CS294-248 Special Topics in Database Theory
Unit 4: AGM Bound, WCOJ

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University of Washington

Outline

- Today: the AGM bound. This is a mathematical formula that gives us $AGM(Q, \boldsymbol{D}) \stackrel{\text{def}}{=} \max_{\boldsymbol{D} \models \text{statistics}} |Q(\boldsymbol{D})|$.

- Thursday: Worst Case Optimal Join, by guest lecturer Hung Ngo. An algorithm that computes $Q(\boldsymbol{D})$ in time $\tilde{O}(AGM(Q, \boldsymbol{D}))$.

Background on Cardinality Estimation

Cardinality Estimation 101 (1/3)

Given:

- Statistics on the input relations $R_1, R_2, \ldots$
- A full conjunctive query Q

"Estimate":

- The size $|Q(\boldsymbol{D})|$.

Numerous applications: query optimization, memory provisioning, data partitioning.

Cardinality Estimation 101 (2/3)

Bottom-up on the query plan:

- Selection $\sigma_p(R)$: assume independence:

$$\boxed{|\sigma_p(R)| \approx \theta_p \cdot |R|} \qquad \boxed{\theta_{p_1 \wedge p_2} \approx \theta_{p_1} \cdot \theta_{p_2}}$$

Histograms, multidimensional histograms.

- Join $J(A, B, C) = R(A, B) \wedge S(B, C)$: assume preservation of values
  - $|J| \approx |R| \cdot \text{avg}(\text{deg}_S(C|B)) = \frac{|R|\cdot|S|}{|\text{Dom}(S.B)|}$.
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Cardinality Estimation 101 (3/3)

- Notoriously hard to estimate cardinality of complex queries.

- No rigorous definition of the *estimate*: there is no probability space.

- How do we combine multiple sources of information?
  - We had two formulas for the join, why choose min?
  - Given $R(A, B, C)$ and histograms on A, B, C, AB, AC, how do we estimate $|\sigma_{A=2,B=4,C=6}(R)|$?

Upper Bound on the Output of a Query

The Output Bound Problem

Given statistics on the input $\boldsymbol{D}$, e.g. cardinalities, # distinct values,

Compute an upper bound B: $$|Q(\boldsymbol{D})| \leq B$$

Challenge: make B tight.

Simple Examples

Assume $|R| \leq N$, $|S| \leq N$, $|T| \leq N$.

- $Q(X, Y, Z) = R(X, Y) \wedge S(Y, Z)$. $\max_{\boldsymbol{D}} |Q(\boldsymbol{D})| = ?$

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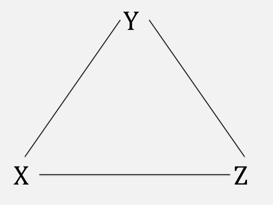
- $Q(X, Y, Z) = R(X, Y) \wedge S(Y, Z) \wedge T(Z, X)$. $\max_{\boldsymbol{D}} |Q(\boldsymbol{D})| = N^{\frac{3}{2}}$
  Here we use a fractional edge cover

AGM Bound: The Statement

Fractional Edge Covers

Query Q to hypegraph $G = (V, E)$.

$R(X, Y) \wedge S(Y, Z) \wedge T(Z, X)$



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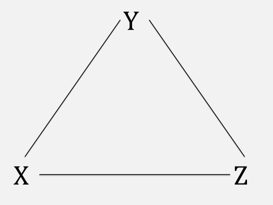
A *fractional edge cover* is $\boldsymbol{w} = (w_e)_{e \in E}$, $w_e \geq 0$:

$$\forall x \in V, \sum_{e \in E : x \in e} w_e \geq 1.$$



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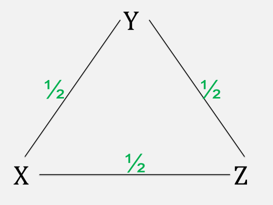

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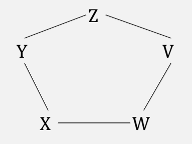

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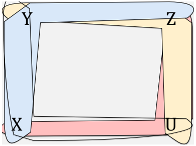

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$$R(X,Y,Z) \wedge S(Y,Z,U)$$
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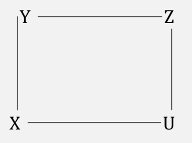

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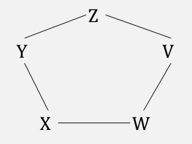

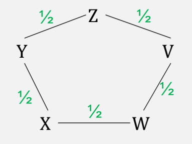

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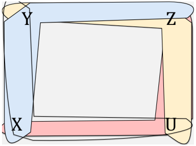

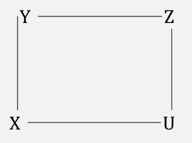

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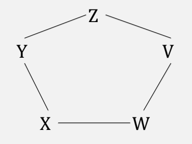

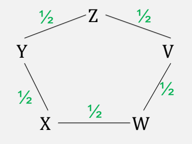

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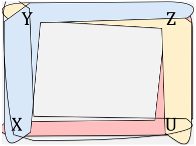

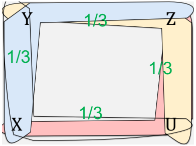

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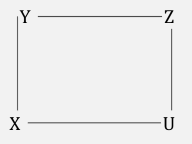

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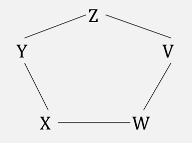

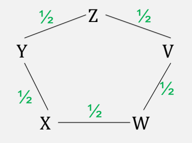

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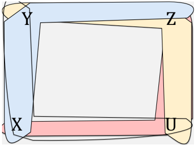

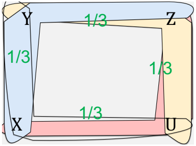

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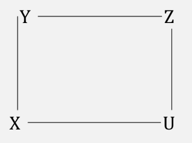

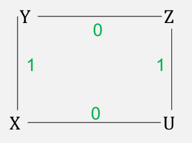

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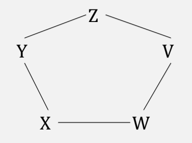

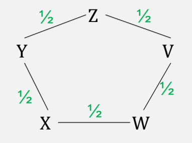

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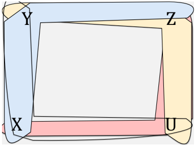

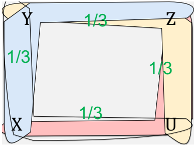

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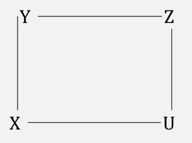

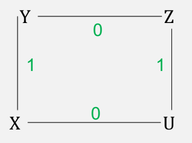

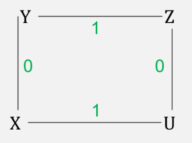

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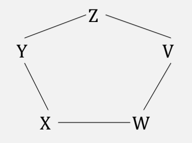

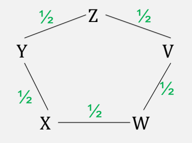

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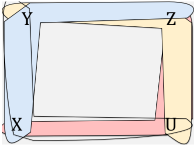

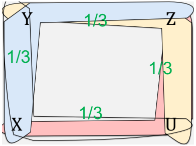

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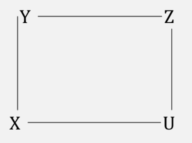

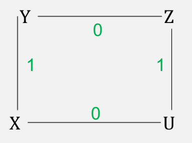

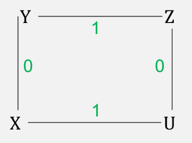

$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ is a convex combination of $(1,0,1,0)$ and $(0,1,0,1)$.

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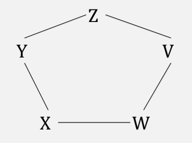

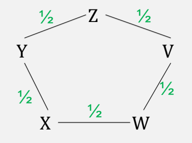

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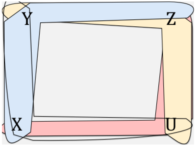

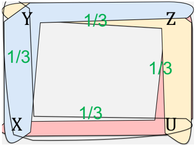

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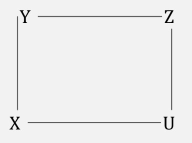

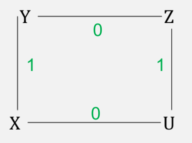

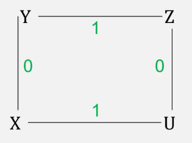

$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is a convex combination of $(1,0,1,0)$ and $(0,1,0,1)$.

Vertex of the edge covering polytope: not $\geq$ convex combination of others.

The AGM Bound [Atserias et al., 2013]

$Q(\boldsymbol{X}) = R_1(\boldsymbol{Y}_1) \wedge \cdots \wedge R_m(\boldsymbol{Y}_m)$

Theorem (Upper Bound)

For every fractional edge cover $\boldsymbol{w}$: $|Q| \leq |R_1|^{w_1} \cdots |R_m|^{w_m}$

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Minimum over vertices of the edge-covering polytope. WHY?

Proof Outline

- Proof of the upper bound: information inequalities (a.k.a. entropic inequalities).

- Proof of the lower bound: construct a worst-case database instance by using strong duality of linear optimization.

Proof of the Upper Bound

Entropic Vectors

Definition

Finite probability space $p : D \to [0, 1]$. X = r.v. with outcomes D.

The *entropy* of X is: $h(X) \stackrel{\text{def}}{=} -\sum_{x \in D} p(x) \log p(x)$

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X	Y	p
a	p	1/4
a	q	1/4
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$h(XY) = \log 4$

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a	p	$1/4$
a	q	$1/4$
b	q	$1/4$
a	m	$1/4$

$h(XY) = \log 4$

X	p
a	$3/4$
b	$1/4$

$h(X) \leq \log 2$

Y	p
p	$1/4$
q	$2/4$
m	$1/4$

$h(Y) \leq \log 3$

$\emptyset$	p
	1

$h(\emptyset) = 0$

Shannon Inequalities

Basic Shannon Inequalities

$$\begin{aligned} h(\emptyset) &= 0 & \\ h(\boldsymbol{U} \cup \boldsymbol{V}) &\geq h(\boldsymbol{U}) & \text{Monotonicity} \\ h(\boldsymbol{U}) + h(\boldsymbol{V}) &\geq h(\boldsymbol{U} \cup \boldsymbol{V}) + h(\boldsymbol{U} \cap \boldsymbol{V}) & \text{Submodularity} \end{aligned}$$

A Shannon inequality is a consequence of these inequalities.

A Shannon Inequality

Example

$$h(XY) + h(YZ) + h(XZ) \geq 2h(XYZ)$$

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$$\begin{aligned} & \underline{h(XY) + h(YZ)} + h(XZ) \\ & \qquad \geq h(XYZ) + h(Y) + h(XZ) \end{aligned}$$

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$$\begin{aligned}\underline{h(XY) + h(YZ)} + h(XZ)& \\ \geq h(XYZ) + &\underline{h(Y) + h(XZ)}\end{aligned}$$

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&\underline{h(XY) + h(YZ)} + h(XZ) \\
&\qquad \geq h(XYZ) + \underline{h(Y) + h(XZ)} \\
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\underline{h(XY) + h(YZ)} + h(XZ) & \\
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&= 2h(XYZ)
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&\qquad \geq 2h(XYZ) + h(\emptyset) \\
&\qquad = 2h(XYZ)
\end{aligned}$$

Note: X is covered 2 times in each expressions. Same for Y, same for Z.

Proof of the AGM Upper Bound: Part 1: $\boxed{|Q| \leq |R_1|^{w_1} \cdots |R_m|^{w_m}}$

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Example

$Q(X, Y, Z) = R(X, Y) \wedge S(Y, Z) \wedge T(Z, X), \qquad |Q| \leq (|R| \cdot |S| \cdot |T|)^{1/2}.$

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$$\begin{aligned}
&\log |R^D| + \log |S^D| + \log |T^D| \\
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\log|R^D| + \log|S^D| + \log|T^D| & \\
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\end{aligned}
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For a general query $Q(\boldsymbol{X}) = R_1(\boldsymbol{Y}_1) \wedge \cdots \wedge R_m(\boldsymbol{Y}_m)$:

If $\boxed{\sum_j w_j h(\boldsymbol{Y}_j) \geq h(\boldsymbol{X})}$ then $\boxed{|R_1|^{w_1} \cdots |R_m|^{w_m} \geq |Q|}$

Proof of the AGM Upper Bound: Part 2: $\boxed{|Q| \leq |R_1|^{w_1} \cdots |R_m|^{w_m}}$

Theorem (Generalized Shearer's Inequality)

If $\boldsymbol{w}$ *is a frac. edge cover, then* $\boxed{w_1 h(\boldsymbol{Y}_1) + \cdots + w_m h(\boldsymbol{Y}_m) \geq h(\boldsymbol{X})}$.

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Proof following [Balister and Bollobás, 2012]. Convert to integers:

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Each variable is "covered $\geq k_0$ times".

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- Every variable remains covered $\geq k_0$ times.
- $\sum_\ell |\boldsymbol{Y}_\ell|^2$ strictly increases (homework!). When do we stop?
- We stop when $\boldsymbol{Y}_1 \supset \boldsymbol{Y}_2 \supset \cdots$
- Thus, $\boldsymbol{Y}_1 = \boldsymbol{X}$ and $k_1 \geq k_0$. $\boxed{\cdots \geq k_1 h(\boldsymbol{Y}_1) \geq k_0 h(\boldsymbol{X})}$

This completes the proof of the Upper AGM Bound.

Discussion

- Shearer's inequality: apply submodularity repeatedly, in any order!
- Shearer inequalities correspond 1-1 to fractional edge covers.
- Any inequality is an upper bound on $|Q|$: $AGM(Q)$ is the smallest.
- How tight is $AGM(Q)$ upper bound? Next

Proof of the Lower Bound

Proof of the AGM Lower Bound

By example:

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Primal program:
Minimize
$w_R \log|R| + w_S \log|S| + w_T \log|T|$
where $\boldsymbol{w}$ is frac. edge cover:

$$\begin{array}{llllll} X: & w_R+ & & w_T & \geq 1 \\ Y: & w_R+ & w_S & & \geq 1 \\ Z: & & w_S+ & w_T & \geq 1 \end{array}$$

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Dual program:
Maximize
$$v_X + v_Y + v_Z$$
where $\boldsymbol{v}$ is "frac. vertex packing":

$$\begin{array}{lllll} R: & v_X + & v_Y & & \leq \log |R| \\ S: & & v_Y + & v_Z & \leq \log |S| \\ T: & v_X + & & v_Z & \leq \log |T| \end{array}$$

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Take optimum $\boldsymbol{v}$, define: $\text{Dom}(X) \stackrel{\text{def}}{=} [\lfloor 2^{v_X} \rfloor]$, $\text{Dom}(Y) \stackrel{\text{def}}{=} [\lfloor 2^{v_Y} \rfloor]$, $\text{Dom}(Z) \stackrel{\text{def}}{=} [\lfloor 2^{v_Z} \rfloor]$.

Proof of the AGM Lower Bound

By example:

$R(X,Y) \wedge S(Y,Z) \wedge T(Z,X)$ $\quad\quad AGM(Q) = \min_{\boldsymbol{w}} |R|^{w_R} \cdot |S|^{w_S} \cdot |T|^{w_T}$

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Proof of the AGM Lower Bound

By example:

$R(X,Y) \wedge S(Y,Z) \wedge T(Z,X)$ $\quad$ $AGM(Q) = \min_{\boldsymbol{w}} |R|^{w_R} \cdot |S|^{w_S} \cdot |T|^{w_T}$

Primal program:
Minimize
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Fix a hypergraph (V, E); $(v_X)_{X \in V} \in \mathbb{R}_+^{|V|}$ is a fractional vertex packing if:

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$$Q = [N^{v_X}] \times [N^{v_Y}] \times [N^{v_Z}]$$

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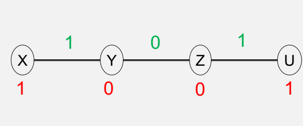

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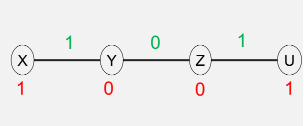

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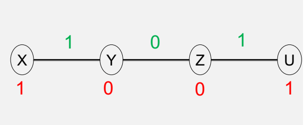

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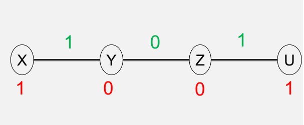

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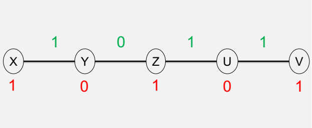

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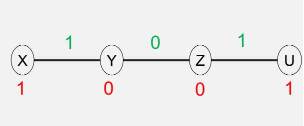

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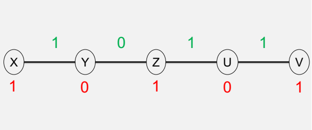

Summary of the AGM Bound

- Upper / lower bound: fractional edge cover / vertex packing.
- Their equality follows from strong duality.
- The worst-case instance of the AGM bound is a Product Database.
- Full CQs only. Otherwise, ignore non-head variables.

Limitation of AGM: only cardinalities. Next: extensions to other stats.

Extensions of the AGM Bound

More Statistics

Statistics for a relation $R(U, V, W, \ldots)$:

- Cardinality $|R|$.
- Same for any projection: $|R.U|, |R.UV|, \ldots$
- Max degree: $\max(\deg_R(VW|U)), \ldots$
- Note: an FD $U \to V$ is $\max(\deg_R(V|U)) = 0$.
- ℓ_p-norm degree sequences: $||\deg_R(V|U)||_2, \ldots$

Simple Functional Dependencies

Given FDs, $|Q| \ll AGM(Q)$.

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$\boldsymbol{U} \to \boldsymbol{V}$ is simple if $|\boldsymbol{U}| = 1$.

Method [Khamis et al., 2016]:

- Expand Q to Q^+ by replacing each atom $R(\boldsymbol{Y})$ with $R'(\boldsymbol{Y}^+)$.
- Return $AGM(Q^+)$.
- This bound is tight. Proof: very useful exercise.

Example

$Q(X, Y, Z) = R(X, Y) \wedge S(Y, Z) \wedge T(Z, X)$
Fractional edge covers: $(1, 1, 0), (1, 0, 1), (0, 1, 1), (1/2, 1/2, 1/2)$

$$\boxed{|Q| \leq \min(|R| \cdot |S|, |R| \cdot |T|, |S| \cdot |T|, \sqrt{|R| \cdot |S| \cdot |T|})}$$

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Assume that $S.Y$ is a key: $Y \rightarrow Z$
$Q^+(X, Y, Z) = R'(X, Y, Z) \wedge S(Y, Z) \wedge T(Z, X)$
Fractional edge covers: $(1, 0, 0), (0, 1, 1)$

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Discussion

The expansion procedure is very easy, but limited only to simple FDs:

$AGM(Q^+)$ is always an upper bound on Q's output, but may not be tight.

Need to use entropic inequalities, beyond Shearer

Conditional Entropy

The *Conditional Entropy*

$$h(\boldsymbol{V}|\boldsymbol{U}) \stackrel{\text{def}}{=} h(\boldsymbol{UV}) - h(\boldsymbol{U})$$

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The submodularity inequality can be written equivalently as:

$$h(\boldsymbol{V}|\boldsymbol{U}) \geq h(\boldsymbol{V}|\boldsymbol{UW})$$

Example of Statistics:

$R =$

U	V	W
a	1	m
a	1	n
a	2	m
a	3	m
b	1	m
b	5	m

$$\begin{aligned} |R| &= 6 \\ |R.U| &= 2 \\ |R.V| &= 4 \\ |R.UV| &= 5 \\ h(UVV) &\leq \log |R| \\ h(U) &\leq \log |R.U| \\ &\cdots \end{aligned}$$

$$\max(\deg_R(VW|U)) = 4$$
$$h(VW|U) \leq \log \max(\deg_R(VW|U))$$

$$\max(\deg_R(V|U)) = 3$$
$$h(V|U) \leq \log \max(\deg_R(V|U))$$

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$$\boxed{|Q| \leq \sqrt{|R| \cdot |S| \cdot |T| \cdot \max(\deg(U|XZ)) \cdot \max(\deg(X|YU))}}$$

Discussion

- AGM/Shearer limited to cardinality statistics.
- More general statistics require general entropic inequalities.
- Everything gets harder: fractional edge cover no longer sufficient, order of the submodularity matters.
- Can we compute the upper bound? Is it tight? Yes and no, it's complicated [Suciu, 2023].
- Do they work in practice? Yes, but you need to do the engineering work [Deeds et al., 2023].

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