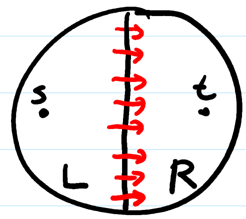
LINEAR PROGRAMS

Minimal Cut:

Def.: Given

- a directed graph $G=(V,E)$
- capacities $c_e \geq 0$, $e \in E$
- a source $s \in V$, a sink $t \in V$



an $\boxed{\text{s-t cut}}$ is a partition $V = L \cup R$ s.th. $s \in L$, $t \in R$

$$\text{capacity}(L,R) = \sum_{\substack{uv \in E \\ u \in L, v \in R}} c_{uv}$$

$$\boxed{\text{Min Cut}} = \min_{(L,R)} \text{capacity}(L,R)$$

Maximum Flow Problem

Maximize $\text{size}(f) := \sum_{v: sv \in E} f_{sv}$ (flow s to t)

s.th. f is a flow, i.e.,

$$0 \leq f_e \leq c_e \qquad \text{for all } e \in E$$

$$\sum_{uv \in E} f_{uv} = \sum_{vw \in E} f_{vw} \qquad \text{for all } v \neq s,t$$

$$\boxed{\text{Max Flow}} = \max_{f \text{ is a flow}} \text{size}(f) \qquad \text{(max. flow from s to t)}$$

Thm: If the capacities are integers

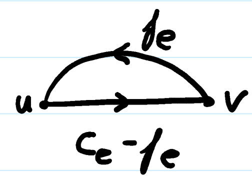
$$\boxed{\text{Max Cut} = \text{Min Flow}} = \text{size}(f)$$

where f is the output of the Ford-Fulkerson algor.

Residual Graph

Def: Given a graph G and a flow f on G, the residual graph G_f is obtained as follows:

For each edge $e = uv \in E$,



- the edge uv has capacity $c_e - f_e$
- we create back-edge vu with capacity f_e

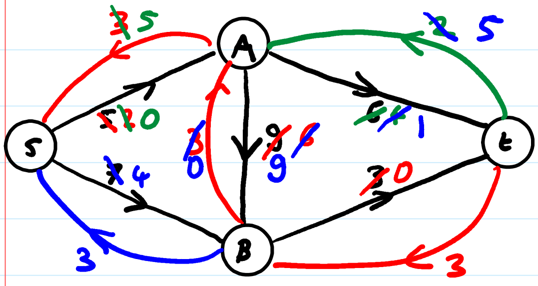
Ford Fulkerson Algorithm

1) Find path P from s to t which is not yet saturated in the residual Graph G_f
2) Add flow $\max_{e \in P} c_e(G_f)$ along P
3) Update the capities in the residual graph
4) Repeat until all paths P from s to t are saturated

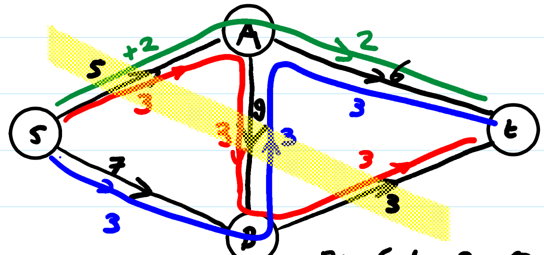
Rem: If the capacities are integers, Ford-Fulkerson assigns integer flows to all edges and terminates using at most $U = \text{Max Flow}$ "augmenting" paths P

Runtime: $\underbrace{\text{\# of augmenting path}}_{\leq U = \text{max-flow}} \times \underbrace{\text{time to find paths}}_{O(n+m)}$ depth first search

Example:



Residual Graph G_f



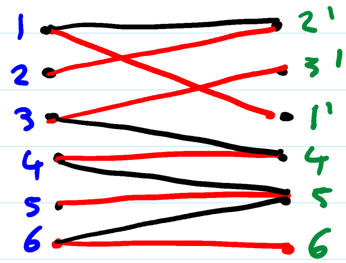
Min Cut = 3+5

Size(Flow) = 2+3+3 = 8

Bipartite Perfect Matching

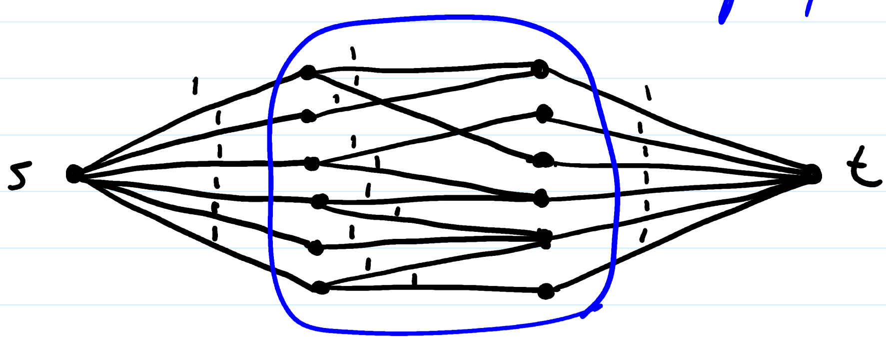
Input: Bipartite graph $G=(L,R,E)$ $|L|=|R|=n$

Output: A perfect matching M from U to V



$M \subseteq E$

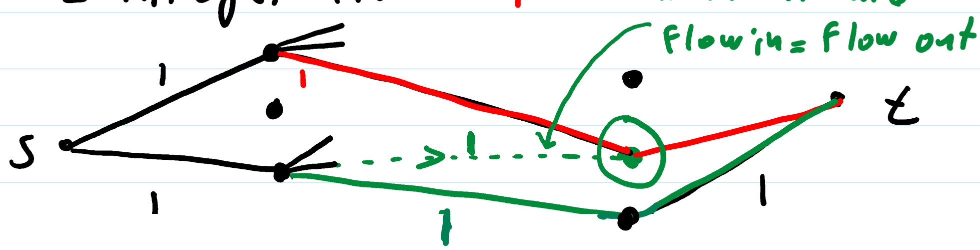
Solution via Max-Flow on new graph G'



G'

$\exists$ perfect matching $\Rightarrow$ $\underbrace{\text{we can send flow } n \text{ from } s \text{ to } t}_{*}$

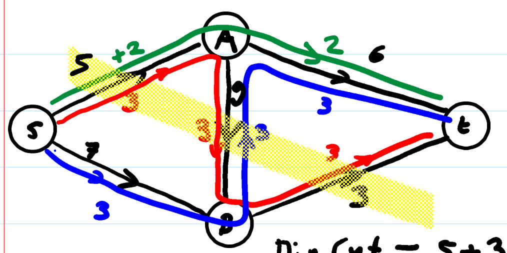
$* \Rightarrow \exists$ integer flow n



$\Rightarrow \exists$ perfect matching

$\Rightarrow$ We can solve the problem by running Ford-Fulkerson on G'

LP Duality



Last time, we used Max Flow $\leq$ Min Cut to prove that size(f) = 8 is optimal

$\boxed{\text{Works much more general!}}$

Example:

$$\max \ 5x_1 + 4x_2$$
s.th. $2x_1 + x_2 \leq 100$
$x_1 \leq 30$
$x_2 \leq 60$
$x_1, x_2 \geq 0$

$\boxed{\text{Solution: } x_1 = 20,\ x_2 = 60 \quad \text{value} = 340}$

$$\max \ 5x_1 + 4x_2$$
s.th. $2x_1 + x_2 \leq 100$
$(x_1 \leq 30)\cdot 5$
$(x_2 \leq 60)\cdot 4$

$$5x_1 + 4x_2 \leq \underbrace{150+240}_{390}$$

$$\max \ 5x_1 + 4x_2$$
s.th. $(2x_1 + x_2 \leq 100)\cdot 3$
$x_1 \leq 30$
$(x_2 \leq 60)\cdot 1$

$$6x_1 + 4x_2 \leq 360$$

$$\max \ 5x_1 + 4x_2$$
s.th. $(2x_1 + x_2 \leq 100)\cdot \frac{5}{2}$
$x_1 \leq 30$
$(x_2 \leq 60)\cdot \frac{3}{2}$

$$5x_1 + 4x_2 \leq \underbrace{250+90}_{340}$$

How did we get these magic numbers $5/2$, $3/2$

Primal LP
$$\begin{cases} \max & 5x_1 + 4x_2 \\ \text{s.th.} & (2x_1 + x_2 \le 100)\ y_1 \\ & (x_1 \le 30)\ y_2 \\ x_1, x_2 \geq 0 & (x_2 \le 60)\ y_3 \end{cases}$$

$$\Rightarrow (2y_1 + y_2)x_1 + (y_1 + y_3)x_2 \le 100y_1 + 30y_2 + 60y_3$$

as long as $y_1, y_2, y_3 \geq 0$

Best upper bd. on $5x_1 + 4x_2$

Dual LP
$$\begin{cases} \min & 100y_1 + 30y_2 + 60y_3 \\ \text{s.th.} & y_1, y_2, y_3 \geq 0 \\ & 2y_1 + y_2 \geq 5 \\ & y_1 + y_3 \geq 4 \end{cases}$$

By Construction: $\boxed{5x_1 + 4x_2 \le 100y_1 + 30y_2 + 60y_3}$

$\Rightarrow$ Primal LP OPT $\le$ Dual LP OPT

General Case:

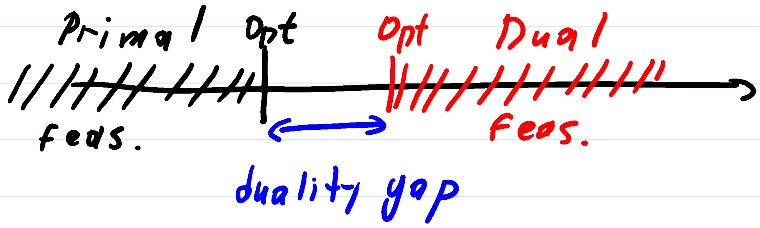
Primal LP	Dual LP
$\max\ c^T x$	$\min\ b^T y$
s.th. $x \geq 0$	s.th. $y \geq 0$
$Ax \le b$	$A^T y \geq c$

Thm: $\forall$ feasable sol. x of primal
$\forall$ —— " —— y of dual

$$\text{value}(x) \leq \text{value}(y)$$

Pf: $c^T x \leq y^T A x \leq y^T b = \sum_i y_i b_i = b^T y$

Cor: Opt Primal $\leq$ Opt Dual (weak duality)



Thm[Strong Duality]: If the primal has a bounded optimum $\Longrightarrow$ Opt Primal = Opt Dual

Ex.: Max Flow = Min Cut

More General Duality

Primal:

$\max c^T x$

s.t. $(Ax)_i \leq b_i \quad i \in I$

$(Ax)_i = b_i \quad i \notin I$

$x_j \geq 0 \quad j \in P$

Dual

$\min b^T y$

s.t. $(A^T y)_j \geq c_j \quad j \in P$

$(A^T y)_j = c_j \quad j \notin P$

$y_i \geq 0 \quad i \in I$

Two Player - Zero Sum Games

Input: Payoff Matrix M

$$\begin{array}{c|ccc} & \text{rock} & \text{pape} & \text{siss.} \\ \hline \text{rock} & 0 & -1 & 1 \\ \text{paper} & 1 & 0 & -1 \\ \text{siss.} & -1 & 1 & 0 \end{array}$$

Row Player: picks row r
Col. Player: picks col c

$\left.\begin{matrix}\text{Row Player: picks row } r \\ \text{Col. Player: picks col } c\end{matrix}\right\}$ Payoff $\begin{cases} M[r,c] \\ -M[r,c] \end{cases}$

2 types of strategies

"Pure strategy": a single row/column

e.g. row always plays rock (beaten by paper)

"Mixed strategy": probability distribution over pure strategies, e.g.

$Pr[\text{Rock}] = \frac{1}{3}$, $Pr[\text{Paper}] = \frac{1}{3}$ $Pr[\text{Siss.}] = \frac{1}{3}$

Note: Average Payoff is 0, no matter what row plays.

Holds for general zero sum game!

Who goes first?

Game 1:

Turn Order:

1. Row player announces mixed strategy $p = (p_1, p_2)$

2. Col player responds w/ mixed strategy $q = (q_1, q_2)$

$p_1 = Pr[\text{Row } 1]$
$p_2 = Pr[\text{Row } 2]$

$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 3 & -1 \\ 2 & -2 & 1 \\ \hline & q_1 & q_2 \end{array}$$

Def: Row player's average score: $\text{Score}(p,q) = 3p_1q_1 - 1p_1q_2 - 2p_2q_1 + 1p_2q_2$

Col player's best response $\quad \underset{\text{mixed strat. } q}{\text{minimize}} \ \text{Score}(p, q)$

$$= \underset{\text{pure strat.}}{\min} \{ \underbrace{3p_1 - 2p_2}_{\text{row 1}}, \underbrace{-p_1 + p_2}_{\text{row 2}} \}$$

Row players best strategy

$$\underset{\text{mixed strat. } p}{\text{maximize}} \ \min \{ 3p_1 - 2p_2, -p_1 + p_2 \}$$

<u>Claim:</u> This is a LP

<u>Pf:</u>

$$\left.\begin{aligned} \max \quad & x \\ \text{s.th.} \quad & x \le 3p_1 - 2p_2 \\ & x \le -p_1 + p_2 \end{aligned}\right\} \quad x = \min\{3p_1 - 2p_2, -p_1 + p_1\}$$

$$p_1, p_2 \ge 0, \ p_1 + p_2 = 1$$

<u>Game 2</u> [col. player goes first]

Given q, payoff row 1: $3q_1 - q_2$

—— " —— 2: $-2q_1 + q_2$

	1	2
1	3	-1
2	-2	1
	q_1	q_2

Row players best resp. $\max\{3q_1 - q_2, -2q_1 + q_2\}$

Col players best strat. $\underset{\text{mixed strat. } q}{\min} \ \max\{3q_1 - q_2, -2q_1 + q_2\}$

LP 2: $\min\ y$

s.th. $y \geq 3q_1 - q_2$

$y \geq -2q_1 + q_2$

$q_1, q_2 \geq 0,\ q_1 + q_2 = 1$

Game 1:
Row player first
Col. — " — 2nd

	1	2
1	3	−1
2	−2	1

Game 2:
Col. player first
Row — " — 2nd

strong duality

$$\max_p \min_q \text{Score}(p,q) \ \cancel{\leq} = \ \min_q \max_p \text{Score}(p,q)$$

weak duality

General Zero-Sum, 2 Player Game

$$\max_p \min_q \sum p_r M[r,c]\, q_c = \min_q \max_p \sum p_r M[r,c]\, q_c$$

LP-formulation for the general case

$$\max_p \min_q \sum_r p_r M[r,c] q_c = \max_p \min_c \sum_r p_r M[r,c]$$

$$= \max_p \max \{ x : x \leq \sum_r p_r M[r,c] \ \forall c \}$$

$$= \boxed{\begin{array}{ll} \max & x \\ \text{s.t.} & x \leq (M^T p)_c \ \ \forall c \\ & \sum p_r = 1,\ p_r \geq 0 \end{array}}$$

(primal LP)

In a similar way, the min max is given by

$$\boxed{\begin{array}{ll} \min & y \\ \text{s.t.} & y \geq (Mq)_r \quad \forall r \\ & \sum_c q_c = 1, \quad q_c \geq 0 \end{array}}$$

(dual LP)