3) $$\sum_{n=1}^{N} \frac{(-1)^{n+1}}{2n+1} = \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} - \cdots$$

i)
```
top=0;
isrt= ~~1*isrt;~~ -1;
for n=1:N
  isrt=-1*isrt;
  top=top+ isrt/(2*n+1);
end
```

ii)
```
top=0;
for n=1:N
  top= top+ (-1)^(n+1)/(2*n+1);
end
ytop= top;
```

4) i)
```
for n=1:N
  if mod(n,2)==0;
    cf=2;
  else
    cf=4;
  end
end
```

ii)
```
for i=1:N
  cf= 3+(-1)^(n+1);
end
```

4.1 Numerical Differentiation

The derivative of the function f at x_0 is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

This formula gives an obvious way to generate an approximation to $f'(x_0)$; simply compute

$$\frac{f(x_0 + h) - f(x_0)}{h}$$

for small values of h. Although this may be obvious, it is not very successful, due to our old nemesis round-off error. But it is certainly a place to start.

To approximate $f'(x_0)$, suppose first that $x_0 \in (a, b)$, where $f \in C^2[a, b]$, and that $x_1 = x_0 + h$ for some $h \neq 0$ that is sufficiently small to ensure that $x_1 \in [a, b]$. We construct the first Lagrange polynomial $P_{0,1}(x)$ for f determined by x_0 and x_1, with its error term:

$$\begin{aligned} f(x) &= P_{0,1}(x) + \frac{(x - x_0)(x - x_1)}{2!} f''(\xi(x)) \\ &= \frac{f(x_0)(x - x_0 - h)}{-h} + \frac{f(x_0 + h)(x - x_0)}{h} + \frac{(x - x_0)(x - x_0 - h)}{2} f''(\xi(x)), \end{aligned}$$

for some $\xi(x)$ between x_0 and x_1. Differentiating gives

$$\begin{aligned} f'(x) &= \frac{f(x_0 + h) - f(x_0)}{h} + D_x \left[\frac{(x - x_0)(x - x_0 - h)}{2} f''(\xi(x)) \right] \\ &= \frac{f(x_0 + h) - f(x_0)}{h} + \frac{2(x - x_0) - h}{2} f''(\xi(x)) \\ &\quad + \frac{(x - x_0)(x - x_0 - h)}{2} D_x(f''(\xi(x))). \end{aligned}$$

Deleting the terms involving $\xi(x)$ gives

$$f'(x) \approx \frac{f(x_0 + h) - f(x_0)}{h}.$$

Difference equations were used and popularized by Isaac Newton in the last quarter of the 17th century, but many of these techniques had previously been developed by Thomas Harriot (1561–1621) and Henry Briggs (1561–1630). Harriot made significant advances in navigation techniques, and Briggs was the person most responsible for the acceptance of logarithms as an aid to computation.

One difficulty with this formula is that we have no information about $D_x f''(\xi(x))$, so the truncation error cannot be estimated. When x is x_0, however, the coefficient of $D_x f''(\xi(x))$ is 0, and the formula simplifies to

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} - \frac{h}{2} f''(\xi). \tag{4.1}$$

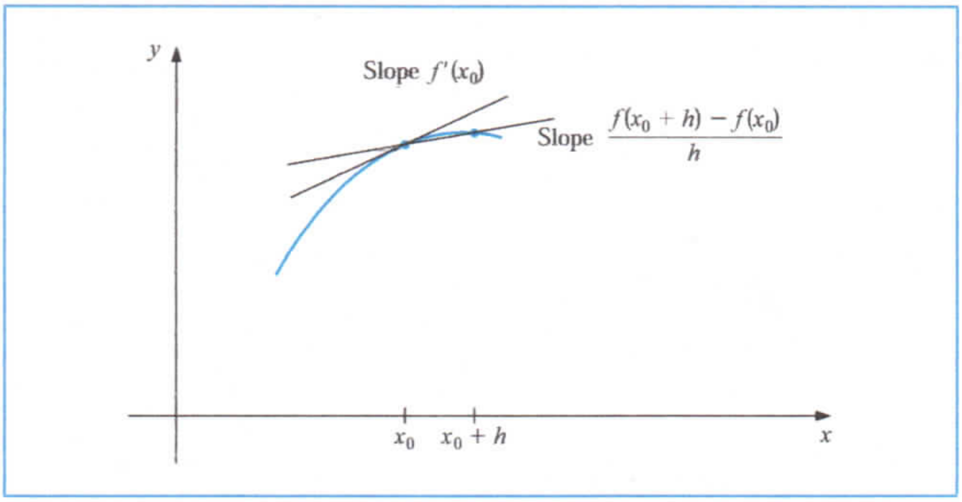
For small values of h, the difference quotient $[f(x_0 + h) - f(x_0)]/h$ can be used to approximate $f'(x_0)$ with an error bounded by $M|h|/2$, where M is a bound on $|f''(x)|$ for x between x_0 and $x_0 + h$. This formula is known as the **forward-difference formula** if $h > 0$ (see Figure 4.1) and the **backward-difference formula** if $h < 0$.

Example 1 Use the forward-difference formula to approximate the derivative of $f(x) = \ln x$ at $x_0 = 1.8$ using $h = 0.1$, $h = 0.05$, and $h = 0.01$, and determine bounds for the approximation errors.

Solution The forward-difference formula

$$\frac{f(1.8 + h) - f(1.8)}{h}$$

Figure 4.1



with $h = 0.1$ gives

$$\frac{\ln 1.9 - \ln 1.8}{0.1} = \frac{0.64185389 - 0.58778667}{0.1} = 0.5406722.$$

Because $f''(x) = -1/x^2$ and $1.8 < \xi < 1.9$, a bound for this approximation error is

$$\frac{|hf''(\xi)|}{2} = \frac{|h|}{2\xi^2} < \frac{0.1}{2(1.8)^2} = 0.0154321.$$

The approximation and error bounds when $h = 0.05$ and $h = 0.01$ are found in a similar manner and the results are shown in Table 4.1.

Table 4.1

h	$f(1.8+h)$	$\frac{f(1.8+h)-f(1.8)}{h}$	$\frac{\lvert h\rvert}{2(1.8)^2}$
0.1	0.64185389	0.5406722	0.0154321
0.05	0.61518564	0.5479795	0.0077160
0.01	0.59332685	0.5540180	0.0015432

Since $f'(x) = 1/x$, the exact value of $f'(1.8)$ is $0.55\overline{5}$, and in this case the error bounds are quite close to the true approximation error. ■

To obtain general derivative approximation formulas, suppose that $\{x_0, x_1, \ldots, x_n\}$ are $(n+1)$ distinct numbers in some interval I and that $f \in C^{n+1}(I)$. From Theorem 3.3 on page 112,

$$f(x) = \sum_{k=0}^{n} f(x_k)L_k(x) + \frac{(x-x_0)\cdots(x-x_n)}{(n+1)!} f^{(n+1)}(\xi(x)),$$

Türev

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0) \text{ idi}$$

h sayısını ve $x_0 = 1.8$ $\Rightarrow$ $h = 0.1$, $h = 0.05$, $h = 0.01$ alarak

$f(x) = \ln(x)$ fonksiyonunun $\frac{\ln(x_0+h) - \ln(x_0)}{h}$ ifadesinin MATLAB'da programını verelim

$$f'(x_0) \cong \frac{\ln(x_0+h) - \ln(x_0)}{h}$$

$f(x) = \ln(x) \Rightarrow f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$ dir.

Hata sınırı $\frac{h}{2} f''(x_0)$ ise

h	$f(1.8+h)$	$\frac{f(1.8+h)-f(1.8)}{h}$	$\frac{h}{2(1.8)^2}$
0.1	0.64185389	0.5406722	0.0154321
0.05	0.61518564	0.5479795	0.0077160
0.01	0.59332685	0.5540180	0.0015432

```
x0=1.8;
disp(' h        f(1.8+h)    (f(1.8+h)-f(1.8))/h     |h|/(2*x0^2) ')
disp('~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~')
for m=1:3
    if m<=2
h=0.1/(2^(m-1));
    else
   h=0.1/10;
    end
    pay= log(1.8+h);
grck=1/x0;
trv= (log(1.8+h)-log(1.8))/h;
hata=abs(h)/(2*(x0)^2);
 fprintf('%5.3f%12.8f%15.7f%19.7f\n',h,pay,trv,hata)
end

%>>turev
% h       f(1.8+h)    (f(1.8+h)-f(1.8))/h     |h|/(2*x0^2)
%~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
%0.100  0.64185389       0.5406722           0.0154321
%0.050  0.61518564       0.5479795           0.0077160
%0.010  0.59332685       0.5540180           0.0015432
```

Taylor Series Expansions

30.11.2021

(1)

- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Taylor series expansions for some common Functions

- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{2n!}$
- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$
- $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

örnek: $e = 1 + \frac{1}{1!} + \frac{1^2}{2!} + \frac{1^3}{3!} + \dots + \frac{1^k}{k!} + \dots$

n	$S_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}$
0	1
1	2
2	2.5
3	2.66666...
4	2.708333...
5	2.716666...
6	2.7180555...
⋮	
15	2.718281828459...

a) $f(x,y) = f(a,b) + h f_x + k f_y + \frac{1}{2!}\left[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right]$

$+ \frac{1}{3!}\left[h^3 f_{xxx} + 3h^2k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy}\right]$

$+ \frac{1}{4!}\left[h^4 f_{xxxx} + 4h^3k f_{xxxy} + 6h^2k^2 f_{xxyy} + 4hk^3 f_{xyyy} + k^4 f_{yyyy}\right] + \cdots$

$h = x-a, \quad k = y-b, \quad f_x = \frac{\partial}{\partial x} f(x,y)\Big|_{\substack{x=a \\ y=b}}$

$f_y = \frac{\partial}{\partial y} f(x,y)\Big|_{\substack{x=a \\ y=b}}$

b)

$f(x) = \ln\left(\frac{1+x}{1-x}\right) = 2\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}$

c) sinhx ve coshx in mcclourin açılımını yazınız.

$\sinh x = \frac{1}{2}\left[e^x - e^{-x}\right] = \frac{1}{2}\left[1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right]$

$- \frac{1}{2}\left[1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right]$

$= \frac{1}{2}\left[2x + 2\frac{x^3}{3!} + \frac{2x^5}{5!} + \cdots\right]$

$= \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\right) = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}$

n=1 ise, Taylor serisi

$$f(x) \simeq f(a) + f'(a)\,h, \qquad h = x-a$$

hata içeren şekli

$$f(x) = f(a) + h f'(a) + O(h^2),$$

$$O(h^2) = f''(a+\xi h)\frac{h^2}{2}, \quad 0<\xi<1.$$

iki boyutlu fonksiyonların taylor serileri

iki değişkenli $f(x,y)$ fonksiyonun, (a,b) civarındaki Taylor açılımı:

$$f(x,y) = f(a,b) + h f_x + g f_y \; \frac{1}{2}\left[h^2 f_{xx} + 2hg f_{xy} + g^2 f_{yy}\right]$$

$$+\frac{1}{6}\left[h^3 f_{xxx} + 3h^2 g f_{xxy} + 3hg^2 f_{xyy} + g^3 f_{yyy}\right]$$

$$+\frac{1}{24}\left[h^4 f_{xxxx} + 4h^3 g f_{xxxy} + 6h^2 g^2 f_{xxyy} + 4hg^3 f_{xyyy} + g^4 f_{yyyy}\right] + \cdots$$

$$h = x-a, \quad g = y-b$$

$$f_x = \frac{\partial}{\partial x} f(x,y)\Big|_{x=a,\, y=b}$$

$$f_y = \frac{\partial}{\partial y} f(x,y)\Big|_{x=a,\, y=b}$$

önek.

(a) Taylor serisi nümerik methodların türetilmesi ve hata analizi için çok önemli bir malzemedir.

(b) $x=0$ civarındaki taylor serisinin açılımına Maclaurin serisi denir.

b) $n=3$ Gauss-Legendre

$$x = \frac{(5-1)t + (5+1)}{2} = 2t+3 \quad dx = 2dt$$

$$2\int_{-1}^{1} \frac{\log((2t+3)^2+1)}{(2t+3)^4+1}\,dt = 2\cdot\left[\frac{5}{9}f\left(-\sqrt{\tfrac{3}{5}}\right) + \frac{8}{9}f(0) + \frac{5}{9}f\left(\sqrt{\tfrac{3}{5}}\right)\right]$$

$$= 0.2897$$

3) $\int_1^3 \frac{dx}{1+e^x}$

a) $n=6$ Simpson 3/8

$$h = \frac{3-1}{6} = \frac{2}{6} = \frac{1}{3}$$

$x_0 = 1$ $f(1) = 0.26894142$ $x_1 = \frac{4}{3}$ $f\left(\frac{4}{3}\right) = 0.20860853$

$x_2 = \frac{5}{3}$ $f\left(\frac{5}{3}\right) = 0.15886910$ $x_3 = 2$ $f(2) = 0.11920292$

$x_4 = \frac{7}{3}$ $f\left(\frac{7}{3}\right) = 0.08839968$ $x_5 = \frac{8}{3}$ $f\left(\frac{8}{3}\right) = 0.06496917$

$x_6 = 3$ $f(3) = 0.04742587$

$$\int_1^3 \frac{dx}{1+e^x} \cong \frac{3h}{8}\left[f(1) + 3f\left(\tfrac{4}{3}\right) + 3f\left(\tfrac{5}{3}\right) + 2f(2) + 3f\left(\tfrac{7}{3}\right) + 3f\left(\tfrac{8}{3}\right) + f(3)\right]$$

$$= 0.264664071777748$$

b) $$\int_1^3 \frac{dx}{1+e^x} = \int_1^3 \frac{1+e^x-e^x}{1+e^x}\,dx = \int_1^3 dx - \int_1^3 \frac{e^x}{1+e^x}\,dx$$

$1+e^x = u$

$e^x dx = du$

$$= x\Big|_1^3 - \int_{1+e^1}^{1+e^3} \frac{du}{u} = 2 - \ln(u)\Big|_{1+e^1}^{1+e^3} = 2 + \ln(1+e) - \ln(1+e^3)$$

$$= 0.264674335944448$$

Hata $= |\text{Gerçek} - \text{Yaklaşık}| = 1.0264167 \times 10^{-5}$

$= 0.0000102642$

$f(x) = \frac{1}{x}$ $\quad x_0 = 1$

$f(3) = \frac{1}{3}$ $\quad f(x) = x^{-1}$ $\quad f'(x) = -x^{-2}$

$f''(x) = (-1)^2 \, 2 \, x^{-3}$

$\vdots$

$f^{(n)}(x) = (-1)^n n! \, x^{-n-1} \qquad n \geq 0$

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!}(x-1)^k = \sum_{k=0}^{n} (-1)^k (x-1)^k$$

$f(3) = \frac{1}{3}$, $P_n(3) =$

n	0	1	2	3	4	5	6	7
$P_n(3)$	1	-1	3	-5	11	-21	43	-85

uzaklaşır.

Lagrange polinomları

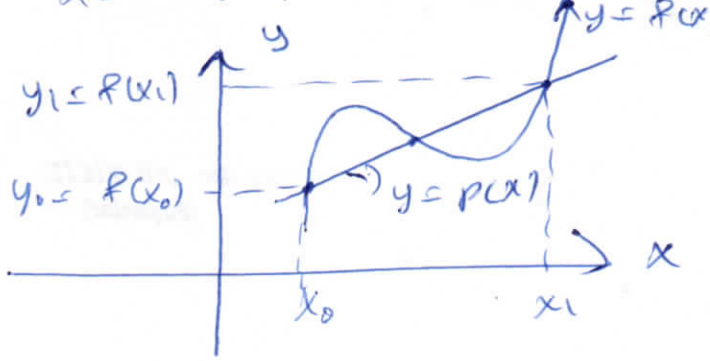
(x_0, y_0) ve (x_1, y_1) 1. dereceden bir polinom düşün.

$y_0 = f(x_0)$
$y_1 = f(x_1)$

$$p(x) = \frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1)$$

$$x = x_0 \to p(x_0) = 1 \cdot f(x_0) + 0 \cdot f(x_1) = f(x_0) = y_0$$

$$x = x_1 \to p(x_1) = 0 \cdot f(x_0) + 1 \cdot f(x_1) = y_1$$



$(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n)) \ldots$

$$p_n(x) = f(x_0) L_{n0}(x) + \ldots + f(x_n) L_{nn}(x) = \sum_{k=0}^{n} f(x_k) L_{nk}(x)$$

$$L_{nk}(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$ (3)

$k=0,1,\ldots,n$

<u>örnek</u> $x_0=2,\ x_1=2.5$ ve $x_2=4$ ve

$f(x)=\frac{1}{x}$ ise $L_0,\ L_1$ ve L_2 bulunur

$$L_0(x) = \frac{(x-2.5)(x-4)}{(2-2.5)(2-4)} = (x-6.5)x+10$$

$$L_1(x) = \frac{(x-2)(x-4)}{(2.5-2)(2.5-4)} = \frac{(-4x+24)x-32}{3}$$

$$L_2(x) = \frac{(x-2.5)(x-2)}{(4-2)(4-2.5)} = \frac{(x-4.5)x+5}{3}$$

$f(x_0)=f(2)=\frac{1}{2}=0.5 \quad f(2.5)=0.4$ ve $f(4)=0.25$

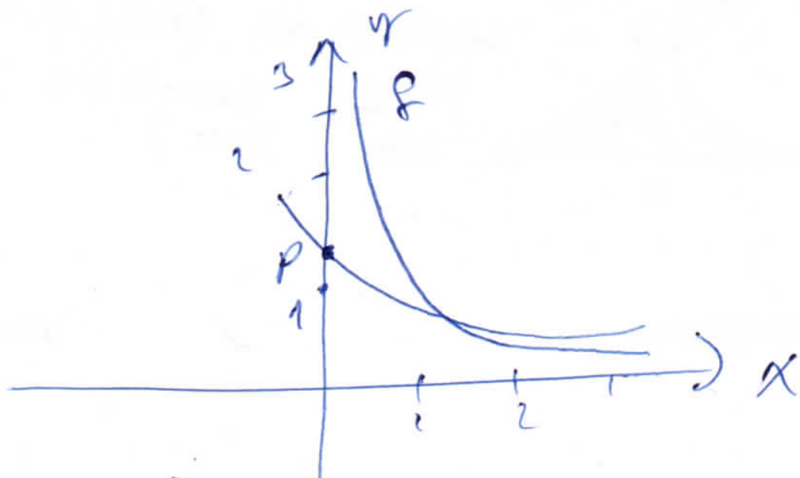
$$P_2(x) = \sum_{k=0}^{2} f(x_k)L_k(x)$$

$$= 0.5\left[(x-6.5)x+10\right] + 0.4\,\frac{(-4x+24)x-32}{3} + 0.25\,\frac{(x-4.5)x+5}{3}$$

$$=(0.05x-0.425)x+1.15$$

$f(3)=\frac{1}{3}$

$P(3)=0.325$



<u>Hata</u>

$$f(x) = P_n(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

$\exists\, \xi(x) \in (x_0,\ldots,x_n)$ ve x

örnek

x	$f(x)$
1.	0.7651977
1.3	0.6200860
1.6	0.4554022
1.9	0.2818186
2.2	0.1103623

$f(1.5) = ?$

(4)

i) $1.3 < 1.5 < 1.6 \Rightarrow$

$$P_1(1.5) = \frac{(1.5-1.6)}{(1.3-1.6)}(0.6200860) + \frac{(1.5-1.3)}{(1.6-1.3)} \cdot (0.4554022)$$

$$= 0.5102968$$

ii) $x_0 = 1.3,\quad x_1 = 1.6,\quad x_2 = 1.9$

$$P_2(1.5) = \frac{(1.5-1.6)(1.5-1.9)}{(1.3-1.6)(1.3-1.9)} \cdot (0.6200860)$$

$$+ \frac{(1.5-1.3)(1.5-1.9)}{(1.6-1.3)(1.6-1.9)} \cdot (0.4554022)$$

$$+ \frac{(1.5-1.3)(1.5-1.6)}{(1.9-1.3)(1.9-1.6)} (0.2818186) = 0.5112857$$

iii) $x_0 = 1.0,\quad x_1 = 1.3,\quad x_2 = 1.6 \Rightarrow \hat{P}_2(1.5) = 0.5124715$