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1 Dynamic programming

- Key concept : Solve a (usually large & difficult) problem via solving a collection of sub problems.

- Bellman's principle of optimality

  "From any point on an optimal trajectory, the remaining trajectory is optimal for the corresponding problem initiated at that point".

- In math : Bellman Equation

$$J(s) = \min\left(J(s') + \text{Reward}(s' \rightarrow s)\right).$$

2 Discrete time LQ (Regulation)

- System:

$$x(k+1) = A\,x(k) + B\,u(k)$$
$$y(k) = C\,x(k)$$

- performance index (Cost function):

$$J = \frac{1}{2} x^T(N)\, S\, x(N) + \frac{1}{2} \sum_{k=k_0}^{N-1} x^T(k)\, Q\, x(k) + u^T(k) R\, u(k)$$

$S, R\ Q$ (PD)

$$J^{\circ} = \min_{u(k_0) \cdots u(N-1)} J$$

- With dynamic programming.

$$J_k^o(x(k)) = \min_{u(k)} \left\{ \frac{1}{2} x^T(k) Q x(k) + \frac{1}{2} u^T(k) R(k) u(k) + J_{k+1}^o(x(k+1)) \right\}$$

boundery condition : $J_N^o(x(N)) = \frac{1}{2} x^T(N) S x(N)$

- Solution (How to introduce this ?)

$$u(k) = - \underbrace{\{[R + B^T P(k+1) B]^{-1} B^T P(k+1) A\}}_{K(k)} x(k)$$

$J_{k_0}^o(x(k_0)) = \frac{1}{2} x_0^T P(0) x_0$; $J_k^o(x(k)) = \frac{1}{2} x^T(k) P(k) x(k)$

- Riccati Equation :

$$P(k) = A^T P(k+1) A - A^T P(k+1) B [R + B^T P(k+1) B]^{-1} B^T P(k+1) A + Q$$

$$S = P(N)$$

- Implementation 1. $P(k)$ is usually generated offline
  2. $u(k)$ is computed online.

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For infinite LQ: (First assume $P_{(k)}$ will converge).

Algebraic Riccati equation

$$P_s = A^T P_s A + Q - A^T P_s B[R+B^T P_s B]^{-1} B^T P_s A$$

$$u^o(k) = -\underbrace{[R + B^T P_s B]^{-1} B^T P_s A}_{K_s}\, x(k)$$

Stability proof:

1 If (A,B) are stabilizable and $S=0$, reccati equation converge to a bounded solution $P_s \succ 0$

(a) $S=0$, $J^o(N) = \min\limits_{u} \left\{\frac{1}{2} \sum_{i=1}^{N} (x^T(i)\, Q\, x(i) + u^T(i)\, R\, u(i))\right\}$ is non-decreasing function on horizon N.

(b) (A,B) is stablilizable, there exists a control make $x_c(k) \to 0$ geometrically, $\Rightarrow \sum_{k=0}^{\infty} \|x_c(k)\|^2 < \infty$ $\Rightarrow J^o(\infty)$ is bounded

(c) $J^o(N)$ is monotone and bounded, $\lim\limits_{N\to\infty} J^o(N) = J^o(\infty)$

~~Since~~ $J^o(\infty) < \infty$,

$J^o(\infty) = \frac{1}{2} x^T(0)\, P\, x(0)$. $\Rightarrow$ P is bounded.

$$x(k+1) = (A - BK_s)\, x(k) = A_{cl}\, x(k)$$

Observability $\Rightarrow P_s > 0$ with $u^o(k) = -K_s x(k)$

$$Q = C^T C$$

$$J = x_0^T P_s x_0 = \sum_{k=0}^{\infty} \{ x^T(k) \underbrace{Q}_{Q = C^T C} x(k) + \underbrace{u^T(k) R u(k)}_{u = -K_s x(k)} \}$$

$$= \sum_{k=0}^{\infty} \left\{ x^T(k) \begin{bmatrix} C \\ R^{\frac{1}{2}} K_s \end{bmatrix}^T \begin{bmatrix} C \\ R^{\frac{1}{2}} K_s \end{bmatrix} x(k) \right\}$$

$$= \sum_{k=0}^{\infty} \left\{ x_0^T (A_{cl}^k)^T \begin{bmatrix} C \\ R^{\frac{1}{2}} K_s \end{bmatrix}^T \begin{bmatrix} C \\ R^{\frac{1}{2}} K_s \end{bmatrix} x(k) \right\}$$

$$= x_0^T \underbrace{W_{cl}}_{\text{observability gramian}} x_0 \qquad W_{cl} = \sum_{k=0}^{\infty} (A_{cl}^k)^T \begin{bmatrix} C \\ R^{\frac{1}{2}} K_s \end{bmatrix}^T \begin{bmatrix} C \\ R^{\frac{1}{2}} K_s \end{bmatrix} A_{cl}^k$$

~~Contr.~~ observability gramian for

$$x(k+1) = A_{cl}\, x(k)$$

$$\tilde{y}(k) = \begin{bmatrix} C \\ R^{\frac{1}{2}} K_s \end{bmatrix} x(k)$$

$$x(k+1) = (A - BK_s)\, x(k) = A_{cl}\, x(k) = Ax - Bu.$$

$$\tilde{y}(k) = \begin{bmatrix} C \\ R^{\frac{1}{2}} K_s \end{bmatrix} x(k)$$

$$u(k) = \begin{bmatrix} 0 & -R^{\frac{1}{2}} \end{bmatrix} \tilde{y}(k)$$