Outline:

- LQ summary
- Probability and stochastic control
- Gaussian distribution
- HW1 hints

1 Discrete-time LQ summary

- system:

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k)$$

Note: A, B, C, D can be time-varying as well.

- performance index:

$$J = \frac{1}{2}x^T(N)Sx(N) + \frac{1}{2}\sum_{k=k_0}^{N-1} x^T(k)Qx(k) + u^T(k)R(k)$$

$$Q = C^T C \qquad J^o = \min_{u(k_0)\cdots u(N-1)} J$$

- key idea in Dynamic Programming:

$$J_k^0(x(k)) = \min_{u(k)} \left\{ \frac{1}{2}x^T(k)Qx(k) + \frac{1}{2}u^T(k)Ru(k) + J_{k+1}^0(x(k+1)) \right\}$$

- boundary condition:

$$P(N) = S \qquad J_N^0 = \frac{1}{2}x^T(N)Sx(N)$$

- solution:

$$u(k) = -\underbrace{\left\{[R + B^T P(k+1)B]^{-1}B^T P(k+1)A\right\}}_{K(k)} x(k)$$

$$J_{k_0}^0(x(k_0)) = \frac{1}{2}x_0^T P(0)x_0; \quad J_k^0(x(k)) = \frac{1}{2}x^T(k)P(k)x(k)$$

- Riccati Eq:

$$P(k) = A^T P(k+1)A - A^T P(k+1)B[R + B^T P(k+1)B]^{-1}B^T P(k+1)A + Q$$

- Implementation of LQ:
  - the Riccati equation can be solved offline (even when the system is time-varying). We can store the matrices $P(k_0), \ldots, P(N)$.
  - $u(k)$ can then be computed online using $P(k+1)$ and $x(k)$.

2 Probability and stochastic control theory

Why are we learning this: we have been very familiar with the following system equation

$$x(k+1) = Ax(k) + Bu(k)$$

However the reality is that the above is often not a whole description of the problem we are considering. Instead, the following might be more helpful from time to time:

$$x(k+1) = Ax(k) + Bu(k) + B_w w(k)$$

where $w(k)$ is the noise term that we have been neglecting.[1] With the introduction of $w(k)$, we need to equip ourselves with some additional tool sets to understand and analyze the problem.

Probability:

- discusses how likely things, or more exactly, events, happen
- there's an overall space, called sample space, that includes all the possible outcomes. One outcome is just one sample of the space
- an event includes some (maybe 1, maybe more, maybe none) outcomes of the sample space. e.g., the event tomorrow won't rain
- to better measure probabilities, we introduce random variables (r.v.'s)

Random variables:

- assign the outcome numerical real values, such that we can now talk about $P(X = x)$ and $P(X \leq x)$, etc.
- expectations: give the average effect of a r.v., e.g.,

$$\begin{aligned} E[X] &: \text{mean} \\ E\left[(X - m_x)^2\right] &: \text{second moment, error energy distributions} \\ E\left[(X - m_x)^3\right] &: \text{third moment} \end{aligned}$$

Random vectors:

- a collection of random variables

2.1 Definitions

Note: Be careful to distinguish between the three phrases: random variable, random vector, and random process. It is not meaningful to talk about things like "autocovariance of a random variable".

Definitions that are valid for random variables and random vectors:

- mean:

$$m_X = E[X] = \begin{cases} \int x p_X(x)\, dx & \text{for a continuous-time r.v. } x \\ \sum x p_X(x) & \text{for a discrete-time r.v. } x \end{cases}$$

Definitions that are valid for random variables:

- variance:

$$Var[X] = E\left[(X - m_X)^2\right]$$

- independent:

$$p_{X,Y}(x, y) = p_X(x)\, p_Y(y)$$

- covariance:

$$Cov(X, Y) = E[(X - m_X)(Y - m_Y)] = E[XY] - E[X]E[Y]$$

[1] This is from the state-space perspective. A block-diagram perspective can also be made.

- correlation coefficient:
$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}$$
- uncorrelated: X and Y are called uncorrelated if $\rho(X,Y) = 0$ (this is a more-easier-to-understand definition than "covariance being zero")
  - independent⇒uncorrelated; uncorrelated however does not infer independent
  - uncorrelated indicates $Cov(X,Y) = E[(X - m_X)(Y - m_Y)] = E[XY] - E[X]E[Y] = 0 \Rightarrow E[XY] = E[X]E[Y]$, which is clearly weaker than X and Y being independent

Definitions that are valid for random vectors:

- covariance matrix: $E\left[[X - m_X][X - m_X]^T\right]$

Definitions that are valid for random process:

- autocovariance:
$$X_{XX}(j,k) = E\left[(X(j) - m_X(j))(X(k) - m_X(k))^T\right]$$
- autocorrelation: this has non uniform definitions (see, e.g., wikipedia.org). In statistics, autocorrelation is often the same as autocovariance. In ME233 and in digital signal processing, autocorrelation is defined as $R_{XX}(j,k) = E\left[X(j)\,X(k)^T\right]$.
- cross covariance:
$$X_{XY}(j,k) = E\left[(X(j) - m_X(j))(Y(k) - m_Y(k))^T\right]$$
- ergodic: all ensemble averages = the corresponding time average. Example: $E[X(k)] = \overline{X(k)} = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{j=-N}^{N} x(j)$
  - in general, it is not easy to test ergodicity. In practice, many processes are however ergodic;
  - ergodicity is extremely important since large samples can be very expensive to collect.
- stationary: tells whether a process changes its statistical properties w.r.t. time
  - strict-sense stationary: probability distribution does not change w.r.t. time
$$Pr(X(k_1) \le x_1, \ldots, X(k_n) \le x_n) = Pr(X(k_1 + l) \le x_1, \ldots, X(k_n + l) \le x_n)$$
  - weak-sense stationary: mean and auto covariance does not dependent on time
$$m_X(k) = m_X;\ X_{XX}(j,k) \text{ only depends on } k - j$$

3 Gaussian (Normal) distribution

- the pdf of a Gaussian random variable is
$$p_x(x) = \frac{1}{\sigma_x\sqrt{2\pi}}\exp\left(-\frac{(x - m_x)^2}{2\sigma_x^2}\right)$$
- the pdf of a Gaussian random vector is
$$p_x(x) = \frac{1}{\sqrt{2\pi}^n\sqrt{|X_x|}}\exp\left(-\frac{1}{2}(x - m_x)^T X_x^{-1}(x - m_x)\right)$$
- the cdf of Gaussian random variables/vectors does not have a simple formula. Instead, we usually use a table to find the numerical values.
- Gaussian distribution has lots of nice properties:

  - the pdf is solely determined by the mean and the variance/covariance matrix
  - if two jointly Gaussian distributed random variables are uncorrelated, then they are independent
  - the output of an LTI system is a Gaussian random process if the input is Gaussian as well
  - linear functions of a Gaussian random process are still Gaussian
  - if X_1 and X_2 are jointly Gaussian, then $X_1|X_2$ and $X_2|X_1$ are also Gaussian

- Gaussian and white
  - they are different concepts
  - there can be Gaussian white noise, Poisson white noise, etc
  - Gaussian white noise is used a lot since it is a good approximation to many practical noises

4 Hints for homework 1

Problem 1:

- start from J_N and write down J_{N-1}. Find the "small sub problem" that we talked about in Discussion 1.
- find out the optimal control law $u^o(N-1)$ using the basic optimization technique in Discussion 1.
- you final solution should at least contain the equation for $u^o(k)$, an equation similar to the Riccati equation, and the recursive equations for $b(k)$. It is encouraged to also find the recursive equation for $c(k)$.

Problem 2:

- note the cost is a product of terms instead of a summation. The key idea is the same as LQ but the equations at each step are different.