ME 233 Advanced Control II

Lecture 20

Stability Analysis of a discrete-time Series-Parallel Least Squares Parameter Identification Algorithm

ARMA Model

Consider the following system

$$A(q^{-1})y(k) = q^{-d}\, B(q^{-1})u(k)$$

Where

- $u(k)$ known ***bounded*** input
- $y(k)$ measured output

ARMA Model

$$A(q^{-1})y(k) = q^{-\mathsf{d}}\,B(q^{-1})u(k)$$

$$A(q^{-1}) = 1 + a_1 q^{-1} + \cdots + a_n q^{-n}$$ **(anti-Schur)**

$$B(q^{-1}) = b_o + b_1 q^{-1} + \cdots + b_m q^{-m}$$

- Orders n and m are ***known***
- a's and b's are ***unknown*** but ***constant*** coefficients

ARMA Model

ARMA model can be written as:

$$y(k+1) = -\sum_{i=1}^{n} a_i\, y(k-i+1) + \sum_{i=0}^{m} b_i\, u(k-i-d+1)$$

$$= \theta^T \phi(k)$$

Where:

$$\theta = \begin{bmatrix} a_1 & \cdots & a_n & b_o \cdots & b_m \end{bmatrix}^T \in \mathcal{R}^{n+m+1}$$

$$\phi(k) = \begin{bmatrix} -y(k) & \cdots & -y(k-n+1) & u(k-d+1) & \cdots & u(k-d-m+1) \end{bmatrix}^T$$

Series-parallel estimation model

A-posteriori series-parallel estimation model

$$\hat{y}(\underline{k+1}) = -\sum_{i=1}^{n} \hat{a}_i(\underline{k+1})\, y(k-i+1) + \sum_{i=0}^{m} \hat{b}_i(\underline{k+1})\, u(k-i-d+1)$$

Where

- $\hat{y}(k)$ a-posteriori estimate of $y(k)$
- $\hat{a}_i(k)$ estimate of a_i at sampling time k
- $\hat{b}_i(k)$ estimate of b_i at sampling time k

Series-parallel estimation model

A-posteriori series-parallel estimation model

$$\hat{y}(k+1) = \hat{\theta}^T(k+1)\,\phi(k)$$

Where

- $\hat{y}(k)$ a-posteriori estimate of $y(k)$

$$\hat{\theta}(k) = \left[\ \hat{a}_1(k) \quad \cdots \quad \hat{a}_n(k) \quad \hat{b}_o(k) \cdots \quad \hat{b}_m(k)\ \right]^T$$

$$\phi(k) = \left[-y(k) \quad \cdots \quad -y(k-n+1) \quad u(k-d+1) \quad \cdots \quad u(k-d-m+1)\right]^T$$

Series-parallel estimation model

A-priori series-parallel estimation model

$$\widehat{y}^{o}(k+1) = \widehat{\theta}^{T}(k)\,\phi(k)$$

Where

- $\widehat{y}^{o}(k)$ a-priori estimate of $y(k)$

$$\widehat{\theta}(k) = \left[\ \hat{a}_1(k) \ \cdots \ \hat{a}_n(k) \ \ \hat{b}_o(k) \cdots \ \hat{b}_m(k)\ \right]^T$$

$$\phi(k) = \left[-y(k) \ \cdots \ -y(k-n+1) \ \ u(k-d+1) \ \cdots \ u(k-d-m+1)\right]^T$$

Additional Notation

- Unknown parameter vector:

$$\theta = \left[\begin{array}{ccccc} a_1 & \cdots & a_n & b_o \cdots & b_m \end{array}\right]^T \in \mathcal{R}^{n+m+1}$$

- Parameter vector estimate:

$$\hat{\theta}(k) = \left[\begin{array}{ccccc} \hat{a}_1(k) & \cdots & \hat{a}_n(k) & \hat{b}_o(k) \cdots & \hat{b}_m(k) \end{array}\right]^T$$

- **Parameter error estimate**:

$$\boxed{\tilde{\theta}(k) = \theta - \hat{\theta}(k)}$$

- Regressor vector:

$$\phi(k) = \left[\begin{array}{cccccc} -y(k) & \cdots & -y(k-n+1) & u(k-d+1) & \cdots & u(k-d-m+1) \end{array}\right]^T$$

Additional Notation

- ***A-posteriori*** output estimation error:

$$
\begin{aligned}
e(k) &= y(k) - \widehat{y}(k) \\
&= \widetilde{\theta}^T(k)\phi(k-1)
\end{aligned}
$$

- ***A-priori*** output estimation error:

$$
\begin{aligned}
e^o(\underline{k}) &= y(k) - \widehat{y}^o(k) \\
&= \widetilde{\theta}^T(\underline{k-1})\phi(k-1)
\end{aligned}
$$

Parameter Adaptation Algorithm (PAA)

A-posteriori version

- **Parameter estimate update**

$$\widehat{\theta}(k+1) = \widehat{\theta}(k) + \frac{1}{\lambda_1(k)} F(k)\phi(k)e(k+1)$$

- **Gain update**

$$F^{-1}(k+1) = \lambda_1(k)\, F^{-1}(k) + \lambda_2(k)\, \phi(k)\phi^T(k)$$

- **We make the restriction**

$$0 \;<\; \lambda_1(k) \leq 1 \qquad\qquad 0 \leq \lambda_2(k) < 2$$

PAA Special Cases

$$\widehat{\theta}(k+1) = \widehat{\theta}(k) + \frac{1}{\lambda_1(k)} F(k)\phi(k)e(k+1)$$

$$F^{-1}(k+1) = \lambda_1(k)\, F^{-1}(k) + \lambda_2(k)\, \phi(k)\phi^T(k)$$

- **Least squares**

$$\lambda_1(k) = 1 \qquad \lambda_2(k) = 1$$

- **Least squares with forgetting factor**

$$0 < \lambda_1(k) < 1 \qquad \lambda_2(k) = 1$$

- **Constant gain**

$$\lambda_1(k) = 1 \qquad \lambda_2(k) = 0$$

Example

- Plant:

$$y(k) = \frac{q^{-1}\,0.1(1+0.5q^{-1})}{(1+0.9q^{-1})(1+0.8q^{-1})}\,u(k)$$

$$y(k+1) \;=\; \theta^T \phi(k)$$

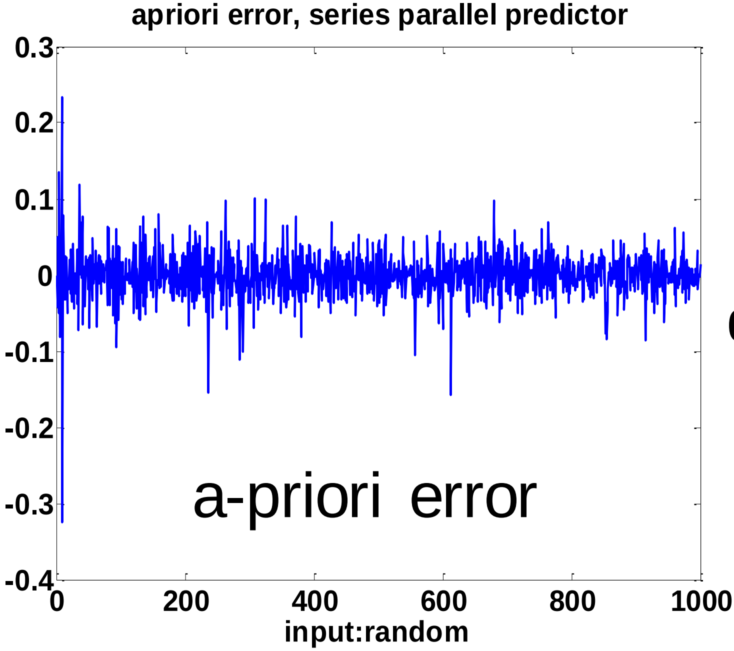
$$\theta = \begin{bmatrix} 1.7 \\ 0.72 \\ 0.1 \\ 0.05 \end{bmatrix} \qquad \phi(k) = \begin{bmatrix} -y(k) \\ -y(k-1) \\ u(k) \\ u(k-1) \end{bmatrix}$$

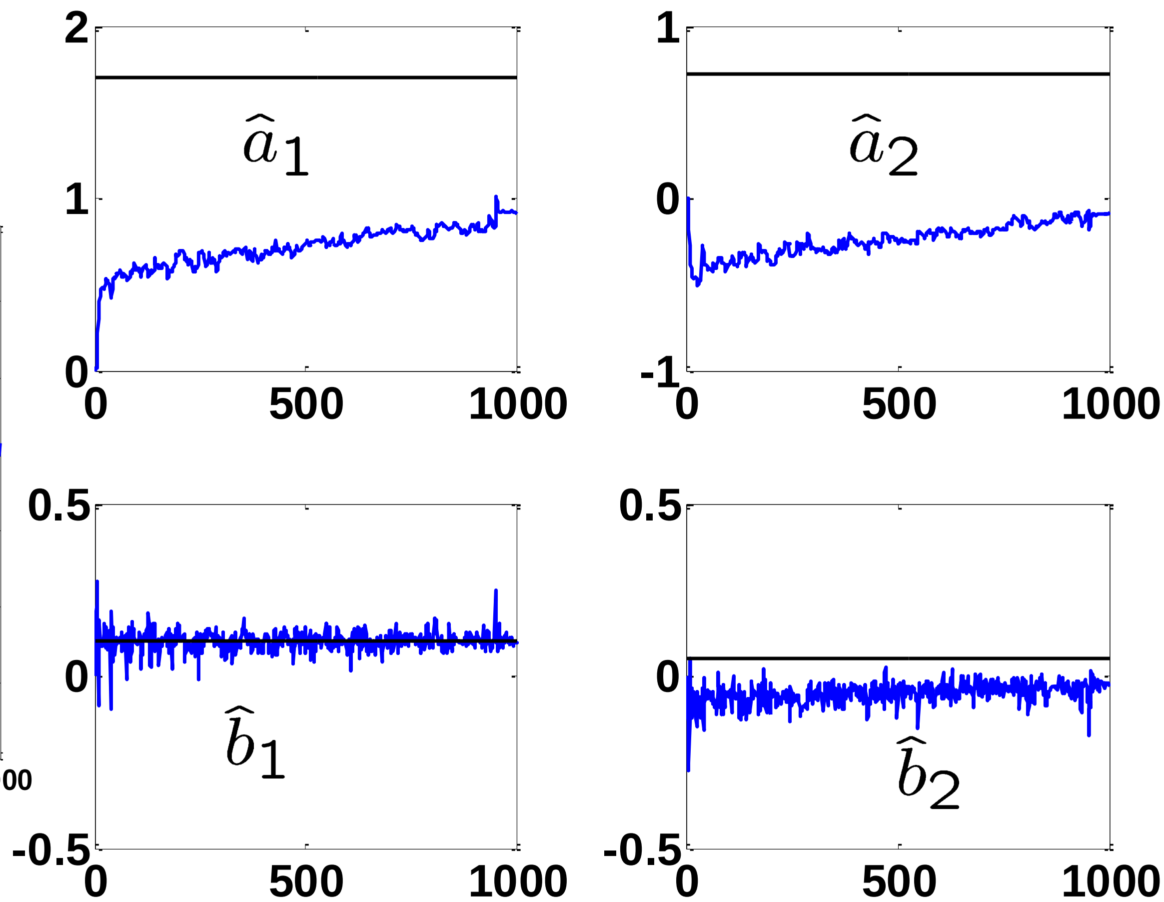
Example: Constant gain

$u(k)$: zero mean uniform white noise between [-1,1]

$$F = 100 * I_4$$

parameter convergence



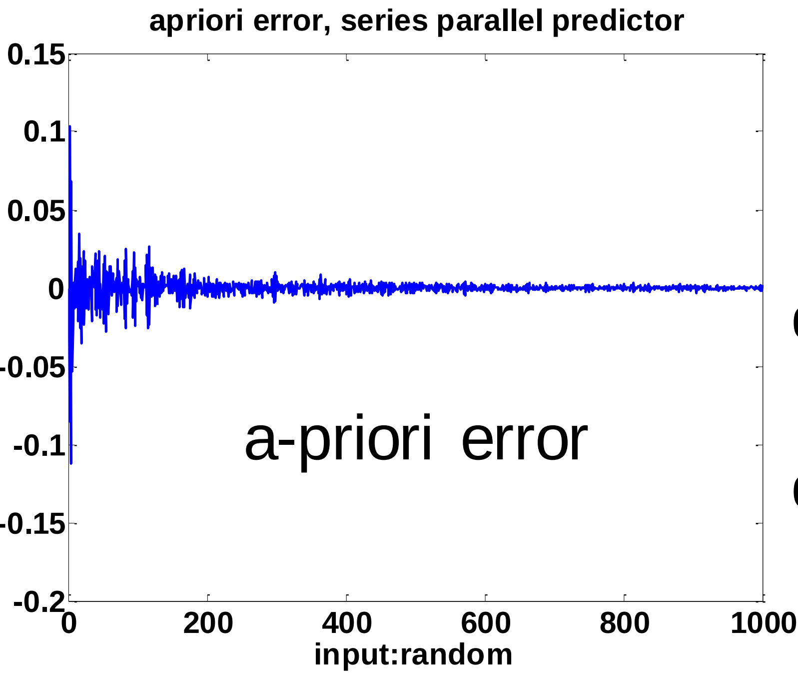


Example: Least Squares

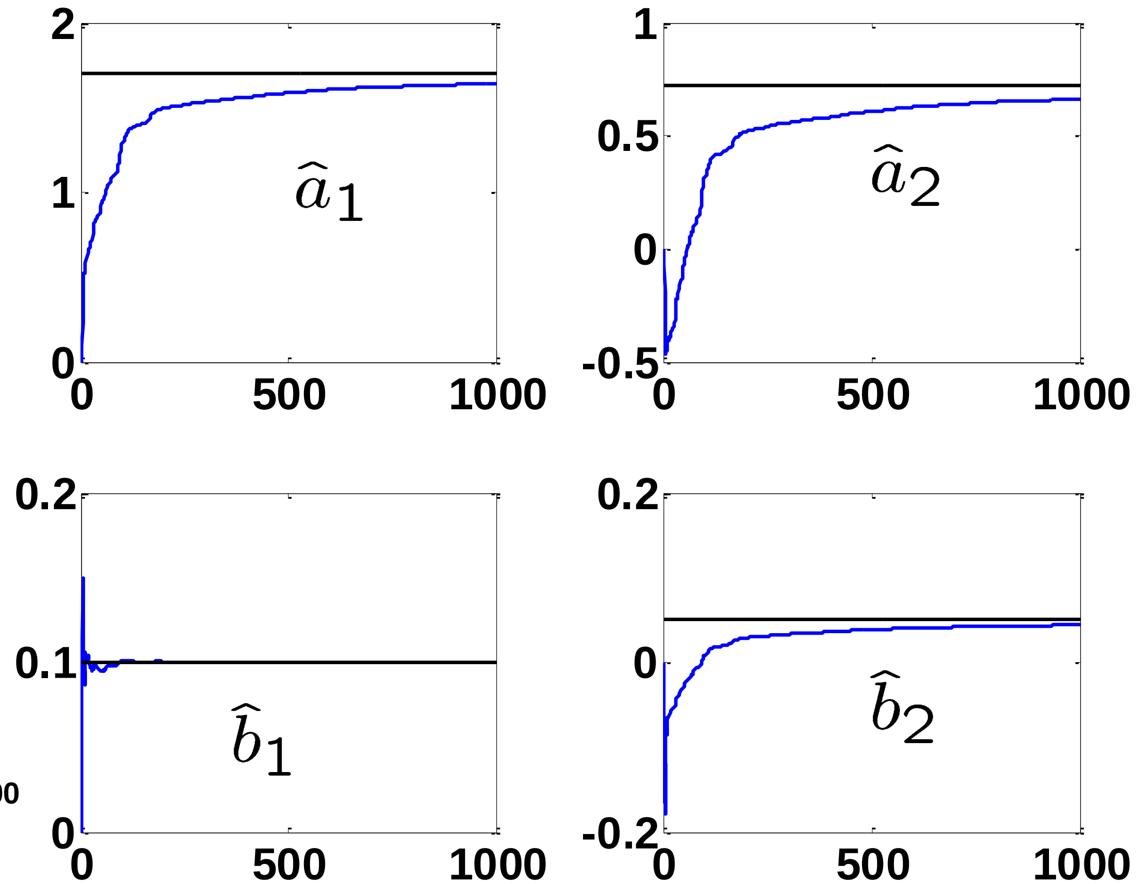
u(k) : zero mean uniform white noise between [-1,1]

$$F(0) = 100 * I_4$$

$$\lambda_1 = 1 \quad \lambda_2 = 1$$



parameter convergence

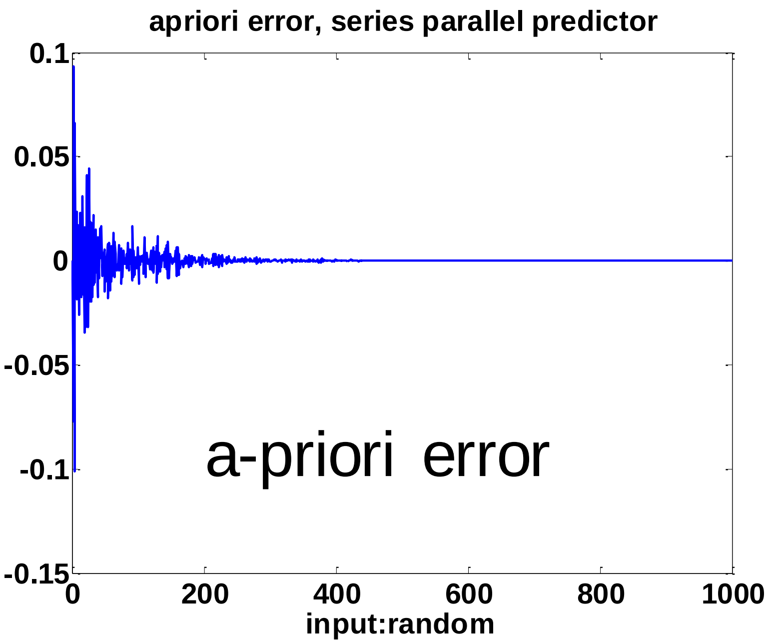


Example: Least Squares & forgetting factor

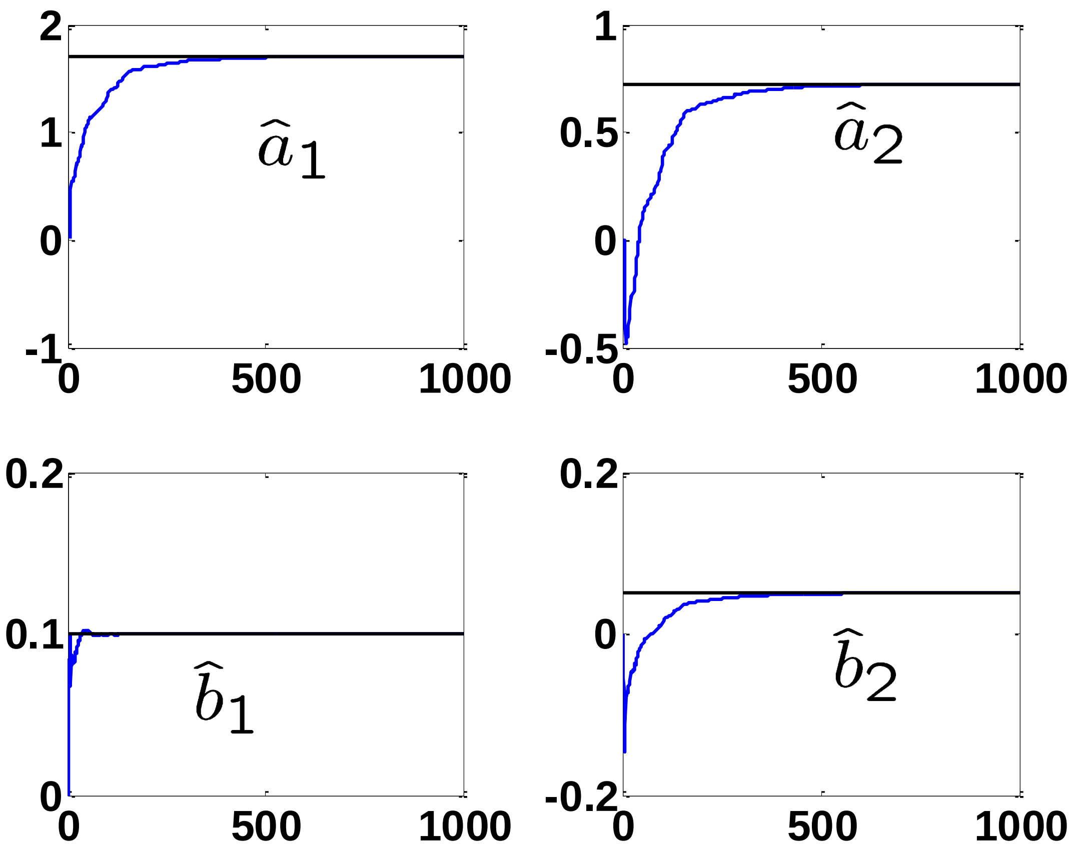
$u(k)$: zero mean uniform white noise between [-1,1]

$$F(0) = 100 * I_4$$

$$\lambda_1 = 0.99 \quad \lambda_2 = 1$$



parameter convergence



Theorem

Under the following conditions:

1. The input $u(k)$ is bounded, i.e. $|u(k)| < \infty$
2. $A(q^{-1})$ is anti-Schur
3. Maximum singular value of $F(k)$ is uniformly bounded

$$\lambda_{\max}\{F(k)\} < K_{max} < \infty\,.$$

$$\boxed{\lim_{k\to\infty} e(k) = 0 \quad \text{and} \quad \lim_{k\to\infty} e^{o}(k) = 0}$$

Parameter Adaptation Algorithm (PAA)

Since the unknown parameters are constant and

$$\tilde{\theta}(k) = \theta - \hat{\theta}(k)$$

the PAA

$$\hat{\theta}(k+1) = \hat{\theta}(k) + \frac{1}{\lambda_1(k)} F(k)\phi(k)e(k+1)$$

implies that

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) - \frac{1}{\lambda_1(k)} F(k)\phi(k)e(k+1)$$

A-posteriori dynamics

- Error dynamics

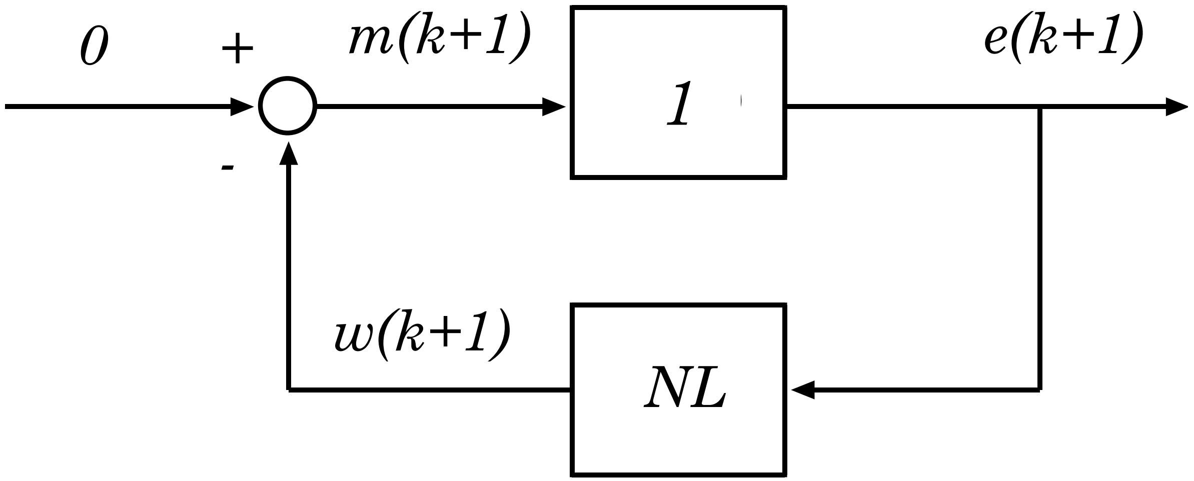
$$e(k+1) = \underbrace{\tilde{\theta}^T(k+1)\phi(k)}_{m(k+1) = -w(k+1)}$$

- PAA

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) - \frac{1}{\lambda_1(k)} F(k)\phi(k)e(k+1)$$

$$F^{-1}(k+1) = \lambda_1(k)\, F^{-1}(k) + \lambda_2(k)\, \phi(k)\phi^T(k)$$

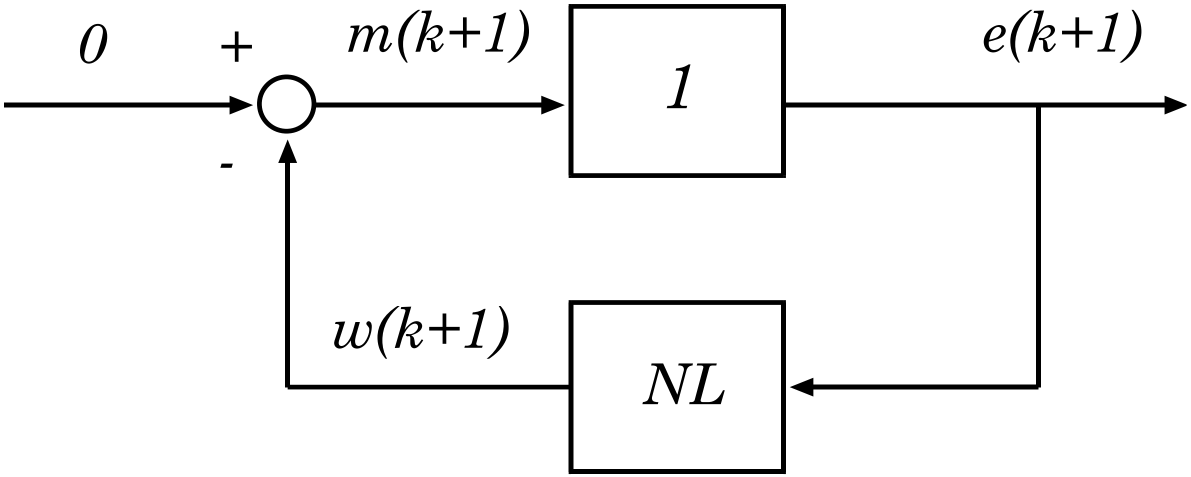
Equivalent Feedback Loop



$$m(k+1) \;=\; \tilde{\theta}^T(k+1)\phi(k) = e(k+1)$$

$$w(k+1) \;=\; -m(k+1)$$

Equivalent Feedback Loop



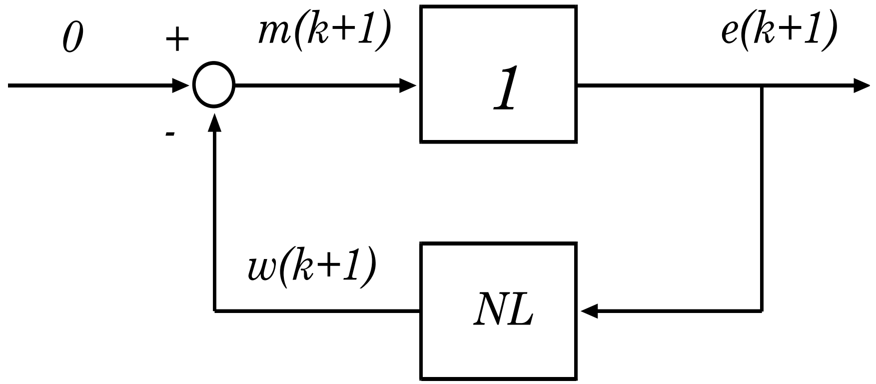
NL:

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) - \frac{1}{\lambda_1(k)} F(k)\phi(k)e(k+1)$$

$$w(k+1) \;=\; -\phi(k)^T \tilde{\theta}(k+1)$$

$$F^{-1}(k+1) = \lambda_1(k)\, F^{-1}(k) + \lambda_2(k)\, \phi(k)\phi^T(k)$$

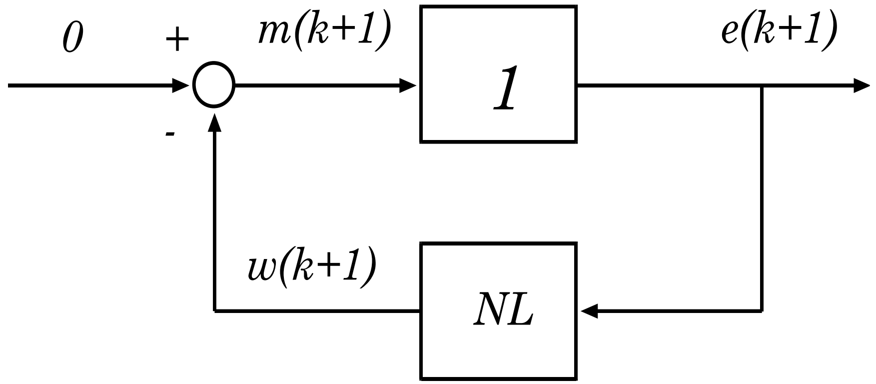
Stability analysis using Hyperstability



1. Verify that the LTI dynamics are SPR

2 Verify that the PAA dynamics are P-class

Good News: LTI ***"very"*** SPR



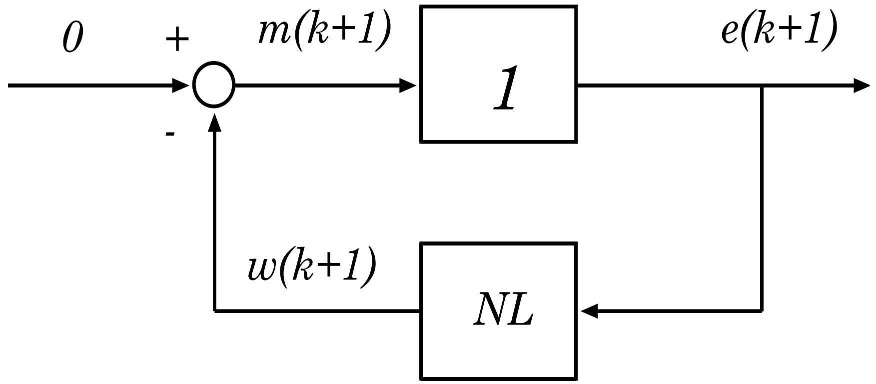
1. Verify that the LTI dynamics are SPR

$$e(k+1) \;=\; m(k+1)$$

$$\boxed{G(z) \;=\; 1}$$

Always SPR

Bad News: NL is ***not*** P-class



- Unfortunately the NL block is **not** P-class

NL:

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) - \frac{1}{\lambda_1(k)} F(k)\phi(k)e(k+1)$$

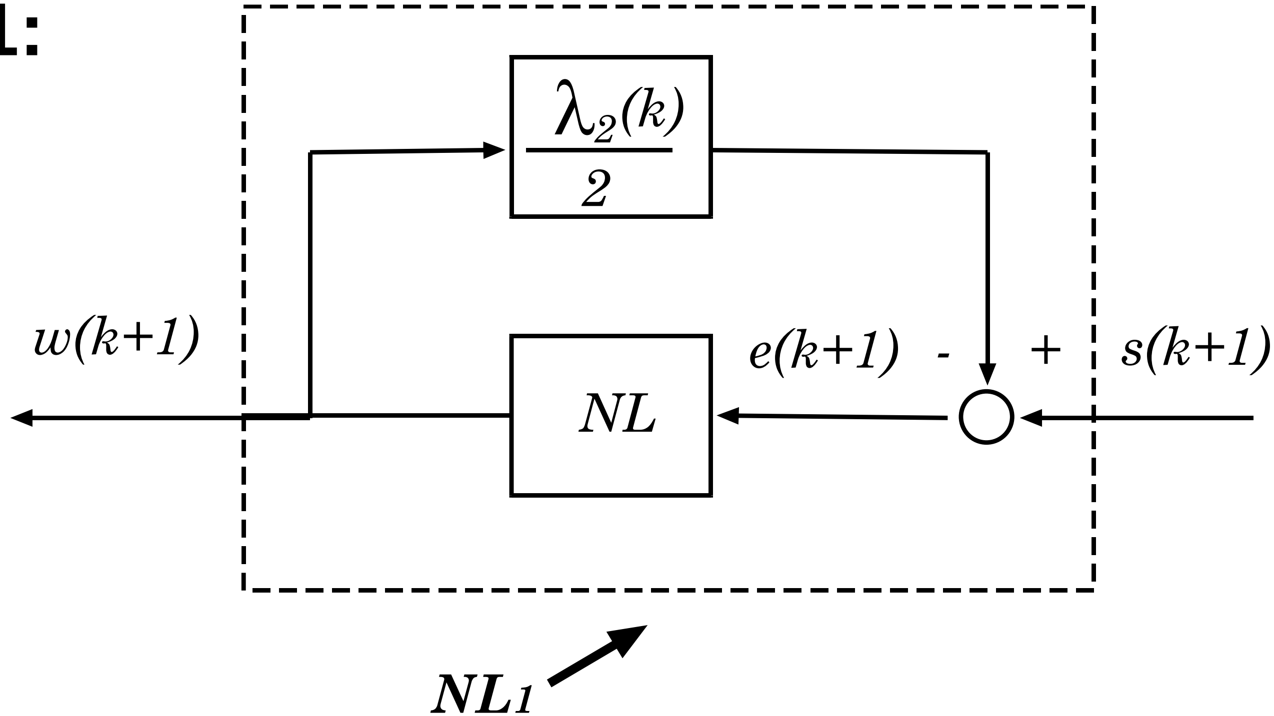
$$w(k+1) \;=\; -\phi(k)^T \tilde{\theta}(k+1)$$

$$F^{-1}(k+1) = \lambda_1(k)\, F^{-1}(k) + \lambda_2(k)\, \phi(k)\phi^T(k)$$

Solution: Modify the NL block

- **Add a feedback term to NL to make it P-class**

NL1:

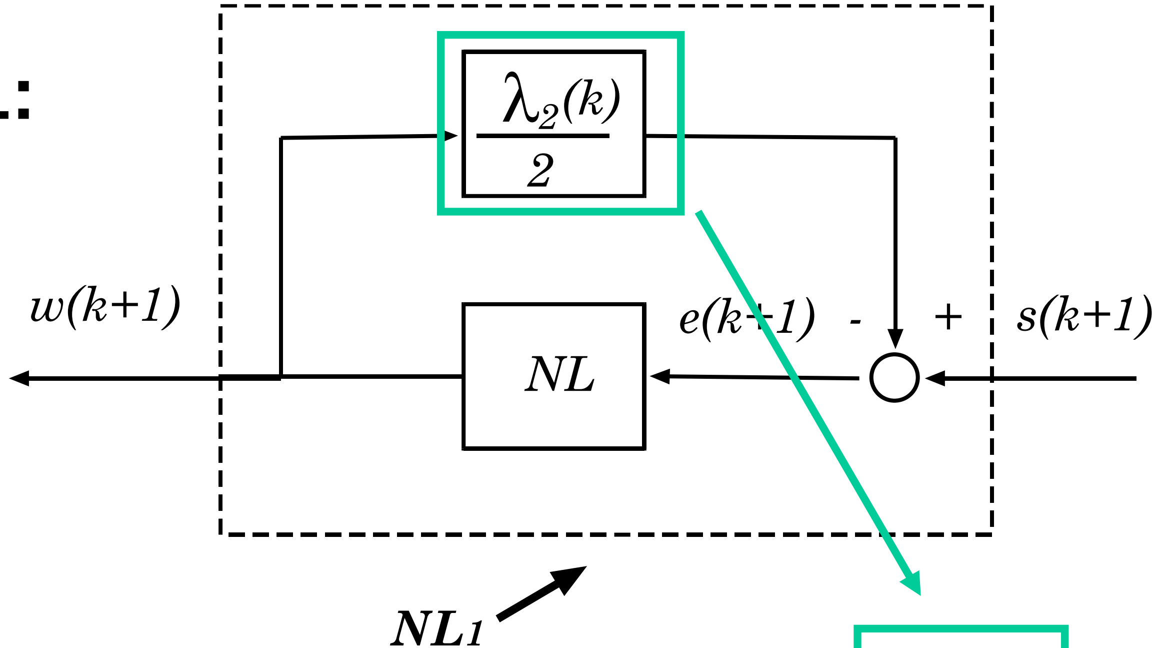


$$\sum_{j=0}^{k} w(j)s(j) \geq -\gamma_o^2$$

Modifying the NL block

- **Add a feedback term to NL to make it P-class**

NL1:

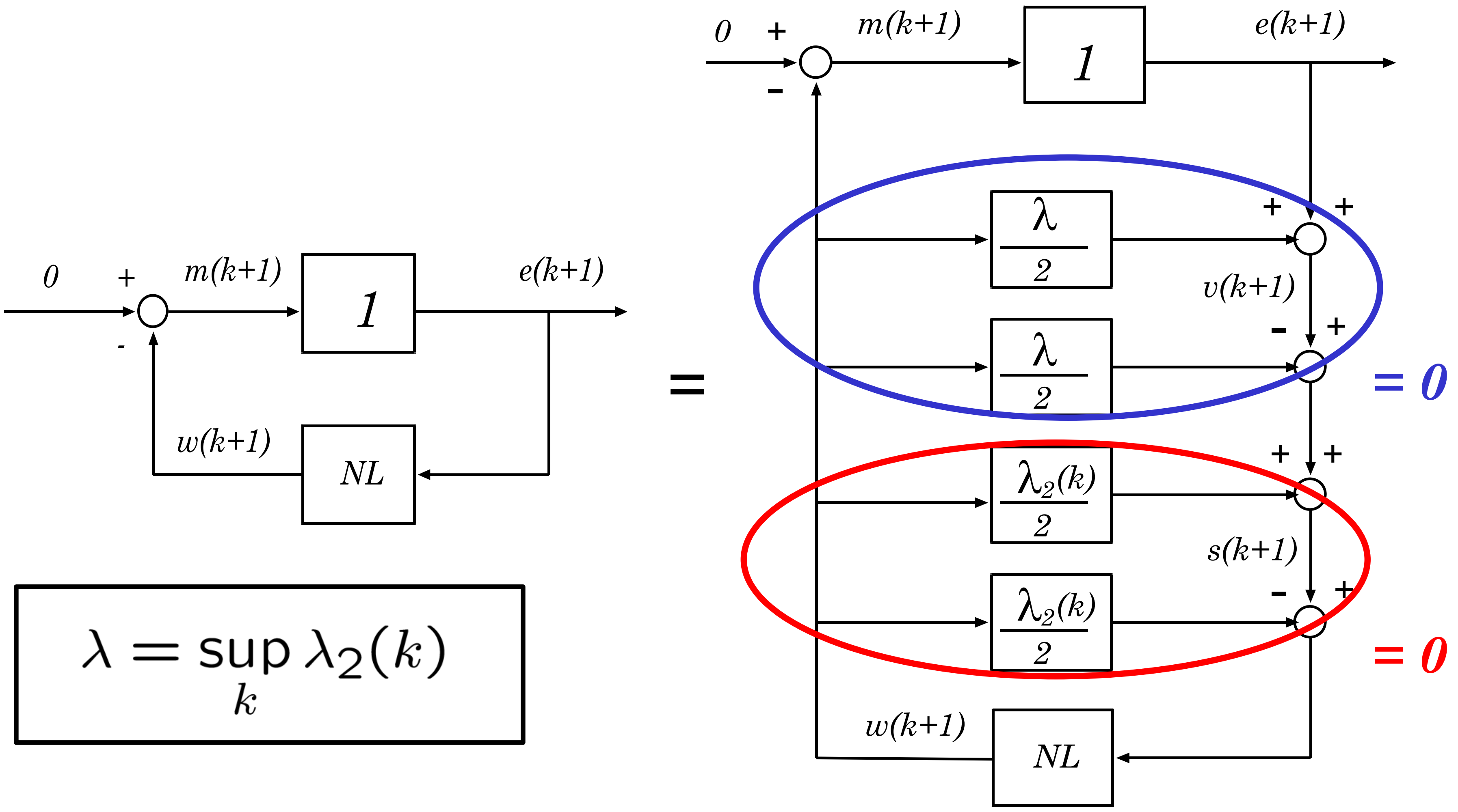


$$F^{-1}(k+1) = \lambda_1(k)\, F^{-1}(k) + \boxed{\lambda_2(k)}\, \phi(k)\phi^T(k)$$

Proof: See Additional Material at end of this lecture (the class notes on bSpace are incorrect)

What happens to the feedback structure?

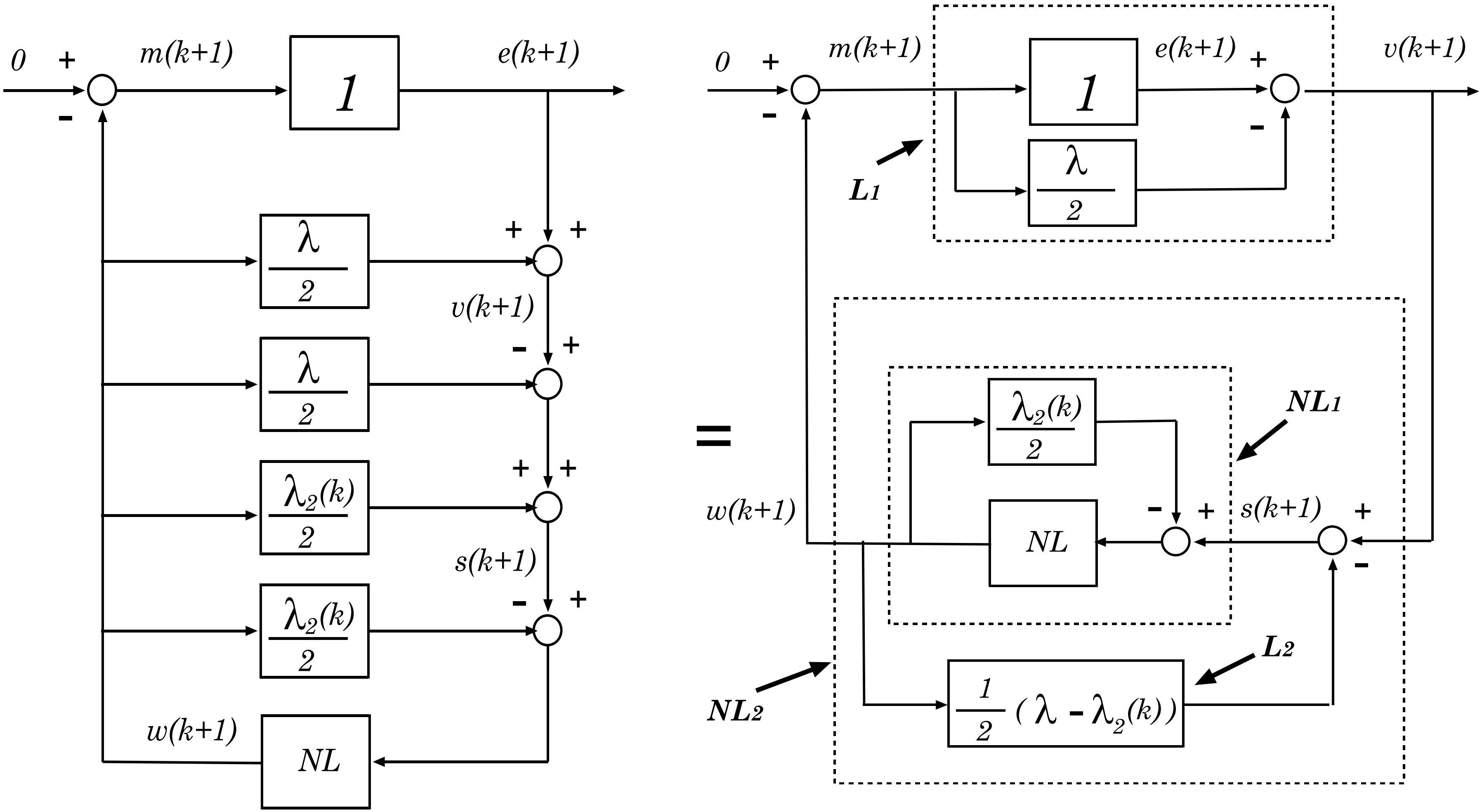
- **Add and subtract the same blocks:**



$$\boxed{\lambda = \sup_{k} \lambda_2(k)}$$

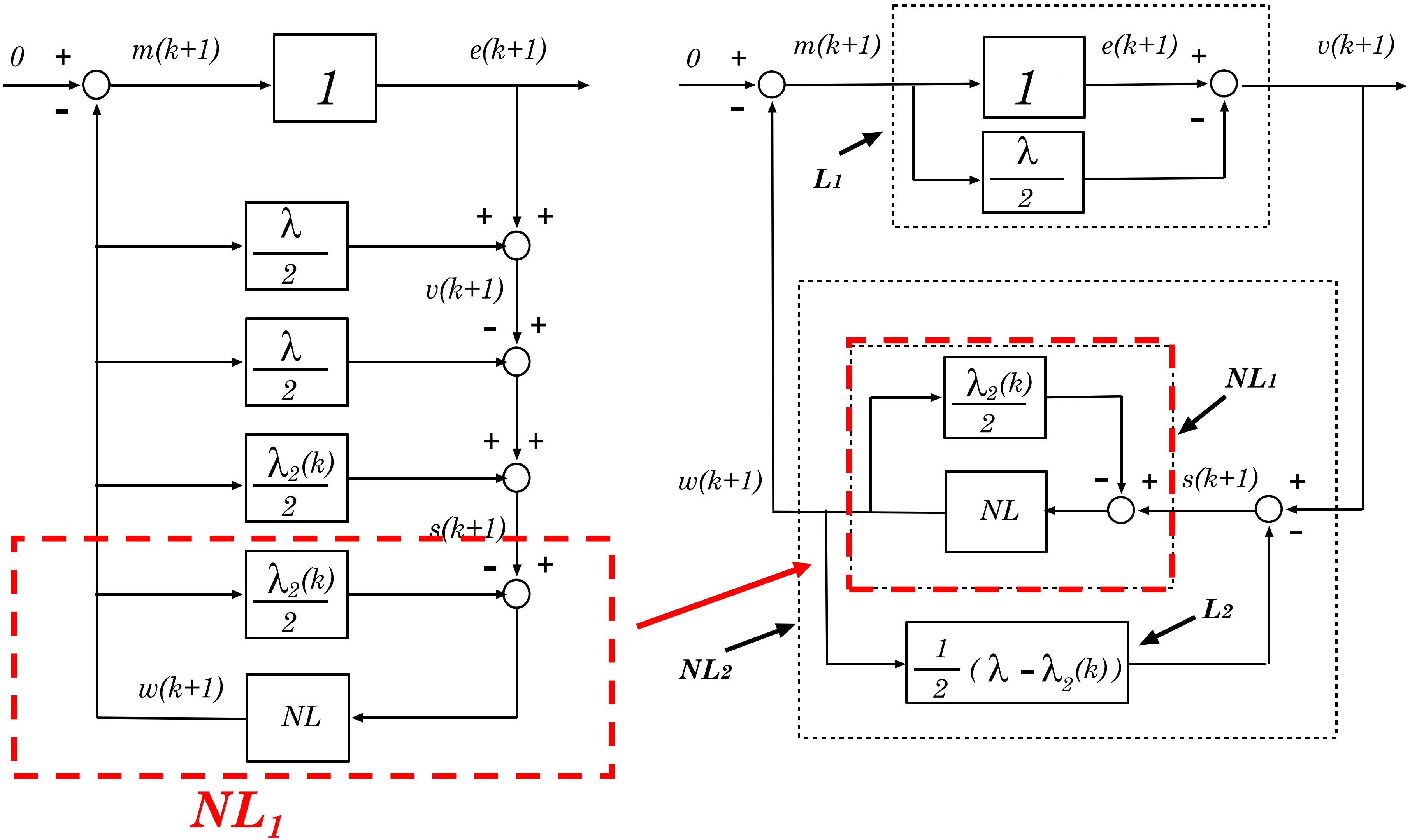
What happens to the feedback structure?

- **Rearranging blocks,**



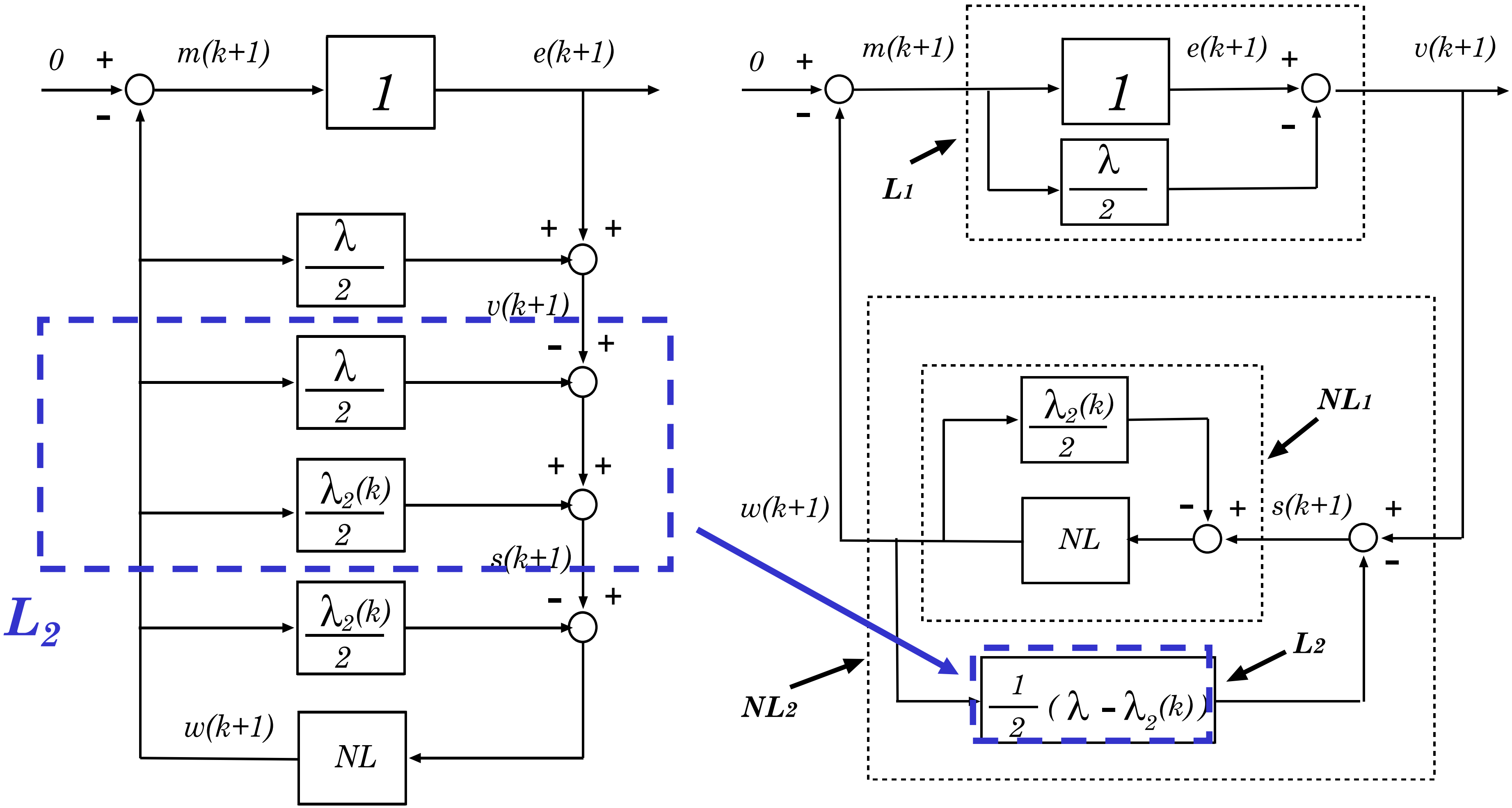
What happens to the feedback structure?

- **Rearranging blocks,**



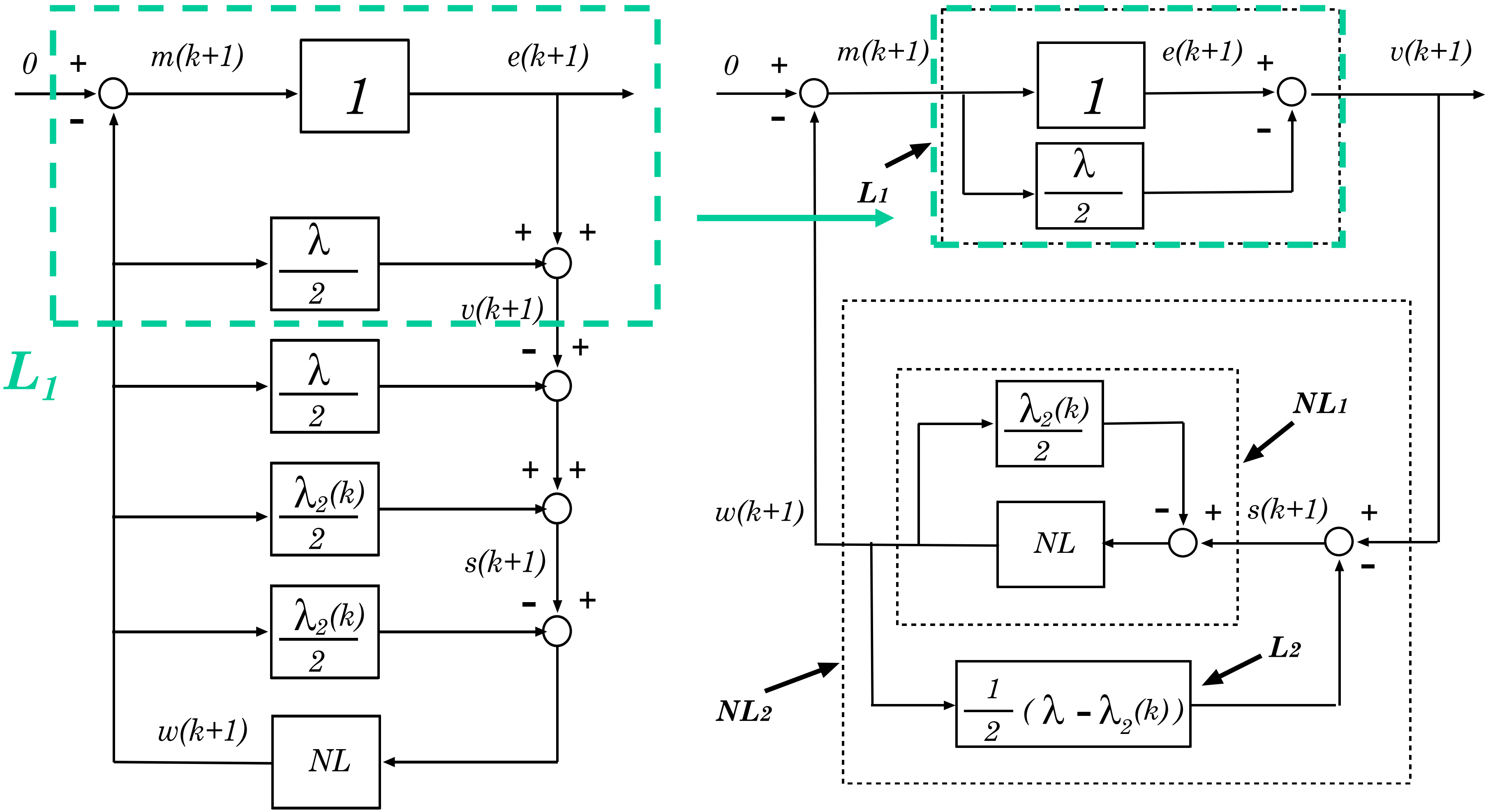
What happens to the feedback structure?

- **Rearranging blocks,**

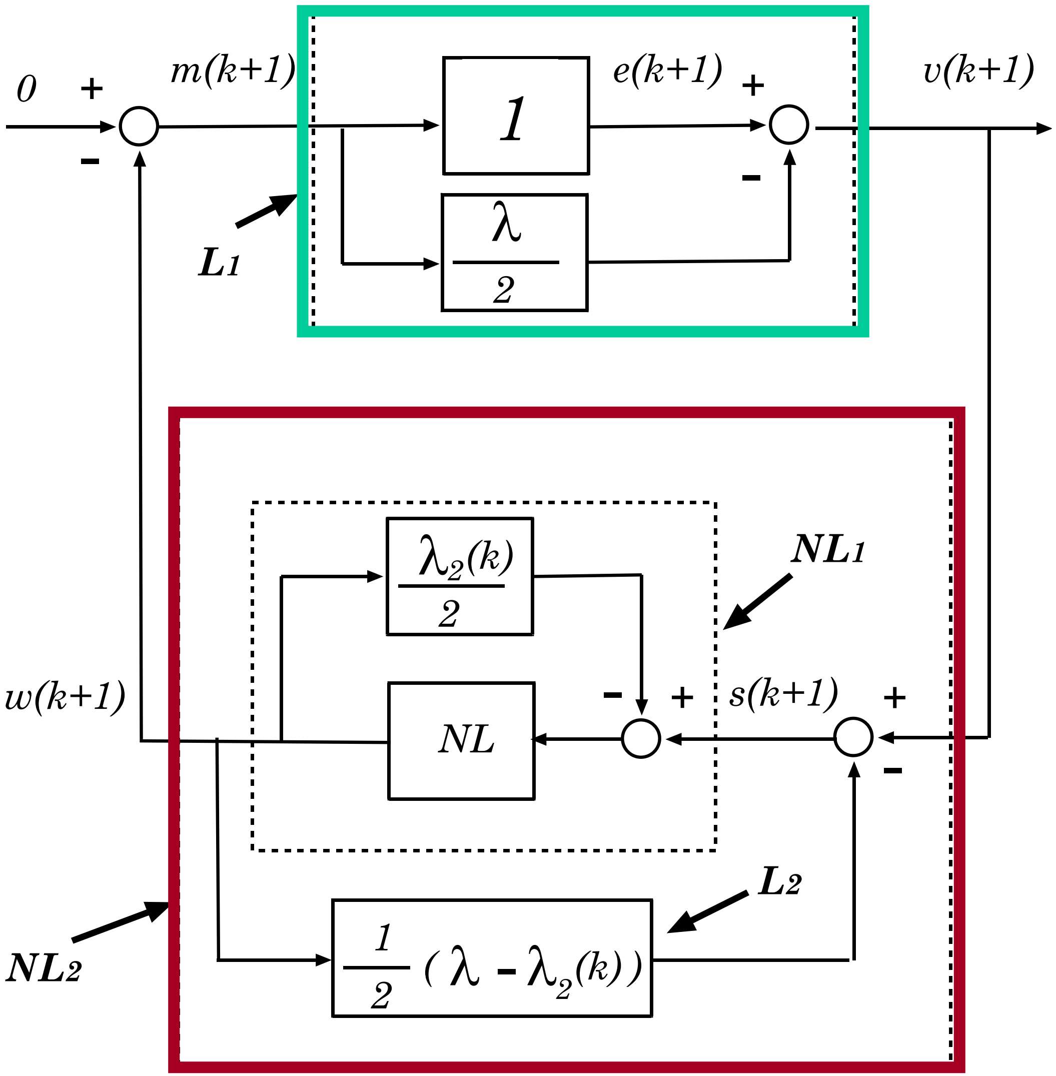


What happens to the feedback structure?

- **Rearranging blocks,**



Can we now use Hyperstability Theory?

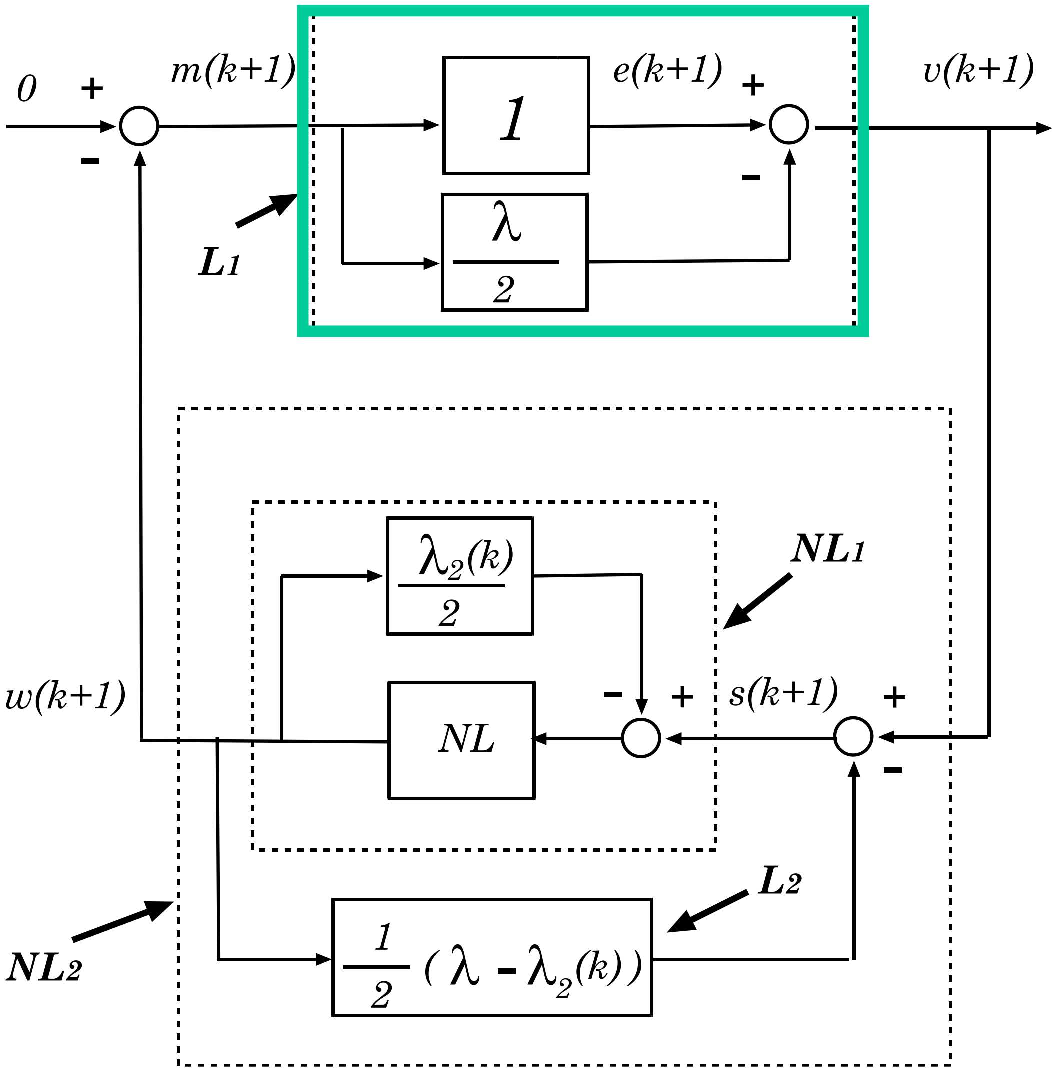


For Asymptotic Hyperstability:

1. L_1 must be SPR

2. NL_2 must be P-class

Linear Block L_1



Since:

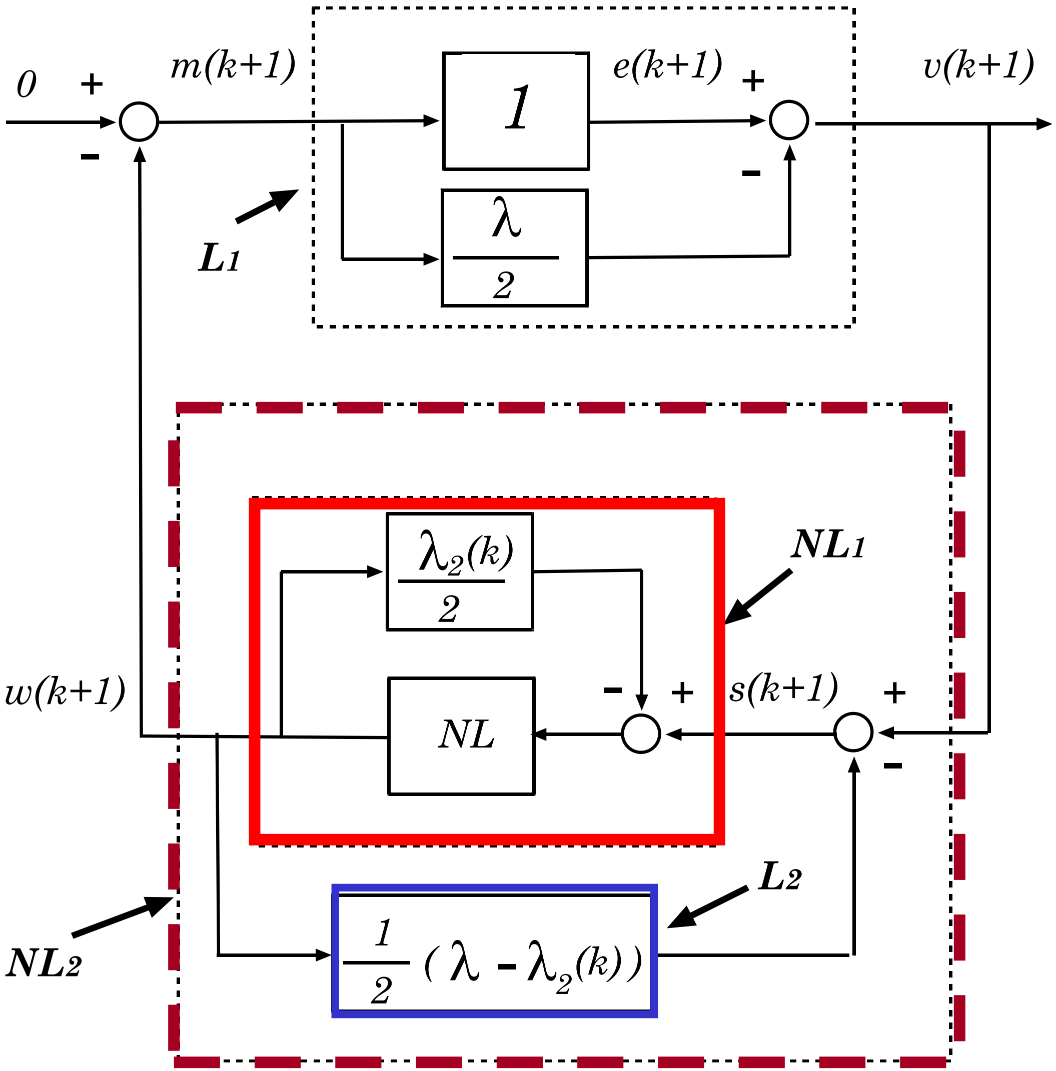
$$L_1: \quad 1 - \frac{\lambda}{2}$$

$$\lambda = \sup_k \lambda_2(k)$$

L_1 is SPR **iff**

$$\sup_k \lambda_2(k) < 2$$

Nonlinear Block NL_2



$$\lambda = \sup_k \lambda_2(k)$$

NL_2 :

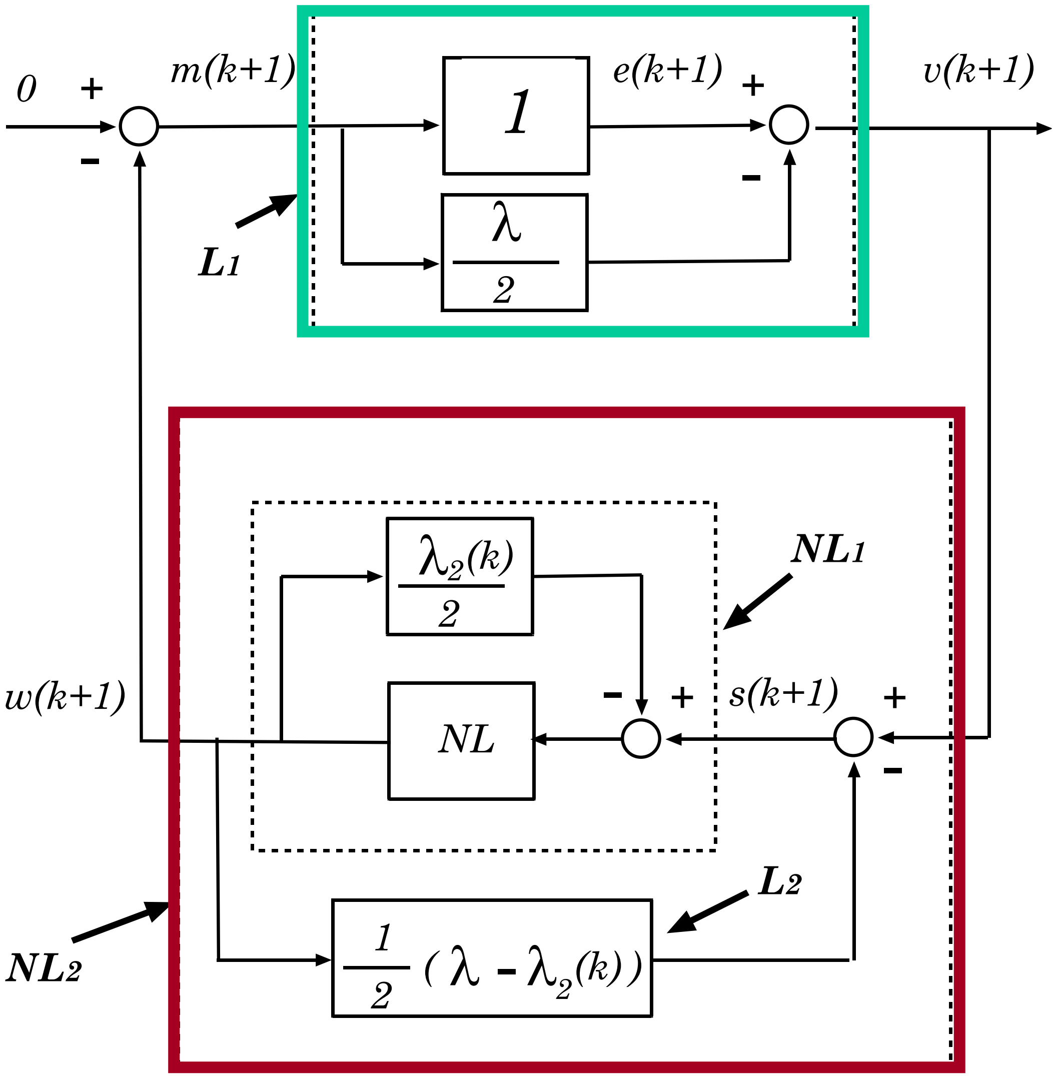
Feedback combination of two blocks:

1. ***NL_1*** **: P-class**

2. ***L_2*** **: P-class**

Therefore NL_2 is P-class

Hyperstability Theorem



If

$$\epsilon < \lambda_1(k) \leq 1$$

$$0 \leq \lambda_2(k) < 2 - \epsilon$$

for some $\epsilon > 0$

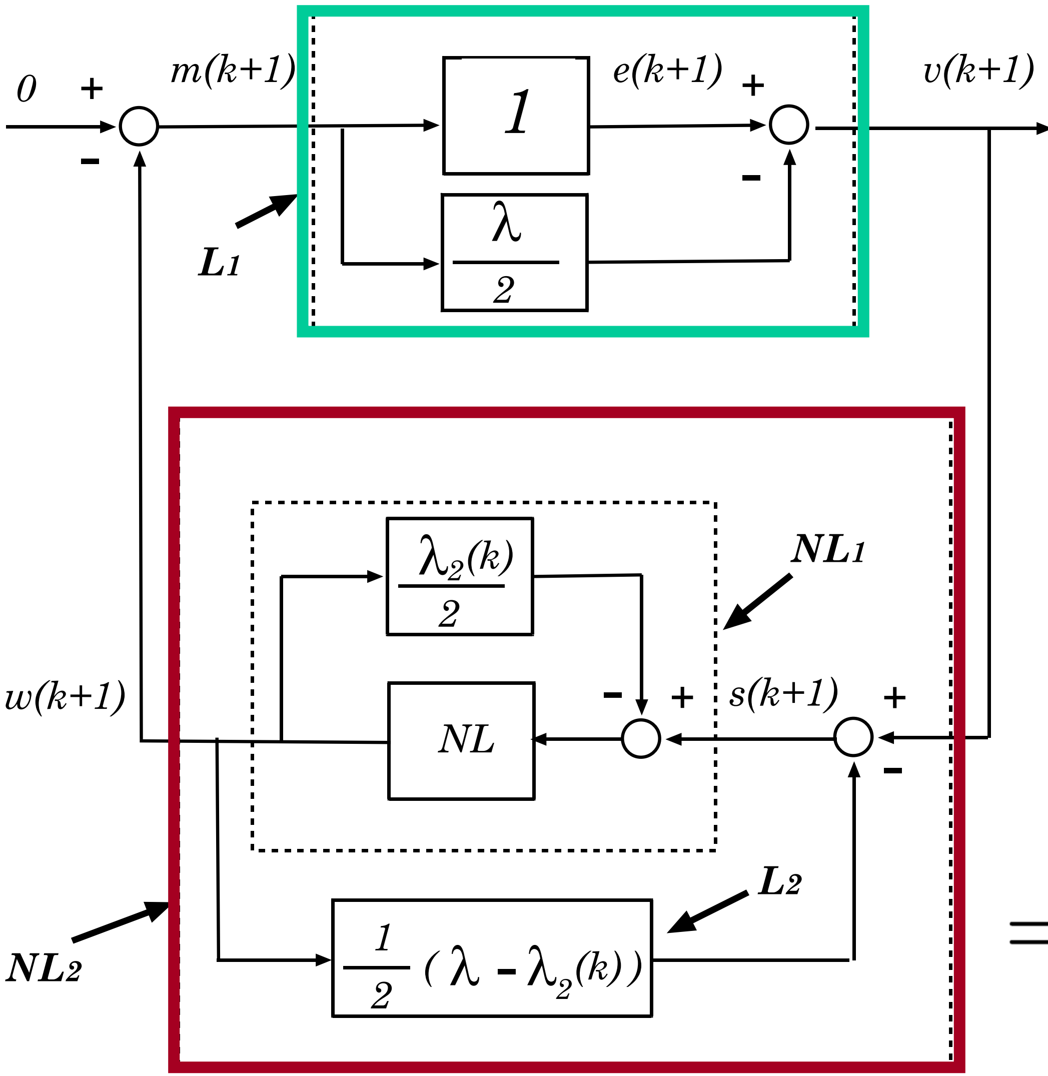
Then

1. L_1 **is SPR**
2. NL_2 **is P-class**

Therefore:

The interconnection is asymptotically hyperstable

Hyperstability Theorem



If

$$\epsilon < \lambda_1(k) \le 1$$

$$0 \le \lambda_2(k) < 2 - \epsilon$$

for some $\epsilon > 0$

Then

$$v(k) \longrightarrow 0$$

$$m(k) \longrightarrow 0$$

$$\Rightarrow \quad \begin{array}{l} e(k) \longrightarrow 0 \\ \hat{y}(k) \longrightarrow y(k) \end{array}$$

A-posteriori error convergence

We have concluded that the a-posteriori output error converges to zero:

$$\lim_{k\to\infty} e(k) = 0$$

where

$$\begin{aligned} e(k) &= y(k) - \widehat{y}(k) \\ &= \tilde{\theta}(k)^T \phi(k) \end{aligned}$$

What about the a-priori output error?

A-posteriori error convergence

- Notice that

$$e(k+1) = \frac{\lambda_1(k)}{\lambda_1(k) + \phi^T(k)F(k)\phi(k)} e^o(k+1)$$

- Therefore, $e(k) \longrightarrow 0$ does not necessarily imply that $e^o(k) \longrightarrow 0$

- To prove $e^o(k) \longrightarrow 0$, we need to first show

$$\|\phi(k)\| < \infty \qquad\qquad \forall\, k \geq 0$$

Bondedness of the regressor vector

Remember that:

$$\phi(k) = \begin{bmatrix} -y(k) & \cdots & -y(k-n+1) & u(k-d+1) & \cdots & u(k-d-m+1) \end{bmatrix}^T$$

Therefore,

$$\|\phi(k)\|^2 = \sum_{i=1}^{n} y^2(k-i+1) + \sum_{j=0}^{m} u^2(k-j-d+1)$$

By assumption,

$$|u(k)| < \infty \qquad \forall\, k \geq 0$$

Bondedness of the regressor vector

Since

$$\|\phi(k)\|^2 = \sum_{i=1}^{n} y^2(k-i+1) + \sum_{j=0}^{m} u^2(k-j-d+1)$$

and,

$$|u(k)| < \infty \qquad \forall\, k \geq 0$$

we only need to show that

$$|y(k)| < \infty \qquad \forall\, k \geq 0$$

Bondedness of the regressor vector

Remember that:

$$y(k) = \frac{q^{-\mathsf{d}} B(q^{-1})}{A(q^{-1})} u(k)$$

and $A(q^{-1})$ is anti-Schur.

Therefore LTI system $G(q^{-1}) = \frac{q^{-\mathsf{d}} B(q^{-1})}{A(q^{-1})}$ is BIBO

Thus,

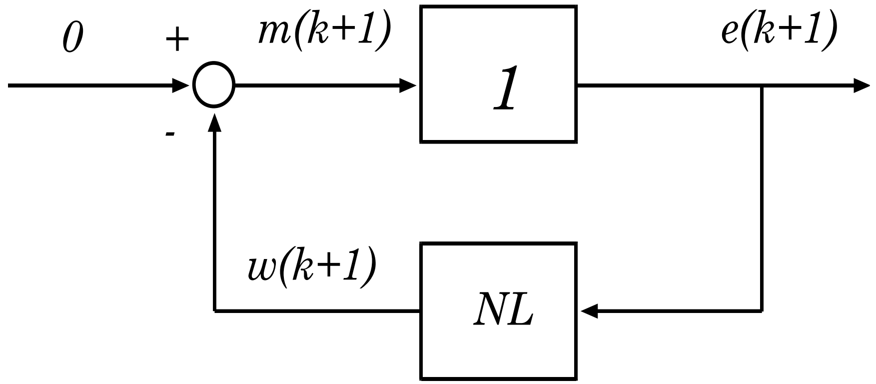
$$|u(k)| < \infty \Rightarrow |y(k)| < \infty$$

■

Additional Material
(you are not responsible for this)

- Proof that NL_1 is P-class

Equivalent feedback loop (review)



- Recall that the *NL* block is **not** P-class

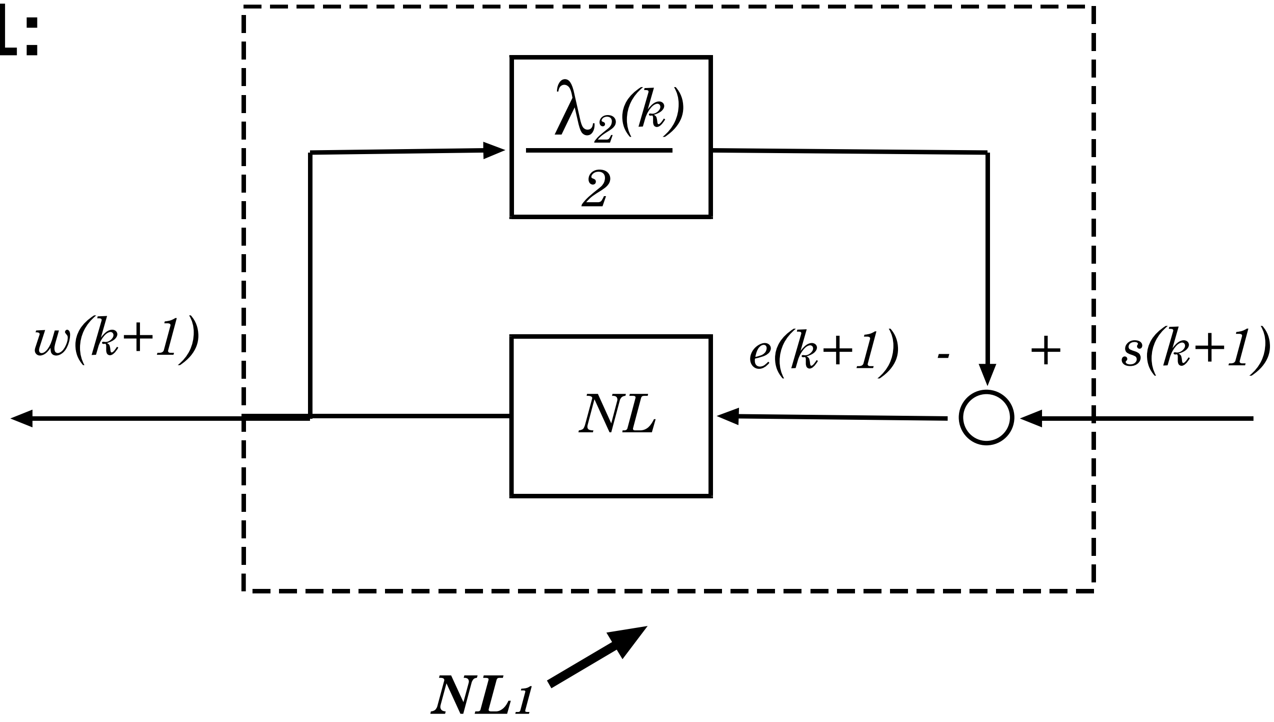
NL:

$$\begin{cases} \tilde{\theta}(k+1) = \tilde{\theta}(k) - \dfrac{1}{\lambda_1(k)} F(k)\phi(k)e(k+1) \\ w(k+1) \;=\; -\phi(k)^T\tilde{\theta}(k+1) \\ F^{-1}(k+1) = \lambda_1(k)\, F^{-1}(k) + \lambda_2(k)\, \phi(k)\phi^T(k) \end{cases}$$

Solution: Modify the *NL* block (review)

- **Add a feedback term to *NL* to make it P-class**

NL1:



We want to show: $$\sum_{j=0}^{k} w(j)s(j) \geq -\gamma_o^2$$

Simplified Notation

$$\widehat{\theta}_k = \widehat{\theta}(k) \qquad \tilde{\theta}_k = \tilde{\theta}(k)$$

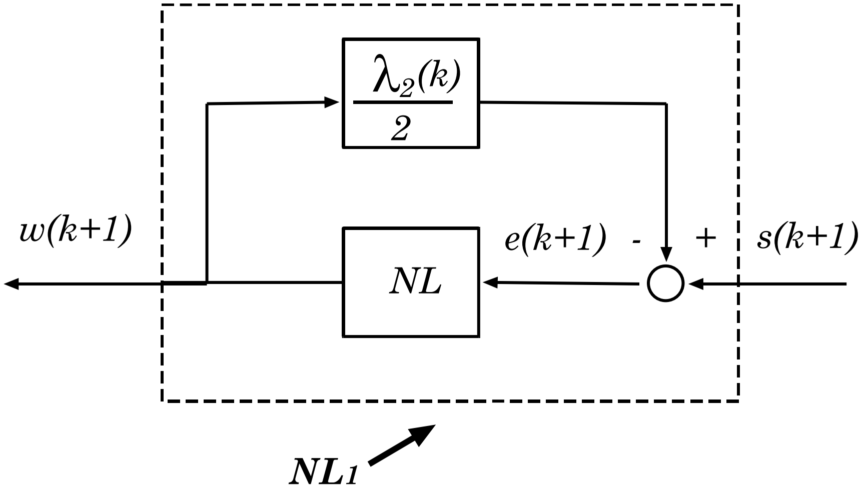
$$\phi_k = \phi(k) \qquad e_k = e(k)$$

$$w_k = w(k) \qquad s_k = s(k)$$

$$\lambda_{1,k} = \lambda_1(k) \qquad \lambda_{2,k} = \lambda_2(k)$$

$$F_k = F(k)$$

Proof that NL_1 is P-class



- From the summing junction, we have

$$e_{k+1} = s_{k+1} - \frac{\lambda_{2,k}}{2} w_{k+1}$$

$$\Downarrow$$

$$s_{k+1} = e_{k+1} + \frac{\lambda_{2,k}}{2} w_{k+1}$$

Proof that NL_1 is P-class

NL:
$$\begin{cases} \tilde{\theta}_{k+1} = \tilde{\theta}_k - \lambda_{1,k}^{-1} F_k \phi_k e_{k+1} \\ w_{k+1} = -\phi_k^T \tilde{\theta}_{k+1} \\ F_{k+1}^{-1} = \lambda_{1,k} F_k^{-1} + \lambda_{2,k} \phi_k \phi_k^T \end{cases}$$

- Multiply the input of NL_1 by its output

$$2w_{k+1}s_{k+1} = \underbrace{w_{k+1}}_{-\tilde{\theta}_{k+1}^T \phi_k} \left[2e_{k+1} + \lambda_{2,k} \underbrace{w_{k+1}}_{-\phi_k^T \tilde{\theta}_{k+1}} \right]$$

$$2w_{k+1}s_{k+1} = \tilde{\theta}_{k+1}^T \underbrace{\left[\lambda_{2,k} \phi_k \phi_k^T \right]}_{F_{k+1}^{-1} - \lambda_{1,k} F_k^{-1}} \tilde{\theta}_{k+1} - 2\tilde{\theta}_{k+1}^T \underbrace{\phi_k e_{k+1}}_{\lambda_{1,k} F_k^{-1} (\tilde{\theta}_k - \tilde{\theta}_{k+1})}$$

Proof that NL_1 is P-class

- From the previous slide

$$2w_{k+1}s_{k+1} = \tilde{\theta}_{k+1}^T \left[F_{k+1}^{-1} - \lambda_{1,k}F_k^{-1}\right]\tilde{\theta}_{k+1} + 2\,\lambda_{1,k}\,\tilde{\theta}_{k+1}^T F_k^{-1}(\tilde{\theta}_{k+1} - \tilde{\theta}_k)$$

$$= \tilde{\theta}_{k+1}^T F_{k+1}^{-1}\tilde{\theta}_{k+1} + \lambda_{1,k}\left[-\tilde{\theta}_{k+1}^T F_k^{-1}\tilde{\theta}_{k+1} + 2\tilde{\theta}_{k+1}^T F_k^{-1}(\tilde{\theta}_{k+1} - \tilde{\theta}_k)\right]$$

Define $\Delta = \tilde{\theta}_k - \tilde{\theta}_{k+1}$

Proof that NL_1 is P-class

- From the previous slide

$$\Delta = \tilde{\theta}_k - \tilde{\theta}_{k+1}$$

$$2w_{k+1}s_{k+1} = \tilde{\theta}_{k+1}^T F_{k+1}^{-1} \tilde{\theta}_{k+1}$$
$$-\lambda_{1,k}\left[\tilde{\theta}_{k+1}^T F_k^{-1} \tilde{\theta}_{k+1} + 2\tilde{\theta}_{k+1}^T F_k^{-1} \Delta\right]$$

$$= \tilde{\theta}_{k+1}^T F_{k+1}^{-1} \tilde{\theta}_{k+1}$$
$$-\lambda_{1,k}\left[\underbrace{\left(\tilde{\theta}_{k+1} + \Delta\right)^T}_{\tilde{\theta}_k^T} F_k^{-1} \underbrace{\left(\tilde{\theta}_{k+1} + \Delta\right)}_{\tilde{\theta}_k} - \Delta^T F_k^{-1} \Delta\right]$$

Proof that NL_1 is P-class

$$\Delta = \tilde{\theta}_k - \tilde{\theta}_{k+1}$$

- From the previous slide

$$2w_{k+1}s_{k+1} = \tilde{\theta}_{k+1}^T F_{k+1}^{-1}\tilde{\theta}_{k+1} \underbrace{-\lambda_{1,k}\tilde{\theta}_k^T F_k^{-1}\tilde{\theta}_k}_{\geq -\tilde{\theta}_k^T F_k^{-1}\tilde{\theta}_k} + \underbrace{\lambda_{1,k}\Delta^T F_k^{-1}\Delta}_{\geq 0}$$

because $\lambda_{1,k} \leq 1$

because $F_k^{-1} \succ 0$, $\lambda_{1,k} > 0$

- Therefore

$$2w_{k+1}s_{k+1} \geq \tilde{\theta}_{k+1}^T F_{k+1}^{-1}\tilde{\theta}_{k+1} - \tilde{\theta}_k^T F_k^{-1}\tilde{\theta}_k$$

Proof that NL_1 is P-class

$$w_{k+1} s_{k+1} \geq \frac{1}{2}\left[\tilde{\theta}_{k+1}^T F_{k+1}^{-1} \tilde{\theta}_{k+1} - \tilde{\theta}_k^T F_k^{-1} \tilde{\theta}_k\right]$$

- Now check the Popov inequality

$$\sum_{i=0}^{k} w_i s_i \geq \frac{1}{2} \sum_{i=0}^{k} \left[\tilde{\theta}_i^T F_i^{-1} \tilde{\theta}_i - \tilde{\theta}_{i-1}^T F_{i-1}^{-1} \tilde{\theta}_{i-1}\right]$$

$$= \frac{1}{2}\left[\tilde{\theta}_k^T F_k^{-1} \tilde{\theta}_k - \tilde{\theta}_{-1}^T F_{-1}^{-1} \tilde{\theta}_{-1}\right]$$

$$\geq -\underbrace{\frac{1}{2} \tilde{\theta}_{-1}^T F_{-1}^{-1} \tilde{\theta}_{-1}}_{\gamma_0^2}$$

$\blacksquare$