1 Review of what we learned so far

Checklist (the list may not be complete):

- Dynamic programming
  - Definition of the optimal cost to go
  - The Bellman's principle of optimality
- Discrete time linear quadratic problem
  - Problem formulation: plant, performance index
  - Solution via dynamic programming (solve the ordinary one variable quadratic optimization problem): optimal control law, backwards discrete time Riccati equation
  - Stationary discrete time linear quadratic problem
  - Solution: the optimal control law, algebraic Riccati equation
- Probability theory basics
  - Definitions: probability axioms, random variable, mean (expectation), variance, uniform and Gaussian distributions, joint distribution, independence, random vector, covariance matrix, conditional mean
  - Jointly Gaussian random vector: joint density function, conditional density function, conditional mean and covariance
- Random process (both discrete time and continuous time)
  - Stationary process and ergodic process
  - Definitions of auto-covariance, auto-correlation, cross-covariance, and cross-correlation
  - Spectral density of a wide sense stationary process
  - White process
  - Filtering a random process by $G(z)$ or $G(s)$
  - State space system disturbed by random processes
- Least squares
  - Least square estimation problem
  - Gaussian case: $\hat{x}$, mean and covariance of $\hat{x}$, mean and covariance of $\tilde{x}$, three properties
- Kalman filter (both discrete time and continuous time)
  - Problem formulation: linear state space system with random noises, conditions
  - Solution: Kalman filter gain, physical meanings of $M(k)$ and $Z(k)$, Kalman filter Riccati equation, duality between LQ control and Kalman filter, innovations process
  - Steady state Kalman filter: Riccati equation, eigenvalues, return difference equality, and symmetric root locus
- Linear Quadratic Gaussian Control
  - Stochastic control with known state (LQ control problem with random input noise)
  - Stochastic control with unknown state (combination of LQ control and Kalman filter)
  - Stationary LQG: conditions (whiteness, observability, controllability, etc.), seperation theorem, continuous-time LQG

Table 1: duality between continuous-time LQ and KF

	Continuous-Time LQ	Continuous-time KF
stationary-case	$A^TP+PA+Q-PBR^{-1}B^TP=0$	$AM_s+M_sA^T+B_wWB_w^T-M_sC^TV^{-1}CM_s=0$
	$K=R^{-1}B^TP$	$F_s=M_sC^TV^{-1}$
non-stationary case	$A^TP+PA+Q-PBR^{-1}B^TP=-\frac{d}{dt}P$	$\frac{dM}{dt}=AM+MA^T+B_wWB_w^T-MC^TV^{-1}CM$

Table 2: duality between discrete-time LQ and KF

Discrete-Time LQ
$A^TPA-P-A^TPB\left[R+B^TPB\right]^{-1}B^TPA+C^TC=0$
$K=\left[R+B^TPB\right]^{-1}B^TPA$
$P(k)=A^TP(k+1)A-A^TP(k+1)B[R+B^TP(k+1)B]^{-1}B^TP(k+1)A+Q$ (non-stationary case)
Discrete-time KF
$AMA^T-M+B_wWB_w^T-AMC^T\left[CMC^T+V\right]^{-1}CMA^T=0$
$F_s=M_sC^T\left[CM_sC^T+V\right]^{-1}$
$M(k+1)=AM(k)A^T+B_wWB_w^T-AM(k)C^T\left[CM(k+1)C^T+V\right]^{-1}CM(k)A^T$ (non-stationary case)

2 Duality between LQ and KF

See Tables I and II.

3 Proof of separation theorem

Key steps (stationary-case as an example):

$$u(k)=-K_sx(k|k) \tag{1}$$

$$x(k+1)=Ax(k)+Bu(k)+B_ww(k) \tag{2}$$

$$\hat{x}(k+1|k+1)=A\hat{x}(k|k)+Bu(k)+F_s\left(y(k+1)-C\hat{x}(k+1|k)\right) \tag{3}$$

Let $\tilde{x}(k|k)=x(k)-\hat{x}(k|k)$

$$\begin{bmatrix}x(k)\\ \hat{x}(k|k)\end{bmatrix}=\begin{bmatrix}1 & 0\\ 1 & -1\end{bmatrix}\begin{bmatrix}x(k)\\ \tilde{x}(k|k)\end{bmatrix} \tag{4}$$

$\Rightarrow$

$$\begin{bmatrix}x(k+1)\\ \tilde{x}(k+1|k+1)\end{bmatrix}=\begin{bmatrix}A-BK_s & BK_s\\ 0 & A-F_sCA\end{bmatrix}\begin{bmatrix}x(k)\\ \tilde{x}(k|k)\end{bmatrix}+\dots \tag{5}$$

$\Rightarrow$ eigen values of the whole system can be decomposed to eigen values of the state feedback LQ problem and the Kalman filter.