CS294-248 Special Topics in Database Theory
Unit 9: Datalog

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University of Washington

Announcement

- Project presentations: Thursday, Nov. 30th, 9:30am, Calvin 146
- By Monday: please add your tentative topic here:
  `https://tinyurl.com/43mdvwzy`
- You can change the topic later, as you wish.

Outline

- Today: Basic Datalog
- Thursday: Extensions with Negation

Review

Motivation

- FO and its fragments cannot express simple, "easy" queries:
  - Transitive closure
  - Parity ("Is $|R|$ even?")
- Datalog: extends CQs with recursion

Datalog Syntax

- A program $P =$ set of rules.
- A rule is a CQ:
  H :- $A_1 \wedge A_2 \wedge \cdots$
- Extensional Database Predicates
  EDBs
- Intensional Database Predicates
  IDBs

$$\boxed{\begin{array}{l} T(X,Y) \text{ :- } E(X,Y) \\ T(X,Y) \text{ :- } T(X,Z) \wedge E(Z,X) \end{array}}$$

Pre-, Post-, and Fixpoints

Poset (partially ordered set) $(P, \leq)$.
We assume P has a minimal element $\perp$.

$f : P \rightarrow P$ is monotone if $x \leq y \Rightarrow f(x) \leq f(y)$.

x is a *pre-fixpoint* if $f(x) \leq x$

x is a *post-fixpoint* if $f(x) \geq x$

x is a *fixpoint* if $f(x) = x$;

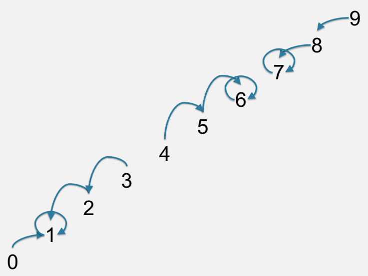
Pre-, Post-, and Fixpoints

What are the pre-, post-, fixpoints?

Pre-fixpoints:

Post-fixpoints:

Fixpoints:



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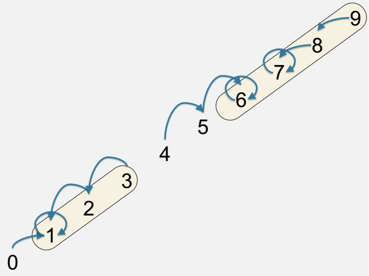

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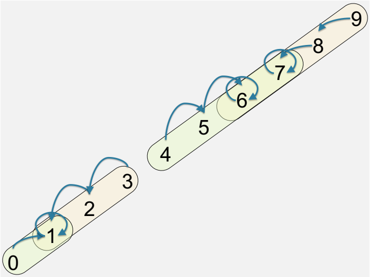

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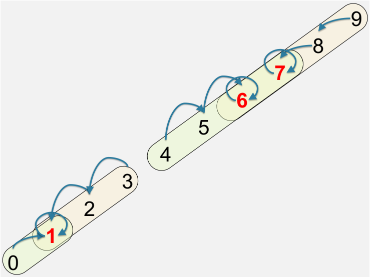

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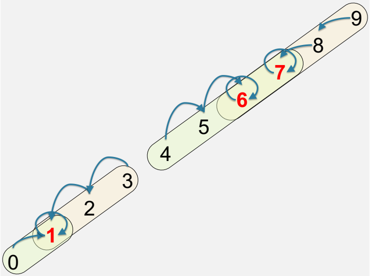

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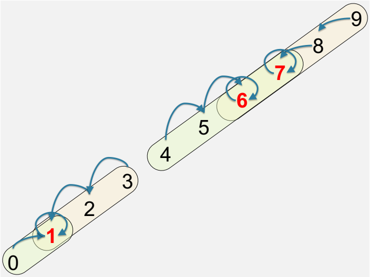

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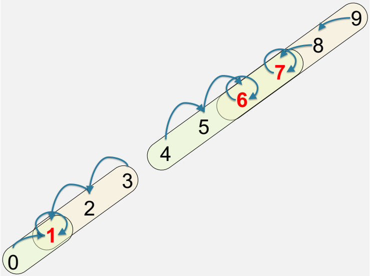

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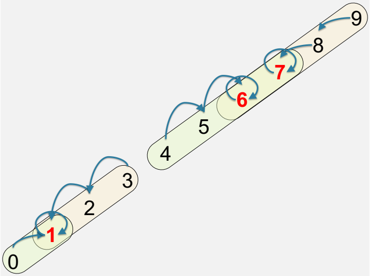

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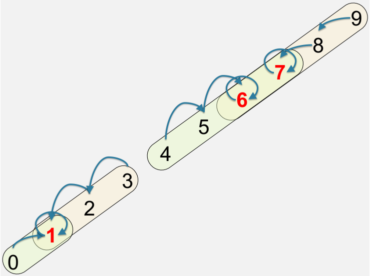

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$f(z) \leq z$

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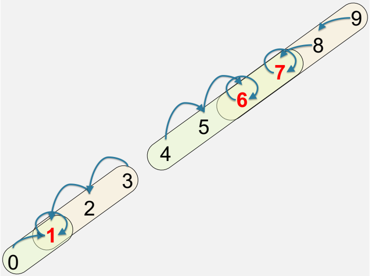

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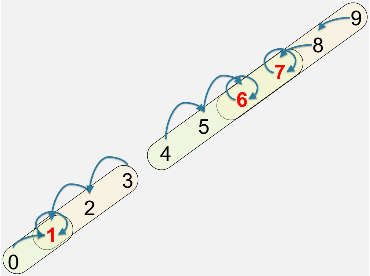

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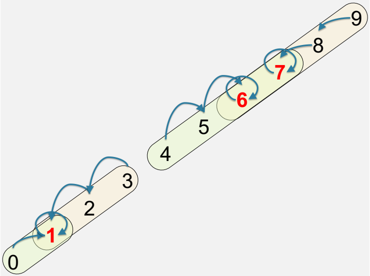

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Kleene's Sequence

$$\boxed{f^{(0)} \stackrel{\text{def}}{=} \perp \qquad f^{(t+1)} \stackrel{\text{def}}{=} f(f^{(t)})} \quad f^{(0)} \leq f^{(1)} \leq f^{(2)} \leq \cdots$$

Fact

If z is any pre-fixpoint, then $f^{(t)} \leq z$ for all t.

Proof by induction: $\perp \leq z$ and $f^{(t+1)} = f(f^{(t)}) \leq f(z) \leq z$.

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Is $\bigvee_{t \geq 0} f^{(t)}$ the least fixpoint?

Not always. Two problems:

- $\bigvee_{t \geq 0} f^{(t)}$ may not exists.
- Even if it exists, we may have $f(\bigvee_{t \geq 0} f^{(t)}) \neq \bigvee_{t \geq 0} f^{(t)}$.

We will circumvent by requiring finite rank

The Rank of a Poset

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- What is the rank of $(P_1, \leq_1) \times (P_2, \leq_2)$? $r = r(P_1) + r(P_2)$.

Fixpoints in Posets of Finite Ranks

$$\boxed{f^{(0)} \stackrel{\text{def}}{=} \perp \qquad f^{(t+1)} \stackrel{\text{def}}{=} f\big(f^{(t)}\big)} \quad f^{(0)} < f^{(1)} < f^{(2)} < \cdots \leq f^{(r)} = f^{(r+1)}$$

Theorem

If P has finite rank r then $\text{lfp}(f) = f^{(r)}$.

Least Fixpoint Semantics of a Datalog Program P

I = an EDB instance, $A \stackrel{\text{def}}{=} \text{ADom}(I)$.

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Immediate Consequence Operator:

$$\boxed{T_P : \mathcal{P}(A^{k_1}) \times \mathcal{P}(A^{k_2}) \times \cdots \rightarrow \mathcal{P}(A^{k_1}) \times \mathcal{P}(A^{k_2}) \times \cdots}$$

The semantics of the datalog program P is $\text{lfp}(T_p)$.

Naive Evaluation Algorithm

$$
\boxed{
\begin{array}{l}
J^{(0)} := \emptyset \\
\texttt{for } t = 0, \infty \\
\quad J^{(t+1)} := T_P(J^{(t)}) \\
\quad \texttt{if } J^{(t+1)} = J^{(t)} \texttt{ break}
\end{array}
}
$$

Notice: $J^{(0)} \subseteq J^{(1)} \subseteq \cdots$ is Kleene's sequence.

Theorem

The Naive Algorithm takes $\leq ADom(I)^k$ iterations, where I is the EDB instance and k is the largest arity of any IDB.

Data complexity is in PTIME.

Examples in Datalog

Overview

- We have seen Transitive Closure. Can we write something different?
- Regular expressions, CFGs.
- Same generation.
- AND/OR reachability.

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Find nodes reachable from 1 with a path labeled with $(aab|aaac)^*$

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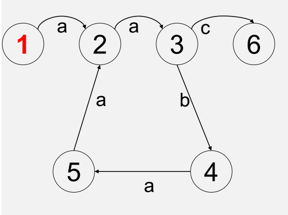
EDB graph:



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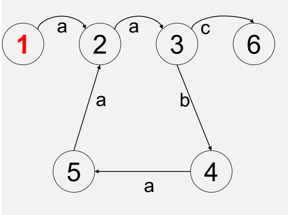

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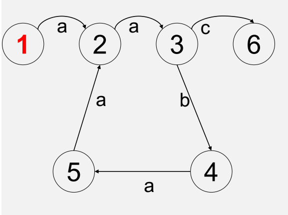

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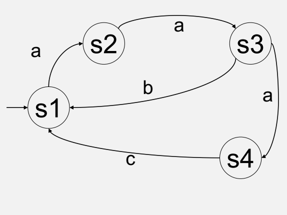

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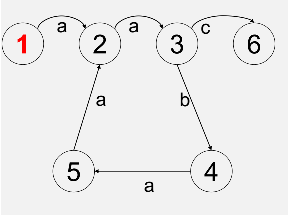

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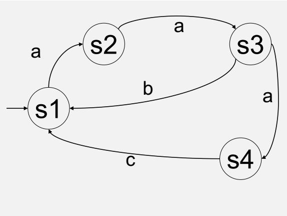

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$$Q1(1) :-$$

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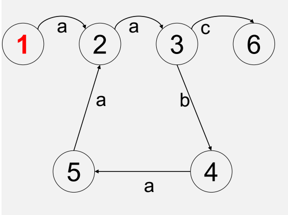

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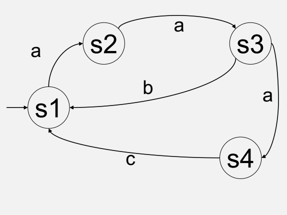

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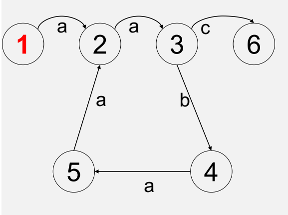

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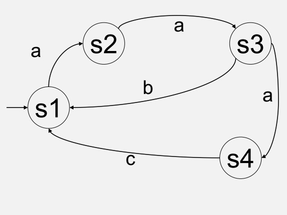

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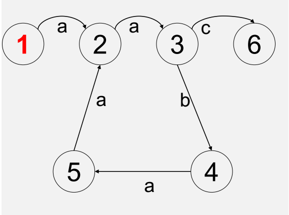

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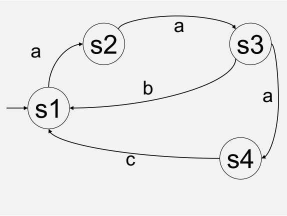

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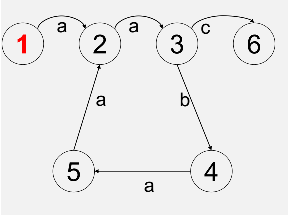

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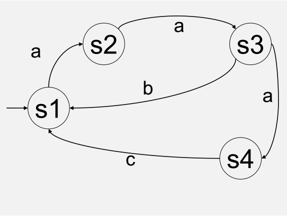

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$$Q4(Y) \text{ :- } Q3(X) \wedge E(X, Y, 'a') \qquad Q1(Y) \text{ :- } Q4(X) \wedge E(X, Y, 'c')$$

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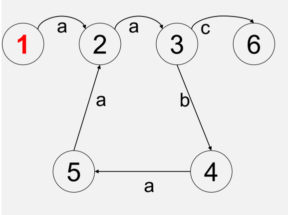

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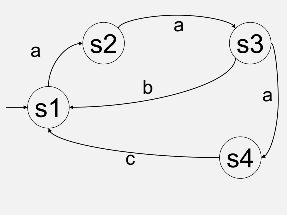

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$$\text{Answer}(X) \text{ :- } Q1(X)$$

Discussion

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- Also CFG. E.g language of parentheses: $S \to \epsilon | SS | aSb$. How?

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- Also CFG. E.g language of parentheses: $S \to \epsilon|SS|aSb$. How?

$$S(X, X) \text{ :- } \text{Node}(X)$$
$$S(X, Y) \text{ :- } S(X, Z) \wedge S(Z, Y)$$
$$S(X, Y) \text{ :- } E(X, U, 'a') \wedge S(U, V) \wedge E(V, Y, 'b')$$

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- Exercise**: non-CFG, e.g. the language $\{a^n b^n c^n \mid n \in \mathbb{N}\}$.

(won't discuss in class)

$$T(X,X,Y,Y,Z,Z) \text{ :- } \text{Node}(X) \wedge \text{Node}(Y) \wedge \text{Node}(Z)$$
$$T(X_1,X_2,Y_1,Y_2,Z_1,Z_2) \text{ :- } T(X_1,X_3,Y_1,Y_3,Z_1,Z_3) \wedge E(X_3,X_2,'a') \wedge E(Y_3,Y_2,'b') \wedge E(Z_3,Z_2,'c')$$
$$\text{Answer}(X,Y) \text{ :- } T(X,U,V,U,V,Y)$$

Same Generation

x, y are at the same generation
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at the same distance.

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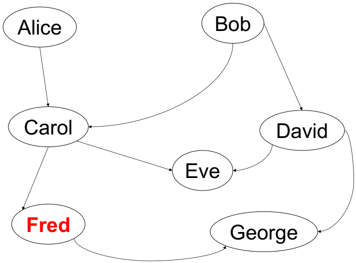

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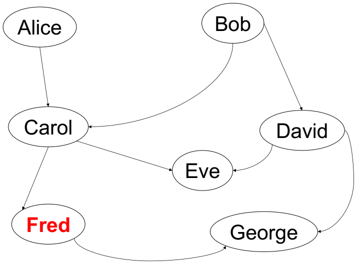

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$SG(X, X)$:- Person(X)
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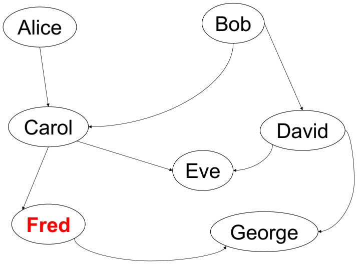

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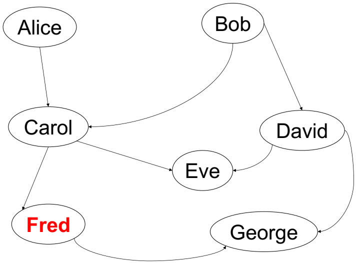

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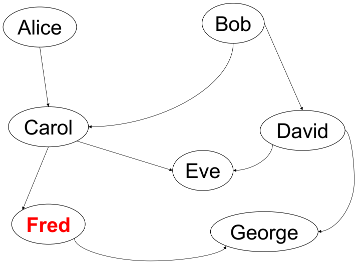

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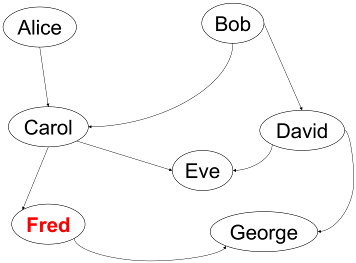
$$SG(X, Y) \text{ :- } SG(U, V) \wedge E(U, X) \wedge E(V, Y)$$
$$\text{Answer}(X) \text{ :- } SG('\text{Fred}', X)$$

EDB graph:



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Discussion

- The examples so far are still just transitive at their essence! why?

- Recall that transitive closure is in NLOGSPACE. The next example goes beyond NLOGSPACE.

AND/OR-Graph Accessibility

OR-nodes: unlimited AND-children

AND-nodes: two OR-children

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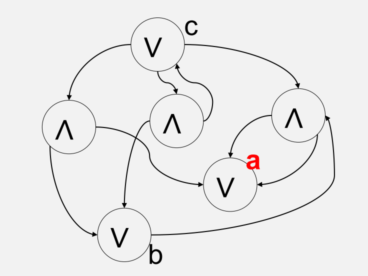

T

X	Y	Z
c	a	b
c	b	c
c	a	a
b	a	a

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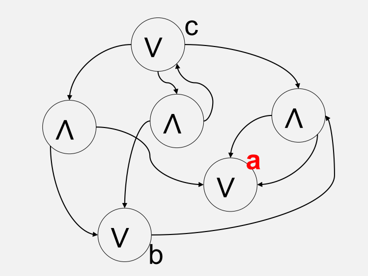

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Answer: a, b, c.

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$A(a)$:-
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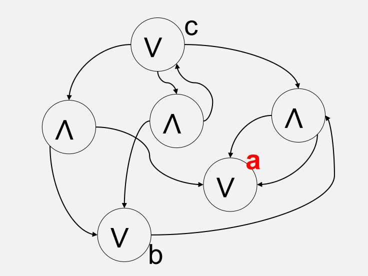

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Answer: a, b, c.

Discussion

- AGAP is PTIME-complete. Recall: NLOGSPACE $\subseteq$ PTIME and inclusion is conjecture to be strict.
- It follows that datalog can express strictly more than transitive closure.
- The data complexity of datalog is in PTIME.
- Limitation of "pure" datalog: monotone queries only.
- Montone queries have huge potential for optimizations (next).

Optimizing Monotone Datalog

Outline

- Semi-naive evaluation.
- Asynchronous execution: also discuss grounding.
- Will not discuss: Magic Set optimization

Naive, and Semi-naive

Naive

$$
\begin{aligned}
&J^{(0)} := \emptyset \\
&\texttt{for}\ \ t = 0, \infty \\
&\quad J^{(t+1)} := T_P(J^{(t)}) \\
&\quad \texttt{if}\ \ J^{(t+1)} = J^{(t)} \\
&\qquad\quad \texttt{break}
\end{aligned}
$$

Naive, and Semi-naive

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$\qquad \texttt{break}$

Semi-naive

$J^{(0)} := \emptyset$

$\texttt{for}\ \ t = 0, \infty$

$\quad \Delta^{(t)} := T_P(J^{(t)}) - J^{(t)}$

$\quad J^{(t+1)} := J^{(t)} \cup \Delta^{(t)}$

$\quad \texttt{if}\ \ \Delta^{(t)} = \emptyset\ \ \texttt{break}$

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w/ incremental computation

$J^{(0)} := \emptyset, \quad \Delta^{(0)} := T_P(\emptyset)$

$\texttt{for}\ \ t = 1, \infty$

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$\qquad\qquad = \Delta T_P(J^{(t-1)}, \Delta^{(t-1)}) - J^{(t)}$

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Naive	Semi-naive	w/ incremental computation
$J^{(0)} := \emptyset$	$J^{(0)} := \emptyset$	$J^{(0)} := \emptyset, \quad \Delta^{(0)} := T_P(\emptyset)$
for $t = 0, \infty$	**for** $t = 0, \infty$	**for** $t = 1, \infty$
$J^{(t+1)} := T_P(J^{(t)})$	$\Delta^{(t)} := T_P(J^{(t)}) - J^{(t)}$	$\Delta^{(t)} := T_P(J^{(t-1)} \cup \Delta^{(t-1)}) - J^{(t)}$
if $J^{(t+1)} = J^{(t)}$	$J^{(t+1)} := J^{(t)} \cup \Delta^{(t)}$	$= \Delta T_P(J^{(t-1)}, \Delta^{(t-1)}) - J^{(t)}$
break	**if** $\Delta^{(t)} = \emptyset$ **break**	$J^{(t+1)} := J^{(t)} \cup \Delta^{(t)}$
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$T(X,Y)$:- $E(X,Y)$

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Naive, and Semi-naive

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Discussion

Semi-naive is implemented by virtually all datalog systems.

Non-linear datalog rules have more complex delta-queries:

- Exponential number of queries:

$$(A \cup \Delta A) \bowtie (B \cup \Delta B) \bowtie (C \cup \Delta C)$$

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- Mix of old/new tables (issue: new tables are bigger):

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Asynchronous Execution

- (Semi-) naive is synchronous: apply all rules to all tuples.
- Asynchronous execution: apply some rules to some tuples.
- Simple principle: fair computation of a fixpoint.

Asynchronous Sequence

Posets $(P_1, \leq), (P_2, \leq)$, finite ranks, $f : P_1 \times P_2 \to P_1$, $g : P_1 \times P_2 \to P_2$.

Goal compute $\text{lfp}(f, g)$: $\boxed{(f(x, y), g(x, y)) = (x, y)}$

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Kleene's sequence:

$$(x^{(0)}, y^{(0)}) \stackrel{\text{def}}{=} (\perp, \perp)$$

$$(x^{(t+1)}, y^{(t+1)}) \stackrel{\text{def}}{=} (f(x^{(t)}, y^{(t)}), g(x^{(t)}, y^{(t)}))$$

Every step is an *fg*-step.

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Asynchronous sequence:

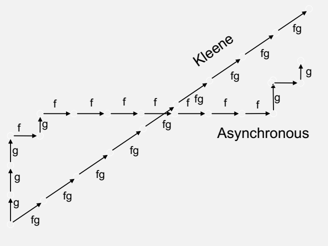
$$(u^{(0)}, v^{(0)}) \stackrel{\text{def}}{=} (\perp, \perp)$$

$$(u^{(k+1)}, v^{(k+1)}) \stackrel{\text{def}}{=} \begin{cases} (f(u^{(k)}, v^{(k)}), g(u^{(k)}, v^{(k)})) & \text{or} \\ (f(u^{(k)}, v^{(k)}), v^{(k)}) & \text{or} \\ (u^{(k)}, g(u^{(k)}, v^{(k)})) & \end{cases}$$

fg-step, or *f*-step, or *g*-step.

Fair Computation of a Fixpoint

Fact 1: for any pre-fixpoint (x, y) of (f, g), $(u^{(k)}, v^{(k)}) \leq (x, y)$.



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Sequence is fair if: $\forall k \exists m > k \exists n > k$ s.t:

- m is an f-step or fg-step, and
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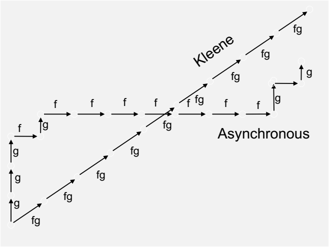

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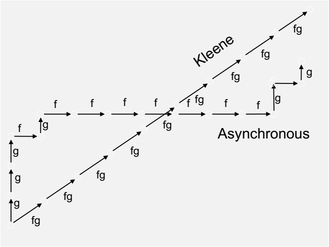

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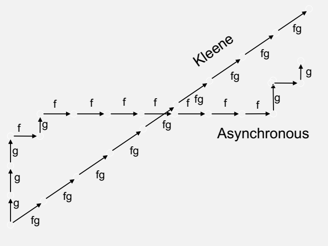

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Proof suffices to prove: $\forall t \exists k$, $(x^{(t)}, y^{(t)}) \leq (u^{(k)}, v^{(k)})$.

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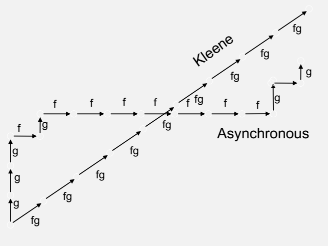

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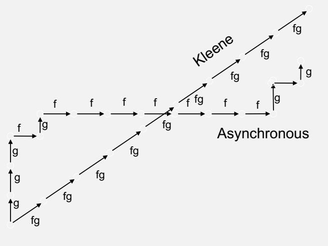
Proof suffices to prove: $\forall t \exists k$, $(x^{(t)}, y^{(t)}) \leq (u^{(k)}, v^{(k)})$.

$$(x^{(t+1)}, y^{(t+1)}) = (f(x^{(t)}, y^{(t)}), g(x^{(t)}, y^{(t)})) \leq (f(u^{(k)}, v^{(k)}), g(u^{(k)}, v^{(k)}))$$

Fair Computation of a Fixpoint

Fact 1: for any pre-fixpoint (x, y) of (f, g), $(u^{(k)}, v^{(k)}) \leq (x, y)$.



Sequence is fair if: $\forall k \exists m > k \exists n > k$ s.t:

- m is an f-step or fg-step, and
- n is a g-step or fg-step.

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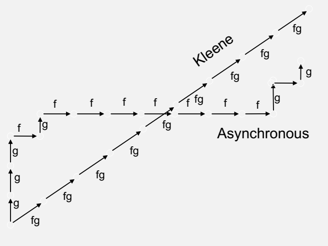

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$$\leq (f(u^{(m)}, v^{(m)}), g(u^{(n)}, v^{(n)})) = (u^{(m+1)}, v^{(n+1)}) \leq (u^{(p)}, v^{(p)})$$

where $p = \max(m, n) + 1$.

Discussion

- Kleene's sequence has rank $\text{rank}(P_1) + \text{rank}(P_2)$; the asynchronous sequence could be as long as $\text{rank}(P_1) \times \text{rank}(P_2)$
- Application: nested recursion

$$\begin{aligned}\text{lfp}(f, g) = \quad & \texttt{let } u = \text{lfp}(\lambda x. \quad \texttt{let } v = \text{lfp}(\lambda y.g(x, y)) \\ & \qquad\qquad\qquad\qquad\quad \texttt{in } (f(x, v), v)) \\ & \texttt{in } (u, \text{lfp}(\lambda y.g(u, y)))\end{aligned}$$

  RHS is asynchronous sequence with steps $ggg \cdots fggg \cdots fggg \cdots$

- Immediate generalization to n posets $(P_1, \leq) \times \cdots (P_n, \leq)$.

Grounding of a Datalog Program

What are the posets $(P_1, \leq), (P_2, \leq), \ldots$ for a datalog program?

- Option 1: P_i is $(\text{ADom}^{k_i}, \subseteq)$ represents an IDB predicate.
- Option 2 (better): P_i is $(\{0, 1\}, \leq)$ represents an IDB tuple.

Example

$R(X)$:- $E(a, X)$
$R(X)$:- $R(Z) \wedge E(Z, X)$

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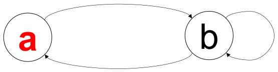
EDB input graph:



Example

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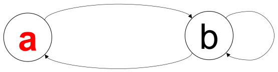
Grounded program:

$R(a)$:- $E(a, a)$
$R(a)$:- $R(a) \wedge E(a, a)$
$R(a)$:- $R(b) \wedge E(b, a)$
$R(b)$:- $E(a, b)$
$R(b)$:- $R(a) \wedge E(a, b)$
$R(b)$:- $R(b) \wedge E(b, b)$

EDB input graph:



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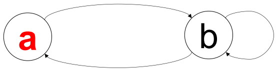


$R(a)$:- $E(a, a) \vee R(a) \wedge E(a, a) \vee R(b) \wedge E(b, a)$,
$R(b)$:- $E(a, b) \vee R(a) \wedge E(a, b) \vee R(b) \wedge E(b, b)$

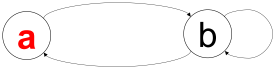
Example

$R(X)$:- $E(\textcolor{red}{a}, X)$
$R(X)$:- $R(Z) \wedge E(Z, X)$

Grounded program:

$R(a)$:- $E(a, a)$
$R(a)$:- $R(a) \wedge E(a, a)$
$R(a)$:- $R(b) \wedge E(b, a)$
$R(b)$:- $E(a, b)$
$R(b)$:- $R(a) \wedge E(a, b)$
$R(b)$:- $R(b) \wedge E(b, b)$

EDB input graph:



$R(a)$:- $E(a, a) \vee R(a) \wedge E(a, a) \vee R(b) \wedge E(b, a)$,
$R(b)$:- $E(a, b) \vee R(a) \wedge E(a, b) \vee R(b) \wedge E(b, b)$

The grounded program allows more fine-grained asynchronous execution.

Summary

- Main purpose of datalog is to add recursion.
- Least-fixpoint semantics; Kleene's sequence; Naive algorithm.
- Cool optimizations: semi-naive, magic-sets (difficult!), asynchronous evaluation.
- Can express PTIME-complete problems (AGAP).
- But limited to monotone queries.

Next lecture: adding negation to datalog.

Stanley, R. P. (1999).
Enumerative combinatorics. Vol. 2, volume 62 of *Cambridge Studies in Advanced Mathematics*.
Cambridge University Press, Cambridge.
With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.