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Probability Review

1 Concepts

$\Omega = \{w_1, \cdots, w_n\}$ — Ω: Sample space; $w_1, \cdots, w_n$: possible outcomes

$S_i \subseteq \Omega$ — S_i: event

Axioms ① $Pr\{S_j\} \geq 0$ ② $Pr\{\Omega\} = 1$ ③ if $S_i \cap S_j = \emptyset$, $Pr\{S_i \cup S_j\} = Pr\{S_i\} + Pr\{S_j\}$

Random variable is a function / mapping $X(w)$. $\Omega \mapsto \mathbb{R}$

① Cumulative density function $F(x) = Pr\{X \leq x\}$

② Probability density function $p(x) = \frac{dF(x)}{dx}$. $Pr(a \leq X \leq b) = \int_a^b P(x)dx$.

③ Mean : $m_X = E(X) = \int_{-\infty}^{\infty} x P_X(x) dx$.

④ Variance : ~~$Var[X] = E[(X - m_X)^2]$~~

$$Var[X] = E[(X - m_X)^2] = \int_{-\infty}^{\infty} (x - m_X)^2 P_X(x) dx.$$

Prove $Var[X] = E[X^2] - (E[X])^2$

$$\begin{aligned} Var[X] &= \int_{-\infty}^{\infty} (x - m_X)^2 P_X(x)\, dx \\ &= \int_{-\infty}^{\infty} x^2 P_X(x)\, dx - 2m_X \int_{-\infty}^{+\infty} x P_X(x)\, dx \\ &\quad + m_X^2 \\ &= E[X^2] - (E[X])^2 \end{aligned}$$

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Multiple random variable

joint : Probability : $Pr(X=x, Y=y)$

Cdf : $F(x,y) = Pr(X \le x, Y \le y)$

pdf : $P(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$

☆ Covariance : $\Sigma_{XY} = E[(X-m_X)(Y-m_Y)]$

uncorrelated : $\Sigma_{XY} = 0$ independent $P(x,y) = P(x)P(y)$

☆ independent $\Rightarrow$ uncorrelated ;

uncorrelated $\not\Rightarrow$ independent

Condition : Probability : $P_X(x|y_1) = P_X(x|Y=y_1) = \frac{P(x,y_1)}{P_Y(y_1)}$

independent : $P_X(x|y_1) = P_X(x)$

Gaussian :

single variable :

$$P_X(x) = \frac{1}{\sigma_X \sqrt{2\pi}} \exp\left(-\frac{(x-m_X)^2}{2\sigma_X^2}\right)$$

vector

$x \in R^n$ $$P_X(x) = \frac{1}{(\sqrt{2\pi})^n \sqrt{\det \Sigma}} \exp\left\{-\frac{1}{2}[x-m_x]^T \Sigma^{-1} [x-m_x]\right\}$$

$$\Sigma = E[(X-m_X)(X-m_X)^T]$$

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Gaussian properties

- The pdf is only determined by mean & variance/covariance matrix.
- Uncorrelation $\Leftrightarrow$ independence.
- Invariant under linear transform (test LTI system)

$\frac{P(x,y)}{P_Y(y)} \Rightarrow$

- If X_1 and X_2 are jointly Gaussian, then $X_1|X_2$ & $X_2|X_1$ are Gaussian.

$$m_{X|Y} = m_X + \Sigma_{XY}\Sigma_{YY}^{-1}(y - m_Y)$$

$$\Sigma_{X|Y} = \Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX}$$

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Random Process

- Auto covariance

$$\Sigma_{XX}(j,k) = E\left[(X(j) - m_X(j))\,(X(k) - m_X(k))^T\right]$$

- Cross covariance

$$\Sigma_{XY}(j,k) = E\left[(X(j) - m_X(j))(Y(k) - m_Y(k))^T\right]$$

- ergodic (for all moments) $E[X(k)] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{j=-N}^{N} X(j) \triangleq \overline{X}(k)$

$$\Sigma_{XX}(j,k) = \overline{[X(j)-m_X][X(k)-m_X]}$$

- Stationary, tells whether a process changes its statistical properties w.r.t. time

  - Strict-sense Stationary: probability distribution does not change w.r.t time.
  - Weak-sense, mean & auto-covariance does depend on time

$\boxed{\text{Ergodic} \Rightarrow \text{Stationary}}$

Stationary: $E[X(k)] = m_X = \text{constant}$

auto-covariance

$$\Sigma_{XX}(k,j) = E[(X(k) - m_X)(X(j) - m_X)] = E[(X(k+q) - m_X)(X(j+q) - m_X)]$$
$$= \Sigma_{XX}(j-k)$$

- White noise: $\Sigma_{XX}(l) = \begin{cases} 0 & l \neq 0 \\ \sigma_{XX}^2 & l = 0 \end{cases}$

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In ME233, Stationary & ergodicity is assumed

Auto covariance $\Sigma_{xy}(l) = \overline{(x(k)-m_x)(y(k+l)-m_y)} = \Sigma_{yx}(-l)$.

Spectral dencity:

$$\Phi_{xx}(w) = \sum_{-\infty}^{+\infty} (\Sigma_{xx}(l)\, e^{-jwl})$$

Why?

$$\Sigma_{yy}(l) = g(l) * \Sigma_{yu}(l)$$

$$\Sigma_{uy}(l) = g(l) * \Sigma_{uu}(l)$$

hard to operate

$$\phi_{yy}(w) = G(e^{-jw})\, G(e^{jw})\, \phi_{uu}(w)$$

$$\Sigma_{uy}(l) = E[(u(k)-m_x)(y(k+l)-m_y)]$$

zero-mean assumed

$$= E[u(k) \cdot \sum_{i=-\infty}^{+\infty} g(i)\, u(k+l-i)]$$

$$= \overline{u(k) \cdot \sum_{-\infty}^{+\infty} g(i)\, u(k+l-i)} = \sum_{i=-\infty}^{+\infty} X_{uu}(l-i)\, g(i)$$

$$= g(l) * X_{uu}(l)$$

In vector sense $\phi_{yy}(e^{jw}) = G(e^{jw}) \cdot \phi_{uu}(e^{jw}) \cdot G^T(e^{-jw})$