Bu açıdan bakıldığında her DZD sistem aslında frekans domeninde bir filtre gibi görülebilir. DZD sistem giriş sinyalinin frekans bileşenlerinden bazılarının genliklerini sıfırlamakta ve dolayısıyla bu bileşenleri süzmektedir. Bileşenlerin faz ve genliklerinin nasıl etkileneceği tamamen DZD sistemin frekans cevabı tarafından belirlenmektedir. Literatürde pek çok filtre tipi mevcuttur. Bunlardan belli başlı olanları şunlardır:

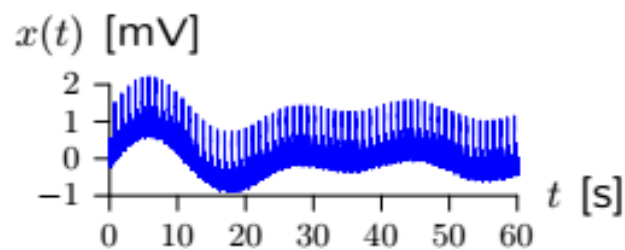
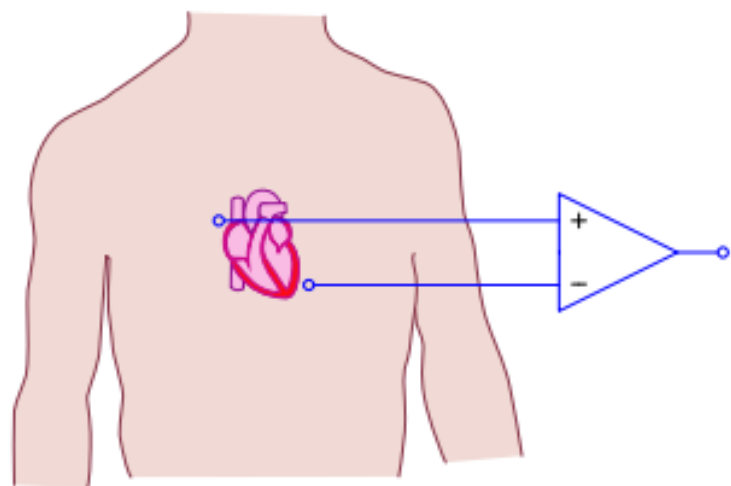
Source-Filter Model of Speech Production

Vibrations of the vocal cords are “filtered” by the mouth and nasal cavities to generate speech.



Filtering Example: Electrocardiogram

An electrocardiogram is a record of electrical potentials that are generated by the heart and measured on the surface of the chest.



In addition to electrical responses of heart, electrodes on the skin also pick up other electrical signals that we regard as “noise.”

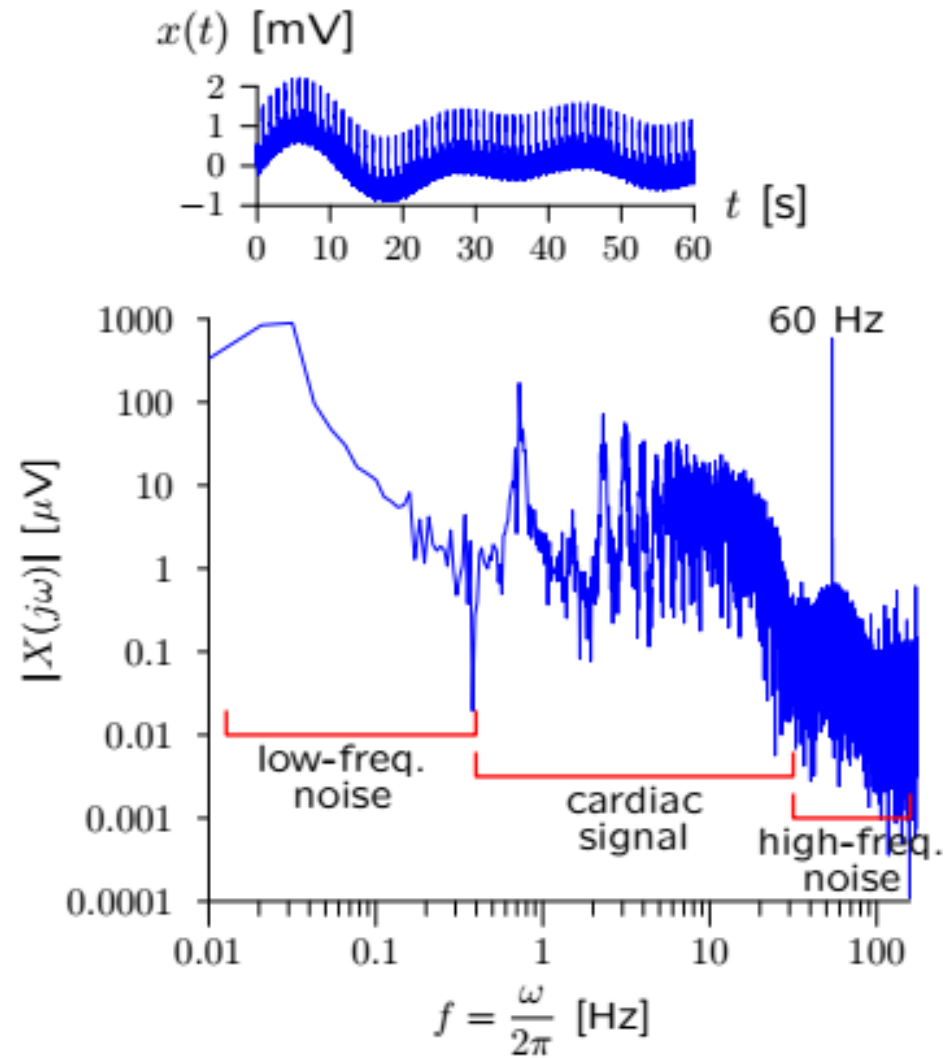
ECG and analysis by T. F. Weiss

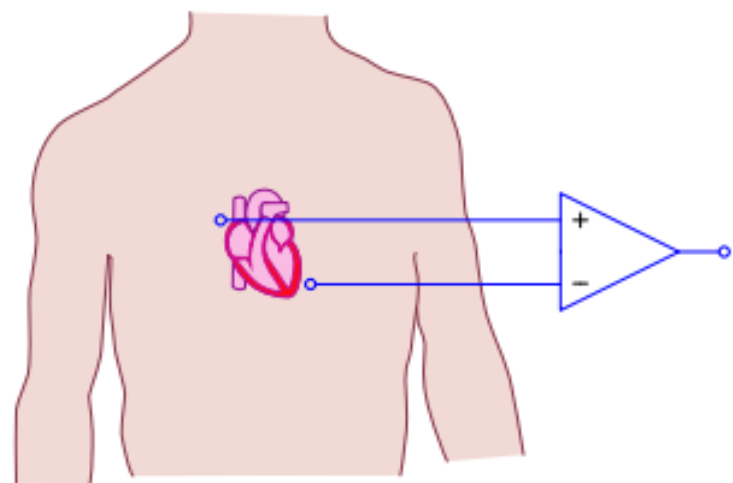
We wish to design a filter to eliminate the noise.



Filtering Example: Electrocardiogram

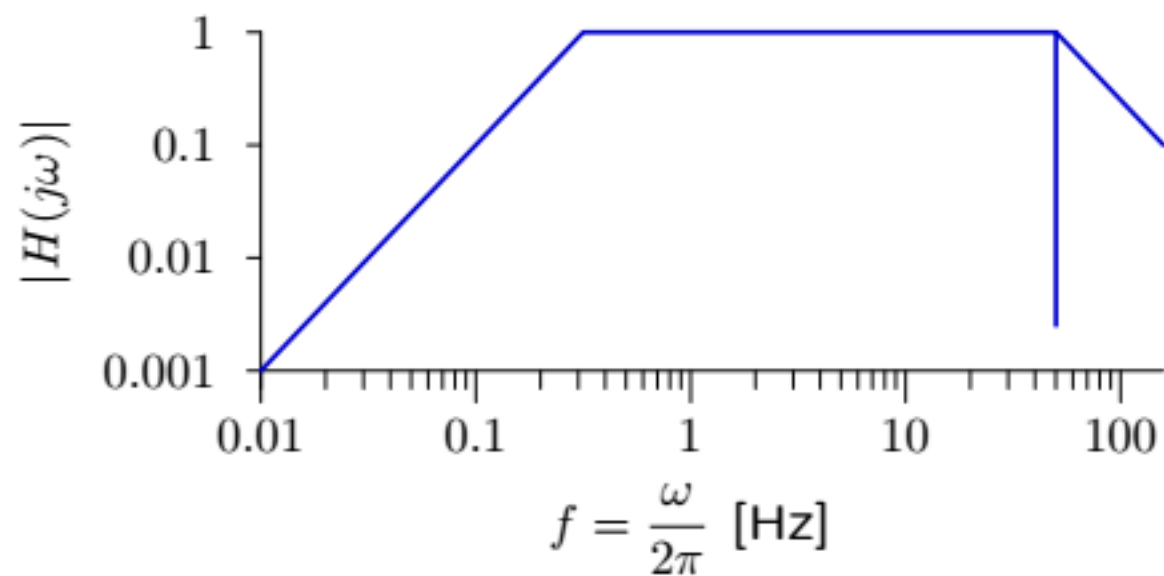
We can identify “noise” using the Fourier transform.





Filtering Example: Electrocardiogram

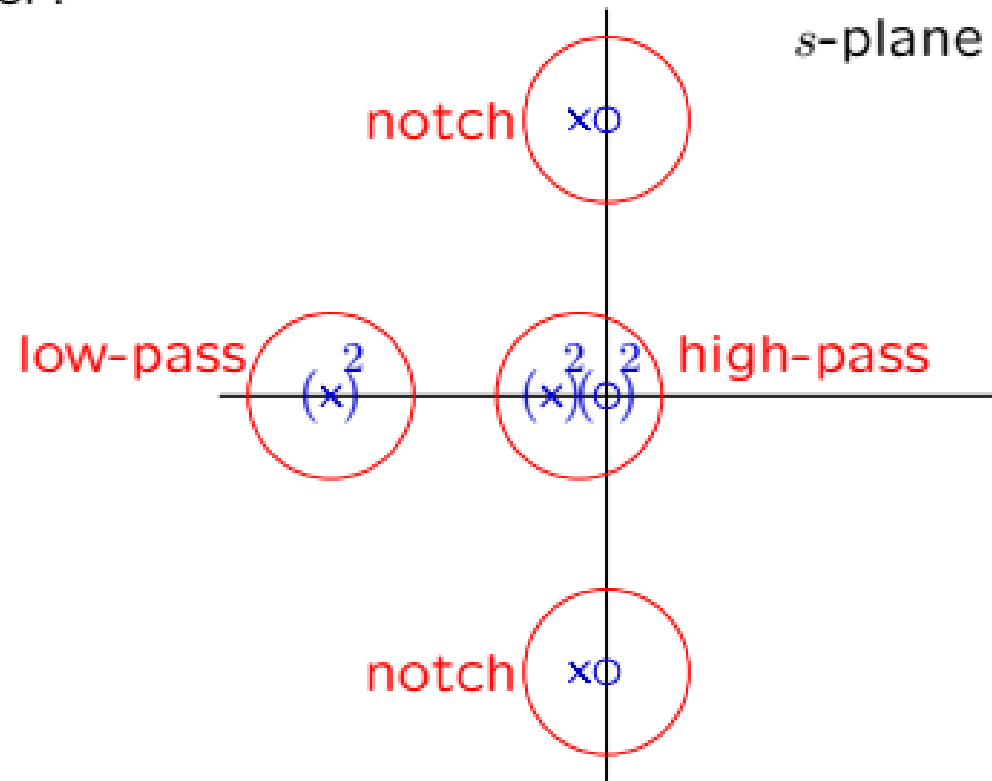
Filter design: low-pass filter + high-pass filter + notch.



Electrocardiogram: Check Yourself

Which poles and zeros are associated with

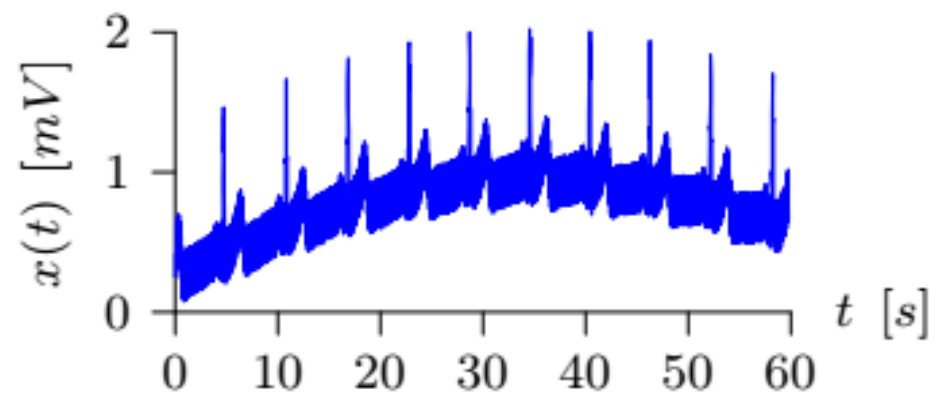
- the high-pass filter?
- the low-pass filter?
- the notch filter?



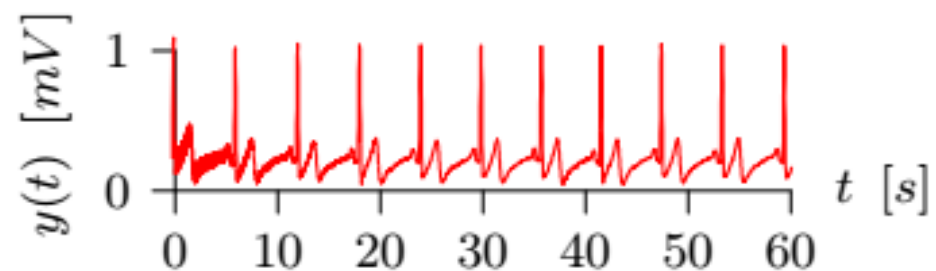
Filtering Example: Electrocardiogram

Filtering is a simple way to reduce unwanted noise.

Unfiltered ECG

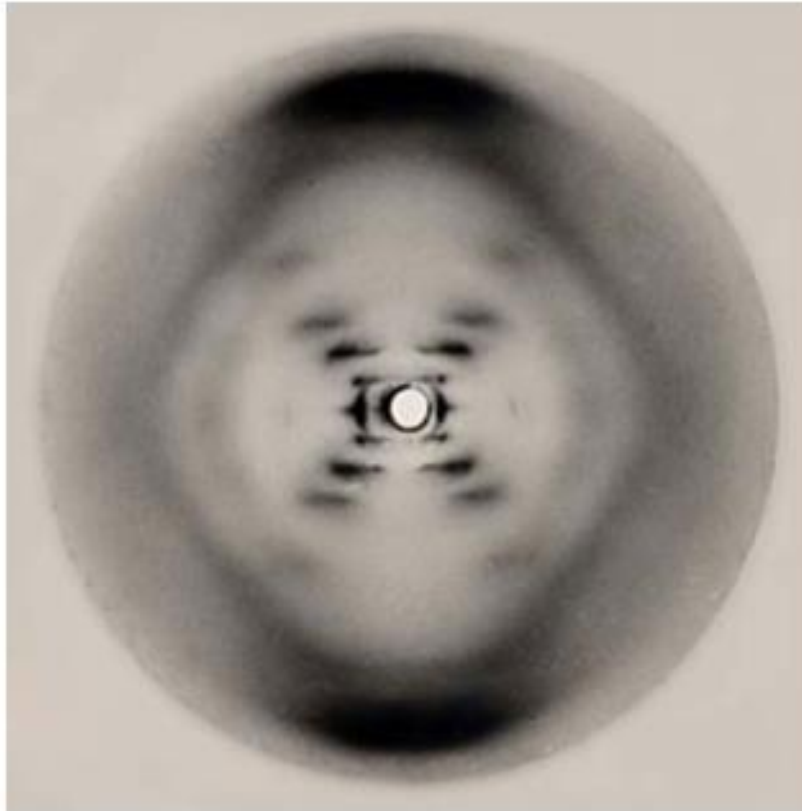


Filtered ECG



An Historic Fourier Transform

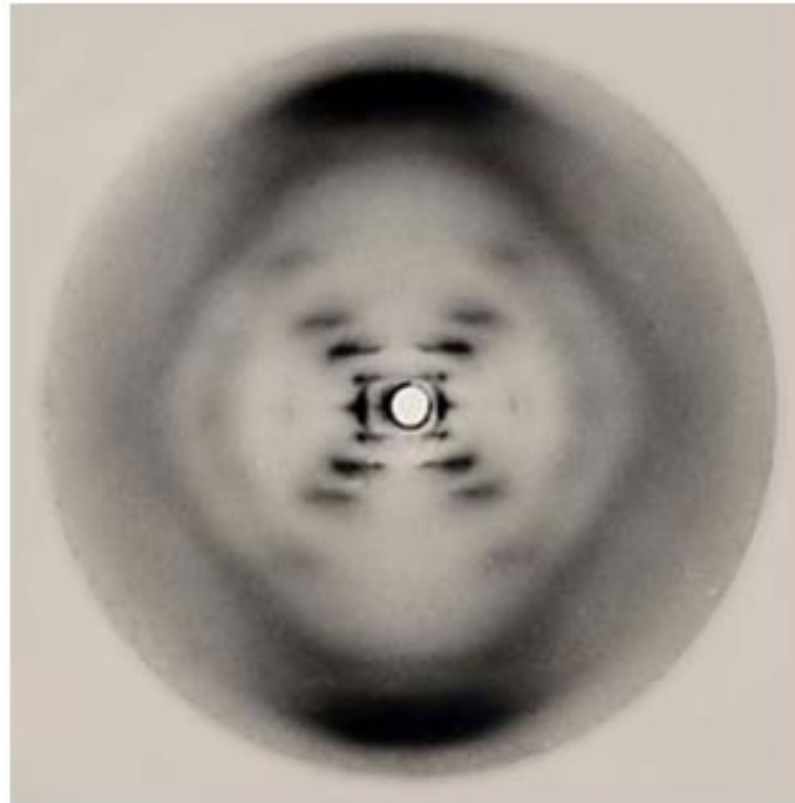
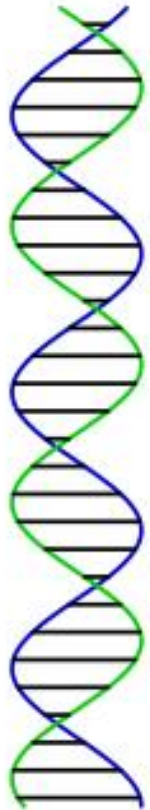
Taken by Rosalind Franklin, this image sparked Watson and Crick's insight into the double helix.



Reprinted by permission from Macmillan Publishers Ltd: Nature.
Source: Franklin, R., and R. G. Gosling. "Molecular Configuration
in Sodium Thymonucleate." *Nature* 171 (1953): 740-741. (c) 1953.

An Historic Fourier Transform

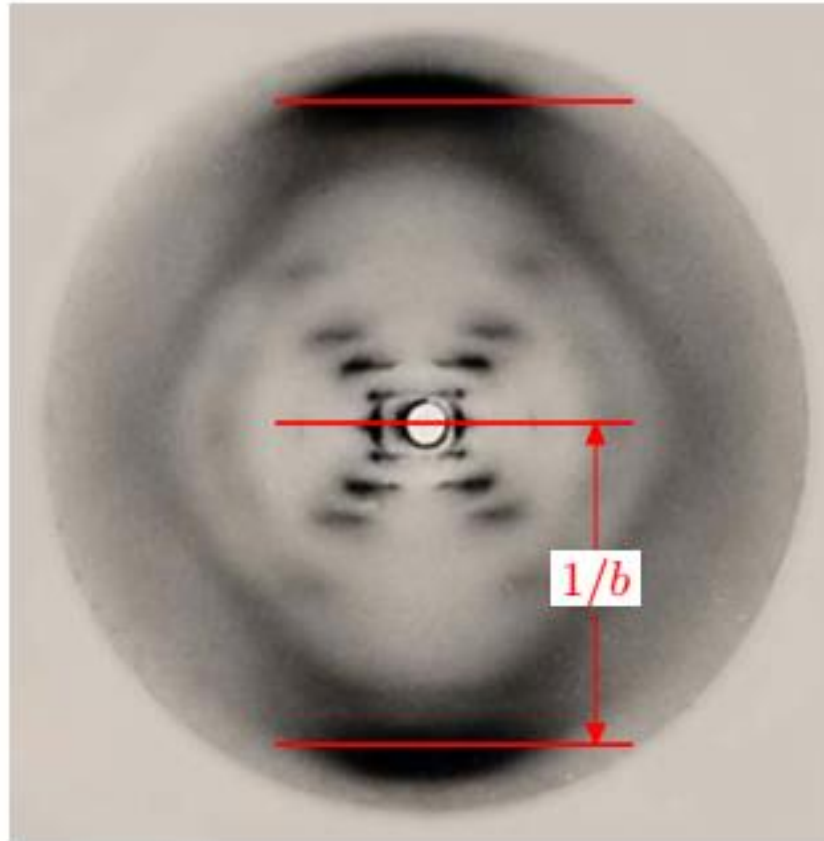
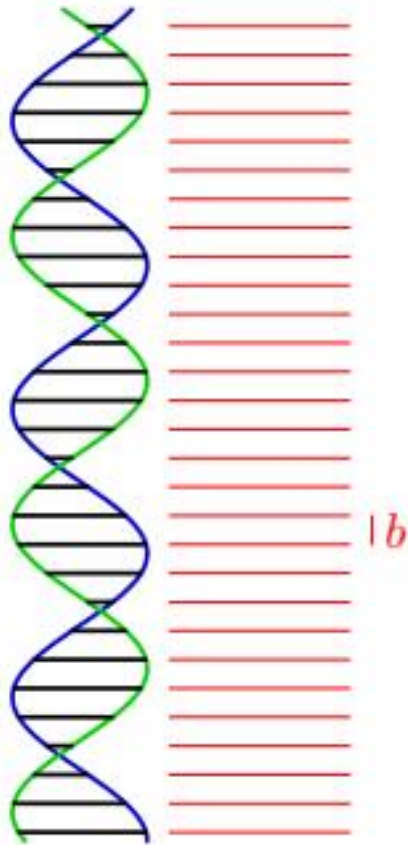
This is an x-ray crystallographic image of DNA, and it shows the Fourier transform of the structure of DNA.



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Source: Franklin, R., and R. G. Gosling. "Molecular Configuration
in Sodium Thymonucleate." *Nature* 171 (1953): 740-741. (c) 1953.

An Historic Fourier Transform

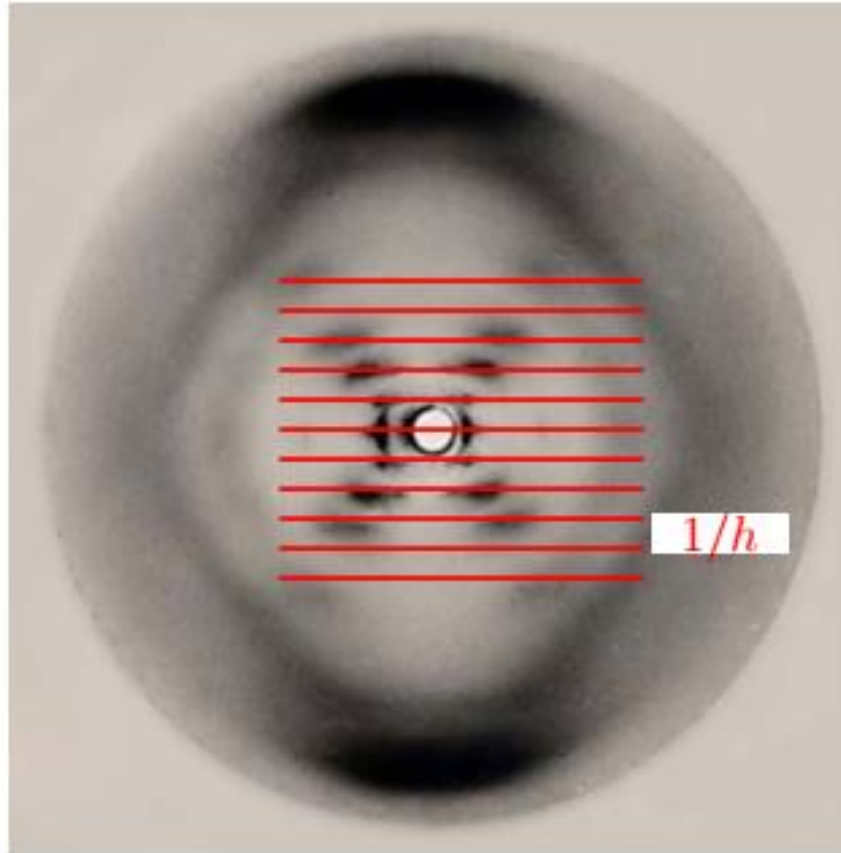
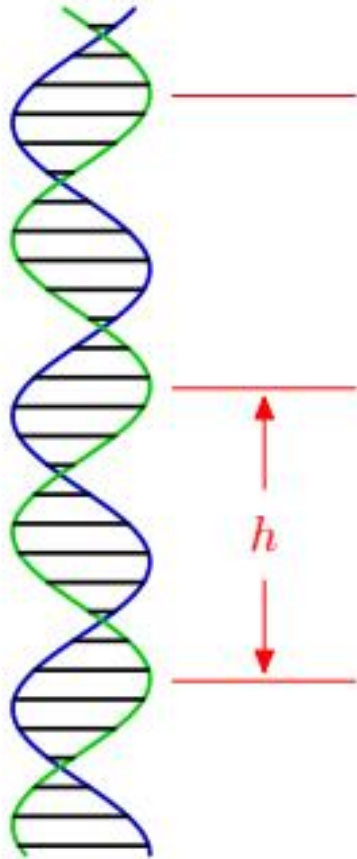
High-frequency bands indicate repeating structure of base pairs.



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Source: Franklin, R., and R. G. Gosling. "Molecular Configuration
in Sodium Thymonucleate." *Nature* 171 (1953): 740-741. (c) 1953.

An Historic Fourier Transform

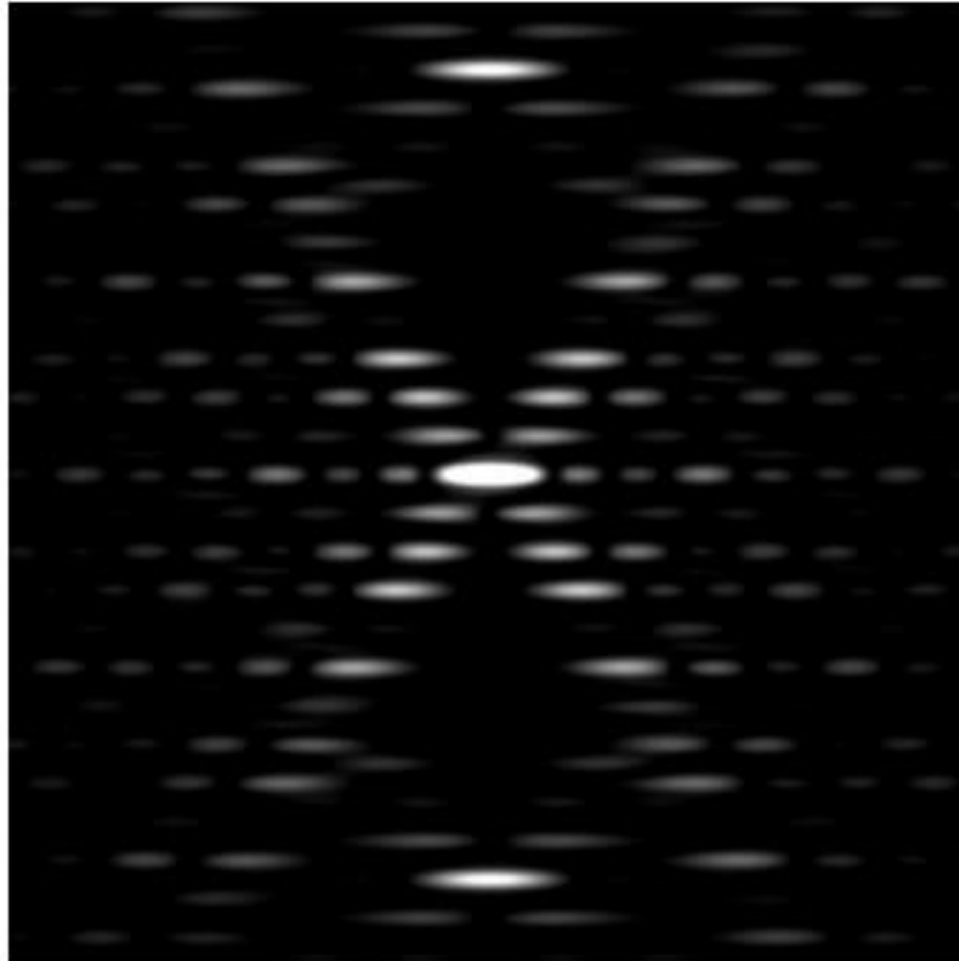
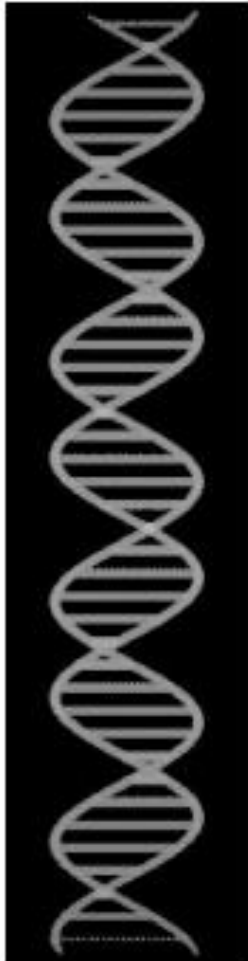
Low-frequency bands indicate a lower frequency repeating structure.



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Source: Franklin, R., and R. G. Gosling. "Molecular Configuration
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Simulation

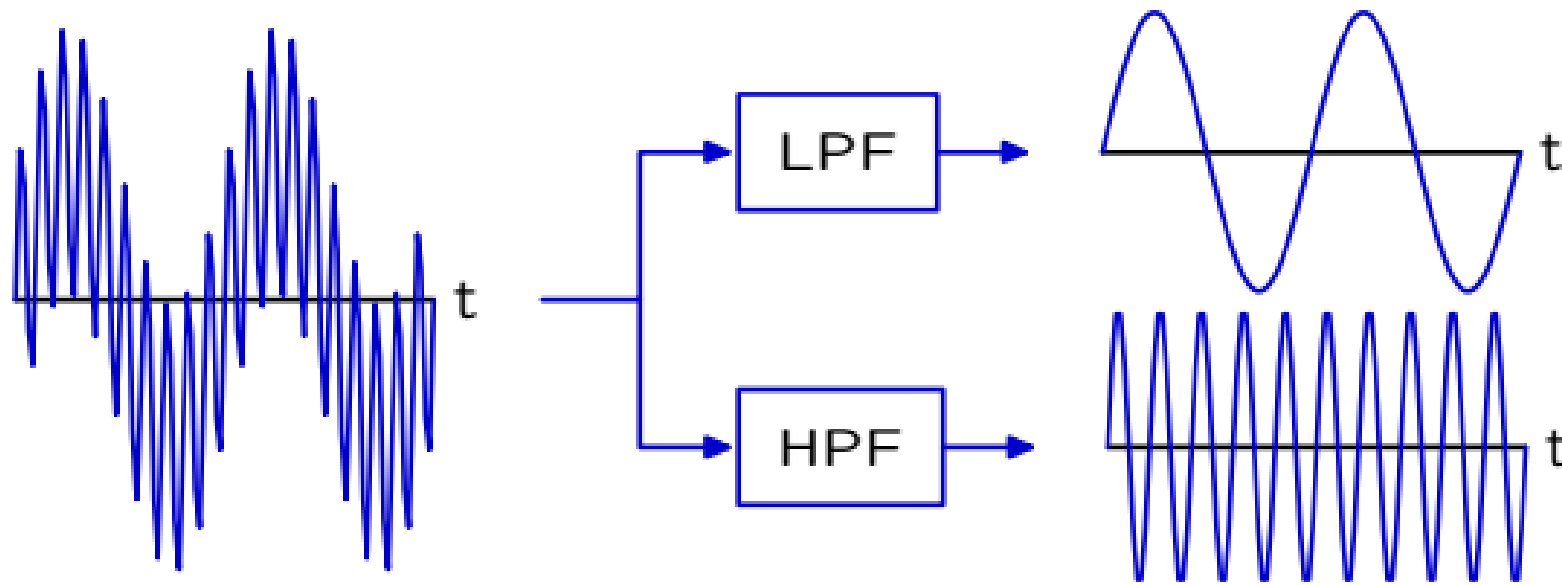
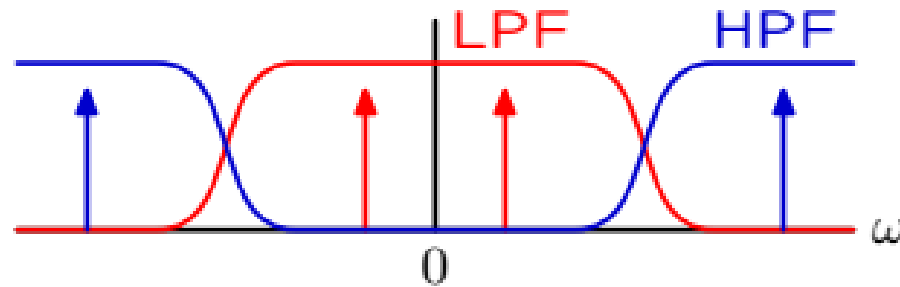
Easy to calculate relation between structure and Fourier transform.



Filtering

Systems can be designed to selectively pass certain frequency bands.

Examples: low-pass filter (LPF) and high-pass filter (HPF).



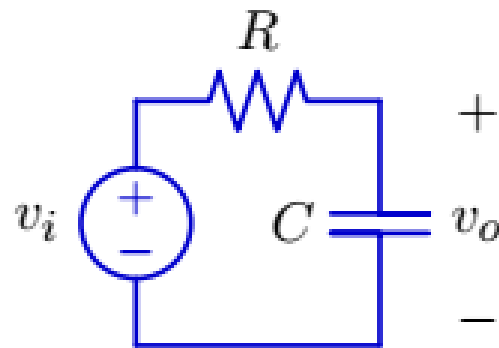
Filtering

Notion of a filter.

LTI systems

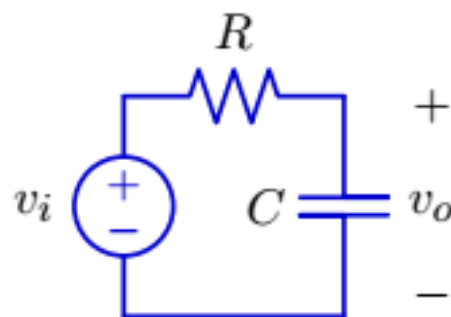
- cannot create new frequencies.
- can scale magnitudes and shift phases of existing components.

Example: Low-Pass Filtering with an RC circuit



Lowpass Filter

Calculate the frequency response of an RC circuit.



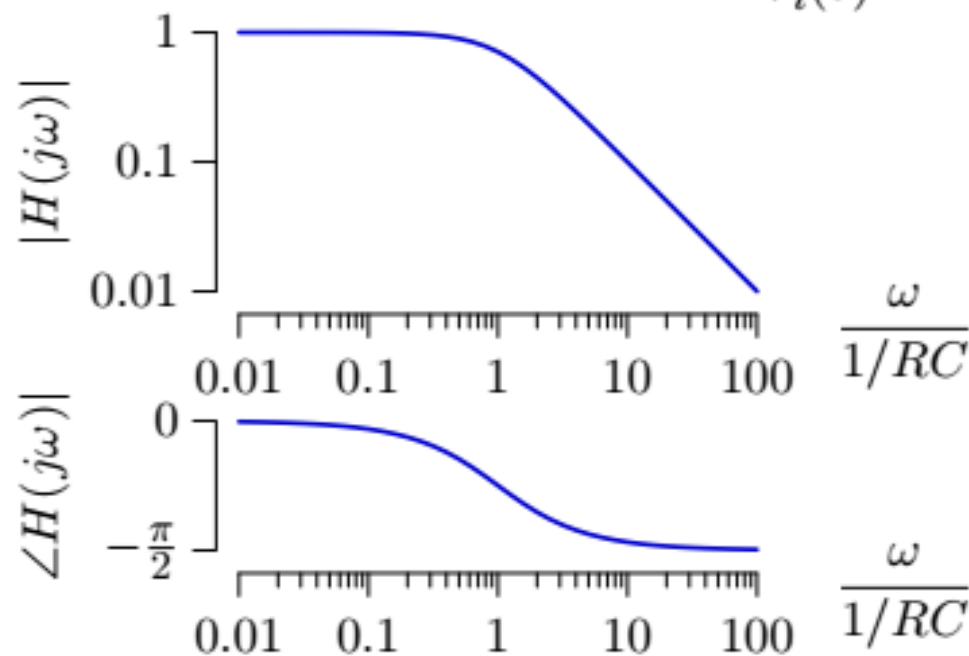
$$\text{KVL: } v_i(t) = Ri(t) + v_o(t)$$

$$\text{C: } i(t) = C\dot{v}_o(t)$$

$$\text{Solving: } v_i(t) = RC\dot{v}_o(t) + v_o(t)$$

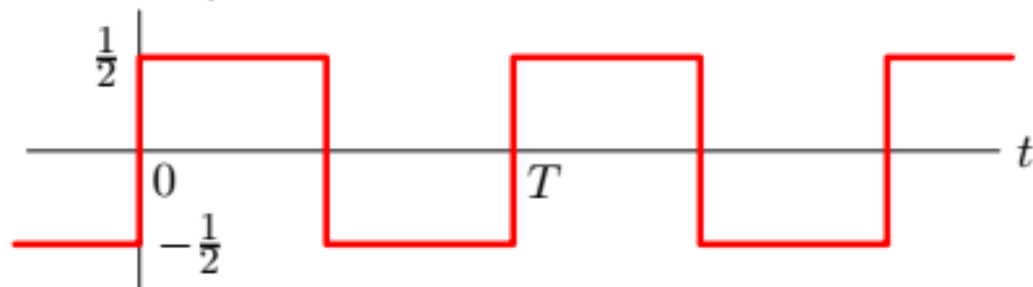
$$V_i(s) = (1 + sRC)V_o(s)$$

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{1 + sRC}$$

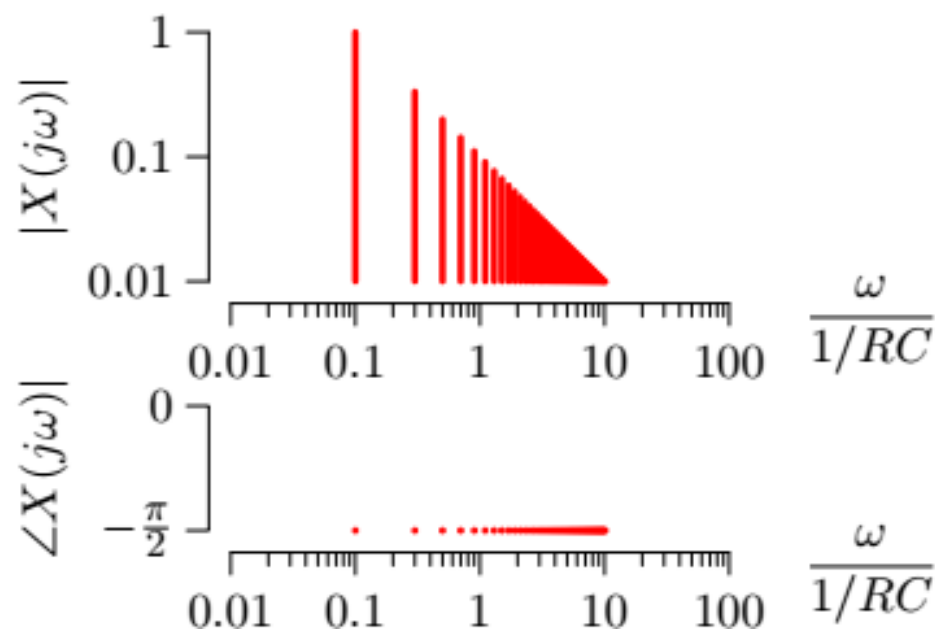


Lowpass Filtering

Let the input be a square wave.

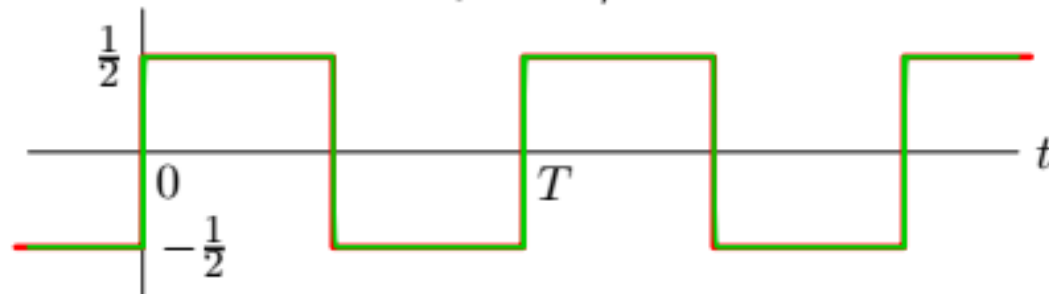


$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 k t} ; \quad \omega_0 = \frac{2\pi}{T}$$

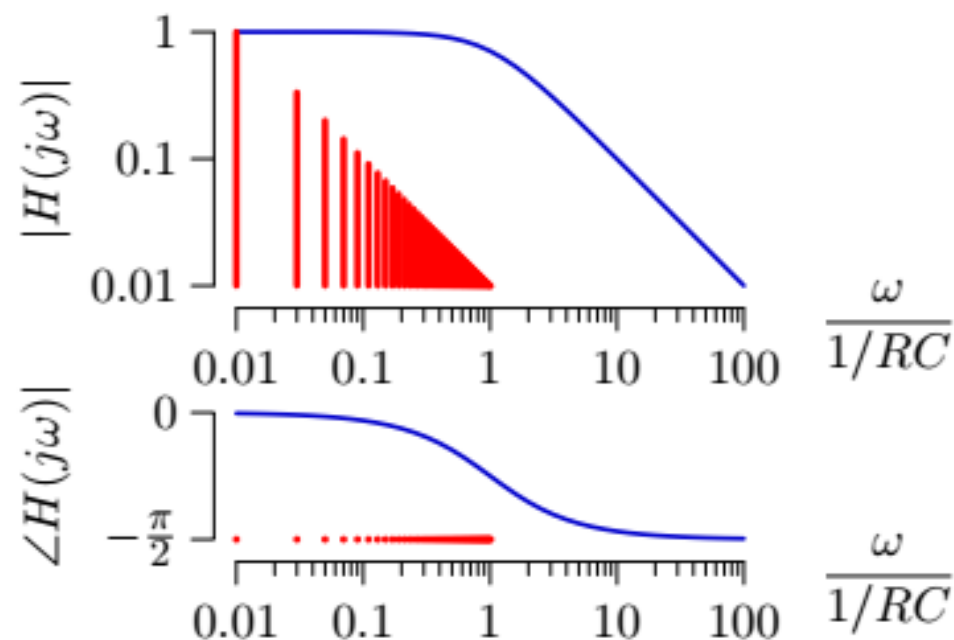


Lowpass Filtering

Low frequency square wave: $\omega_0 \ll 1/RC$.

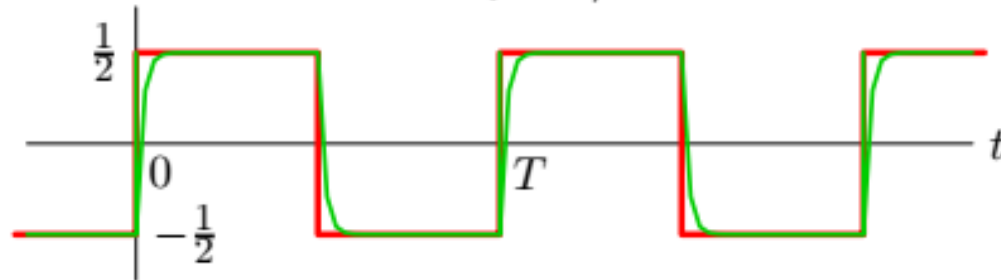


$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 k t} ; \quad \omega_0 = \frac{2\pi}{T}$$

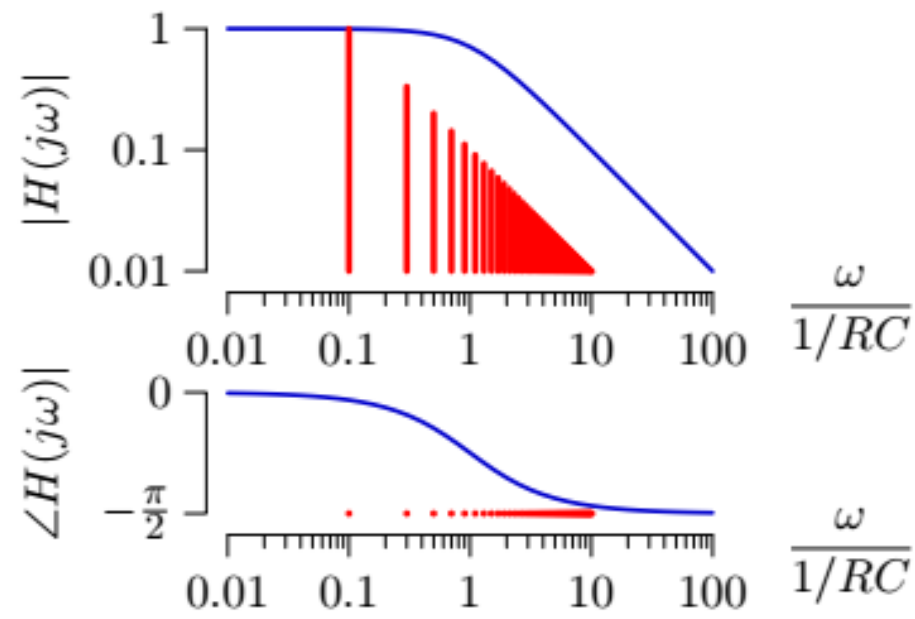


Lowpass Filtering

Higher frequency square wave: $\omega_0 < 1/RC$.

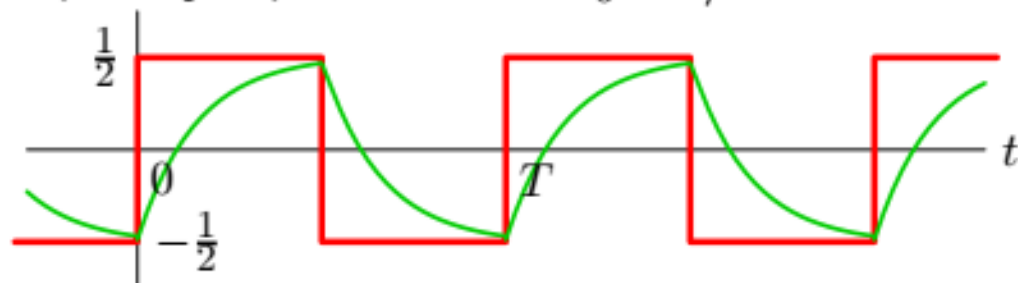


$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 k t} ; \quad \omega_0 = \frac{2\pi}{T}$$

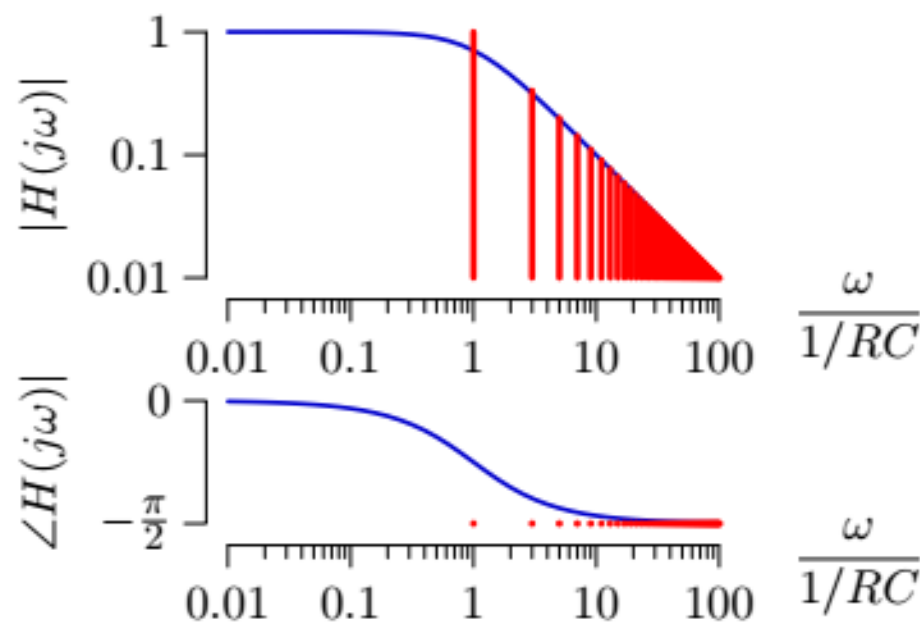


Lowpass Filtering

Still higher frequency square wave: $\omega_0 = 1/RC$.

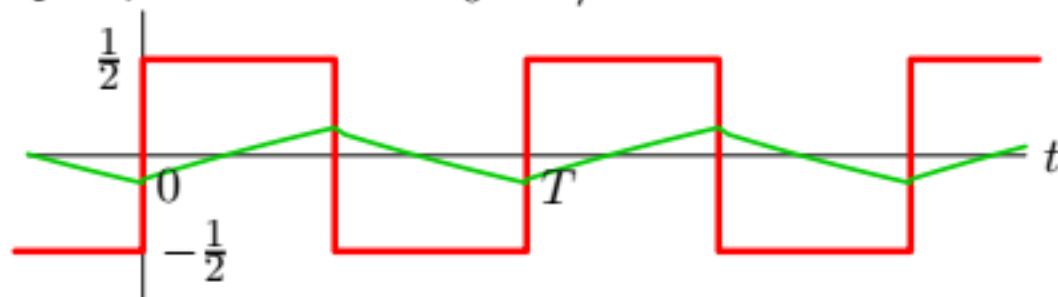


$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 k t} ; \quad \omega_0 = \frac{2\pi}{T}$$

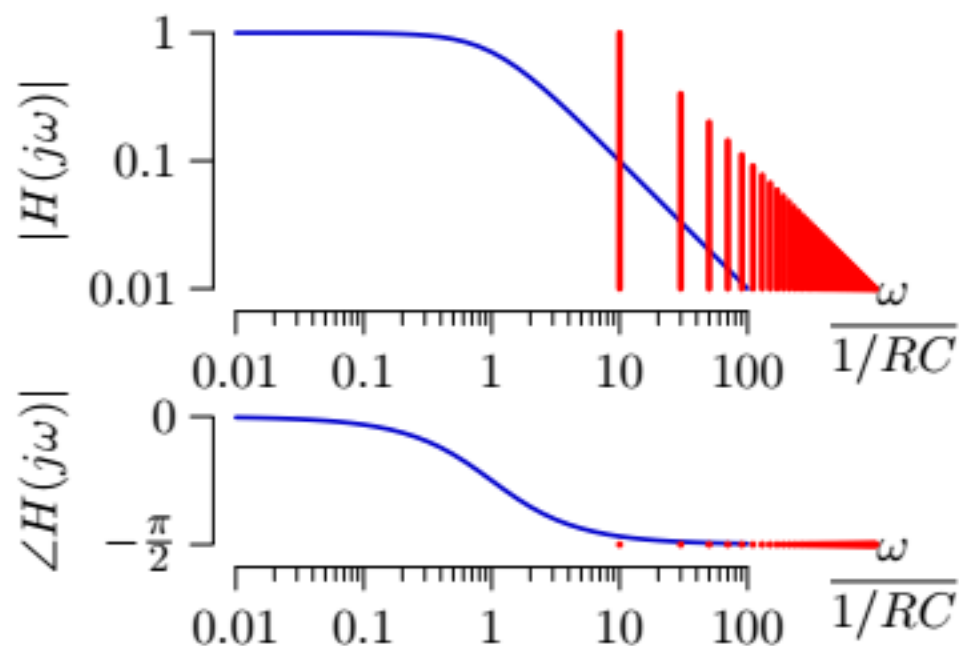


Lowpass Filtering

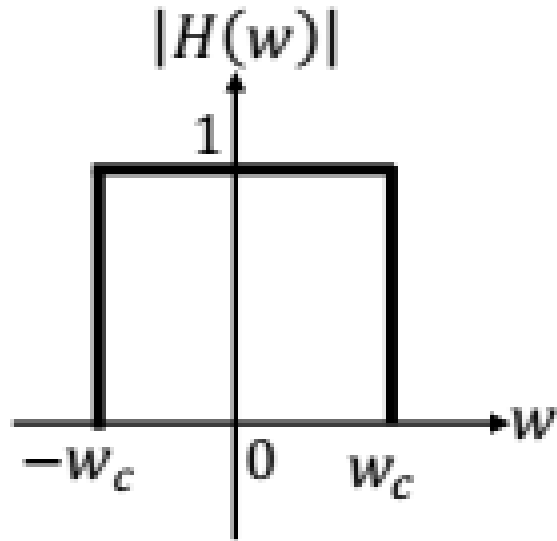
High frequency square wave: $\omega_0 > 1/RC$.



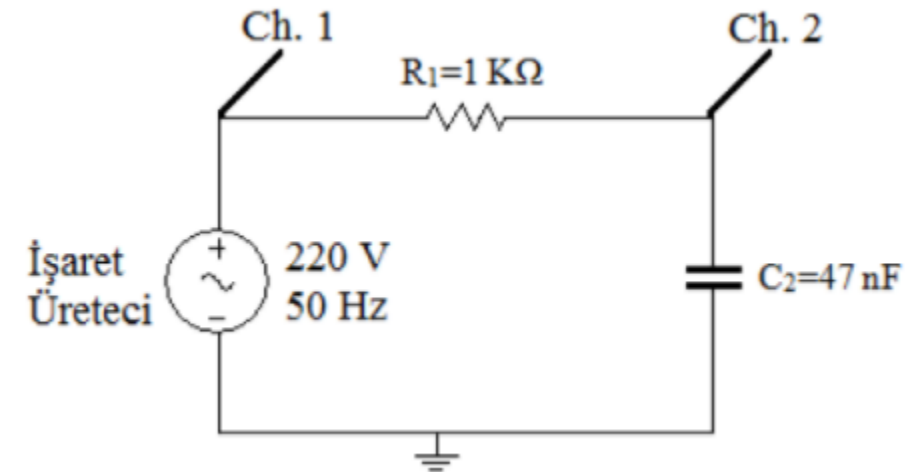
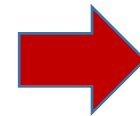
$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 k t} ; \quad \omega_0 = \frac{2\pi}{T}$$



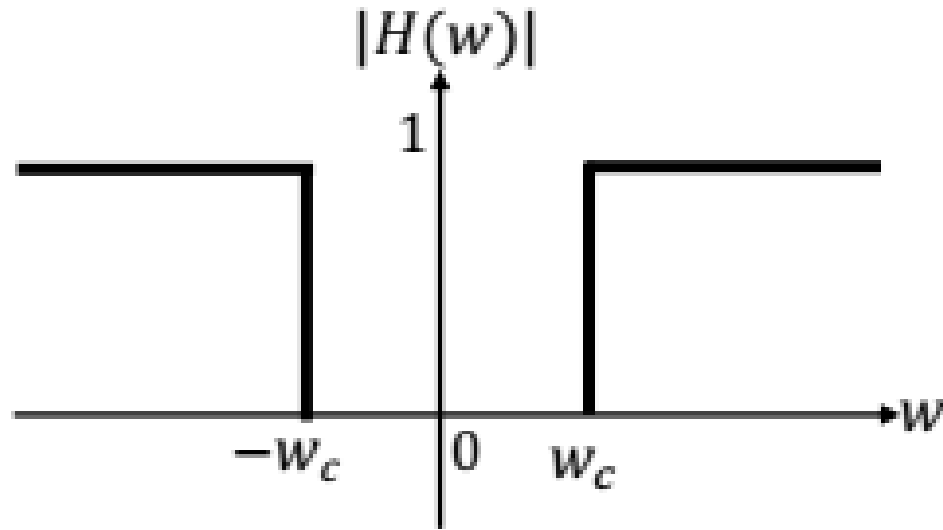
İdeal Alçak Geçiren Filtre



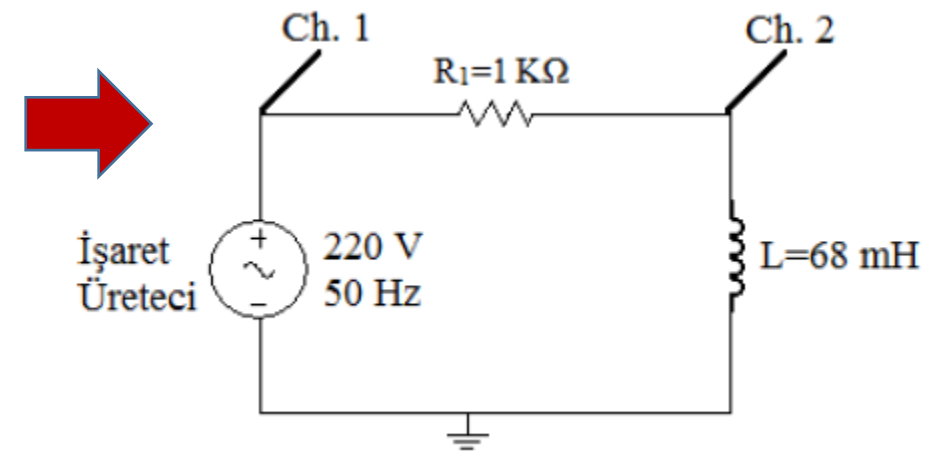
$$|H(\omega)| = P_a(\omega_c) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$



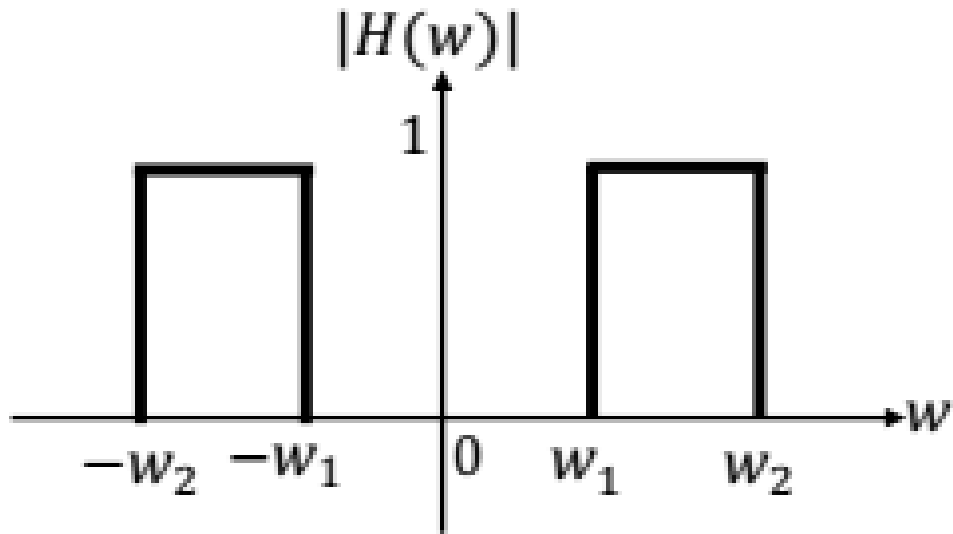
İdeal Yüksek Geçiren Filtre



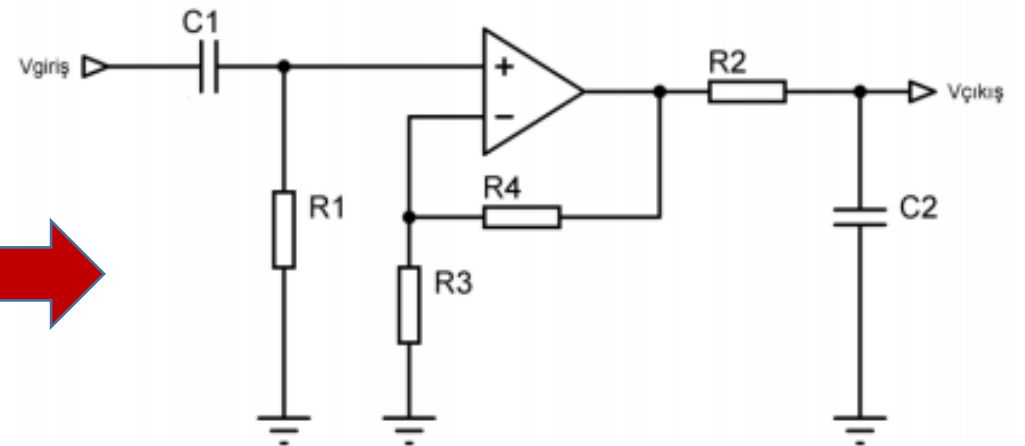
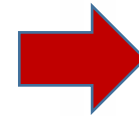
$$|H(\omega)| = P_a(\omega_c) = \begin{cases} 1 & |\omega| > \omega_c \\ 0 & |\omega| \leq \omega_c \end{cases}$$



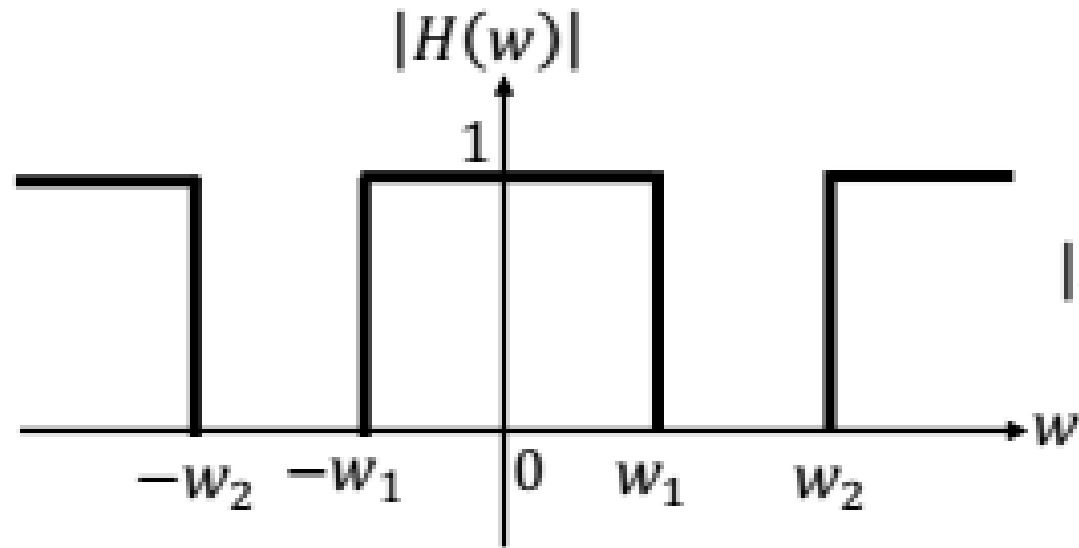
İdeal Bant Geçiren Filtre



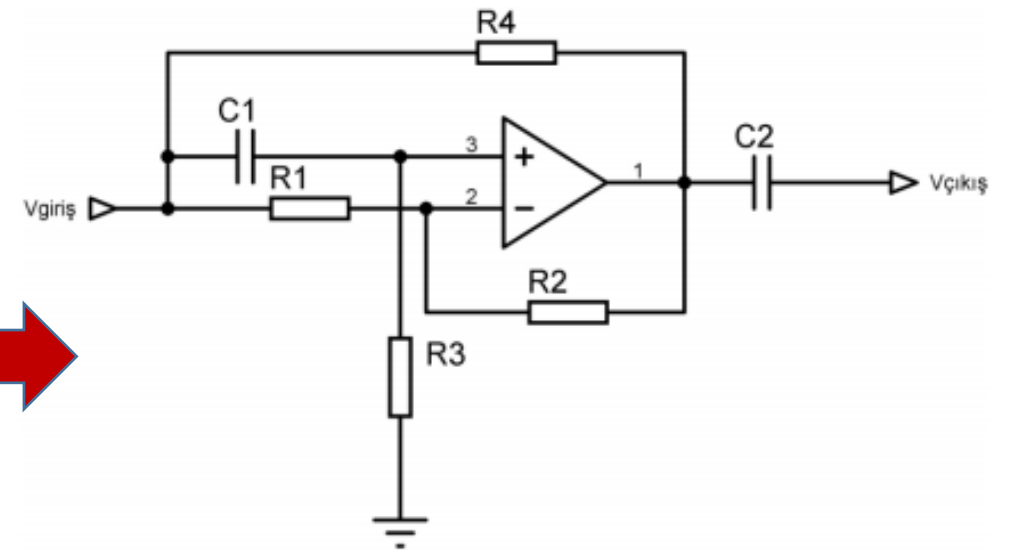
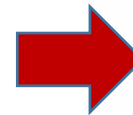
$$|H(w)| = P_a(w_c) = \begin{cases} 1 & w_1 < |w| < w_2 \\ 0 & \text{diger} \end{cases}$$



İdeal Bant Söndüren Filtre



$$|H(w)| = P_a(w_c) = \begin{cases} 0 & w_1 < |w| < w_2 \\ 1 & \text{diger} \end{cases}$$



Diferansiyel Denklemlerle İlişkisi

$$a_0 y(t) + a_1 \dot{y}(t) + a_2 \ddot{y}(t) + \dots + a_N y^{(N)}(t) = b_0 x(t) + b_1 \dot{x}(t) + b_2 \ddot{x}(t) + \dots + b_M x^{(M)}(t)$$

buradaki a_k ve b_k katsayıları reel ve sabit katsayılardır. Şimdi, giriş-çıkış ilişkisi bu şekildeki bir denklem ile ifade edilen DZD bir sistemin frekans cevabını bulalım: İlk olarak denklemin her iki tarafının Fourier dönüşümünü alalım:

$$\begin{aligned} a_0 Y(w) + a_1 jw Y(w) + a_2 (jw)^2 Y(w) + \dots + a_N (jw)^N Y(w) &= b_0 X(w) + b_1 jw X(w) + \dots + b_M (jw)^M X(w) \\ (a_0 + a_1 jw + a_2 (jw)^2 + \dots + a_N (jw)^N) Y(w) &= (b_0 + b_1 jw + \dots + b_M (jw)^M) X(w) \end{aligned}$$

Şimdi frekans cevabını yazabiliriz:

$$H(w) = \frac{Y(w)}{X(w)} = \frac{b_0 + b_1 jw + \dots + b_M (jw)^M}{a_0 + a_1 jw + a_2 (jw)^2 + \dots + a_N (jw)^N}$$

Örnek: Giriş-çıkış ilişkisi aşağıdaki gibi sabit katsayılı doğrusal bir denklem ile ifade edilen nedensel bir DZD sistemi ele alalım. Sistemin girişinden uygulanan $x(t) = e^{-t}u(t)$ sinyaline karşı sistemin $y(t)$ çıkışını bulunuz. Not: Sistemin başlangıç koşulları sıfırdır.

$$x(t) = e^{-t}u(t) \rightarrow \boxed{\ddot{y}(t) + 1.6\dot{y}(t) + y(t) = 3\ddot{x}(t) + 6.2\dot{x}(t) + 4x(t)} \rightarrow y(t) = ?$$

İlk olarak denklemin her iki tarafının Fourier dönüşümünü alalım:

$$(jw)^2Y(w) + 1.6jwY(w) + Y(w) = 3(jw)^2X(w) + 6.2jwX(w) + 4X(w).$$

Şimdi frekans cevabını yazabiliriz:

$$H(w) = \frac{Y(w)}{X(w)} = \frac{3(jw)^2 + 6.2jw + 4}{(jw)^2 + 1.6jw + 1}$$

Giriş sinyalinin Fourier dönüşümü

$$x(t) = e^{-t}u(t) \leftrightarrow X(w) = \frac{1}{jw + 1}$$

şeklindedir, böylece çıkışın Fourier dönüşümü şu şekilde bulunur:

$$\begin{aligned} Y(w) &= H(w)X(w) \\ &= \frac{3(jw)^2 + 6.2jw + 4}{(jw)^2 + 1.6jw + 1} \frac{1}{jw + 1} \end{aligned}$$

$Y(j\omega)$ 'yi kısmi kesirlere ayırılım:

$$\begin{aligned} Y(j\omega) &= \frac{3(j\omega)^2 + 6.2j\omega + 4}{(j\omega)^2 + 1.6j\omega + 1} \frac{1}{j\omega + 1} \\ &= \frac{2}{j\omega + 1} + \frac{j\omega + 2}{(j\omega)^2 + 1.6j\omega + 1} \\ &= \frac{2}{j\omega + 1} + \frac{s + 2}{(j\omega + 0.8)^2 + 0.36} \\ &= \frac{2}{j\omega + 1} + \frac{j\omega + 0.8}{(j\omega + 0.8)^2 + (0.6)^2} + 2 \frac{0.6}{(j\omega + 0.8)^2 + (0.6)^2} \end{aligned}$$

$Y(s)$ 'nin ters Fourier dönüşümünü alırsak çıkış sinyali $y(t)$ bulunur:

Tablo 5.2 Bazı Fourier Dönüşüm Çiftleri

$x(t)$	$X(w)$	$X(s)$
$\delta(t)$	1	1
$u(t)$	$\pi\delta(w) + \frac{1}{jw}$	$\frac{1}{s}$
$-u(-t)$	$\pi\delta(w) - \frac{1}{jw}$	$\frac{1}{s}$
1	$2\pi\delta(w)$	
$\text{sgn}(t)$	$\frac{2}{jw}$	
$tu(t)$		$\frac{1}{s^2}$
$e^{-at}u(t)$	$\frac{1}{jw + a}$	$\frac{1}{s + a}$
$-e^{-at}u(-t)$	$\frac{1}{jw + a}$	$\frac{1}{s + a}$
$te^{-at}u(t)$	$\frac{1}{(jw + a)^2}$	$\frac{1}{(s + a)^2}$
$-te^{-at}u(-t)$	$\frac{1}{(jw + a)^2}$	$\frac{1}{(s + a)^2}$

$x(t)$	$X(w)$	$X(s)$
$e^{-at}\cos(w_0t)u(t)$	$\frac{jw + a}{(jw + a)^2 + w_0^2}$	$\frac{s + a}{(s + a)^2 + w_0^2}$
$e^{-at}\sin(w_0t)u(t)$	$\frac{w_0}{(jw + a)^2 + w_0^2}$	$\frac{w_0}{(s + a)^2 + w_0^2}$
$e^{\mp jw_0t}$	$2\pi\delta(w \pm w_0)$	
$\cos(w_0t)$	$\pi\delta(w - w_0) + \pi\delta(w + w_0)$	
$\sin(w_0t)$	$-j\pi\delta(w - w_0) + j\pi\delta(w + w_0)$	
$e^{-a t }$	$\frac{2a}{a^2 + w^2}$	
$\frac{1}{a^2 + t^2}$	$\frac{\pi}{a}e^{-a w }$	
$P_a(t)$	$2\frac{\sin(aw)}{w}$	
$\frac{\sin(at)}{\pi t}$	$P_a(w)$	
e^{-at^2}	$\sqrt{\frac{\pi}{a}}e^{-\frac{w^2}{4a}}$	
$\sum_{k=-\infty}^{\infty} \delta(t - kT_0)$	$w_0 \sum_{k=-\infty}^{\infty} \delta(w - kw_0)$	

$Y(j\omega)$ 'yi kısmi kesirlere ayırılım:

$$\begin{aligned} Y(j\omega) &= \frac{3(j\omega)^2 + 6.2j\omega + 4}{(j\omega)^2 + 1.6j\omega + 1} \frac{1}{j\omega + 1} \\ &= \frac{2}{j\omega + 1} + \frac{j\omega + 2}{(j\omega)^2 + 1.6j\omega + 1} \\ &= \frac{2}{j\omega + 1} + \frac{s + 2}{(j\omega + 0.8)^2 + 0.36} \\ &= \frac{2}{j\omega + 1} + \frac{j\omega + 0.8}{(j\omega + 0.8)^2 + (0.6)^2} + 2 \frac{0.6}{(j\omega + 0.8)^2 + (0.6)^2} \end{aligned}$$

$Y(s)$ 'nin ters Fourier dönüşümünü alırsak çıkış sinyali $y(t)$ bulunur:

$$y(t) = 2e^{-t}u(t) + e^{-0.8t} \cos 0.6t u(t) + 2e^{-0.8t} \sin 0.6t u(t)$$

Bu ders notu için faydalanılan kaynaklar

EEEN343 Sinyaller ve Sistemler **Ders Notları**

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Prof. Dr. Serdar İplikçi
Pamukkale Üniversitesi
Mühendislik Fakültesi
Elektrik-Elektronik Mühendisliği

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