CS294-248 Special Topics in Database Theory
Unit 4: AGM Bound, WCOJ

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Outline

- Today: the AGM bound. This is a mathematical formula that gives us $AGM(Q, \boldsymbol{D}) \stackrel{\text{def}}{=} \max_{\boldsymbol{D} \models \text{statistics}} |Q(\boldsymbol{D})|$.

- Thursday: Worst Case Optimal Join, by guest lecturer Hung Ngo. An algorithm that computes $Q(\boldsymbol{D})$ in time $\tilde{O}(AGM(Q, \boldsymbol{D}))$.

Background on Cardinality Estimation

Cardinality Estimation 101 (1/3)

Given:

- Statistics on the input relations $R_1, R_2, \ldots$
- A full conjunctive query Q

"Estimate":

- The size $|Q(\boldsymbol{D})|$.

Numerous applications: query optimization, memory provisioning, data partitioning.

Cardinality Estimation 101 (2/3)

Bottom-up on the query plan:

- Selection $\sigma_p(R)$: assume independence:

$$\boxed{|\sigma_p(R)| \approx \theta_p \cdot |R|} \qquad \boxed{\theta_{p_1 \wedge p_2} \approx \theta_{p_1} \cdot \theta_{p_2}}$$

  Histograms, multidimensional histograms.

- Join $J(A, B, C) = R(A, B) \wedge S(B, C)$: assume preservation of values
  - $|J| \approx |R| \cdot \text{avg}(\deg_S(C|B)) = \frac{|R| \cdot |S|}{|\text{Dom}(S.B)|}$.
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  - Heuristic: take the *minimum*: $\boxed{|J| \approx \frac{|R| \cdot |S|}{\max(|\text{Dom}(R.B)| \cdot |\text{Dom}(S.B)|)}}$

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Cardinality Estimation 101 (3/3)

- Notoriously hard to estimate cardinality of complex queries.
- No rigorous definition of the *estimate*: there is no probability space.
- How do we combine multiple sources of information?
  - We had two formulas for the join, why choose min?
  - Given $R(A, B, C)$ and histograms on A, B, C, AB, AC, how do we estimate $|\sigma_{A=2,B=4,C=6}(R)|$?

Upper Bound on the Output of a Query

The Output Bound Problem

Given statistics on the input $\boldsymbol{D}$, e.g. cardinalities, # distinct values,

Compute an upper bound B: $\boxed{|Q(\boldsymbol{D})| \leq B}$

Challenge: make B tight.

Simple Examples

Assume $|R| \leq N$, $|S| \leq N$, $|T| \leq N$.

- $Q(X, Y, Z) = R(X, Y) \wedge S(Y, Z)$. $\max_{\boldsymbol{D}} |Q(\boldsymbol{D})| = ?$

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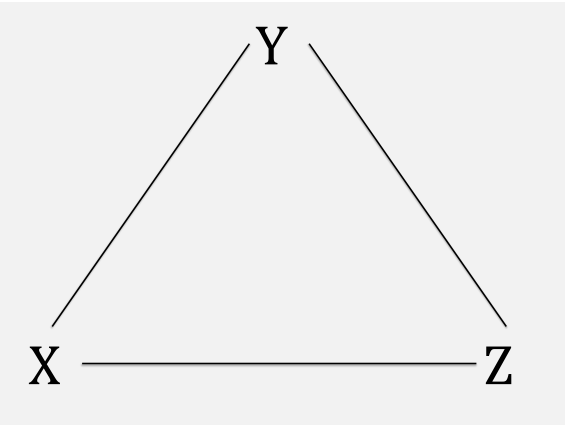
- $Q(X, Y, Z) = R(X, Y) \wedge S(Y, Z) \wedge T(Z, X)$. $\max_{\boldsymbol{D}} |Q(\boldsymbol{D})| = N^{\frac{3}{2}}$
  Here we use a fractional edge cover

AGM Bound: The Statement

Fractional Edge Covers

Query Q to hypegraph $G = (V, E)$. $R(X, Y) \wedge S(Y, Z) \wedge T(Z, X)$



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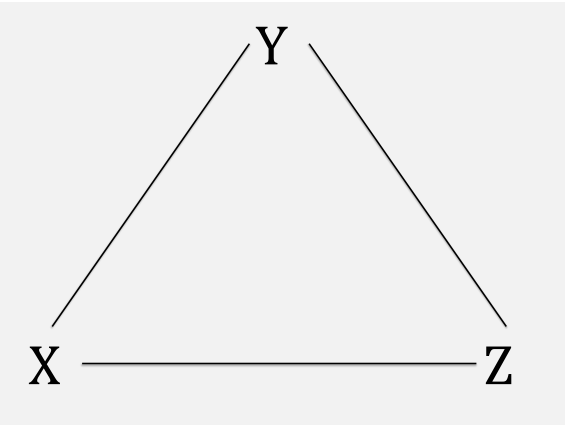
A *fractional edge cover* is $\boldsymbol{w} = (w_e)_{e \in E}$, $w_e \geq 0$:

$$\forall x \in V, \sum_{e \in E : x \in e} w_e \geq 1.$$



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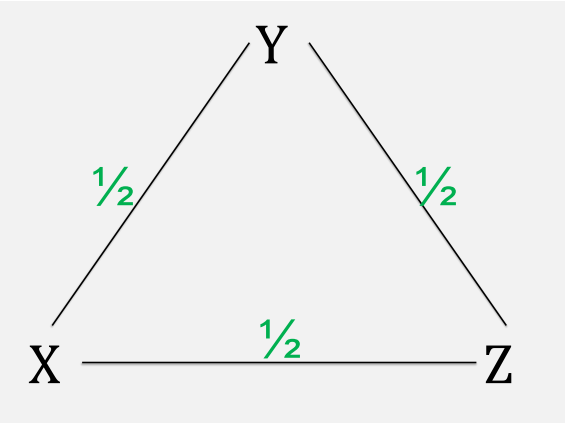

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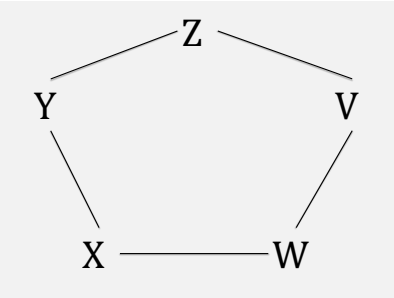

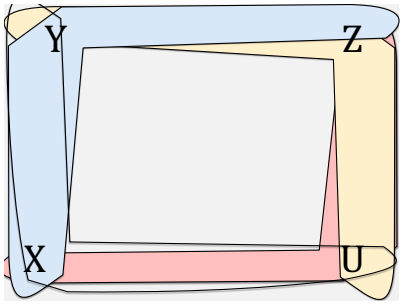

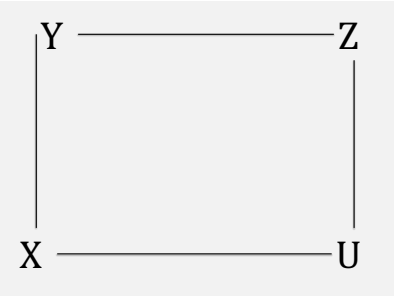

5-cycle

Loomis-Whitney:

$$R(X,Y,Z) \wedge S(Y,Z,U)$$
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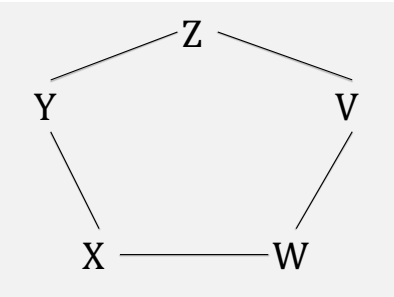

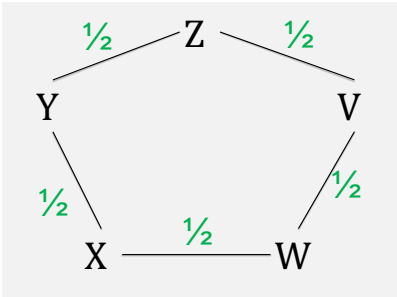

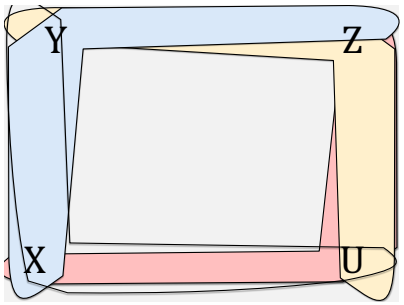

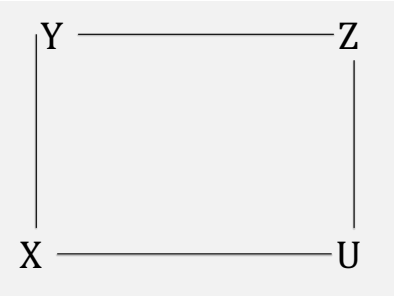

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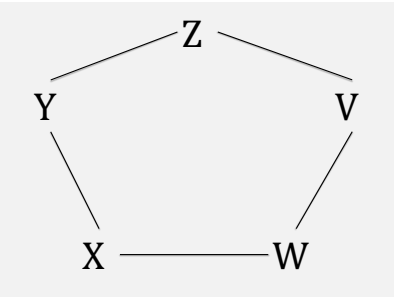

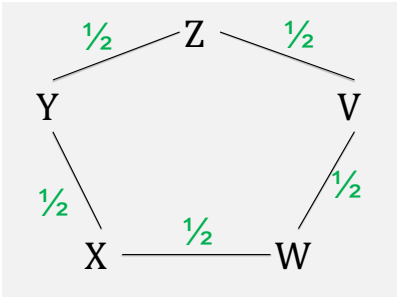

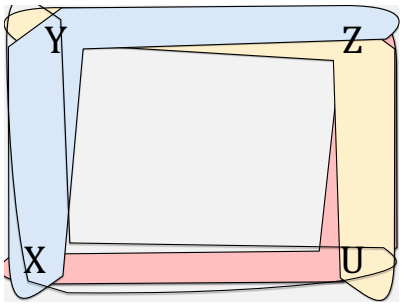

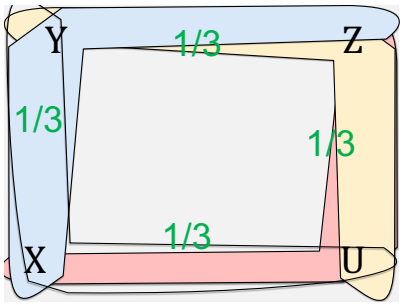

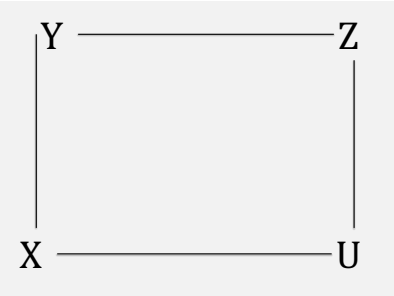

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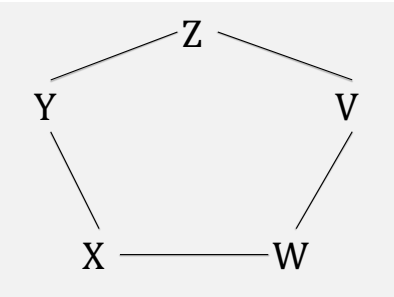

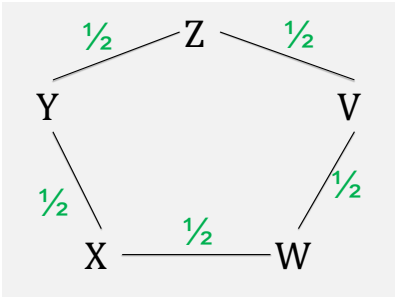

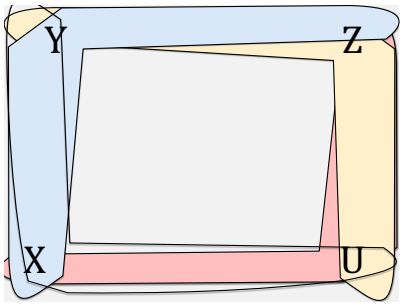

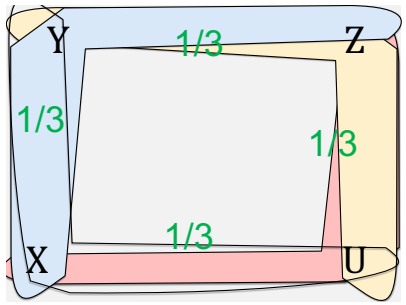

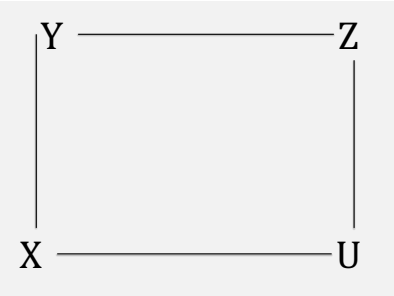

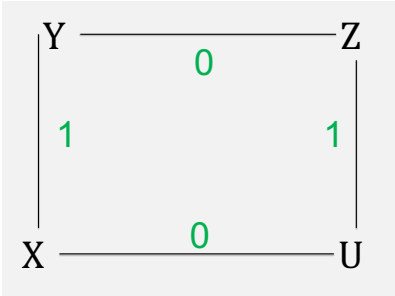

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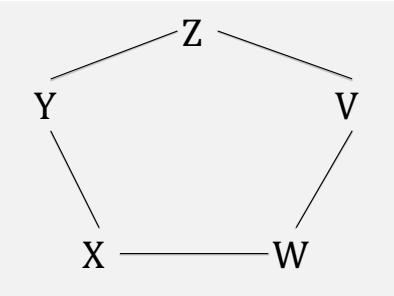

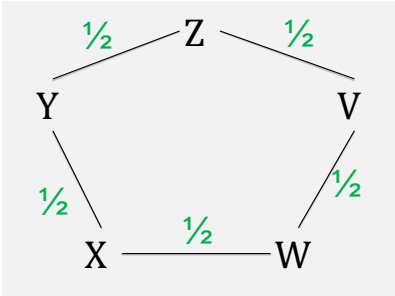

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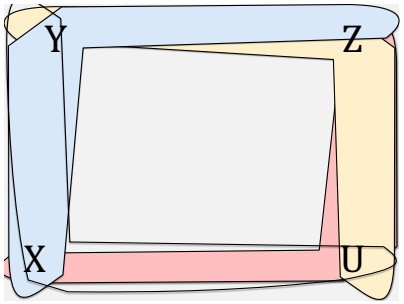

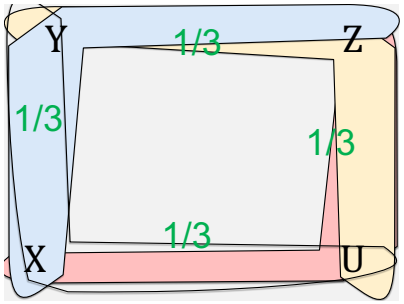

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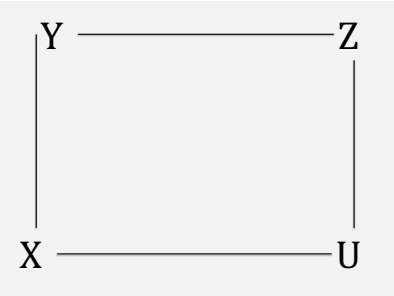

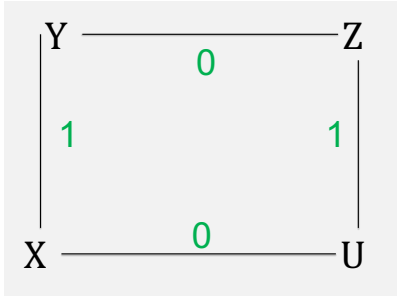

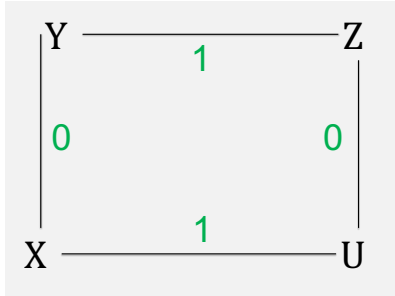

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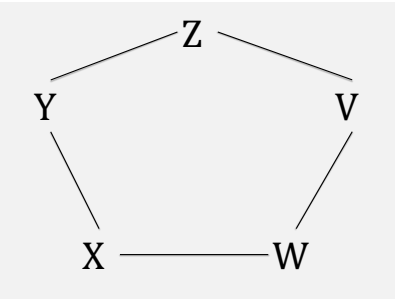
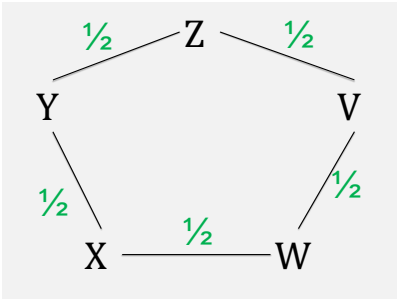
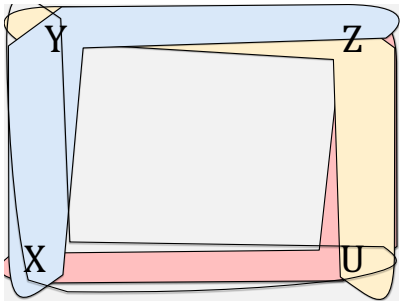
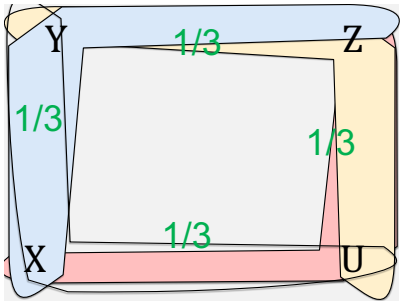




5-cycle





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$$R(X,Y,Z) \wedge S(Y,Z,U)$$
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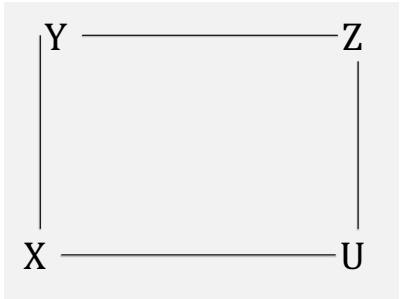

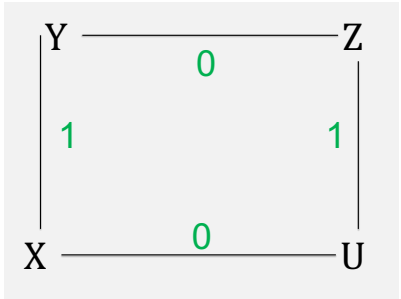

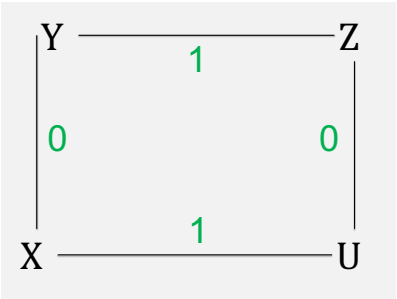

$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is a convex combination of $(1,0,1,0)$ and $(0,1,0,1)$.

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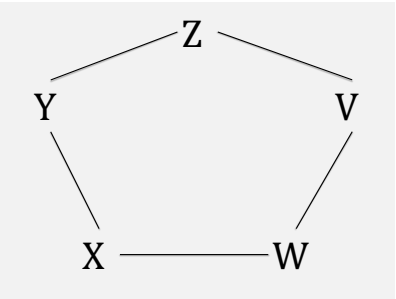

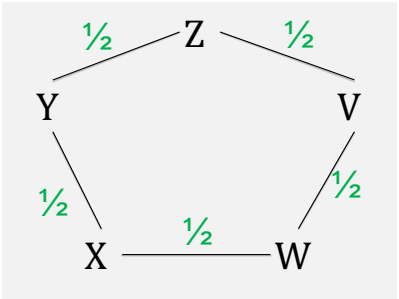

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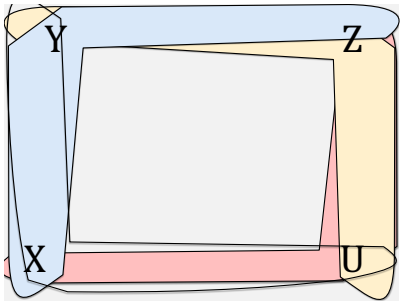

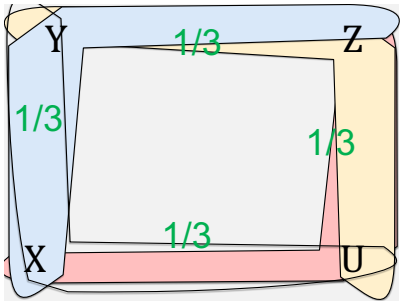

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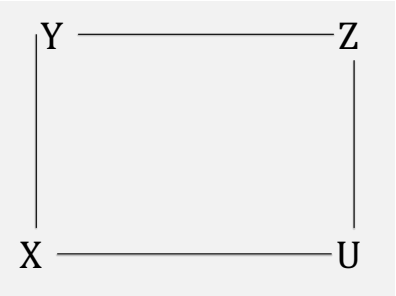

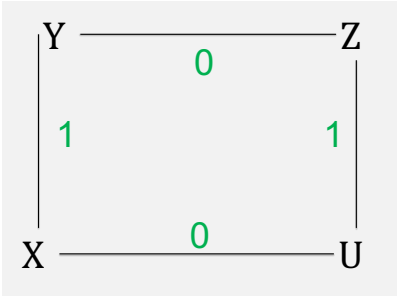

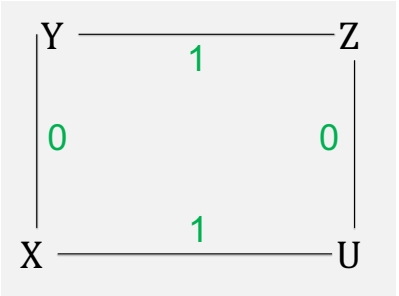

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Vertex of the edge covering polytope: not $\geq$ convex combination of others.

The AGM Bound [Atserias et al., 2013]

$$Q(\boldsymbol{X}) = R_1(\boldsymbol{Y}_1) \wedge \cdots \wedge R_m(\boldsymbol{Y}_m)$$

Theorem (Upper Bound)

For every fractional edge cover $\boldsymbol{w}$*:* $|Q| \leq |R_1|^{w_1} \cdots |R_m|^{w_m}$

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Minimum over vertices of the edge-covering polytope. WHY?

Proof Outline

- Proof of the upper bound: information inequalities (a.k.a. entropic inequalities).

- Proof of the lower bound: construct a worst-case database instance by using strong duality of linear optimization.

Proof of the Upper Bound

Entropic Vectors

Definition

Finite probability space $p : D \to [0, 1]$. $X =$ r.v. with outcomes D.

The *entropy* of X is: $h(X) \stackrel{\text{def}}{=} -\sum_{x \in D} p(x) \log p(x)$

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a	p	$1/4$
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$h(XY) = \log 4$

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$h(XY) = \log 4$

X	p
a	$3/4$
b	$1/4$

$h(X) \leq \log 2$

Y	p
p	$1/4$
q	$2/4$
m	$1/4$

$h(Y) \leq \log 3$

$\emptyset$	p
	1

$h(\emptyset) = 0$

Shannon Inequalities

Basic Shannon Inequalities

$$h(\emptyset) = 0$$

$$h(\boldsymbol{U} \cup \boldsymbol{V}) \geq h(\boldsymbol{U}) \qquad \text{Monotonicity}$$

$$h(\boldsymbol{U}) + h(\boldsymbol{V}) \geq h(\boldsymbol{U} \cup \boldsymbol{V}) + h(\boldsymbol{U} \cap \boldsymbol{V}) \qquad \text{Submodularity}$$

A Shannon inequality is a consequence of these inequalities.

A Shannon Inequality

Example

$$h(XY) + h(YZ) + h(XZ) \geq 2h(XYZ)$$

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Note: X is covered 2 times in each expressions. Same for Y, same for Z.

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For a general query $Q(\boldsymbol{X}) = R_1(\boldsymbol{Y}_1) \wedge \cdots \wedge R_m(\boldsymbol{Y}_m)$:

If $\boxed{\sum_j w_j h(\boldsymbol{Y}_j) \geq h(\boldsymbol{X})}$ then $\boxed{|R_1|^{w_1} \cdots |R_m|^{w_m} \geq |Q|}$

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When do we stop?

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When do we stop?

Proof of the AGM Upper Bound: Part 2: $\boxed{|Q| \leq |R_1|^{w_1} \cdots |R_m|^{w_m}}$

Theorem (Generalized Shearer's Inequality)

If $\boldsymbol{w}$ *is a frac. edge cover, then* $\boxed{w_1 h(\boldsymbol{Y}_1) + \cdots + w_m h(\boldsymbol{Y}_m) \geq h(\boldsymbol{X})}$.

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- Thus, $\boldsymbol{Y}_1 = \boldsymbol{X}$ and $k_1 \geq k_0$. $\boxed{\cdots \geq k_1 h(\boldsymbol{Y}_1) \geq k_0 h(\boldsymbol{X})}$

This completes the proof of the Upper AGM Bound.

Discussion

- Shearer's inequality: apply submodularity repeatedly, in any order!
- Shearer inequalities correspond 1-1 to fractional edge covers.
- Any inequality is an upper bound on $|Q|$: $AGM(Q)$ is the smallest.
- How tight is $AGM(Q)$ upper bound? Next

Proof of the Lower Bound

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where $\boldsymbol{w}$ is frac. edge cover:

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Special Case: $|R| = |S| = \cdots = N$

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Fix a hypergraph (V, E); $(v_X)_{X \in V} \in \mathbb{R}_+^{|V|}$ is a fractional vertex packing if:

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Then: $R = [N^{v_X}] \times [N^{v_Y}]$, $S = [N^{v_Y}] \times [N^{v_Z}]$, $T = [N^{v_X}] \times [N^{v_Z}]$.

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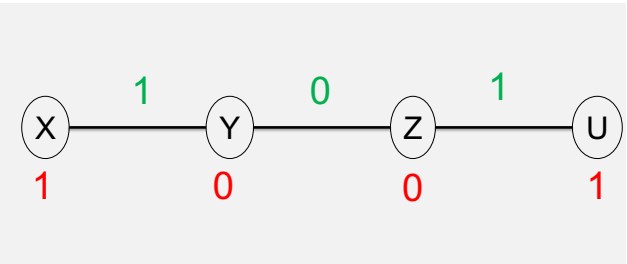

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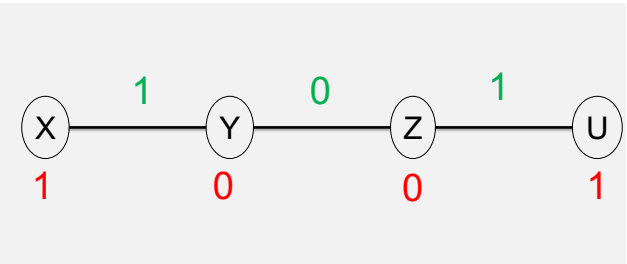

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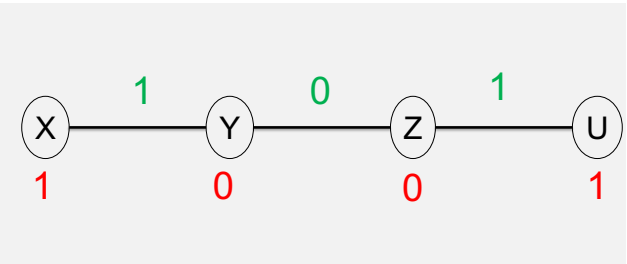

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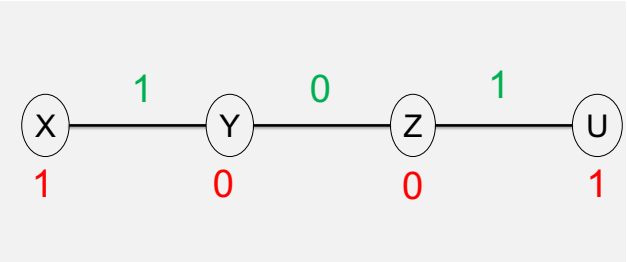

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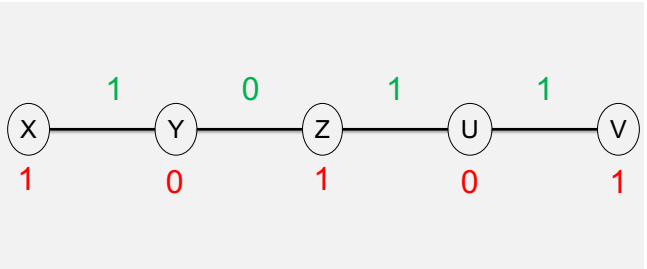

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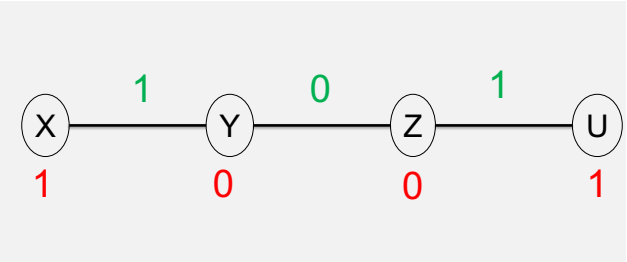

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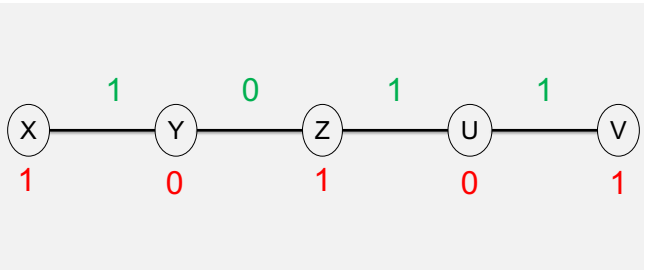

Summary of the AGM Bound

- Upper / lower bound: fractional edge cover / vertex packing.
- Their equality follows from strong duality.
- The worst-case instance of the AGM bound is a Product Database.
- Full CQs only. Otherwise, ignore non-head variables.

Limitation of AGM: only cardinalities. Next: extensions to other stats.

Extensions of the AGM Bound

More Statistics

Statistics for a relation $R(U, V, W, \ldots)$:

- Cardinality $|R|$.
- Same for any projection: $|R.U|, |R.UV|, \ldots$
- Max degree: $\max(\deg_R(VW|U)), \ldots$
- Note: an FD $U \to V$ is $\max(\deg_R(V|U)) = 0$.
- ℓ_p-norm degree sequences: $||\deg_R(V|U)||_2, \ldots$

Simple Functional Dependencies

Given FDs, $|Q| \ll AGM(Q)$.

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Method [Khamis et al., 2016]:

- Expand Q to Q^+ by replacing each atom $R(\boldsymbol{Y})$ with $R'(\boldsymbol{Y}^+)$.
- Return $AGM(Q^+)$.
- This bound is tight. Proof: very useful exercise.

Example

$Q(X, Y, Z) = R(X, Y) \wedge S(Y, Z) \wedge T(Z, X)$
Fractional edge covers: $(1, 1, 0), (1, 0, 1), (0, 1, 1), (1/2, 1/2, 1/2)$

$$\boxed{|Q| \leq \min(|R| \cdot |S|, |R| \cdot |T|, |S| \cdot |T|, \sqrt{|R| \cdot |S| \cdot |T|})}$$

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$Q^+(X, Y, Z) = R'(X, Y, Z) \wedge S(Y, Z) \wedge T(Z, X)$
Fractional edge covers: $(1, 0, 0), (0, 1, 1)$

$$\boxed{|Q| \leq \min(|R|, |S| \cdot |T|)}$$

Discussion

The expansion procedure is very easy, but limited only to simple FDs:

$AGM(Q^+)$ is always an upper bound on Q's output, but may not be tight.

Need to use entropic inequalities, beyond Shearer

Conditional Entropy

The *Conditional Entropy*

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The submodularity inequality can be written equivalently as:

$$h(\boldsymbol{V}|\boldsymbol{U}) \geq h(\boldsymbol{V}|\boldsymbol{UW})$$

Example of Statistics:

$R =$

U	V	W
a	1	m
a	1	n
a	2	m
a	3	m
b	1	m
b	5	m

$$
\begin{aligned}
|R| &= 6 \\
|R.U| &= 2 \\
|R.V| &= 4 \\
|R.UV| &= 5 \\
h(UVV) &\le \log |R| \\
h(U) &\le \log |R.U| \\
&\dots
\end{aligned}
$$

$$\max(\deg_R(VW|U)) = 4$$
$$h(VW|U) \le \log \max(\deg_R(VW|U))$$

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$$\boxed{\log|R| + \log|S| + \log|T| + \log \max \deg_A(U|XZ) + \log \max \deg_B(X|YU) \geq}$$

$$\geq h(XY) + h(YZ) + h(ZU) + h(U|XZ) + h(X|YU)$$

$$\geq h(XYZ) + \underline{h(Y) + h(ZU)} + h(U|XZ) + h(X|YU)$$

Example of Upper Bound

$$Q = R(X,Y) \wedge S(Y,Z) \wedge T(Z,U) \wedge A(X,Z,U) \wedge B(X,Y,U)$$

Assume $|R| = |S| = |T| = N$: $AGM(Q) = N^2$.

If the FDs $XZ \to U$ and $YU \to X$ hold: $|Q| \leq N^{3/2}$.

$$\boxed{\log|R| + \log|S| + \log|T| + \log \max \deg_A(U|XZ) + \log \max \deg_B(X|YU) \geq}$$

$$\begin{aligned}
&\geq h(XY) + h(YZ) + h(ZU) + h(U|XZ) + h(X|YU) \\
&\geq h(XYZ) + \underline{h(Y) + h(ZU)} + h(U|XZ) + h(X|YU) \\
&\geq h(XYZ) + h(YZU) + h(U|XZ) + h(X|YU)
\end{aligned}$$

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Example of Upper Bound

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$$\begin{aligned}
&\geq h(XY) + h(YZ) + h(ZU) + h(U|XZ) + h(X|YU) \\
&\geq h(XYZ) + h(Y) + h(ZU) + h(U|XZ) + h(X|YU) \\
&\geq h(XYZ) + h(YZU) + \underline{h(U|XZ)} + \underline{h(X|YU)} \\
&\geq h(XYZ) + h(YZU) + h(U|XYZ) + h(X|YZU) \\
&= 2h(XYZU) = \boxed{2\log|Q|}
\end{aligned}$$

Example of Upper Bound

$$Q = R(X,Y) \wedge S(Y,Z) \wedge T(Z,U) \wedge A(X,Z,U) \wedge B(X,Y,U)$$

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$$\begin{aligned}
&\geq h(XY) + h(YZ) + h(ZU) + h(U|XZ) + h(X|YU) \\
&\geq h(XYZ) + h(Y) + h(ZU) + h(U|XZ) + h(X|YU) \\
&\geq h(XYZ) + h(YZU) + h(U|XZ) + h(X|YU) \\
&\geq h(XYZ) + h(YZU) + h(U|XYZ) + h(X|YZU) \\
&= 2h(XYZU) = \boxed{2\log|Q|}
\end{aligned}$$

$$\boxed{|Q| \leq \sqrt{|R| \cdot |S| \cdot |T| \cdot \max(\deg(U|XZ)) \cdot \max(\deg(X|YU))}}$$

Discussion

- AGM/Shearer limited to cardinality statistics.
- More general statistics require general entropic inequalities.
- Everything gets harder: fractional edge cover no longer sufficient, order of the submodularity matters.
- Can we compute the upper bound? Is it tight? Yes and no, it's complicated [Suciu, 2023].
- Do they work in practice? Yes, but you need to do the engineering work [Deeds et al., 2023].


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