CS294-248 Special Topics in Database Theory
Unit 9: Datalog (Part 2)

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Announcements

- Next Tuesday, Nov. 28: office hours 2pm-4:30pm.
- Please submit a short report on your project by Wednesday, Nov. 29.
- Project presentations: Thursday, Nov. 30, 9:30am, Calvin 146. More details TBD.

Recursion and Negation

Recap: Datalog

- Datalog = set of rules.
- Immediate consequence operator
- Least fixpoint semantics
- Naive algorithm $J^{(0)} \subseteq J^{(1)} \subseteq \cdots$

Example:

$$\boxed{\begin{array}{l} T(X,Y) \text{ :- } E(X,Y) \\ T(X,Y) \text{ :- } T(X,Z) \wedge E(Z,X) \end{array}}$$

Non-example:

$$\boxed{C(X) \text{ :- } A(X) \wedge \neg B(X)}$$

What happens if we allow negation?

Recursion and Negation Don't Mix

EDB is $S = \{1\}$

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Pre-fixpoint 3: $A = B = \{1\}$

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A simpler example:

$$A \text{ :- } \neg B$$
$$B \text{ :- } \neg A$$

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ICO not monotone! Need new semantics

Outline

- Semi-positive, stratified datalog
- Semantics motivated by logic.
- Semantics motivated by computation.

Mostly based on [Abiteboul et al., 1995].

Stratified Datalog

Three Examples

Transitive closure
of the complement graph:

$$\boxed{\begin{array}{l} EC(X,Y) \text{ :- } V(X) \wedge V(Y) \wedge \neg E(X,Y) \\ T(X,Y) \text{ :- } EC(X,Y) \\ T(X,Y) \text{ :- } T(X,Z) \wedge EC(Z,X) \end{array}}$$

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Complement of the transitive closure:

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The Win-Move Game:

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W(X) \text{ :- } E(X,Y) \wedge \neg W(Y)
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(will explain it later)

Semi-positive Datalog

EDB atoms may be positive or negated.

IDB atoms can only be positive.

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The Immediate Consequence Operator is monotone.

Semantics: least fixpoint of the ICO.

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- Semantics: for each stratum $s = 1, 2, \ldots$, view it as a semi-positive datalog program, compute its fixpoint.
- The output is called perfect model; it is not a minimal model!

Example

$$\boxed{\begin{array}{l} T(X,Y) \text{ :- } E(X,Y) \\ T(X,Y) \text{ :- } T(X,Z) \wedge E(Z,X) \\ \text{Answ}(X,Y) \text{ :- } V(X) \wedge V(Y) \wedge \neg T(X,Y) \end{array}}$$

Stratum 1: T

Stratum 2: Answ

[1] Assuming no isolated nodes

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Semantics:

T = transitive closure, Answ = its complement

This is not the least fixpoint (minimal model) why??

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Stratum 1: T

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Semantics:

$T =$ transitive closure, Answ $=$ its complement

This is not the least fixpoint (minimal model) why??

The following is also a fixpoint:[1]

$T = V \times V$, Answ $= \emptyset$

[1] Assuming no isolated nodes

Discussion

- Stratified datalog is by far the most popular extension of datalog with negation.
- It is limited: it completely prevents the interleaving of recursion and negation. The following is not allowed:

$$A \text{ :- } \neg B$$
$$B \text{ :- } \neg A$$

Logic-Based Extensions

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- Stable Models

- Well Founded Model

Representative example: the Win-Move Game (next)

The Win-Move Game

- Players I, II take turns moving a pebble in a graph.
- Player who cannot move loses.
- For each node X, does Player I have a winning strategy?

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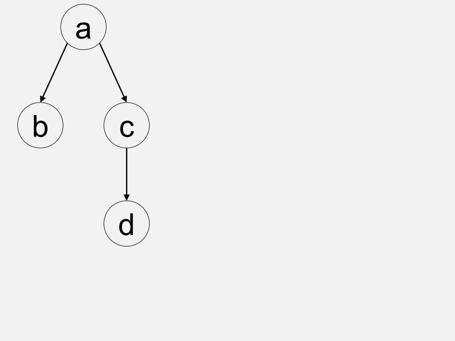

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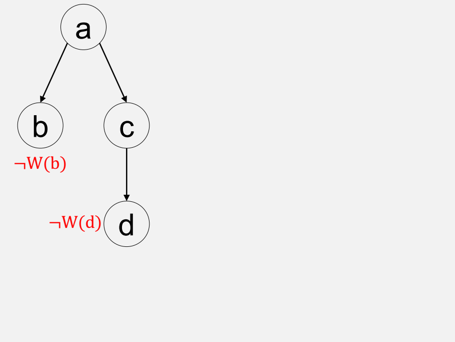

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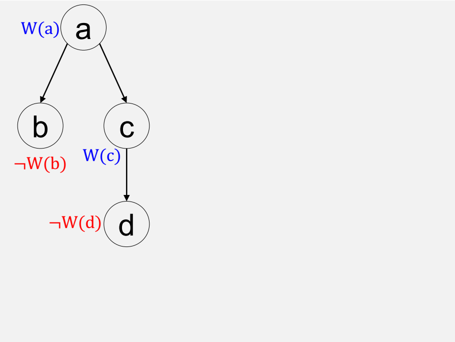

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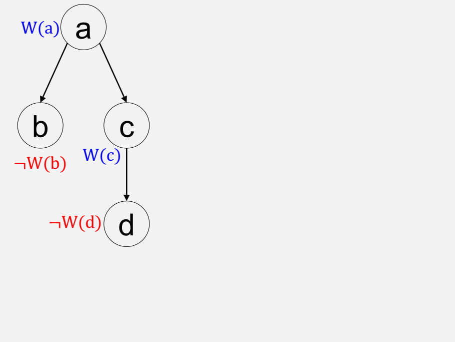

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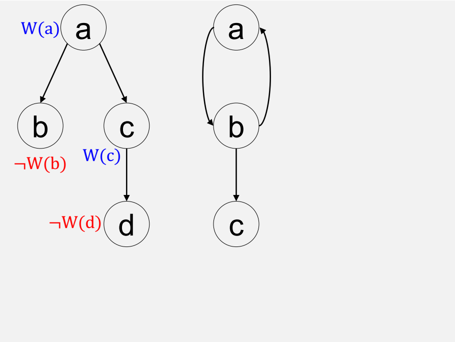

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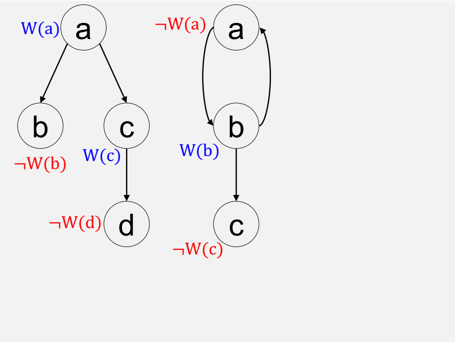

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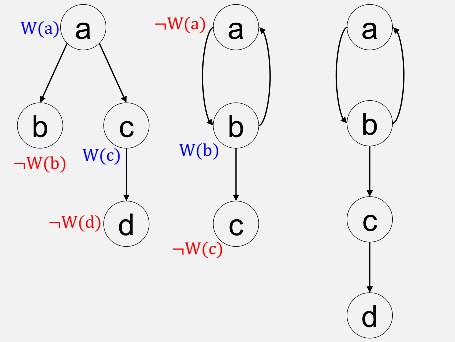

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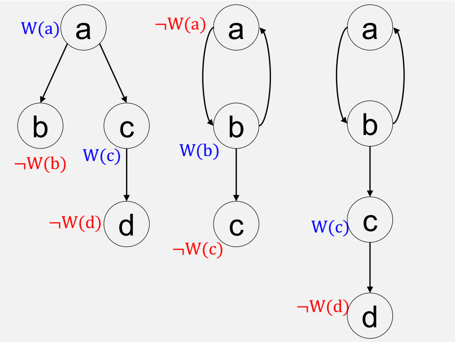

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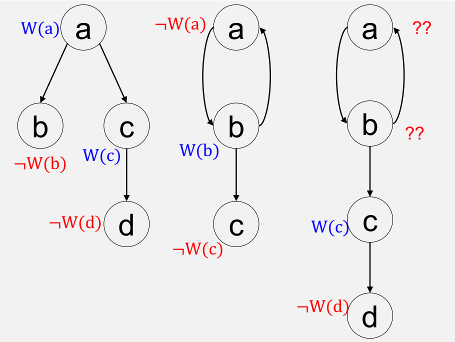

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Discussion

- Least Fixpoint Logic (LFP) is FO extended with monotone fixpoint. E.g. the win-move game:

$$\boxed{\mathrm{lfp}_{W(x)}(\exists y(E(x,y) \Rightarrow \forall z(E(y,z) \wedge W(z))))}$$

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- Hence: need datalog extensions beyond stratified datalog.

Before that we discuss two simple technical constructs:

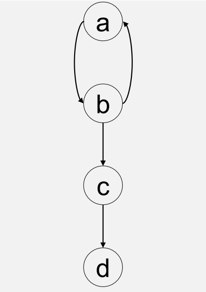
grounded program and reduct

The Grounded Datalog Program

A grounded atom, or a fact, is an atom without variables

A grounded rule is a rule whose atoms are grounded.

The grounding of a program P consists of all possible groundings of its rules



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The Reduct

$P \stackrel{\text{def}}{=}$ the grounded program, $J =$ any set of grounded atoms;

The reduct, P_J is obtained as follows:

- Remove all rules with a negated atom in J.
- Remove all remaining negated atoms.

P_J is monotone; $\text{lfp}(P_J)$ exists; $J_1 \subseteq J_2$ implies $\text{lfp}(P_{J_1}) \supseteq \text{lfp}(P_{J_2})$

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$J = \{W(a), W(d)\}$;

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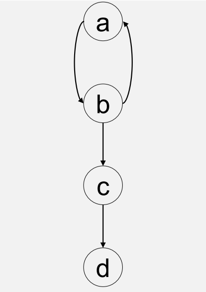
Stable Models

J is a stable model if $J = \text{lfp}(P_J)$



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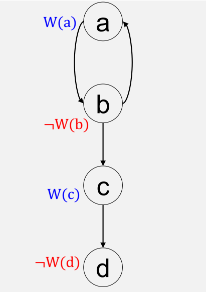

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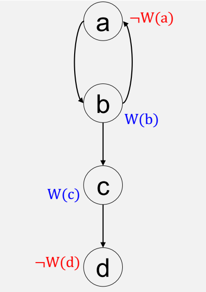

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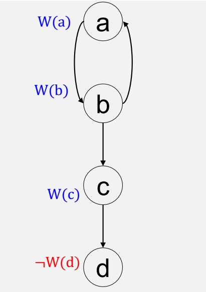

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Non-example: $J = \{W(a), W(b), W(c)\}$ why??

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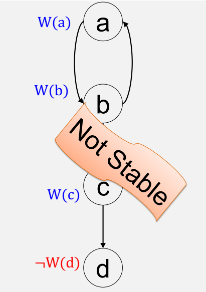
Stable Models

J is a stable model if $J = \text{lfp}(P_J)$

Example: $J = \{W(a), W(c)\}$

Example: $J = \{W(b), W(c)\}$

Non-example: $J = \{W(a), W(b), W(c)\}$ why??



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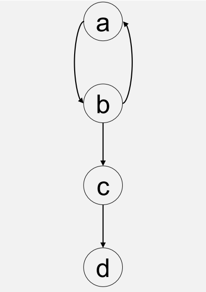

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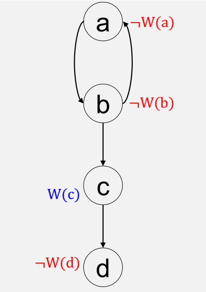

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Discussion

- Stable models introduced by [Gelfond and Lifschitz, 1988]
- Elegant, principled definition.
- But: NP-hard to check if there exists any stable model.

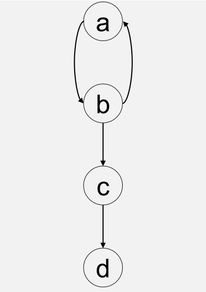
A stratified program has a unique stable model, which is the perfect model.

$A(1)$:-
$B(1)$:- $\neg A(1)$
$C(1)$:- $A(1)$
$C(1)$:- $C(1) \wedge \neg B(1)$

Perfect model: $J = \{A(1), C(1)\}$

Not stable: $J = \{A(1), B(1), C(1)\}$ why?

Well-Founded Model



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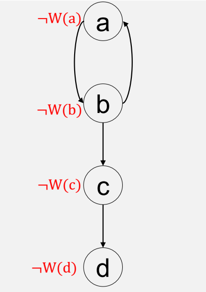

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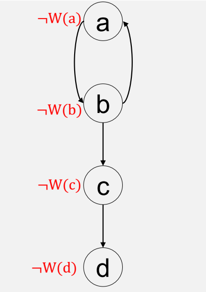

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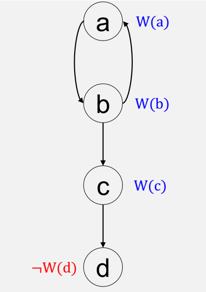

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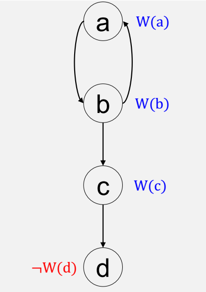

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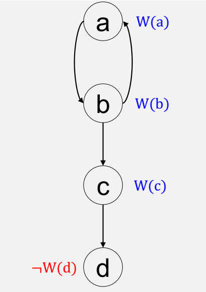

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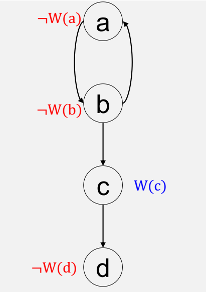

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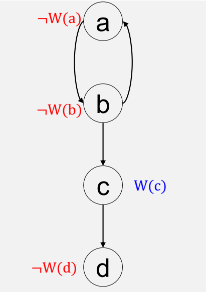

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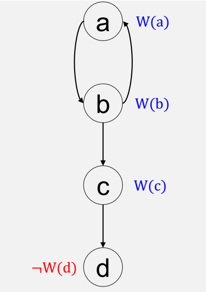

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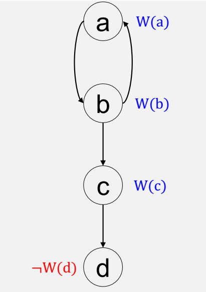

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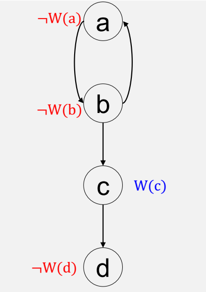

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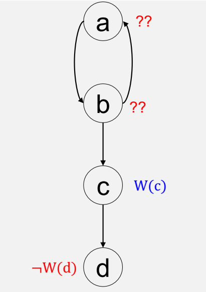

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Certain facts: $W(c)$;

possible facts: $W(a), W(b)$.

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Discussion

- Well-founded models can be computed in PTIME.

- Yet, I don't know of any system that supports it.
  Maybe because of the 3-valued logic?

Next: two other semantics motivated by computation.

Computation-Based Extensions

Computation-Based Extensions

- Datalog with inflationary fixpoint semantics.

- Datalog with partial fixpoint semantics.

Inflationary Fixpoint

Let P be a datalog$^\neg$ program, T_P its ICO.

The inflationary fixpoint is $\boxed{\text{ifp}(P) \stackrel{\text{def}}{=} \bigcup_{t \geq 0} J_t}$, where:

$$J_0 \stackrel{\text{def}}{=} \emptyset, \quad J_{t+1} \stackrel{\text{def}}{=} J_t \cup T_P(J_t)$$

Fact

ifp(P) can be computed in PTIME in the size of the EDB I.

why?

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Fact

ifp(P) can be computed in PTIME in the size of the EDB I.

why? Because $J_0 \subseteq J_1 \subseteq \cdots \subseteq (\text{ADom}(I))^k$

Partial Fixpoint

The partial fixpoint is:

$$\text{pfp}(P) \stackrel{\text{def}}{=} \begin{cases} J_{t_0} & \text{if } J_{t_0} = J_{t_0+1} \\ \emptyset & \text{if } J_t \neq J_{t+1}, \forall t \end{cases}$$

where

$$J_0 \stackrel{\text{def}}{=} \emptyset, \quad J_{t+1} \stackrel{\text{def}}{=} T_P(J_t)$$

Fact

pfp(P) can be computed in PSPACE in the size of the EDB I.

why?

Partial Fixpoint

The partial fixpoint is:

$$\mathrm{pfp}(P) \stackrel{\text{def}}{=} \begin{cases} J_{t_0} & \text{if } J_{t_0} = J_{t_0+1} \\ \emptyset & \text{if } J_t \neq J_{t+1}, \forall t \end{cases}$$

where

$$J_0 \stackrel{\text{def}}{=} \emptyset, \quad J_{t+1} \stackrel{\text{def}}{=} T_P(J_t)$$

Fact

$pfp(P)$ can be computed in PSPACE in the size of the EDB I.

why? each $|J_t|$ has size polynomial in $\mathrm{ADom}(I)$.

Detect non-termination using a counter.

How To Express Negation

It's harder than one may think!

Complement of the TC:

$$
\begin{aligned}
T(X,Y) &\text{ :- } E(X,Y) \\
T(X,Y) &\text{ :- } T(X,Z) \wedge E(Z,X) \\
\text{Answ}(X,Y) &\text{ :- } V(X) \wedge V(Y) \\
&\qquad \wedge \neg T(X,Y)
\end{aligned}
$$

ifp(P) is incorrect!

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ifp(P) is incorrect!

Detect the last step:

$$
\begin{aligned}
T(X,Y) &\text{ :- } E(X,Y)\\
T(X,Y) &\text{ :- } T(X,Z) \wedge E(Z,X)\\
T_{\text{prev}}(X,Y) &\text{ :- } T(X,Y)\\
C() &\text{ :- } T(U,V) \wedge \neg T_{\text{prev}}(U,V)\\
D() &\text{ :- } T(U,V) \wedge \neg C()\\
\text{Answ}(X,Y) &\text{ :- } V(X) \wedge V(Y)\\
&\quad \wedge \neg T(X,Y) \wedge D()
\end{aligned}
$$

Descriptive Complexity

- Datalog$^\neg$ cannot express parity, no matter which semantics we adopt.

[2] Exercise: express succ(X, Y), min(X), max(Y) using $<$.

Descriptive Complexity

- Datalog$^\neg$ cannot express parity, no matter which semantics we adopt.
- If we have access to an order relation $<$ then we can express parity as:[2]

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Descriptive Complexity

- Datalog$^\neg$ cannot express parity, no matter which semantics we adopt.
- If we have access to an order relation $<$ then we can express parity as:[2]

$$
\begin{aligned}
E(X, Y) &\text{ :- } \mathrm{succ}(X, Z) \wedge \mathrm{succ}(Z, Y) \\
E(X, Y) &\text{ :- } E(X, Z) \wedge E(Z, Y) \quad // \text{ even-length distance} \\
\mathrm{Even}() &\text{ :- } R(X) \wedge \min(X) \wedge E(X, Y) \wedge \max(Y) \wedge R(Y)
\end{aligned}
$$

Theorem (Descriptive Complexity [Vardi, 1982, Immerman, 1986])

- *Datalog$^\neg$($<$, ifp) expresses precisely queries in PTIME.*
- *Datalog$^\neg$($<$, pfp) expresses precisely queries in PSPACE.*

[2] Exercise: express $\mathrm{succ}(X, Y)$, $\min(X)$, $\max(Y)$ using $<$.

Discussion

- Datalog: simple, elegant, appealing. New resurgence after a 40 years history.
- Stratified datalog$^{\neg}$ is a simple and practical extension.
- Beyond that, it becomes questionable.
- But the theory is beautiful. A famous result:

Theorem ([Abiteboul et al., 1992])

datalog$^{\neg}$(ifp) = datalog$^{\neg}$(pfp) iff PTIME=PSPACE.

Abiteboul, S., Hull, R., and Vianu, V. (1995).
Foundations of Databases.
Addison-Wesley.

Abiteboul, S., Vardi, M. Y., and Vianu, V. (1992).
Fixpoint logics, relational machines, and computational complexity.
In *Proceedings of the Seventh Annual Structure in Complexity Theory Conference, Boston, Massachusetts, USA, June 22-25, 1992*, pages 156–168. IEEE Computer Society.

Chandra, A. K. and Harel, D. (1985).
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