

# CS294-248 Special Topics in Database Theory

## Unit 2: Conjunctive Queries

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# Query Evaluation for CQ

# Motivation

We already know that the data complexity is in  $AC^0$ .

What is the expression complexity? The combined complexity?

Will answer both, and also discuss the expression/combined complexity for FO (which we left out).

Importantly: we will define query evaluation for CQ in terms of  
**Homomorphisms**

# Equivalent Concepts

- A Conjunctive Query:

$$R(x, y, z) \wedge S(x, u) \wedge S(y, v) \wedge S(z, w) \wedge R(u, v, w)$$

- A database instance:

$$R(A, B, C) =$$

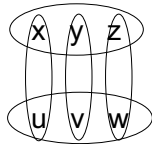
A	B	C
x	y	z
u	v	w

$$S(D, E) =$$

D	E
x	u
y	v
z	w

- A labeled hypergraph,  $G = (V, E)$ , where

$V = \{x, y, z, u, v, w\}$ ,  $E = \{\{x, y, z\}, \{u, v, w\}, \{x, u\}, \{y, v\}, \{z, w\}\}$   
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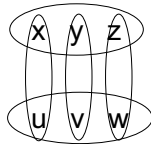
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# Homomorphisms

$$Q(\mathbf{x}_0) = R_1(\mathbf{x}_1) \wedge \cdots \wedge R_m(\mathbf{x}_m), \quad Q'(\mathbf{y}_0) = S_1(\mathbf{y}_1) \wedge \cdots \wedge S_n(\mathbf{y}_n).$$

## Definition

A **homomorphism**  $h : Q' \rightarrow Q$  is a function

$h : \text{Const}(Q') \cup \text{Vars}(Q') \rightarrow \text{Const}(Q) \cup \text{Vars}(Q)$  s.t.:

- $\forall c \in \text{Const}(Q'), h(c) = c.$
- $S_j(\mathbf{y}_j) \in \text{Atoms}(Q'), \exists R_i(\mathbf{x}_i) \in \text{Atoms}(Q)$  such that  $R_i = S_j$  (they are the same relation name) and  $h(\mathbf{y}_j) = \mathbf{x}_i.$
- $h$  maps head vars to head vars:  $h(\mathbf{y}_0) = \mathbf{x}_0.$

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# Query Evaluation for CQ and Homomorphisms

Computing  $Q(\mathbf{D})$  consists of finding all homomorphisms  $h : Q \rightarrow D$  and returning  $h(\text{Head}(Q))$ .

$$Q(x) = R(x) \wedge S(x, y) \wedge T(y, 'a')$$

$R =$	<table><tr><th><math>x</math></th></tr><tr><td>1</td></tr><tr><td>2</td></tr></table>	$x$	1	2	$S =$	<table><tr><th><math>x</math></th><th><math>y</math></th></tr><tr><td>1</td><td>10</td></tr><tr><td>1</td><td>20</td></tr><tr><td>2</td><td>20</td></tr></table>	$x$	$y$	1	10	1	20	2	20	$T =$	<table><tr><th><math>y</math></th><th><math>z</math></th></tr><tr><td>10</td><td><math>a</math></td></tr><tr><td>10</td><td><math>b</math></td></tr><tr><td>20</td><td><math>a</math></td></tr></table>	$y$	$z$	10	$a$	10	$b$	20	$a$
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We list all homomorphisms:

	$x(= \text{Head}(Q))$	$y$	$a$
$h =$	1	10	$a$
	1	20	$a$
	2	20	$a$

Final answer after duplicate elimination:  $Q(\mathbf{D}) = \{1, 2\}$ .



# The Combined Complexity for UCQ is in NP

## Theorem

*The combined complexity for UCQ is in NP.*

**Proof:** Fix a UCQ  $Q = Q_1 \vee Q_2 \vee \dots$  and a database  $D$ .

To check  $D \models Q$ :

- “guess” a CQ  $Q_i$ , and
- “guess” a homomorphism  $h : Q_i \rightarrow D$

# The Expression Complexity for CQ is NP-hard

## Theorem

*There exists a database  $D$  for which the expression complexity of CQ queries is NP complete.*

Thus, the expression complexity is also NP-complete.

**Proof** Many proofs are possible (will explain shortly why). We will use reduction from 3SAT, because we will reuse it a few times.

Given a 3CNF formula  $\Phi$  we construct  $Q_\Phi, D$  such that:

$\Phi$  is satisfiable iff  $\exists h : Q_\Phi \rightarrow D$ .

Notice that  $D$  is independent of  $\Phi$ .

Details next.

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## Reduction from 3SAT to CQ Evaluation

Given a 3CNF formula  $\Phi$  we construct  $Q_\Phi, \mathbf{D}$  such that:

$\Phi$  is satisfiable iff  $\exists h : Q_\Phi \rightarrow \mathbf{D}$ .

$Q_\Phi$  has one atom for each clause  $C$  in  $\Phi$ :

- If  $C = (X_i \vee X_j \vee X_k)$  then  $Q_\Phi$  contains  $A(x_i, x_j, x_k)$ .
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$\mathbf{D}$  has 4 tables with 7 tuples each **which tuple is missing?**

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# Combined Complexity for FO

Recall that the combined complexity of FO is in PSPACE.

## Theorem

*There exists a database  $\mathbf{D}$  for which the expression complexity of FO queries is PSPACE complete.*

Thus, the combined complexity is also PSPACE-complete.

**Proof:** Reduction from the Quantified Boolean Formula Satisfiability:

$$Q_1 X_1 \ Q_2 X_2 \ \cdots \ Q_n X_n \ \Phi$$

where  $\Phi$  is 3CNF.

Use the same  $Q_\Phi, \mathbf{D}$  before, but add appropriate quantifiers to  $Q_\Phi$ :

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## Discussion: CQ and CSP

The generalized Constraint Satisfaction Problem is:

Definition ([Kolaitis and Vardi, 1998])

Given two classes of finite structures  $\mathcal{A}, \mathcal{B}$ , the  $CSP(\mathcal{A}, \mathcal{B})$  problem is:  
Given  $A \in \mathcal{A}, B \in \mathcal{B}$ , is there a homomorphism  $h : A \rightarrow B$ ?

Standard CSP restricts the right-hand side,  $CSP(-, B)$ .

What is  $B$  for 3SAT? For 3-colorability? For Hamiltonian path?

Query evaluation restricts the left-hand side,  $CSP(Q, -)$

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# Summary

- Evaluating  $Q(\mathbf{D})$  consists of finding homomorphisms  $h : Q \rightarrow \mathbf{D}$ .
- This problem is in NP, in fact it is the very definition of NP.
- If  $Q$  is fixed, then the problem is in PTIME in  $|\mathbf{D}|$ . **Data complexity**
- If  $Q$  is part of the input (i.e. can be huge) then NP-complete.  
**Expression complexity**



# Acyclic Queries

# Motivation

How efficiently can we compute a conjunctive query  $Q$  on a database  $\mathbf{D}$ ?  
 $N \stackrel{\text{def}}{=} |\text{ADom}(\mathbf{D})|$ ,  $M \stackrel{\text{def}}{=} \max_i |R_i^{\mathbf{D}}|$ .

- Nested for-loops:

```
for  $x_1$  in ADom
  for  $x_2$  in ADom
    ...
```

Runtime:  $O(N^{|\text{Vars}(Q)|})$ .

- Joins:

$$(\dots (R_1 \bowtie R_2) \bowtie R_2 \dots) \bowtie R_m$$

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Both are  $O(|\text{Input}|^{O(1)})$ . We would like:  $\tilde{O}(|\text{Input}| + |\text{Output}|)$

Semijoin reduction, and tree decomposition.

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<sup>1</sup>Recall:  $\tilde{O}(f(N))$  means  $O(f(N) \log N)$ .

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# Joins, Semijoins

Suppose relations  $A(\mathbf{x}, \mathbf{y})$ ,  $B(\mathbf{x}, \mathbf{z})$  have common variables  $\mathbf{x}$ .

## Definition

Join  $A \bowtie B$ :  $J(\mathbf{x}, \mathbf{y}, \mathbf{z}) = A(\mathbf{x}, \mathbf{y}) \wedge B(\mathbf{x}, \mathbf{z})$ .

(Left) Semi-join  $SJ = A \ltimes B$ :  $SJ(\mathbf{x}, \mathbf{y}) = A(\mathbf{x}, \mathbf{y}) \wedge B(\mathbf{x}, \mathbf{z})$ .

## Fact

$A \bowtie B$  can be computed in time  $\tilde{O}(|A| + |B| + |A \bowtie B|)$ .

$A \ltimes B$  can be computed in time  $\tilde{O}(|A| + |B|)$ .

# Joins, Semijoins: Properties

- $A \bowtie B \subseteq A$ .
- $A \bowtie B = (A \bowtie B) \bowtie B$ .
- $A \bowtie B = \Pi_{\text{Vars}(A)}(A \bowtie B)$ .

$A := A \bowtie B$  doesn't increase size.

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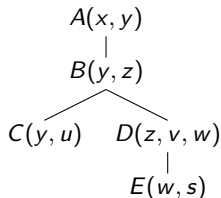
# Acyclic Query

## Definition

$Q$  is **acyclic** if it admits a **join tree**, which is a tree  $T$  where:

- The nodes in  $T$  are in 1-1 correspondence with the atoms in  $Q$ .
- $T$  satisfies the **running intersection property**: for any variable, the set of nodes that contain it forms a connected component.

Acyclic:  $Q = A(x, y) \wedge B(y, z) \wedge C(y, u) \wedge D(z, v, w) \wedge E(w, s)$





# Acyclic Query

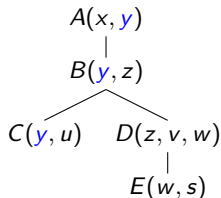
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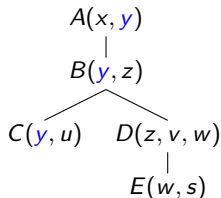
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Not acyclic:  $A(x, y) \wedge B(y, z) \wedge C(z, x)$ . **why?**



# Acyclic Query - GYO

## GYO Acyclicity Test (Graham and Yu-Oszoyoglu)

Repeat:

- Remove an **isolated variable** (i.e. occurs in only one atom).
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$Q$  is a acyclic iff result is one empty edge.

**Proof: exercise.**

Which var is **isolated**?  $Q = A(x, y) \wedge B(y, z) \wedge C(y, u) \wedge D(z, v, w) \wedge E(w, s)$

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# Yannakakis' Algorithm: Boolean Query

Boolean, acyclic query  $Q() = \exists x_1 \exists x_2 \dots$ , join tree  $T$ .

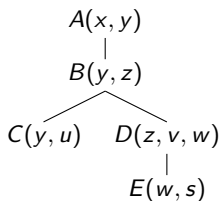
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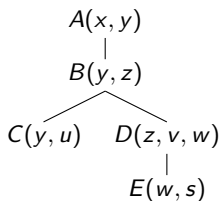


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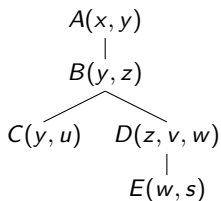
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Correctness:

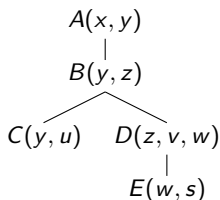
- $A \bowtie (\dots) \neq \emptyset$  iff  $A \bowtie (\dots) \neq \emptyset$ .
- $A \bowtie (B \bowtie (\dots)) = A \bowtie (B \bowtie (\dots))$  running intersection property.
- Etc.

# Yannakakis' Algorithm: Full Conjunctive Query

Full CQ  $Q$ , join tree  $T$ , database  $D$ .

Want to compute  $Q(D)$  in time  $O(|\text{Input}| + |\text{Output}|)$ .

Can we simply compute all the joins, in some order?

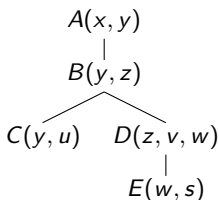


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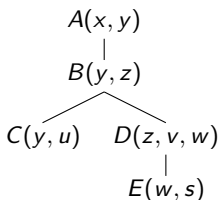
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**NO:** intermediate results  $\gg |\text{Output}|$ .

# Yannakakis' Algorithm: Full Conjunctive Query

Full CQ  $Q$ , join tree  $T$ , database  $\mathbf{D}$ . Choose an arbitrary root in  $T$ .

## Phase 1: Semijoin Reduction.

- Traverse the tree bottom-up and set  $R_n := R_n \bowtie R_{\text{child}(n)}$ .
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**Phase 2: Join Computation.** Initialize  $\text{Out}_0 := \{()\}$  (empty tuple).

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**Return**  $\text{Out}_m$ .



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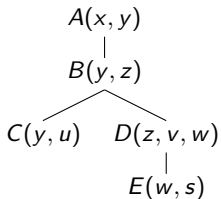
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## Theorem

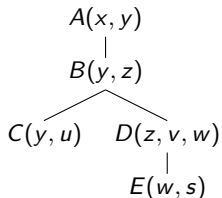
*Yannakakis' algorithm is correct and runs in time  $O(|\text{Input}| + |\text{Output}|)$*

Before the proof, let's see an example.

# Yannakakis' Algorithm: Example



# Yannakakis' Algorithm: Example



## Semijoin Reduction

Bottom-up:

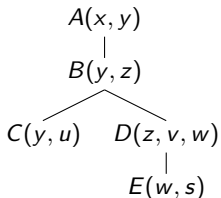
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## Semijoin Reduction

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Top-down:

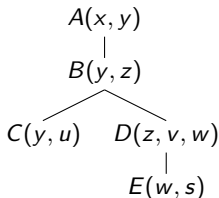
$$B := B \ltimes A$$

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# Yannakakis' Algorithm: Example



## Semijoin Reduction

Bottom-up:

$$D := D \bowtie E$$

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Top-down:

$$B := B \bowtie A$$

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## Join Computation

$$\text{Out}_0 := \{()\}$$

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$$Q := \text{Out}_4 \bowtie E$$

# Yannakakis' Algorithm: Proof

Many proofs are done using informal arguments.

But database optimizers do not understand informal arguments: they are based on [identities](#), or [rewrite rules](#).

Yannakakis' algorithm uses Joins and Semijoins, and we know what identities they satisfy.

Let's prove the correctness and runtime of the algorithm using only those identities.

# Yannakakis' Algorithm: Proof

## Theorem

Yannakakis' algorithm is *correct* and runs in time  $O(|Input| + |Output|)$

## Correctness

- If we run only Phase 2, then correctness by assoc./commutativity:

$$\text{E.g. } (((D \bowtie A) \bowtie C) \bowtie E) \bowtie B = (((A \bowtie B) \bowtie C) \bowtie D) \bowtie E$$

- Phase 1 harmless because  $R_i := R_i \bowtie R_j$  does not affect the join.

$$\text{E.g. } (((A \bowtie B) \bowtie C) \bowtie D) \bowtie E = (((A \bowtie B) \bowtie (C \bowtie B)) \bowtie D) \bowtie E$$

This proves correctness.

# Yannakakis' Algorithm: Proof

## Theorem

Yannakakis' algorithm is correct and *runs in time*  $O(|Input| + |Output|)$

## Runtime

Call  $R$  reduced w.r.t.  $Q$  if  $R = R \bowtie Q$ . The runtime follows from:

- **Claim 1** After Phase 1, every  $R_n$  is reduced w.r.t. the output  $Q$ .
- **Claim 2** During Phase 2, every  $Out_i$  is reduced w.r.t. the output  $Q$ .

Runtime of Phase 1 is  $O(|Input|)$ .

Runtime of Phase 2 is  $O(|Input| + \sum_i |Out_i|) = O(|Input| + |Output|)$ .

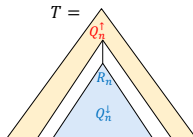


# Proof of Claim 1

For  $n \in \text{Nodes}(T)$  define:

$$Q_n^\downarrow \stackrel{\text{def}}{=} \bowtie_{i \in \text{descendants}(n)} R_i$$

$$Q_n^\uparrow \stackrel{\text{def}}{=} \bowtie_{i \notin \text{descendants}(n)} R_i$$

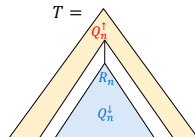


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We prove on the next slide:

- After Bottom-up:  $\forall n, R_n = R_n \bowtie Q_n^\downarrow$
- After Top-down:  $\forall n, R_n = R_n \bowtie Q_n^\uparrow$

Therefore, after Phase 1, by distributivity:

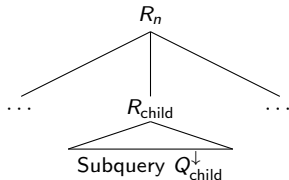
$$R_n \bowtie Q = R_n \bowtie (Q_n^\downarrow \bowtie Q_n^\uparrow) = (R_n \bowtie Q_n^\downarrow) \cap (R_n \bowtie Q_n^\uparrow) = R_n \cap R_n = R_n$$

## Details

After Bottom-up,  $R_n$  is reduced w.r.t.  $Q_n^\downarrow$ :  $R_n = R_n \bowtie Q_n^\downarrow$

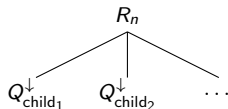
If  $R_{\text{child}}$  reduced for  $Q_{\text{child}}^\downarrow$ , then so is  $R_n^{\text{new}} := R_n \bowtie R_{\text{child}}$ :

$$\begin{aligned}
 R_n^{\text{new}} \bowtie Q_{\text{child}}^\downarrow &= (R_n \bowtie R_{\text{child}}) \bowtie Q_{\text{child}}^\downarrow \\
 &= \left( R_n \bowtie (R_{\text{child}} \bowtie Q_{\text{child}}^\downarrow) \right) \bowtie Q_{\text{child}}^\downarrow \text{ induction} \\
 &= \left( R_n \bowtie (R_{\text{child}} \bowtie Q_{\text{child}}^\downarrow) \right) \bowtie Q_{\text{child}}^\downarrow \text{ cascading} \\
 &= \left( R_n \bowtie (R_{\text{child}} \bowtie Q_{\text{child}}^\downarrow) \right) \bowtie (R_{\text{child}} \bowtie Q_{\text{child}}^\downarrow) \\
 &= R_n \bowtie (R_{\text{child}} \bowtie Q_{\text{child}}^\downarrow) = R_n \bowtie R_{\text{child}} = R_n^{\text{new}}
 \end{aligned}$$



If  $R_n$  is reduced for each  $Q_{\text{child}_i}^\downarrow$  then is reduced for  $\bowtie_i Q_{\text{child}_i}^\downarrow$ :

$$\begin{aligned}
 R_n \bowtie \left( \bowtie_i Q_{\text{child}_i}^\downarrow \right) &= \bigcap_i (R_n \bowtie Q_{\text{child}_i}^\downarrow) \quad \text{Distributivity} \\
 &= R_n
 \end{aligned}$$



After Top-down,  $R_n$  is reduced w.r.t.  $Q_n^\uparrow$ :  $R_n = R_n \bowtie Q_n^\uparrow$ . Exercise.

## Proof of Claim 2

During Phase 2,  $\text{Out}_i$  is reduced w.r.t.  $Q$ :  $\text{Out}_i = \text{Out}_i \bowtie Q$ .

By induction on  $i$ :

Assuming:

- Induction hypothesis:  $\boxed{\text{Out}_i = \text{Out}_i \bowtie Q}$
- By Claim 1:  $\boxed{R_n = R_n \bowtie Q}$

prove that  $\text{Out}_{i+1} := \text{Out}_i \bowtie R_n$  is reduced w.r.t.  $Q$ . Need to show:

$$\boxed{\text{Out}_i \bowtie R_n = (\text{Out}_i \bowtie R_n) \bowtie Q}$$

Does the following hold in general?  $(A \bowtie B) \bowtie Q = (A \bowtie Q) \bowtie (B \bowtie Q)$ ?

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Does the following hold in general?  $(A \bowtie B) \bowtie Q = (A \bowtie Q) \bowtie (B \bowtie Q)$ ?

NO!

On Homework 2: complete the proof of Claim 2.

## Discussion: is the Semi-join Reduction Necessary?

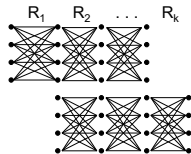
Yes! Otherwise, intermediate results can be much larger than final result:

$$\text{E.g. } Q(x_0, x_1, \dots, x_k) = R_1(x_0, x_1) \wedge \dots \wedge R_k(x_{k-1}, x_k)$$

$$|R_0 \bowtie \dots \bowtie R_{k-1}| = \Omega(N^k)$$

$$|R_1 \bowtie \dots \bowtie R_k| = \Omega(N^k)$$

$$R_0 \bowtie R_1 \bowtie \dots \bowtie R_k = \emptyset$$



$$|\text{Input}| = O(N^2), |\text{Output}| = 0.$$

If we join directly, then the runtime is  $O(N^k) \neq O(|\text{Input}| + |\text{Output}|)$ .

# Yannakakis Algorithm for General CQ

$$Q(x_1, \dots, x_p) = \exists x_{p+1} \cdots \exists x_k (A_1 \wedge \cdots \wedge A_m)$$

## Definition

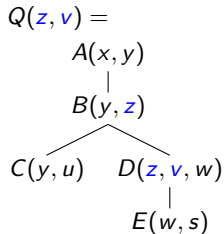
$Q$  is **acyclic free-connex** if it is acyclic after we add an atom **Out**( $x_1, \dots, x_p$ ).

## Theorem

*Yannakis' algorithm computes  $Q$  in time  $O(|Input| + |Output|)$ .*

Phase 1 is unchanged. In Phase 2 the elimination order is towards the new atom **Out**( $x_1, \dots, x_p$ ).

## Example of a Free-Connex Query

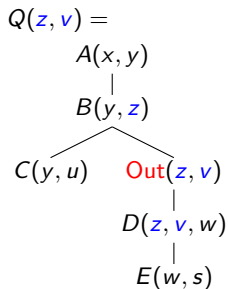


Where do we place

$\text{Out}(z, v)$ ?

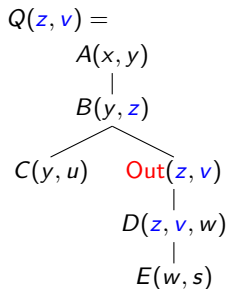


## Example of a Free-Connex Query



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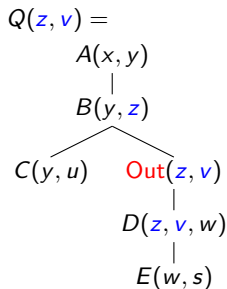
# Example of a Free-Connex Query



**Semijoin Reduction**

As before.

# Example of a Free-Connex Query



## Join Computation

$$\begin{aligned}
 T_1(y) &:= A(x, y) \\
 T_2(y, z) &:= T_1(y) \bowtie B(y, z) \\
 T_3(y) &:= C(y, u) \\
 T_4(z) &:= T_2(y, z) \bowtie T_3(y) \\
 T_5(w) &:= E(w, s) \\
 T_6(z, v) &:= T_5(w) \bowtie D(z, v, w) \\
 T_7(z, v) &:= T_6(z, v) \bowtie T_4(z)
 \end{aligned}$$

Return  $T_7(z, v)$ .

## Semijoin Reduction

As before.

The tree traversal is from the leaves towards  $\text{Out}(z, v)$ .  
 Each  $T_i$  is either a subset of some input relation, or of the output  $Q(z, v)$ , hence  $\text{Time} = O(|\text{Input}| + |\text{Output}|)$

## Non Free-Connex Acyclic Queries

If  $Q$  is acyclic but not free-connex, unlikely to be computable in time  $O(|\text{Input}| + |\text{Output}|)$

### Conjecture

The Boolean matrix multiplication conjecture: if  $A, B$  are  $N \times N$  Boolean matrices, then there exists no algorithm for computing  $A \cdot B$  in times  $O(N^2)$ .

$$Q(i, k) = \exists j (A(i, j) \wedge B(j, k))$$

Cannot compute in time  $O(|A| + |B| + |\text{Output}|) = O(N^2)$ .

# Summary

- Yannakakis' algorithm: Semijoin reduction (up, then down), then joins.
  - ▶ Requires the query to be acyclic.
  - ▶ Works for full CQs, for Boolean CQs, and for “free-connext” CQs.
  - ▶ Related to the [Junction-tree Algorithm](#) in graphical models.
- Most SQL queries in practice are acyclic.
- [Discussion in class](#) Do database engines run Yannakakis algorithm? If not, why not?

# Hypertree Decomposition

# Motivation

What do we do when the query is not acyclic?  $R(x, y) \wedge S(y, z) \wedge T(z, x)$ .

We compute a [tree decomposition](#) then (1) we compute each node of the tree, (2) run Yannakakis' algorithm on the results.

# Hypertree Decomposition

## Definition

A hypertree decomposition of a query (hypergraph)  $Q$  is  $(T, \chi)$  where  $T$  is a tree and  $\chi : \text{Nodes}(T) \rightarrow 2^{\text{Vars}(Q)}$  such that:

- Running intersection property:  $\forall x \in \text{Vars}(Q)$ , the set  $\{n \in \text{Nodes}(T) \mid x \in \chi(n)\}$  is connected.
- Every atom  $R_i(\mathbf{x}_i)$  is covered:  $\exists n \in \text{Nodes}(T)$  s.t.  $\mathbf{x}_i \subseteq \chi(n)$

A set  $\chi(n)$  for  $n \in \text{Nodes}(T)$  is called a **bag**.

$$Q = R(x, y) \wedge S(y, z) \wedge T(z, u) \wedge K(u, x)$$

$$T = \begin{array}{c} \{x, y, z\} \\ | \\ \{x, u, z\} \end{array}$$



## Hypertree Width

A **edge-cover** of a set of variables  $\mathbf{z} \subseteq \text{Vars}(Q)$  is a set  $\mathcal{C} \subseteq \text{Atoms}(Q)$  such that  $\mathbf{z} \subseteq \bigcup_{R(\mathbf{x}) \in \mathcal{C}} \mathbf{x}$ .

The **edge-cover number** of  $\mathbf{z}$  is  $\rho(\mathbf{z}) \stackrel{\text{def}}{=} \min_{\mathcal{C}} |\mathcal{C}|$  where  $\mathcal{C}$  ranges over all edge-covers.

### Definition

The **hypertree width** of a tree is  $\text{HTW}(T) \stackrel{\text{def}}{=} \max_{n \in \text{Nodes}(T)} \rho(\chi(n))$ .

The **hypertree width** of a query is  $\text{HTW}(Q) \stackrel{\text{def}}{=} \min_T \text{HTW}(T)$  where  $T$  ranges over tree decompositions of  $Q$ .

Warning: some text use the term *generalized* hypertree width.

What is  $\text{HTW}(Q)$ ?

$$Q = R(x, y) \wedge S(y, z) \wedge T(z, u) \wedge K(u, x) \quad \begin{array}{c} \{x, y, z\} \\ | \\ \{x, u, z\} \end{array}$$

# Discussion: Structural Optimization of Conjunctive Queries

Assume  $Q$  is a full conjunctive query:

- Find a tree decomposition with minimum  $\text{HTW}(T)$ .
- Compute every bag using a left-deep join plan  $(R_1 \bowtie R_2) \bowtie \dots$  and materialize it.  
(We will discuss a better method, Worst-Case Optimal Joins, in a few weeks. Don't miss it!)
- Run Yannakakis' algorithm on the result.

# Query Containment, Equivalence, Minimization

# Motivation

Query equivalence means  $Q_1(\mathbf{D}) = Q_2(\mathbf{D})$  for any input database  $\mathbf{D}$ .

This is the most important static analysis problem.

Will show that equivalence is undecidable for FO,  
but is decidable for CQ, UCQ, and extensions with inequalities ( $\leq, \neq$ ).

# Query Equivalence

## Definition (Equivalence)

$Q_1, Q_2$  are **equivalent** if  $\forall \mathbf{D}, Q_1(\mathbf{D}) = Q_2(\mathbf{D})$ . Notation:  $Q_1 \equiv Q_2$ .

It suffices to study equivalence of Boolean queries, because of the following:

## Fact

$Q_1(\mathbf{x}) \equiv Q_2(\mathbf{y})$  iff they have the same arity ( $|\mathbf{x}| = |\mathbf{y}|$ ), and for some constants  $\mathbf{c}$  not occurring in  $Q_1, Q_2$ ,  $Q_1[\mathbf{c}/\mathbf{x}] \equiv Q_2[\mathbf{c}/\mathbf{y}]$ .

# Query Containment

## Definition (Containment)

$Q_1$  is **contained** in  $Q_2$  if  $\forall \mathbf{D}, Q_1(\mathbf{D}) \subseteq Q_2(\mathbf{D})$ .

It suffices to assume  $Q_1, Q_2$  are Boolean. Then  $Q_1 \subseteq Q_2$  same as  $Q_1 \Rightarrow Q_2$ .

## Fact

Equivalence and containment are (almost) the same problem:

$$\boxed{Q_1 \equiv Q_2} \text{ iff } \boxed{Q_1 \Rightarrow Q_2 \text{ and } Q_2 \Rightarrow Q_1}$$

$$\boxed{Q_1 \Rightarrow Q_2} \text{ iff}^2 \boxed{Q_1 \equiv Q_1 \wedge Q_2}$$

---

<sup>2</sup>Language must be closed under  $\wedge$ .

# Containment for FO is Undecidable

## Theorem

*The problem Given  $Q_1, Q_2$ , check whether  $Q_1 \subseteq Q_2$  is undecidable.*

**Proof** By reduction from  $\text{SAT}_{\text{fin}}$ .

Let  $\Phi$  be any sentence. (We want to check  $\text{SAT}_{\text{fin}}(\Phi)$ .)

Define  $Q_1 \stackrel{\text{def}}{=} \Phi$  and  $Q_2 \stackrel{\text{def}}{=} \text{false}$ . Then  $Q_1 \subseteq Q_2$  iff  $\neg \text{SAT}_{\text{fin}}(\Phi)$ .

# Containment for CQs

The containment problem for CQ is decidable; More precisely, NP-complete.

This is one of the oldest, most celebrated result in database theory [Chandra and Merlin, 1977].



## Containment for CQs

Assume CQs Boolean queries; extension to non-Boolean is immediate.

### Definition (Canonical Database)

The **canonical database** associated to a CQ  $Q$  is the following: its domain is  $\text{Vars}(Q)$ , and its tuples are the atoms of  $Q$ . Notation:  $D_Q$ .

### Theorem

*The following are equivalent:*

- *Containment holds:  $Q_1 \subseteq Q_2$*
- *There exists a homomorphism  $h : Q_2 \rightarrow Q_1$*
- *$Q_2(D_{Q_1}) = \text{true}$ .*

**Proof** in class.

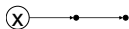
## Examples

Which pairs of queries are contained? Equivalent?

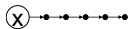
$$Q_1(x) = \exists y \exists z \exists w (E(x, y) \wedge E(y, z) \wedge E(x, w))$$



$$Q_2(x) = \exists u \exists v (E(x, u) \wedge E(u, v))$$



$$Q_3(x) = \exists u_1 \cdots \exists u_5 (E(x, u_1) \wedge E(u_1, u_2) \wedge \cdots \wedge E(u_4, u_5))$$



$$Q_4(x) = \exists y (E(x, y) \wedge E(y, x))$$



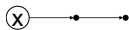
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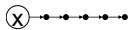
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$$Q_4(x) = \exists y (E(x, y) \wedge E(y, x))$$



$$Q_4 \subseteq Q_3 \subsetneq Q_1 \equiv Q_2$$

# Containment of UCQs

## Theorem

Let  $Q = Q_1 \vee Q_2 \vee \dots$ ,  $Q' = Q'_1 \vee Q'_2 \vee \dots$ . The following are equivalent:

- Containment holds:  $Q \subseteq Q'$
- Every  $Q_i$  is contained in some  $Q_j$ :  $\forall i \exists j, Q_i \subseteq Q'_j$ .

**Proof** in class.

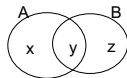
# Join/Semi-join Identities: Idempotence

$$A \bowtie B = (A \bowtie B) \bowtie B$$

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Denote  $\mathbf{x}, \mathbf{y}, \mathbf{z}$  the set of variables:



$$Q_1(\mathbf{x}, \mathbf{y}, \mathbf{z}) = A(\mathbf{x}, \mathbf{y}) \wedge B(\mathbf{y}, \mathbf{z})$$

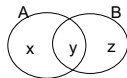
$$\begin{aligned} Q_2(\mathbf{x}, \mathbf{y}, \mathbf{z}) &= (\exists \mathbf{z} (A(\mathbf{x}, \mathbf{y}) \wedge B(\mathbf{y}, \mathbf{z}))) \wedge B(\mathbf{y}, \mathbf{z}) \\ &= \exists \mathbf{u} A(\mathbf{x}, \mathbf{y}) \wedge B(\mathbf{y}, \mathbf{u}) \wedge B(\mathbf{y}, \mathbf{z}) \end{aligned}$$

We renamed  $\exists \mathbf{z}$  to  $\exists \mathbf{u}$  so it doesn't clash with the head variable  $\mathbf{z}$ .

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$h_1 : Q_1 \rightarrow Q_2$  maps  $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \mapsto (\mathbf{x}, \mathbf{y}, \mathbf{z})$ .

$h_2 : Q_2 \rightarrow Q_1$  maps  $(\mathbf{x}, \mathbf{u}, \mathbf{y}, \mathbf{z}) \mapsto (\mathbf{x}, \mathbf{z}, \mathbf{y}, \mathbf{z})$ .

Therefore,  $Q_1 \equiv Q_2$ .

# Join/Semi-join Identities: Cascading

If  $\text{Vars}(A) \cap \text{Vars}(C) \subseteq \text{Vars}(B)$ .

then  $A \bowtie (B \bowtie C) = A \bowtie (B \bowtie C)$

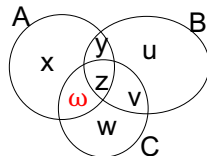


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Variables (notice that  $\omega$  doesn't exist):



$$Q_1(x, y, z, \omega) = \exists u, v, w (A(x, y, z, \omega) \wedge (B(y, z, u, v) \wedge C(z, v, w, \omega)))$$

$$Q_2(x, y, z, \omega) = \exists u, v (A(x, y, z, \omega) \wedge \exists w, \alpha (B(y, z, u, v) \wedge C(z, v, w, \alpha)))$$

If  $\omega$  doesn't exist, then  $Q_1 \equiv Q_2$ .

# Query Minimization

A CQ  $Q$  may be equivalent to many other CQs  $Q \equiv Q_2 \equiv Q_3 \equiv \dots$ .

## Definition (Minimal Query)

A CQ  $Q$  is **minimal** if  $Q \equiv Q'$  implies  $|\text{Atoms}(Q)| \leq |\text{Atoms}(Q')|$ .

The **minimization problem** is: given  $Q$ , find  $Q_{\min} \equiv Q$  s.t.  $Q_{\min}$  is minimal.

E.g. minimize:  $Q(x) = \exists y \exists z \exists w (E(x, y) \wedge E(y, z) \wedge E(x, w))$

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## Theorem

*The minimal query is unique up to isomorphism.*

**Proof:** Let  $Q, Q'$  minimal and  $Q \equiv Q'$ ; then  $\exists h : Q \rightarrow Q', h' : Q' \rightarrow Q$ .  
 $h' \circ h : Q \rightarrow Q$  is surjective, otherwise  $Q \equiv \text{Im}(h' \circ h)$  violating minimality.  
Thus,  $h' \circ h$  is an isomorphism (since its domain is finite).

# The Core of a CQ

## Definition

The **core** of  $Q$  is a subquery  $Q_0$  (meaning: a subset of atoms) such that

- (1) there exists a homomorphism  $h : Q \rightarrow Q_0$ , and
- (2) there is no strict subquery of  $Q_0$  with this property.

Note: the term **core** is commonly used for graphs.

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Note: the term **core** is commonly used for graphs.

## Theorem

*The core of  $Q$  is a minimal query equivalent to  $Q$ .*

Minimization Algorithm: Repeatedly remove an atom  $A$  from  $Q$  as long as  $\exists h : Q \rightarrow Q - \{A\}$ .

# Minimizing UCQ

A UCQ query  $Q = Q_1 \vee Q_2 \vee \dots$  is minimal if:

- each CQ  $Q_i$  is minimal
- for all  $i, j$ ,  $Q_i \subseteq Q_j$  implies  $i = j$ .

(Discussion in class)

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(Discussion in class)

Query minimization:

- Minimize each  $Q_i$  for  $i = 1, 2, \dots$
- Remove  $Q_i$  whenever  $\exists j \neq i$  s.t.  $Q_i \subseteq Q_j$ .



# Summary

- Query containment/minimization is the poster child of database theory.
- In practice? Not so much. Real queries have bag semantics query minimization does not apply:  $Q_1(x) = R(x) \wedge R(x)$  is not equivalent to  $Q_2(x) = R(x)$ .
- However the theory becomes quite relevant for reasoning about semi-joins and query rewriting using views, which is a major topic for database systems.
- Next: adding inequalities  $\leq, \neq$ . The query containment/minimization problem becomes surprisingly subtle!

Adding Inequalities:  $<$ ,  $\leq$ ,  $\neq$

# Inequalities

Extend CQ with  $<, \leq, \neq$ . E.g.  $Q(x, y, z) = R(x, y) \wedge R(x, z) \wedge y \neq z$ .

The extend languages is denoted  $CQ^<$ , or  $CQ^{\leq, \neq}$ , or  $CQ(\leq, \neq)$ .

The domain of a database instance ***D*** is densely ordered, e.g. a subset of  $\mathbb{Q}$ .

**Problems:** containment, minimization.

## Homomorphism is Sufficient

A homomorphism  $h : Q' \rightarrow Q$  is now required to map an inequality  $t_1 \text{ op } t_2$  in  $Q'$  to one implied by  $Q$ , i.e.  $Q \models h(t_1) \text{ op } h(t_2)$ .

### Fact

If there exists a homomorphism  $Q' \rightarrow Q$  then  $Q \subseteq Q'$ .

Proof by example.  $Q, Q'$  are Boolean queries (dropping  $\exists$ ):

$$Q = R(x, y, z) \wedge x < y \wedge y < z$$

$$Q' = R(u, v, w) \wedge u \leq w$$

The homomorphism  $(u, v, w) \mapsto (x, y, z)$  maps  $u \leq w$  to  $x \leq z$ .  
We have  $Q \models x \leq z$ , therefore,  $Q \subseteq Q'$

# Homomorphism is Not Necessary

## Fact

A homomorphism  $Q' \rightarrow Q$  is a sufficient, but not a necessary condition for  $Q \subseteq Q'$ .

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A homomorphism  $Q' \rightarrow Q$  is a sufficient, but not a necessary condition for  $Q \subseteq Q'$ .

Example: (Boolean queries):

$$Q = S(x, y) \wedge S(y, z) \wedge x < z$$

$$Q' = S(u, v) \wedge u < v$$

There is no homomorphism  $Q' \rightarrow Q$ , yet  $Q \subseteq Q'$ . Why?

# Preorder Relations

A relation  $\preceq$  on a set  $V$  is called a **preorder** if:

- It is **reflexive**:  $x \preceq x$ .
- It is **transitive**:  $x \preceq y, y \preceq z$  implies  $x \preceq z$ .

Write  $a \equiv b$  for  $a \preceq b$  and  $b \preceq a$ .

The preorder is **total** if  $\forall a, b \in V$ , either  $a \preceq b$  or  $b \preceq a$  or both hold.

# Preorder Relations

A relation  $\preceq$  on a set  $V$  is called a **preorder** if:

- It is **reflexive**:  $x \preceq x$ .
- It is **transitive**:  $x \preceq y, y \preceq z$  implies  $x \preceq z$ .

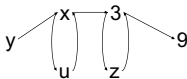
Write  $a \equiv b$  for  $a \preceq b$  and  $b \preceq a$ .

The preorder is **total** if  $\forall a, b \in V$ , either  $a \preceq b$  or  $b \preceq a$  or both hold.

For a preorder  $\preceq$  on  $\text{Vars}(Q) \cup \text{Const}(Q)$ ,  $Q_{\preceq} \stackrel{\text{def}}{=}$  is its extension with  $\preceq$ .

E.g.  $Q = R(x, y, 3) \wedge S(y, z, u, 9) \wedge u \leq x$

Total preorder:  $y \prec x \equiv u \prec 3 \equiv z \prec 9$



$$Q_{\preceq} = R(x, y, 3) \wedge S(y, z, u, 9) \wedge y < x \wedge x = u \wedge x < 3 \wedge 3 = z \wedge \dots$$



# A Necessary and Sufficient Condition

## Theorem ([Klug, 1988])

Let  $Q, Q'$  be  $CQ^{<, \leq, \neq}$  queries. The following conditions are equivalent:

- $Q \subseteq Q'$
- For any consistent total preorder  $\preceq$  on  $Q$ ,  $\exists h : Q' \rightarrow Q_{\preceq}$ .

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**Proof:** If  $Q(\mathbf{D}) = \text{true}$ , then there exists a homomorphism:

$$h_0 : Q \rightarrow \mathbf{D}$$

This induces a total preorder  $\preceq$  on  $Q$ . Let  $h$  be a homomorphism:

$$h : Q' \rightarrow Q_{\preceq}$$

Their composition is a homomorphism  $Q' \rightarrow \mathbf{D}$ , proving  $Q'(\mathbf{D}) = \text{true}$ .

## Example

$$Q = S(x, y) \wedge S(y, z) \wedge x < z$$

$$Q' = S(u, v) \wedge u < v$$

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3 consistent total preorders on  $Q$ :

$$Q_1 = S(x, y) \wedge S(y, z) \wedge x = y \wedge y < z$$

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In each case, either  $(u, v) \mapsto (x, y)$  or  $(u, v) \mapsto (y, z)$  is a homomorphism.

Notice: we need to check **both** homomorphisms.

# Complexity

Theorem ([Klug, 1988, van der Meyden, 1997])

*The problem given  $Q, Q'$  in  $CQ^{<, \leq, \neq}$  determine whether  $Q \subseteq Q'$  is  $\Pi_2^P$ -complete.*

**Proof:** Membership in  $\Pi_2^P$  follows from the fact that  $Q \subseteq Q'$  if for all refinements of  $Q$ , there exists a homomorphism  $Q' \rightarrow Q$ .

For hardness we will discuss a simpler proof than [van der Meyden, 1997].

# Proof of $\Pi_2^p$ -Hardness

Reduction from  $\forall 3CNF$ :  $\Psi = \forall X_1 \cdots \forall X_k \exists X_{k+1} \cdots \exists X_n \Phi$ ,  $\Phi$  is 3CNF.

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Recall the reduction from 3SAT to query containment  $Q \subseteq Q'$ :

- $Q$  has 4 relations  $A, B, C, D$  each with 7 tuples.
- $Q'_\phi$  has one atom/clause. E.g.  $(X_i \vee \neg X_j \vee X_k)$  becomes  $B(x_i, x_k, x_j)$ .
- $\exists X_1 \cdots \exists X_n \Phi$  iff  $\exists h : Q'_\phi \rightarrow Q$ .



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- $\exists X_1 \cdots \exists X_n \Phi$  iff  $\exists h : Q'_\phi \rightarrow Q$ .

For each universal variable  $x_i$ , add the following atoms:

- Add  $S(0, u_i, v_i) \wedge S(1, v_i, w_i) \wedge u_i < w_i$  to  $Q$ .
- Add  $S(x_i, a_i, b_i) \wedge a_i < b_i$  to  $Q'_\phi$ .

$\boxed{Q \subseteq Q'_\phi}$  holds iff **both**  $x_i \mapsto 0, x_i \mapsto 1$  lead to a homomorphisms.

# Summary

- The big question: what other extensions of CQ can we allow and still be able to decide containment?
- The following have been studied: inequalities, safe negation  $\neg$ , certain aggregates sum, min, max, count.
- The elegant containment/minimization theory for standard CQs quickly becomes very involved.



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