CSE 321
Introduction to Algorithm Design
Homework 1

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1-

a) $\lim\limits_{n\to\infty} \dfrac{\log_2 n^2+1}{n} = \dfrac{(2\log n+1)'}{n'} = \dfrac{2\cdot\dfrac{n'}{n.\ln 2}}{1} = \dfrac{2}{n.\ln 2}$

$= \lim\limits_{n\to\infty} \dfrac{2}{\ln 2}\cdot\dfrac{1}{n} = 0$, thus, the statement is True

b) $\lim\limits_{n\to\infty} \dfrac{\sqrt{n^2+n}}{n} = \dfrac{\sqrt{n^2(1+\frac{1}{n})}}{n} = \dfrac{\cancel{n}.\sqrt{1+\frac{1}{n}}}{\cancel{n}} = 1$, thus, True

c) $\lim\limits_{n\to\infty} \dfrac{n^n.n^{-1}}{n^n} = \dfrac{1}{n} \to 0$, thus, False

d) $\lim\limits_{n\to\infty} \dfrac{2^n+n^3}{4^n} = \dfrac{2^n}{4^n} + \cancel{\dfrac{n^3}{4^n}} = \dfrac{1}{2^n} \to 0$, thus, True

e) $\lim\limits_{n\to\infty} \dfrac{\frac{2}{3}\log_3 n}{6\log_2 n} = \dfrac{1}{9}\cdot\dfrac{\log_3 n}{\log_2 n} = \dfrac{1}{9}\cdot\dfrac{\log_n 2}{\log_n 3} = \dfrac{\log_3 2}{9}$, False

f) $\lim\limits_{n\to\infty} \dfrac{\frac{1}{2}\cdot\cancel{\log_2 n}}{(\log_2 n)^{\cancel{2}}} = \dfrac{1}{2\log n} = 0$, False

2- $[10^n > 2^n > 8^{\log n} = n^3 > n^2\log n > n^2 \; \sqrt{n} > \log n]$

- $\lim\limits_{n\to\infty} \dfrac{10^n}{2^n} = 5^n \to \infty$

- $\lim\limits_{n\to\infty} \dfrac{2^n}{8^{\log n}} = \dfrac{2^n}{n^{\log 8}} = \dfrac{(2^n)'}{(n^3)'} = \dfrac{2^n.\log e}{3n^2} = \dots = \infty$

- $\lim\limits_{n\to\infty} \dfrac{8^{\log n}}{n^3} = \dfrac{n^{\log 8}}{n^3} = \dfrac{\cancel{n^3}}{\cancel{n^3}} = 1$

- $\lim\limits_{n\to\infty} \dfrac{n^3}{n^2\log n} = \dfrac{(n)'}{(\log n)'} = \dfrac{1}{\dfrac{1}{n.\ln 2}} = n.\ln 2 = \infty$

- $\lim_{n\to\infty} \frac{n^2 \log n}{n^2} = \log n = \underline{\underline{\infty}}$

- $\lim_{n\to\infty} \frac{n^2}{\sqrt{n}} = n^{\frac{3}{2}} = \underline{\underline{\infty}}$

- $\lim_{n\to\infty} \frac{(\sqrt{n})'}{(\log n)'} = \frac{\frac{1}{2} \cdot n^{-\frac{1}{2}}}{\frac{1}{n \cdot \ln 2}} = \frac{\sqrt{n} \cdot \ln 2}{2} = \underline{\underline{\infty}}$

3-

a) Since neither of the statements change the value of i and also again neither of the statements contain another loop, the function loops always n times. If we analyze in detail, we can see that, the operation count made by the function, depending on the best case and worst case scenarios, will be between 4n (for an array in descending-order) and n+6 (for an array in ascending-order). So the complexity is $\underline{\underline{O(n)}}$.

b) We can sum up this loops iteration as every two (which is constant so not important) iterations, $i = i^2 + i$ and we can discard i because we have i^2 which has the highest degree. So, for x iterations we have the equation,

$$n = i^{2^x}$$

Since we know that i is equal to 2,

$$n = 2^{2^x}$$

So, for a list (or limit) n, the complexity of iteration count x for reaching to the end of a list would be,

$$2^{2^x} = n \rightarrow \log 2^x = \log n \rightarrow \underline{\underline{x = \log \log n}}$$

So the complexity is $\underline{\underline{O(\log \log n)}}$

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4-

a) Non Decreasing:

$$\int_0^n f(x).dx \le f(x) \le \int_1^{n+1} f(x).dx$$

$$\int_b^a x^2 \log x . dx \rightarrow \quad u = \log x \quad dv = x^2$$

$$du = \frac{\log e}{x} \quad v = \frac{x^3}{3}$$

$$= \log x . \frac{x^3}{3} - \int_b^a \frac{x^{3^2}}{3} . \frac{\log e}{\cancel{x}} = \log x . \frac{x^3}{3} - \frac{x^3}{9} . \log e \Big|_b^a$$

$$= \frac{x^3 (3\log x - \log e)}{9} \Big|_b^a$$

$$\underbrace{\frac{x^3(3\log x - \log e)}{9} \Big|_0^n}_{\text{Undefined for } x=0} \le f(n) \le \underbrace{\frac{x^3(3\log x - \log e)}{9} \Big|_1^{n+1}}$$

$$f(n) \le \frac{(n+1)^3 (3\log(n+1) - \log e)}{9} - \frac{-\log e}{9}$$

So,

Lowerbond: $f(n) \in O(n^3 \log n)$,
Upperbond; $f(n) \in \Omega(n^3 \log n)$,
$f(n) \in \Theta(n^3 \log n)$

b) Non Decreasing

$$\int_0^n x^3 dx \le f(x) \le \int_1^{n+1} x^3 dx \rightarrow \frac{x^4}{4}\Big|_0^n \le f(x) \le \frac{x^4}{4}\Big|_1^{n+1}$$

$$\frac{n^4}{4} \le f(x) \le \frac{(n+1)^4 - 1}{4}, \quad f(n) \in \Theta(n^4)$$

c) Non Increasing

$$\int_0^n \frac{1}{2} . x^{-\frac{1}{2}} . dx \le f(x) \le \int_1^{n+1} \frac{1}{2} . x^{-\frac{1}{2}} . dx \quad \frac{\sqrt{x}}{4}\Big|_1^{n+1} \le f(x) \le \frac{\sqrt{x}}{4}\Big|_0^n$$

$$\frac{\sqrt{n+1} - 1}{4} \le f(n) \le \frac{\sqrt{n}}{4} \longrightarrow f(n) \in \Theta(\sqrt{n})$$

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d) Non Increasing

$$\underbrace{\int_1^{n+1} \frac{1}{x}\,dx}_{} \le f(n) \le \underbrace{\int_0^{n} \frac{1}{x}\,dx}_{\text{undefined for } x=0}$$

$\ln(n+1) \le f(n)$

- $f(n) \le 1 + \int_1^{n} \frac{1}{x}\,dx \to 1 + \ln x \Big|_1^n \to f(n) \le 1 + \ln(n)$

So,
$f(n) \in \Omega(\log n)$
$f(n) \in O(\log n)$
$f(n) \in \Theta(\log n)$

5- Since programs and compilers treats every element of an array as if they were distinct elements even if some of them are not and there is no relation between iteration of a linear array and contents of the elements, we will make all our calculations considering not the distinct object count but total object count. So, for an array of n elements,

$f(n) \in \Theta(1)$ for best case, which is the possiblelity of the object being the first element,

$g(n) \in \Theta(n)$ for worst case, which is the possiblelity of the object being the last element of the array.