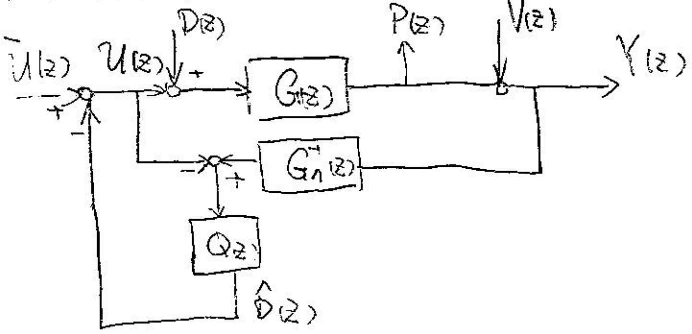
Disturbance Observer.



closed loop dynamics

$$P = \frac{G(1-Q)}{1+Q(G_n^{-1}G-1)} D + \frac{G}{1+Q(G_n^{-1}G-1)} \bar{U}$$

$$- \frac{GQG_n^{-1}}{1+Q(G_n^{-1}G-1)} V.$$

$G(z) = G_n(z)(1+\Delta(z))$ $\Delta(z)$ stable

1 $Q(z) \cdot G_n^{-1}(z)$ have to be causal

2. Disturbance rejection

$Q(e^{jw}) \approx 1$ $\cdot \frac{G_n(1+\Delta)(1-Q)}{1+Q\Delta} D$

at desired frequency range

3. Sensor noise, $|Q(e^{jw})|$ small for sensor noise active frequencies

4. $\|\Delta(e^{jw}) Q(e^{jw})\| \le 1$ for all w.

With a closed loop controller, how to guarantee stability?

1. Controller design

2. minimum phase $G(z)$

If there is no mis-match?

Frequency shaped LQ.

$\boxed{\text{Discrete case}}$

Cost fcn

$$J = \sum_{k=0}^{\infty} \left\{ x^T(k) Q x(k) + u^T(k) R u(k) \right\}$$

1 cost fcn in frequency domain

$$J = \int_{-\pi}^{\pi} X^T(e^{-jw}) Q X(e^{jw}) + U^T(e^{-jw}) R U(e^{jw}) \, dw$$

Instead of Keeping Q, R constant, design

$$Q(e^{jw}) = Q_f^T(e^{-jw}) Q_f(e^{jw})$$

$$R(e^{jw}) = R_f^T(e^{-jw}) R_f(e^{jw})$$

$$J = \int_{-\pi}^{\pi} X^T(e^{jw}) Q_f^T(e^{-jw}) Q_f(e^{jw}) X(e^{jw}) + U^T(e^{-jw}) R_f^T(e^{-jw}) R_f(e^{jw}) U(e^{jw}) \, dw$$

2. Let

replace e^{jw} with z

$$x(k) \longrightarrow \boxed{Q_f(z)} \longrightarrow x_f(k) \qquad u(k) \longrightarrow \boxed{R_f(z)} \longrightarrow u_f(k)$$

$$z_1(k+1) = A_1 z_1(k) + B_1 x(k) \qquad z_2(k+1) = A_2 z_2(k) + B_2 u(t)$$

$$x_f(k) = C_1 z_1(k) + D_1 x(k) \qquad u_f(k) = C_2 z_2(k) + D_2 u(k)$$

* Size of A_1 = Size of Q_f

Column of B = # of element of x

3. $J = \sum_{k=0}^{\infty} \left\{ x_f^T(k) x_f(k) + u_f^T(k) u_f(k) \right\}$

$$\underbrace{\begin{bmatrix} x(k+1) \\ z_1(k+1) \\ z_2(k+1) \end{bmatrix}}_{x_e(k+1)} = \underbrace{\begin{bmatrix} A & 0 & 0 \\ B_1 & A_1 & 0 \\ 0 & 0 & A_2 \end{bmatrix}}_{A_e} \begin{bmatrix} x(k) \\ z_1(k) \\ z_2(k) \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \\ B_2 \end{bmatrix}}_{B_e} u(k)$$

$$\begin{bmatrix} x_f(k) \\ u_f(k) \end{bmatrix} = \overbrace{\begin{bmatrix} D_1 & C_1 & 0 \\ 0 & 0 & C_2 \end{bmatrix}}^{C_e} \begin{bmatrix} x(k) \\ z_1(k) \\ z_2(k) \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ D_2 \end{bmatrix}}^{D_e} u(k)$$

$$J = \sum_{k=0}^{\infty} \left\{ \begin{bmatrix} x_e(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} C_e^T \\ D_e^T \end{bmatrix} [C_e \ D_e] \begin{bmatrix} x_e(k) \\ u(k) \end{bmatrix} \right\}$$

$$u = [B_e^T P B_e + D_e^T D_e]^{-1} [B_e^T P A_e + D_e^T C_e]$$

$$P = A_e^T P A_e + C_e^T C_e - [A_e^T P B_e + C_e^T D_e][B_e^T P B_e + D_e^T D_e]^{-1} [B_e^T P A_e + D_e^T C_e]$$

$\boxed{\text{Continuous case}}$

cost fcn

$$J = \int_0^{\infty} (x^T(t) Q x(t) + u^T(t) R u(t)) \, dt$$

1 Cost fcn in frequency domain

$$J = \frac{1}{2\pi} \int_{-\infty}^{\infty} (X^T(-jw) Q X(jw) + U^T(jw) R U(jw)) \, d$$

Instead of keeping Q & R constant, design

$$Q(jw) = Q_f^T(-jw) Q_f(jw)$$

$$R(jw) = R_f^T(-jw) R_f(jw)$$

$$J = \frac{1}{2\pi} \int_{-\infty}^{\infty} (X^T(-jw) Q_f^T(-jw) Q_f(jw) X(jw) + U^T(-jw) R_f^T(-jw) R_f(jw) U(jw)) \, dw$$

2. Let $x(t) \longrightarrow \boxed{Q_f} \longrightarrow x_f(t) \qquad u(t) \longrightarrow \boxed{R_f} \longrightarrow u_f(t)$

$$\dot{z}_1(t) = A_1 z_1(t) + B_1 x(t) \qquad \dot{z}_2(t) = A_2 z_2(t) + B_2 u(t)$$

$$x_f(t) = C_1 z_1(t) + D_1 x(t) \qquad u_f(t) = C_2 z_2(t) + D_2 u(t)$$

* Size of A_1 = Size of Q_f

Column of B_1 = # of element of x

3. $J = \int_0^{\infty} (x_f^T(t) x_f(t) + u_f^T(t) u_f(t)) \, dt$

$$\frac{d}{dt} \underbrace{\begin{bmatrix} x(t) \\ z_1(t) \\ z_2(t) \end{bmatrix}}_{x_e(t)} = \underbrace{\begin{bmatrix} A & 0 & 0 \\ B_1 & A_1 & 0 \\ 0 & 0 & A_2 \end{bmatrix}}_{A_e} \begin{bmatrix} x(t) \\ z_1(t) \\ z_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \\ B_2 \end{bmatrix}}_{B_e} u$$

$$x_f = \underbrace{[D_1 \ C_1 \ 0]}_{C_e} x_e \qquad u_f = [0 \ 0 \ C_2] x_e + D_2 u$$

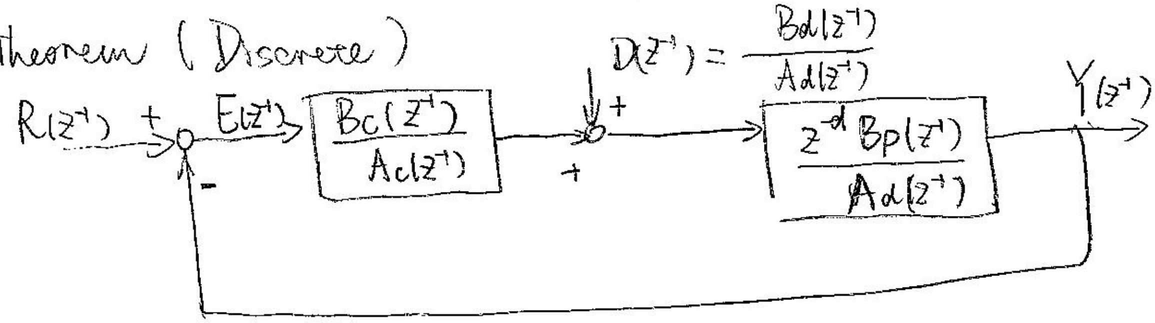
$$J = \int_0^{\infty} x_e^T \underbrace{\begin{bmatrix} D_1^T D_1 & D_1^T C_1 & 0 \\ C_1^T D_1 & C_1^T C_1 & 0 \\ 0 & 0 & C_2^T C_2 \end{bmatrix}}_{Q_e} x_e + 2u^T \underbrace{[0 \ 0 \ D_2^T C_2}_{N_e} + u^T \underbrace{D_2^T D_2}_{R_e} u \, dt$$

$$u = -R_e^{-1} (B_e^T P_e + N_e) x_e$$

$$A_e^T P_e + P_e A_e - (B_e^T P_e + N_e)^T R_e^{-1} (B_e^T P_e + N_e) + (Q_e$$

Internal Model Principle & Repetitive control

Theorem (Discrete)



Assume $B_p(z^{-1}) = 0$ and $A_d(z^{-1}) = 0$ do not have common zeros. If the closed loop is asymptotically stable, and $A_c(z^{-1})$ can be factorized as $A_c(z^{-1}) = A_d(z^{-1}) A_c'(z^{-1})$ then the disturbance is asymptotically rejected.

$d(k)$	$A_d(z^{-1})$
dc const	$1 - z^{-1}$
$\cos(w_0 k)$ & $\sin(w_0 k)$	$1 - 2z^{-1}\cos(w_0) + z^{-2}$
$d(k) = \alpha k + \beta$	$1 - 2z^{-1} - z^{-2}$
$d(k) = d(k-N)$	$1 - z^{-N}$

Procedure in design a controller to reject repetitive disturbance

1. For a disturbance, find $D(z^{-1})$
2. By internal model principle, the denominator of controller should include $A_d(z^{-1})$, i.e. controller $G_c(z^{-1}) = \frac{R(z^{-1})}{A_d(z^{-1}) S(z^{-1})}$, R & S are to be designed
3. design close loop characteristic for $A_{closed}(z^{-1})$
4. solve R & S by Diophantine eqn

$$A_d(z^{-1}) A_p(z^{-1}) S(z^{-1}) + z^{-d} B_p(z^{-1}) R(z^{-1}) = A_{closed}(z^{-1})$$