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Kalman Filter

Notation : $\quad Y_K \triangleq \{y(0), y(1), \cdots y(k)\}$

- a priori estimation $\quad \hat{x}(k|k-1) = E[x(k) \mid Y_{k-1}]$
- a posteriori estimation $\quad \hat{x}(k|k) = E[x(k) \mid Y_k]$
- a priori covariance $\quad M(k) = \Sigma_{\tilde{x}(k|k-1)|Y_{k-1}\, \tilde{x}(k|k-1)|Y_{k-1}}$
- a posteriori covariance $\quad Z(k) = \Sigma_{\tilde{x}(k)|Y_k\, \tilde{x}(k)|Y_k}$

System model

$$x(k+1) = A x(k) + B u(k) + B_w w(k)$$

$$y(k) = C x(k) + v(k)$$

$$E[w(k)] = 0 \quad E[v(k)] = 0 \quad \Sigma_{ww}(k,j) = W(k)\, \delta_{kj}$$

$$\Sigma_{vv}(k,j) = V(k)\, \delta_{kj} \quad \Sigma_{wv}(k,j) = 0$$

Kalman filter

$$\hat{x}(k+1|k+1) = \hat{x}(k+1|k) + F(k+1)\left[y(k+1) - C\hat{x}(k+1|k)\right]$$

$$F(k+1) = M(k+1)\, C^T \left[C\, M(k+1)\, C^T + V(k+1)\right]^{-1}$$

$$\hat{x}(k+1|k) = A\,\hat{x}(k|k) + B u(k)$$

$$M(k+1) = A\, Z(k)\, A^T + B_w\, W(k)\, B_w^T$$

$$Z(k+1) = M(k+1) - M(k+1)\, C^T \left(C M(k+1) C^T + V(k+1)\right)^{-1} C M(k+1)$$

Implementation

$x(0|-1) \triangleq x_0$

Initial condition $\quad M(0) = \Sigma_{x_0 x_0} \quad M(1) \quad M(2) \cdots$

$u(0) \rightarrow \hat{x}(0|0) \qquad u(0) \rightarrow \hat{x}(1|1)$

$y(0) \qquad y(1)$

$\hat{x}_0 = \hat{x}(0|-1) \qquad \hat{x}(1|0) \qquad \hat{x}(2|1)$

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- Dynamic update $\hat{x}(k|k) \longrightarrow \hat{x}(k+1|k)$.

$$- \quad \hat{x}(k|k-1) = A\hat{x}(k-1|k-1) + Bu(k-1)$$

$$\tilde{x}(k|k-1) = A\tilde{x}(k-1|k-1) + B_w w(k-1)$$

$$M(k) = \Sigma_{\tilde{x}(k|k-1)\tilde{x}(k|k-1)} = AZ(k-1)A^T + B_w W(k) B_w^T$$

$$M(0) = \Sigma_{x_0 x_0}$$

- Measurement update (Least square) $\hat{x}(k+1|k) \longrightarrow \hat{x}(k+1|k+1)$

$$\hat{x}(k+1) = E[x(k)|Y_k] = E[x(k)|Y_{k-1}, y(k)]$$

$$= E[x(k)|Y_{k-1}] + E[\tilde{x}(k)|Y_{k-1} \,|\, \tilde{y}(k)|Y_{k-1}]$$

$$= E[x(k)|Y_{k-1}] + \Sigma_{\tilde{x}(k)|Y_{k-1}\,\tilde{y}(k)|Y_{k-1}} \Sigma^{-1}_{\tilde{y}(k)|Y_{k-1}\,\tilde{y}(k)|Y_{k-1}} \cdot (\tilde{y}(k)|Y_{k-1} - E[\tilde{y}(k)|Y_{k-1}])$$

$$\tilde{y}(k)|Y_{k-1} = C\tilde{x}(k)|Y_{k-1} + v(k) = C\tilde{x}(k|k-1) + v(k)$$

$$\Sigma_{\tilde{x}(k)|Y_{k-1}\,\tilde{y}(k)|Y_{k-1}} = M(k)C^T \qquad (\Sigma_{\tilde{x}(k|k-1)\tilde{x}(k|k-1)} = M(k))$$

$$\Sigma_{\tilde{y}(k)|Y_{k-1}\,\tilde{y}(k)|Y_{k-1}} = CM(k)C^T + V(k)$$

$$\hat{x}(k|k) = \hat{x}(k|k-1) + \underbrace{M(k)C^T(CM(k)C^T + V(k))^{-1}}_{F(k)} (y(k) - C\hat{x}(k|k-1))$$

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— Variance update $M(k) \longrightarrow Z(k)$.

$$\Sigma_{\tilde{x}(k)|Y_k\, \tilde{x}(k)|Y_k} \hat{=} Z(k)$$

$$= \Sigma_{\tilde{x}(k)|Y_{k-1}, y(k)\, \tilde{x}(k)|Y_{k-1}, y(k)}$$

$$= \Sigma_{\tilde{x}(k)|Y_{k-1}\, \tilde{x}(k)|Y_{k-1}} - \Sigma_{\tilde{x}(k)|Y_{k-1}\, \tilde{y}(k)|Y_{k-1}} \Sigma^{-1}_{\tilde{y}(k)|Y_{k-1}\, \tilde{y}(k)|Y_{k-1}} \Sigma_{\tilde{y}(k)|Y_{k-1}\, \tilde{x}(k)|Y_{k-1}}.$$

$$= M(k) - M(k) C^T (C M(k) C^T + V(k))^{-1} C M(k)$$

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Multirate KF.

For a system $x(k+1) = Ax(k) + Bu(k) + B_w w(k)$

$$m_{x_0} = x_0 \quad \Sigma_{\tilde{x}_0 \tilde{x}_0} = X_0$$

$$m_w = 0 \quad \Sigma_{ww}^{(k,j)} = W \delta_{kj}$$

The measurement $y(k) = Cx(k) + v(k)$ is taken $k = 0, 2, 4, \ldots$

$$m_v = 0 \quad \Sigma_{vv}(k,j) = V \delta_{k,j}.$$

Sol: Dynamics

$$x(k+2) = A^2 x(k) + [AB, B]\begin{bmatrix} u(k) \\ u(k+1) \end{bmatrix} + [AB_w \; B_w]\begin{bmatrix} w(k) \\ w(k+1) \end{bmatrix}$$

$$= \bar{A} x(k) + \bar{B}\bar{u}(k) + \bar{B}_w \bar{w}(k)$$

$$y(k+2) = Cx(k+2) + v(k+2)$$

$$E[\bar{w}(k)] = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad E[\bar{w}(k)\bar{w}^T(j)] = \begin{bmatrix} W & 0 \\ 0 & W \end{bmatrix}$$

KF: $\hat{x}(k+2|k+2) = \hat{x}(k+2|k) + F(k+2)[y(k+2) - C\hat{x}(k+2|k)]$

$$\hat{x}(k+2|k) = \bar{A}\hat{x}(k|k) + \bar{B}\bar{u}(k)$$

$$\hat{x}(0|-2) = x_0.$$

$$F(k+2) = M(k+2)C^T[CM(k+2)C^T + V]^{-1}$$

$$M(k+2) = \bar{A}Z(k)\bar{A}^T + \bar{B}_w \begin{bmatrix} W & 0 \\ 0 & W \end{bmatrix} \bar{B}_w^T \quad M(0) = X_0$$

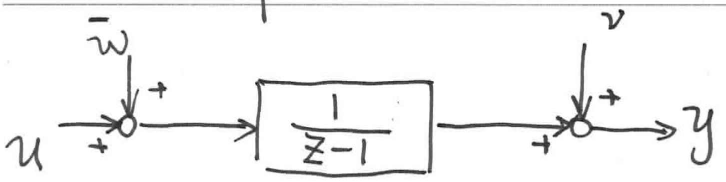
$$Z(k+2) = M(k+2) - M(k+2)C^T[CM(k+2)C^T + V]^{-1} CM(k+2)$$

W is white, unit variance

$$\Sigma_{ww}(l) = W\delta(l) = 1\,\delta(l) \begin{cases} 1 & l=0 \\ 0. & l\neq 0 \end{cases}$$

$$\phi_{ww}(w) = \sum_{-\infty}^{+\infty} \delta(l)\, e^{-jwl} = 1:$$

Colored input noise

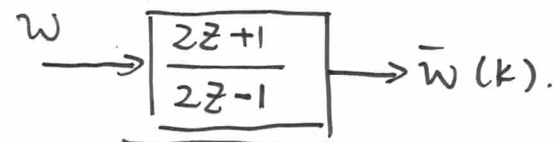


$\bar{w}$ is zero mean, spectral density

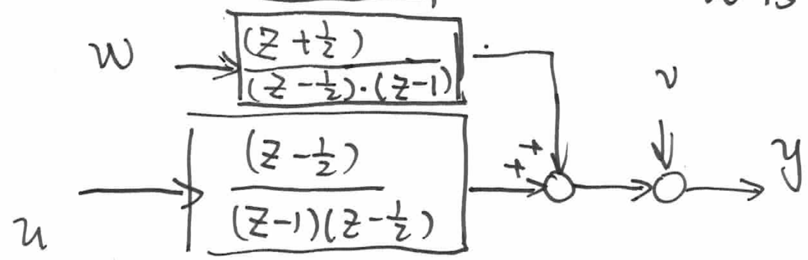
$$\phi_{\bar{w}\bar{w}}(w) = \frac{5+4\cos w}{5-4\cos w}$$

$$= \frac{5+2e^{jw}+2e^{-jw}}{5-2e^{jw}-2e^{-jw}} = \frac{2e^{jw}+1}{2e^{jw}-1}\,\frac{2e^{-jw}+1}{2e^{-jw}-1}$$

$$= G(e^{jw})\, G(e^{-jw})$$



~~W is~~
W is white with unit variance



$$x(k+1) = \begin{bmatrix} 3/2 & 1 \\ -1/2 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} u(k) + \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} w(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) + v(k).$$