UNIVERSITY OF CALIFORNIA AT BERKELEY
Department of Mechanical Engineering
ME233 Advanced Control Systems II
Spring 2016

Homework #4

Assigned:	Mar. 30 (Wed)
Due:	Apr. 5 (Tu)

1. Consider the state-space realization $G(z) = C(zI - A)^{-1}B + D$, where $D^T D$ is invertible. Assume that the dimensions of the matrices are given by $A \in \mathcal{R}^{n_x \times n_x}$, $B \in \mathcal{R}^{n_x \times n_u}$, $C \in \mathcal{R}^{n_y \times n_x}$, and $D \in \mathcal{R}^{n_y \times n_u}$. In this problem, we will establish the relationship between the transmission zeros of this realization and the unobservable modes of $(\hat{C}, \hat{A})$, where

$$\begin{aligned} \hat{A} &= A - B(D^T D)^{-1} D^T C \\ \hat{C} &= C - D(D^T D)^{-1} D^T C \, . \end{aligned}$$

   (a) Show that, for any matrix M, the columns of the matrix X are linearly independent if and only if the columns of the matrix

$$Z := \begin{bmatrix} I \\ M \end{bmatrix} X$$

   are linearly independent

   Hint: A good way to start is by showing that the null space of X is equal to the null space of Z.

   (b) Using the result from part (a), prove that the following conditions are equivalent:

   - $\begin{bmatrix} \hat{A} - \lambda I \\ \hat{C} \end{bmatrix} X = 0$ and the columns of X are linearly independent
   - $\exists Y$ such that $\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = 0$ and the columns of $\begin{bmatrix} X \\ Y \end{bmatrix}$ are linearly independent

   (c) Using the result from part (b), prove that

$$\text{nullity} \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} = \text{nullity} \begin{bmatrix} \hat{A} - \lambda I \\ \hat{C} \end{bmatrix}, \qquad \forall \lambda \in \mathcal{C} \, .$$

   (d) Using the result from part (c), find the relationship between the transmission zeros of the state-space realization $G(z) = C(zI - A)^{-1}B + D$ and the unobservable modes of $(\hat{C}, \hat{A})$.

   Hint: First convert the condition in part (b) into a condition relating the rank of the two matrices. Then show that

$$\text{rank} \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} < \text{normalrank} \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} \quad \Leftrightarrow \quad \text{rank} \begin{bmatrix} \hat{A} - \lambda I \\ \hat{C} \end{bmatrix} < n_x \, .$$

2. When designing an infinite-horizon LQR for the discrete-time system

$$x(k+1) = Ax(k) + Bu(k)$$

under the cost function

$$J = \sum_{k=0}^{\infty} p^T(k)p(k)$$

where $p(k) = Cx(k) + Du(k)$ and D^TD is invertible, we can guarantee that the optimal solution exists provided that (A, B) is stabilizable and a condition involving the transmission zeros of the state space realization $C(zI - A)^{-1}B + D$ holds. Using the result of problem 1, show that when $C^TD = 0$, the transmission zeros of the state space realization $C(zI - A)^{-1}B + D$ are the unobservable modes of (C, A).

3. Consider a discrete-time process described by

$$x(k+1) = Ax(k) + Bu(k) + B_w w(k)$$

where the stationary, Gaussian white noise $w \in \mathcal{R}^s$ satisfies

$$E\{w(k)\} = 0, \qquad E\{w(k+j)w^T(k)\} = W\delta(j) \ .$$

The signals x and u respectively have dimension n and m.

There are two sensors configurations we will be considering.

Sensor Configuration A: In this configuration, the output equation is

$$y(k) = Cx(k) + v_A(k)$$

where $y \in \mathcal{R}^r$ is output vector and $v_A \in \mathcal{R}^r$ is a stationary Gaussian measurement noise. The measurement noise is independent from the initial state and input noise, and the following quantities are given

$$E\{v_A(k)\} = 0, \qquad E\{v_A(k+j)v_A^T(k)\} = V_A\delta(j) \ .$$

Sensor Configuration B: For the purpose of preparing for any sensor failures, this configuration uses two identical sets of sensors and measures the output twice. With this sensor configuration the measurement vector is $2r$ dimensional, and it is given by

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} C \\ C \end{bmatrix} x(k) + \begin{bmatrix} v_{B1}(k) \\ v_{B2}(k) \end{bmatrix}$$

where the measurement noises are independent from the initial state and the input noise, and the following quantities are given for $i = 1, 2$:

$$E\{v_{Bi}(k)\} = 0, \quad E\{v_{Bi}(k+j)v_{Bi}^T(k)\} = V_B\delta(j), \quad E\{v_{B1}(k+j)v_{B2}^T(k)\} = 0$$

While the duplication of the output measurement increases the hardware cost in Sensor Configuration B, it is also true that the specification for each sensor may be relaxed if the same output is measured twice and the two measurements are used in the Kalman filter.

(a) List a set of conditions that guarantee that the stationary Kalman filter exists for Sensor Configuration A. Show that these conditions also guarantee that the stationary Kalman filter exists for Sensor Configuration B. (Do not assume that A is Schur.)

(b) Let the assumptions you listed in part (a) hold and assume that you design and use a Kalman filter for each sensor configuration. Determine a relationship between the sensor noise covariances V_A and V_B so that $M_A = M_B$, where M_A is the steady state a-priori state estimation error covariance for Sensor Configuration A, and M_B is the steady state a-priori state estimation error covariance for Sensor Configuration B.

Hint: The following relationships for stationary Kalman filters are helpful in solving this problem:

- $(I + MC^T V^{-1} C)^{-1} = I - MC^T(CMC^T + V)^{-1}C$
- $A - LC = A\left[I - MC^T(CMC^T + V)^{-1}C\right]$
- $M = (A - LC)MA^T + B_w W B_w^T$

The first relationship comes from the matrix inversion lemma, the second relationship expresses $A - LC$ as a function of M, and the third relationship is a restatement of the discrete algebraic Riccati equation.

4. Define the matrices

$$A_e := \begin{bmatrix} A & 0 & 0 \\ B_1 & A_1 & 0 \\ 0 & 0 & A_2 \end{bmatrix} \qquad B_e := \begin{bmatrix} B \\ 0 \\ B_2 \end{bmatrix}$$

$$C_e := \begin{bmatrix} D_1 & C_1 & 0 \\ 0 & 0 & C_2 \end{bmatrix} \qquad D_e := \begin{bmatrix} 0 \\ D_2 \end{bmatrix}$$

For the frequency-shaped linear quadratic regulator (FSLQR) problem considered in the first half of Lecture 14, we derived the following set of conditions that guarantee the existence of the optimal FSLQR:

A1: (A_e, B_e) is stabilizable

A2: The state space realization $C_e(zI - A_e)^{-1}B_e + D_e$ has no transmission zeros on the unit circle.

It was subsequently stated (but not proved) that the following conditions imply that (A1)–(A2) hold:

B1: (A, B) is stabilizable
B2: A_1 and A_2 are Schur
B3: nullity $\begin{bmatrix} A_2 - \lambda I & B_2 \\ C_2 & D_2 \end{bmatrix} = 0$ whenever $|\lambda| = 1$
B4: nullity $\begin{bmatrix} A_1 - \lambda I & B_1 \\ C_1 & D_1 \end{bmatrix} = 0$ whenever λ is a unit circle eigenvalue of A

Prove that conditions (B1)–(B4) imply conditions (A1)–(A2).

Hint: You will find it useful to use the following characterizations:

- (A, B) is stabilizable if and only if nullity $\begin{bmatrix} A^T - \lambda I \\ B^T \end{bmatrix} = 0$ whenever $|\lambda| \geq 1$
- λ is a transmission zero of the realization $C_e(zI - A_e)^{-1}B_e + D_e$ if and only if nullity $\begin{bmatrix} A_e - \lambda I & B_e \\ C_e & D_e \end{bmatrix} > 0$

The second characterization is a result of $D_e^T D_e = D_2^T D_2 \succ 0$ holding for the FSLQR problem.