Outlines

- LQG/LTR
- FSLQ

1 LQG/Loop Transfer Recover (LTR)

History and notes:

- LQG/LTR should be regarded as one word
- a robust control design method that uses LQG control structure
- not an optimal control design method
- not even a stochastic control design method
- uses Fictitious KF

Setup: we have the continuous-time plant $G(s)$ with state-space matrices A, B, C, ($D = 0$); and the stationary LQG controller

$$\frac{d\hat{x}(t|t)}{dt} = A\hat{x}(t|t) + Bu(t) + F\left(y(t) - C\hat{x}(t|t)\right) = (A - BK - FC)\hat{x}(t|t) + Fy(t)$$

$$u(t) = -K\hat{x}(t|t)$$

Converting the above to an equivalent feedback controller in the transfer-function form, we have

$$-y \to u: \; G_c(s) = K(sI - A + BK + FC)^{-1}F \tag{1}$$

Conditions: the plant $G(s)$ has only minimum-phase transmission zeros, (A, B) is controllable, (A, C) is observable.
Result: if we choose the LQ cost function properly, the loop transfer function $G_p(s)G_c(s)$ converges point-wise in s to the Kalman filter loop transfer function $C(sI - A)^{-1}F$:

$$\left\{[C(sI - A)^{-1}B][K(sI - A + BK + FC)^{-1}F]\right\} \xrightarrow{\rho \to 0} C(sI - A)^{-1}F \tag{2}$$

Here the "properly chosen" cost function is

$$J = \int_0^\infty \left(x^T(t)C^TCx(t) + \rho u^T(t)Nu(t)\right)dt \tag{3}$$

Great benefit for feedback control design: recall that

- Continuous-time LQ problems have the robust stability results of infinite gain margin, no less than 60 degree phase margin, etc. This is from the return difference equality $\left[I + K\left(-sI - A\right)^{-1}B\right]^T R \left[I + K\left(sI - A\right)^{-1}B\right] = R + G\left(-s\right)^T G\left(s\right)$. And $I + K\left(sI - A\right)^{-1}B$ defines the closed-loop property.
- Kalman filters and LQ are dual problems, and in Kalman filter we learned the return difference equality

$$\left[I + C\left(sI - A\right)^{-1}F\right] V \left[I + F^T\left(-sI - A\right)^{-T}C^T\right] = V + G_w\left(s\right)WG_w^T\left(-s\right)$$

- Looking now at the KF return difference equality, we see the term $C\left(sI - A\right)^{-1}F$ on the left hand side of the equality, is our target loop transfer function in LQG/LTR! Hence, we can use LQG/LTR to approximate this target loop transfer function, which will give lots of benefits in the sense of disturbance rejection and closed-loop robustness.
- Now $C(sI - A)^{-1}F$ is a target loop transfer function (which we design), KF is hence not an actual optimal state estimator, and a fictitious one.
- The plant with fictitious noise input is:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Lw(t) \\ y(t) &= Cx(t) + v(t) \end{aligned} \tag{4}$$

where

$$\begin{aligned} E[w(t)w^T(t+\tau)] &= I\delta(\tau) \\ E[v(t)v^T(t+\tau)] &= \mu I\delta(\tau) \end{aligned} \tag{5}$$

The steady-state KF gain and Riccati equation for the above fictitious system is

$$F = \frac{1}{\mu}MC^T, \quad AM + M^TA + LL^T - \frac{1}{\mu}MC^TCM = 0 \tag{6}$$

The return difference equality can be computed to be given by

$$\left[I + C\left(sI - A\right)^{-1}F\right]\left[I + F^T\left(-sI - A\right)^{-T}C^T\right] = I + \frac{1}{\mu}G_w\left(s\right)WG_w^T\left(-s\right)$$

from the above equation several nice properties can be derived (see the course reader for details)

- $\sigma_{max}S(j\omega) = \sigma_{max}[I + G_F(j\omega)]^{-1} \leq 1 \Rightarrow$no disturbance amplification at any frequency
- $\sigma_{max}T(j\omega) = \sigma_{max}\{[I + G_F(j\omega)]^{-1}G_F(j\omega)\} \leq 2 \Rightarrow$closed loop stable if the plant uncertainty satisfies $\sigma_{max}\Delta(j\omega) \leq 1/2$

1.1 Step-by-step design and implementation

Plant:

$$\begin{cases} \dot{x}_p(t) &= A_px_p(t) + B_pu_p(t) \\ y_p(t) &= C_px_p(t) \end{cases}$$

Compensator for additional feedback properties (e.g. an integral action):

$$\begin{cases} \dot{x}_c(t) &= A_cx_c(t) + B_cu(t) \\ u_p(t) &= C_cx_c(t) + D_cu(t) \end{cases}$$

Step 1: enlarged overall plant (the "plant" we are considering in the fictitious KF design):

$$\underbrace{\frac{d}{dt}\begin{bmatrix} x_p(t) \\ x_c(t) \end{bmatrix}}_{x_e(t)} = \underbrace{\begin{bmatrix} A_p & B_pC_c \\ 0 & A_c \end{bmatrix}}_{A_e}\begin{bmatrix} x_p(t) \\ x_c(t) \end{bmatrix} + \underbrace{\begin{bmatrix} B_pD_c \\ B_c \end{bmatrix}}_{B_e}u(t) \tag{7}$$

$$y(t) = [C_p\ 0][x_p^T(t)\ x_c^T(t)]^T = \underbrace{[C_p\ 0]}_{C_e}x_e(t) + \underbrace{0}_{D_e}u(t) \tag{8}$$

Step 2: fictitious Kalman filter for the enlarged plant (we now add the fictitious noise terms)

$$\begin{cases} \dot{x}_e(t) = A_ex_e(t) + B_eu\left(t\right) + Lw(t) \\ y(t) = C_ex_e(t) + v(t) \end{cases} \tag{9}$$

$$E\left[w\left(t\right)w\left(t+\tau\right)^T\right] = I\delta(t),\ E\left[v\left(t\right)v\left(t+\tau\right)^T\right] = \mu I\delta(t)$$

Choose L (e.g. $L = B_e$), and μ. Get $F_e = \frac{1}{\mu}M_eC_e^T$ from

$$A_eM_e + M_eA_e^T - \frac{1}{\mu}M_eC_e^TC_eM_e + LL^T = 0 \tag{10}$$

then we have target feedback loop transfer function:

$$C_e\left(sI - A_e\right)^{-1}F_e$$

Here the choice of μ might be iterated to find a good gain cross over frequency for the target feedback loop. Keep in mind that L and μ are design parameters here to reach the target loop transfer function.

Step 3: choose ρ, solve the LQ problem, and get $K_e = \frac{1}{\rho} N^{-1} B_e^T P_e$

$$J = \int_0^\infty \left(x_e^T(t) C_e^T C_e x_e(t) + \rho u^T(t) N u(t) \right) dt, \ e.g.\ N = I \tag{11}$$

$$\dot{x}_e(t) = A_e x_e(t) + B_e u(t)$$

$$A_e^T P_e + P_e A_e - \frac{1}{\rho} P_e B_e N^{-1} B_e^T P_e + C_e^T C_e = 0 \tag{12}$$

Reduce ρ until the actual loop transfer function recovers the target feedback loop at the desired frequency region.

Results:

$$G_{LTR}(s) = K_e (sI - A_e + B_e K_e + F_e C_e)^{-1} F_e, \quad G_p(s) = C_e \left(sI - A_e\right)^{-1} B_e, \quad G_{\text{recover}}(s) = G_p(s) G_{LTR}(s) \tag{13}$$

where $G_{\text{recover}}(s)$ is the recovered loop transfer function that approximates the target loop $C_e \left(sI - A_e\right)^{-1} F_e$.

2 LQR with Frequency Shaped LQ (FSLQ)

We step back from LQG a bit and consider a generalized idea of optimal control.

In the standard continuous-time LQ problem we have the cost function

$$J = \int_0^\infty \left(x^T(t) Q x(t) + \rho u^T(t) R u(t) \right) dt \tag{14}$$

FSLQ is derived based on the following intuitions:

1, Frequency-domain interpretation of the cost function: From the Parseval's Theorem, the time-domain quadratic cost function in (14) is equivalent to the following frequency-domain quadratic cost function

$$J = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(X^T(-j\omega) Q X(j\omega) + \rho U^T(-j\omega) R U(j\omega) \right) d\omega$$

Instead of keeping Q and R constants, now we wish to have more freedom in these quantities. This is not very intutive to design in the time domain but easy to achieve in the frequency domain. Making Q and R frequency-dependent in the form of $Q(j\omega) = Q_f\left(-j\omega\right)^T Q_f\left(j\omega\right)$ and $R(j\omega) = R_f\left(-j\omega\right)^T R_f\left(j\omega\right)$, we have

$$J = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(X^T(-j\omega) Q_f^T(-j\omega) Q_f(j\omega) X(j\omega) + \rho U^T(-j\omega) R_f^T(-j\omega) R_f(j\omega) U(j\omega) \right) d\omega$$

2, Let $x_f = Q_f x,\ i.e.,\ x(t) \rightarrow \boxed{Q_f} \rightarrow x_f(t)$, where Q_f is selected to meet the desired control action and the performance requirements. Pick a state-space realization of this filtering process:

$$\begin{cases} \dot{z}_1(t) = A_1 z_1(t) + B_1 x(t) \\ x_f(t) = C_1 z_1(t) + D_1 x(t) \end{cases}$$

Note that this is an MIMO system. We need to be careful about the matrix dimensions:

step 1: check the order of Q_f, and decide the order of A_1. We need the row/column number of A_1 to be equal to the order of Q_f

step 2: check the order of x, and decide the order of B_1. We need the column number of B_1 to be equal to the number of elements in x

step 3: decide the order of D_1. We should have, in the notation of MATLAB commands $size(D_1) == size(x_f, 1) \times size(x, 1)$

3, Let $u_f = R_f u,\ i.e.,\ u(t) \rightarrow \boxed{R_f} \rightarrow u_f(t)$, where R_f is selected to meet the robustness requirements.

$$\begin{aligned} \dot{z}_2(t) &= A_2 z_2(t) + B_2 u(t) \\ u_f(t) &= C_2 z_2(t) + D_2 u(t) \end{aligned} \tag{15}$$

4, Write

$$J=\int_0^\infty \left(x_f^T(t)x_f(t)+\rho u_f^T(t)u_f(t)\right)dt \tag{16}$$

And figure out the relationship between x_f and the states in the following enlarged system:

$$\frac{d}{dt}\underbrace{\begin{bmatrix} x(t)\\ z_1(t)\\ z_2(t)\end{bmatrix}}_{x_e(t)}=\underbrace{\begin{bmatrix} A & 0 & 0\\ B_1 & A_1 & 0\\ 0 & 0 & A_2\end{bmatrix}}_{A_e}\begin{bmatrix} x(t)\\ z_1(t)\\ z_2(t)\end{bmatrix}+\underbrace{\begin{bmatrix} B\\ 0\\ B_2\end{bmatrix}}_{B_e}u(t)$$

$\Rightarrow$

$$x_f=\underbrace{[D_1\ C_1\ 0]}_{C_e}[x\ z_1\ z_2]^T \quad u_f=[0\ 0\ C_2]x_e+D_2u(t)$$

$\Rightarrow$

$$J=\int_0^\infty \Big(x_e^T(t)\underbrace{\begin{bmatrix} D_1^TD_1 & D_1^TC_1 & 0\\ C_1^TD_1 & C_1^TC_1 & 0\\ 0 & 0 & \rho C_2^TC_2\end{bmatrix}}_{Q_e}x_e(t)+2u^T\underbrace{\begin{bmatrix} 0 & 0 & \rho D_2^TC_2\end{bmatrix}}_{N_e}x_e+u^T(t)\underbrace{\rho D_2^TD_2}_{R_e}u(t)\Big)dt$$

The solution of the above LQ problem is

$$u=-R_e^{-1}(B_e^TP_e+N_e)x_e=-Kx-K_1z_1-K_2z_2 \tag{17}$$

with the Riccati equation

$$A_e^TP_e+P_eA_e-(B_e^TP_e+N_e)^TR_e^{-1}(B_e^TP_e+N_e)+Q_e=0$$

Here P_e can be obtained from the MATLAB command “Pe=care(Ae, Be, Qe, Re, Ne')”