Introduction

Big picture
Syllabus
Requirements

Big picture

ME 233 talks about advanced and practical control theories, including but not limited to:

- dynamic programming
- optimal estimation (Kalman Filter) and stochastic control
- SISO and MIMO feedback design principles
- digital control: implementation and design
- feedforward design techniques: preview control, zero phase error tracking, etc
- feedback design techniques: LQG/LTR, internal model principle, repetitive control, disturbance observer
- system identification
- adaptive control
- ...

Teaching staff and class notes

- instructor:
  - Xu Chen, 2013 UC Berkeley Ph.D., maxchen@berkeley.edu
  - office hour: Tu Thur 1pm-2:30pm at 5112 Etcheverry Hall
- teaching assistant:
  - Changliu Liu, changliuliu@berkeley.edu
  - office hour: M, W 10:00am – 11:00am in 136 Hesse Hall
- class notes:
  - ME233 Class Notes by M. Tomizuka (Parts I and II); Both can be purchased at Copy Central, 48 Shattuck Square, Berkeley

Requirements and evaluations

- website (case sensitive):
  - www.me.berkeley.edu/ME233/sp14
  - bcourses.berkeley.edu
- prerequisites: ME C 232 or its equivalence
- lectures: Tu Thur 8-9:30am, 3113 Etcheverry Hall
- discussions: Fri. 10-11am, 1165 Etcheverry Hall
- homework (20%)
- two in-class midterms (20% each): Mar. 4, 2014 and Apr. 15, 2014; one-page handwritten summary sheets allowed
- one final exam (40%): May 15 2014 (Th), 7 pm -10 pm; open notes

Prerequisites (ME 232 table of contents)

- Laplace and Z transformations
- Models and Modeling of linear dynamical systems: transfer functions, state space models
- Solutions of linear state equations
- Stability: poles, eigenvalues, Lyapunov stability
- Controllability and observability
- State and output feedbacks, pole assignment via state feedback
- State estimation and observer, observer state feedback control
- Linear Quadratic (LQ) Optimal Control, LQR properties, Riccati equation

Remark

ME233 will be webcasted:

- Berkeley's YouTube channel (http://www.youtube.com/ucberkeley)
- iTunes U (http://itunes.berkeley.edu/)
- webcast.berkeley (http://webcast.berkeley.edu)

links will be posted on course website when available

References (also on course website)

- Probability
  - Bertsekas, Introduction to Probability, Athena Scientific
  - Yates and Goodman, Probability and Stochastic Processes, second edition, Willey
- Linear Quadratic Optimal Control
  - Anderson and Moore, Optimal Control: Linear Quadratic Methods, Dover Books on Engineering (paperback), 2007. A PDF can be downloaded from: http://users.rsise.anu.edu.au/%7Ejohn/papers/index.html
  - Lewis and Syrmos, Vassilis L., Optimal Control, Wiley-IEEE, 1995
  - Bryson and Ho, Applied Optimal Control: Optimization, Estimation, and Control, Wiley
- Stochastic Control Theory and Optimal Filtering
  - Brown and Hwang, Introduction to Random Signals and Applied Kalman Filtering, Third Edition, Willey
  - Lewis and Xie and Popa, Optimal and Robust Estimation, Second Edition CRC
  - Grewal and Andrews, Kalman Filter, Theory and Practice, Prentice Hall
  - Anderson, and Moore, Optimal Filtering, Dover Books on Engineering (paperback), New York, 2005. A PDF can be downloaded from: http://users.rsise.anu.edu.au/%7Ejohn/papers/index.html
  - Astrom, Introduction to Stochastic Control Theory, Dover Books on Engineering (paperback), New York, 2006
- Adaptive Control
  - Astrom and Wittenmark, Adaptive Control, Addison Wesley, 2nd Ed., 1995
  - Goodwin and Sin, Adaptive Filtering Prediction and Control, Prentice Hall, 1984
  - Krstic, Kanellakopoulos, and Kokotovic, Nonlinear and Adaptive Control Design, Willey

Lecture 1: Dynamic Programming

General problem
Multivariable derivative
Discrete-time LQ

Dynamic programming (DP)

introduction:

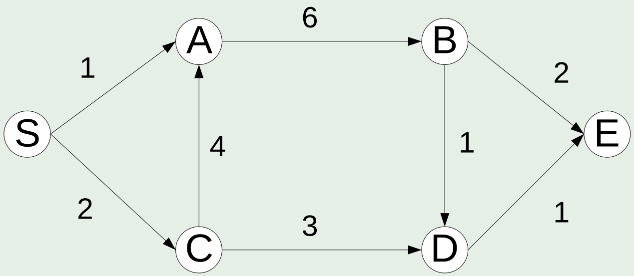
- history: developed in the 1950's by Richard Bellman
- "programming": ∼"planning" (has nothing to do with computers)
- a useful concept with lots of applications
- IEEE Global History Network: "A breakthrough which set the stage for the application of functional equation techniques in a wide spectrum of fields..."

Essentials of dynamic programming

- key idea: solve a complex and difficult problem via solving a collection of sub problems

Example (Path planning)

goal: obtain minimum cost path from S to E



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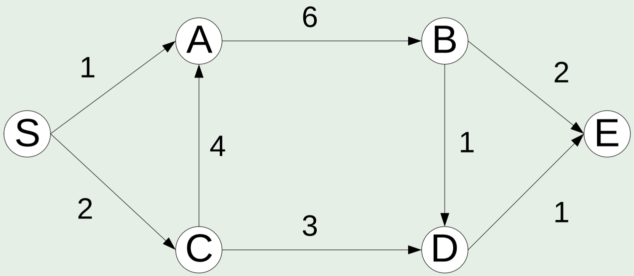
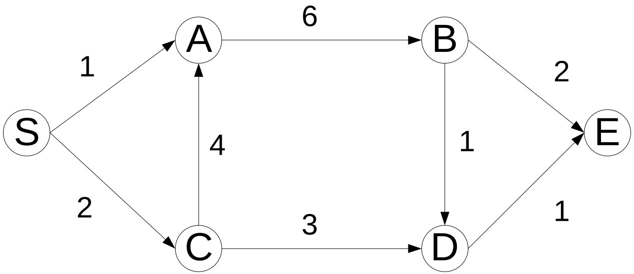


- observation: if node C is on the optimal path, the then path from node C to node E must be optimal as well

Essentials of dynamic programming



$dist\,(E) \triangleq$ minimum cost $S \to E$

- solution:
  backward analysis

$$dist\,(E) = \min\{dist\,(B)+2, dist\,(D)+1\}$$

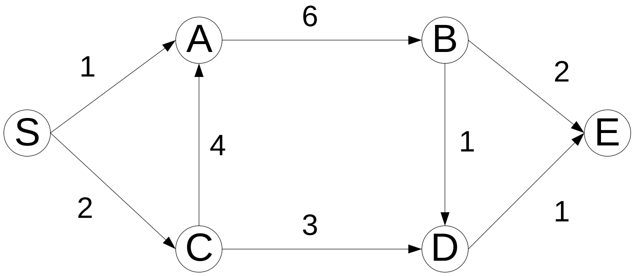
$$dist\,(B) = dist\,(A)+6$$

$$dist\,(D) = min\{dist\,(B)+1, dist\,(C)+3\}$$

$$dist\,(C) = 2$$

$$dist\,(A) = \min\{1, dist\,(C)+4\}$$

Essentials of dynamic programming



$$\mathit{dist}\,(E) \triangleq \text{minimum cost } S \to E$$

- solution:
  backward analysis
  forward computation

$$\mathit{dist}\,(E) = \min\{\mathit{dist}\,(B) + 2, \mathit{dist}\,(D) + 1\} \qquad \mathit{dist}\,(C) = 2$$

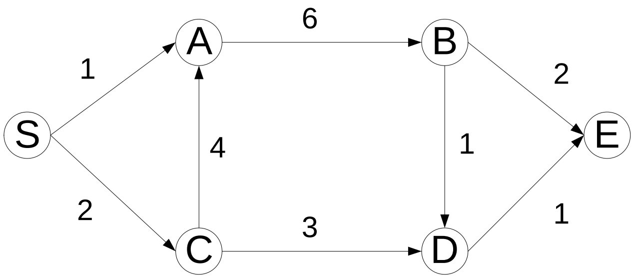
$$\mathit{dist}\,(B) = \mathit{dist}\,(A) + 6 \qquad \mathit{dist}\,(A) = 1$$

$$\mathit{dist}\,(D) = \mathit{min}\{\mathit{dist}\,(B) + 1, \mathit{dist}\,(C) + 3\} \qquad \mathit{dist}\,(B) = 1 + 6 = 7$$

$$\mathit{dist}\,(C) = 2 \qquad \mathit{dist}\,(D) = 5$$

$$\mathit{dist}\,(A) = \min\{1, \mathit{dist}\,(C) + 4\} \qquad \mathit{dist}\,(E) = 6$$

Essentials of dynamic programming



- summary (Bellman's principle of optimality): "From any point on an optimal trajectory, the remaining trajectory is optimal for the corresponding problem initiated at that point."

General optimal control problems

- general discrete-time plant:

$$x(k+1) = f(x(k), u(k), k)$$

$$\text{state constraint: } x(k) \in X \subset \mathbf{R}^n$$

$$\text{input constraint: } u(k) \in U \subset \mathbf{R}^m$$

- performance index:

$$J = S(x(N)) + \sum_{k=0}^{N-1} L(x(k), u(k), k)$$

S & L–real, scalar-valued functions; N–final time (optimization horizon)

- goal: obtain the optimal control sequence

$$\{u^o(0), u^o(1), \ldots, u^o(N-1)\}$$

Dynamic programming for optimal control

- define: $U_k \triangleq \{u(k), u(k+1), \ldots, u(N-1)\}$
- optimal cost to go at time k:

$$
\begin{aligned}
J_k^o(x(k)) &\triangleq \min_{U_k} \left\{ S(x(N)) + \sum_{j=k}^{N-1} L(x(j), u(j), j) \right\} \\
&= \min_{u(k)} \min_{U_{k+1}} \left\{ L(x(k), u(k), k) + \left[S(x(N)) + \sum_{j=k+1}^{N-1} L(x(j), u(j), j) \right] \right\} \\
&= \min_{u(k)} \left\{ L(x(k), u(k), k) + \min_{U_{k+1}} \left[S(x(N)) + \sum_{j=k+1}^{N-1} L(x(j), u(j), j) \right] \right\} \\
&= \min_{u(k)} \left\{ L(x(k), u(k), k) + J_{k+1}^o(x(k+1)) \right\} \qquad (1)
\end{aligned}
$$

- boundary condition: $J_N^o(x(N)) = S(x(N))$
- The problem can now be solved by solving a sequence of problems $J_{N-1}^o,\ J_{N-2}^o, \ldots, J_1^o,\ J^o$.

Solving discrete-time finite-horizon LQ via DP

- system dynamics:

$$x(k+1) = A(k)x(k) + B(k)u(k), \; x(k_0) = x_o \qquad (2)$$

- performance index:

$$J = \frac{1}{2}x^T(N)Sx(N) + \frac{1}{2}\sum_{k=k_0}^{N-1}\left\{x^T(k)Q(k)x(k) + u^T(k)R(k)u(k)\right\}$$

$$Q(k) = Q^T(k) \succeq 0, \; S = S^T \succeq 0, \; R(k) = R^T(k) \succ 0$$

- optimal cost to go:

$$J_k^o(x(k)) = \min_{u(k)}\left\{\frac{1}{2}x^T(k)Q(k)x(k) + \frac{1}{2}u^T(k)R(k)u(k) + J_{k+1}^o(x(k+1))\right\}$$

with boundary condition: $J_N^o(x(N)) = \frac{1}{2}x^T(N)Sx(N)$

Facts about quadratic functions

- consider

$$f(u) = \frac{1}{2}u^T M u + p^T u + q, \; M = M^T \tag{3}$$

- optimality (maximum when M is negative definite; minimum when M is positive definite) is achieved when

$$\frac{\partial f}{\partial u} = Mu + p = 0 \Rightarrow u^o = -M^{-1}p \tag{4}$$

- and the optimal cost is

$$f^o = f(u^o) = -\frac{1}{2}p^T M^{-1} p + q \tag{5}$$

From J_N^o to J_{N-1}^o in discrete-time LQ

- by definition:

$$J_{N-1}^o(x(N-1)) = \min_{u(N-1)} \left\{ \frac{1}{2} x^T(N) S x(N) \right.$$
$$\left. + \frac{1}{2} \left[x^T(N-1) Q(N-1) x(N-1) + u^T(N-1) R(N-1) u(N-1) \right] \right\}$$

- using the system dynamics (2) gives

$$J_{N-1}^o(x(N-1)) = \frac{1}{2} \min_{u(N-1)} \{ x^T(N-1) Q(N-1) x(N-1)$$
$$+ u^T(N-1) R(N-1) u(N-1) + [A(N-1)x(N-1) + B(N-1)u(N-1)]^T$$
$$\times S[A(N-1)x(N-1) + B(N-1)u(N-1)]\}$$

- optimal control by letting $\partial J_{N-1}/\partial u(N-1) = 0$:

$$u^o(N-1) = -\underbrace{\left[R(N-1) + B^T(N-1) S B(N-1) \right]^{-1} B^T(N-1) S A(N-1)}_{\text{state feedback gain: } K(N-1)} x(N-1)$$

⋆Optimality at N and $N-1$

at time N: optimal cost is

$$J_N^o(x(N)) = \frac{1}{2}x^T(N)Sx(N) \triangleq \frac{1}{2}x^T(N)P(N)x(N)$$

at time $N-1$:

$$J_{N-1}^o(x(N-1)) = \frac{1}{2}\min_{u(N-1)}\{x^T(N-1)Q(N-1)x(N-1)$$

$$+u^T(N-1)R(N-1)u(N-1)+[A(N-1)x(N-1)+B(N-1)u(N-1)]^T$$
$$\times S[A(N-1)x(N-1)+B(N-1)u(N-1)]\}$$

optimal cost to go [by using (5)] is

$$J_{N-1}^o(x(N-1)) = \frac{1}{2}x^T(N-1)\Big\{Q(N-1)+A^T(N-1)SA(N-1)$$

$$-(\underline{\ldots})^T\left[R(N-1)+B^T(N-1)SB(N-1)\right]^{-1}\underline{B^T(N-1)SA(N-1)}\Big\}x(N-1)$$

$$\triangleq \frac{1}{2}x^T(N-1)P(N-1)x(N-1)$$

Summary: from N to $N-1$

at N:

$$J_N^o(x(N)) = \frac{1}{2}x^T(N)Sx(N) = \frac{1}{2}x^T(N)P(N)x(N)$$

at $N-1$:

$$J_{N-1}^o(x(N-1)) = \frac{1}{2}x^T(N-1)P(N-1)x(N-1)$$

with (S has been replaced with $P(N)$ here)

$$\begin{aligned}
&P(N-1) = Q(N-1) + A^T(N-1)P(N)A(N-1) \\
&-(\underline{\ldots})^T\Big[R(N-1)+B^T(N-1)P(N)B(N-1)\Big]^{-1}\underline{B^T(N-1)P(N)A(N-1)}
\end{aligned}$$

and state-feedback law

$$\begin{aligned}
u^o(N-1) = -&\Big[R(N-1)+B^T(N-1)P(N)B(N-1)\Big]^{-1} \\
&\times B^T(N-1)P(N)A(N-1)x(N-1)
\end{aligned}$$

Induction from $k+1$ to k

- assume at $k+1$:

$$J_{k+1}^{o}(x(k+1)) = \frac{1}{2}x^{T}(k+1)P(k+1)x(k+1)$$

- analogous as the case from N to $N-1$, we can get, at k:

$$J_{k}^{o}(x(k)) = \frac{1}{2}x^{T}(k)P(k)x(k)$$

  with Riccati equation

$$\begin{aligned}P(k) &= A^{T}(k)P(k+1)A(k)+Q(k)\\ &-A^{T}(k)P(k+1)B(k)\left[R(k)+B^{T}(k)P(k+1)B(k)\right]^{-1}B^{T}(k)P(k+1)A(k)\end{aligned}$$

  and state-feedback law

$$u^{o}(k) = -\left[R(k)+B^{T}(k)P(k+1)B(k)\right]^{-1}B^{T}(k)P(k+1)A(k)x(k)$$

Implementation

- optimal state-feedback control law:

$$u^o(k) = -\left[R(k) + B^T(k)P(k+1)B(k)\right]^{-1} B^T(k)P(k+1)A(k)x(k)$$

- Riccati equation:

$$P(k) = A^T(k)P(k+1)A(k) + Q(k)$$
$$-A^T(k)P(k+1)B(k)\left[R(k) + B^T(k)P(k+1)B(k)\right]^{-1} B^T(k)P(k+1)A(k)$$

  with the boundary condition $P(N) = S$.
- $u^o(k)$ depends on
  - the state vector $x(k)$
  - system matrices $A(k)$ and $B(k)$ and the cost matrix $R(k)$
  - $P(k+1)$, which depends on $Q(k+2)$, $A(k+1)$, $B(k+1)$, and $P(k+2)$...
- iterating gives: $u(0)$ depends on $\{A(k), B(k), R(k), Q(k+1)\}_{k=0}^{N-1}$
  In practice, $P(k)$ can be computed offline since they do not require information of $x(k)$.

Lecture 3: Review of Probability Theory

Connection with control systems
Random variable, distribution
Multiple random variables
Random process, filtering a random process

Big picture

why are we learning this:

We have been very familiar with deterministic systems:

$$x(k+1) = Ax(k) + Bu(k)$$

In practice, we commonly have:

$$x(k+1) = Ax(k) + Bu(k) + B_w w(k)$$

where $w(k)$ is the noise term that we have been neglecting. With the introduction of $w(k)$, we need to equip ourselves with some additional tool sets to understand and analyze the problem.

Sample space, events and probability axioms

- experiment: a situation whose outcome depends on chance
- trial: each time we do an experiment we call that a trial

Example (Throwing a fair dice)

possible outcomes in one trail: getting a ONE, getting a TWO, ...

- sample space Ω: includes all the possible outcomes

- probability: discusses how likely things, or more formally, events, happen
- an event S_i: includes some (maybe 1, maybe more, maybe none) outcomes of the sample space. e.g., the event that it won't rain tomorrow; the event that getting odd numbers when throwing a dice

Sample space, events and probability axioms

probability axioms

- $\Pr\{S_j\} \geq 0$
- $\Pr\{\Omega\} = 1$
- if $S_i \cap S_j = \emptyset$ (empty set), then $\Pr\{S_i \cup S_j\} = \Pr\{S_i\} + \Pr\{S_j\}$

Example (Throwing a fair dice)

the sample space:

$$\Omega = \{\underbrace{\text{getting a ONE}}_{\omega_1}, \underbrace{\text{getting a TWO}}_{\omega_2}, \ldots, \underbrace{\text{getting a SIX}}_{\omega_6}\}$$

the event S_1 of observing an even number:

$$S_1 = \{\omega_2, \omega_4, \omega_6\}$$
$$\Pr\{S_1\} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

Random variables

to better measure probabilities, we introduce random variables (r.v.'s)

- r.v.: a real valued function $X(\omega)$ defined on Ω; $\forall x \in \mathbf{R}$ there defined the *(probability) cumulative distribution function (cdf)*

$$F(x) = \Pr\{X \leq x\}$$

- cdf $F(x)$: non-decreasing, $0 \leq F(x) \leq 1$, $F(-\infty) = 0$, $F(\infty) = 1$

Example (Throwing a fair dice)

can define X: the obtained number of the dice

$$X(\omega_1) = 1,\ X(\omega_2) = 2,\ X(\omega_3) = 3,\ X(\omega_4) = 4, \ldots$$

can also define X: indicator of whether the obtained number is even

$$X(\omega_1) = X(\omega_3) = X(\omega_5) = 0,\ X(\omega_2) = X(\omega_4) = X(\omega_6) = 1$$

Probability density and moments of distributions

- probability density function (pdf):

$$p(x) = \frac{\mathrm{d}F(x)}{\mathrm{d}x}$$

$$\Pr(a < X \leq b) = \int_a^b p(x)\,\mathrm{d}x, \ a < b$$

  sometimes we write $p_X(x)$ to emphasize that it is for the r.v. X
- mean, or expected value (first moment):

$$m_X = \mathrm{E}[X] = \int_{-\infty}^{\infty} x p_X(x)\,\mathrm{d}x$$

- variance (second moment):

$$\mathrm{Var}[X] = \mathrm{E}\left[(X - m_X)^2\right] = \int_{-\infty}^{\infty} (x - m_X)^2 p_X(x)\,\mathrm{d}x$$

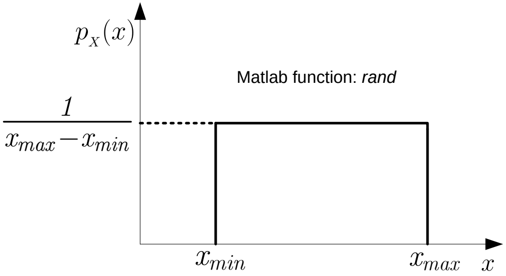
- standard deviation (std): $\sigma = \sqrt{\mathrm{Var}[X]}$
- exercise: prove that $\mathrm{Var}[X] = \mathrm{E}\left[X^2\right] - (\mathrm{E}[X])^2$

Example distributions

uniform distribution

- a r.v. uniformly distributed between x_{min} and x_{max}
- probability density function:

$$p(x) = \frac{1}{x_{\mathrm{max}} - x_{\mathrm{min}}}$$



- cumulative distribution function:

$$F(x) = \frac{x - x_{\mathrm{min}}}{x_{\mathrm{max}} - x_{\mathrm{min}}}, \; x_{\mathrm{min}} \le x \le x_{\mathrm{max}}$$

- mean and variance:

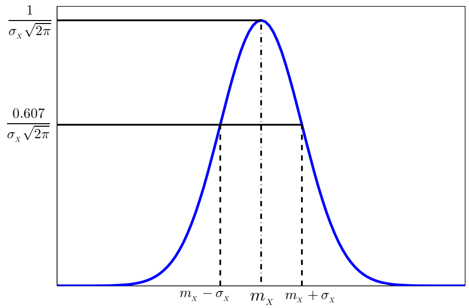
$$\mathrm{E}[X] = \frac{1}{2}(x_{\mathrm{max}} + x_{\mathrm{min}}), \; \mathrm{Var}[X] = \frac{(x_{\mathrm{max}} - x_{\mathrm{min}})^2}{12}$$

Example distributions

Gaussian/normal distribution

- importance: sum of independent r.v.s $\rightarrow$ a Gaussian distribution
- probability density function:

$$p(x) = \frac{1}{\sigma_X\sqrt{2\pi}} \exp\left(-\frac{(x-m_X)^2}{2\sigma_X^2}\right)$$



- pdf fully characterized by m_X and σ_X. Hence a normal distribution is usually denoted as $N(m_X, \sigma_X)$
- nice properties: if X is Gaussian and Y is a linear function of X, then Y is Gaussian

Example distributions

Gaussian/normal distribution

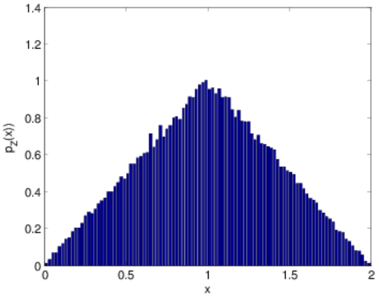
Central Limit Theorem: if X_1, X_2, ... are independent identically distributed random variables with mean m_X and variance σ_X^2, then

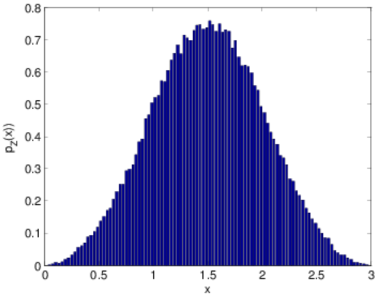
$$Z_n = \frac{\sum_{k=1}^{n}(X_k - m_X)}{\sqrt{n\sigma_X^2}}$$

converges in distribution to a normal random variable $X \sim N(0,1)$

example: sum of uniformly distributed random variables in $[0,1]$

```
X1 = rand(1,1e5);
X2 = rand(1,1e5);
X3 = rand(1,1e5);
Z = X1 + X2;
[fz,x] = hist(Z,100);
w_fz = x(end)/length(fz);
fz = fz/sum(fz)/w_fz;
figure, bar(x,fz)
xlabel 'x'; ylabel 'p_Z(x))';
Y = X1 + X2 + X3;
% ...
```





Multiple random variables

joint probability

for the same sample space Ω, multiple r.v.'s can be defined

- joint probability: $\Pr(X = x, Y = y)$
- joint cdf:

$$F(x,y) = \Pr(X \le x, Y \le y)$$

- joint pdf: $p(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$
- covariance:

$$\begin{aligned}\mathrm{Cov}(X,Y) = \Sigma_{XY} = \mathrm{E}[(X - m_X)(Y - m_Y)] &= \mathrm{E}[XY] - \mathrm{E}[X]E[Y] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x - m_X)(y - m_Y)\,p(x,y)\,\mathrm{d}x\mathrm{d}y\end{aligned}$$

- uncorrelated: $\Sigma_{XY} = 0$
- independent random variables satisfy:

$$F(x,y) = \Pr(X \le x, Y \le y) = \Pr(X \le x)\Pr(Y \le y) = F_X(x)F_Y(y)$$

$$p(x,y) = p_X(x)\,p_Y(y)$$

Multiple random variables

more about correlation

correlation coefficient:

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$$

X and Y are uncorrelated if $\rho(X,Y) = 0$

- independent$\Rightarrow$uncorrelated; uncorrelated$\not\Rightarrow$independent
- uncorrelated indicates $\mathrm{Cov}(X,Y) = \mathrm{E}[XY] - \mathrm{E}[X]\mathrm{E}[Y] = 0$, which is weaker than X and Y being independent

Example

X–uniformly distributed on $[-1,1]$. Construct Y: if $X \le 0$ then $Y = -X$; if $X > 0$ then $Y = X$. X and Y are uncorrelated due to

- $\mathrm{E}[X] = 0$, $\mathrm{E}[Y] = \frac{1}{2}$
- $\mathrm{E}[XY] = 0$

however X and Y are clearly dependent

Multiple random variables

random vector

- vector of r.v.'s:

$$Z = \begin{bmatrix} X \\ Y \end{bmatrix}$$

- mean:

$$m_Z = \begin{bmatrix} m_X \\ m_Y \end{bmatrix}$$

- covariance matrix:

$$\Sigma = \mathrm{E}\left[(Z - m_Z)(Z - m_Z)^T\right] = \begin{bmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{bmatrix}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \begin{bmatrix} (X - m_X)^2 & (X - m_X)(Y - m_Y) \\ (Y - m_Y)(X - m_X) & (Y - m_Y)^2 \end{bmatrix} p(x, y)\,\mathrm{d}x\mathrm{d}y$$

Conditional distributions

- joint pdf to single pdf:

$$p_X(x) = \int_{-\infty}^{\infty} p(x,y)\,\mathrm{d}y$$

- conditional pdf:

$$p_X(x|y_1) = p_X(x|Y = y_1) = \frac{p(x,y_1)}{p_Y(y_1)}$$

- conditional mean:

$$\mathrm{E}[X|y_1] = \int_{-\infty}^{\infty} x p_X(x|y_1)\,\mathrm{d}x$$

- note: independent$\Rightarrow p_X(x|y_1) = p_X(x)$
- properties of conditional mean:

$$\underset{y}{\mathrm{E}}[\mathrm{E}[X|y]] = \mathrm{E}[X]$$

Multiple random variables

Gaussian random vectors

Gaussian r.v. is particularly important and interesting as its pdf is mathematically sound

Special case: two independent Gaussian r.v. X_1 and X_2

$$p(x_1,x_2) = p_{X_1}(x_1)\,p_{X_2}(x_2) = \frac{1}{\sigma_{X_1}\sqrt{2\pi}} e^{-\left(x_1 - m_{X_1}\right)^2/\left(2\sigma_{X_1}^2\right)} \frac{1}{\sigma_{X_2}\sqrt{2\pi}} e^{-\left(x_2 - m_{X_2}\right)^2/\left(2\sigma_{X_2}^2\right)}$$

$$= \frac{1}{\sigma_{X_1}\sigma_{X_2}\left(\sqrt{2\pi}\right)^2} \exp\left\{ -\frac{1}{2} \begin{bmatrix} x_1 - m_{X_1} \\ x_2 - m_{X_2} \end{bmatrix}^T \begin{bmatrix} \sigma_{X_1}^2 & 0 \\ 0 & \sigma_{X_2}^2 \end{bmatrix}^{-1} \begin{bmatrix} x_1 - m_{X_1} \\ x_2 - m_{X_2} \end{bmatrix} \right\}$$

We can use the random vector notation: $X = [X_1, X_2]^T$

$$\Sigma = \begin{bmatrix} \sigma_{X_1}^2 & 0 \\ 0 & \sigma_{X_2}^2 \end{bmatrix}$$

and write

$$p_X(x) = \frac{1}{\left(\sqrt{2\pi}\right)^2 \sqrt{\det \Sigma}} \exp\left\{ -\frac{1}{2} [X - m_X]^T \Sigma^{-1} [X - m_X] \right\}$$

General Gaussian random vectors

pdf for a n-dimensional jointly distributed Gaussian random vector X:

$$p_X(x) = \frac{1}{\left(\sqrt{2\pi}\right)^n \sqrt{\det \Sigma}} \exp\left\{-\frac{1}{2}[X - m_X]^T \Sigma^{-1} [X - m_X]\right\} \quad (1)$$

joint pdf for 2 Gaussian random vectors X (n-dimensional) and Y (m-dimensional):

$$p(x,y) = \frac{1}{\left(\sqrt{2\pi}\right)^{n+m} \sqrt{\det \Sigma}} \exp\left\{-\frac{1}{2}\begin{bmatrix} x - m_X \\ y - m_Y \end{bmatrix}^T \Sigma^{-1} \begin{bmatrix} x - m_X \\ y - m_Y \end{bmatrix}\right\} \quad (2)$$

$$\Sigma = \begin{bmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{bmatrix}$$

where Σ_{XY} is the cross covariance (matrix) between X and Y

$$\Sigma_{XY} = \mathrm{E}\left[(X - m_X)(Y - m_Y)^T\right] = \mathrm{E}\left[(Y - m_Y)(X - m_X)^T\right]^T = \Sigma_{YX}^T$$

General Gaussian random vectors

conditional mean and covariance

important facts about conditional mean and covariance:

$$m_{X|y} = m_X + \Sigma_{XY}\Sigma_{YY}^{-1}[y - m_Y]$$

$$\Sigma_{X|y} = \Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX}$$

proof uses $p(x,y) = p(x|y)p(y)$, (1), and (2)

- getting $\det\Sigma$ and the inverse Σ^{-1}: do a transformation

$$\begin{bmatrix} I & -\Sigma_{XY}\Sigma_{YY}^{-1} \\ 0 & I \end{bmatrix}\begin{bmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{bmatrix}\begin{bmatrix} I & 0 \\ -\Sigma_{YY}^{-1}\Sigma_{YX} & I \end{bmatrix} = \begin{bmatrix} \Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX} & 0 \\ 0 & \Sigma_{YY} \end{bmatrix} \quad (3)$$

hence

$$\det\Sigma = \det\Sigma_{YY}\det\left(\Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX}\right) \quad (4)$$

General Gaussian random vectors

inverse of the covariance matrix

computing the inverse Σ^{-1}:

–(3) gives

$$\Sigma^{-1} = \begin{bmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} I & 0 \\ -\Sigma_{YY}^{-1}\Sigma_{YX} & I \end{bmatrix} \begin{bmatrix} \Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX} & 0 \\ 0 & \Sigma_{YY} \end{bmatrix}^{-1} \begin{bmatrix} I & -\Sigma_{XY}\Sigma_{YY}^{-1} \\ 0 & I \end{bmatrix}$$

–hence in (2):

$$\begin{bmatrix} x - m_X \\ y - m_Y \end{bmatrix}^T \Sigma^{-1} \begin{bmatrix} x - m_X \\ y - m_Y \end{bmatrix}$$

$$= \begin{bmatrix} \star \end{bmatrix}^T \begin{bmatrix} \Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX} & 0 \\ 0 & \Sigma_{YY} \end{bmatrix}^{-1} \underbrace{\begin{bmatrix} x - \left(m_X + \Sigma_{XY}\Sigma_{YY}^{-1}[y - m_Y]\right) \\ y - m_Y \end{bmatrix}}_{[\star]} \quad (5)$$

General Gaussian random vectors

$p(x,y) = p(x|y)\,p(y) \Rightarrow p(x|y) = p(x,y)\,/\,p(y)$

- using (4) and (5) in (2), we get

$$p(x|y) = \frac{p(x,y)}{p(y)} = \frac{1}{\left(\sqrt{2\pi}\right)^n \sqrt{\det \underbrace{\left(\Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX}\right)}_{[\star\star]}}}$$

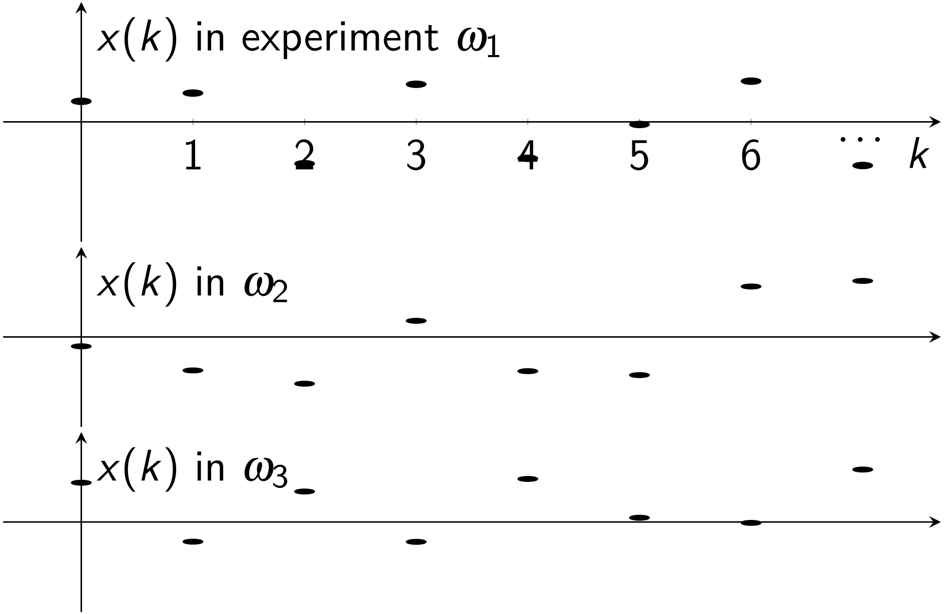
$$\times \exp\left\{ -\frac{1}{2}\left[\underline{\ldots}\right]^T [\star\star]^{-1} \left[\underline{x - \left(m_X + \Sigma_{XY}\Sigma_{YY}^{-1}[y - m_Y]\right)}\right] \right\}$$

hence $X|y$ is also Gaussian, with

$$m_{X|y} = m_X + \Sigma_{XY}\Sigma_{YY}^{-1}\left[y - m_Y\right]$$
$$\Sigma_{X|y} = \Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX}$$

Random process

- discrete-time random process: a random variable evolving with time $\{x(k),\ k = 1, 2, \dots\}$
- a stack of random vectors: $x(k) = [x(1), x(2), \dots]^T$



$\vdots$

Random process

$x(k) = [x(1), x(2), \ldots]^T$:

- complete probabilistic properties defined by the joint pdf $p(x(1), x(2), \ldots)$, which is usually difficult to get
- usually sufficient to know the mean $\mathrm{E}[x(k)] = m_x(k)$ and *auto-covariance*:

$$\mathrm{E}[(x(j) - m_x(j))(x(k) - m_x(k))] = \Sigma_{xx}(j, k) \tag{6}$$

- sometimes $\Sigma_{xx}(j, k)$ is also written as $X_{xx}(j, k)$

Random process

let $x(k)$ be a 1-d random process

- time average of $x(k)$:

$$\overline{x(k)} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{j=-N}^{N} x(j)$$

- ensemble average:

$$\mathrm{E}[x(k)] = m_{x(k)}$$

- *ergodic* random process: for all moments of the distribution, the ensemble averages equal the time averages

$$\mathrm{E}[x(k)] = \overline{x(k)},\ \Sigma_{xx}(j,k) = \overline{[x(j) - m_x][x(k) - m_x]}, \ldots$$

- ergodicity: not easy to test but many processes in practice are ergodic; extremely important as large samples can be expensive to collect in practice
- one necessary condition for ergodicity is stationarity

Random process

stationarity: tells whether the statistics characteristics changes w.r.t. time

- *stationary in the strict sense*: probability distribution does not change w.r.t. time

$$Pr\{x(k_1) \leq x_1, \ldots, x(k_n) \leq x_n\} = Pr\{x(k_1 + l) \leq x_1, \ldots, x(k_n + l) \leq x_n\}$$

- *stationary in the week/wide sense*: mean does not dependent on time

$$\mathrm{E}[x(k)] = m_x = \text{costant}$$

  and the auto-covariance (6) depends only on the time difference $l = j - k$
- can hence write

$$\mathrm{E}[(x(k) - m_x)(x(k+l) - m_x)] = \Sigma_{xx}(l) = X_{xx}(l)$$

- for stationary and ergodic random processes:

$$\Sigma_{xx}(l) = \mathrm{E}[(x(k) - m_x)(x(k+l) - m_x)] = \overline{(x(k) - m_x)(x(k+l) - m_x)}$$

Random process

covariance and correlation for stationary ergodic processes

- we will assume stationarity and ergodicity unless otherwise stated
- *auto-correlation*: $R_{xx}(l) = \mathrm{E}[x(k)x(k+l)]$.
- *cross-covariance*:

$$\Sigma_{xy}(l) = X_{xy}(l) = \mathrm{E}[(x(k)-m_x)(y(k+l)-m_y)]$$

- property (using ergodicity):

$$\begin{aligned}\Sigma_{xy}(l) = X_{xy}(l) &= \overline{(x(k)-m_x)(y(k+l)-m_y)} \\ &= \overline{(y(k+l)-m_y)(x(k)-m_x)} = X_{yx}(-l) = \Sigma_{yx}(-l)\end{aligned}$$

Random process

white noise

- *white* noise: a purely random process with $x(k)$ not correlated with $x(j)$ at all if $k \neq j$:

$$X_{xx}(0) = \sigma_{xx}^2,\ X_{xx}(l) = 0\ \forall l \neq 0$$

- non-stationary zero mean white noise:

$$\mathrm{E}[x(k)x(j)] = Q(k)\delta_{kj},\ \delta_{kj} = \begin{cases} 1 & ,\ k = j \\ 0 & ,\ k \neq j \end{cases}$$

Random process

auto-covariance and spectral density

- *spectral density*: the Fourier transform of auto-covariance

$$\Phi_{xx}(\omega) = \sum_{l=-\infty}^{\infty} X_{xx}(l)\, e^{-j\omega l},\; X_{xx}(l) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega l}\Phi_{xx}(\omega)\,\mathrm{d}\omega$$

- *cross spectral density*:

$$\Phi_{xy}(\omega) = \sum_{l=-\infty}^{\infty} X_{xy}(l)\, e^{-j\omega l},\; X_{xy}(l) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega l}\Phi_{xy}(\omega)\,\mathrm{d}\omega$$

properties:

- the variance of x is the area under the spectral density curve

$$\mathrm{Var}[x] = \mathrm{E}\left[(x-\mathrm{E}[x])^2\right] = X_{xx}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{xy}(\omega)\,\mathrm{d}\omega$$

- $X_{xx}(0) \geq |X_{xx}(l)|,\ \forall l$

Filtering a random process

passing a random process $u(k)$ through an LTI system (convolution) generates another random process:

$$y(k) = g(k) * u(k) = \sum_{i=-\infty}^{\infty} g(i)\, u(k-i)$$

- if u is *zero mean* and ergodic, then

$$X_{uy}(l) = \overline{u(k) \sum_{i=-\infty}^{\infty} u(k+l-i)\, g(i)}$$

$$= \sum_{i=-\infty}^{\infty} \overline{u(k)\, u(k+l-i)}\, g(i) = \sum_{i=-\infty}^{\infty} X_{uu}(l-i)\, g(i) = g(l) * X_{uu}(l)$$

similarly

$$X_{yy}(l) = \sum_{i=-\infty}^{\infty} X_{yu}(l-i)\, g(i) = g(l) * X_{yu}(l)$$

- in pictures:

$$X_{uu}(l) \longrightarrow \boxed{G(z)} \longrightarrow X_{uy}(l); \quad X_{yu}(l) \longrightarrow \boxed{G(z)} \longrightarrow X_{yy}(l)$$

Filtering a random process

input-output spectral density relation

for a general LTI system

$$u(k) \longrightarrow \boxed{G(z) = \frac{b_n z^n + b_{n-1} z^{n-1} + \cdots + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_0}} \longrightarrow y(k)$$

$$Y(z) = G(z)\,U(z) \Leftrightarrow Y\left(e^{j\omega}\right) = G\left(e^{j\omega}\right) U\left(e^{j\omega}\right)$$

- auto-covariance relation in the last slide:

$$X_{uu}(l) \longrightarrow \boxed{G(z)} \longrightarrow X_{uy}(l); \quad X_{yu}(l) \longrightarrow \boxed{G(z)} \longrightarrow X_{yy}(l)$$

$$X_{yu}(l) = X_{uy}(-l) = g(-l) * X_{uu}(-l) = g(-l) * X_{uu}(l)$$

hence

$$\boxed{\Phi_{yy}(\omega) = G\left(e^{j\omega}\right) G\left(e^{-j\omega}\right) \Phi_{uu}(\omega) = \left|G\left(e^{j\omega}\right)\right|^2 \Phi_{uu}(\omega)}$$

Filtering a random process

MIMO case:

- if u and y are vectors, $G(z)$ becomes a transfer function matrix
- dimensions play important roles:

$$X_{uy}(l) = \mathrm{E}\left[(u(k) - m_u)(y(k+l) - m_y)^T\right] = X_{yu}(-l)^T$$

$$X_{uu}(l) \longrightarrow \boxed{G(z)} \longrightarrow X_{uy}(l); \quad X_{yu}(l) \longrightarrow \boxed{G(z)} \longrightarrow X_{yy}(l)$$

$$\begin{aligned} X_{yy}(l) &= g(l) * X_{yu}(l) = g(l) * X_{uy}^T(-l) \\ &= g(l) * [g(-l) * X_{uu}(-l)]^T \end{aligned}$$

$$\boxed{\Phi_{yy}\left(e^{j\omega}\right) = G\left(e^{j\omega}\right) \cdot \Phi_{uu}\left(e^{j\omega}\right) G^T\left(e^{-j\omega}\right)}$$

Filtering a random process in state space

consider: $w(k)$–zero mean, white, $\mathrm{E}\left[w(k)w(k)^T\right]=W(k)$ and

$$x(k+1) = A(k)x(k) + B_w(k)w(k) \tag{7}$$

assume random initial state $x(0)$ (uncorrelated to $w(k)$):

$$\mathrm{E}[x(0)] = m_{x_0},\ \mathrm{E}\left[(x(0) - m_{x_0})(x(0) - m_{x_0})^T\right] = X_0$$

- mean of state vector $x(k)$:

$$\boxed{m_x(k+1) = A(k)m_x(k)},\ m_x(0) = m_{x_0} \tag{8}$$

- covariance $X(k)=X_{xx}(k,k)$: (7)-(8)$\Rightarrow$

$$\boxed{X(k+1) = A(k)X(k)A^T(k) + B_w(k)W(k)B_w^T(k)},\ X(0) = X_o$$

- intuition: covariance is a "second-moment" statistical property

Filtering a random process in state space

dynamics of the mean:

$$\boxed{m_x(k+1) = A(k)\, m_x(k)},\ m_x(0) = m_{x_0}$$

dynamics of the covariance:

$$\boxed{X(k+1) = A(k)\, X(k)\, A^T(k) + B_w(k)\, W(k)\, B_w^T(k)},\ X(0) = X_o$$

- (steady state) if $A(k) = A$ and is stable, $B_w(k) = B_w$, and $w(k)$ is stationary $W(k) = W$, then

$$m_x(k) \to 0,\ X(k) \to \text{a steady state } X_{ss}$$

$$X_{ss} = AX_{ss}A^T + B_w W B_w^T:\ \text{discrete-time Lyapunov Eq.} \qquad (9)$$

$$X_{ss}(l) = E\left[x(k)\, x^T(k+l)\right] = X_{ss}\left(A^T\right)^l$$

$$X_{ss}(-l) = X_{ss}(l)^T = A^l X_{ss}$$

Filtering a random process in state space

Example

first-order system

$$x(k+1) = ax(x) + \sqrt{1-a^2}w(k),\ \mathrm{E}[w(k)] = 0,\ \mathrm{E}[w(k)w(j)] = W\delta_{kj}$$

with $|a| < 1$ and $x(0)$ uncorrelated with $w(k)$.
steady-state variance equation (9) becomes

$$X_{ss} = a^2 X_{ss} + \left(1-a^2\right) W \Rightarrow X_{ss} = W$$

and

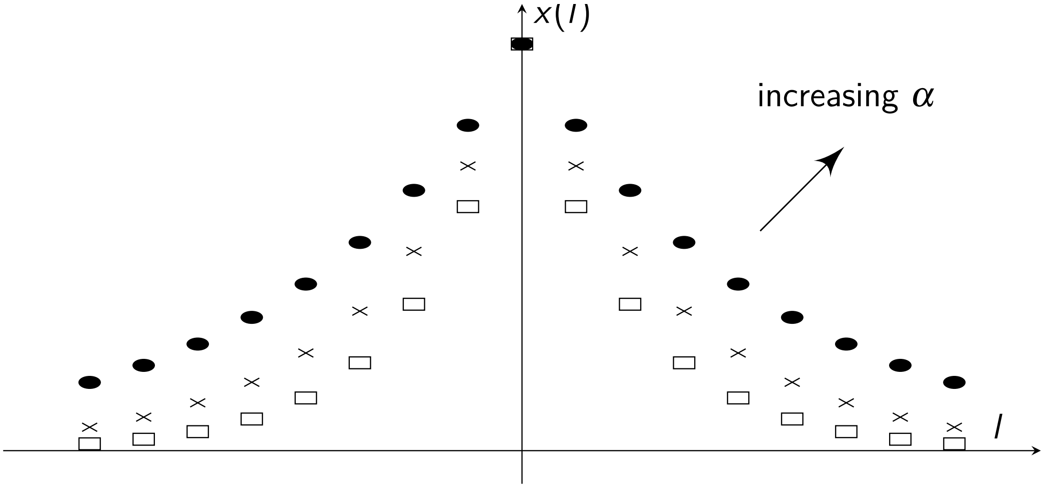
$$X(l) = X(-l) = a^l X_{ss} = a^l W$$

Filtering a random process in state space

Example

$$x(k+1) = ax(x) + \sqrt{1-a^2}w(k),\ \mathrm{E}[w(k)] = 0,\ \mathrm{E}[w(k)w(j)] = W\delta_{kj}$$

$$X(l) = X(-l) = a^l X_{ss} = a^l W$$



Filtering a random process

continuous-time case

similar results hold in the continuous-time case:

$$u(t) \longrightarrow \boxed{G(s)} \longrightarrow y(t)$$

- spectral density (SISO case):

$$\Phi_{yy}(j\omega) = G(j\omega)\,G(-j\omega)\,\Phi_{uu}(j\omega) = |G(j\omega)|^2\,\Phi_{uu}(j\omega)$$

- mean and covariance dynamics:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + B_w w(t),\ \mathrm{E}[w(t)] = 0,\ \mathrm{Cov}[w(t)] = W$$

$$\frac{\mathrm{d}m_x(t)}{\mathrm{d}t} = Am_x(t),\ m_x(0) = m_{x_0}$$

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = AX + XA^T + B_w W B_w^T$$

- steady state: $X_{ss}(\tau) = X_{ss} e^{A^T\tau};\ X_{ss}(-\tau) = e^{A\tau} X_{ss}$ where

$$AX_{ss} + X_{ss}A^T = -B_w W B_w^T:\ \text{continuout-time Lyapunov Eq.}$$

Appendix: Lyapunov equations

- discrete-time case:
$$A^T PA - P = -Q$$
has the following unique solution iff $\lambda_i(A)\lambda_j(A) \neq 1$ for all $i, j = 1, \ldots, n$:
$$P = \sum_{k=0}^{\infty} \left(A^T\right)^k QA^k$$

- continuous-time case:
$$A^T P + PA = -Q$$
has the following unique solution iff $\lambda_i(A) + \bar{\lambda}_j(A) \neq 0$ for all $i, j = 1, \ldots, n$:
$$P = \int_0^{\infty} e^{A^T t} Q e^{At} \mathrm{d}t$$

Summary

1. Big picture

2. Basic concepts: sample space, events, probability axioms, random variable, pdf, cdf, probability distributions

3. Multiple random variables

   random vector, joint probability and distribution, conditional probability

   Gaussian case

4. Random process

Lecture 4: Least Squares (LS) Estimation

Background and general solution
Solution in the Gaussian case
Properties
Example

Big picture

general least squares estimation:

- given: jointly distributed x (n-dimensional) & y (m-dimensional)
- goal: find the optimal estimate $\hat{x}$ that minimizes

$$\mathrm{E}\left[||x-\hat{x}||^2 \middle| y=y_1 \right] = \mathrm{E}\left[(x-\hat{x})^T (x-\hat{x}) \middle| y=y_1 \right]$$

- *solution*: consider

$$J(z) = \mathrm{E}\left[||x-z||^2 \middle| y=y_1 \right] = \mathrm{E}\left[x^T x \middle| y=y_1 \right] - 2z^T \mathrm{E}\left[x \middle| y=y_1 \right] + z^T z$$

which is quadratic in z. For optimal cost,

$$\frac{\partial}{\partial z} J(z) = 0 \Rightarrow z = \mathrm{E}\left[x|y=y_1 \right] \triangleq \hat{x}$$

hence

$$\boxed{\hat{x} = \mathrm{E}\left[x|y=y_1 \right] = \int_{-\infty}^{\infty} x p_{x|y}(x|y_1)\,\mathrm{d}x}$$

$$\boxed{J_{\min} = J(\hat{x}) = \mathrm{Tr}\left\{ \mathrm{E}\left[(x-\hat{x})(x-\hat{x})^T |y=y_1 \right] \right\}}$$

Big picture

general least squares estimation:

$$\boxed{\hat{x} = \mathrm{E}\left[x|y = y_1\right] = \int_{-\infty}^{\infty} x p_{x|y}\left(x|y_1\right) \mathrm{d}x}$$

achieves the minimization of

$$\mathrm{E}\left[\,||x - \hat{x}||^2 \middle|\, y = y_1\right]$$

solution concepts:

- the solution holds for any probability distribution in y
- for each y_1, $\mathrm{E}\left[x|y = y_1\right]$ is different
- if no specific value of y is given, $\hat{x}$ is a function of the random vector/variable y, written as

$$\hat{x} = \mathrm{E}\left[x|y\right]$$

Least square estimation in the Gaussian case

Why Gaussian?

- Gaussian is common in practice:
  - macroscopic random phenomena = superposition of microscopic random effects (Central limit theorem)
- Gaussian distribution has nice properties that make it mathematically feasible to solve many practical problems:
  - pdf is solely determined by the mean and the variance/covariance
  - linear functions of a Gaussian random process are still Gaussian
  - the output of an LTI system is a Gaussian random process if the input is Gaussian
  - if two jointly Gaussian distributed random variables are uncorrelated, then they are independent
  - X_1 and X_2 jointly Gaussian$\Rightarrow X_1|X_2$ and $X_2|X_1$ are also Gaussian

Least square estimation in the Gaussian case

Why Gaussian?

Gaussian and white:

- they are different concepts
- there can be Gaussian white noise, Poisson white noise, etc
- Gaussian white noise is used a lot since it is a good approximation to many practical noises

Least square estimation in the Gaussian case

the solution

problem (re-stated): x, y–Gaussian distributed

$$\text{minimize } \mathrm{E}\left[||x-\hat{x}||^2 \middle| y \right]$$

solution: $\boxed{\hat{x} = \mathrm{E}[x|y] = \mathrm{E}[x] + X_{xy}X_{yy}^{-1}(y - \mathrm{E}[y])}$

properties:

- the estimation is unbiased: $\boxed{\mathrm{E}[\hat{x}] = \mathrm{E}[x]}$
- y is Gaussian$\Rightarrow\hat{x}$ is Gaussian; and $x - \hat{x}$ is also Gaussian
- covariance of $\hat{x}$:

$$\mathrm{E}\left[(\hat{x}-\mathrm{E}[\hat{x}])(\hat{x}-\mathrm{E}[\hat{x}])^T\right] = \mathrm{E}\left\{(y-\mathrm{E}[y])\left[X_{xy}X_{yy}^{-1}(y-\mathrm{E}[y])\right]^T\right\} = X_{xy}X_{yy}^{-1}X_{yx}$$

- estimation error $\tilde{x} \triangleq x - \hat{x}$: zero mean and

$$\mathrm{Cov}[\tilde{x}] = \underbrace{\mathrm{E}\left[(x-\mathrm{E}[x|y])(x-\mathrm{E}[x|y])^T\right]}_{\text{conditional covariance}} = X_{xx} - X_{xy}X_{yy}^{-1}X_{yx}$$

Least square estimation in the Gaussian case

$$\boxed{\hat{x} = \mathrm{E}[x|y] = \mathrm{E}[x] + X_{xy}X_{yy}^{-1}(y - \mathrm{E}[y])}$$

$\mathrm{E}[x|y]$ is a better estimate than $\mathrm{E}[x]$:

- the estimation is unbiased: $\mathrm{E}[\hat{x}] = \mathrm{E}[x]$
- estimation error $\tilde{x} \triangleq x - \hat{x}$: zero mean and

$$\mathrm{Cov}[x - \hat{x}] = X_{xx} - X_{xy}X_{yy}^{-1}X_{yx} \preceq \mathrm{Cov}[x - \mathrm{E}[X]]$$

Properties of least square estimate (Gaussian case)

two random vectors x and y

Property 1:
(i) the estimation error $\tilde{x} = x - \hat{x}$ is uncorrelated with y
(ii) $\tilde{x}$ and $\hat{x}$ are orthogonal:

$$\mathrm{E}\left[(x-\hat{x})^T\hat{x}\right] = 0$$

proof of (i):

$$\begin{aligned}\mathrm{E}\left[\tilde{x}(y-m_y)^T\right] &= \mathrm{E}\left[\left(x-\mathrm{E}[x]-X_{xy}X_{yy}^{-1}(y-m_y)\right)(y-m_y)^T\right] \\ &= X_{xy} - X_{xy}X_{yy}^{-1}X_{yy} = 0\end{aligned}$$

Properties of least square estimate (Gaussian case)

two random vectors x and y

proof of (ii): $\mathrm{E}\left[\tilde{x}^T\hat{x}\right] = \mathrm{E}\left[(x-\hat{x})^T\left(\mathrm{E}[x] + X_{xy}X_{yy}^{-1}(y-m_y)\right)\right] =$
$\mathrm{E}\left[\tilde{x}^T\right]\mathrm{E}[x] + \mathrm{E}\left[(x-\hat{x})^T X_{xy}X_{yy}^{-1}(y-m_y)\right]$ where $\mathrm{E}\left[\tilde{x}^T\right] = 0$ and

$$\begin{aligned}
\mathrm{E}\left[(x-\hat{x})^T X_{xy}X_{yy}^{-1}(y-m_y)\right] &= \mathrm{Tr}\left\{\mathrm{E}\left[X_{xy}X_{yy}^{-1}(y-m_y)(x-\hat{x})^T\right]\right\} \\
&= \mathrm{Tr}\left\{X_{xy}X_{yy}^{-1}\,\mathrm{E}\left[(y-m_y)(x-\hat{x})^T\right]\right\} = 0 \text{ because of (i)}
\end{aligned}$$

- note: $\mathrm{Tr}\{BA\} = \mathrm{Tr}\{AB\}$. Consider, e.g. $A = [a, b]$, $B = \begin{bmatrix} c \\ d \end{bmatrix}$

Properties of least square estimate (Gaussian case)

two random vectors x and y

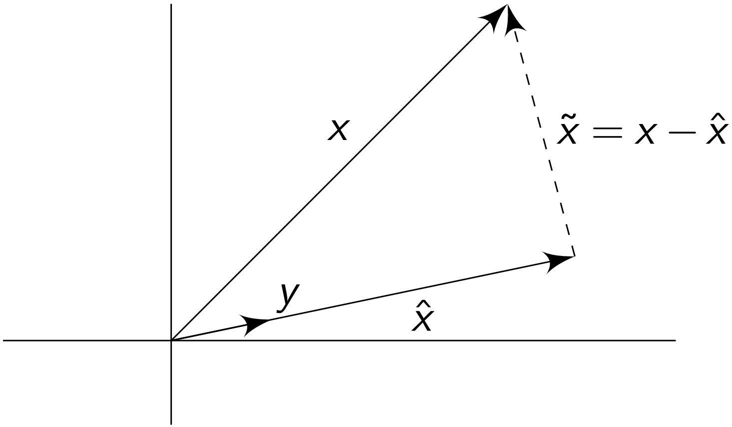
Property 1 (re-stated):

(i) the estimation error $\tilde{x} = x - \hat{x}$ is uncorrelated with y

(ii) $\tilde{x}$ and $\hat{x}$ are orthogonal:

$$\mathrm{E}\left[(x - \hat{x})^T \hat{x}\right] = 0$$

- intuition: least square estimation is a projection



Properties of least square estimate (Gaussian case)

three random vectors x y and z, where y and z are uncorrelated

Property 2: let y and z be Gaussian and uncorrelated, then

(i) the optimal estimate of x is

$$\begin{aligned} \mathrm{E}[x|y,z] &= \mathrm{E}[x] + \overbrace{(\mathrm{E}[x|y] - \mathrm{E}[x])}^{\text{first improvement}} + \overbrace{(\mathrm{E}[x|z] - \mathrm{E}[x])}^{\text{second improvement}} \\ &= \mathrm{E}[x|y] + (\mathrm{E}[x|z] - \mathrm{E}[x]) \end{aligned}$$

Alternatively, let $\boxed{\hat{x}_{|y} \triangleq \mathrm{E}[x|y]}$, $\boxed{\tilde{x}_{|y} \triangleq x - \mathrm{E}[x|y] = x - \hat{x}_{|y}}$, then

$$\boxed{\mathrm{E}[x|y,z] = \mathrm{E}[x|y] + \mathrm{E}\left[\tilde{x}_{|y}|z\right]}$$

(ii) the estimation error covariance is

$$X_{xx} - X_{xy}X_{yy}^{-1}X_{yx} - X_{xz}X_{zz}^{-1}X_{zx} = X_{\tilde{x}\tilde{x}} - X_{xz}X_{zz}^{-1}X_{zx} = \underline{X_{\tilde{x}\tilde{x}} - X_{\tilde{x}z}X_{zz}^{-1}X_{z\tilde{x}}}$$

where $X_{\tilde{x}\tilde{x}} = \mathrm{E}\left[\tilde{x}_{|y}\tilde{x}_{|y}^T\right]$ and $X_{\tilde{x}z} = \mathrm{E}\left[\tilde{x}_{|y}(z - m_z)^T\right]$

Properties of least square estimate (Gaussian case)

three random vectors x y and z, where y and z are uncorrelated

proof of (i): let $w = [y, z]^T$

$$\mathrm{E}[x|w] = \mathrm{E}[x] + \begin{bmatrix} X_{xy} & X_{xz} \end{bmatrix} \begin{bmatrix} X_{yy} & X_{yz} \\ X_{zy} & X_{zz} \end{bmatrix}^{-1} \begin{bmatrix} y - \mathrm{E}[y] \\ z - \mathrm{E}[z] \end{bmatrix}$$

Using $X_{yz} = 0$ yields

$$\begin{aligned} \mathrm{E}[x|w] &= \mathrm{E}[x] + \underbrace{X_{xy} X_{yy}^{-1} (y - \mathrm{E}[y])}_{\mathrm{E}[x|y]-\mathrm{E}[x]} + \underbrace{X_{xz} X_{zz}^{-1} (z - \mathrm{E}[z])}_{\mathrm{E}[x|z]-\mathrm{E}[x]} \\ &= \mathrm{E}[x|y] + \mathrm{E}\left[\left(\hat{x}_{|y} + \tilde{x}_{|y}\right)|z\right] - \mathrm{E}[x] \\ &= \mathrm{E}[x|y] + \mathrm{E}\left[\tilde{x}_{|y}|z\right] \end{aligned}$$

where $\mathrm{E}\left[\hat{x}_{|y}|z\right] = \mathrm{E}[\mathrm{E}[x|y]|z] = \mathrm{E}[x]$ as y and z are independent

Properties of least square estimate (Gaussian case)

three random vectors x y and z, where y and z are uncorrelated

proof of (ii): let $w = [y, z]^T$, the estimation error covariance is

$$X_{xx} - X_{xw} X_{ww}^{-1} X_{wx} = X_{xx} - X_{xy} X_{yy}^{-1} X_{yx} - X_{xz} X_{zz}^{-1} X_{zx}$$

additionally

$$\begin{aligned} X_{xz} &= \mathrm{E}\left[\left(\underline{x} - \mathrm{E}[x]\right)\left(z - \mathrm{E}[z]\right)^T\right] = \mathrm{E}\left[\left(\underline{\hat{x}_{|y} + \tilde{x}_{|y}} - \mathrm{E}[x]\right)\left(z - \mathrm{E}[z]\right)^T\right] \\ &= \mathrm{E}\left[\left(\hat{x}_{|y} - \mathrm{E}[x]\right)\left(z - \mathrm{E}[z]\right)^T\right] + \mathrm{E}\left[\tilde{x}_{|y}\left(z - \mathrm{E}[z]\right)^T\right] \end{aligned}$$

but $\hat{x}_{|y} - \mathrm{E}[x]$ is a linear function of y, which is uncorrelated with z, hence $\mathrm{E}\left[\left(\hat{x}_{|y} - \mathrm{E}[x]\right)\left(z - \mathrm{E}[z]\right)^T\right] = 0$ and $X_{xz} = X_{\tilde{x}_{|y} z}$

Properties of least square estimate (Gaussian case)

three random vectors x y and z, where y and z are uncorrelated

Property 2 (re-stated): let y and z be Gaussian and uncorrelated

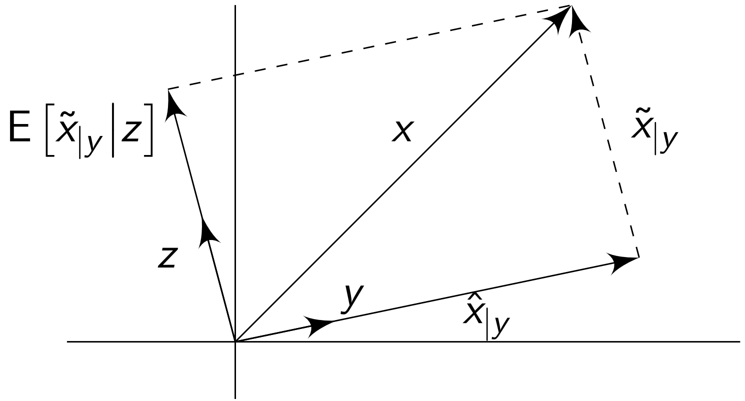
(i) the optimal estimate of x is

$$\mathrm{E}\left[x|y,z\right] = \mathrm{E}\left[x|y\right] + \mathrm{E}\left[\tilde{x}_{|y}|z\right]$$

(ii) the estimation error covariance is

$$X_{\tilde{x}\tilde{x}} - X_{\tilde{x}z}X_{zz}^{-1}X_{z\tilde{x}}$$

- intuition:



Properties of least square estimate (Gaussian case)

three random vectors x y and z, where y and z are correlated

Property 3: let y and z be Gaussian and correlated, then

(i) the optimal estimate of x is

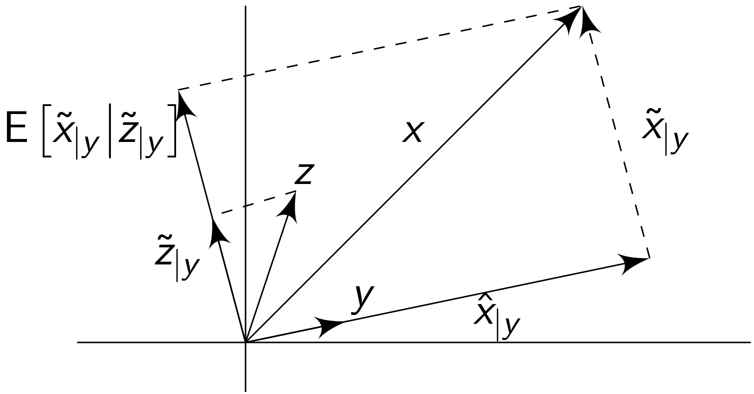
$$\mathrm{E}\left[x|y,z\right] = \mathrm{E}\left[x|y\right] + \mathrm{E}\left[\tilde{x}_{|y}|\tilde{z}_{|y}\right]$$

where $\tilde{z}_{|y} = z - \hat{z}_{|y} = z - \mathrm{E}\left[z|y\right]$ and $\tilde{x}_{|y} = x - \hat{x}_{|y} = x - \mathrm{E}\left[x|y\right]$

(ii) the estimation error covariance is

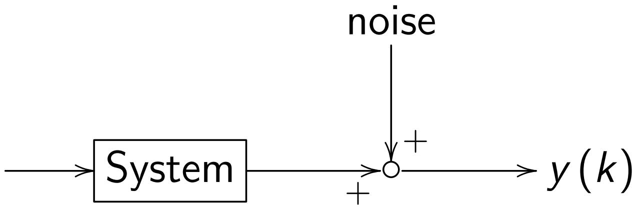
$$X_{\tilde{x}_{|y}\tilde{x}_{|y}} - X_{\tilde{x}_{|y}\tilde{z}_{|y}} X^{-1}_{\tilde{z}_{|y}\tilde{z}_{|y}} X_{\tilde{z}_{|y}\tilde{x}_{|y}}$$

- intuition:



Application of the three properties

Consider



Given $[y(0), y(1), \ldots, y(k)]^T$, we want to estimate the state $x(k)$

- the properties give a recursive way to compute

$$\hat{x}(k) \,|\, \{y(0), y(1), \ldots, y(k)\}$$

Example

Consider estimating the velocity x of a motor, with

$$\begin{aligned} \mathrm{E}\left[x\right] = m_x = 10 \text{ rad/s} \\ \mathrm{Var}\left[x\right] = 2 \text{ rad}^2/\text{s}^2 \end{aligned}$$

There are two (tachometer) sensors available:

- $y_1 = x + v_1$: $\mathrm{E}\left[v_1\right] = 0$, $\mathrm{E}\left[v_1^2\right] = 1 \text{ rad}^2/\text{s}^2$
- $y_2 = x + v_2$: $\mathrm{E}\left[v_2\right] = 0$, $\mathrm{E}\left[v_2^2\right] = 1 \text{ rad}^2/\text{s}^2$

where v_1 and v_2 are independent, Gaussian, $\mathrm{E}\left[v_1 v_2\right] = 0$ and x is independent of v_i, $\mathrm{E}\left[\left(x - \mathrm{E}\left[x\right]\right) v_i\right] = 0$

Example

- best estimate of x using only y_1:

$$\begin{aligned} X_{xy_1} &= \mathrm{E}\left[(x - m_x)(y_1 - m_{y_1})\right] = \mathrm{E}\left[(x - m_x)(x - m_x + v_1)\right] \\ &= X_{xx} + \mathrm{E}\left[(x - m_x)\, v_1\right] = 2 \end{aligned}$$

$$\begin{aligned} X_{y_1 y_1} &= \mathrm{E}\left[(y_1 - m_{y_1})(y_1 - m_{y_1})\right] = \mathrm{E}\left[(x - m_x + v_1)(x - m_x + v_1)\right] \\ &= X_{xx} + \mathrm{E}\left[v_1^2\right] = 3 \end{aligned}$$

$$\hat{x}_{|y_1} = \mathrm{E}[x] + X_{xy_1} X_{y_1 y_1}^{-1} (y_1 - \mathrm{E}[y_1]) = 10 + \frac{2}{3}(y_1 - 10)$$

- similarly, best estimate of x using only y_2: $\hat{x}_{|y_2} = 10 + \frac{2}{3}(y_2 - 10)$

Example

- best estimate of x using y_1 and y_2 (direct approach): let $y = [y_1, y_2]^T$

$$X_{xy} = \mathrm{E}\left[(x - m_x)\begin{bmatrix} y_1 - m_{y_1} \\ y_2 - m_{y_2} \end{bmatrix}^T\right] = [2, 2]$$

$$X_{yy} = \mathrm{E}\left[\begin{bmatrix} y_1 - m_{y_1} \\ y_2 - m_{y_2} \end{bmatrix}\begin{bmatrix} y_1 - m_{y_1} & y_2 - m_{y_2} \end{bmatrix}\right] = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

$$\hat{x}_{|y} = \mathrm{E}[x] + X_{xy}X_{yy}^{-1}(y - m_y) = 10 + [2, 2]\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}^{-1}\begin{bmatrix} y_1 - 10 \\ y_2 - 10 \end{bmatrix}$$

- note: X_{yy}^{-1} is expensive to compute at high dimensions

Example

- best estimate of x using y_1 and y_2 (alternative approach using Property 3):

$$\mathrm{E}\left[x|y_1,y_2\right] = \mathrm{E}\left[x|y_1\right] + \mathrm{E}\left[\tilde{x}_{|y_1}|\tilde{y}_{2|y_1}\right]$$

which involves just the scalar computations:

$$\mathrm{E}\left[x|y_1\right] = 10 + \frac{2}{3}(y_1 - 10),\ \tilde{x}_{|y_1} = x - \mathrm{E}\left[x|y_1\right] = \frac{1}{3}(x-10) + \frac{2}{3}v_1$$

$$\tilde{y}_{2|y_1} = y_2 - \mathrm{E}\left[y_2|y_1\right] = y_2 - \left[\mathrm{E}\left[y_2\right] + X_{y_2 y_1}\frac{1}{X_{y_1 y_1}}(y_1 - m_{y_1})\right] = (y_2 - 10) - \frac{2}{3}(y_1 - 10)$$

$$X_{\tilde{x}_{|y_1}\tilde{y}_{2|y_1}} = \mathrm{E}\left[\left(\frac{1}{3}(x-10) + \frac{2}{3}v_1\right)\left((y_2-10) - \frac{2}{3}(y_1-10)\right)^T\right] = \frac{1}{9}\mathrm{Var}\left[x\right] + \frac{4}{9}\mathrm{Var}\left[v_1\right] = \frac{2}{3}$$

$$X_{\tilde{y}_{2|y_1}\tilde{y}_{2|y_1}} = \frac{1}{9}\mathrm{Var}\left[x\right] + \mathrm{Var}\left[v_2\right] + \frac{4}{9}\mathrm{Var}\left[v_1\right] = \frac{5}{3}$$

$$\begin{aligned}\mathrm{E}\left[\tilde{x}_{|y_1}|\tilde{y}_{2|y_1}\right] &= \mathrm{E}\left[\tilde{x}_{|y_1}\right] + X_{\tilde{x}_{|y_1}\tilde{y}_{2|y_1}}\frac{1}{X_{\tilde{y}_{2|y_1}\tilde{y}_{2|y_1}}}\left[\tilde{y}_{2|y_1} - \mathrm{E}\left[\tilde{y}_{2|y_1}\right]\right] \\ &= 10 + \frac{2}{5}(y_1 - 10) + \frac{2}{5}(y_2 - 10)\end{aligned}$$

Summary

1. Big picture

   $\hat{x} = \mathrm{E}[x|y]$ minimizes $J = \mathrm{E}\left[||x - \hat{x}||^2 \middle| y \right]$

2. Solution in the Gaussian case

   Why Gaussian?

   $\hat{x} = \mathrm{E}[x|y] = \mathrm{E}[x] + X_{xy} X_{yy}^{-1} (y - \mathrm{E}[y])$

3. Properties of least square estimate (Gaussian case)

   two random vectors x and y

   three random vectors x y and z: y and z are uncorrelated

   three random vectors x y and z: y and z are correlated

* Appendix: trace of a matrix

- the trace of a $n \times n$ matrix is given by $\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii}$
- trace is the matrix inner product:

$$\langle A, B\rangle = \mathrm{Tr}\left(A^T B\right) = \mathrm{Tr}\left(B^T A\right) = \langle B, A\rangle \tag{1}$$

- take a three-column example: write the matrices in the column vector form $B = [b_1, b_2, b_3]$, $A = [a_1, a_2, a_3]$, then,

$$A^T B = \begin{bmatrix} a_1^T b_1 & * & * \\ * & a_2^T b_2 & * \\ * & * & a_3^T b_3 \end{bmatrix} \tag{2}$$

$$\mathrm{Tr}\left(A^T B\right) = a_1^T b_1 + a_2^T b_2 + a_3^T b_3 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}^T \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{3}$$

  which is the inner product of the two long stacked vectors.
- we frequently use the inner-product equality $\langle A, B\rangle = \langle B, A\rangle$

Lecture 5: Stochastic State Estimation (Kalman Filter)

Big picture

why are we learning this?

- state estimation in deterministic case:

$$\text{Plant: } x(k+1) = Ax(k) + Bu(k),\ y(k) = Cx(k)$$
$$\text{Observer: } \hat{x}(k+1) = A\hat{x}(k) + Bu(k) + L(y(k) - C\hat{x}(k))$$

- L designed based on the error $(e(k) = x(k) - \hat{x}(k))$ dynamics:

$$e(k+1) = (A - LC)e(k) \tag{1}$$

  to reach fast convergence of $\lim_{k\to\infty} e(k) = 0$

- L is not optimal when there is noise in the plant; actually $\lim_{k\to\infty} e(k) = 0$ isn't even a valid goal when there is noise
- Kalman Filter provides optimal state estimation under input and output noises

Problem statement

plant:
$$x(k+1) = A(k)x(k) + B(k)u(k) + B_w(k)w(k)$$
$$y(k) = C(k)x(k) + v(k)$$

- $w(k)$–s-dimensional input noise; $v(k)$–r-dimensional measurement noise; $x(0)$–unknown initial state
- assumptions: $x(0)$, $w(k)$, and $v(k)$ are independent and Gaussian distributed; $w(k)$ and $v(k)$ are white:

$$\mathrm{E}[x(0)] = x_o,\ \mathrm{E}\left[(x(0)-x_o)(x(0)-x_o)^T\right] = X_0$$
$$\mathrm{E}[w(k)] = 0,\ \mathrm{E}[v(k)] = 0,\ \mathrm{E}\left[w(k)v^T(j)\right] = 0\ \forall k,j$$
$$\mathrm{E}\left[w(k)w^T(j)\right] = W(k)\delta_{kj},\ \mathrm{E}\left[v(k)v^T(j)\right] = V(k)\delta_{kj}$$

Problem statement

- goal:

$$\text{minimize } \mathrm{E}\left[\left. ||x(k) - \hat{x}(k)||^2 \right|_{Y_j} \right], \;\; Y_j = \{y(0), y(1), \ldots, y(j)\}$$

- solution:

$$\hat{x}(k) = \mathrm{E}\left[x(k) \,|\, Y_j\right]$$

- three classes of problems:
  - $k > j$: prediction problem
  - $k = j$: filtering problem
  - $k < j$: smoothing problem

History

Rudolf Kalman:

- obtained B.S. in 1953 and M.S. in 1954 from MIT, and Ph.D. in 1957 from Columbia University, all in Electrical Engineering
- developed and implemented Kalman Filter in 1960, during the Apollo program, and furthermore in various famous programs including the NASA Space Shuttle, Navy submarines, etc.
- was awarded the National Medal of Science on Oct. 7, 2009 from U.S. president Barack Obama

Useful facts

assume x is Gaussian distributed

- if $y = Ax + B$ then

$$\begin{cases} X_{xy} = \mathrm{E}\left[(x - \mathrm{E}[x])(y - \mathrm{E}[y])^T\right] & = X_{xx}A^T \\ X_{yy} = \mathrm{E}\left[(y - \mathrm{E}[y])(y - \mathrm{E}[y])^T\right] & = AX_{xx}A^T \end{cases} \tag{2}$$

- if $y = Ax + B$ and $y' = A'x + B'$ then

$$X_{yy'} = AX_{xx}\left(A'\right)^T,\ X_{y'y} = A'X_{xx}A^T \tag{3}$$

- if $y = Ax + Bv$; v is Gaussian and independent of x, then

$$X_{yy} = AX_{xx}A^T + BX_{vv}B^T \tag{4}$$

- if $y = Ax + Bv$, $y' = A'x + B'v$; v is Gaussian and dependent of x, then

$$X_{yy'} = AX_{xx}\left(A'\right)^T + AX_{xv}\left(B'\right)^T + BX_{vx}\left(A'\right)^T + BX_{vv}\left(B'\right)^T \tag{5}$$

Derivation of Kalman Filter

- goal:

$$\text{minimize } \mathrm{E}\left[||x(k) - \hat{x}(k)||^2 \Big|_{Y_k} \right], \quad Y_k = \{y(0), y(1), \dots, y(k)\}$$

- the best estimate is the conditional expectation

$$\begin{aligned} \mathrm{E}[x(k)|Y_k] &= \mathrm{E}[x(k)|\{Y_{k-1}, y(k)\}] \\ &= \mathrm{E}[x(k)|Y_{k-1}] + \mathrm{E}\left[\tilde{x}(k)|_{Y_{k-1}} \middle| \tilde{y}(k)|_{Y_{k-1}} \right] \end{aligned}$$

- introduce some notations:

$$\text{a priori estimation } \hat{x}(k|k-1) = \mathrm{E}[x(k)|Y_{k-1}] = \hat{x}(k)|_{y(0),\dots y(k-1)}$$

$$\text{a posteriori estimation } \hat{x}(k|k) = \mathrm{E}[x(k)|Y_k] = \hat{x}(k)|_{y(0),\dots y(k)}$$

$$\text{a priori covariance } M(k) = \mathrm{E}\left[\tilde{x}(k)|_{Y_{k-1}} \tilde{x}^T(k)|_{Y_{k-1}} \right]$$

$$\text{a posteriori covariance } Z(k) = \mathrm{E}\left[\tilde{x}(k)|_{Y_k} \tilde{x}^T(k)|_{Y_k} \right]$$

Derivation of Kalman Filter

KF gain update

to get $\mathrm{E}\left[\tilde{x}(k)|_{Y_{k-1}}\middle|\tilde{y}(k)|_{Y_{k-1}}\right]$ in

$$\mathrm{E}[x(k)|Y_k] = \mathrm{E}[x(k)|Y_{k-1}] + \mathrm{E}\left[\tilde{x}(k)|_{Y_{k-1}}\middle|\tilde{y}(k)|_{Y_{k-1}}\right]$$

we need $X_{\tilde{x}(k)|_{Y_{k-1}}\tilde{y}(k)|_{Y_{k-1}}}$ and $X^{-1}_{\tilde{y}(k)|_{Y_{k-1}}\tilde{y}(k)|_{Y_{k-1}}}$

$y(k) = C(k)x(k) + v(k)$ gives

$$\hat{y}(k)|_{Y_{k-1}} = C(k)\hat{x}(k|k-1) + \hat{v}(k)|_{Y_{k-1}} = C(k)\hat{x}(k|k-1)$$
$$\Rightarrow \tilde{y}(k)|_{Y_{k-1}} = C(k)\tilde{x}(k|k-1) + v(k)$$

hence

$$X_{\tilde{x}(k)|_{Y_{k-1}}\tilde{y}(k)|_{Y_{k-1}}} = M(k)C^T(k) \tag{6}$$
$$X_{\tilde{y}(k)|_{Y_{k-1}}\tilde{y}(k)|_{Y_{k-1}}} = C(k)M(k)C^T(k) + V(k) \tag{7}$$

Derivation of Kalman Filter

KF gain update

$$\tilde{y}(k)|_{Y_{k-1}} = C(k)\tilde{x}(k|k-1) + v(k)$$

unbiased estimation: $\mathrm{E}[\hat{x}(k|k-1)] = \mathrm{E}[x] \Rightarrow$

$$\mathrm{E}\left[\tilde{y}(k)|_{Y_{k-1}}\right] = \mathrm{E}\left[\tilde{x}(k)|_{Y_{k-1}}\right] + \mathrm{E}\left[v(k)|_{Y_{k-1}}\right] = 0$$

thus

$$\begin{aligned}
&\mathrm{E}\left[\tilde{x}(k)|_{Y_{k-1}} \middle| \tilde{y}(k)|_{Y_{k-1}}\right] \\
&= \cancelto{0}{\mathrm{E}\left[\tilde{x}(k)|_{Y_{k-1}}\right]} + X_{\tilde{x}(k)|_{Y_{k-1}}\tilde{y}(k)|_{Y_{k-1}}} X^{-1}_{\tilde{y}(k)|_{Y_{k-1}}\tilde{y}(k)|_{Y_{k-1}}} \left(\tilde{y}(k)|_{Y_{k-1}} - 0\right) \\
&= M(k)C^T(k)\left[C(k)M(k)C^T(k) + V(k)\right]^{-1}\left(y(k) - \hat{y}(k)|_{Y_{k-1}}\right)
\end{aligned}$$

Derivation of Kalman Filter

KF gain update

$$\mathrm{E}\left[x(k)\,|\,Y_k\right] = \mathrm{E}\left[x(k)\,|\,Y_{k-1}\right] + \mathrm{E}\left[\tilde{x}(k)|_{Y_{k-1}}\,\middle|\,\tilde{y}(k)|_{Y_{k-1}}\right]$$

now becomes

$$\hat{x}(k|k) = \hat{x}(k|k-1) + \underbrace{M(k)\,C^T\left(CM(k)\,C^T + V(k)\right)^{-1}}_{F(k)}\left(y(k) - C\hat{x}(k|k-1)\right)$$

namely

$$\boxed{\begin{cases} \hat{x}(k|k) & = \hat{x}(k|k-1) + F(k)\left(y(k) - C(k)\,\hat{x}(k|k-1)\right) \\ F(k) & = M(k)\,C^T(k)\left(C(k)\,M(k)\,C^T(k) + V(k)\right)^{-1} \end{cases}} \qquad (8)$$

Derivation of Kalman Filter

KF covariance update

now for the variance update:

$$
\begin{aligned}
\mathrm{E}\left[\tilde{x}(k)|_{Y_k}\tilde{x}(k)^T|_{Y_k}\right] &= \mathrm{E}\left[\tilde{x}(k)|_{\{Y_{k-1},y(k)\}}\tilde{x}(k)^T|_{\{Y_{k-1},y(k)\}}\right] \\
= \mathrm{E}&\left[\tilde{x}(k)|_{Y_{k-1}}\tilde{x}(k)^T|_{Y_{k-1}}\right] \\
&\quad - X_{\tilde{x}(k)|_{Y_{k-1}}\tilde{y}(k)|_{Y_{k-1}}} X^{-1}_{\tilde{y}(k)|_{Y_{k-1}}\tilde{y}(k)|_{Y_{k-1}}} X_{\tilde{y}(k)|_{Y_{k-1}}\tilde{x}(k)|_{Y_{k-1}}}
\end{aligned}
$$

or, using the introduced notations,

$$
\boxed{Z(k) = M(k) - M(k)\,C^T(k)\left(C(k)\,M(k)\,C^T(k) + V(k)\right)^{-1} C(k)\,M(k)}
$$

Derivation of Kalman Filter

KF covariance update

the connection between $Z(k)$ and $M(k)$:

$$
\begin{aligned}
x(k) &= A(k-1)x(k-1) + B(k-1)u(k-1) + B_w(k-1)w(k-1) \\
\Rightarrow \hat{x}(k|k-1) &= A(k-1)\hat{x}(k-1|k-1) + B(k-1)u(k-1) \\
\Rightarrow \tilde{x}(k|k-1) &= A(k-1)\tilde{x}(k-1|k-1) + B_w(k-1)w(k-1)
\end{aligned}
$$

thus $M(k) = \mathrm{Cov}[\tilde{x}(k|k-1)]$ is [using uesful fact (4)]

$$\boxed{M(k) = A(k-1)Z(k-1)A^T(k-1) + B_w(k-1)W(k-1)B_w^T(k-1)}$$

with $M(0) = \mathrm{E}\left[\tilde{x}(0|-1)\tilde{x}(0|-1)^T\right] = X_0$

The full set of KF equations

$$\hat{x}(k|k) = \hat{x}(k|k-1) + F(k)\overbrace{\left[y(k) - C(k)\hat{x}(k|k-1)\right]}^{e_y(k)}$$

$$\hat{x}(k|k-1) = A(k-1)\hat{x}(k-1|k-1) + B(k-1)u(k-1)$$

$$F(k) = M(k)C^T(k)\left[C(k)M(k)C^T(k) + V(k)\right]^{-1}$$

$$M(k) = A(k-1)Z(k-1)A^T(k-1) + B_w(k-1)W(k-1)B_w^T(k-1)$$

$$Z(k) = M(k) - M(k)C^T(k)\ldots$$

$$\times\left(C(k)M(k)C^T(k) + V(k)\right)^{-1}C(k)M(k)$$

with initial conditions $\hat{x}(0|-1) = x_o$ and $M(0) = X_0$.

The full set of KF equations

in a shifted index:

$$\hat{x}(k+1|k+1) = \hat{x}(k+1|k) + F(k+1)\left[y(k+1) - C(k+1)\hat{x}(k+1|k)\right]$$

$$\hat{x}(k+1|k) = A(k)\hat{x}(k|k) + B(k)u(k)$$

$$F(k+1) = M(k+1)C^T(k+1)\left[C(k+1)M(k+1)C^T(k+1) + V(k+1)\right]^{-1}$$

$$M(k+1) = A(k)Z(k)A^T(k) + B_w(k)W(k)B_w^T(k) \tag{9}$$

$$Z(k+1) = M(k+1) - M(k+1)C^T(k+1)\ldots \tag{10}$$

$$\times\left(C(k+1)M(k+1)C^T(k+1) + V(k+1)\right)^{-1} C(k+1)M(k+1)$$

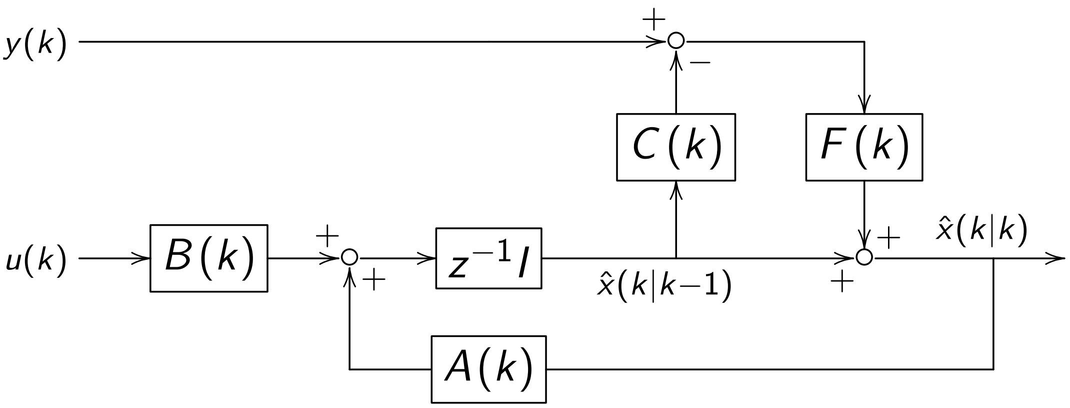
combining (9) and (10) gives the Riccati equation:

$$M(k+1) = A(k)M(k)A^T(k) + B_w(k)W(k)B_w^T(k)$$
$$-A(k)M(k)C^T(k)\left[C(k)M(k)C^T(k) + V(k)\right]^{-1} C(k)M(k)A^T(k) \tag{11}$$

The full set of KF equations

Several remarks

- $F(k)$, $M(k)$, and $Z(k)$ can be obtained offline first
- Kalman Filter (KF) is linear, and optimal for Gaussian. More advanced nonlinear estimation won't improve the results here.
- KF works for time-varying systems
- the block diagram of KF is:



Steady-state KF

assumptions:

- system is time-invariant: A, B, B_w, and C are constant;
- noise is stationary: $V \succ 0$ and $W \succ 0$ do not depend on time.

KF equations become:

$$\begin{aligned}
\hat{x}(k+1|k+1) &= \hat{x}(k+1|k) + F(k+1)\left[y(k+1) - C\hat{x}(k+1|k)\right] \\
&= A\hat{x}(k|k) + Bu(k) + F(k+1)\left[y(k+1) - C\hat{x}(k+1|k)\right] \\
F(k+1) &= M(k+1)C^T\left[CM(k+1)C^T + V\right]^{-1} \\
M(k+1) &= AZ(k)A^T + B_w W B_w^T; \quad M(0) = X_0 \\
Z(k+1) &= M(k+1) - M(k+1)C^T\left[CM(k+1)C^T + V\right]^{-1} CM(k+1)
\end{aligned}$$

with Riccati equation (RE):

$$M(k+1) = AM(k)A^T + B_w W B_w^T - AM(k)C^T\left[CM(k)C^T + V\right]^{-1} CM(k)A^T$$

Steady-state KF

if
- (A, C) is observable or detectable
- (A, B_w) is controllable (disturbable) or stabilizable

then $M(k)$ in the RE converges to some steady-state value M_s and KF can be implemented by

$$
\begin{aligned}
\hat{x}(k+1|k+1) &= \hat{x}(k+1|k) + F_s\left[y(k+1) - C\hat{x}(k+1|k)\right] \\
\hat{x}(k+1|k) &= A\hat{x}(k|k) + Bu(k) \\
F_s &= M_s C^T \left[C M_s C^T + V\right]^{-1}
\end{aligned}
$$

M_s is the positive definite solution of the algebraic Riccati equation:

$$
M_s = A M_s A^T + B_w W B_w^T - A M_s C^T \left[C M_s C^T + V\right]^{-1} C M_s A^T
$$

Duality with LQ

The steady-state condition is obtained by comparing the RE in LQ and KF

discrete-time LQ:

$$P(k) = A^T P(k+1)A - A^T P(k+1)B[R + B^T P(k+1)B]^{-1} B^T P(k+1)A + Q$$

discrete-time KF (11):

$$M(k+1) = AM(k)A^T - AM(k)C^T\left[CM(k)C^T + V\right]^{-1} CM(k)A^T + B_w W B_w^T$$

discrete-time LQ	discrete-time KF
A	A^T
B	C^T
C	B_w
R	V
$Q = C^T C$	$B_w W B_w^T$
P	M
backward recursion	forward recursion

Duality with LQ

discrete-time LQ	discrete-time KF
A	A^T
B	C^T
C	B_w
$Q = C^T C$	$B_w W B_w^T$

steady-state conditions for discrete-time LQ:

- (A, B) controllable or stabilizable
- (A, C) observable or detectable

steady-state conditions for discrete-time KF:

- $\left(A^T, C^T\right)$ controllable or stabilizable$\Leftrightarrow$$(A, C)$ observable or detectable
- $\left(A^T, B_w^T\right)$ observable or detectable$\Leftrightarrow$$(A, B_w)$ controllable or stabilizable

Duality with LQ

discrete-time LQ	discrete-time KF
A	A^T
B	C^T
C	B_w
R	V
$Q = C^T C$	$B_w W B_w^T$
P	M
backward recursion	forward recursion

- LQ: stable closed-loop "A" matrix is

$$A - BK_s = \underline{A - B[R + B^T P_s B]^{-1} B^T P_s A}$$

- KF: stable KF "A" matrix is

$$\begin{aligned}\hat{x}(k+1|k) &= A\hat{x}(k|k) + Bu(k)\\ &= A\hat{x}(k|k-1) + AF_s[y(k) - C\hat{x}(x|k-1)] + Bu(k)\\ &= \underline{\left[A - AM_s C^T\left(CM_s C^T + V\right)^{-1} C\right]}\hat{x}(k|k-1) + \dots\end{aligned}$$

Purpose of each condition

- (A, C) observable or detectable: assures the existence of the steady-state Riccati solution
- (A, B_w) controllable or stabilizable: assures that the steady-state solution is positive definite and that the KF dynamics is stable

Remark

- KF: stable KF "A" matrix is

$$\hat{x}(k+1|k) = \left[A - AM_sC^T\left(CM_sC^T + V\right)^{-1}C\right]\hat{x}(k|k-1) + \dots$$
$$= \underline{(A - AF_sC)}\hat{x}(k|k-1) + \dots$$

in the form of $\hat{x}(k|k)$ dynamics:

$$\hat{x}(k+1|k+1) = \hat{x}(k+1|k) + F_s\left[y(k+1) - C\hat{x}(k+1|k)\right]$$
$$= \underline{(A - F_sCA)}\hat{x}(k|k) + (I - F_sC)Bu(k) + F_sy(k+1)$$
$$= \left[A - M_sC^T\left(CM_sC^T + V\right)^{-1}CA\right]\hat{x}(k|k) + \dots$$

- can show that $eig(A - AF_sC) = eig(A - F_sCA)$

hint: $\det(I + MN) = \det(I + NM) \Rightarrow \det\left[I - z^{-1}A(I - F_sC)\right] = \det\left[I - (I - F_sC)z^{-1}A\right]$

Remark

intuition of guaranteed KF stability: ARE $\Rightarrow$ Lyapunov equation

$$
\begin{aligned}
M_s &= AM_sA^T + B_wWB_w^T - AM_sC^T\left[CM_sC^T+V\right]^{-1}CM_sA^T \\
&= AM_sA^T + B_wWB_w^T - A\underbrace{M_sC^T\left[CM_sC^T+V\right]^{-1}}_{F_s}\left[CM_sC^T+V\right]\underbrace{\left[CM_sC^T+V\right]^{-1}CM_s}_{F_s^T}A^T \\
&= (A-AF_sC)\,M_s\,(A-AF_sC)^T + 2AF_sCM_sA^T - AF_sCM_sC^TF_s^TA^T \\
&\qquad + B_wWB_w^T - AF_s\left[CM_sC^T+V\right]F_s^TA^T \\
&= (A-AF_sC)\,M_s\,(A-AF_sC)^T + AF_sVF_s^TA^T + B_wWB_w^T
\end{aligned}
$$

$$
\Longleftrightarrow (A-AF_sC)\,M_s\,(A-AF_sC)^T - M_s = -AF_sVF_s^TA^T - B_wWB_w^T
$$

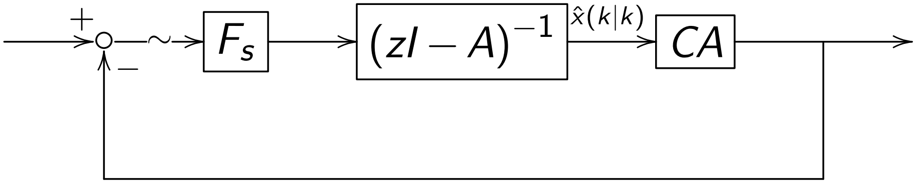
which is a Lyapunov equation with the right hand side being negative semidefinite and $M_s \succ 0$.

Return difference equation

KF dynamics

$$\begin{aligned}\hat{x}(k+1|k+1) &= \underline{(A - F_s CA)}\hat{x}(k|k) + (I - F_s C)\,Bu(k) + F_s y(k+1)\\ &= A\hat{x}(k|k) - F_s CA\hat{x}(k|k) + (I - F_s C)\,Bu(k) + F_s y(k+1)\end{aligned}$$

$$[zI - A]\,\hat{x}\,(k|k) = F_s y(k+1) + (I - F_s C)\,Bu(k) - F_s CA\hat{x}(k|k)$$



let $G(z) = C(zI - A)^{-1}B_w$
ARE $\Rightarrow$ return difference equation (RDE) (see ME232 reader)

$$\left[I + CA(zI - A)^{-1}F_s\right](V + CM_sC^T)\left[I + CA(z^{-1}I - A)^{-1}F_s\right]^T = V + G(z)WG^T(z^{-1})$$

Symmetric root locus for KF

- KF eigenvalues:

$$\det\left[I + CA\underline{(zI-A)^{-1}F_s}\right] = \det\left[I + \underline{(zI-A)^{-1}F_s}CA\right]$$
$$= \frac{\det\left(zI - A + F_sCA\right)}{\det\left(zI-A\right)} \triangleq \frac{\beta\left(z\right)}{\phi\left(z\right)}$$

- taking determinants in RDE gives

$$\beta(z)\beta(z^{-1}) = \phi(z)\phi(z^{-1})\frac{\det\left(V + G(z)WG^T(z^{-1})\right)}{\det\left(V + CMC^T\right)}$$

- single-output case: KF poles come from $\beta(z)\beta(z^{-1}) = 0$, i.e.

$$\det\left(V + G(z)WG^T(z^{-1})\right) = V\left(1 + G(z)\frac{W}{V}G^T(z^{-1})\right) = 0$$

- $W/V \to 0$: KF poles $\to$ stable poles of $G\left(z\right)G^T\left(z^{-1}\right)$
- $W/V \to \infty$: KF poles $\to$ stable zeros of $G\left(z\right)G^T\left(z^{-1}\right)$

Continuous-time KF

summary of solutions

system:

$$\dot{x}(t) = Ax(t) + Bu(t) + B_w w(t)$$
$$y(t) = Cx(t) + v(t)$$

assumptions: same as discrete-time KF

aim: minimize $J = \left. \|x(t) - \hat{x}(t)\|_2^2 \right|_{\{y(\tau):0\le\tau\le t\}}$

continuous-time KF:

$$\frac{\mathrm{d}\hat{x}(t|t)}{\mathrm{d}t} = A\hat{x}(t|t) + Bu(t) + F(t)\left[y(t) - C\hat{x}(t|t)\right], \; \hat{x}(0|0) = x_0$$

$$F(t) = M(t)C^T V^{-1}$$

$$\frac{\mathrm{d}M(t)}{\mathrm{d}t} = AM(t) + M(t)A^T + B_w W B_w^T - M(t)C^T V^{-1} CM(t), \; M(0) = X_0$$

Continuous-time KF: steady state

assumptions: (A, C) observable or detectable;
(A, B_w) controllable or stabilizable

asymptotically stable steady-state KF:

$$\frac{\mathrm{d}\hat{x}(t|t)}{\mathrm{d}t} = A\hat{x}(t|t) + Bu(t) + F_s[y(t) - C\hat{x}(t|t)]$$

$$F_s = M_s C^T V^{-1}$$

$$AM_s + M_s A^T + B_w W B_w^T - M_s C^T V^{-1} C M_s = 0$$

duality with LQ:

Continuous-Time LQ
$A^T P_s + P_s A + Q - P_s B R^{-1} B^T P_s = 0$
$K = R^{-1} B^T P_s$

Continuous-time KF: return difference equality

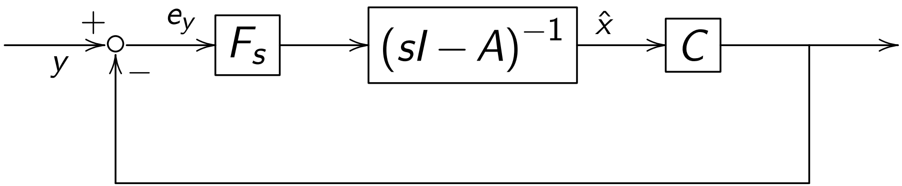
analogy to LQ gives the return difference equality:

$$\left[I + C(sI - A)^{-1}F_s\right] V \left[I + F_s^T(-sI - A)^{-T}C^T\right] = V + G(s)WG^T(-s)$$

where $G(s) = C(sI - A)^{-1}B_w$, hence:

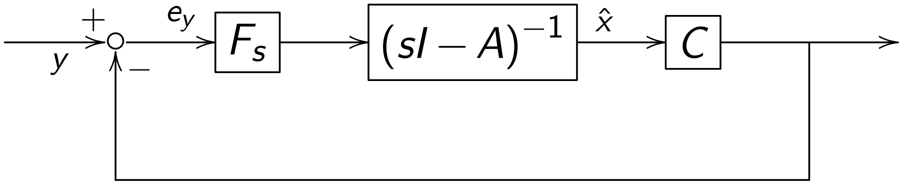
$$\left[I + C(j\omega I - A)^{-1}F_s\right] V \left[I + C(-j\omega I - A)^{-1}F_s\right]^T = V + G(j\omega)WG^T(-j\omega)$$

observation 1: $\frac{\mathrm{d}\hat{x}(t|t)}{\mathrm{d}t} = A\hat{x}(t|t) + Bu(t) + F_s\underbrace{[y(t) - C\hat{x}(t|t)]}_{e_y(t)}$



Continuous-time KF: properties

observation 1:



- transfer function from y to e_y: $\left[I + C(j\omega I - A)^{-1} F_s\right]^{-1}$
- spectral density relation:

$$\Phi_{e_y e_y}(\omega) = \left[I + C(j\omega I - A)^{-1} F_s\right]^{-1} \Phi_{yy}(\omega) \left\{\left[I + C(-j\omega I - A)^{-1} F_s\right]^{-1}\right\}^T$$

Continuous-time KF: properties

observation 2:

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) + B_w w(t) \\ y(t) &= Cx(t) + v(t) \end{cases} \Rightarrow \Phi_{yy}(\omega) = G(j\omega)WG^T(-j\omega) + V$$

from observations 1 and 2:

$$\left[I + C(j\omega I - A)^{-1} F_s\right] V \left[I + C(-j\omega I - A)^{-1} F_s\right]^T = V + G(j\omega)WG^T(-j\omega)$$

thus says

$$\begin{aligned} \Phi_{e_y e_y}(\omega) &= \left[I + C(j\omega I - A)^{-1} F_s\right]^{-1} \Phi_{yy}(\omega) \left\{\left[I + C(-j\omega I - A)^{-1} F_s\right]^{-1}\right\}^T \\ &= V \end{aligned}$$

namely, the estimation error is white!

Continuous-time KF: symmetric root locus

taking determinants of RDE gives:

$$\det\left[I + C(sI - A)^{-1} F_s\right] \det V \det\left[I + C(-sI - A)^{-1} F_s\right]^T$$
$$= \det\left[V + G(s) W G^T(-s)\right]$$

for single-output systems:

$$\det\left[I + C(sI - A)^{-1} F_s\right] \det\left[I + C(-sI - A)^{-1} F_s\right]^T = 1 + G(s)\frac{W}{V} G^T(-s)$$

Continuous-time KF: symmetric root locus

the left hand side of

$$\det\left[I + C(sI - A)^{-1}F_s\right]\det\left[I + C(-sI - A)^{-1}F_s\right]^T = 1 + G(s)\frac{W}{V}G^T(-s)$$

determines the KF eigenvalues:

$$\begin{aligned}\det\left[I + C(sI - A)^{-1}F_s\right] &= \det\left[I + (sI - A)^{-1}F_sC\right]\\ &= \det\left[(sI - A)^{-1}\right]\det\left[sI - A + F_sC\right]\\ &= \frac{\det\left[sI - (A - F_sC)\right]}{\det(sI - A)}\end{aligned}$$

hence looking at $1 + G(s)\frac{W}{V}G^T(-s)$, we have:

- $W/V \to 0$: KF poles $\to$ stable poles of $G(s)G^T(-s)$
- $W/V \to \infty$: KF poles $\to$ stable zeros of $G(s)G^T(-s)$

Summary

1. Big picture

2. Problem statement

3. Discrete-time KF

   Gain update
   Covariance update
   Steady-state KF
   Duality with LQ

4. Continuous-time KF

   Solution
   Steady-state solution and conditions
   Properties: return difference equality, symmetric root locus...

Lecture 6: Linear Quadratic Gaussian (LQG) Control

Big picture
LQ when there is Gaussian noise
LQG
Steady-state LQG

Big picture

in deterministic control design:

- state feedback: arbitrary pole placement for controllable systems
- observer provides (when system is observable) state estimation when not all states are available
- separation principle for observer state feedback control

we have now learned:

- LQ: optimal state feedback which minimizes a quadratic cost about the states
- KF: provides optimal state estimation

in stochastic control:

- the above two give the linear quadratic Gaussian (LQG) controller

Big picture

plant:

$$x(k+1) = Ax(k) + Bu(k) + B_w w(k)$$
$$y(k) = Cx(k) + v(k)$$

assumptions:

- $w(k)$ and $v(k)$ are independent, zero mean, white Gaussian random processes, with

$$\mathrm{E}[w(k)w^T(k)] = W, \ \mathrm{E}[v(k)v^T(k)] = V$$

- $x(0)$ is a Gaussian random vector independent of $w(k)$ and $v(k)$, with

$$\mathrm{E}[x(0)] = x_0, \ \mathrm{E}[(x(0)-x_0)(x(0)-x_0)^T] = X_0$$

LQ when there is noise

Assume all states are accessible in the plant

$$x(k+1) = Ax(k) + Bu(k) + B_w w(k)$$

The original LQ cost

$$2J = x^T(N)Sx(N) + \sum_{j=0}^{N-1}\left\{x^T(j)Qx(j) + u^T(j)Ru(j)\right\}$$

is no longer valid due to the noise term $w(k)$.
Instead, consider a stochastic performance index:

$$\boxed{J = \underset{\{x(0),w(0),\dots,w(N-1)\}}{\mathrm{E}}\left\{x^T(N)Sx(N) + \sum_{j=0}^{N-1}[x^T(j)Qx(j) + u^T(j)Ru(j)]\right\}}$$

with $S \succeq 0$, $Q \succeq 0$, $R \succ 0$

LQ with noise and exactly known states

solution via stochastic dynamic programming:

Define "cost to go":

$$J_k(x(k)) \triangleq \mathop{\mathrm{E}}_{W_k^+}\left\{x^T(N)Sx(N) + \sum_{j=k}^{N-1}\left[x^T(j)Qx(j) + u^T(j)Ru(j)\right]\right\},$$

$$W_k^+ = \{w(k), \ldots, w(N-1)\}$$

We look for the optima under control $U_k^+ = \{u(k), \ldots, u(N-1)\}$:

$$J_k^o(x(k)) = \min_{U_k^+} J_k(x(k))$$

- the ultimate optimal cost is

$$J^o = \mathop{\mathrm{E}}_{x(0)}\left[\min_{U_0^+} J_0(x(0))\right]$$

LQ with noise and exactly known states

solution via stochastic dynamic programming:

iteration on *optimal* cost to go:

$$J_k^o(x(k)) = \min_{U_k^+} \mathop{\mathrm{E}}_{W_k^+} \left\{ x^T(N)Sx(N) + x^T(k)Qx(k) + u^T(k)Ru(k) + \sum_{j=k+1}^{N-1} [x^T(j)Qx(j) + u^T(j)Ru(j)] \right\}$$

$$= \min_{U_{k+1}^+} \min_{u(k)} \mathop{\mathrm{E}}_{W_k^+} \left\{ x^T(N)Sx(N) + x^T(k)Qx(k) + u^T(k)Ru(k) + \sum_{j=k+1}^{N-1} [x^T(j)Qx(j) + u^T(j)Ru(j)] \right\} \quad (1)$$

$$= \min_{U_{k+1}^+} \min_{u(k)} \left\{ x^T(k)Qx(k) + u^T(k)Ru(k) + \mathop{\mathrm{E}}_{W_k^+} \left[x^T(N)Sx(N) + \sum_{j=k+1}^{N-1} [x^T(j)Qx(j) + u^T(j)Ru(j)] \right] \right\} \quad (2)$$

$$= \min_{u(k)} \left\{ x^T(k)Qx(k) + u^T(k)Ru(k) + \min_{U_{k+1}^+} \mathop{\mathrm{E}}_{w(k)} \mathop{\mathrm{E}}_{W_{k+1}^+} \left[x^T(N)Sx(N) + \sum_{j=k+1}^{N-1} [x^T(j)Qx(j) + u^T(j)Ru(j)] \right] \right\} \quad (3)$$

$$= \min_{u(k)} \left\{ x^T(k)Qx(k) + u^T(k)Ru(k) + \mathop{\mathrm{E}}_{w(k)} \min_{U_{k+1}} \mathop{\mathrm{E}}_{W_{k+1}^+} \left[x^T(N)Sx(N) + \sum_{j=k+1}^{N-1} [x^T(j)Qx(j) + u^T(j)Ru(j)] \right] \right\} \quad (4)$$

$$= \min_{u(k)} \left\{ x^T(k)Qx(k) + u^T(k)Ru(k) + \mathop{\mathrm{E}}_{w(k)} \left[J_{k+1}^o(x(k+1)) \right] \right\} \quad (5)$$

- (1) to (2): $x(k)$ does not depend on $w(k)$, $w(k+1)$,..., $w(N-1)$

LQ with noise and exactly known states

solution via stochastic dynamic programming: induction

$$\boxed{J_k^o(x(k)) = \min_{u(k)} \left\{ x^T(k)Qx(k) + u^T(k)Ru(k) + \mathop{\mathrm{E}}_{w(k)} \left[J_{k+1}^o(x(k+1)) \right] \right\}}$$

at time N: $$J_N^o(x(N)) = x^T(N)Sx(N)$$

assume at time $k+1$:

$$J_{k+1}^o(x(k+1)) = \underbrace{x^T(k+1)P(k+1)x(k+1)}_{\text{cost in a standard LQ}} + \underbrace{b(k+1)}_{\text{due to noise}}$$

then at time k:

$$J_k^o(x(k)) = \min_{u(k)} \left(x^T(k)Qx(k) + u^T(k)Ru(k) + \mathop{\mathrm{E}}_{w(k)} \left[x^T(k+1)P(k+1)x(k+1) + b(k+1) \right] \right)$$

next: use system dynamics $x(k+1) = Ax(k) + Bu(k) + B_w w(k)$...

LQ with noise and exactly known states

after some algebra:

$$J_k^o(x(k)) = \underset{w(k)}{\mathrm{E}} \min_{u(k)} \{x^T(k)\left[Q + A^T P(k+1) A\right] x(k)$$

$$+ u^T(k)\left[R + B^T P(k+1) B\right] u(k) + 2x^T(k) A^T P(k+1) B u(k) + 2x^T(k) A^T P(k+1) B_w w(k)$$

$$+ 2u^T(k) B^T P(k+1) B_w w(k) + w(k)^T B_w^T P(k+1) B_w w(k) + b(k+1)\}$$

$w(k)$ is white and zero mean $\Rightarrow$:

$$\underset{w(k)}{\mathrm{E}} \left\{2x^T(k) A^T P(k+1) B_w w(k) + 2u^T(k) B^T P(k+1) B_w w(k)\right\} = 0$$

$\underset{w(k)}{\mathrm{E}} \left\{w(k)^T B_w^T P(k+1) B_w w(k)\right\}$ equals

$$\mathrm{Tr}\left\{\underset{w(k)}{\mathrm{E}}\left[B_w^T P(k+1) B_w w(k) w(k)^T\right]\right\} = \mathrm{Tr}\left[B_w^T P(k+1) B_w W\right]$$

other terms: not random w.r.t. $w(k)$; can be taken outside of $\mathrm{E}_{w(k)}$

LQ with noise and exactly known states

therefore

$$J_k^o(x(k)) = \min_{u(k)} \{ x^T(k) \left[Q + A^T P(k+1) A \right] x(k)$$
$$+ u^T(k) \left[R + B^T P(k+1) B \right] u(k) + 2x^T(k) A^T P(k+1) Bu(k) \}$$
$$+ \mathrm{Tr} \left[B_w^T P(k+1) B_w W \right] + b(k+1)$$

note: the term inside the minimization is a quadratic (actually convex) function of $u(k)$. Optimization is easily done.

Recall: facts of quadratic functions

- consider

$$f(u) = \frac{1}{2}u^T M u + p^T u + q, \ M = M^T \tag{6}$$

- optimality (maximum when M is negative definite; minimum when M is positive definite) is achieved when

$$\frac{\partial f}{\partial u^o} = M u^o + p = 0 \Rightarrow u^o = -M^{-1} p \tag{7}$$

- and the optimal cost is

$$f^o = f(u^o) = -\frac{1}{2}p^T M^{-1} p + q \tag{8}$$

LQ with noise and exactly known states

$$J_k^o(x(k)) = \min_{u(k)} \Big\{ u^T(k) \left[R + B^T P(k+1) B\right] u(k) + 2x^T(k) A^T P(k+1) Bu(k) + x^T(k) \left[Q + A^T P(k+1) A\right] x(k) \Big\} + \mathrm{Tr}\left[B_w^T P(k+1) B_w W\right] + b(k+1)$$

- optimal control law [by using (7)]:

$$\boxed{u^o(k) = -\left[R + B^T P(k+1) B\right]^{-1} B^T P(k+1) Ax(k)}$$

- optimal cost [by using (8)]:

$$J_k^o(x(k)) = \Big\{ -x^T(k) A^T P(k+1) B \left[R + B^T P(k+1) B\right]^{-1} B^T P(k+1) Ax(k) + x^T(k) \left[Q + A^T P(k+1) A\right] x(k) \Big\} + \mathrm{Tr}\left[B_w^T P(k+1) B_w W\right] + b(k+1)$$

LQ with noise and exactly known states

Riccati equation:

the optimal cost

$$J_k^o(x(k)) = \left\{-x^T(k)A^TP(k+1)B\left[R+B^TP(k+1)B\right]^{-1}B^TP(k+1)Ax(k) + x^T(k)\left[Q+A^TP(k+1)A\right]x(k)\right\} + \mathrm{Tr}\left[B_w^TP(k+1)B_wW\right] + b(k+1)$$

can be written as

$$\boxed{J_k^o(x(k)) = x^T(k)P(k)x(k) + b(k)}$$

with the Riccati equation

$$\boxed{P(k) = A^TP(k+1)A - A^TP(k+1)B\left[R+B^TP(k+1)B\right]^{-1}B^TP(k+1)A + Q}$$

and

$$\boxed{b(k) = \mathrm{Tr}\left[B_w^TP(k+1)B_wW\right] + b(k+1)}$$

boundary conditions: $P(N) = S$ and $b(N) = 0$

LQ with noise and exactly known states

observations:

- optimal control law and Riccati equation are the same as those in the regular LQ problem
- addition cost is due to $B_w w(k)$:

$$b(k) = \mathrm{Tr}\left[B_w^T P(k+1) B_w W\right] + b(k+1),\ b(N) = 0$$

- the final optimal cost is

$$\begin{aligned}
J^o(x(0)) &= \mathop{\mathrm{E}}_{x(0)}\left[x^T(0) P(0) x(0) + b(0)\right] \\
&= \mathop{\mathrm{E}}_{x(0)}\left[(x_o + x(0) - x_o)^T P(0) (x_o + x(0) - x_o) + b(0)\right] \\
&= x_o^T P(0) x_o + \mathrm{Tr}(P(0) X_o) + b(0) \qquad (9)
\end{aligned}$$

where

$$b(0) = \sum_{j=0}^{N-1} \mathrm{Tr}\left[B_w^T P(j+1) B_w W\right]$$

LQG: LQ with noise and inexactly known states

notice that

- not all states may be available and there is usually output noise:

$$x(k+1) = Ax(k) + Bu(k) + B_w w(k)$$
$$y(k) = Cx(k) + v(k)$$

- when u is a function of y, the cost has to also consider the randomness from $V_k^+ = \{v(k), \ldots, v(N-1)\}$

$$\boxed{J = \mathop{\mathrm{E}}_{x(0), W_0^+, V_0^+} \left\{ x^T(N)Sx(N) + \sum_{j=0}^{N-1} [x^T(j)Qx(j) + u^T(j)Ru(j)] \right\}} \tag{10}$$

these motivate the linear quadratic Gaussian (LQG) control problem

LQG solution

only $y(k)$ is accessible instead of $x(k)$, some connection has to be built to connect the cost to $Y_k = \{y(0), \ldots, y(k)\}$:

$$
\begin{aligned}
&\mathrm{E}\left[x^T(k)\,Qx(k)\right] \\
=&\mathrm{E}\left\{\mathrm{E}\left[x^T(k)\,Qx(k)\middle|\,Y_k\right]\right\} \\
=&\mathrm{E}\left\{\mathrm{E}\left[(x(k)-\hat{x}(k|k)+\hat{x}(k|k))^T Q(x(k)-\hat{x}(k|k)+\hat{x}(k|k))\middle|\,Y_k\right]\right\} \\
=&\mathrm{E}\left\{\mathrm{E}\left[(x(k)-\hat{x}(k|k))^T Q(x(k)-\hat{x}(k|k))\middle|\,Y_k+\hat{x}^T(k|k)\,Q\hat{x}(k|k)\middle|\,Y_k\right.\right. \\
&\qquad\left.\left.+2(x(k)-\hat{x}(k|k))^T Q\hat{x}(k|k)\middle|\,Y_k\right]\right\} \qquad (11)
\end{aligned}
$$

LQG solution

but $\mathrm{E}[x(k)|Y_k] = \hat{x}(k|k)$ and $\hat{x}(k|k)$ is orthogonal to $\tilde{x}(k|k)$ (property of least square estimation), so

$$\begin{aligned}\mathrm{E}\left\{\mathrm{E}\left[(x(k)-\hat{x}(k|k))^T Q\hat{x}(k|k)\middle| Y_k\right]\right\} &= \mathrm{E}\left[(x(k)-\hat{x}(k|k))^T Q\hat{x}(k|k)\right] \\ &= \mathrm{Tr}\,\mathrm{E}\left[Q\hat{x}(k|k)\tilde{x}^T(k|k)\right] = 0\end{aligned}$$

yielding

$$\begin{aligned}&\underline{\mathrm{E}\left[x^T(k)Qx(k)\right]} \\ =&\mathrm{E}\left\{\mathrm{E}\left[(x(k)-\hat{x}(k|k))^T Q(x(k)-\hat{x}(k|k))\middle| Y_k + \hat{x}^T(k|k)Q\hat{x}(k|k)\middle| Y_k\right]\right\} \\ =&\mathrm{E}\left[\hat{x}^T(k|k)Q\hat{x}(k|k)\middle| Y_k\right] \\ &+\mathrm{E}\left\{\mathrm{E}\left[\mathrm{Tr}\left\{Q(x(k)-\hat{x}(k|k))(x(k)-\hat{x}(k|k))^T\right\}\middle| Y_k\right]\right\} \\ =&\underline{\mathrm{E}\left[\hat{x}^T(k|k)Q\hat{x}(k|k)\right]+\mathrm{Tr}\{QZ(k)\}}\end{aligned}$$

LQG solution

the LQG cost (10) is thus

$$J = \overbrace{\mathrm{E}\left\{\hat{x}^T(N|N)S\hat{x}(N|N) + \sum_{j=0}^{N-1}\left[\hat{x}^T(j|j)Q\hat{x}(j|j) + u^T(j)Ru(j)\right]\right\}}^{\hat{J}}$$

$$+ \underbrace{\mathrm{Tr}\left\{SZ(N)\right\} + \sum_{j=0}^{N-1}\mathrm{Tr}\left\{QZ(j)\right\}}_{\text{independent of the control input}}$$

hence

$$\boxed{\min_{\{u(0),\ldots,u(N-1)\}} J \Longleftrightarrow \min_{\{u(0),\ldots,u(N-1)\}} \hat{J}}$$

LQG is equivalent to an LQ with exactly know states

consider the equivalent problem to minimize:

$$\hat{J} = \mathrm{E}\left\{\hat{x}^T(N|N)S\hat{x}(N|N) + \sum_{j=0}^{N-1}[\hat{x}^T(j|j)Q\hat{x}(j|j) + u^T(j)Ru(j)]\right\}$$

- $\hat{x}(k|k)$ is fully accessible, with the dynamics:

$$\begin{aligned}\hat{x}(k+1|k+1) &= \hat{x}(k+1|k) + F(k+1)\,e_y(k+1)\\ &= A\hat{x}(k|k) + Bu(k) + F(k+1)\,e_y(k+1)\end{aligned}$$

- from KF results, $e_y(k+1)$ is white, Gaussian with covariance:

$$V + CM(k+1)\,C^T$$

LQG is equivalent to LQ with exactly know states

LQ with exactly known states:

$$J = \mathrm{E}\left\{x^T(N)Sx(N) + \sum_{j=0}^{N-1}\left[x^T(j)Qx(j) + u^T(j)Ru(j)\right]\right\}$$

$$x(k+1) = Ax(k) + Bu(k) + B_w w(k)$$

$$u^o(k) = -\left[R + B^T P(k+1) B\right]^{-1} B^T P(k+1) Ax(k)$$

LQG:

$$\hat{J} = \mathrm{E}\left\{\hat{x}^T(N|N)S\hat{x}(N|N) + \sum_{j=0}^{N-1}\left[\hat{x}^T(j|j)Q\hat{x}(j|j) + u^T(j)Ru(j)\right]\right\}$$

$$\hat{x}(k+1|k+1) = A\hat{x}(k|k) + Bu(k) + F(k+1)e_y(k+1)$$

the solution of LQG is thus:

$$\boxed{u^o(k) = -\left[R + B^T P(k+1) B\right]^{-1} B^T P(k+1) A\hat{x}(k|k)} \tag{12}$$

$$\boxed{P(k) = A^T P(k+1) A - A^T P(k+1) B\left[R + B^T P(k+1) B\right]^{-1} B^T P(k+1) A + Q}$$

Optimal cost of LQG control

- LQ with known states (see (9)):

$$x(k+1) = Ax(k) + Bu(k) + B_w w(k)$$

$$J^o = x_o^T P(0) x_o + \mathrm{Tr}(P(0) X_o) + \underbrace{\sum_{j=0}^{N-1} \mathrm{Tr}\left[B_w^T P(j+1) B_w W\right]}_{b(0)}$$

- LQG:

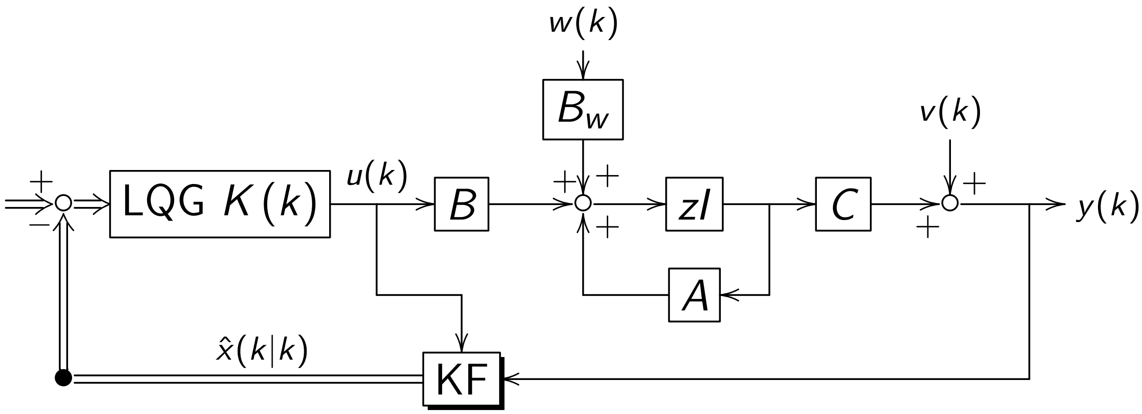
$$\hat{x}(k+1|k+1) = A\hat{x}(k|k) + Bu(k) + F(k+1) e_y(k+1)$$

$$\hat{J}^o = x_o^T P(0) x_o + \mathrm{Tr}[P(0)Z(0)]$$
$$+ \sum_{j=0}^{N-1} \mathrm{Tr}\left\{F^T(j+1)P(j+1)F(j+1)[V + CM(k+1)C^T]\right\} \quad (13)$$

$$J_o = \hat{J}_o + \sum^{N-1} \mathrm{Tr}\{QZ(j)\} + \mathrm{Tr}\{SZ(N)\}$$

Separation theorem in LQG

KF: an (optimal) **observer**
LQ: an (optimal) **state feedback** control
Separation theorem in observer state feedback holds–the closed-loop dynamics contains two **separated** parts: LQ dynamics plus KF dynamics



Stationary LQG problem

Assumptions: system is time invariant; weighting matrices in performance index is time-invariant; noises are white, Gaussian, wide sense stationary.
Equivalent problem: minimize

$$J' = \lim_{N\to\infty} \frac{J}{N} = \lim_{N\to\infty} \mathrm{E}\left\{ \frac{x^T(N)Sx(N)}{N} + \frac{1}{N}\sum_{j=0}^{N-1}[x^T(j)Qx(j)+u^T(j)Ru(j)] \right\}$$

$$= \mathrm{E}\left[x^T(k)Qx(k)+u^T(k)Ru(k)\right]$$

Solution of stationary LQG problem

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) + B_w w(k) \\
y(k) &= Cx(k) + v(k) \\
J' &= \mathrm{E}\left[x^T(k)Qx(k) + u^T(k)Ru(k)\right]
\end{aligned}
$$

the solution is $u = -K_s\hat{x}(k|k)$: steady-state LQ + steady-state KF

$$
\begin{aligned}
K_s &= \left[R + B^T P_s B\right]^{-1} B^T P_s A \\
P_s &= A^T P_s A - A^T P_s B\left[R + B^T P_s B\right]^{-1} B^T P_s A + Q \\
F_s &= M_s C^T\left[C M_s C^T + V\right]^{-1} \\
M_s &= A M_s A^T - A M_s C^T\left[C M_s C^T + V\right]^{-1} C M_s A^T + B_w W B_w^T
\end{aligned}
$$

stability and convergence conditions of the Riccati equations:

- (A, B_w) and (A, B): controllable or stabilizable
- (A, C_q) and (A, C): observable or detectable $(Q = C_q^T C_q)$

Solution of stationary LQG problem

- stability conditions: guaranteed closed-loop stability and KF stability
- separation theorem: closed-loop eigenvalues come from
  - the n eigenvalues of LQ state feedback: $A - BK_s$
  - the n eigenvalues of KF: $A - AF_s C$ (or equivalently $A - F_S CA$)
- optimal cost:

$$J_\infty^o = \mathrm{Tr}\left[P_s\left(BK_sZ_sA^T + B_wWB_w^T\right)\right] \tag{14}$$

- exercise: prove (14)

Continuous-time LQG

- plant:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + B_w w(t) \\ y(t) &= Cx(t) + v(t) \end{aligned}$$

- assumptions: $w(t)$ and $v(t)$ are Gaussian and white; $x(0)$ is Gaussian
- cost:

$$J = \mathrm{E}\left\{ x^T(t_f)\, S x(t_f) + \int_{t_0}^{t_f} \left[x^T(t)\, Q(t)\, x(t) + u^T(t)\, R(t)\, u(t) \right] \mathrm{d}t \right\}$$

where $S \succeq 0$, $Q(t) \succeq 0$, and $R(t) \succ 0$ and the expectation is taken over all random quantities $\{x(0), w(t), v(t)\}$

Continuous-time LQG solution

- Continuous-time LQ:

$$u(t) = -R^{-1}B^T P(t)\hat{x}(t|t) \tag{15}$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = A^T P + PA - PBR^{-1}B^T P + Q, \;\; P(t_f) = S \tag{16}$$

- Continuous-time KF:

$$\frac{\mathrm{d}\hat{x}(t|t)}{\mathrm{d}t} = A\hat{x}(t|t) + Bu(t) + F(t)\left(y(t) - C\hat{x}(t|t)\right) \tag{17}$$

$$F(t) = M(t)C^T V^{-1}, \;\; \hat{x}(t_0|t_0) = x_o \tag{18}$$

$$\frac{\mathrm{d}M}{\mathrm{d}t} = AM + MA^T - MC^T V^{-1} CM + B_w W B_W^T, \;\; M(t_0) = X_o \tag{19}$$

Summary

1. Big picture
2. Stochastic control with exactly known state
3. Stochastic control with inexactly known state
4. Steady-state LQG
5. Continuous-time LQG problem

Lecture 7: Principles of Feedback Design

MIMO closed-loop analysis
Robust stability
MIMO feedback design

Big picture

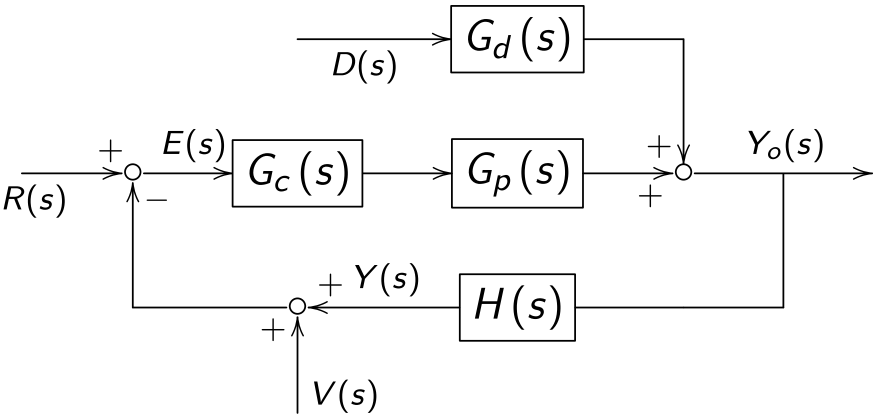
- we are pretty familiar with SISO feedback system design and analysis
- state-space designs (LQ, KF, LQG,...): time-domain; good mathematical formulation and solutions based on rigorous linear algebra
- frequency-domain and transfer-function analysis: builds intuition; good for properties such as stability robustness

MIMO closed-loop analysis

signals and transfer functions are vectors and matrices now:

- r (reference) and y (plant output): m-dimensional
- $G_p(s)$: p by m transfer function matrix

$$
\begin{aligned}
E(s) &= R(s) - (H(s)\,Y_o(s) + V(s)) \\
&= R(s) - \{H(s)\,G_p(s)\,G_c(s)\,E(s) + H(s)\,G_d(s)\,D(s) + V(s)\} \quad (1)
\end{aligned}
$$



MIMO closed-loop analysis

(1) gives

$$E(s) = (I_m + G_{\text{open}}(s))^{-1} R(s)$$
$$- (I_m + G_{\text{open}}(s))^{-1} H(s) G_d(s) D(s) - (I_m + G_{\text{open}}(s))^{-1} V(s)$$

where the *loop transfer function*

$$\boxed{G_{\text{open}}(s) = H(s) G_p(s) G_c(s)}$$

We want to minimize $E^*(s) \triangleq R(s) - Y(k) = E(s) + V(s)$

$$E^*(s) = \underline{(I_m + G_{\text{open}}(s))^{-1}} R(s)$$
$$- (I_m + G_{\text{open}}(s))^{-1} H(s) G_d(s) D(s) + \underline{(I_m + G_{\text{open}}(s))^{-1} G_{\text{open}}(s)} V(s)$$

Sensitivity and complementary sensitivity functions:

$$S(s) \triangleq (I_m + G_{\text{open}}(s))^{-1}$$
$$T(s) \triangleq (I_m + G_{\text{open}}(s))^{-1} G_{\text{open}}(s)$$

Fundamental limitations in feedback design

$$
\begin{aligned}
E^*(s) &= S(s)R(s) + T(s)V(s) - S(s)H(s)G_d(s)D(s) \\
Y(s) &= R(s) - E^*(s) = T(s)R(s) + \dots
\end{aligned}
$$

- sensitivity function $S(s)$: explains disturbance-rejection ability
- complementary sensitivity function $T(s)$: explains reference tracking and sensor-noise rejection abilities
- fundamental constraint of feedback design:

$$\boxed{S(s) + T(s) = I_m}$$

equivalently

$$S(j\omega) + T(j\omega) = I_m$$

- cannot do well in all aspects: e.g., if $S(j\omega) \approx 0$ (good disturbance rejection), $T(j\omega)$ will be close to identity (bad sensor-noise rejection)

Goals of SISO control design

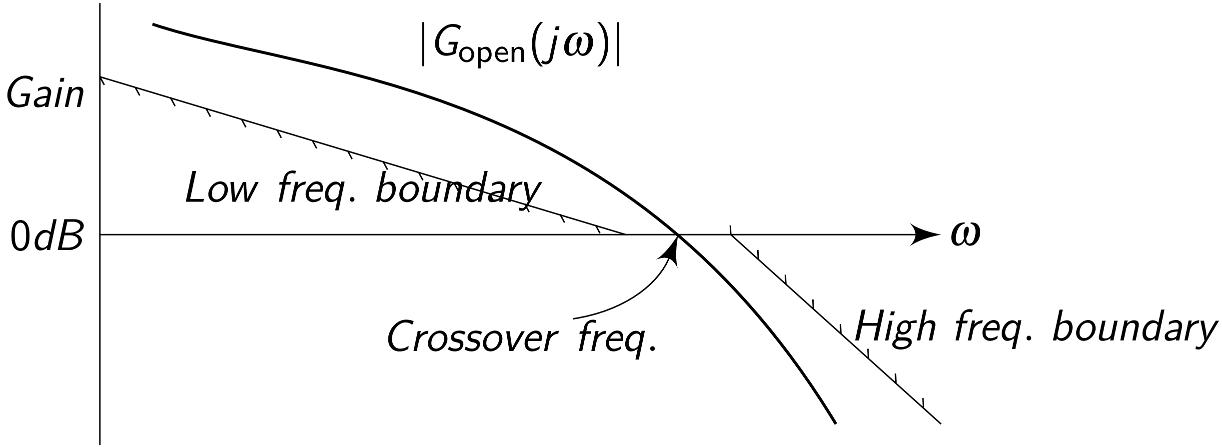
single-input single-output (SISO) control design:

$$S(j\omega) = \frac{1}{1 + G_{\text{open}}(j\omega)}, \quad T(j\omega) = \frac{G_{\text{open}}(j\omega)}{1 + G_{\text{open}}(j\omega)}$$

- goals:
  1. nominal stability
  2. stability robustness
  3. command following and disturbance rejection
  4. sensor-noise rejection
- feedback achieves: 1 (Nyquist theorem), 2 (sufficient (gain and phase) margins), and
  - 3: small $S(j\omega)$ at relevant frequencies (usually low frequency)
  - 4: small $T(j\omega)$ at relevant frequencies (usually high frequency)
- additional control design for meeting the performance goals: feedforward, predictive, preview controls, etc

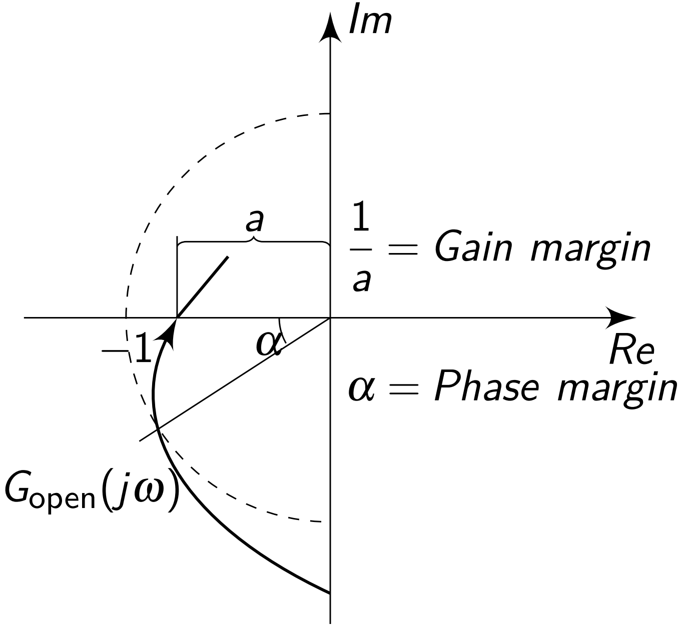
SISO loop shaping

typical loop shape (magnitude response of G_{open}):



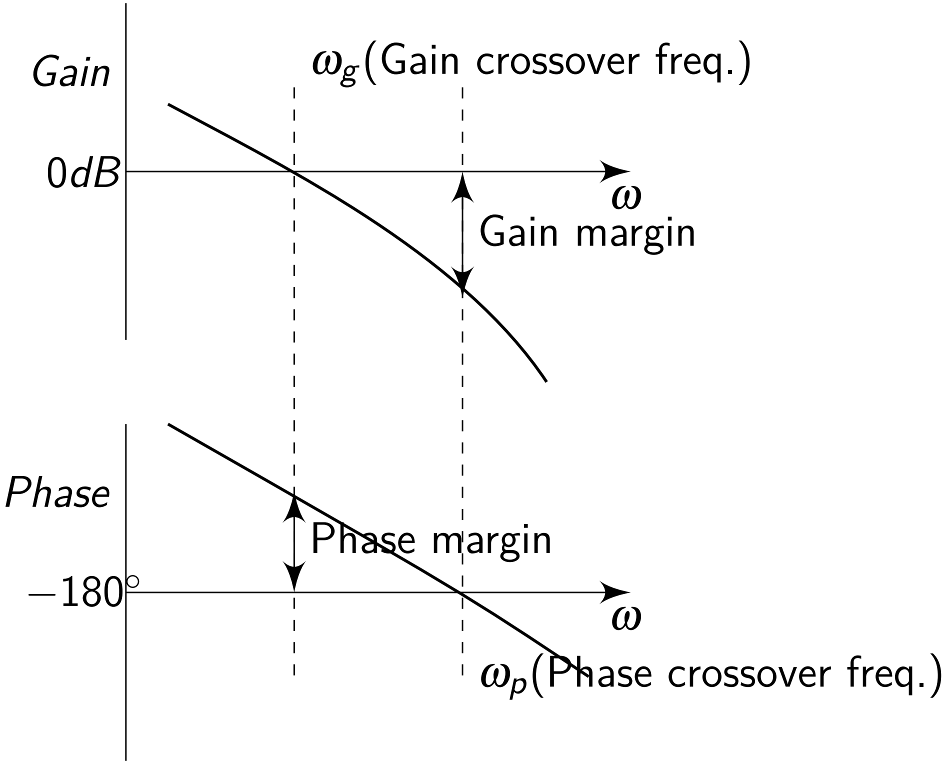
SISO loop shaping: stability robustness

the idea of stability margins:



SISO loop shaping: stability robustness

the idea of stability margins:

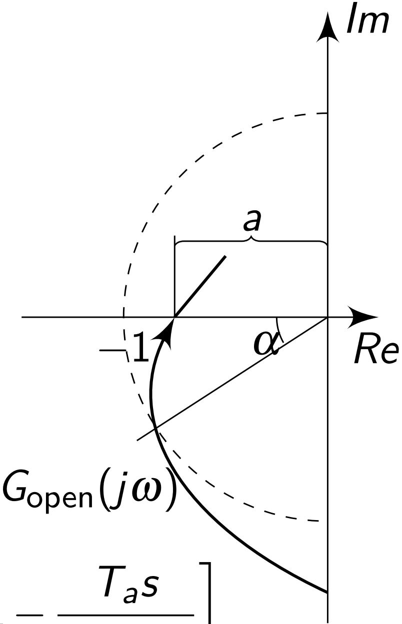


SISO loop shaping: stability robustness

$G_{\text{open}}(j\omega)$ should be sufficiently far away from $(-1,0)$ for robust stability.
Commonly there are uncertainties and the actual case is

$$\tilde{G}_{\text{open}}(s) = G_{\text{open}}(s)\left[1+\Delta(s)\right]$$

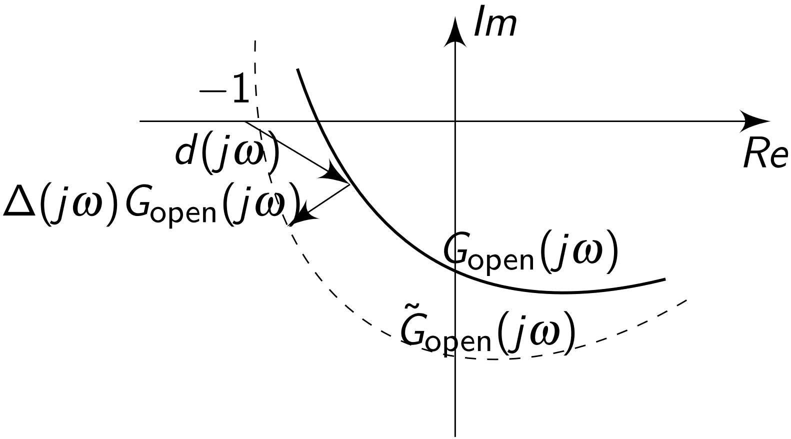
e.g. ignored actuator dynamics in a positioning system:



$$\tilde{G}_{\text{open}}(s) = G_{\text{open}}(s)\frac{1}{T_a s+1} = G_{\text{open}}(s)\left[1-\frac{T_a s}{T_a s+1}\right]$$

$$\Delta(j\omega) = -\frac{T_a j\omega}{T_a j\omega+1}$$

SISO loop shaping: stability robustness



if nominal stability holds, robust stability needs

$$\left|\Delta(j\omega)\, G_{open}(j\omega)\right| = \left|\tilde{G}_{open}(j\omega) - G_{open}(j\omega)\right| < \overbrace{\left|1 + G_{open}(j\omega)\right|}^{|d(j\omega)|}$$

$$\Leftrightarrow \boxed{\left|\Delta(j\omega)\frac{G_{open}(j\omega)}{1+G_{open}(j\omega)}\right| < 1 \Leftrightarrow \left|\Delta(j\omega)\, T(j\omega)\right| < 1,\ \forall\omega}$$

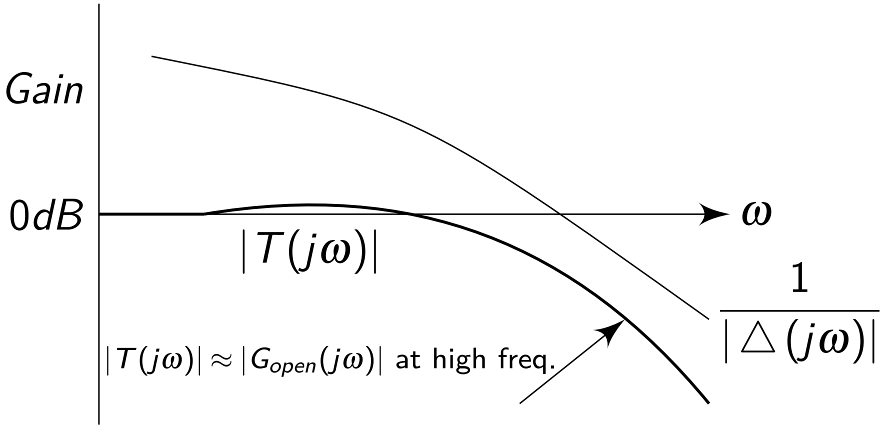
SISO loop shaping: stability robustness

if $|G_{open}(j\omega)| \ll 1$ then

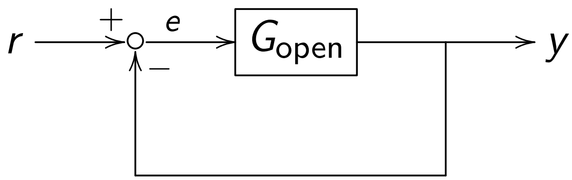
$$\boxed{\left|\Delta(j\omega)\frac{G_{open}(j\omega)}{1+G_{open}(j\omega)}\right| < 1}$$

approximately means

$$|G_{open}(j\omega)| < \frac{1}{|\Delta(j\omega)|}$$



MIMO Nyquist criterion



- assume G_{open} is $m \times m$ and realized by

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + Be(t), \; x \in \mathbb{R}^{m\times 1}$$
$$y(t) = Cx(t)$$

- the closed-loop dynamics is

$$\begin{cases} \frac{\mathrm{d}x(t)}{\mathrm{d}t} & = (A - BC)x(t) + Br(t) \\ y(t) & = Cx(t) \end{cases} \tag{2}$$

MIMO Nyquist criterion

(2) gives the closed-loop transfer function

$$G_{\text{closed}}(s) = C(sI - A + BC)^{-1} B$$

- closed-loop stability depends on the eigenvalues $\text{eig}(A - BC)$, which come from

$$\begin{aligned}\phi_{\text{closed}}(s) &= \det(sI - A + BC) = \det\left\{(sI - A)\left[I + (sI - A)^{-1} BC\right]\right\} \\ &= \det(sI - A)\det\left(I + C(sI - A)^{-1} B\right) \\ &= \underbrace{\det(sI - A)}_{\text{open loop } \phi_{\text{open}}(s)} \det(I + G_{\text{open}}(s))\end{aligned}$$

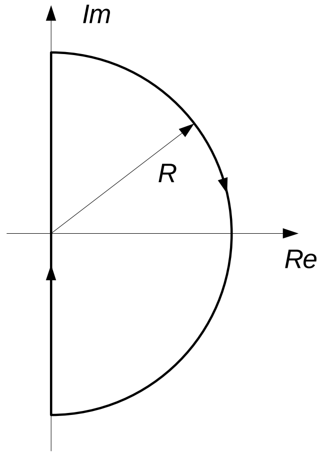
- hence

$$\boxed{\frac{\phi_{\text{closed}}(s)}{\phi_{\text{open}}(s)} = \det(I + G_{\text{open}}(s))}$$

MIMO Nyquist criterion

$$\boxed{\frac{\phi_{\text{closed}}(s)}{\phi_{\text{open}}(s)} = \det\left(I + G_{\text{open}}(s)\right) = \frac{\prod_{j=1}^{n_1}(s - p_{\text{cl}})}{\prod_{i=1}^{n_2}(s - p_{\text{ol}})}}$$

- evaluate $\det\left(I + G_{\text{open}}(s)\right)$ along the D contour $(R \to \infty)$
- Z closed-loop "unstable" eigen values in $\prod_{j=1}^{n_1}(s - p_{\text{cl}})$ contribute to $2\pi Z$ net increase in phase
- P open-loop "unstable" eigen values in $\prod_{j=1}^{n_2}(s - p_{\text{ol}})$ contribute to $-2\pi P$ net increase in phase
- stable eigen values do not contribute to net phase change



MIMO Nyquist criterion

the number of counter clockwise encirclements of the origin by $\det(I + G_{\text{open}}(s))$ is:

$$\boxed{\mathrm{N}(0, \det(I + G_{\text{open}}(s)), \mathrm{D}) = P - Z}$$

stability condition: $Z = 0$

Theorem (Multivariable Nyquist Stability Criterion)

the closed-loop system is asymptotically stable if and only if

$$\mathrm{N}(0, \det(I + G_{\text{open}}(s)), \mathrm{D}) = P$$

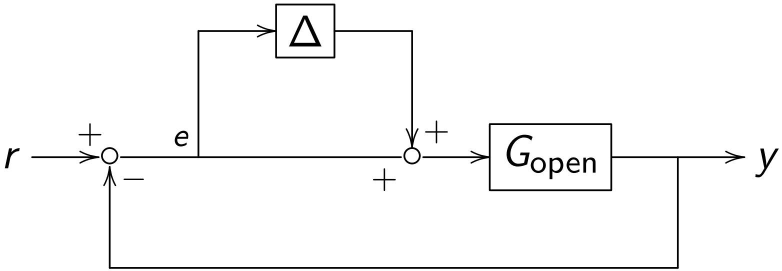
i.e., the number of counterclockwise encirclements of the origin by $\det(I + G_{\text{open}}(s))$ *along the* D *contour equals the number of open-loop unstable eigen values (of the A matrix).*

MIMO robust stability

Given the nominal model G_{open}, let the actual open loop be perturbed to

$$\tilde{G}_{\text{open}}\,(j\omega) = G_{\text{open}}(j\omega)\,[I + \Delta(j\omega)]$$

where $\Delta(j\omega)$ is the uncertainty (bounded by $\sigma(\Delta(j\omega)) \leq \bar{\sigma}$)



- what properties should the nominal system possess in order to have robust stability?

MIMO robust stability

- obviously need a stable nominal system to start with:

$$\mathrm{N}\left(0, \det\left(I + G_{\text{open}}\left(s\right)\right), \mathrm{D}\right) = P$$

- for robust stability, we need

$$\mathrm{N}\left(0, \det\left(I + G_{\text{open}}\left(s\right)\left(1 + \Delta\left(s\right)\right)\right), \mathrm{D}\right) = P \text{ for all possible } \Delta$$

- under nominal stability, we need the boundary condition

$$\boxed{\det\left(I + G_{\text{open}}\left(j\omega\right)\left(1 + \Delta\left(j\omega\right)\right)\right) \neq 0}$$

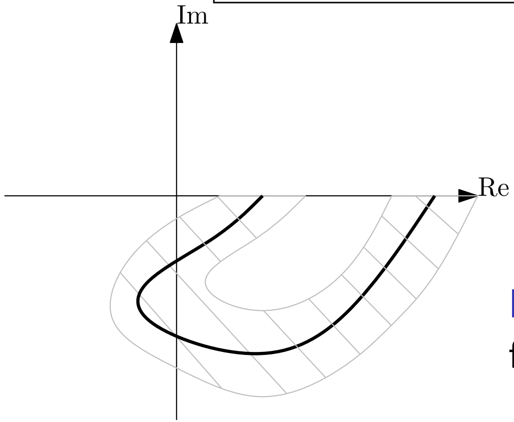


Figure: Example Nyquist plot for robust stability analysis

MIMO robust stability

- note the determinant equivalence:

$$\det\left(I + G_{\text{open}}(j\omega)\left(1 + \Delta(j\omega)\right)\right) = \det\left(I + G_{\text{open}}(j\omega)\right) \\ \times \det\left[I + \left(I + G_{\text{open}}(j\omega)\right)^{-1} G_{\text{open}}(j\omega)\Delta(j\omega)\right]$$

- as the system is open-loop asymptotically stable, no poles are on the imaginary, i.e.,

$$\det\left(I + G_{\text{open}}(j\omega)\right) \neq 0$$

- hence $\det\left(I + G_{\text{open}}(j\omega)\left(1 + \Delta(j\omega)\right)\right) \neq 0 \Longleftrightarrow$

$$\det\left[I + \underbrace{\left(I + G_{\text{open}}(j\omega)\right)^{-1} G_{\text{open}}(j\omega)}_{T(j\omega)}\Delta(j\omega)\right] \neq 0 \qquad (3)$$

MIMO robust stability

- intuitively, (3) means $T(j\omega)\Delta(j\omega)$ should be "smaller than" I
- mathematically, (3) will be violated if $\exists\ x \neq 0$ that achieves

$$\begin{aligned}[I + T(j\omega)\Delta(j\omega)]x &= 0 \\ \Leftrightarrow T(j\omega)\Delta(j\omega)x &= -x \end{aligned} \tag{4}$$

which will make the singular value

$$\sigma_{\max}[T(j\omega)\Delta(j\omega)] = \max_{v\neq 0} \frac{||T(j\omega)\Delta(j\omega)v||_2}{||v||_2} \geq \frac{||T(j\omega)\Delta(j\omega)x||_2}{||x||_2}$$

- as this cannot happen, we must have

$$\sigma_{\max}[T(j\omega)\Delta(j\omega)] < 1$$

It turns out this is both necessary and sufficient if $\Delta(j\omega)$ is unstructured (can 'attack' from any directions). Message: we can design G_{open} such that $\sigma_{\max}[\Delta(j\omega)] < \sigma_{\min}\left[T^{-1}(j\omega)\right]$.

Summary

1. Big picture

2. MIMO closed-loop analysis

3. Loop shaping
   SISO case

4. MIMO stability and robust stability
   MIMO Nyquist criterion
   MIMO robust stability

Lecture 8: Discretization and Implementation of Continuous-time Design

Big picture
Discrete-time frequency response
Discretization of continuous-time design
Aliasing and anti-aliasing

Big picture

why are we learning this:

- nowadays controllers are implemented in discrete-time domain
- implementation media: digital signal processor, field-programmable gate array (FPGA), etc
- either: controller is designed in continuous-time domain and implemented digitally
- or: controller is designed directly in discrete-time domain

Frequency response of LTI SISO digital systems

$$a\sin(\omega T_s k) \longrightarrow \boxed{G(z)} \longrightarrow b\sin(\omega T_s k+\phi) \text{ at steady state}$$

- sampling time: T_s
- $\phi\left(e^{j\omega T_s}\right)$: phase difference between the output and the input
- $M\left(e^{j\omega T_s}\right) = b/a$: magnitude difference

continuous-time frequency response:

$$G(j\omega) = G(s)|_{s=j\omega} = |G(j\omega)|\, e^{j\angle G(j\omega)}$$

discrete-time frequency response:

$$G\left(e^{j\omega T_s}\right) = G(z)|_{z=e^{j\omega T_s}} = \left|G\left(e^{j\omega T_s}\right)\right| e^{j\angle G\left(e^{j\omega T_s}\right)}$$
$$= M\left(e^{j\omega T_s}\right) e^{j\phi\left(e^{j\omega T_s}\right)}$$

Sampling

sufficient samples must be collected (i.e., fast enough sampling frequency) to recover the frequency of a continuous-time sinusoidal signal (with frequency ω in rad/sec)

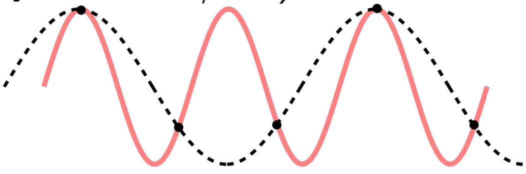


Figure: Sampling example (source: Wikipedia.org)

- the sampling frequency $= \frac{2\pi}{T_s}$
- Shannon's sampling theorem: the Nyquist frequency $(\triangleq \frac{\pi}{T_s})$ must satisfy

$$-\frac{\pi}{T_s} < \omega < \frac{\pi}{T_s}$$

Approximation of continuous-time controllers

bilinear transform

formula:

$$\boxed{s = \frac{2}{T_s}\frac{z-1}{z+1} \qquad z = \frac{1+\frac{T_s}{2}s}{1-\frac{T_s}{2}s}} \tag{1}$$

intuition:

$$z = \mathrm{e}^{sT_s} = \frac{\mathrm{e}^{sT_s/2}}{\mathrm{e}^{-sT_s/2}} \approx \frac{1+\frac{T_s}{2}s}{1-\frac{T_s}{2}s}$$

implementation: start with $G(s)$, obtain the discrete implementation

$$G_d(z) = G(s)\big|_{s=\frac{2}{T_s}\frac{z-1}{z+1}} \tag{2}$$

Bilinear transformation maps the closed left half s-plane to the closed unit ball in z-plane
Stability reservation: $G(s)$ stable $\Longleftrightarrow$ $G_d(z)$ stable

Approximation of continuous-time controllers

history

Bilinear transform is also known as Tustin transform.
Arnold Tustin (16 July 1899 – 9 January 1994):

- British engineer, Professor at University of Birmingham and at Imperial College London
- served in the Royal Engineers in World War I
- worked a lot on electrical machines

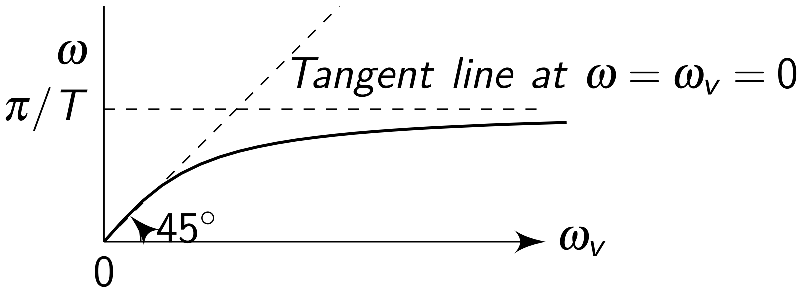
Approximation of continuous-time controllers

frequency mismatch in bilinear transform

$$\frac{2}{T_s}\frac{z-1}{z+1}\bigg|_{z=e^{j\omega T_s}} = \frac{2}{T_s}\frac{e^{j\omega T_s/2}\left(e^{j\omega T_s/2}-e^{-j\omega T_s/2}\right)}{e^{j\omega T_s/2}\left(e^{j\omega T_s/2}+e^{-j\omega T_s/2}\right)} = j\overbrace{\frac{2}{T_s}\tan\left(\frac{\omega T_s}{2}\right)}^{\omega_v}$$

$G(s)|_{s=j\omega}$ is the true frequency response at ω; yet bilinear implementation gives,

$$G_d\left(e^{j\omega T_s}\right) = G(s)|_{s=j\omega_v} \neq G(s)|_{s=j\omega}$$



Approximation of continuous-time controllers

bilinear transform with prewarping

goal: extend bilinear transformation such that

$$G_d(z)\big|_{z=e^{j\omega T_s}} = G(s)\big|_{s=j\omega}$$

at a particular frequency ω_p

solution:

$$\boxed{s = p\frac{z-1}{z+1}, \qquad z = \frac{1+\frac{1}{p}s}{1-\frac{1}{p}s}, \qquad p = \frac{\omega_p}{\tan\left(\frac{\omega_p T_s}{2}\right)}}$$

which gives

$$G_d(z) = G(s)\big|_{s=\frac{\omega_p}{\tan(\frac{\omega_p T}{2})}\frac{z-1}{z+1}}$$

and

$$\left.\frac{\omega_p}{\tan\left(\frac{\omega_p T_s}{2}\right)}\frac{z-1}{z+1}\right|_{z=e^{j\omega_p T_s}} = j\frac{\omega_p}{\cancel{\tan\left(\frac{\omega_p T_s}{2}\right)}}\cancel{\tan\left(\frac{\omega_p T_s}{2}\right)}$$

Approximation of continuous-time controllers

bilinear transform with prewarping

choosing a prewarping frequency ω_p:

- must be below the Nyquist frequency:

$$0 < \omega_p < \frac{\pi}{T_s}$$

- standard bilinear transform corresponds to the case where $\omega_p = 0$
- the best choice of ω_p depends on the important features in control design

example choices of ω_p:

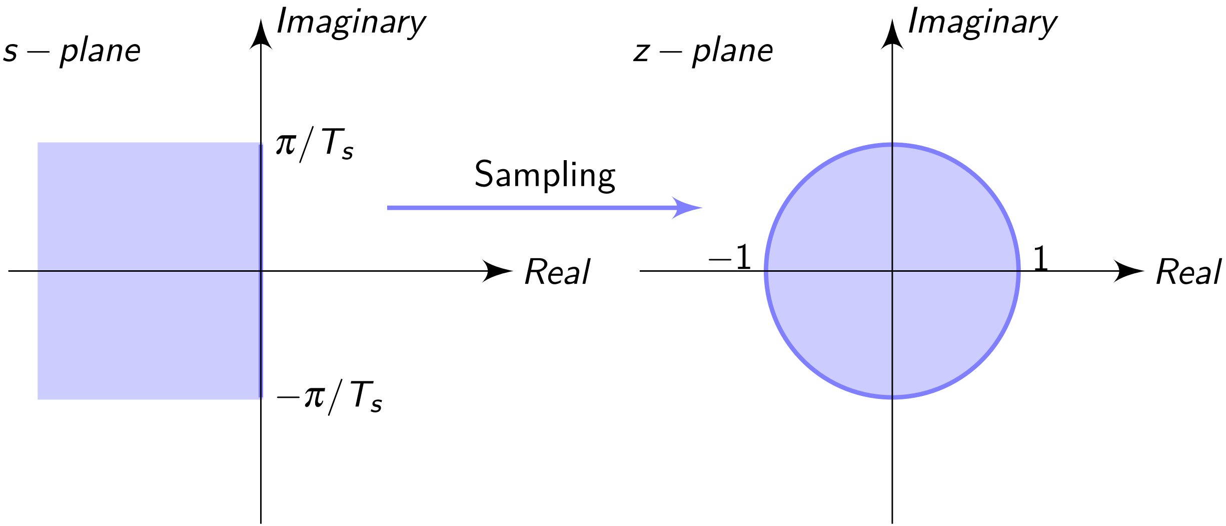
- at the cross-over frequency (which helps preserve phase margin)
- at the frequency of a critical notch for compensating system resonances

Sampling and aliasing

sampling maps the continuous-time frequency

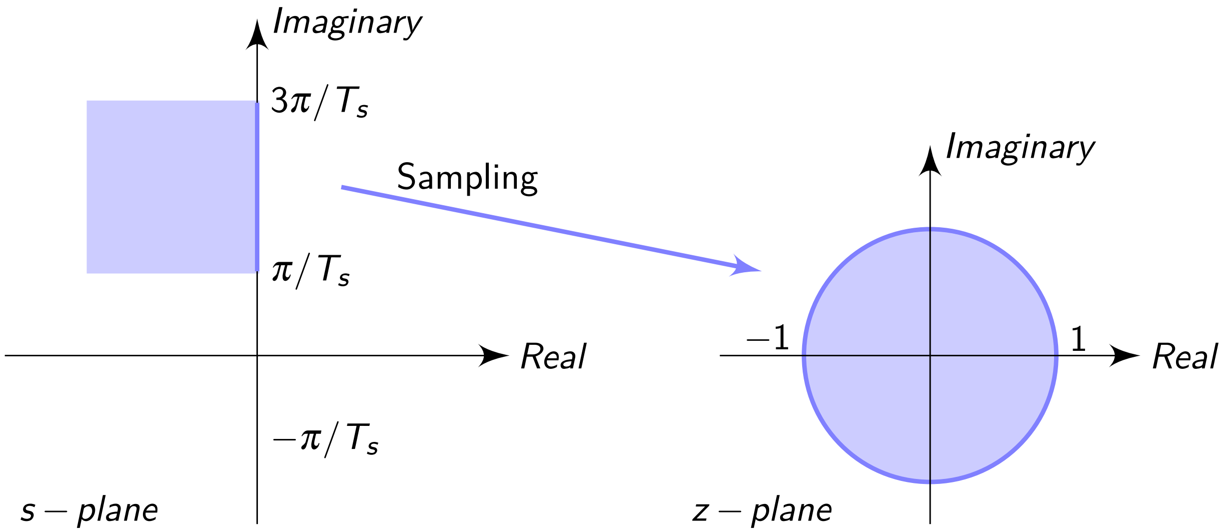
$$-\frac{\pi}{T_s} < \omega < \frac{\pi}{T_s}$$

onto the unit circle



Sampling and aliasing

sampling also maps the continuous-time frequencies $\frac{\pi}{T_s} < \omega < 3\frac{\pi}{T_s}$, $3\frac{\pi}{T_s} < \omega < 5\frac{\pi}{T_s}$, etc, onto the unit circle



Sampling and aliasing

Example (Sampling and Aliasing)

T_s=1/60 sec (Nyquist frequency 30 Hz).

a continuous-time 10-Hz signal [10 Hz$\leftrightarrow 2\pi\times 10$ rad/sec $\in(-\pi/T_s,\pi/T_s)$]

$$y_1(t) = \sin(2\pi\times 10t)$$

is sampled to

$$y_1(k) = \sin\left(2\pi\times\frac{10}{60}k\right) = \sin\left(2\pi\times\frac{1}{6}k\right)$$

a 70-Hz signal [$2\pi\times 70$ rad/sec $\in(\pi/T_s,3\pi/T_s)$]

$$y_2(t) = \sin(2\pi\times 70t)$$

is sampled to

$$y_2(k) = \sin\left(2\pi\times\frac{70}{60}k\right) = \sin\left(2\pi\times\frac{1}{6}k\right) \equiv y_1(k)!$$

Anti-aliasing

need to avoid the negative influence of *aliasing* beyond the Nyquist frequencies

- sample faster: make π/T_s large; the sampling frequency should be high enough for good control design
- anti-aliasing: perform a low-pass filter to filter out the signals $|\omega| > \pi/T_s$

Summary

1. Big picture
2. Discrete-time frequency response
3. Approximation of continuous-time controllers
4. Sampling and aliasing

Sampling example

- continuous-time signal

$$y(t) = \begin{cases} e^{-at}, & t \geq 0 \\ 0, & t < 0 \end{cases}, \quad a > 0$$

$$\mathscr{L}\{y(t)\} = \frac{1}{s+a}$$

- discrete-time sampled signal

$$y(k) = \begin{cases} e^{-aT_s k}, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

$$\mathscr{Z}\{y(k)\} = \frac{1}{1 - z^{-1}e^{-aT_s}}$$

- sampling maps the continuous-time pole $s_i = -a$ to the discrete-time pole $z_i = e^{-aT_s}$, via the mapping

$$z_i = e^{s_i T_s}$$

Lecture 9: LQG/Loop Transfer Recovery (LTR)

Big picture
Loop transfer recovery
Target feedback loop
Fictitious KF

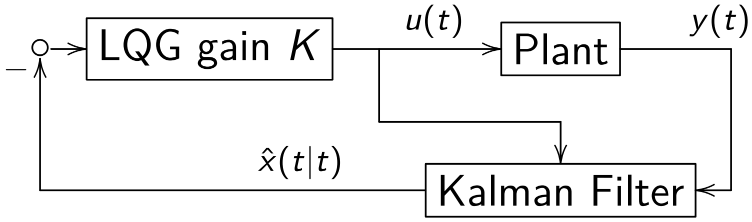
Big picture

Where are we now?

- LQ: optimal control, guaranteed robust stability under basic assumptions in stationary case
- KF: optimal state estimation, good properties from the duality between LQ and KF
- LQG: LQ+KF with separation theorem
- frequency-domain feedback design principles and implementations

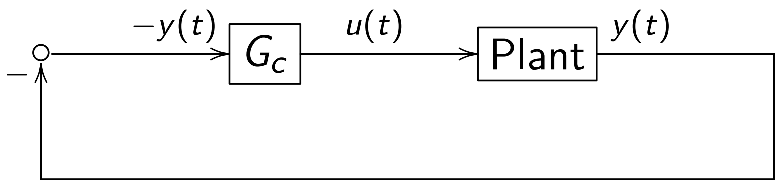
Stability robustness of LQG was discussed in one of the homework problems: the nice robust stability in LQ (good gain and phase margins) is lost in LQG.
LQG/LTR is one combined scheme that uses many of the concepts learned so far.

Continuous-time stationary LQG solution



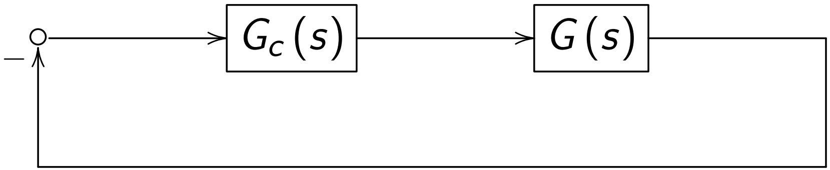
$$
\begin{aligned}
u(t) &= -K\hat{x}(t|t) \\
\frac{\mathrm{d}\hat{x}(t|t)}{\mathrm{d}t} &= A\hat{x}(t|t) + Bu(t) + F\left(y(t) - C\hat{x}(t|t)\right) \\
&= (A - BK - FC)\hat{x}(t|t) + Fy(t)
\end{aligned}
$$

$\Longleftrightarrow$



$$
G_c(s) = K(sI - A + BK + FC)^{-1}F \tag{1}
$$

Loop transfer recovery (LTR)



Theorem (Loop Transfer Recovery (LTR))

If a $m \times m$ dimensional $G(s)$ has only minimum phase transmission zeros, then the open-loop transfer function

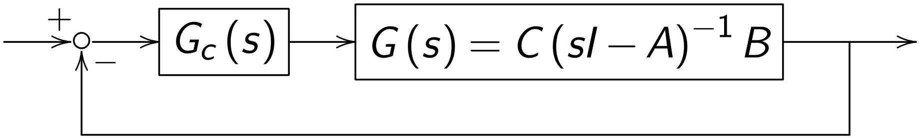
$$G(s)\,G_c(s) = \left[C(sI - A)^{-1}B\right]\left[K(sI - A + BK + FC)^{-1}F\right] \xrightarrow{\rho \to 0} C(sI - A)^{-1}F \quad (2)$$

K and ρ are from the LQ [(A, B) controllable, (A, C) observable]

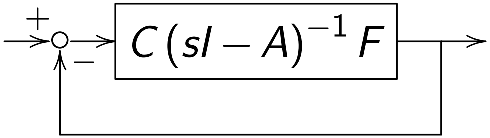
$$J = \int_0^\infty \left(x^T(t)C^TCx(t) + \rho u^T(t)Nu(t)\right)dt \quad (3)$$

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (4)$$

Loop transfer recovery (LTR)



converges, as $\rho \to 0$, to the *target feedback loop*



key concepts:

- regard LQG as an output feedback controller
- will design F such that $C(sI - A)^{-1}F$ has a good loop shape

- not a conventional optimal control problem
- not even a stochastic control design method

Selection of F for the target feedback loop

standard KF procedure: given noise properties (W, V, etc), KF gain F comes from RE

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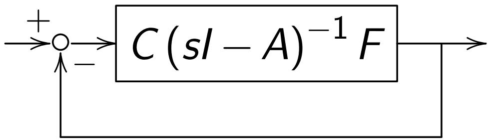
fictitious KF for target feedback loop design: want to have good behavior in



select W and V to get a desired F (hence a *fictitious* KF problem):

$$\dot{x}(t) = Ax(t) + Lw(t), \qquad E[w(t)w^T(t+\tau)] = I\delta(\tau)$$
$$y(t) = Cx(t) + v(t), \qquad E[v(t)v^T(t+\tau)] = \mu I\delta(\tau)$$

which gives

$$F = \frac{1}{\mu}MC^T, \quad AM + M^TA + LL^T - \frac{1}{\mu}MC^TCM = 0,\ M \succ 0 \quad (5)$$

The target feedback loop from fictitious KF

$$\dot{x}(t) = Ax(t) + Lw(t), \qquad E[w(t)w^T(t+\tau)] = I\delta(\tau)$$
$$y(t) = Cx(t) + v(t), \qquad E[v(t)v^T(t+\tau)] = \mu I\delta(\tau)$$

Return difference equation for the fictitious KF is

$$[I_m + G_F(s)][I_m + G_F(-s)]^T = I_m + \frac{1}{\mu}[C\Phi(s)L][C\Phi(-s)L]^T$$

where $G_F(s) = C(sI - A)^{-1}F$ and $\Phi(s) = (sI - A)^{-1}$. Then

$$\sigma[I_m + G_F(j\omega)] = \sqrt{\lambda\left\{[I_m + G_F(j\omega)][I_m + G_F(-j\omega)]^T\right\}}$$
$$= \sqrt{1 + \frac{1}{\mu}\{\sigma[C\Phi(j\omega)L]\}^2} \geq 1$$

The (nice) target feedback loop from fictitious KF

$$\sigma\left[I_m + G_F(j\omega)\right] = \sqrt{\lambda\left\{\left[I_m + G_F(j\omega)\right]\left[I_m + G_F(-j\omega)\right]^T\right\}}$$
$$= \sqrt{1 + \frac{1}{\mu}\left\{\sigma\left[C\Phi(j\omega)L\right]\right\}^2} \geq 1$$

gives:

- $\sigma_{\max} S(j\omega) = \sigma_{\max}[I + G_F(j\omega)]^{-1} \leq 1$, namely

  no disturbance amplification at any frequency

- $\sigma_{\max} T(j\omega) = \sigma_{\max}[I - S(j\omega)] \leq 2$, hence,

  guaranteed closed loop stable if $\sigma_{\max}\Delta(j\omega) < 1/2$

Lecture 10: LQ with Frequency Shaped Cost Function (FSLQ)

Background
Parseval's Theorem
Frequency-shaped LQ cost function
Transformation to a standard LQ

Big picture

why are we learning this:

- in standard LQ, Q and R are constant matrices in the cost function

$$J=\int_0^{\infty}\left(x^T(t)Qx(t)+\rho u^T(t)Ru(t)\right)\mathrm{d}t \tag{1}$$

- how can we introduce more design freedom for Q and R?

Connection between time and frequency domains

Theorem (Parseval's Theorem)

For a square integrable signal $f(t)$ defined on $[0, \infty)$

$$\int_0^\infty f^T(t) f(t) \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^\infty F^T(-j\omega) F(j\omega) \mathrm{d}\omega$$

1D case:
$$\int_0^\infty |f(t)|^2 \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^\infty |F(j\omega)|^2 \mathrm{d}\omega$$

Intuition: energy in time-domain equals energy in frequency domain

For the general case, $f(t)$ can be acausal. We have

$$\int_{-\infty}^\infty f^T(t) f(t) \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^\infty F^T(-j\omega) F(j\omega) \mathrm{d}\omega$$

Discrete-time version:

$$\sum_{k=-\infty}^{\infty} f^T(k) f(k) = \frac{1}{2\pi} \int_{-\infty}^\infty F^T\left(\mathrm{e}^{-j\omega}\right) F\left(\mathrm{e}^{j\omega}\right) \mathrm{d}\omega$$

History

Marc-Antoine Parseval (1755-1836):

- French mathematician
- published just five (but important) mathematical publications in total (source: Wikipedia.org)

Frequency-domain LQ cost function

From Parseval's Theorem, the LQ cost in frequency domain is

$$J = \int_0^{\infty} \left(x^T(t) Q x(t) + \rho u^T(t) R u(t) \right) \mathrm{d}t \tag{2}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(X^T(-j\omega) Q X(j\omega) + \rho U^T(-j\omega) R U(j\omega) \right) \mathrm{d}\omega \tag{3}$$

Frequency-shaped LQ expands Q and R to frequency-dependent functions:

$$J = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(X^T(-j\omega) Q(j\omega) X(j\omega) + \rho U^T(-j\omega) R(j\omega) U(j\omega) \right) \mathrm{d}\omega \tag{4}$$

Frequency-domain LQ cost function

Let

$$Q(j\omega) = Q_f^T(-j\omega)Q_f(j\omega) \succeq 0, \;\; X_f(j\omega) = Q_f(j\omega)X(j\omega)$$
$$R(j\omega) = R_f^T(-j\omega)R_f(j\omega) \succ 0, \;\; U_f(j\omega) = R_f(j\omega)U(j\omega)$$

(4) becomes

$$J = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(X_f^T(-j\omega)X_f(j\omega) + \rho U_f^T(-j\omega)U_f(j\omega)\right)\mathrm{d}\omega$$

which is equivalent to (using Parseval's Theorem again)

$$\boxed{J = \int_0^{\infty}\left(x_f^T(t)x_f(t) + \rho u_f^T(t)u_f(t)\right)\mathrm{d}t} \tag{5}$$

Frequency-domain LQ cost function

Summarizing, we have:

- plant:

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{cases} \tag{6}$$

- new cost:

$$J = \int_0^\infty \left(x_f^T(t) x_f(t) + \rho\, u_f^T(t) u_f(t) \right) \mathrm{d}t \tag{7}$$

- filtered states and inputs:

$$x(t) \longrightarrow \boxed{Q_f(s)} \longrightarrow x_f(t)\,, \quad u(t) \longrightarrow \boxed{R_f(s)} \longrightarrow u_f(t)$$

We just need to translate the problem to a standard one [which we know (very well) how to solve]

Frequency-domain weighting filters

state filtering

$$x(t) \longrightarrow \boxed{Q_f(s)} \longrightarrow x_f(t)$$

- a MIMO process in general: if $x(t) \in \mathbb{R}^n$ and $x_f(t) \in \mathbb{R}^q$, then $Q_f(s)$ is a $q \times n$ transfer function matrix
- $Q_f(s)$: state filter; designer's choice; can be selected to meet the desired control action and the performance requirements
- write $Q_f(s) = C_1(sI - A_1)^{-1} B_1 + D_1$ in the general state-space realization:

$$\begin{cases} \dot{z}_1(t) = A_1 z_1(t) + B_1 x(t) \\ x_f(t) = C_1 z_1(t) + D_1 x(t) \end{cases} \tag{8}$$

Frequency-domain weighting filters

input filtering

$$u(t) \longrightarrow \boxed{R_f(s)} \longrightarrow u_f(t)$$

- $R_f(s)$: input filter; designer's choice; can be selected to meet the robustness requirements
- write $R_f(s) = C_2 (sI - A_2)^{-1} B_2 + D_2$ in the general state-space realization:

$$\begin{cases} \dot{z}_2(t) & = A_2 z_2(t) + B_2 u(t) \\ u_f(t) & = C_2 z_2(t) + D_2 u(t) \end{cases} \tag{9}$$

Back to time-domain design

Combining (6), (8) and (9) gives the enlarged system

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\begin{bmatrix} x(t) \\ z_1(t) \\ z_2(t) \end{bmatrix}}_{x_e(t)} = \underbrace{\begin{bmatrix} A & 0 & 0 \\ B_1 & A_1 & 0 \\ 0 & 0 & A_2 \end{bmatrix}}_{A_e} \begin{bmatrix} x(t) \\ z_1(t) \\ z_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \\ B_2 \end{bmatrix}}_{B_e} u(t)$$

and

$$x_f(t) = \underbrace{[D_1 \;\; C_1 \;\; 0]}_{C_e} \begin{bmatrix} x(t) \\ z_1(t) \\ z_2(t) \end{bmatrix}$$

$$u_f(t) = [0 \;\; 0 \;\; C_2] x_e(t) + D_2 u(t)$$

Summary of solution

With the enlarged system, the cost

$$J = \int_0^\infty \left(x_f^T(t) x_f(t) + \rho u_f^T(t) u_f(t) \right) \mathrm{d}t \tag{10}$$

translates to

$$J = \int_0^\infty \left(x_e^T(t) Q_e x_e(t) + 2u^T(t) \underbrace{\begin{bmatrix} 0 & 0 & \rho D_2^T C_2 \end{bmatrix}}_{N_e} x_e(t) + u^T(t) \underbrace{\rho D_2^T D_2}_{R_e} u(t) \right) \mathrm{d}t$$

$$Q_e = \begin{bmatrix} D_1^T D_1 & D_1^T C_1 & 0 \\ C_1^T D_1 & C_1^T C_1 & 0 \\ 0 & 0 & \rho C_2^T C_2 \end{bmatrix}$$

- solution (see appendix for more details):

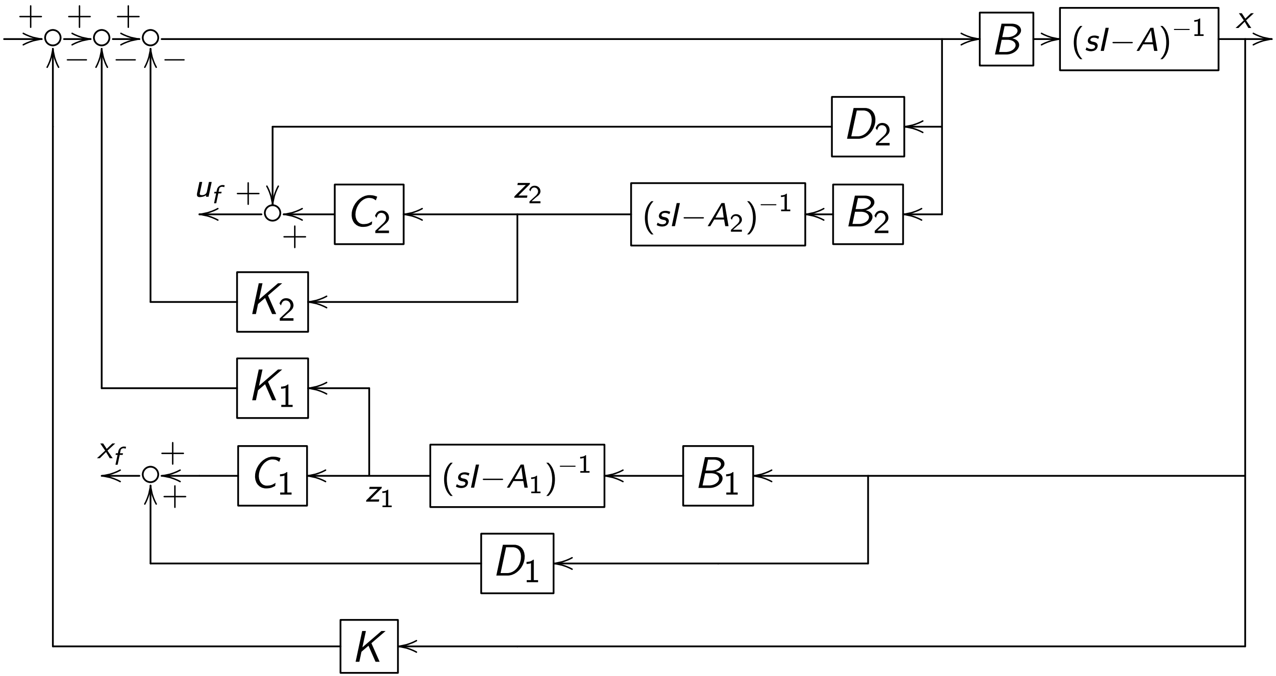
$$\boxed{u(t) = -R_e^{-1}(B_e^T P_e + N_e) x_e(t) = -Kx(t) - K_1 z_1(t) - K_2 z_2(t)}$$

- algebraic Riccati equation:

$$\boxed{A_e^T P_e + P_e A_e - (B_e^T P_e + N_e)^T R_e^{-1} (B_e^T P_e + N_e) + Q_e = 0}$$

Implementation

structure of the FSLQ system:



Appendix: general LQ solution

Consider LQ problems with cost

$$J = \int_0^\infty \left(x^T(t) \underbrace{C^T C}_{Q} x(t) + 2u^T(t) N x(t) + u^T(t) R u(t) \right) \mathrm{d}t \quad (11)$$

and system dynamics

$$\dot{x}(t) = Ax(t) + Bu(t)$$

- assume (A, B) is controllable/stabilizable and (A, C) is observable/detectable
- the solution of the problem is

$$u(t) = -R^{-1}(B^T P + N)x(t)$$

$$A^T P + PA - (B^T P + N)^T R^{-1}(B^T P + N) + Q = 0$$

Appendix: general LQ solution

Intuition: under the assumptions, we know we can stabilize the system and drive $x(t)$ to zero. Consider Lyapunov function $V(t) = x^T(t)Px(t)$, $P = P^T \succ 0$

$$\cancelto{0}{V(\infty)} - V(0) = \int_0^\infty \dot{V}(t)\mathrm{d}t$$
$$= \int_0^\infty \left(x^T(t)\left(PA + A^T P\right)x(t) + 2x^T(t)PBu(t)\right)\mathrm{d}t$$

Adding (11) on both sides yields

$$V(\infty) - V(0) + J =$$
$$\int_0^\infty \left(x^T(t)\left(Q + PA + A^T P\right)x(t) + 2x^T(t)\left(PB + N^T\right)u(t) + u^T(t)Ru(t)\right)\mathrm{d}t \tag{12}$$

- to minimize the cost, we are going to re-organize the terms in (12) into some "squared" terms

Appendix: general LQ solution

"completing the squares":

$$\begin{aligned}2x^T(t)\left(PB+N^T\right)u(t)+u^T(t)Ru(t) &= \left\| R^{1/2}u(t)+R^{-1/2}\left(B^TP+N\right)x(t)\right\|_2^2 \\ &\quad - x^T(t)\left(PB+N^T\right)R^{-1}\left(B^TP+N\right)x(t)\end{aligned}$$

hence (12) is actually

$$\begin{aligned}&\cancelto{0}{V(\infty)} - V(0) + J \\ &= \int_0^\infty \Big[x^T(t)\left(Q+PA+A^TP-\left(PB+N^T\right)R^{-1}\left(B^TP+N\right)\right)x(t) \\ &\qquad + \left\| R^{1/2}u(t)+R^{-1/2}\left(B^TP+N\right)x(t)\right\|_2^2 \Big] \mathrm{d}t\end{aligned}$$

hence $J_{\min} = V(0) = x^T(0)Px(0)$ is achieved when

$$Q+PA+A^TP-\left(PB+N^T\right)R^{-1}\left(B^TP+N\right)=0$$

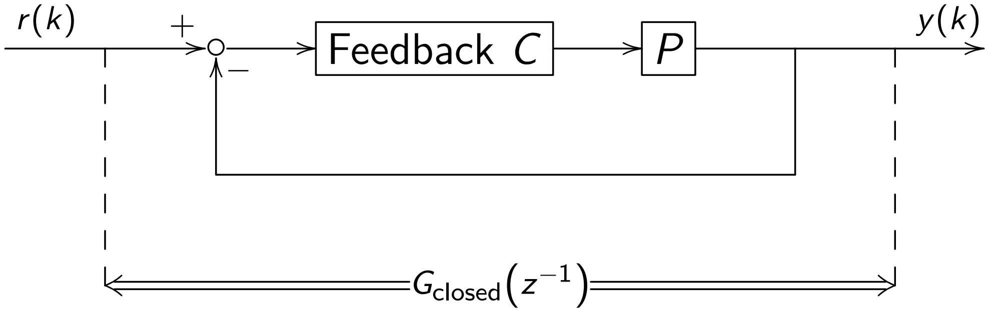
$$\text{and } u(t) = -R^{-1}\left(B^TP+N\right)x(t)$$

Lecture 11: Feedforward Control
Zero Phase Error Tracking

Big picture
Stable pole-zero cancellation
Phase error
Zero phase error tracking

Big picture

why are we learning this:

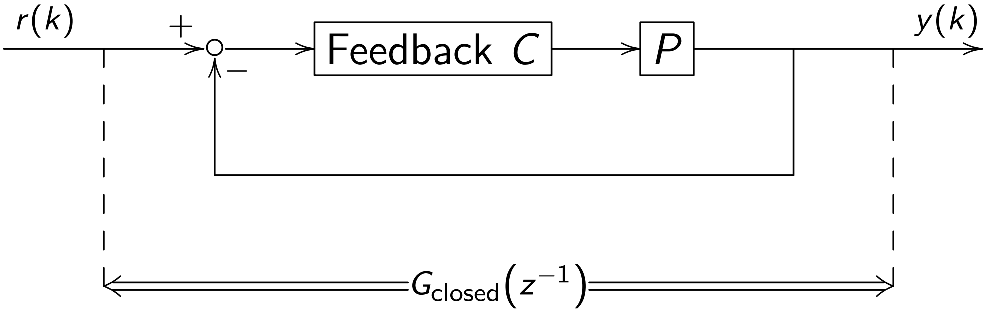


- two basic control problems: tracking (the reference) and regulation (against disturbances)
- feedback control has performance limitations
- For tracking $r(k)$, ideally we want

$$G_{\text{closed}}\left(z^{-1}\right) = 1$$

which is **not attainable** by feedback. We thus need **feedforward** control.

Big picture



- notation:

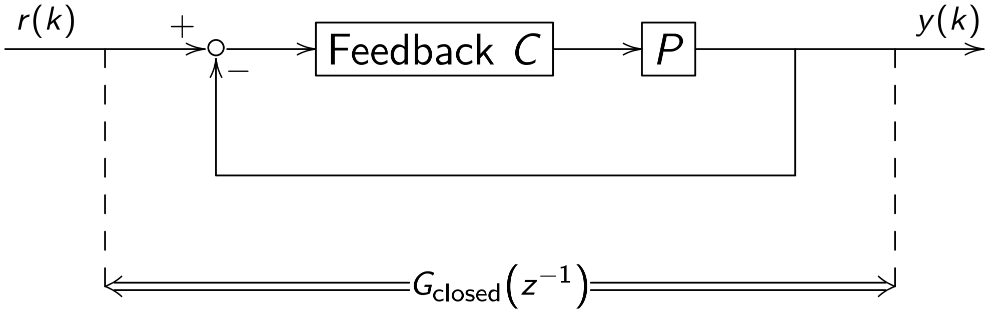
$$G_{\text{closed}}\left(z^{-1}\right) = \frac{z^{-d}B_c\left(z^{-1}\right)}{A_c\left(z^{-1}\right)}$$

where

$$B_c\left(z^{-1}\right) = b_{c0} + b_{c1}z^{-1} + \cdots + b_{cm}z^{-m}, \;\; b_{co} \neq 0$$
$$A_c\left(z^{-1}\right) = 1 + a_{c1}z^{-1} + \cdots + a_{cn}z^{-n}$$

- z^{-1}: one-step delay operator. $z^{-1}r(k) = r(k-1)$

Big picture



one naive approach: to let $y(k)$ track $y_d(k)$, we can do

$$r(k) = G_{\text{closed}}^{-1}(z^{-1})\, y_d(k) = \frac{z^d A_c(z^{-1})}{B_c(z^{-1})} y_d(k) = \frac{A_c(z^{-1})}{B_c(z^{-1})} y_d(k+d) \quad (1)$$

- **causality**: (1) requires knowledge of $y_d(k)$ for at least d steps ahead (usually not an issue)
- **stability**: poles of $G_{\text{closed}}^{-1}(z^{-1})$, i.e., zeros of $G_{\text{closed}}(z^{-1})$, must be all stable (usually an issue)
- **robustness**: the model $G_{\text{closed}}(z^{-1})$ needs to be accurate

The cancellable parts in $G_{\text{closed}}\left(z^{-1}\right)$

$$y_d(k) \longrightarrow \boxed{\text{Feedforward}} \longrightarrow \boxed{G_{\text{closed}}(z^{-1}) = \frac{z^{-d}B_c(z^{-1})}{A_c(z^{-1})}} \longrightarrow y(k)$$

- $G_{\text{closed}}(z^{-1})$ is always stable $\Rightarrow A_c\left(z^{-1}\right)$ can be fully canceled
- $B_c\left(z^{-1}\right)$ may contain *uncancellable parts* (zeros on or outside the unit circle)
- partition $G_{\text{closed}}(z^{-1})$ as

$$G_{\text{closed}}(z^{-1}) = \frac{z^{-d}B_c(z^{-1})}{A_c(z^{-1})} = \frac{z^{-d}\overbrace{B_c^{+}(z^{-1})}^{\text{cancellable}}\ \overbrace{B_c^{-}(z^{-1})}^{\text{uncancellable}}}{A_c(z^{-1})} \qquad (2)$$

Stable pole-zero cancellation

$$y_d(k) \longrightarrow \boxed{\text{Feedforward}} \longrightarrow \boxed{G_{\text{closed}}(z^{-1}) = \frac{z^{-d}B_c(z^{-1})}{A_c(z^{-1})}} \longrightarrow y(k)$$

feedforward via stable pole-zero cancellation:

$$G_{\text{spz}}\left(z^{-1}\right) = \frac{z^d A_c\left(z^{-1}\right)}{B_c^+\left(z^{-1}\right)} \frac{1}{B_c^-\left(1\right)} \tag{3}$$

where $B_c^-(1) = B_c^-\left(z^{-1}\right)\big|_{z^{-1}=1}$

- $B_c^-(1)$ makes the overall DC gain from $y_d(k)$ to $y(k)$ to be one:

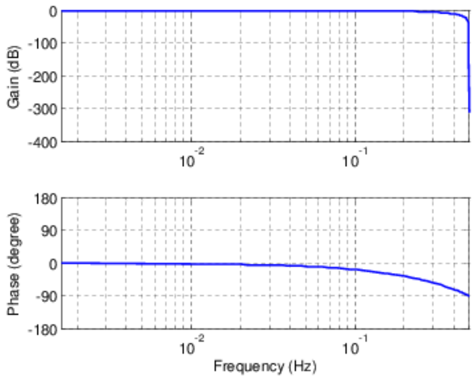
$$G_{y_d \to y}\left(z^{-1}\right) = G_{\text{spz}}\left(z^{-1}\right) G_{\text{closed}}\left(z^{-1}\right) = \frac{B_c^-(z^{-1})}{B_c^-(1)}$$

- example: $B_c^-\left(z^{-1}\right) = 1 + z^{-1}$, $B_c^-(1) = 2$, then

$$G_{y_d \to y}\left(z^{-1}\right) = \frac{1 + z^{-1}}{2}\text{: a moving-average low-pass filter}$$

Stable pole-zero cancellation

properties of $G_{y_d \to y}\left(z^{-1}\right) = \frac{1+z^{-1}}{2}$:



- there is always a **phase error** in tracking
- example: if $y_d(k) = \alpha k$ (a ramp signal)

$$y(k) = G_{y_d \to y}\left(z^{-1}\right) y_d(k) = \alpha k - \frac{\alpha}{2}$$

which is always delayed by a factor of $\alpha/2$

Zero Phase Error Tracking (ZPET)

$$y_d(k) \longrightarrow \boxed{\text{Feedforward}} \longrightarrow \boxed{G_{\text{closed}}(z^{-1}) = \frac{z^{-d}B_c^{+}(z^{-1})B_c^{-}(z^{-1})}{A_c(z^{-1})}} \longrightarrow y(k)$$

Zero Phase Error Tracking (ZPET): extend (3) by adding a $B_c^{-}(z)$ part

$$\boxed{G_{\text{ZPET}}(z^{-1}) = \frac{z^{d}A_c(z^{-1})}{B_c^{+}(z^{-1})}\frac{B_c^{-}(z)}{B_c^{-}(1)^2}} \tag{4}$$

where $B_c^{-}(z) = b_{c0}^{-} + b_{c1}^{-}z + \cdots + b_{cs}^{-}z^{s}$ if $B_c^{-}(z^{-1}) = b_{c0}^{-} + b_{c1}^{-}z^{-1} + \cdots + b_{cs}^{-}z^{-s}$

- overall dynamics between $y(k)$ and $y_d(k)$:

$$G_{y_d\to y}\left(z^{-1}\right) = G_{\text{closed}}\left(z^{-1}\right)G_{\text{ZPET}}\left(z^{-1}\right) = \frac{B_c^{-}(z)B_c^{-}(z^{-1})}{\left[B_c^{-}(1)\right]^2} \tag{5}$$

Zero Phase Error Tracking (ZPET)

understanding (5):

- the frequency response always has **zero phase error**:

$$B_c^-\left(e^{j\omega}\right) = \overline{B_c^-\left(e^{-j\omega}\right)} \text{ (a complex conjugate pair)}$$

- $B_c^-(1)^2$ normalizes $G_{y_d \to y}$ to have unity DC gain:

$$G_{y_d\to y}\left(e^{-j\omega}\right)\Big|_{\omega=0} = \frac{B_c^-(e^{j\omega})\big|_{\omega=0}\,B_c^-(e^{-j\omega})\big|_{\omega=0}}{\left[B_c^-(1)\right]^2} = \cancelto{1}{\frac{[B_c^-(1)]^2}{\left[B_c^-(1)\right]^2}}$$

- example: $B_c^-\left(z^{-1}\right) = 1 + z^{-1}$, then

$$G_{y_d\to y}\left(z^{-1}\right) = \frac{(1+z)\left(1+z^{-1}\right)}{4}$$

- if $y_d(k) = \alpha k$, then $y(k) = \alpha k$!
- fact: ZPET provides perfect tracking to step and ramp signals at steady state (see ME 233 course reader)

Zero Phase Error Tracking (ZPET)

Example: $B_c^-\left(z^{-1}\right) = 1 + 2z^{-1}$

$$G_{y_d \to y}\left(z^{-1}\right) = \frac{(1+2z)(1+2z^{-1})}{9} = \frac{2z+5+2z^{-1}}{9}$$

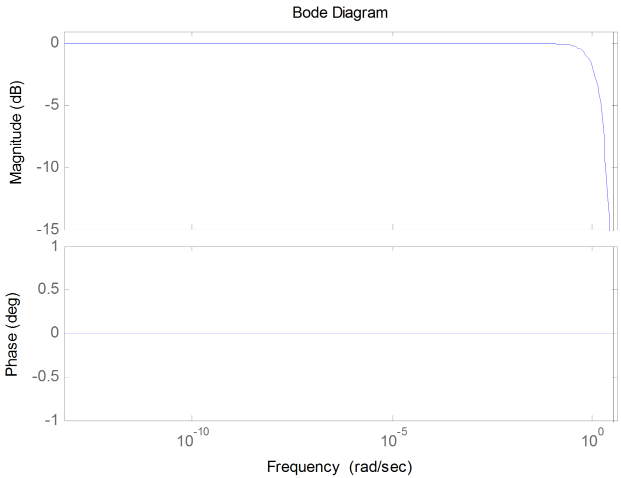


Figure: Bode Plot of $\frac{2z+5+2z^{-1}}{9}$

Implementation

$$y_d(k) \longrightarrow \boxed{G_{\text{ZPET}}\left(z^{-1}\right)} \xrightarrow{r(k)} \boxed{G_{\text{closed}}(z^{-1}) = \frac{z^{-d}B_c^+(z^{-1})B_c^-(z^{-1})}{A_c(z^{-1})}} \longrightarrow y(k)$$

$$r(k) = \left[\frac{z^d A_c(z^{-1})}{B_c^+(z^{-1})}\frac{B_c^-(z)}{B_c^-(1)^2}\right] y_d(k)$$

- z^d is not causal $\Rightarrow$ do instead

$$r(k) = \left[\frac{A_c(z^{-1})}{B_c^+(z^{-1})}\frac{B_c^-(z)}{B_c^-(1)^2}\right] y_d(k+d)$$

- $B_c^-(z) = b_{c0}^- + b_{c1}^- z + \cdots + b_{cs}^- z^s$ is also not causal $\Rightarrow$ do instead

$$r(k) = \left[\frac{A_c(z^{-1})}{B_c^+(z^{-1})}\frac{z^{-s}B_c^-(z)}{B_c^-(1)^2}\right] y_d(k+d+s)$$

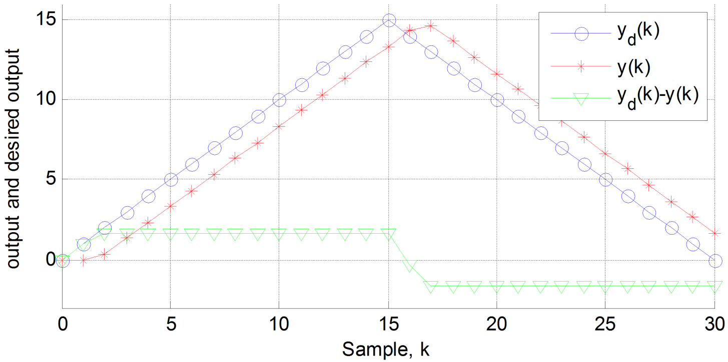
- at time k, requires $y_d(k+d+s)$: $d+s$ steps preview of the desired output

Implementation

Example:

$$G_{\text{closed}}(z^{-1}) = \frac{z^{-1}(1+2z^{-1})}{3}$$

- without feedforward control:

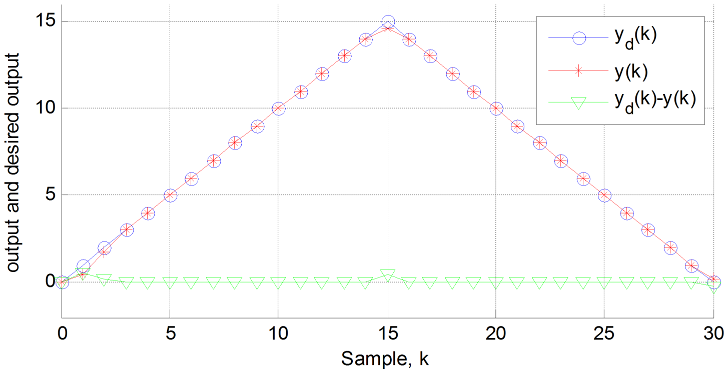


Implementation

Example:

$$G_{\text{closed}}(z^{-1}) = \frac{z^{-1}(1+2z^{-1})}{3}$$

- with ZPET feedforward:



Implementation

ZPET extensions:

- standard form:

$$G_{\text{ZPET}}(z^{-1}) = \frac{z^d A_c(z^{-1})}{B_c^+(z^{-1})} \frac{B_c^-(z)}{B_c^-(1)^2}$$

- extended bandwidth (ref: B. Haack and M. Tomizuka, "The effect of adding zeros to feedforward controllers," *ASME J. Dyn. Syst. Meas. Control*, vol. 113, pp. 6-10, 1991):

$$G_{\text{ZPET}}(z^{-1}) = \frac{z^d A_c(z^{-1})}{B_c^+(z^{-1})} \frac{B_c^-(z)}{B_c^-(1)^2} \frac{\left(z^{-1}-b\right)(z-b)}{(1-b)^2},\ 0<b<1$$

- remark: $\left(z^{-1}-b\right)(z-b)/(1-b)^2$, $0<b<1$ is a high-pass filter with unity DC gain

Lecture 12: Preview Control

Big picture
Problem formulation
Relationship to LQ
Solution

Review: optimal tracking

We consider controlling the system

$$x(k+1) = Ax(k) + Bu(k) \quad (1)$$
$$y(k) = Cx(k)$$

where

$$x \in \mathbb{R}^n, \; u \in \mathbb{R}^m, \; y \in \mathbb{R}^r$$

Optimal tracking with full reference information (homework 1):

$$\begin{aligned}\min_{U_0} J := & \frac{1}{2}\left[y_d(N) - y(N)\right]^T S \left[y_d(N) - y(N)\right] \\ & + \frac{1}{2}\sum_{k=0}^{N-1}\Big(\left[y_d(k) - y(k)\right]^T Q_y \left[y_d(k) - y(k)\right] + u(k)^T R u(k)\Big) \quad (2)\end{aligned}$$

$$u^o(k) = -\left[R + B^T P(k+1) B\right]^{-1} B^T \left[P(k+1)Ax(k) + b^T(k+1)\right] \quad (3)$$

$$J_k^o(x(k)) = \frac{1}{2}x^T(k)P(k)x(k) + b(k)x(k) + c(k) \quad (4)$$

Overview of preview control

Preview control considers the same cost-function structure, with:

- **a N_p-step preview window**: the desired output signals in this window are known
- **post preview window**: after the preview window we assume we no longer know the desired output (due to, e.g., limited vision in the example of vehicle driving), but we assume the reference is generated from some models.
- e.g. (deterministic model)

$$y_d(k+N_p+l) = y_d(k+N_p), \; l > 0 \tag{5}$$

- or (stochastic model):

$$\begin{aligned} x_d(k+1) &= A_d x_d(k) + B_d w_d(k) \\ y_d(k) &= C_d x_d(k) \end{aligned} \tag{6}$$

where $w_d(k)$ is white and Gaussian distributed. Note: if $A_d = I$, $B_d = 0$, $C_d = I$, $x_d(k+N_p) = y_d(k+N_p)$, then (6)$\Leftrightarrow$(5).

Structuring the future knowledge

Knowledge of the future trajectory can be built into

$$\underbrace{\begin{bmatrix} y_d(k+1) \\ y_d(k+2) \\ \vdots \\ y_d(k+N_p) \\ \hline x_d(k+N_p+1) \end{bmatrix}}_{Y_d(k+1)} = \underbrace{\left[\begin{array}{cccc|c} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 & C_d \\ \hline 0 & \dots & 0 & 0 & A_d \end{array}\right]}_{A_{Y_d}} \underbrace{\begin{bmatrix} y_d(k) \\ y_d(k+1) \\ \vdots \\ y_d(k+N_p-1) \\ \hline x_d(k+N_p) \end{bmatrix}}_{Y_d(k)}$$

$$+ \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \hline B_d \end{bmatrix}}_{B_{Y_d}} \underbrace{w_d(k+N_p)}_{\bar{w}_d(k)} \quad (7)$$

The cost function

At time k

$$J_k = \frac{1}{1+N} \mathrm{E}\Big\{ (y(N+k) - y_d(N+k))^T S_y (y(N+k) - y_d(N+k)) + \sum_{j=0}^{N-1} \Big[(y(j+k) - y_d(j+k))^T Q_y (y(j+k) - y_d(j+k)) + u(j+k)^T R u(j+k) \Big] \Big\} \quad (8)$$

- a moving horizon cost
- only $u(k)$ is applied to the plant after we find a solution to minimize J_k.
- in deterministic formulation, we remove the expectation sign. In stochastic formulation, expectation is taken with respect to

$$\{ w_d(k+N_p), w_d(k+N_p+1), \ldots, w_d(k+N-1) \}$$

for the minimization of J_k.

Augmenting the system

Augmenting the plant with the reference model yields

$$\boxed{\begin{bmatrix} x(k+1) \\ Y_d(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 \\ 0 & A_{Y_d} \end{bmatrix}}_{A_e} \underbrace{\begin{bmatrix} x(k) \\ Y_d(k) \end{bmatrix}}_{x_e(k)} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{B_e} u(k) + \underbrace{\begin{bmatrix} 0 \\ B_{Y_d} \end{bmatrix}}_{B_{w,e}} \bar{w}_d(k)} \tag{9}$$

and

$$\begin{aligned} y(j+k) - y_d(j+k) &= Cx(k+j) - [I, 0, \ldots, 0]\, Y_d(k+j) \\ &= \underbrace{[C, -I, 0, \ldots, 0]}_{C_e} x_e(k+j) \end{aligned}$$

Translation to a standard LQ

$$y(j+k) - y_d(j+k) = \underbrace{[C, -I, 0, \ldots, 0]}_{C_e} x_e(k+j)$$

Hence

$$J_k = \frac{1}{1+N} \mathrm{E} \Big\{ (y(N+k) - y_d(N+k))^T S_y (y(N+k) - y_d(N+k))$$
$$+ \sum_{j=0}^{N-1} \Big[(y(j+k) - y_d(j+k))^T Q_y (y(j+k) - y_d(j+k)) + u(j+k)^T R u(j+k) \Big] \Big\}$$

is nothing but

$$J_k = \frac{1}{1+N} \mathrm{E} \Bigg\{ x_e(N+k)^T C_e^T S_y C_e x_e(N+k)$$
$$+ \sum_{j=0}^{N-1} \Big[x_e(j+k)^T C_e^T Q_y C_e x_e(j+k) + u(j+k)^T R u(j+k) \Big] \Bigg\} \quad (10)$$

Solution of the preview control problem

The equivalent formulation

$$x_e(k+1) = A_e x_e(k) + B_e u(k) + B_{w,e}\bar{w}_d(k)$$

$$J_k = \frac{1}{1+N} \mathrm{E}\left\{ x_e(N+k)^T C_e^T S_y C_e x_e(N+k) \right.$$
$$\left. + \sum_{j=0}^{N-1} \left[x_e(j+k)^T C_e^T Q_y C_e x_e(j+k) + u(j+k)^T R u(j+k) \right] \right\}$$

is a standard LQ (deterministic formulation) or a standard LQG problem with exactly known state (stochastic formulation). Hence

$$u^o(k) = -\left[B_e^T P(k+1) B_e + R \right]^{-1} B_e^T P(k+1) A_e x_e(k)$$

$$P(k) = -A_e^T P(k+1) B_e \left[B_e^T P(k+1) B_e + R \right]^{-1} B_e^T P(k+1) A_e$$
$$+ A_e^T P(k+1) A_e + C_e^T Q_y C_e$$

where $P(k+N) = C_e^T S_y C_e$

Remark

Let $u^o(k) = K_e x_e(k) = \begin{bmatrix} K_{e1}(k) & K_{e2}(k) \end{bmatrix} x_e(k)$, the closed-loop matrix is

$$
\begin{aligned}
A_e - B_e K_e(k) &= \begin{bmatrix} A & 0 \\ 0 & A_{Y_d} \end{bmatrix} - \begin{bmatrix} B \\ 0 \end{bmatrix} \begin{bmatrix} K_{e1}(k) & K_{e2}(k) \end{bmatrix} \\
&= \begin{bmatrix} A - BK_{e1}(k) & -BK_{e2}(k) \\ 0 & A_{Y_d} \end{bmatrix}
\end{aligned}
$$

- the closed-loop eigenvalue from A_{Y_d} will not be changed.
- The Riccati equation may look ill conditioned if A_{Y_d} contains marginally stable eigenvalues. This, however, does not cause a problem. For additional details, see the course reader or come to the instructor's office hour .

Summary

1. Big picture

2. Formulation of the optimal control problem

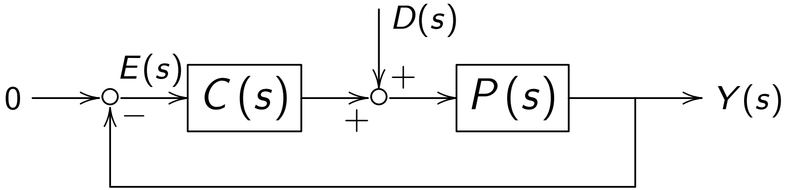
3. Translation to a standard LQ

Lecture 13: Internal Model Principle and Repetitive Control

Big picture

review of integral control in PID design

example:



where

$$P(s) = \frac{1}{ms+b}, \; C(s) = k_p + k_i \frac{1}{s} + k_d s, \; k_p, k_i, k_d > 0$$

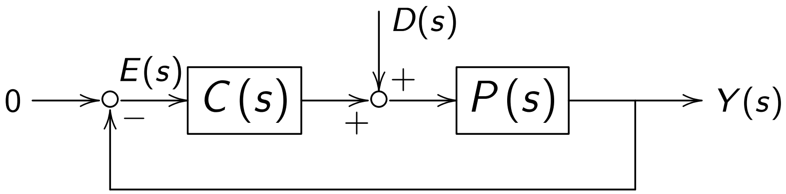
- the integral action in PID control perfectly rejects (asymptotically) constant disturbances ($D(s) = d_o/s$):

$$E(s) = \frac{-P(s)}{1+P(s)C(s)} D(s) = \frac{-d_o}{(m+k_d)s^2 + (k_p+b)s + k_i}$$

$$\Rightarrow e(t) \to 0$$

Big picture

review of integral control in PID design



the "structure" of the reference/disturbance is built into the integral controller:

- controller:

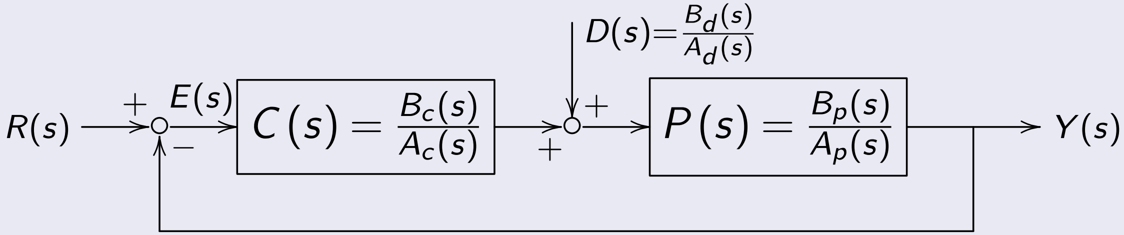
$$C(s) = k_p + k_i \frac{1}{s} + k_d s = \boxed{\frac{1}{s}} \left(k_d s^2 + k_p s + k_i\right)$$

- constant disturbance:

$$d(t) = d_o \Leftrightarrow \mathscr{L}\{d(t)\} = \boxed{\frac{1}{s}} d_o$$

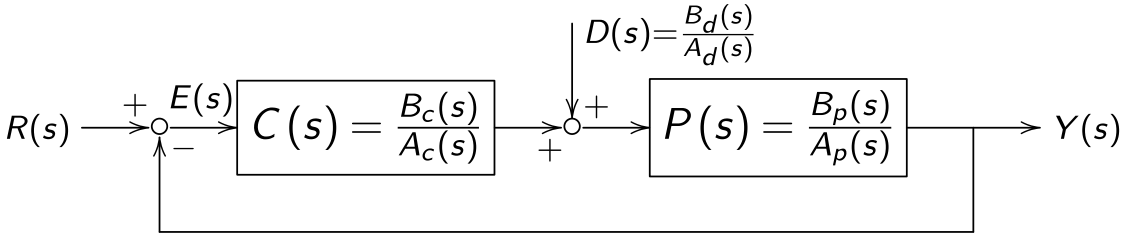
General case: internal model principle (IMP)

Theorem (Internal Model Principle)



Assume $B_p(s)=0$ and $A_d(s)=0$ do not have common roots.
If the closed loop is asymptotically stable,
and $A_c(s)$ can be factorized as $A_c(s)=A_d(s)A_c^{'}(s)$,
then the disturbance is asymptotically rejected.

General case: internal model principle (IMP)



Proof: The steady-state error response to the disturbance is

$$E(s) = \frac{-P(s)}{1+P(s)C(s)}D(s) = \frac{-B_p(s)A_c(s)}{A_p(s)A_c(s)+B_p(s)B_c(s)}\frac{B_d(s)}{A_d(s)}$$

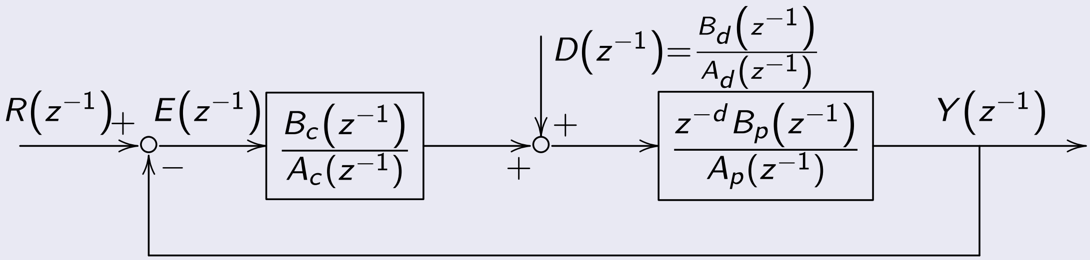
$$= \frac{-B_p(s)A_c'(s)B_d(s)}{A_p(s)A_c(s)+B_p(s)B_c(s)}$$

where all roots of $A_p(s)A_c(s)+B_p(s)B_c(s)=0$ are on the left half plane. Hence $e(t)\to 0$

Internal model principle

discrete-time case:

Theorem (Discrete-time IMP)

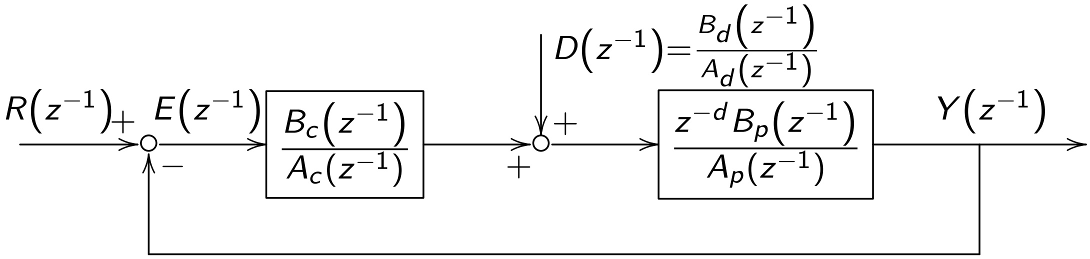


Assume $B_p(z^{-1}) = 0$ *and* $A_d(z^{-1}) = 0$ *do not have common zeros.*
If the closed loop is asymptotically stable,
and $A_c(z^{-1})$ *can be factorized as* $A_c(z^{-1}) = A_d(z^{-1})A_c'(z^{-1})$*,*
then the disturbance is asymptotically rejected.

Proof: analogous to the continuous-time case.

Internal model principle

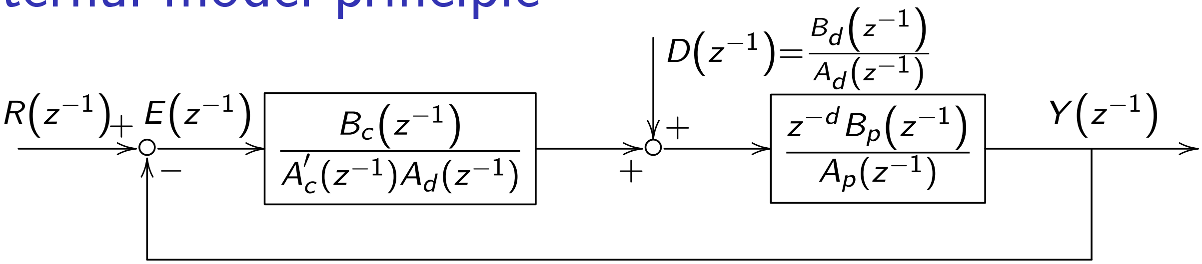
the disturbance structure:



example disturbance structures:

$d(k)$	$A_d(z^{-1})$
constant d_o	$1 - z^{-1}$
$\cos(\omega_0 k)$ and $\sin(\omega_0 k)$	$1 - 2z^{-1}\cos(\omega_0) + z^{-2}$
shifted ramp signal $d(k) = \alpha k + \beta$	$1 - 2z^{-1} + z^{-2}$
periodic: $d(k) = d(k - N)$	$1 - z^{-N}$

Internal model principle



observations:

- the controller contains a "counter disturbance" generator
- high-gain control: the open-loop frequency response

$$P\left(e^{-j\omega}\right)C\left(e^{-j\omega}\right) = \frac{e^{-dj\omega}B_p\left(e^{-j\omega}\right)B_c\left(e^{-j\omega}\right)}{A_p\left(e^{-j\omega}\right)A_c'\left(e^{-j\omega}\right)A_d\left(e^{-j\omega}\right)}$$

  is large at frequencies where $A_d(e^{-j\omega}) = 0$
- $D\left(z^{-1}\right) = B_d\left(z^{-1}\right)/A_d\left(z^{-1}\right)$ means $d\left(k\right)$ is the impulse response of $B_d\left(z^{-1}\right)/A_d\left(z^{-1}\right)$:

$$A_d\left(z^{-1}\right)d\left(k\right) = B_d\left(z^{-1}\right)\delta\left(k\right)$$

Outline

1. Big Picture
   review of integral control in PID design

2. Internal Model Principle
   theorems
   typical disturbance structures

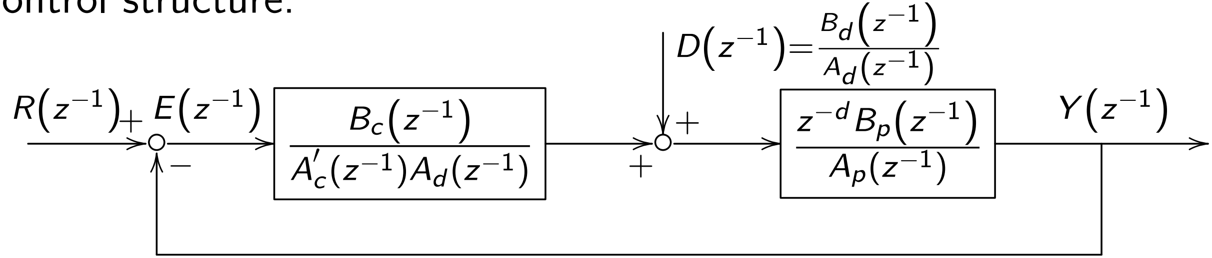
3. Repetitive Control
   use of internal model principle
   design by pole placement
   design by stable pole-zero cancellation

Repetitive control

Repetitive control focus on attenuating periodic disturbances with

$$A_d\left(z^{-1}\right) = 1 - z^{-N}$$

Control structure:



It remains to design $B_c\left(z^{-1}\right)$ and $A_c'\left(z^{-1}\right)$. We discuss two methods:

- **pole placement**
- (partial) cancellation of plant dynamics: **prototype repetitive control**

1, Pole placement: prerequisite

Theorem

Consider $G(z) = \frac{\beta(z)}{\alpha(z)} = \frac{\beta_1 z^{n-1} + \beta_2 z^{n-2} + \cdots + \beta_n}{z^n + \alpha_1 z^{n-1} + \cdots + \alpha_n}$. $\alpha(z)$ *and* $\beta(z)$ *are coprime (no common roots) iff* S *(Sylvester matrix) is nonsingular:*

$$S = \begin{bmatrix} 1 & 0 & \cdots & 0 & \beta_1 & 0 & \cdots & \cdots & 0 \\ \alpha_1 & 1 & \ddots & \vdots & \beta_2 & \beta_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \alpha_1 & 1 & \beta_{n-1} & & \ddots & \ddots & 0 \\ \alpha_{n-1} & & & \alpha_1 & \beta_n & \ddots & & \ddots & \beta_1 \\ \alpha_n & \ddots & & \vdots & 0 & \beta_n & \ddots & & \beta_2 \\ 0 & \alpha_n & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \alpha_{n-1} & \vdots & & \ddots & \beta_n & \beta_{n-1} \\ 0 & \cdots & 0 & \alpha_n & 0 & \cdots & \cdots & 0 & \beta_n \end{bmatrix}_{(2n-1)\times(2n-1)}$$

1, Pole placement: prerequisite

Example:

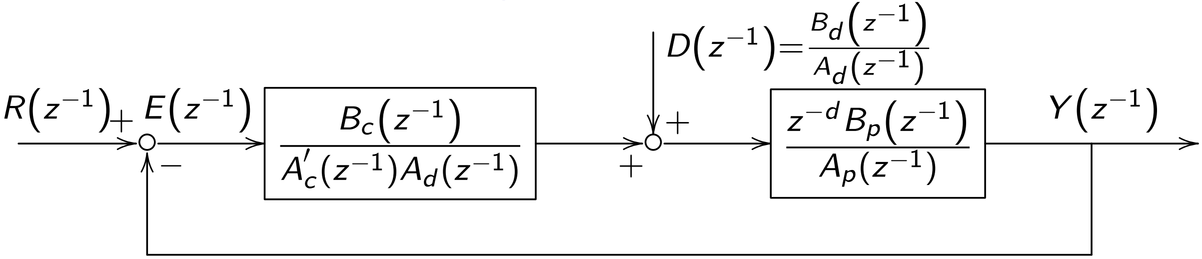
$$G(z) = \frac{\beta_1 z^{n-1} + \beta_2 z^{n-2} + \cdots + \beta_n}{z^n + \alpha_1 z^{n-1} + \cdots + \alpha_n} = \frac{z^{n-1} + \alpha_1 z^{n-2} + \cdots + \alpha_{n-1}}{z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1} z + 0}$$

i.e.

$$\beta_1 = 1$$
$$\beta_i = \alpha_{i-1} \ \forall i \geq 2$$
$$\alpha_n = 0$$

$\alpha(z)$ and $\beta(z)$ are not coprime, and S is clearly singular.

1, Pole placement: big picture



Disturbance model: $A_d\left(z^{-1}\right) = 1 - z^{-N}$

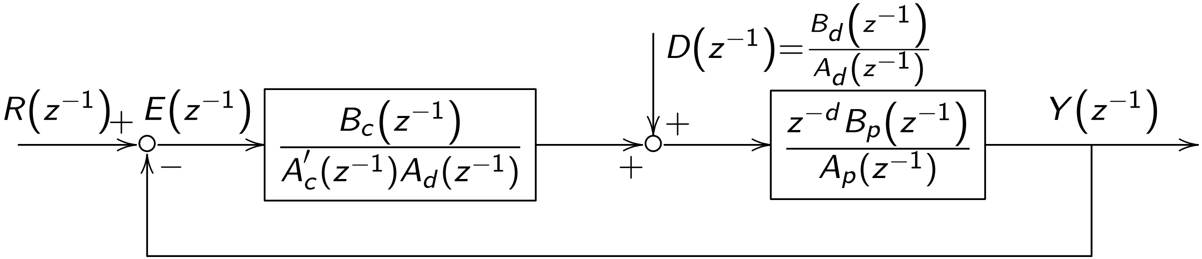
Pole placement assigns the closed-loop characteristic equation:

$$z^{-d}B_p\left(z^{-1}\right)B_c\left(z^{-1}\right) + A_p\left(z^{-1}\right)A_c'\left(z^{-1}\right)A_d\left(z^{-1}\right)$$
$$= \underbrace{1 + \eta_1 z^{-1} + \eta_2 z^{-2} + \cdots + \eta_q z^{-q}}_{\eta(z^{-1})}$$

which is in the structure of a *Diophantine equation*.

Design procedure: specify the desired closed-loop dynamics $\eta\left(z^{-1}\right)$; match coefficients of z^{-i} on both sides to get $B_c\left(z^{-1}\right)$ and $A_c'\left(z^{-1}\right)$.

1, Pole placement: big picture



Diophantine equation in Pole placement:

$$z^{-d}B_p\left(z^{-1}\right)B_c\left(z^{-1}\right)+A_p\left(z^{-1}\right)A_c'\left(z^{-1}\right)A_d\left(z^{-1}\right) = \underbrace{1+\eta_1 z^{-1}+\eta_2 z^{-2}+\cdots+\eta_q z^{-q}}_{\eta\left(z^{-1}\right)}$$

Questions:

- what are the constraints for choosing $\eta\left(z^{-1}\right)$?
- how to guarantee unique solution in Diophantine equation?

Design and analysis procedure

General procedure of control design:

- Problem definition
- Control design for solution (current stage)
- Prove stability
- Prove stability robustness
- Case study or implementation results

1, Pole placement: details

Theorem (Diophantine equation)

Given

$$\eta\left(z^{-1}\right) = 1 + \eta_1 z^{-1} + \eta_2 z^{-2} + \cdots + \eta_q z^{-q}$$
$$\alpha\left(z^{-1}\right) = 1 + \alpha_1 z^{-1} + \cdots + \alpha_n z^{-n}$$
$$\beta\left(z^{-1}\right) = \beta_1 z^{-1} + \beta_2 z^{-2} + \cdots + \beta_n z^{-n}$$

The Diophantine equation

$$\alpha\left(z^{-1}\right)\sigma\left(z^{-1}\right) + \beta\left(z^{-1}\right)\gamma\left(z^{-1}\right) = \eta\left(z^{-1}\right)$$

can be solved uniquely for $\sigma\left(z^{-1}\right)$ *and* $\gamma\left(z^{-1}\right)$

$$\sigma\left(z^{-1}\right) = 1 + \sigma_1 z^{-1} + \cdots + \sigma_{n-1} z^{-(n-1)}$$
$$\gamma\left(z^{-1}\right) = \gamma_0 + \gamma_1 z^{-1} + \cdots + \gamma_{n-1} z^{-(n-1)}$$

if the numerators of $\alpha\left(z^{-1}\right)$ *and* $\beta\left(z^{-1}\right)$ *are coprime and*
$\deg\left(\eta\left(z^{-1}\right)\right) = q \leq 2n-1$

1, Pole placement: details

Proof of Diophantine equation Theorem (key ideas):

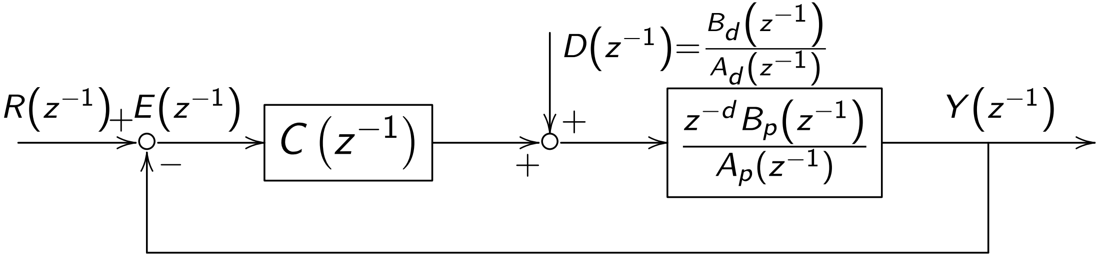
$$\alpha\left(z^{-1}\right)\underbrace{\sigma\left(z^{-1}\right)}_{\text{unknown}}+\beta\left(z^{-1}\right)\underbrace{\gamma\left(z^{-1}\right)}_{\text{unknown}}=\eta\left(z^{-1}\right)$$

Matching the coefficients of z^{-i} gives (see one numerical example in course reader)

$$S\begin{bmatrix}\sigma_1\\ \sigma_2\\ \vdots\\ \sigma_{n-1}\\ \gamma_0\\ \vdots\\ \gamma_{n-1}\end{bmatrix}+\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\\ 0\\ \vdots\\ 0\end{bmatrix}=\begin{bmatrix}\eta_1\\ \eta_2\\ \vdots\\ \eta_{n-1}\\ \eta_n\\ \vdots\\ \eta_{2n-1}\end{bmatrix}$$

The coprime condition assures S is invertible. $\deg\eta\left(z^{-1}\right)\leq 2n-1$ assures the proper dimension on the right hand side of the equality.

2, Prototype repetitive control: simple case



$$A_d\left(z^{-1}\right) = 1 - z^{-N}$$

If all poles and zeros of the plant are stable, then prototype repetitive control uses

$$C\left(z^{-1}\right) = \frac{k_r z^{-N+d} A_p\left(z^{-1}\right)}{\left(1 - z^{-N}\right) B_p\left(z^{-1}\right)}$$

Theorem (Prototype repetitive control)

Under the assumptions above, the closed-loop system is asymptotically stable for $0 < k_r < 2$

2, Prototype repetitive control: stability

Proof of Theorem on prototype repetitive control:
From

$$1+\frac{k_r z^{-N+d} A_p\left(z^{-1}\right)}{\left(1-z^{-N}\right) B_p\left(z^{-1}\right)} \frac{z^{-d} B_p\left(z^{-1}\right)}{A_p\left(z^{-1}\right)}=0$$

the closed-loop characteristic equation is

$$A_p\left(z^{-1}\right) B_p\left(z^{-1}\right)\left[1-\left(1-k_r\right) z^{-N}\right]=0$$

- roots of $A_p\left(z^{-1}\right) B_p\left(z^{-1}\right)=0$ are all stable
- roots of $1-\left(1-k_r\right) z^{-N}=0$ are

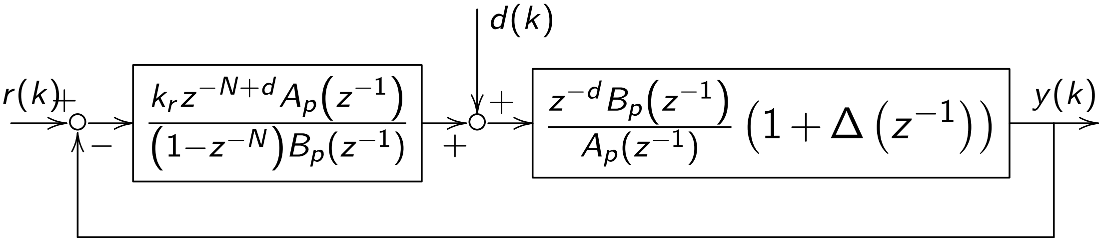
$$\left|1-k_r\right|^{\frac{1}{N}} e^{j \frac{2\pi i}{N}}, \ i=0, \pm 1, \ldots, \text{ for } 0<k_r \leq 1$$

$$\left|1-k_r\right|^{\frac{1}{N}} e^{j\left(\frac{2\pi i}{N}+\frac{\pi}{N}\right)}, \ i=0, \pm 1, \ldots, \text{ for } 1<k_r$$

which are all inside the unit circle

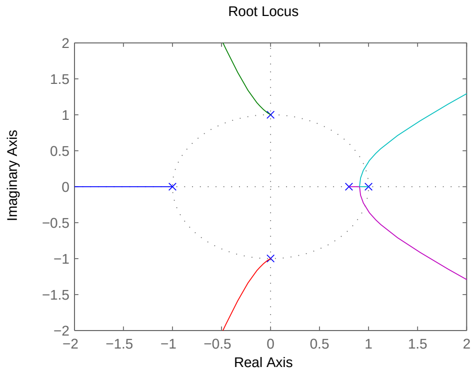
2, Prototype repetitive control: stability robustness

Consider the case with plant uncertainty



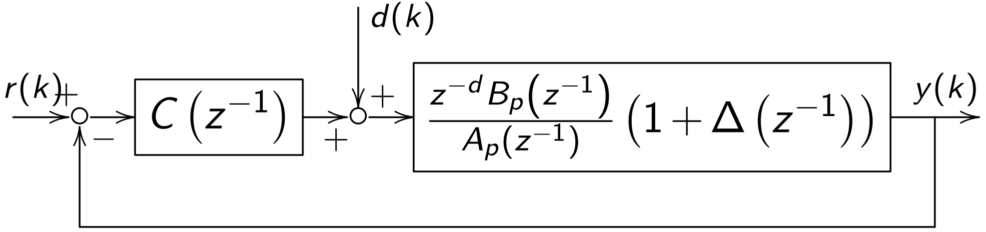
N open-loop poles on the unit circle

Root locus example: $N = 4$, $1+\Delta\left(z^{-1}\right) = q/(z-p)$



$\forall\ k_r > 0$, the closed-loop system is now unstable!

2, Prototype repetitive control: stability robustness



To make the controller robust to plant uncertainties, do instead

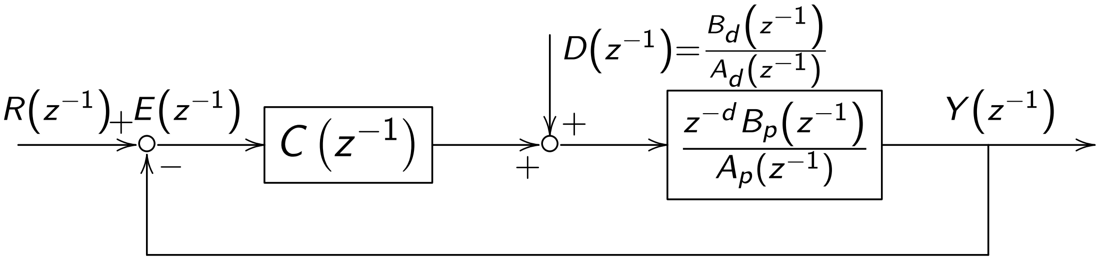
$$C\left(z^{-1}\right) = \frac{k_r q(z, z^{-1}) z^{-N+d} A_p\left(z^{-1}\right)}{\left(1 - q(z, z^{-1}) z^{-N}\right) B_p\left(z^{-1}\right)} \tag{1}$$

$q(z, z^{-1})$: low-pass filter. e.g. zero-phase low pass $\dfrac{\alpha_1 z^{-1} + \alpha_0 + \alpha_1 z}{\alpha_0 + 2\alpha_1}$

which shifts the marginary stable open-loop poles to be inside the unit circle:

$$A_p\left(z^{-1}\right) B_p\left(z^{-1}\right) \left[1 - (1 - k_r)\, q(z, z^{-1}) z^{-N}\right] = 0$$

2, Prototype repetitive control: extension



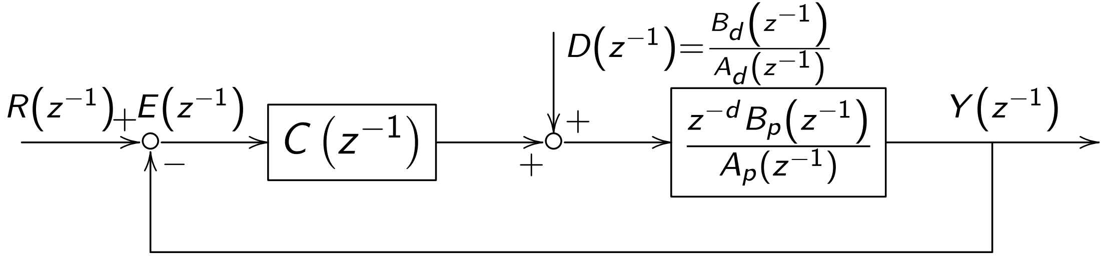
If poles of the plant are stable but **NOT all zeros are stable**, let $B_p(z^{-1}) = B_p^-(z^{-1})B_p^+(z^{-1})$ [$B_p^-(z^{-1})$—the uncancellable part] and

$$C(z^{-1}) = \frac{k_r z^{-N+\mu} A_p(z^{-1}) B_p^-(z) z^{-\mu}}{(1 - z^{-N}) B_p^+(z^{-1}) z^{-d} b}, \ b > \max_{\omega \in [0,\pi]} |B_p^-(e^{j\omega})|^2 \quad (2)$$

Similar as before, can show that the closed-loop system is stable (in-class exercise).

2, Prototype repetitive control: extension

Exercise: analyze the stability of



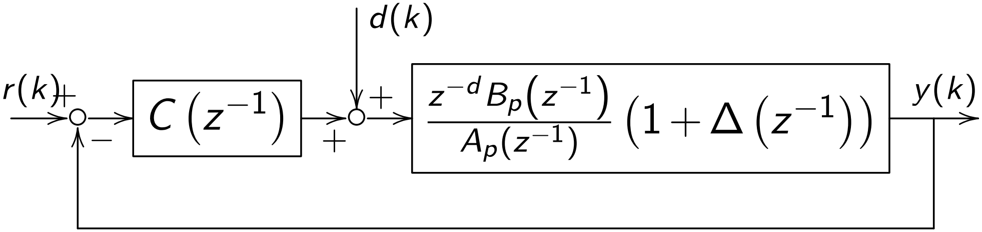
$$C\left(z^{-1}\right) = \frac{k_r z^{-N+\mu} A_p(z^{-1}) B_p^-(z) z^{-\mu}}{(1-z^{-N}) B_p^+(z^{-1}) z^{-d} b}, \ b > \max_{\omega \in [0,\pi]} |B_p^-(e^{j\omega})|^2 \quad (3)$$

Key steps: $\left|\frac{B_p^-\left(e^{j\omega}\right)B_p^-\left(e^{-j\omega}\right)}{b}\right| < 1$; $\left|\frac{k_r B_p^-\left(e^{j\omega}\right)B_p^-\left(e^{-j\omega}\right)}{b} - 1\right| < 1$; all roots from

$$z^{-N}\left[\frac{k_r B_p^-(z) B_p^-\left(z^{-1}\right)}{b} - 1\right] + 1 = 0$$

are inside the unit circle.

2, Prototype repetitive control: extension



Robust version in the presence of plant uncertainties:

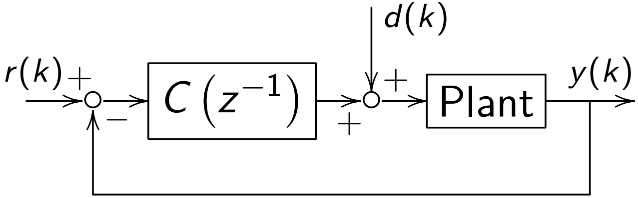
$$C\left(z^{-1}\right) = \frac{k_r z^{-N+\mu} q(z,z^{-1}) A_p(z^{-1}) B_p^{-}(z) z^{-\mu}}{(1 - q(z,z^{-1}) z^{-N}) B_p^{+}(z^{-1}) z^{-d} b} \tag{4}$$

where

$q(z,z^{-1})$: low-pass filter. e.g. zero-phase low pass $\dfrac{\alpha_1 z^{-1} + \alpha_0 + \alpha_1 z}{\alpha_0 + 2\alpha_1}$

and μ is the order of $B_p^{-}(z)$

Example



disturbance period: $N = 10$; nominal plant:

$$\frac{z^{-d}B_p\left(z^{-1}\right)}{A_p\left(z^{-1}\right)} = \frac{z^{-1}}{\left(1-0.8z^{-1}\right)\left(1-0.7z^{-1}\right)}$$

$$C\left(z^{-1}\right) = k_r \frac{\left(1-0.8z^{-1}\right)\left(1-0.7z^{-1}\right)q(z,z^{-1})z^{-10}}{z^{-1}\left(1-q(z,z^{-1})z^{-10}\right)}$$

Additional reading

- ME233 course reader
- X. Chen and M. Tomizuka, "An Enhanced Repetitive Control Algorithm using the Structure of Disturbance Observer," in *Proceedings of 2012 IEEE/ASME International Conference on Advanced Intelligent Mechatronics*, Taiwan, Jul. 11-14, 2012, pp. 490-495
- X. Chen and M. Tomizuka, "New Repetitive Control with Improved Steady-state Performance and Accelerated Transient," *IEEE Transactions on Control Systems Technology*, vol. 22, no. 2, pp. 664-675 (12 pages), Mar. 2014

Summary

1. Big Picture
   review of integral control in PID design

2. Internal Model Principle
   theorems
   typical disturbance structures

3. Repetitive Control
   use of internal model principle
   design by pole placement
   design by stable pole-zero cancellation

Lecture 14: Disturbance Observer

Big picture

Disturbance and uncertainties in mechanical systems:

- system models are important in design: e.g., in ZPET, observer, and preview controls
- inevitable to have uncertainty in actual mechanical systems
- system is also subjected to disturbances

Related control design:

- robust control
- adaptive control

Disturbance observer is one example of robust control.

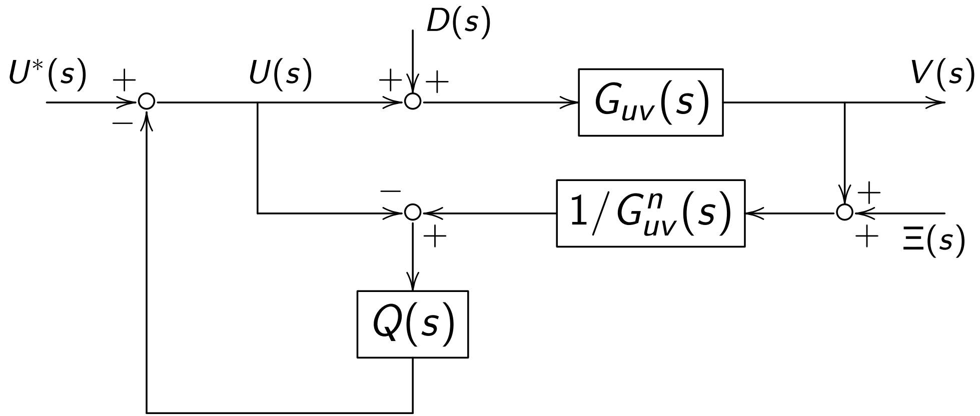
Disturbance observer (DOB)

- introduced by Ohnishi (1987) and refined by Umeno and Hori (1991)

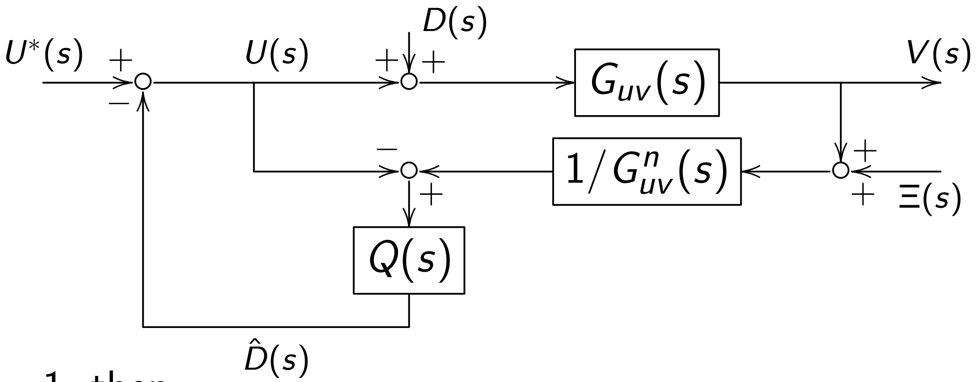
System:

$$V(s) = G_{uv}(s)\left[U(s) + D(s)\right]$$

Assumptions: $u(t)$–input; $d(t)$–disturbance; $v(t)$–output; $G_{uv}(s)$–actual plant dynamics between u and v; $G^n_{nv}(s)$–nominal model



DOB intuition



if $Q(s) = 1$, then

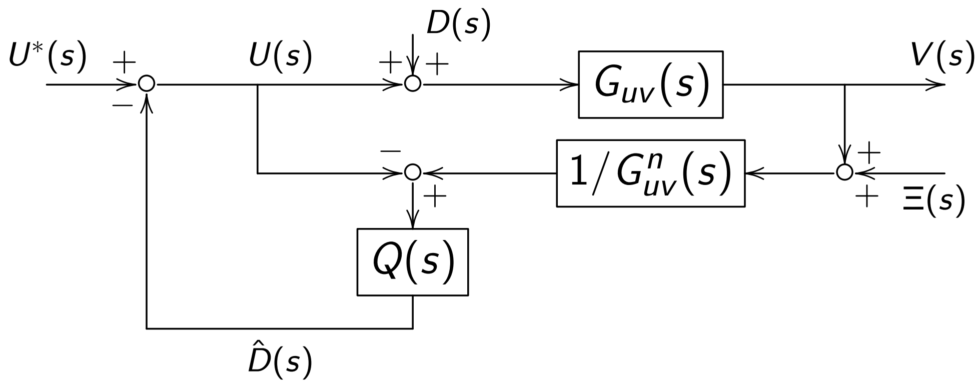
$$U(s) = U^*(s) - \left[\frac{G_{uv}(s)}{G_{uv}^n(s)}(U(s) + D(s)) + \frac{1}{G_{uv}^n(s)}\Xi(s) - U(s)\right]$$

$$\Rightarrow U(s) = \frac{G_{uv}^n(s)}{G_{uv}(s)}U^*(s) - D(s) - \frac{1}{G_{uv}(s)}\Xi(s)$$

$$V(s) = G_{uv}^n(s)U^*(s) - \Xi(s)$$

i.e., dynamics between $U^*(s)$ and $V(s)$ follows the nominal model; and disturbance is rejected

DOB intuition

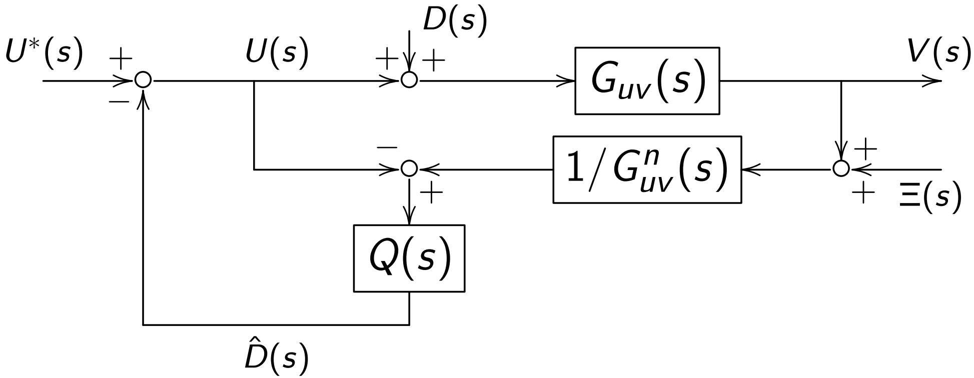


if $Q(s) = 1$, then

$$\hat{D}(s) = \left(\frac{G_{uv}(s)}{G_{uv}^n(s)} - 1 \right) U(s) + \frac{1}{G_{uv}^n(s)} \Xi(s) + \frac{G_{uv}(s)}{G_{uv}^n(s)} D(s)$$
$$\approx \frac{1}{G_{uv}(s)} \Xi(s) + D(s) \text{ if } G_{uv}(s) = G_{uv}^n(s)$$

i.e., disturbance $D(s)$ is estimated by $\hat{D}(s)$.

DOB details: causality

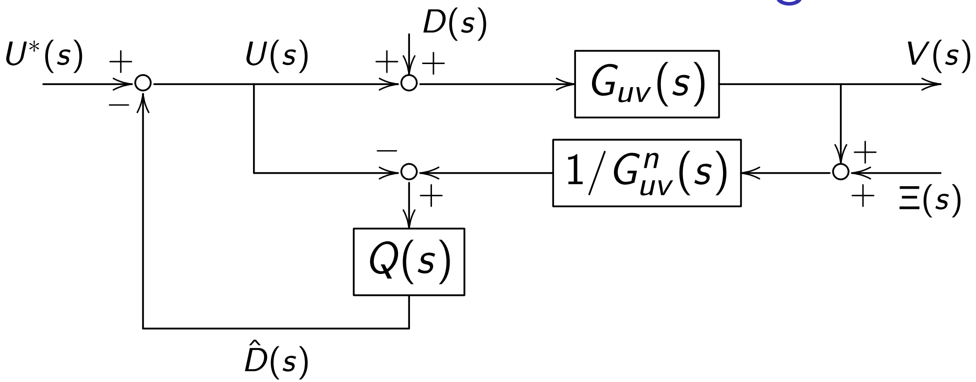


It is impractical to have $Q(s) = 1$.
e.g., if $G_{uv}(s) = 1/s^2$, then $1/G_{uv}^n(s) = s^2$ (not realizable)
$Q(s)$ should be designed such that $Q(s)/G_{nv}^n(s)$ is causal. e.g. (low-pass filter)

$$Q(s) = \frac{1 + \sum_{k=1}^{N-r} a_k (\tau s)^k}{1 + \sum_{k=1}^{N} a_k (\tau s)^k},\ Q(s) = \frac{3\tau s + 1}{(\tau s + 1)^3},\ Q(s) = \frac{6(\tau s)^2 + 4\tau s + 1}{(\tau s + 1)^4}$$

where τ determines the filter bandwidth

DOB details: nominal model following



Block diagram analysis gives

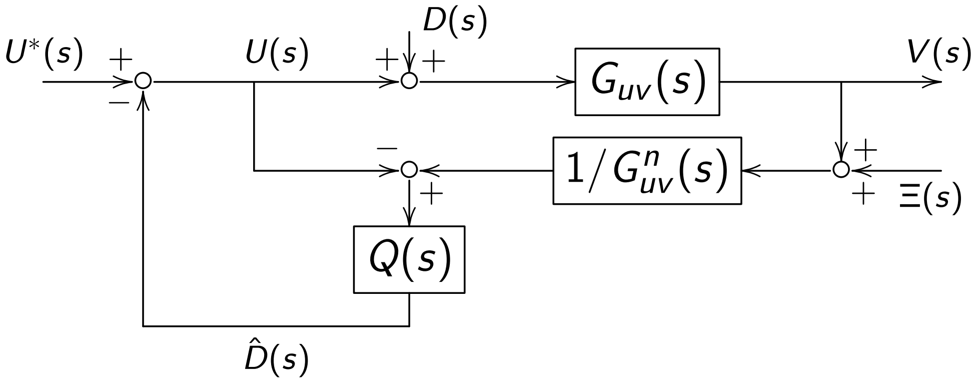
$$V(s) = G_{uv}^o(s)\,U^*(s) + G_{dv}^o(s)\,D(s) + G_{\xi v}^o(s)\,\Xi(s)$$

where

$$G_{uv}^o = \frac{G_{uv}G_{uv}^n}{G_{uv}^n + (G_{uv} - G_{uv}^n)\,Q},\quad G_{dv}^o = \frac{G_{uv}G_{uv}^n(1-Q)}{G_{uv}^n + (G_{uv} - G_{uv}^n)\,Q}$$

$$G_{\xi v}^o = -\frac{G_{uv}Q}{G_{uv}^n + (G_{uv} - G_{uv}^n)\,Q}$$

DOB details: nominal model following



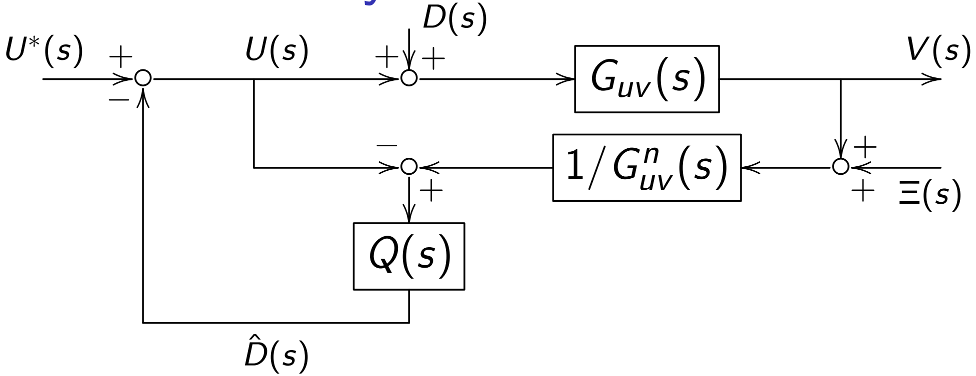
if $Q(s) \approx 1$, we have disturbance rejection and nominal model following:

$$G_{uv}^o \approx G_{uv}^n, \; G_{dv}^o \approx 0, \; G_{\xi v}^o = -1$$

if $Q(s) \approx 0$, DOB is cut off:

$$G_{uv}^o \approx G_{uv}, \; G_{dv}^o \approx G_{uv}, \; G_{\xi v}^o \approx 0$$

DOB details: stability robustness



$$G_{uv}^o = \frac{G_{uv}G_{uv}^n}{G_{uv}^n + (G_{uv} - G_{uv}^n)Q},\ G_{dv}^o = \frac{G_{uv}G_{uv}^n(1-Q)}{G_{uv}^n + (G_{uv} - G_{uv}^n)Q},\ G_{\xi v}^o = -\frac{G_{uv}Q}{G_{uv}^n + (G_{uv} - G_{uv}^n)Q}$$

closed-loop characteristic equation:

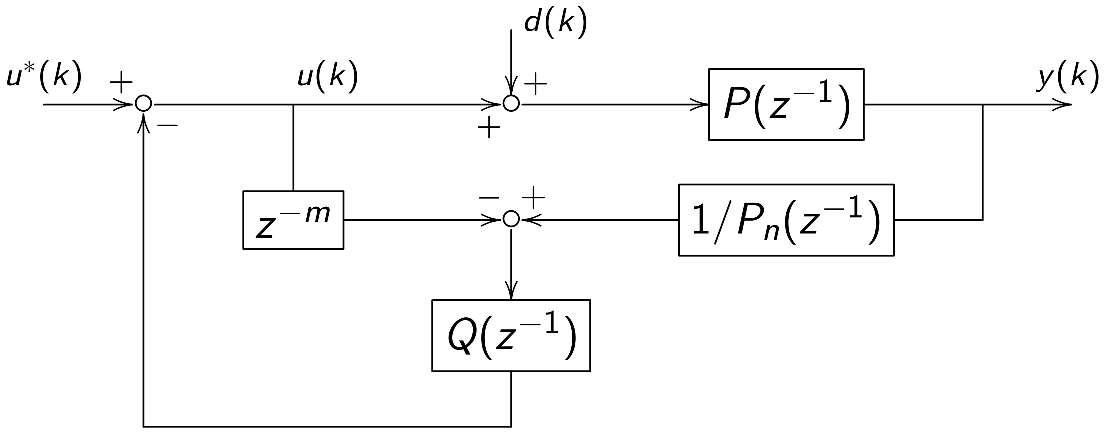
$$G_{uv}^n(s) + (G_{uv}(s) - G_{uv}^n(s))Q(s) = 0$$
$$\Leftrightarrow G_{uv}^n(s)(1 + \Delta(s)Q(s)) = 0, \text{ if } G_{uv}(s) = G_{uv}^n(s)(1 + \Delta(s))$$

robust stability condition: stable zeros for $G_{nv}^n(s)$, plus

$$||\Delta(j\omega)Q(j\omega)|| < 1,\ \forall\omega$$

Application example

Discrete-time case



where $P\left(z^{-1}\right) \approx z^{-m} P_n\left(z^{-1}\right)$

see more details in, e.g., X. Chen and M. Tomizuka, "Optimal Plant Shaping for High Bandwidth Disturbance Rejection in Discrete Disturbance Observers," in Proceedings of the 2010 American Control Conference, Baltimore, MD, Jun. 30-Jul. 02, 2010, pp. 2641-2646

Lecture 15: System Identification and Recursive Least Squares

Big picture

We have been assuming knwoledge of the plant in controller design.
In practice, plant models come from:

- modeling by physics: Newton's law, conservation of energy, etc
- (input-output) data-based system identification

The need for system identification and adaptive control come from

- unknown plants
- time-varying plants
- known disturbance structure but unknown disturbance parameters

System modeling

Consider the input-output relationship of a plant:

$$u(k) \longrightarrow \boxed{G_p\left(z^{-1}\right) = \frac{z^{-1}B\left(z^{-1}\right)}{A(z^{-1})}} \longrightarrow y(k)$$

or equivalently

$$u(k) \longrightarrow \boxed{\frac{B\left(z^{-1}\right)}{A(z^{-1})}} \longrightarrow y(k+1) \tag{1}$$

where

$$B(z^{-1}) = b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}; \quad A(z^{-1}) = 1 + a_1 z^{-1} + \cdots + a_n z^{-n}$$

- $y(k+1)$ is a linear combination of $y(k)$, ... , $y(k+1-n)$ and $u(k)$, ..., $u(k-m)$:

$$y(k+1) = -\sum_{i=1}^{n} a_i y(k+1-i) + \sum_{i=0}^{m} b_i u(k-i) \tag{2}$$

System modeling

Define *parameter vector* θ and *regressor vector* $\phi(k)$:

$$\theta \triangleq [a_1, a_2, \cdots a_n, b_0, b_1, \cdots, b_m]^T$$

$$\phi(k) \triangleq [-y(k), \cdots, -y(k+1-n), u(k), u(k-1), \cdots, u(k-m)]^T$$

- (2) can be simply written as:

$$\boxed{y(k+1) = \theta^T \phi(k)} \tag{3}$$

- $\phi(k)$ and $y(k+1)$ are known or measured
- **goal**: estimate the unknown θ

Parameter estimation

Suppose we have an estimate of the parameter vector:

$$\hat{\theta} \triangleq [\hat{a}_1, \hat{a}_2, \cdots \hat{a}_n, \hat{b}_0, \hat{b}_1, \cdots, \hat{b}_m]^T$$

At time k, we can do estimation:

$$\boxed{\hat{y}(k+1) = \hat{\theta}^T(k)\phi(k)} \tag{4}$$

where $\hat{\theta}(k) \triangleq [\hat{a}_1(k), \hat{a}_2(k), \cdots \hat{a}_n(k), \hat{b}_0(k), \hat{b}_1(k), \cdots, \hat{b}_m(k)]^T$

Parameter identification by least squares (LS)

At time k, the least squares (LS) estimate of θ minimizes:

$$\boxed{J_k = \sum_{i=1}^{k} \left[y(i) - \hat{\theta}^T(k)\phi(i-1) \right]^2} \tag{5}$$

Solution:

$$J_k = \sum_{i=1}^{k} \left(y(i)^2 + \hat{\theta}^T(k)\phi(i-1)\phi^T(i-1)\hat{\theta}(k) - 2y(i)\phi^T(i-1)\hat{\theta}(k) \right)$$

Letting $\partial J_k / \partial \hat{\theta}(k) = 0$ yields

$$\boxed{\hat{\theta}(k) = \underbrace{\left[\sum_{i=1}^{k} \phi(i-1)\phi^T(i-1) \right]^{-1}}_{F(k)} \sum_{i=1}^{k} \phi(i-1)y(i)} \tag{6}$$

Recursive least squares (RLS)

At time $k+1$, we know $u(k+1)$ and have one more measurement $y(k+1)$.

Instead of (5), we can do better by minimizing

$$J_{k+1} = \sum_{i=1}^{k+1}\left[y(i) - \hat{\theta}^T(k+1)\phi(i-1)\right]^2$$

whose solution is

$$\hat{\theta}(k+1) = \overbrace{\left[\sum_{i=1}^{k+1}\phi(i-1)\phi^T(i-1)\right]^{-1}}^{F(k+1)}\sum_{i=1}^{k+1}\phi(i-1)y(i) \qquad (7)$$

recursive least squares (RLS): no need to do the matrix inversion in (7), $\hat{\theta}(k+1)$ can be obtained by

$$\boxed{\hat{\theta}(k+1) = \hat{\theta}(k) + [\text{correction term}]} \qquad (8)$$

RLS parameter adaptation

Goal: to obtain $\boxed{\hat{\theta}(k+1) = \hat{\theta}(k) + [\text{correction term}]}$ (9)

Derivations:

$$F(k+1)^{-1} = \sum_{i=1}^{k+1} \phi(i-1)\phi^T(i-1) = F(k)^{-1} + \phi(k)\phi^T(k)$$

$$\begin{aligned}
\hat{\theta}(k+1) &= F(k+1)\sum_{i=1}^{k+1}\phi(i-1)y(i) \\
&= F(k+1)\left[\sum_{i=1}^{k}\phi(i-1)y(i) + \phi(k)y(k+1)\right] \\
&= F(k+1)\left[F(k)^{-1}\hat{\theta}(k) + \phi(k)y(k+1)\right] \\
&= F(k+1)\left[\left(F(k+1)^{-1} - \phi(k)\phi^T(k)\right)\hat{\theta}(k) + \phi(k)y(k+1)\right] \\
&= \hat{\theta}(k) + F(k+1)\phi(k)\left[y(k+1) - \hat{\theta}^T(k)\phi(k)\right] \qquad (10)
\end{aligned}$$

RLS parameter adaptation

Define

$$\begin{aligned}
\hat{y}^o(k+1) &= \hat{\theta}^T(k)\phi(k) \\
\varepsilon^o(k+1) &= y(k+1)-\hat{y}^o(k+1)
\end{aligned}$$

(10) is equivalent to

$$\boxed{\hat{\theta}(k+1)=\hat{\theta}(k)+F(k+1)\phi(k)\varepsilon^o(k+1)} \tag{11}$$

RLS adaptation gain recursion

$F(k+1)$ is called the adaptation gain, and can be updated by

$$\boxed{F(k+1) = F(k) - \frac{F(k)\phi(k)\phi^T(k)F(k)}{1+\phi^T(k)F(k)\phi(k)}} \tag{12}$$

Proof:

- matrix inversion lemma: if A is nonsingular, B and C have compatible dimensions, then

$$(A+BC)^{-1} = A^{-1} - A^{-1}B\left(CA^{-1}B+I\right)^{-1}CA^{-1}$$

- note the algebra

$$F(k+1) = \left[\sum_{i=1}^{k+1}\phi(i-1)\phi^T(i-1)\right]^{-1} = \left[F(k)^{-1}+\phi(k)\phi^T(k)\right]^{-1}$$

$$= F(k) - F(k)\phi(k)\left(\phi^T(k)F(k)\phi(k)+1\right)^{-1}\phi^T(k)F(k)$$

which gives (12)

RLS parameter adaptation

An alternative representation of adaptation law (11):

$$(12) \Rightarrow F(k+1)\phi(k) = F(k)\phi(k) - \frac{F(k)\phi(k)\phi^T(k)F(k)}{1+\phi^T(k)F(k)\phi(k)}\phi(k)$$
$$= \frac{F(k)\phi(k)}{1+\phi^T(k)F(k)\phi(k)}$$

Hence we have the parameter adaptation algorithm (PAA):

$$\boxed{\begin{aligned}\hat{\theta}(k+1) &= \hat{\theta}(k) + F(k+1)\phi(k)\varepsilon^o(k+1)\\ &= \hat{\theta}(k) + \frac{F(k)\phi(k)}{1+\phi^T(k)F(k)\phi(k)}\varepsilon^o(k+1)\\ F(k+1) &= F(k) - \frac{F(k)\phi(k)\phi^T(k)F(k)}{1+\phi^T(k)F(k)\phi(k)}\end{aligned}}$$

PAA implementation

- $\hat{\theta}(0)$: initial guess of parameter vector

$$\hat{\theta}(k+1) = \hat{\theta}(k) + \frac{F(k)\phi(k)}{1+\phi^T(k)F(k)\phi(k)}\varepsilon^o(k+1)$$

- $F(0) = \sigma I$: σ is a large number, as $F(k)$ is always none-increasing

$$F(k+1) = F(k) - \frac{F(k)\phi(k)\phi^T(k)F(k)}{1+\phi^T(k)F(k)\phi(k)}$$

RLS parameter adaptation

Up till now we have been using the *a priori* prediction and *a priori* prediction error:

$$
\begin{aligned}
\hat{y}^o(k+1) &= \hat{\theta}^T(k)\phi(k): \text{ after measurement of } y(k) \\
\varepsilon^o(k+1) &= y(k+1) - \hat{y}^o(k+1)
\end{aligned}
$$

Further analysis (e.g., convergence of $\hat{\theta}(k)$) requires the new definitions of *a posteriori* prediction and *a posteriori* prediction error:

$$
\begin{aligned}
\hat{y}(k+1) &= \hat{\theta}^T(k+1)\phi(k): \textit{ after measurement of } y(k+1) \\
\varepsilon(k+1) &= y(k+1) - \hat{y}(k+1)
\end{aligned}
$$

Relationship between $\varepsilon(k+1)$ and $\varepsilon^o(k+1)$

Note that

$$\hat{\theta}(k+1) = \hat{\theta}(k) + \frac{F(k)\phi(k)}{1+\phi^T(k)F(k)\phi(k)}\varepsilon^o(k+1)$$

$$\Rightarrow \underbrace{\phi^T(k)\hat{\theta}(k+1)}_{\hat{y}(k+1)} = \underbrace{\phi^T(k)\hat{\theta}(k)}_{\hat{y}^o(k+1)} + \frac{\phi^T(k)F(k)\phi(k)}{1+\phi^T(k)F(k)\phi(k)}\varepsilon^o(k+1)$$

$$\Rightarrow \underbrace{y(k+1) - \hat{y}(k+1)}_{\varepsilon(k+1)} = \underbrace{y(k+1) - \hat{y}^o(k+1)}_{\varepsilon^o(k+1)} - \frac{\phi^T(k)F(k)\phi(k)}{1+\phi^T(k)F(k)\phi(k)}\varepsilon^o(k+1)$$

Hence

$$\boxed{\varepsilon(k+1) = \frac{\varepsilon^o(k+1)}{1+\phi^T(k)F(k)\phi(k)}} \qquad (13)$$

- note: $|\varepsilon(k+1)| \leq |\varepsilon^o(k+1)|$ ($\hat{y}(k+1)$ is more accurate than $\hat{y}^o(k+1)$)

A posteriori RLS parameter adaptation

With (13), we can write the PAA in the *a posteriori* form

$$\boxed{\hat{\theta}(k+1) = \hat{\theta}(k) + F(k)\phi(k)\varepsilon(k+1)} \tag{14}$$

which is not implementable but is needed for stability analysis.

Forgetting factor

motivation

- previous discussions assume the actual parameter vector θ is constant
- adaptation gain $F(k)$ keeps decreasing, as

$$F^{-1}(k+1) = F^{-1}(k) + \phi(k)\phi^T(k)$$

- this means adaptation becomes weaker and weaker
- for time-varying parameters, we need a mechanism to "forget" the "old' data

Forgetting factor

Consider a new cost

$$J_k = \sum_{i=1}^{k} \lambda^{k-i} \left[y(i) - \hat{\theta}^T(k)\phi(i-1) \right]^2, \; 0 < \lambda \leq 1$$

- past errors are less weighted:

$$J_k = \left[y(k) - \hat{\theta}^T(k)\phi(k-1) \right]^2 + \lambda \left[y(k-1) - \hat{\theta}^T(k)\phi(k-2) \right]^2 + \lambda^2 \left[y(k-2) - \hat{\theta}^T(k)\phi(k-3) \right]^2 + \dots$$

- the solution is:

$$\hat{\theta}(k) = \overbrace{\left[\sum_{i=1}^{k} \lambda^{k-i} \phi(i-1)\phi^T(i-1) \right]^{-1}}^{F(k)} \sum_{i=1}^{k} \lambda^{k-i} \phi(i-1) y(i) \quad (15)$$

Forgetting factor

- in (15), the recursion of the adaptation gain is:

$$F(k+1)^{-1} = \lambda F(k)^{-1} + \phi(k)\phi(k)^T$$

or, equivalently

$$F(k+1) = \frac{1}{\lambda}\left[F(k) - \frac{F(k)\phi(k)\phi^T(k)F(k)}{\lambda + \phi^T(k)F(k)\phi(k)}\right] \qquad (16)$$

Forgetting factor

The weighting can be made more flexible:

$$F(k+1) = \frac{1}{\lambda_1(k)} \left[F(k) - \frac{F(k)\phi(k)\phi^T(k)F(k)}{\lambda_1(k) + \phi^T(k)F(k)\phi(k)} \right]$$

which corresponds to the cost function

$$J_k = \left[y(k) - \hat{\theta}^T(k)\phi(k-1) \right]^2 + \lambda_1(k-1)\left[y(k-1) - \hat{\theta}^T(k)\phi(k-2) \right]^2$$
$$+ \lambda_1(k-1)\lambda_1(k-2)\left[y(k-2) - \hat{\theta}^T(k)\phi(k-3) \right]^2 + \dots$$

i.e. (assuming $\prod_{j=k}^{k-1} \lambda_1(j) = 1$)

$$J_k = \sum_{i=1}^{k} \left\{ \left(\prod_{j=i}^{k-1} \lambda_1(j) \right) \left[y(i) - \hat{\theta}^T(k)\phi(i-1) \right]^2 \right\}$$

Forgetting factor

The general form of the adaptation gain is:

$$F(k+1) = \frac{1}{\lambda_1(k)}\left[F(k) - \frac{F(k)\phi(k)\phi^T(k)F(k)}{\frac{\lambda_1(k)}{\lambda_2(k)} + \phi^T(k)F(k)\phi(k)}\right] \tag{17}$$

which comes from:

$$F(k+1)^{-1} = \lambda_1(k)\,F(k)^{-1} + \lambda_2(k)\,\phi(k)\,\phi^T(k)$$

with $0 < \lambda_1(k) \leq 1$ and $0 \leq \lambda_2(k) \leq 2$ (for stability requirements, will come back to this soon).

$\lambda_1(k)$	$\lambda_2(k)$	PAA
1	0	constant adaptation gain
1	1	least square gain
< 1	1	least square gain with forgetting factor

*Influence of the initial conditions

If we initialize $F(k)$ and $\hat{\theta}(k)$ at F_0 and θ_0, we are actually minimizing

$$J_k = \left(\hat{\theta}(k) - \theta_0\right)^T F_0^{-1} \left(\hat{\theta}(k) - \theta_0\right) + \sum_{i=1}^{k} \alpha_i \left[y(i) - \hat{\theta}^T(k)\phi(i-1)\right]^2$$

where α_i is the weighting for the error at time i. The least square parameter estimate is

$$\hat{\theta}(k) = \left[F_0^{-1} + \sum_{i=1}^{k} \alpha_i \phi(i-1)\phi^T(i-1)\right]^{-1} \left[F_0^{-1}\theta_0 + \sum_{i=1}^{k} \alpha_i \phi(i-1)y(i)\right]$$

We see the relative importance of the initial values decays with time.

*Influence of the initial conditions

If it is possible to wait a few samples before the adaptation, proper initial values can be obtained if the recursion is started at time k_0 with

$$F(k_0) = \left[\sum_{i=1}^{k_0} \alpha_i \phi(i-1)\phi^T(i-1)\right]^{-1}$$

$$\hat{\theta}(k_0) = F(k_0)\sum_{i=1}^{k_0} \alpha_i \phi(i-1)\, y(i)$$

Lecture 16: Stability of Parameter Adaptation Algorithms

Big picture

- For
$$\hat{\theta}(k+1) = \hat{\theta}(k) + [\text{correction term}]$$
we haven't talked about whether $\hat{\theta}(k)$ will converge to the true value θ if $k \to \infty$. We haven't even talked about whether $\hat{\theta}(k)$ will stay bounded or not!

- Tools of stability evaluation: Lyapunov-based analysis, or hyperstability theory (topic of this lecture)

Outline

Hyperstability theory

history

Vasile M. Popov:

- born in 1928, Romania
- retired from University of Florida in 1993
- developed hyperstability theory independently from Lyapnov theory

Hyperstability theory

Consider a closed-loop system in Fig. 1

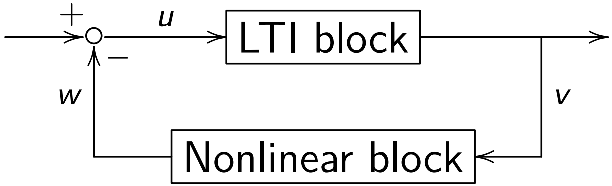


Figure 1: Block diagram for hyperstability analysis

The linear time invariant (LTI) block is realized by

continuous-time case:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$v(t) = Cx(t) + Du(t)$$

discrete-time case:

$$x(k+1) = Ax(k) + Bu(k)$$
$$v(k) = Cx(k) + Du(k)$$

Hyperstability discusses conditions for “nice” behaviors in x.

Passive systems

Definition (Passive system).

The system $v \longrightarrow \boxed{\text{System}} \longrightarrow w$ is called passive if

$$\int_0^{t_1} w^T(t)\,v(t)\,\mathrm{d}t \geq -\gamma^2,\ \forall t_1 \geq 0 \text{ or } \sum_{k=0}^{k_1} w^T(k)\,v(k) \geq -\gamma^2,\ \forall k_1 \geq 0$$

where δ and γ depends on the initial conditions.

- intuition: $\int_0^{t_1} w^T(t)\,v(t)\,\mathrm{d}t$ is the work/supply done to the system. By conservation of energy,

$$E(t_1) \leq E(0) + \int_0^{t_1} w^T(t)\,v(t)\,\mathrm{d}t$$

Strictly passive systems

If the equality is *strict* in the passivity definition, with

$$\int_0^{t_1} w^T(t)\, v(t)\, \mathrm{d}t \geq -\gamma^2 + \delta \int_0^{t_1} v^T(t)\, v(t)\, \mathrm{d}t + \varepsilon \int_0^{t_1} w^T(t)\, w(t)\, \mathrm{d}t,\ \forall t_1 \geq 0$$

or in the discrete-time case

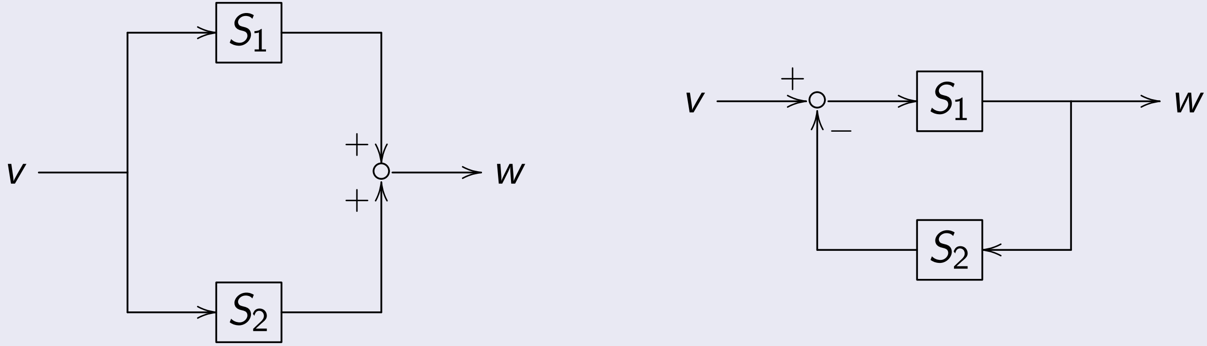
$$\sum_{k=0}^{k_1} w^T(k)\, v(k) \geq -\gamma^2 + \delta \sum_{k=0}^{k_1} v^T(k)\, v(k) + \varepsilon \sum_{k=0}^{k_1} w^T(k)\, w(k),\ \forall k_1 \geq 0$$

where $\delta \geq 0$, $\varepsilon \geq 0$, but not both zero, the system is *strictly passive.*

Passivity of combined systems

Fact (Passivity of connected systems).

If two systems S_1 and S_2 are both passive, then the following parallel and feedback combination of S_1 and S_2 are also passive



Hyperstability theory

Definition (Hyperstability).

The feedback system in Fig. 1 is hyperstable if and only if there exist positive constants $\delta > 0$ and $\gamma > 0$ such that

$$\|x(t)\| < \delta \left[\|x(0)\| + \gamma\right], \ \forall t > 0 \text{ or } \|x(k)\| < \delta \left[\|x(0)\| + \gamma\right], \ \forall k > 0$$

for all feedback blocks that satisfy the *Popov inequality*

$$\int_0^{t_1} w^T(t)\, v(t)\, \mathrm{d}t \geq -\gamma^2, \ \forall t_1 \geq 0 \text{ or } \sum_{k=0}^{k_1} w^T(k)\, v(k) \geq -\gamma^2, \ \forall k_1 \geq 0$$

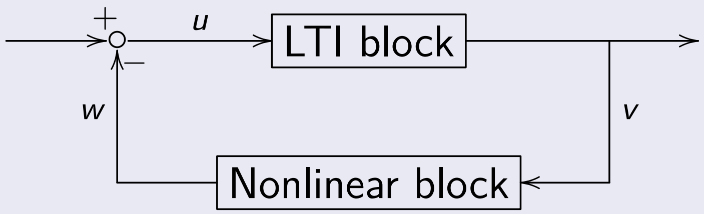
In other words, the LTI block is bounded in states for any initial conditions for any *passive* nonlinear blocks.

Hyperstability theory

Definition (Asymptotic hyperstability).

The feedback system below is asymptotically hyperstable if and only if it is hyperstable and for all *bounded* w satisfying the Popov inequality we have

$$\lim_{k\to\infty} x(k) = 0$$



Hyperstability theory

Theorem (Hyperstability).

The feedback system in Fig. 1 is hyperstable if and only if the nonlinear block satisfies ***Popov inequality*** *(i.e., it is passive) and the LTI transfer function is* ***positive real****.*

Theorem (Asymptotical hyperstability).

The feedback system in Fig. 1 is asymptotically hyperstable if and only if the nonlinear block satisfies ***Popov inequality*** *and the LTI transfer function is* ***strictly positive real****.*

intuition: a strictly passive system in feedback connection with a passive system gives an asymptotically stable closed loop.

Positive real and strictly positive real

Positive real transfer function (continuous-time case): a SISO transfer function $G(s)$ is called *positive real* (**PR**) if

- $G(s)$ is real for real values of s
- $\mathrm{Re}\{G(s)\} > 0$ for $\mathrm{Re}\{s\} > 0$

The above is intuitive but not practical to evaluate. Equivalently, $G(s)$ is PR if

1. $G(s)$ does not possess any pole in $\mathrm{Re}\{s\} > 0$ (no unstable poles)
2. any pole on the imaginary axis $j\omega_0$ does not repeat and the associated residue (i.e., the coefficient appearing in the partial fraction expansion) $\lim_{s\to j\omega_0}(s - j\omega_0)G(s)$ is non-negative
3. $\forall\omega \in \mathbb{R}$ where $s = j\omega$ is not a pole of $G(s)$, $G(j\omega) + G(-j\omega) = 2\mathrm{Re}\{G(j\omega)\} \geq 0$

Positive real and strictly positive real

Strictly positive real transfer function (continuous-time case): a SISO transfer function $G(s)$ is *strictly positive real* (**SPR**) if

1. $G(s)$ does not possess any pole in $\mathrm{Re}\{s\} \geq 0$
1. $\forall \omega \in \mathbb{R},\ G(j\omega) + G(-j\omega) = 2\,\mathrm{Re}\{G(j\omega)\} > 0$

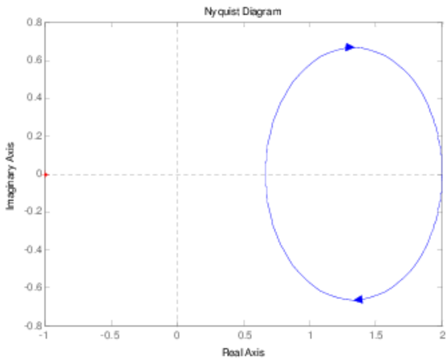


Figure: example Nyquist plot of a SPR transfer function

Positive real and strictly positive real

discrete-time case

A SISO discrete-time transfer function $G(z)$ is positive real (**PR**) if:

1. it does not possess any pole outside of the unit circle
2. any pole on the unit circle does not repeat and the associated redsidue is non-negative
3. $\forall |\omega| \leq \pi$ where $z = e^{j\omega}$ is not a pole of $G(z)$,
   $G(e^{-j\omega}) + G(e^{j\omega}) = 2\,\mathrm{Re}\left\{G(e^{j\omega})\right\} \geq 0$

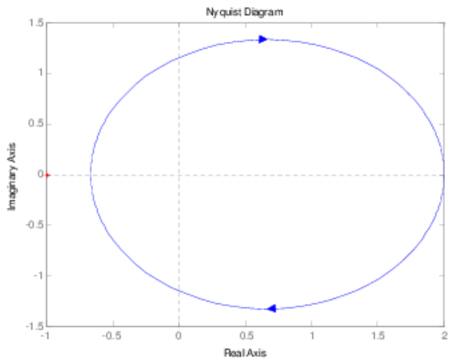
$G(z)$ is strictly positive real (**SPR**) if:

1. $G(z)$ does not possess any pole outside of or on the unit circle on z-plane
2. $\forall |\omega| < \pi,\ G(e^{-j\omega}) + G(e^{j\omega}) = 2\,\mathrm{Re}\left\{G(e^{j\omega})\right\} > 0$

Examples of PR and SPR transfer functions

- $G(z) = c$ is SPR if $c > 0$
- $G(z) = \frac{1}{z-a}$, $|a| < 1$ is asymptotically stable but not PR:

$$2\,\mathrm{Re}\left\{G\left(e^{j\omega}\right)\right\} = \frac{1}{e^{j\omega}-a} + \frac{1}{e^{-j\omega}-a}$$
$$= 2\frac{\cos\omega - a}{1 + a^2 - 2a\cos\omega}$$



- $G(z) = \frac{z}{z-a}$, $|a| < 1$ is asymptotically stable and SPR

Strictly positive real implies strict passivity

It turns out [see Appendix (to prove on board at the end of class)]:
Lemma: the LTI system $G(s) = C(sI - A)^{-1}B + D$ (in minimal realization)

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

is

- passive if $G(s)$ is positive real
- strictly passive if $G(s)$ is strictly positive real

Analogous results hold for discrete-time systems.

Outline

1. Big picture

2. Hyperstability theory

   Passivity

   Main results

   Positive real and strictly positive real

   Understanding the hyperstability theorem

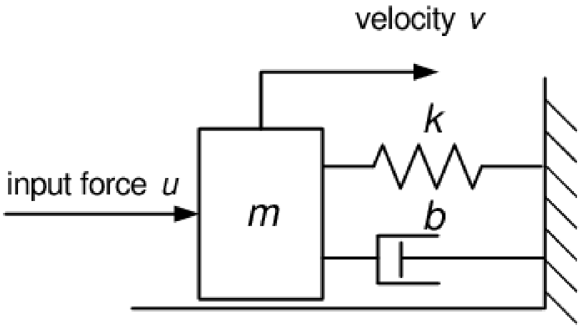
3. Procedure of PAA stability analysis by hyperstability theory

4. Appendix

   Strictly positive real implies strict passivity

Understanding the hyperstability theorem

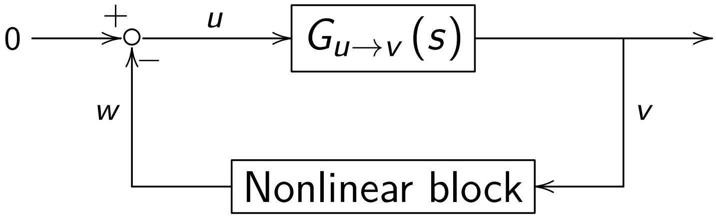
Example: consider a mass-spring-damper system



$$m\ddot{x} + b\dot{x} + kx = u \Rightarrow$$

$$G_{u\to x}(s) = \frac{1}{ms^2 + bs + k}$$
$$G_{u\to v}(s) = \frac{s}{ms^2 + bs + k}$$

with a general nonlinear feedback control law



- $\int_0^{t_1} u(t)\,v(t)\,\mathrm{d}t$ is the total energy supplied to the system

Understanding the hyperstability theorem

- if the nonlinear block satisfies the Popov inequality

$$\int_0^{t_1} w(t)\,v(t)\,\mathrm{d}t \geq -\gamma_0^2,\ \forall t_1 \geq 0$$

then from $u(t) = -w(t)$, the energy supplied to the system is bounded by

$$\int_0^{t_1} u(t)\,v(t)\,\mathrm{d}t \leq \gamma_0^2$$

- energy conservation (assuming $v(0) = v_0$ and $x(0) = x_0$):

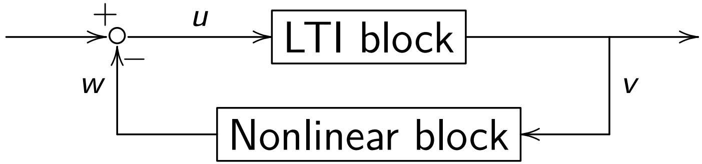
$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 - \frac{1}{2}mv_0^2 - \frac{1}{2}kx_0^2 = \int_0^{t_1} u(t)\,v(t)\,\mathrm{d}t \leq \gamma_0^2$$

- define state vector $x = [x_1, x_2]^T$, $x_1 = \sqrt{\frac{k}{2}}x$, $x_2 = \sqrt{\frac{m}{2}}v$, then

$$||x(t)||_2^2 \leq ||x(0)||_2^2 + \gamma_0^2 \leq (||x(0)||_2 + \gamma_0)^2$$

which is a special case in the hyperstability definition

Understanding the hyperstability theorem

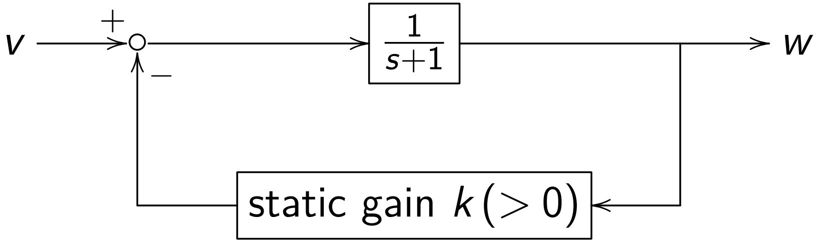


intuition from the example:
The nonlinear block satisfying Popov inequality assures bounded supply to the LTI system. Based on energy conservation, the energy of the LTI system is bounded. If the energy function is positive definite with respect to the states, then the states will be bounded.

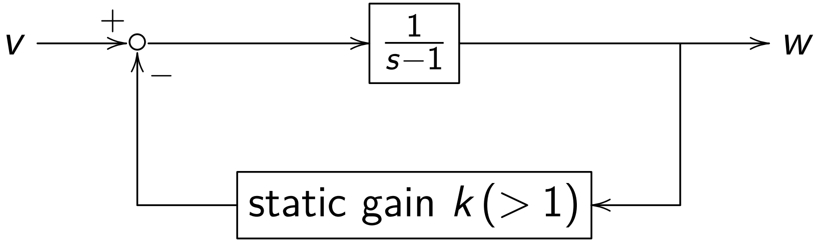
more intuition:
If the LTI system is strictly passive, it consumes energy. The bounded supply will eventually be all consumed, hence the convergence to zero for the states.

A remark about hyperstability

An example of a system that is asymptotically hyperstable and stable:



Stable systems may however be not hyperstable: for instance



is stable but not hyperstable ($\frac{1}{s-1}$ is unstable and hence not SPR)

Outline

1. Big picture

2. Hyperstability theory

   Passivity

   Main results

   Positive real and strictly positive real

   Understanding the hyperstability theorem

3. Procedure of PAA stability analysis by hyperstability theory

4. Appendix

   Strictly positive real implies strict passivity

PAA stability analysis by hperstability theory

- step 1: translate the adaptation algorithm to a feedback combination of a LTI block and a nonlinear block, as shown in Fig. 1
- step 2: verify that the feedback block satisfies the Popov inequality
- step 3: check that the LTI block is strictly positive real
- step 4: show that the output of the feedback block is bounded. Then from the definition of asymptotic hyperstability, we conclude that the state x converges to zero

Example: hyperstability of RLS with constant adaptation gain

Recall PAA with recursive least squares:

- *a priori* parameter update

$$\hat{\theta}(k+1) = \hat{\theta}(k) + \frac{F(k)\phi(k)}{1+\phi^T(k)F(k)\phi(k)}\varepsilon^o(k+1)$$

- *a posteriori* parameter update

$$\hat{\theta}(k+1) = \hat{\theta}(k) + F(k)\phi(k)\varepsilon(k+1)$$

We use the *a posteriori* form to prove that the RLS with $F(k) = F \succ 0$ is **always asymptotically hyperstable.**

Example cont'd

step 1: transformation to a feedback structure

$$\hat{\theta}(k+1) = \hat{\theta}(k) + F\phi(k)\varepsilon(k+1)$$

parameter estimation error (vector) $\tilde{\theta}(k) = \hat{\theta}(k) - \theta$:

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) + F\phi(k)\varepsilon(k+1)$$

a posteriori prediction error $\varepsilon(k+1) = y(k+1) - \hat{\theta}^T(k+1)\phi(k)$:

$$\begin{aligned}\varepsilon(k+1) &= \theta^T\phi(k) - \hat{\theta}^T(k+1)\phi(k) \\ &= -\tilde{\theta}^T(k+1)\phi(k)\end{aligned}$$

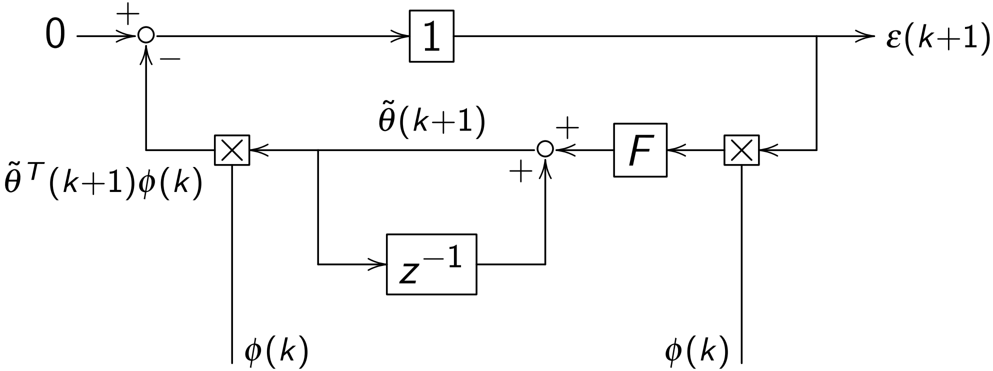
Example cont'd

step 1: transformation to a feedback structure

PAA equations:

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) + F\phi(k)\varepsilon(k+1)$$
$$\varepsilon(k+1) = -\tilde{\theta}^T(k+1)\phi(k)$$

equivalent block diagram:



Example cont'd

step 2: Popov inequality

for the feedback nonlinear block, need to prove

$$\sum_{k=0}^{k_1} \tilde{\theta}^T(k+1)\phi(k)\varepsilon(k+1) \geq -\gamma_0^2, \ \forall k_1 \geq 0$$

$\tilde{\theta}(k+1) - \tilde{\theta}(k) = F\phi(k)\varepsilon(k+1)$ gives

$$\sum_{k=0}^{k_1} \tilde{\theta}^T(k+1)\phi(k)\varepsilon(k+1)$$
$$= \sum_{k=0}^{k_1} \left(\tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k+1) - \tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k)\right)$$

Example cont'd

step 2: Popov inequality

"adding and substracting terms" gives

$$
\begin{aligned}
&\sum_{k=0}^{k_1} \tilde{\theta}^T(k+1)\phi(k)\varepsilon(k+1) \\
&\qquad = \sum_{k=0}^{k_1} \left(\tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k+1) - \tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k) \right) \\
&\qquad\quad = \sum_{k=0}^{k_1} \Big(\tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k+1) \pm \tilde{\theta}^T(k)F^{-1}\tilde{\theta}(k) \\
&\qquad\qquad\qquad\qquad\qquad - \tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k) \Big)
\end{aligned}
$$

Example cont'd

step 2: Popov inequality

Combining terms yields

$$\sum_{k=0}^{k_1}\left(\tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k+1)\pm\tilde{\theta}^T(k)F^{-1}\tilde{\theta}(k)-\tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k)\right)$$

$$=\underline{\sum_{k=0}^{k_1}\frac{1}{2}\left(\tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k+1)-\tilde{\theta}^T(k)F^{-1}\tilde{\theta}(k)\right)}$$

$$+\sum_{k=0}^{k_1}\frac{1}{2}\underbrace{\left(\tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k+1)-2\tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k)+\tilde{\theta}^T(k)F^{-1}\tilde{\theta}(k)\right)}_{[\bigstar]}$$

- $[\bigstar]$ is equivalent to

$$\left(F^{-1/2}\tilde{\theta}(k+1)-F^{-1/2}\tilde{\theta}(k)\right)^T\left(F^{-1/2}\tilde{\theta}(k+1)-F^{-1/2}\tilde{\theta}(k)\right)\geq 0$$

Example cont'd

step 2: Popov inequality

- the underlined term is also lower bounded:

$$\sum_{k=0}^{k_1} \frac{1}{2}\left(\tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k+1) - \tilde{\theta}^T(k)F^{-1}\tilde{\theta}(k)\right)$$

$$= \frac{1}{2}\tilde{\theta}^T(k_1+1)F^{-1}\tilde{\theta}(k_1+1) - \frac{1}{2}\tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0)$$

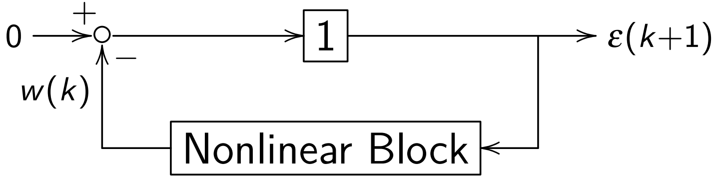
$$\geq -\frac{1}{2}\tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0)$$

hence

$$\boxed{\sum_{k=0}^{k_1} \tilde{\theta}^T(k+1)\phi(k)\varepsilon(k+1) \geq -\frac{1}{2}\tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0)}$$

Example cont'd

step 3: SPR condition

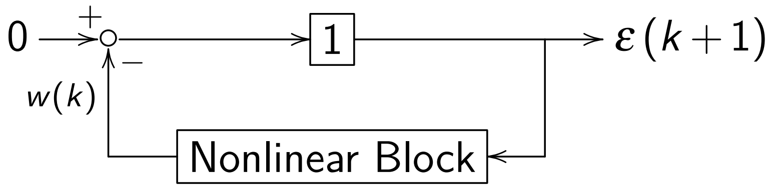


the identity block $G\left(z^{-1}\right) = 1$ is always SPR

- from steps 1-3, we conclude the adaptation system is asymptotically hyperstable
- this means $\varepsilon(k+1)$ will be bounded, and if $w(k)$ is further shown to be bounded, $\varepsilon(k+1)$ converge to zero

Example cont'd

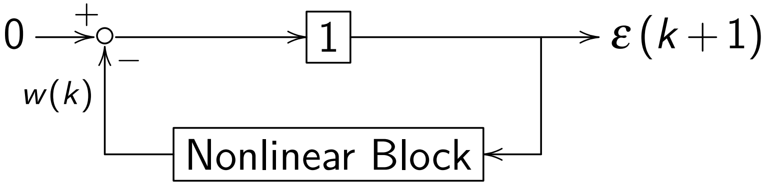
step 4: boundedness of the signal



- $\varepsilon(k+1) = -w(k)$, so $w(k)$ is bounded if $\varepsilon(k+1)$ is bounded
- thus hyperstability theorem gives that $\varepsilon(k+1)$ converges to zero

Example cont'd

intuition



For this simple case, we can intuitively see why $\varepsilon(k+1) \to 0$: Popov inequality gives $\sum_{k=0}^{k_1} \varepsilon(k+1) w(k) \geq -\gamma_0^2$; as $w(k) = -\varepsilon(k+1)$, so

$$\sum_{k=0}^{k_1} \varepsilon^2(k+1) \leq \gamma_0^2$$

Let $k_1 \to \infty$. $\varepsilon(k+1)$ must converge to 0 to ensure the boundedness.

One remark

Recall

$$\varepsilon(k+1) = \frac{\varepsilon^o(k+1)}{1+\phi^T(k)F\phi(k)}$$

- $\varepsilon(k+1) \to 0$ does not necessarily mean $\varepsilon^o(k+1) \to 0$
- need to show $\phi(k)$ is bounded: for instance, the plant needs to be input-output stable for $y(k)$ to be bounded
- see details in ME 233 course reader

There are different PAAs with different stability and convergence requirements

Summary

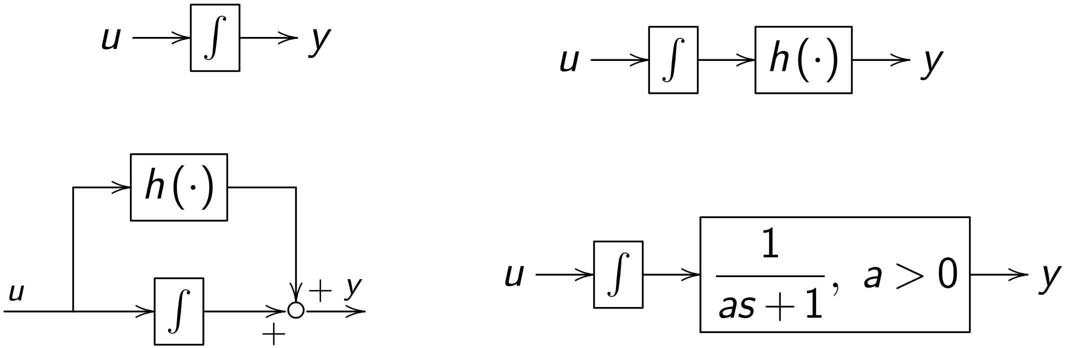
1. Big picture
2. Hyperstability theory
   - Passivity
   - Main results
   - Positive real and strictly positive real
   - Understanding the hyperstability theorem
3. Procedure of PAA stability analysis by hyperstability theory
4. Appendix
   - Strictly positive real implies strict passivity

Exercise

In the following block diagrams, u and y are respectively the input and output of the overall system; $h(\cdot)$ is a sector nonlinearity satisfying

$$2|x| < |h(x)| < 5|x|$$

Check whether they satisfy the Popov inequality.



*Kalman Yakubovich Popov Lemma

Kalman Yakubovich Popov (KYP) Lemma connects frequency-domain SPR conditions and time-domain system matrices:
Lemma: Consider $G(s) = C(sI - A)^{-1}B + D$ where (A, B) is controllable and (A, C) is observable. $G(s)$ is **strictly positive real** if and only if there exist matrices $P = P^T \succ 0$, L, and W, and a positive constant ε such that

$$\begin{aligned} PA + A^T P &= -L^T L - \varepsilon P \\ PB &= C^T - L^T W \\ W^T W &= D + D^T \end{aligned}$$

Proof: see H. Khalil, "Nonlinear Systems", Prentice Hall

*Kalman Yakubovich Popov Lemma

Discrete-time version of KYP lemma: replace s with z and replace the matrix equalities with

$$A^T PA - P = -L^T L - \varepsilon P$$
$$B^T PA - C = -K^T L$$
$$D + D^T - B^T PB = K^T K$$

*Strictly positive real implies strict passivity

From KYP lemma, the following result can be shown:
Lemma: the LTI system $G(s) = C(sI - A)^{-1}B + D$ (in minimal realization)

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

is

- passive if $G(s)$ is positive real
- strictly passive if $G(s)$ is strictly positive real

Analogous results hold for discrete-time systems.

*Strictly positive real implies strict passivity

Proof: Consider $V = \frac{1}{2}x^T P x$:

$$V(x(T)) - V(x(0)) = \int_0^T \dot{V}\mathrm{d}t = \int_0^T \left[\frac{1}{2}x^T\left(A^T P + PA\right)x + u^T B^T P x\right]\mathrm{d}t$$

Let u and y be the input and the output of $G(s)$. KYP lemma gives

$$V(x(T)) - V(x(0)) = \int_0^T \left[-\frac{1}{2}x^T\left(L^T L + \varepsilon P\right)x + u^T B^T P x\right]\mathrm{d}t$$

$$\begin{aligned}\int_0^T u^T y\mathrm{d}t &= \int_0^T u^T (Cx + Du)\mathrm{d}t = \int_0^T \left[u^T\left(B^T P + W^T L\right)x + u^T D u\right]\mathrm{d}t \\ &= \int_0^T \left[u^T\left(B^T P + W^T L\right)x + \frac{1}{2}u^T\left(D + D^T\right)u\right]\mathrm{d}t \\ &= \int_0^T \left[u^T\left(B^T P + W^T L\right)x + \frac{1}{2}u^T W^T W u\right]\mathrm{d}t\end{aligned}$$

*Strictly positive real implies strict passivity

hence

$$
\begin{aligned}
&\int_0^T u^T y \mathrm{d}t - V(x(T)) + V(x(0)) \\
=&\int_0^T \left[u^T \left(B^T P + W^T L \right) x + \frac{1}{2} u^T W^T W u + \frac{1}{2} x^T \left(L^T L + \varepsilon P \right) x - u^T B^T P x \right] \mathrm{d}t \\
=&\frac{1}{2} \int_0^T (Lx + Wu)^T (Lx + Wu) \mathrm{d}t + \frac{1}{2} \varepsilon x^T P x \geq \frac{1}{2} \varepsilon x^T P x > 0
\end{aligned}
$$

Lecture 17: PAA with Parallel Predictors

Big picture: we know now...

$$u(k) \longrightarrow \boxed{\dfrac{B\left(z^{-1}\right)}{A\left(z^{-1}\right)}} \longrightarrow y(k+1)$$

simply means:

$$\begin{aligned} y(k+1) &= B\left(z^{-1}\right)u(k) - \left(A\left(z^{-1}\right) - 1\right)y(k+1) \\ &= \theta^T \phi(k) \end{aligned}$$

In RLS:

$$\hat{y}^o(k+1) = \hat{\theta}^T(k)\phi(k) = \hat{B}\left(z^{-1},k\right)u(k) - \left(\hat{A}\left(z^{-1},k\right) - 1\right)y(k+1)$$

Understanding the notation: if $B\left(z^{-1}\right) = b_o + b_1 z^{-1} + \cdots + b_m z^{-m}$,
then $\hat{B}\left(z^{-1},k\right) = \hat{b}_o(k) + \hat{b}_1(k)z^{-1} + \cdots + \hat{b}_m(k)z^{-m}$
Remark: z^{-1}–shift operator; some references use q^{-1} instead

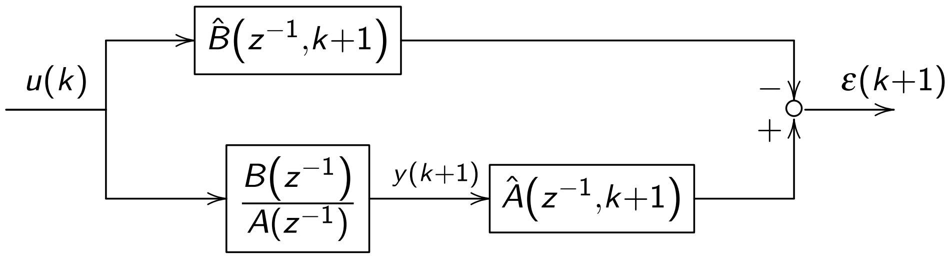
RLS is a series-parallel adjustable system

RLS in *a posteriori* form:

$$\hat{y}(k+1) = \hat{B}\left(z^{-1}, k+1\right) u(k) - \left(\hat{A}\left(z^{-1}, k+1\right) - 1\right) y(k+1)$$

prediction error:

$$\varepsilon(k+1) = y(k+1) - \hat{y}(k+1) = \hat{A}\left(z^{-1}, k+1\right) y(k+1) - \hat{B}\left(z^{-1}, k+1\right) u(k)$$



A series-parallel structure: $\hat{A}\left(z^{-1}, k+1\right)$–in series with plant; $\hat{B}\left(z^{-1}, k+1\right)$–in parallel with the plant

Observation

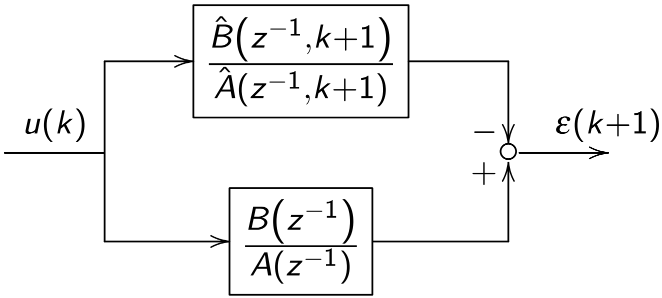
If hyperstability holds such that $\varepsilon(k+1) \to 0$, $\hat{y}(k+1) \to y(k+1)$, it seems fine to do instead:

$$\hat{y}(k+1) = \hat{B}\left(z^{-1}, k+1\right) u(k) - \left(\hat{A}\left(z^{-1}, k+1\right) - 1\right) \hat{y}(k+1) \quad (1)$$

i.e.

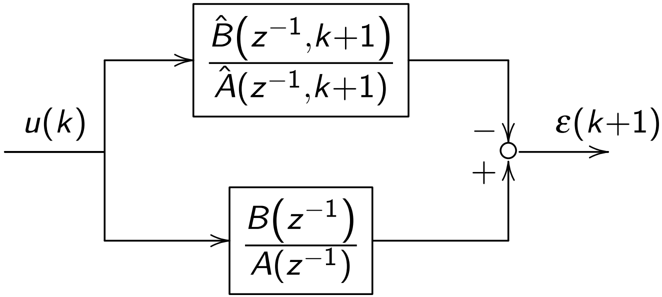
$$u(k) \longrightarrow \boxed{\frac{\hat{B}\left(z^{-1},k+1\right)}{\hat{A}(z^{-1},k+1)}} \longrightarrow \hat{y}(k+1)$$

then we have a parallel structure

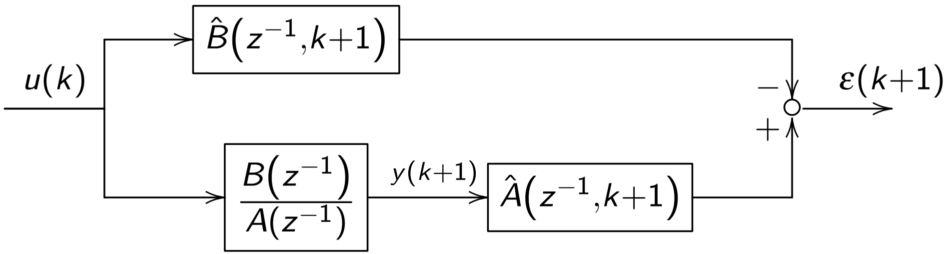


- it turns out this brings certain advantages

Other names



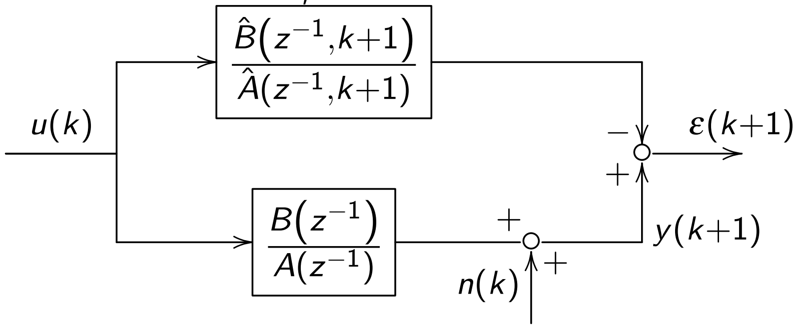
is also called an output-error method



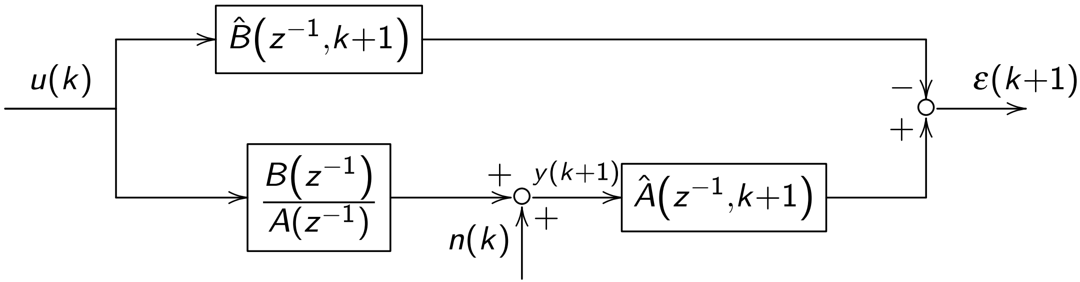
is also called an equation-error method

Benefits of parallel algorithms

Intuition: when there is noise,



provides better convergence of $\hat{\theta}$ than



We will talk about the PAA convergence in a few more lectures.

Outline

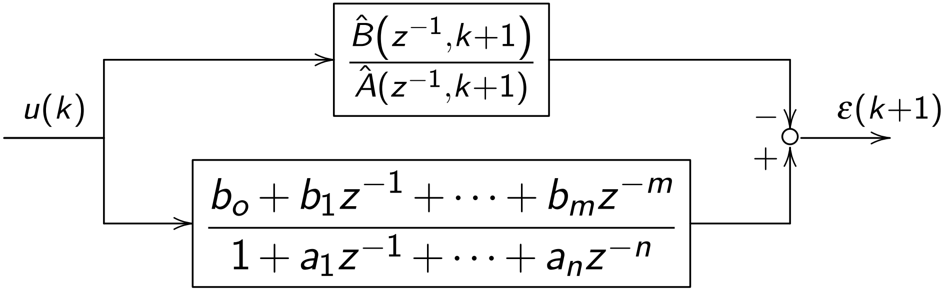
1. Big picture
   Series-parallel adjustable system (equation-error method)
   Parallel adjustable system (output-error method)

2. RLS-based parallel PAA
   Formulas
   Stability requirement for PAAs with fixed adaptation gain
   Stability requirement for PAAs with time-varying adaptation gain

3. Parallel PAAs with relaxed SPR requirements

4. PAAs with time-varying adaptation gains (revisit)

RLS based parallel PAA



PAA summary:

- *a priori*
$$\hat{\theta}(k+1) = \hat{\theta}(k) + \frac{F(k)\phi(k)}{1+\phi^T(k)F(k)\phi(k)}\varepsilon^o(k+1)$$
- *a posteriori*
$$\hat{\theta}(k+1) = \hat{\theta}(k) + F(k)\phi(k)\varepsilon(k+1)$$
$$F^{-1}(k+1) = \lambda_1(k)F^{-1}(k) + \lambda_2(k)\phi(k)\phi^T(k)$$

$$\boxed{\phi^T(k) = [-\hat{y}(k), -\hat{y}(k-1), \ldots, -\hat{y}(k+1-n), u(k), \ldots, u(k-m)]}$$

Stability of RLS based parallel PAA

step 1: transformation to a feedback structure

parameter estimation error :

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) + F(k)\phi(k)\varepsilon(k+1)$$

a posteriori prediction error : $y(k+1) = \frac{B(z^{-1})}{A(z^{-1})}u(k)$ gives

$$B\left(z^{-1}\right)u(k) = A\left(z^{-1}\right)y(k+1)$$

$$\hat{B}\left(z^{-1},k+1\right)u(k) = \hat{A}\left(z^{-1},k+1\right)\hat{y}(k+1)$$

hence

$$A\left(z^{-1}\right)y(k+1) - \hat{A}\left(z^{-1},k+1\right)\hat{y}(k+1)\boxed{\pm A\left(z^{-1}\right)\hat{y}(k+1)}$$
$$= B\left(z^{-1}\right)u(k) - \hat{B}\left(z^{-1},k+1\right)u(k)$$

i.e. $$A\left(z^{-1}\right)\varepsilon(k+1) = [B\left(z^{-1}\right) - \hat{B}\left(z^{-1},k+1\right)]u(k)$$
$$-[A\left(z^{-1}\right) - \hat{A}\left(z^{-1},k+1\right)]\hat{y}(k+1)$$

Stability of RLS based parallel PAA

step 1: transformation to a feedback structure

a posteriori prediction error (cont'd):

$$A\left(z^{-1}\right)\varepsilon(k+1) = \overbrace{\left[B\left(z^{-1}\right) - \hat{B}\left(z^{-1}, k+1\right)\right]u(k)}^{[\bigstar]} - \left[A\left(z^{-1}\right) - \hat{A}\left(z^{-1}, k+1\right)\right]\hat{y}(k+1)$$

Look at [★]: $B\left(z^{-1}\right) = b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}$ gives

$$\left[B\left(z^{-1}\right) - \hat{B}\left(z^{-1}, k+1\right)\right]u(k)$$

$$= \begin{bmatrix} b_0 - \hat{b}_0(k+1) \\ b_1 - \hat{b}_1(k+1) \\ \vdots \\ b_m - \hat{b}_m(k+1) \end{bmatrix}^T \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-m} \end{bmatrix} u(k) = \begin{bmatrix} b_0 - \hat{b}_0(k+1) \\ b_1 - \hat{b}_1(k+1) \\ \vdots \\ b_m - \hat{b}_m(k+1) \end{bmatrix}^T \begin{bmatrix} u(k) \\ u(k-1) \\ \vdots \\ u(k-m) \end{bmatrix}$$

Stability of RLS based parallel PAA

step 1: transformation to a feedback structure

Similarly, for $A\left(z^{-1}\right)=1+a_1z^{-1}+\cdots+a_nz^{-n}$

$$\left[\hat{A}\left(z^{-1},k+1\right)-A\left(z^{-1}\right)\right]\hat{y}\left(k+1\right)=\begin{bmatrix} a_1-\hat{a}_1\left(k+1\right)\\ a_2-\hat{a}_2\left(k+1\right)\\ \vdots\\ a_n-\hat{a}_n\left(k+1\right)\end{bmatrix}^T\begin{bmatrix} -\hat{y}\left(k\right)\\ -\hat{y}\left(k-1\right)\\ \vdots\\ -\hat{y}\left(k+1-n\right)\end{bmatrix}$$

Recall: $\theta^T=\left[a_1,a_2,\cdots a_n,b_0,b_1,\cdots,b_m\right]^T$

$$\phi\left(k\right)=\left[-\hat{y}\left(k\right),-\hat{y}\left(k-1\right),\ldots,-\hat{y}\left(k+1-n\right),u\left(k\right),\ldots,u\left(k-m\right)\right]$$

hence

$$A\left(z^{-1}\right)\varepsilon\left(k+1\right)=\left[B\left(z^{-1}\right)-\hat{B}\left(z^{-1},k+1\right)\right]u\left(k\right)$$
$$-\left[A\left(z^{-1}\right)-\hat{A}\left(z^{-1},k+1\right)\right]\hat{y}\left(k+1\right)=\underline{-\tilde{\theta}^T\left(k+1\right)\phi\left(k\right)}$$

Stability of RLS based parallel PAA

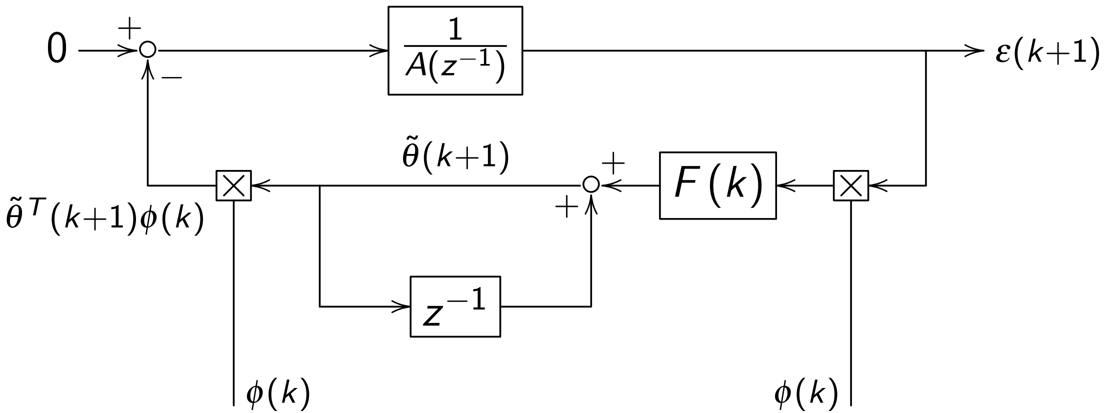
step 1: transformation to a feedback structure

PAA equations:

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) + F(k)\phi(k)\varepsilon(k+1)$$

$$A\left(z^{-1}\right)\varepsilon(k+1) = -\tilde{\theta}^T(k+1)\phi(k)$$

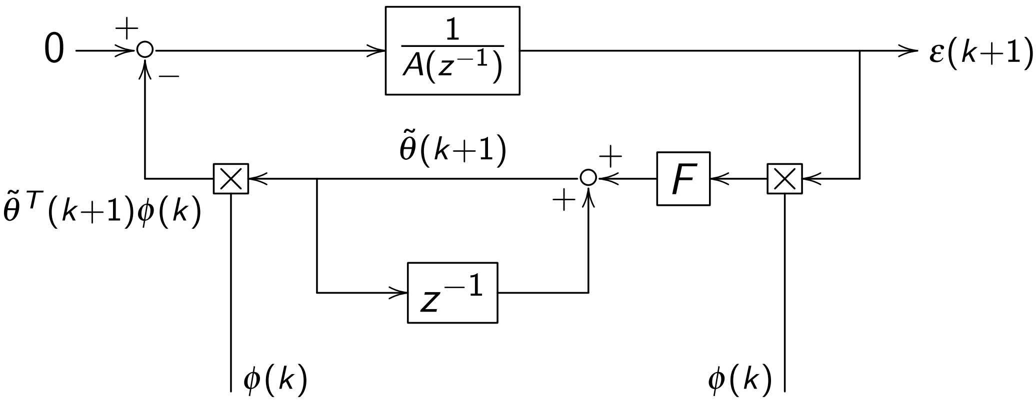
equivalent block diagram:



Stability of RLS based parallel PAA

step 2: Popov inequality

We will consider a simplified case with $\underline{F(k) = F \succ 0}$:

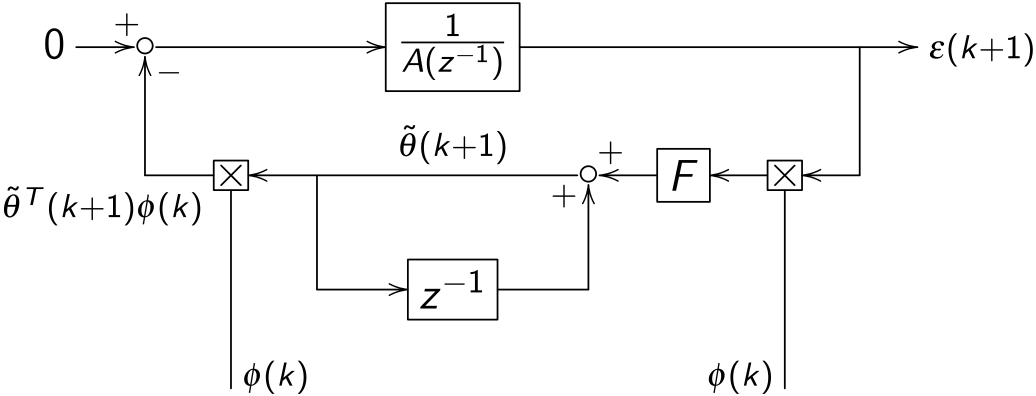


The nonlinear block is exactly the same as that in RLS, hence satisfying Popov inequality:

$$\sum_{k=0}^{k_1} \tilde{\theta}^T(k+1)\phi(k)\varepsilon(k+1) \geq -\frac{1}{2}\tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0)$$

Stability of RLS based parallel PAA

step 3: SPR condition



If $G\left(z^{-1}\right) = \frac{1}{A(z^{-1})}$ is SPR, then the PAA is asmptotically hyperstable

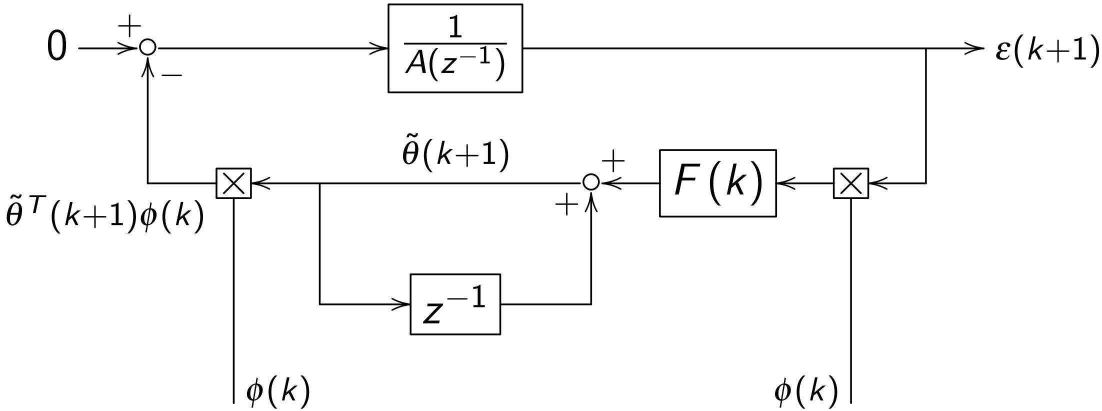
Remarks:

- RLS has an identity block: $G\left(z^{-1}\right) = 1$ which is independent of the plant
- $1/A\left(z^{-1}\right)$ depends on the plant (usually not SPR)
- several other PAAs are developed to relax the SPR condition

Stability of RLS based parallel PAA: extension

For the case of a time-varying $F(k)$ with

$$F^{-1}(k+1) = \lambda_1(k) F^{-1}(k) + \lambda_2(k)\phi(k)\phi^T(k)$$



the nonlinear block is more involved; we'll prove later, that it requires

$$\frac{1}{A(z^{-1})} - \frac{1}{2}\lambda, \text{ where } \lambda = \max_k \lambda_2(k) < 2, \text{ to be SPR}$$

Outline

1. Big picture
   Series-parallel adjustable system (equation-error method)
   Parallel adjustable system (output-error method)

2. RLS-based parallel PAA
   Formulas
   Stability requirement for PAAs with fixed adaptation gain
   Stability requirement for PAAs with time-varying adaptation gain

3. Parallel PAAs with relaxed SPR requirements

4. PAAs with time-varying adaptation gains (revisit)

Parallel algorithm with a fixed compensator

Instead of:

$$\hat{\theta}(k+1) = \hat{\theta}(k) + \frac{F(k)\phi(k)}{1+\phi^T(k)F(k)\phi(k)}\varepsilon^o(k+1)$$

$$F^{-1}(k+1) = \lambda_1(k)F^{-1}(k) + \lambda_2(k)\phi(k)\phi^T(k)$$

$$\phi^T(k) = [-\hat{y}(k), -\hat{y}(k-1), \ldots, -\hat{y}(k+1-n), u(k), \ldots, u(k-m)]$$

do: $$\hat{\theta}(k+1) = \hat{\theta}(k) + \frac{F(k)\phi(k)}{1+\phi^T(k)F(k)\phi(k)}\nu^o(k+1)$$

where

$$\nu(k+1) = C\left(z^{-1}\right)\varepsilon(k+1) = \left(c_0 + c_1 z^{-1} + \ldots c_n z^{-n}\right)\varepsilon(k+1)$$

$$\nu^o(k+1) = c_0\varepsilon^o(k+1) + c_1\varepsilon(k) + \ldots c_n\varepsilon(k-n+1)$$

Parallel algorithm with a fixed compensator

The SPR requirement becomes

$$\frac{C\left(z^{-1}\right)}{A\left(z^{-1}\right)} - \frac{\lambda}{2},\ \lambda = \max_k \lambda_2\left(k\right) < 2 \tag{2}$$

should be SPR.

Remark:

- if c_i's are close to a_i's, (2) approximates $1-\lambda/2>0$, and hence is likely to be SPR
- problem: $A\left(z^{-1}\right)$ is unknown *a priori* for the assigning of $C\left(z^{-1}\right)$
- solution: make $C\left(z^{-1}\right)$ to be adjustable as well

Parallel algorithm with an adjustable compensator

If $A\left(z^{-1}\right)=1+a_1z^{-1}+\cdots+a_nz^{-n}$, let $\hat{C}\left(z^{-1}\right)=1+\hat{c}_1z^{-1}+\cdots+\hat{c}_nz^{-n}$ and

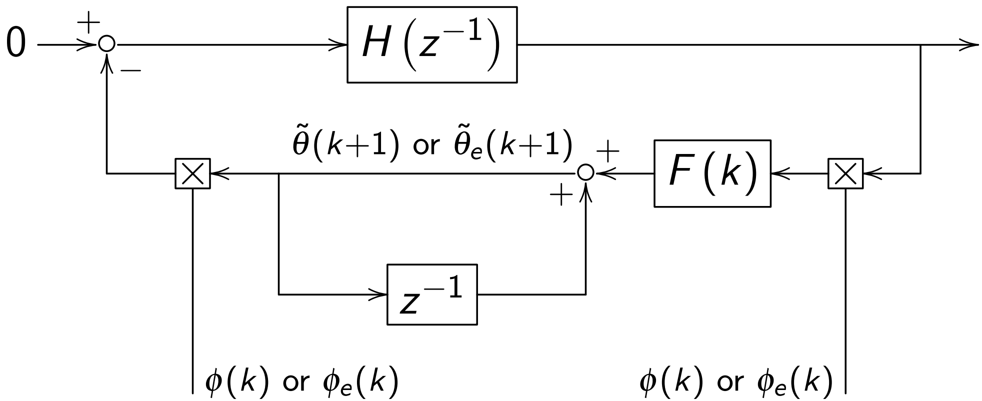
$$\nu(k+1)=\hat{C}\left(z^{-1},k+1\right)\varepsilon(k+1)$$
$$\nu^o(k+1)=\varepsilon^o(k+1)+\sum_{i=1}^{n}\hat{c}_i(k)\varepsilon(k+1-i)$$

do

$$\hat{\theta}_e(k+1)=\hat{\theta}_e(k)+\frac{F_e(k)\phi_e(k)}{1+\phi_e^T(k)F_e(k)\phi_e(k)}\nu^o(k+1)$$
$$\hat{\theta}_e^T(k)=\left[\hat{\theta}^T(k),\hat{c}_1(k),\ldots,\hat{c}_n(k)\right]$$
$$\phi_e^T(k)=\left[\phi^T(k),-\varepsilon(k),\ldots,-\varepsilon(k+1-n)\right]$$
$$F_e^{-1}(k+1)=\lambda_1(k)F_e^{-1}(k)+\lambda_2(k)\phi_e(k)\phi_e^T(k)$$

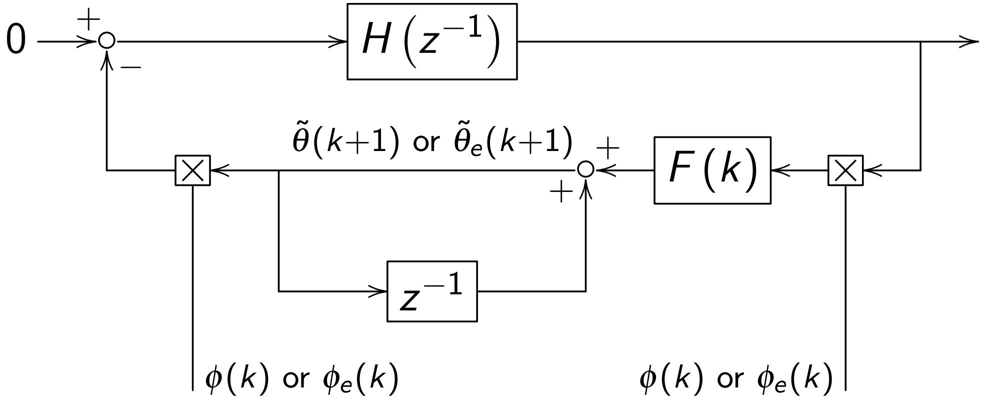
which has guaranteed **asymptotical stablility.**

General PAA block diagram



$H\left(z^{-1}\right)$	PAA
1	RLS/parallel predictor with adjustable compensator
$1/A\left(z^{-1}\right)$	parallel predictor
$C\left(z^{-1}\right)/A\left(z^{-1}\right)$	parallel predictor with fixed compensator

General PAA block diagram



- if $F(k) = F$, $H\left(z^{-1}\right)$ being SPR is sufficient for asymptotic stability
- if $F(k)$ is time-varying, we will show next: $H\left(z^{-1}\right) - \frac{1}{2}\lambda$ being SPR is sufficient for asymptotic stability

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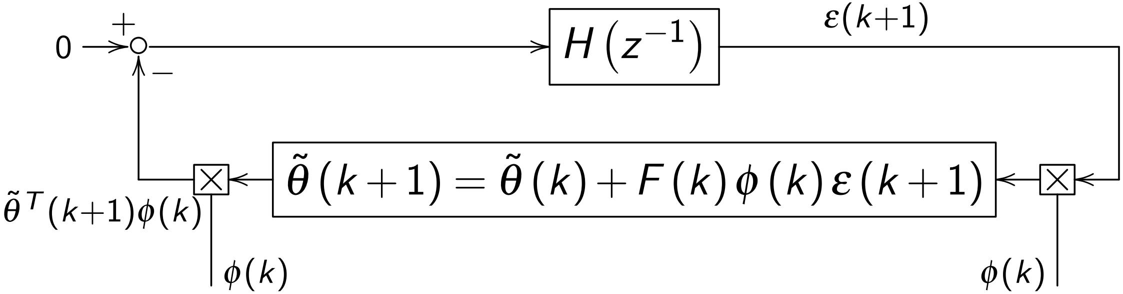
PAA with time-varying adaptation gains

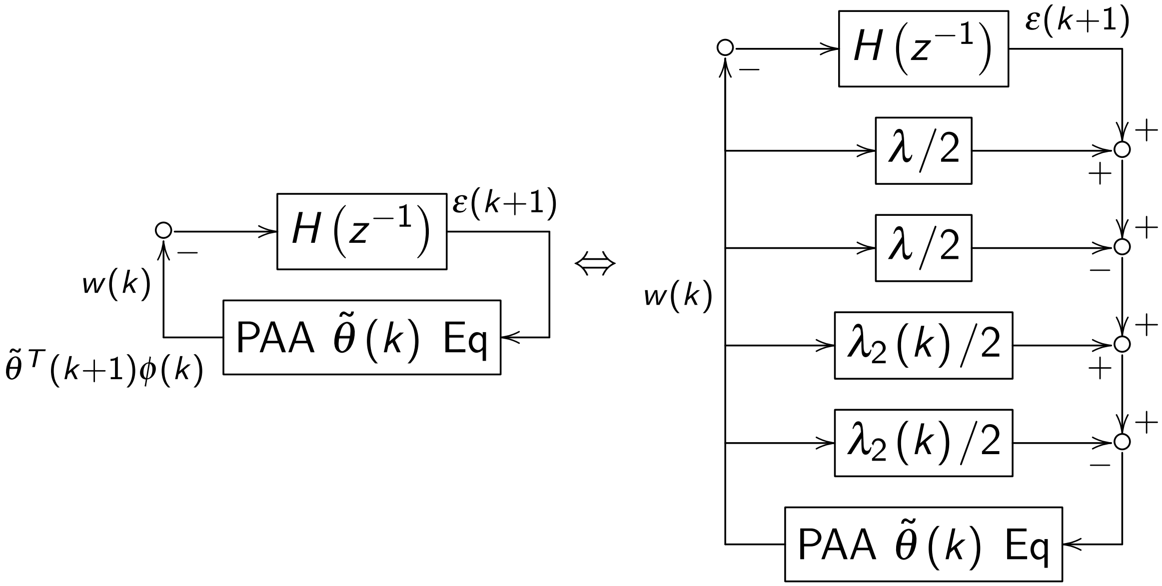


where $F^{-1}(k+1)=\lambda_1(k)F^{-1}(k)+\lambda_2(k)\phi^T(k)\phi(k)$

- unfortunately, the nonlinear block does not satisfy Popov inequality (not passive)

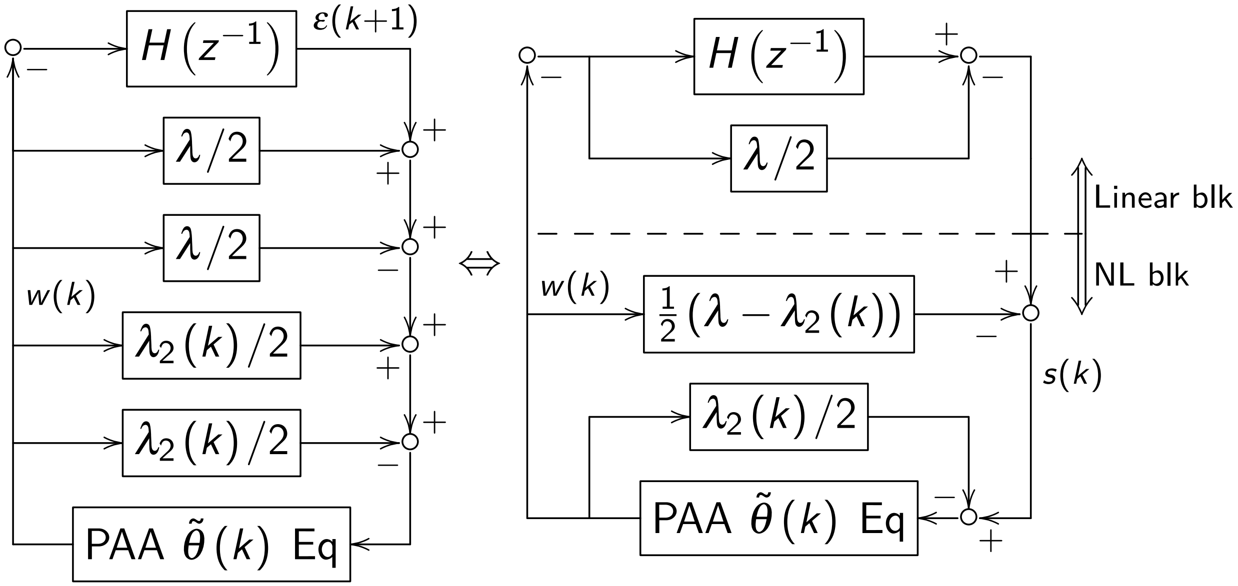
PAA with time-varying adaptation gains

a modification can re-gain the passivity of the feedback block



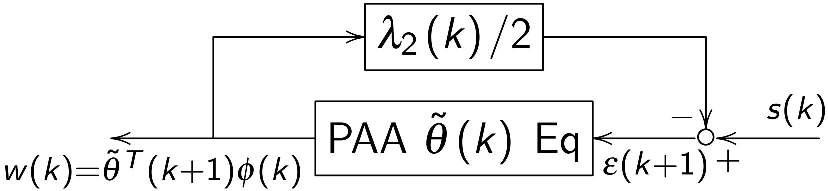
PAA with time-varying adaptation gains

a modification can re-gain the passivity of the feedback block

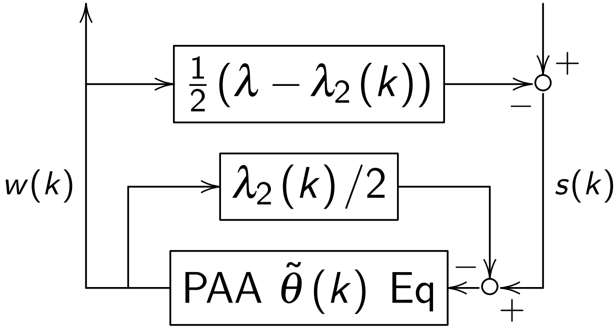


PAA with time-varying adaptation gains

step 1: show that the following is passive



step 2: the following is then passive

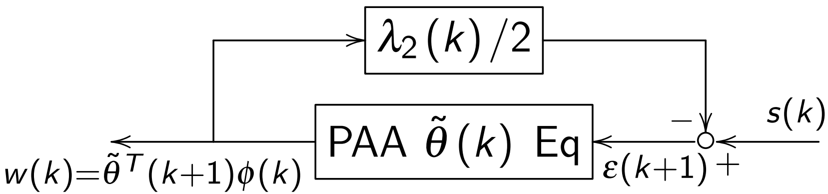


note that it is a feedback connection of a passive block with $\frac{1}{2}(\lambda - \lambda_2(k)) \geq 0$

step 3: SPR condition for the linear block $H\left(z^{-1}\right) - \frac{\lambda}{2}$

Passivity of the sub nonlinear block

Consider:



$s(k) = \varepsilon(k+1) + \frac{\lambda_2(k)}{2}\tilde{\theta}^T(k+1)\phi(k)$ gives

$$\sum_{k=0}^{k_1} w(k)s(k)$$

$$= \sum_{k=0}^{k_1} \tilde{\theta}^T(k+1)\phi(k)\left[\varepsilon(k+1) + \frac{\lambda_2(k)}{2}\tilde{\theta}^T(k+1)\phi(k)\right]$$

$\Downarrow$note that $F^{-1}(k+1) = \lambda_1(k)F^{-1}(k) + \lambda_2(k)\phi(k)\phi^T(k)$

$$= \sum_{k=0}^{k_1} \tilde{\theta}^T(k+1)\phi(k)\varepsilon(k+1) + \frac{1}{2}\tilde{\theta}^T(k+1)\left[F^{-1}(k+1) - \lambda_1(k)F^{-1}(k)\right]\tilde{\theta}(k+1)$$

which is no less than $-\frac{1}{2}\tilde{\theta}^T(0)F^{-1}(0)\tilde{\theta}(0)$ as shown next.

Proof of passivity of the sub nonlinear block

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) + F(k)\phi(k)\varepsilon(k+1)$$

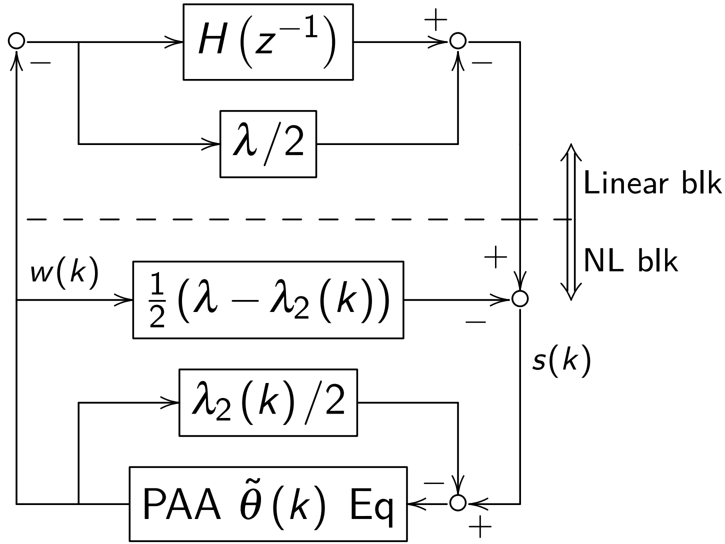
hence

$$\sum_{k=0}^{k_1} \tilde{\theta}^T(k+1)\phi(k)\varepsilon(k+1) = \sum_{k=0}^{k_1} \tilde{\theta}^T(k+1)F^{-1}(k)\left(\tilde{\theta}(k+1) - \tilde{\theta}(k)\right)$$

Combining terms and after some algebra (see appendix), we get

$$\begin{aligned}\sum_{k=0}^{k_1} w(k)s(k) = &\sum_{k=0}^{k_1} \frac{1}{2}\tilde{\theta}^T(k+1)(1-\lambda_1(k))F^{-1}(k)\tilde{\theta}(k+1) \\ &+ \sum_{k=0}^{k_1} \frac{1}{2}\left[\tilde{\theta}(k+1) - \tilde{\theta}(k)\right]^T F^{-1}(k)\left[\tilde{\theta}(k+1) - \tilde{\theta}(k)\right] \\ &+ \underbrace{\sum_{k=0}^{k_1} \frac{1}{2}\left[\tilde{\theta}^T(k+1)F^{-1}(k)\tilde{\theta}(k+1) - \tilde{\theta}^T(k)F^{-1}(k)\tilde{\theta}(k)\right]}_{\frac{1}{2}\tilde{\theta}^T(k_1+1)F^{-1}(k_1)\tilde{\theta}(k_1+1) - \frac{1}{2}\tilde{\theta}^T(0)F^{-1}(0)\tilde{\theta}(0) \geq -\frac{1}{2}\tilde{\theta}^T(0)F^{-1}(0)\tilde{\theta}(0)}\end{aligned} \quad (3)$$

Summary



In summary, the NL block indeed satisfies Popov inequality.
For stability of PAA, it is sufficient that

$$H\left(z^{-1}\right) - \frac{\lambda}{2} \text{ is SPR}$$

Appendix: derivation of (3)

$$
\begin{aligned}
&\sum_{k=0}^{k_1} \underline{\tilde{\theta}^T(k+1)F^{-1}(k)\left(\tilde{\theta}(k+1)-\tilde{\theta}(k)\right)} + \frac{1}{2}\tilde{\theta}^T(k+1)\left[F^{-1}(k+1)-\lambda_1(k)F^{-1}(k)\right]\tilde{\theta}(k+1) \\
=&\sum_{k=0}^{k_1} \underline{\tilde{\theta}^T(k+1)F^{-1}(k)\tilde{\theta}(k+1)-\tilde{\theta}^T(k+1)F^{-1}(k)\tilde{\theta}(k)} + \frac{1}{2}\tilde{\theta}^T(k+1)\left[F^{-1}(k+1)-\lambda_1(k)F^{-1}(k)\right]\tilde{\theta}(k+1) \\
=&\sum_{k=0}^{k_1} \underline{\tilde{\theta}^T(k+1)F^{-1}(k)\tilde{\theta}(k+1)} - \boxed{\tilde{\theta}^T(k+1)F^{-1}(k)\tilde{\theta}(k)} + \frac{1}{2}\tilde{\theta}^T(k+1)\left[F^{-1}(k+1)-\underline{\lambda_1(k)F^{-1}(k)}\right]\tilde{\theta}(k+1) \\
=&\sum_{k=0}^{k_1} \underline{\frac{1}{2}\tilde{\theta}^T(k+1)(1-\lambda_1(k))F^{-1}(k)\tilde{\theta}(k+1)+\frac{1}{2}\tilde{\theta}^T(k+1)F^{-1}(k)\tilde{\theta}(k+1)} - \boxed{\tilde{\theta}^T(k+1)F^{-1}(k)\tilde{\theta}(k)} \\
&\quad + \frac{1}{2}\tilde{\theta}^T(k+1)F^{-1}(k+1)\tilde{\theta}(k+1)
\end{aligned} \tag{4}
$$

The term $\frac{1}{2}\tilde{\theta}^T(k+1)(1-\lambda_1(k))F^{-1}(k)\tilde{\theta}(k+1)$ is always none-negative if $1-\lambda_1(k)\geq 0$, which is the assumption in the forgetting factor definition. We only need to worry about

$$
\sum_{k=0}^{k_1} \frac{1}{2}\tilde{\theta}^T(k+1)F^{-1}(k)\tilde{\theta}(k+1)-\tilde{\theta}^T(k+1)F^{-1}(k)\tilde{\theta}(k)+\frac{1}{2}\tilde{\theta}^T(k+1)F^{-1}(k+1)\tilde{\theta}(k+1) \tag{5}
$$

Appendix: derivation of (3)

The underlined terms are already available in (5). Adding and substracting terms in (5) gives

$$\sum_{k=0}^{k_1} \frac{1}{2}\tilde{\theta}^T(k+1)F^{-1}(k)\tilde{\theta}(k+1) - \frac{1}{2}\underline{\tilde{\theta}^T(k)F^{-1}(k)\tilde{\theta}(k)}$$

$$+\frac{1}{2}\tilde{\theta}^T(k+1)F^{-1}(k+1)\tilde{\theta}(k+1) - \tilde{\theta}^T(k+1)F^{-1}(k)\tilde{\theta}(k) + \underline{\frac{1}{2}\tilde{\theta}^T(k)F^{-1}(k)\tilde{\theta}(k)}$$

$$= \sum_{k=0}^{k_1} \frac{1}{2}\tilde{\theta}^T(k+1)F^{-1}(k)\tilde{\theta}(k+1) - \frac{1}{2}\tilde{\theta}^T(k)F^{-1}(k)\tilde{\theta}(k)$$

$$+ \underbrace{\frac{1}{2}\left[\tilde{\theta}(k+1) - \tilde{\theta}(k)\right]^T F^{-1}(k)\left[\tilde{\theta}(k+1) - \tilde{\theta}(k)\right]}_{\geq 0}$$

Appendix: derivation of (3)

Summarizing, we get

$$\sum_{k=0}^{k_1} w(k)s(k) = \sum_{k=0}^{k_1} \frac{1}{2}\tilde{\theta}^T(k+1)(1-\lambda_1(k))F^{-1}(k)\tilde{\theta}(k+1)$$

$$+\sum_{k=0}^{k_1}\frac{1}{2}\left[\tilde{\theta}(k+1)-\tilde{\theta}(k)\right]^T F^{-1}(k)\left[\tilde{\theta}(k+1)-\tilde{\theta}(k)\right]$$

$$+\underbrace{\sum_{k=0}^{k_1}\frac{1}{2}\left[\tilde{\theta}^T(k+1)F^{-1}(k)\tilde{\theta}(k+1)-\tilde{\theta}^T(k)F^{-1}(k)\tilde{\theta}(k)\right]}_{\frac{1}{2}\tilde{\theta}^T(k_1+1)F^{-1}(k_1)\tilde{\theta}(k_1+1)-\frac{1}{2}\tilde{\theta}^T(0)F^{-1}(0)\tilde{\theta}(0)}$$

hence

$$\sum_{k=0}^{k_1} w(k)s(k) \geq -\frac{1}{2}\tilde{\theta}^T(0)F^{-1}(0)\tilde{\theta}(0)$$

Summary

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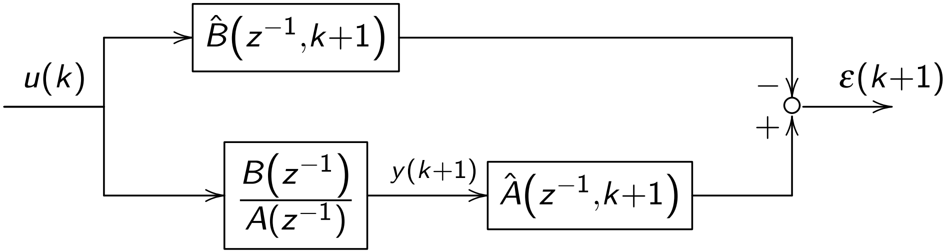
Lecture 18: Parameter Convergence in PAAs

Big picture

why are we learning this:

Consider a series-parallel PAA



where the plant is stable.

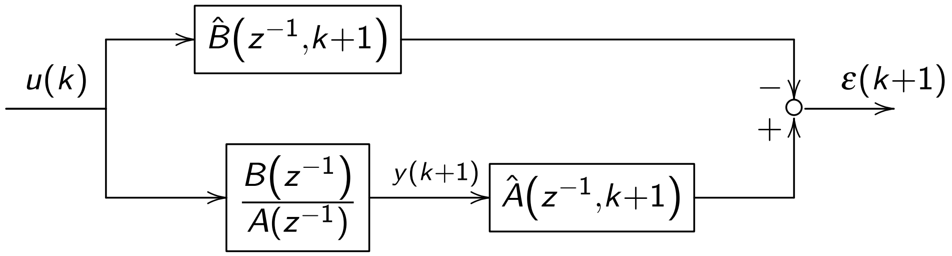
(Hyper)stability of PAA gives

$$\lim_{k\to\infty} \varepsilon(k) = \lim_{k\to\infty} \left\{ -\tilde{\theta}^T(k)\phi(k-1) \right\} = 0$$

But this does not guarantee

$$\lim_{k\to\infty} \tilde{\theta}(k) = 0 \Longleftrightarrow \lim_{k\to\infty} \hat{\theta}(k) = \theta$$

Parameter convergence condition



$\varepsilon(k) \to 0$ means

$$\hat{A}\left(z^{-1},k+1\right)\frac{B\left(z^{-1}\right)}{A\left(z^{-1}\right)}u(k) - \hat{B}\left(z^{-1},k+1\right)u(k) \to 0$$

$$\Rightarrow \left[\hat{A}\left(z^{-1},k+1\right)B\left(z^{-1}\right) - A\left(z^{-1}\right)\hat{B}\left(z^{-1},k+1\right)\right]u(k) \to 0$$

$$\Leftrightarrow \left[\hat{A}\left(z^{-1}\right)B\left(z^{-1}\right) \pm A\left(z^{-1}\right)B\left(z^{-1}\right) - A\left(z^{-1}\right)\hat{B}\left(z^{-1}\right)\right]u(k) \to 0$$

$$\Leftrightarrow \left[\tilde{A}\left(z^{-1}\right)B\left(z^{-1}\right) - A\left(z^{-1}\right)\tilde{B}\left(z^{-1}\right)\right]u(k) \to 0$$

where $\tilde{A}\left(z^{-1}\right) = \hat{A}\left(z^{-1}\right) - A\left(z^{-1}\right)$.

Parameter convergence condition

Consider

$$\overbrace{\left[\tilde{A}\left(z^{-1}\right)B\left(z^{-1}\right)-A\left(z^{-1}\right)\tilde{B}\left(z^{-1}\right)\right]}^{\text{a new polynomial } \alpha_0+\alpha_1 z^{-1}+\cdots+\alpha_{m+n}z^{-m-n}}u(k)\to 0$$

$$\tilde{B}\left(z^{-1}\right)=\tilde{b}_0+\tilde{b}_1 z^{-1}+\cdots+\tilde{b}_m z^{-m} \qquad B\left(z^{-1}\right)=b_0+b_1 z^{-1}+\cdots+b_m z^{-m}$$

$$A\left(z^{-1}\right)=1+a_1 z^{-1}+\cdots+a_n z^{-n} \qquad \tilde{A}\left(z^{-1}\right)=\tilde{a}_1 z^{-1}+\cdots+\tilde{a}_n z^{-n}$$

Two questions we are going to discuss for assuring $\tilde{\theta}=0$:

- is $\alpha_i=0$ true iff $\tilde{a}_i=0,\ \tilde{b}_i=0$ (i.e., $\{\alpha_i\}=0\Leftrightarrow\tilde{\theta}=0$)?
- if $\alpha_i\neq 0$, can $\left[\alpha_0+\alpha_1 z^{-1}+\cdots+\alpha_{m+n}z^{-m-n}\right]u(k)=0$?

Parameter convergence condition

Qs 1: $\alpha_i = 0 \iff \tilde{a}_i = 0,\ \tilde{b}_i = 0$? Ans: yes if $B\left(z^{-1}\right)$ and $A\left(z^{-1}\right)$ are coprime

$$\alpha_0 + \alpha_1 z^{-1} + \cdots + \alpha_{m+n} z^{-m-n} = \tilde{A}\left(z^{-1}\right) B\left(z^{-1}\right) - A\left(z^{-1}\right) \tilde{B}\left(z^{-1}\right)$$

- the right hand side is composed of terms of $\tilde{a}_i b_j$ and $a_p \tilde{b}_q$
- comparing coefficients of z^{-k} gives

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \vdots \\ \vdots \\ \alpha_{m+n} \end{bmatrix} = S \begin{bmatrix} \tilde{b}_0 \\ \tilde{b}_1 \\ \vdots \\ \tilde{b}_m \\ \tilde{a}_1 \\ \vdots \\ \tilde{a}_n \end{bmatrix},\ S: \text{a square matrix composed of } \{a_i, b_j\}$$

- turns out S is non-singular if and only if $B\left(z^{-1}\right)$ and $A\left(z^{-1}\right)$ are coprime (recall the theorem discussed in repetitive control)

Parameter convergence condition

Qs 2: if $\alpha_i \neq 0$, can $\left[\alpha_0 + \alpha_1 z^{-1} + \cdots + \alpha_{m+n} z^{-m-n}\right] u(k) = 0$?

Simple example with $n+m=2$, $u(k) = \cos(\omega k) = \mathrm{Re}\left\{e^{j\omega k}\right\}$:

$$\left[\alpha_0 + \alpha_1 z^{-1} + \alpha_2 z^{-2}\right] u(k) \to 0$$
$$\Leftarrow \left[\alpha_0 + \alpha_1 z^{-1} + \alpha_2 z^{-2}\right] e^{j\omega k} \to 0$$

which can be achieved either by $\alpha_0 = \alpha_1 = \alpha_2 = 0$ (the desired case) or by

$$\left(1 - e^{-j\omega} z^{-1}\right)\left(1 - e^{j\omega} z^{-1}\right) e^{j\omega k}$$
$$= \left[1 - 2\cos(\omega) z^{-1} + z^{-2}\right] e^{j\omega k} \to 0$$

Parameter convergence condition

Qs 2: if $\alpha_i \neq 0$, can $\left[\alpha_0 + \alpha_1 z^{-1} + \cdots + \alpha_{m+n} z^{-m-n}\right] u(k) = 0$?

If, however,

$$u(k) = c_1 \cos(\omega_1 k) + c_2 \cos(\omega_2 k) = \text{Re}\left\{c_1 e^{j\omega_1 k} + c_2 e^{j\omega_2 k}\right\}$$

then

$$\left[\alpha_0 + \alpha_1 z^{-1} + \alpha_2 z^{-2}\right] u(k) \to 0$$

can only be achieved by $\alpha_0 = \alpha_1 = \alpha_2 = 0$ (the desired case).

Observations:

- complex roots of $\alpha_0 + \alpha_1 z^{-1} + \alpha_2 z^{-2}$ always come as pairs
- impossible for $\alpha_0 + \alpha_1 z^{-1} + \alpha_2 z^{-2}$ to have four roots at $e^{\pm j\omega_1}$ and $e^{\pm j\omega_2}$
- if the total number of parameters $n + m = 3$, $u(k)$ should contain at least $2 \left(= \frac{n+m+1}{2}\right)$ frequency components

Parameter convergence condition

general case:

$$\alpha_0 + \alpha_1 z^{-1} + \cdots + \alpha_{m+n} z^{-m-n} = 0$$

- number of the pairs of roots $= (m+n)/2$, if $m+n$ is even
- number of the pairs of roots $= (m+n-1)/2$ if $m+n$ is odd

Theorem (Persistant of excitation for PAA convergence)

For PAAs with a series-parallel predictor, the convergence

$$\lim_{k\to\infty} \hat{\theta}_i(k) = \theta_i(k)$$

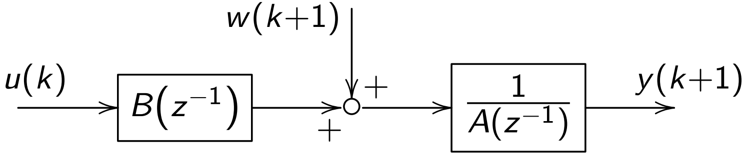
is assured if
1, the plant transfer function is irreducible
2, the input signal contains at least $1+(m+n)/2$ *(for* $n+m$ *even) or* $(m+n+1)/2$ *(for* $m+n$ *odd) independent frequency components.*

Outline

1. Big picture
2. Parameter convergence conditions
3. Effect of noise on parameter identification
4. Convergence improvement in the presence of stochastic noises
5. Effect of deterministic disturbances

Effect of noise on parameter identification

Noise modeling:

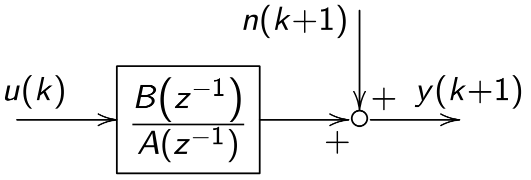


i.e.

$$A\left(z^{-1}\right)y(k+1) = B\left(z^{-1}\right)u(k) + w(k+1)$$

$$y(k+1) = \theta^T \phi(k) + w(k+1)$$

or



i.e.

$$y(k+1) = \theta^T \phi(k) + A\left(z^{-1}\right)n(k+1)$$

which is equivalent to $w(k+1) = A\left(z^{-1}\right)n(k+1)$ in the first case

Effect of noise on parameter identification

plant output: $$y(k+1) = \theta^T \phi(k) + w(k+1)$$

predictor output: $$\hat{y}(k+1) = \hat{\theta}^T(k+1)\phi(k)$$

Effect of noise on parameter identification

plant output: $y(k+1) = \theta^T \phi(k) + w(k+1)$

predictor output: $\hat{y}(k+1) = \hat{\theta}^T(k+1)\phi(k)$

a posteriori prediction error:

$$\varepsilon(k+1) = y(k+1) - \hat{y}(k+1) = \overbrace{-\tilde{\theta}^T(k+1)\phi(k)}^{\underline{\varepsilon}(k+1)\text{: error without noise}} + w(k+1)$$

PAA:
$$\begin{aligned}\hat{\theta}(k+1) &= \hat{\theta}(k) + F(k)\phi(k)\varepsilon(k+1)\\ &= \hat{\theta}(k) + F(k)\phi(k)\underline{\varepsilon}(k+1) + F(k)\phi(k)w(k+1)\end{aligned}$$

Effect of noise on parameter identification

plant output: $y(k+1) = \theta^T \phi(k) + w(k+1)$

predictor output: $\hat{y}(k+1) = \hat{\theta}^T(k+1)\phi(k)$

a posteriori prediction error:

$$\varepsilon(k+1) = y(k+1) - \hat{y}(k+1) = \overbrace{-\tilde{\theta}^T(k+1)\phi(k)}^{\underline{\varepsilon}(k+1):\ \text{error without noise}} + w(k+1)$$

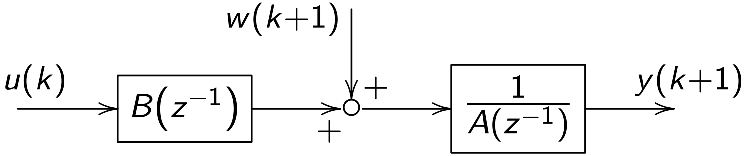
PAA: $$\hat{\theta}(k+1) = \hat{\theta}(k) + F(k)\phi(k)\varepsilon(k+1)$$
$$= \hat{\theta}(k) + F(k)\phi(k)\underline{\varepsilon}(k+1) + F(k)\phi(k)w(k+1)$$

- $F(k)\phi(k)w(k+1)$ is integrated by PAA
- need: $\boxed{\mathrm{E}[\phi(k)w(k+1)] = 0}$

and a vanishing adaptation gain $F(k)$:

$F^{-1}(k+1) = \lambda_1(k)F^{-1}(k) + \lambda_2(k)\phi(k)\phi^T(k)$, $\lambda_1(k) \xrightarrow{k\to\infty} 1$ and $0 < \lambda_2(k) < 2$

Series-parallel PAA convergence condition



$$\hat{\theta}(k+1) = \hat{\theta}(k) + F(k)\phi(k)\underline{\varepsilon}(k+1) + F(k)\phi(k)w(k+1)$$

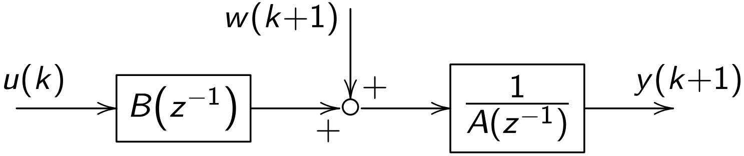
In series-parallel PAA:

$$\phi(k) = [-y(k), -y(k-1), \ldots, -y(k-n+1), \\ u(k), u(k-1), \ldots, u(k-m)]^T$$

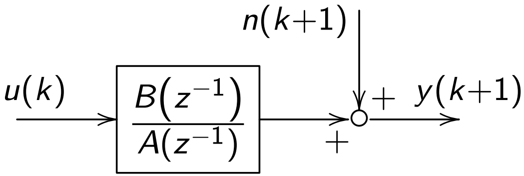
$\boxed{\mathrm{E}[\phi(k)w(k+1)] = 0}$ is achieved if

- $w(k+1)$ is white, and
- $u(k)$ and $w(k+1)$ are independent

Series-parallel PAA convergence condition



Issues: $w(k+1)$ is rarely white, e.g.,



where the output measurement noise $n(k+1)$ is usually white but

$$y(k+1) = \theta^T \phi(k) + \overbrace{A\left(z^{-1}\right) n(k+1)}^{w(k+1)}$$

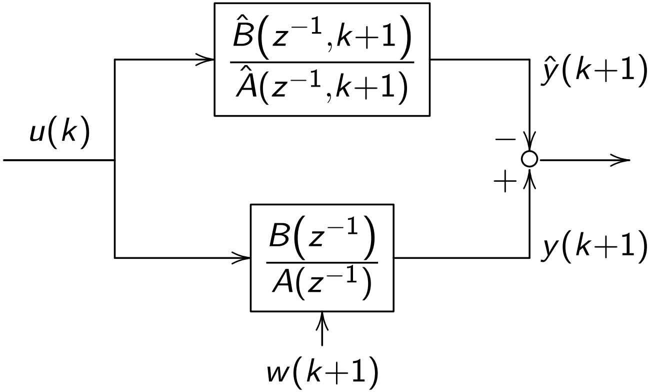
so $w(k+1)$ is not white.

Parallel PAA convergence condition

In parallel PAA:

$$\phi(k) = [-\hat{y}(k), -\hat{y}(k-1), \ldots, -\hat{y}(k-n+1), \\ u(k), u(k-1), \ldots, u(k-m)]^T$$

$\boxed{E[\phi(k)w(k+1)] = 0}$ does not require $w(k+1)$ to be white as $\hat{y}(k)$ does not depend on $w(k+1)$ by design



Summary

Theorem (Series-parallel PAA convergence condition)

When the predictor is of series-parallel type, the PAA with a vanishing adaptation gain has unbiased convergence when
i. $u(k)$ *is rich in frequency (persistent excitation) and is independent from the noise* $w(k+1)$
ii. $w(k+1)$ *is white*

Theorem (Parallel PAA convergence condition)

When the predictor is of parallel type, the PAA with vanishing adaptation gain has unbiased convergence when
i. $u(k)$ *satisfies the persistent excitation condition*
ii. $u(k)$ *is independent from* $w(k+1)$

Note: parallel predictors have more strict stability requirements

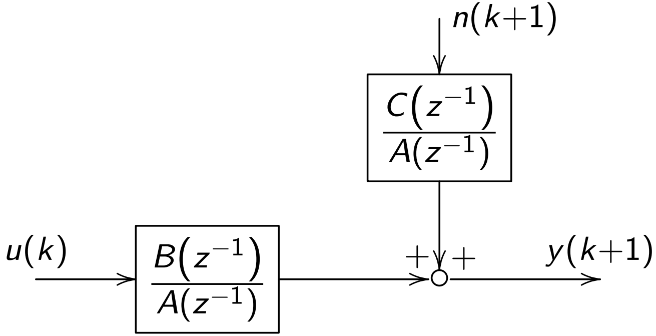
Outline

1. Big picture
2. Parameter convergence conditions
3. Effect of noise on parameter identification
4. Convergence improvement in the presence of stochastic noises
5. Effect of deterministic disturbances

Convergence improvement when there is noise

extended least squares

If the effect of noise can be expressed as



i.e. $w(k+1) = C(z^{-1})n(k+1) = \left[1 + c_1 z^{-1} + \ldots c_{n_C} z^{-n_C}\right] n(k+1)$

where $n(k+1)$ is white, then

$$y(k+1) = \theta^T \phi(k) + C\left(z^{-1}\right) n(k+1) = \theta_e^T \phi_e(k) + n(k+1)$$

$$\theta_e^T = \left[\theta^T, c_1, \ldots, c_{n_C}\right]$$

$$\phi_e^T(k) = \left[\phi^T(k), n(k), \ldots, n(k - n_C + 1)\right]$$

Convergence improvement when there is noise

extended least squares

a posteriori prediction

$$\hat{y}(k+1) = \hat{\theta}_e^T(k+1)\phi_e(k)$$
$$\phi_e^T(k) = \left[\phi^T(k), n(k), \ldots, n(k-n_C+1)\right]$$

but $n(k), \ldots, n(k-n_C+1)$ are not measurable. However, if $\hat{\theta}_e$ is close to θ_e, then

$$\varepsilon(k+1) = y(k+1) - \hat{y}(k+1) \approx n(k+1)$$

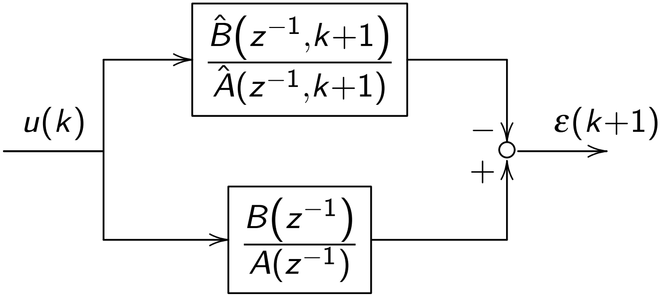
extended least squares uses

$$\hat{y}(k+1) = \hat{\theta}_e^T(k+1)\phi_e^*(k)$$
$$\phi_e^*(k) = \left[\phi^T(k), \varepsilon(k), \ldots, \varepsilon(k-n_C+1)\right]^T$$

where $\varepsilon(k) = y(k) - \hat{y}(k)$

Convergence improvement when there is noise

output error method with adjustable compensator



If $A\left(z^{-1}\right) = 1 + a_1 z^{-1} + \cdots + a_n z^{-n}$, let $\hat{C}\left(z^{-1}\right) = 1 + \hat{c}_1 z^{-1} + \cdots + \hat{c}_n z^{-n}$ and

$$\nu(k+1) = \hat{C}\left(z^{-1}, k+1\right) \varepsilon(k+1)$$

$$\nu^o(k+1) = \varepsilon^o(k+1) + \sum_{i=1}^{n} \hat{c}_i(k) \varepsilon(k+1-i)$$

construct PAA with $\theta_e^T = \left[\theta^T, a_1, \ldots, a_n\right]$ and $\nu(k+1)$ as the adaptation error.

Convergence improvement when there is noise

output error method with adjustable compensator

$$\hat{\theta}_e(k+1) = \hat{\theta}_e(k) + \frac{F_e(k)\phi_e(k)}{1+\phi_e^T(k)F_e(k)\phi_e(k)}\nu^o(k+1)$$

$$\hat{\theta}_e^T(k) = \left[\hat{\theta}^T(k), \hat{c}_1(k), \dots, \hat{c}_n(k)\right]$$

$$\phi_e^T(k) = \left[\phi^T(k), -\varepsilon(k), \dots, -\varepsilon(k+1-n)\right]$$

$$F_e^{-1}(k+1) = \lambda_1(k)F_e^{-1}(k) + \lambda_2(k)\phi_e(k)\phi_e^T(k)$$

Stability condition:

$$1 - \frac{\lambda}{2} \text{ is SPR; } \lambda = \max_k \lambda_2(k) < 2$$

Convergence condition: depend on properties of the disturbance and $A\left(z^{-1}\right)$; see details in ME233 reader

Different recursive identification algorithms

- there are more PAAs for improved convergence

- each algorithm suits for a certain model of plant + disturbance

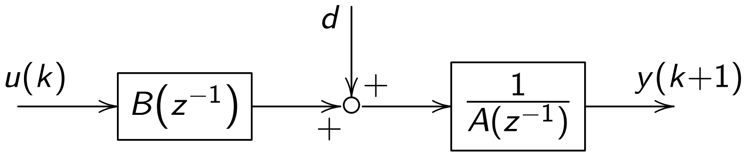
Outline

1. Big picture
2. Parameter convergence conditions
3. Effect of noise on parameter identification
4. Convergence improvement in the presence of stochastic noises
5. Effect of deterministic disturbances

Effect of deterministic disturbances

Intuition: if the disturbance structure is known, it can be included in PAA for improved performance.

Example (constant disturbance):



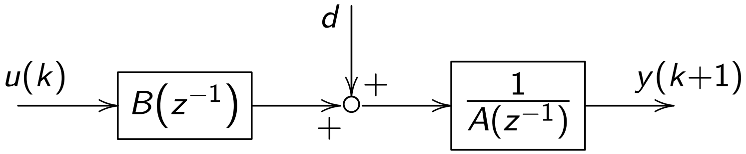
$$y(k+1) = -\sum_{i=1}^{n} a_i y(k+1-i) + \sum_{i=0}^{m} b_i u(k-i) + d = \theta^T \phi(k) + d$$

Approach 1: enlarge the model as

$$y(k+1) = \left[\theta^T, d\right] \begin{bmatrix} \phi(k) \\ 1 \end{bmatrix} = \theta_e^T \phi_e(k)$$

and construct PAA on θ_e.

Effect of deterministic disturbances



$$y(k+1) = -\sum_{i=1}^{n} a_i y(k+1-i) + \sum_{i=0}^{m} b_i u(k-i) + d = \theta^T \phi(k) + d$$

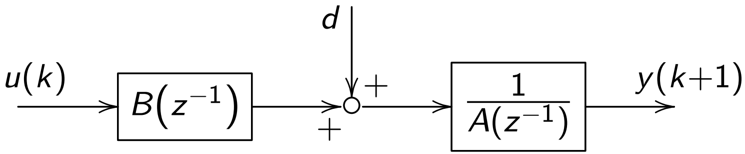
Approach 2: notice that $\left(1-z^{-1}\right) d = 0$, we can do

$$y(k+1) \longrightarrow \boxed{1-z^{-1}} \longrightarrow y_f(k+1)\ ; \quad u(k+1) \longrightarrow \boxed{1-z^{-1}} \longrightarrow u_f(k+1)\ ;$$

and have a new "disturbance-free" model for PAA:

$$y_f(k+1) = -\sum_{i=1}^{n} a_i y_f(k+1-i) + \sum_{i=0}^{m} b_i u_f(k-i)$$

Effect of deterministic disturbances



Similar considerations can be applied to the cases when d is sinusoidal, repetitive, etc

Lecture 19: Adaptive Control based on Pole Assignment

Big picture

reasons for adaptive control:

- unknown or time-varying plants
- unknown or time-varying disturbance (with known structure but unknown coefficients)

two main steps:

- decide the controller structure
- design PAA to adjust the controller parameters

two ways of adaptation process:

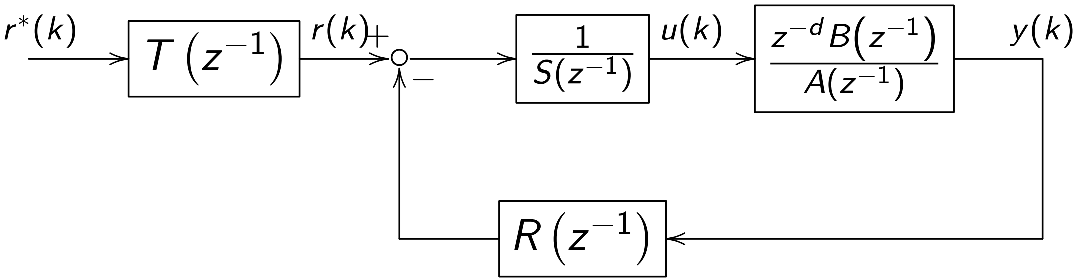
- indirect adaptive control: adapt the plant parameters and use them in the updated controller
- direct adaptive control: directly adapt the controller parameters

RST control structure

Plant:

$$G\left(z^{-1}\right)=\frac{z^{-d}B\left(z^{-1}\right)}{A(z^{-1})} \qquad \begin{aligned} B\left(z^{-1}\right) &= b_0+b_1z^{-1}+\cdots+b_mz^{-m},\ b_0\neq 0 \\ A\left(z^{-1}\right) &= 1+a_1z^{-1}+\cdots+a_nz^{-n} \end{aligned}$$

Consider RST type controller:



Closed-loop transfer function:

$$\frac{Y\left(z^{-1}\right)}{R\left(z^{-1}\right)}=\frac{z^{-d}B\left(z^{-1}\right)}{A(z^{-1})\,S\left(z^{-1}\right)+z^{-d}B\left(z^{-1}\right)R\left(z^{-1}\right)}$$

Pole placement

Closed-loop pole assignment via:

$$z^{-d}B(z^{-1})R(z^{-1})+S(z^{-1})A(z^{-1})=D(z^{-1})$$

- this is a polynominal (Diophantine) equation
- design $D\left(z^{-1}\right)$, find $S\left(z^{-1}\right)$ and $R\left(z^{-1}\right)$ by coefficient matching

Pole placement for plants with stable zeros

If zeros of plant are all stable, they can be cancelled. We can do

$$S\left(z^{-1}\right)=S^{'}(z^{-1})B(z^{-1})$$
$$D(z^{-1})=D^{'}(z^{-1})B(z^{-1})$$

yielding

$$z^{-d}R(z^{-1})+S^{'}(z^{-1})A(z^{-1})=D^{'}(z^{-1}) \tag{1}$$

where the polynomials should match order:

$$S^{'}\left(z^{-1}\right)=1+s_1^{'}z^{-1}+\cdots+s_{d-1}^{'}z^{-(d-1)}$$
$$R\left(z^{-1}\right)=r_0+r_1z^{-1}+\cdots+r_{n-1}z^{-(n-1)}$$

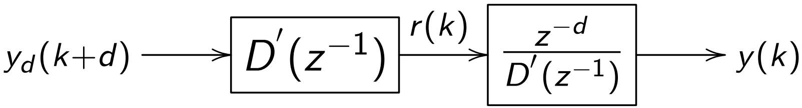
The transfer function from $r(k)$ to $y(k)$ is thus

$$G_{r\to y}(z^{-1})=\frac{z^{-d}B(z^{-1})}{S(z^{-1})A(z^{-1})+z^{-d}B(z^{-1})R(z^{-1})}=\frac{z^{-d}}{D^{'}(z^{-1})}$$

Pole placement for plants with stable zeros

Hence we can let

$$T(z^{-1}) = D^{'}(z^{-1}),\ r^{*}(k) = y_d(k+d)$$



which means

$$D^{'}\left(z^{-1}\right)\left[y\left(k+d\right) - y_d\left(k+d\right)\right] = 0$$

- this is the desired control goal, you can compare it with the goal in system identification: $y\left(k+1\right) - \hat{y}\left(k+1\right) = 0$
- next we express $D^{'}\left(z^{-1}\right) y\left(k+d\right)$ and $D^{'}\left(z^{-1}\right) y_d\left(k+d\right)$ in forms similar to "$\theta^{T}\phi\left(k\right)$"

Pole placement for plants with stable zeros

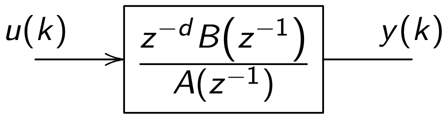
the $D'(z^{-1})\, y(k+d)$ term

For a tuned pole placement with known plant model:

- $z^{-d}R(z^{-1}) + S'(z^{-1})A(z^{-1}) = D'(z^{-1})$ yields

$$A(z^{-1})S'(z^{-1})y(k+d) = D'(z^{-1})y(k+d) - z^{-d}R(z^{-1})y(k+d)$$

- and the plant model



gives

$$A(z^{-1})y(k+d) = B(z^{-1})u(k)$$

Combining the two gives

$$D'(z^{-1})y(k+d) = B(z^{-1})S'(z^{-1})u(k) + R(z^{-1})y(k) \qquad (2)$$

Pole placement for plants with stable zeros

the $D^{'}\left(z^{-1}\right)y\left(k+d\right)$ term

We will now simplify (2). Note first:

$$S\left(z^{-1}\right) = B\left(z^{-1}\right)S^{'}\left(z^{-1}\right) = s_0 + s_1 z^{-1} + \cdots + s_{d+m-1} z^{-(d+m-1)}$$

hence

$$\begin{aligned}\underline{D^{'}(z^{-1})y(k+d)} &= \overbrace{B(z^{-1})S^{'}(z^{-1})}^{S(z^{-1})}u(k) + R(z^{-1})y(k) \\ &= \underline{\theta_c^T \phi(k)}\end{aligned}$$

where

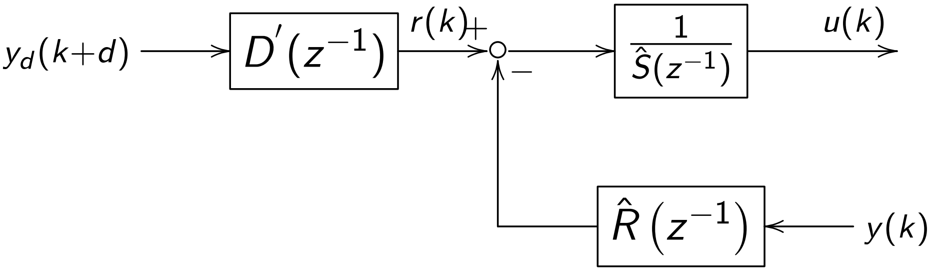
$$\theta_c^T = [s_0, s_1, \ldots, s_{d+m-1}, r_0, \ldots, r_{n-1}]$$

$$\phi(k) = [u(k), u(k-1), \ldots, u(k-d-m+1), y(k), \ldots, y(k-n+1)]^T$$

Pole placement for plants with stable zeros

the $D^{'}\left(z^{-1}\right)y_d\left(k+d\right)$ term

For the actual adaptive $S\left(z^{-1}\right)$ and $R\left(z^{-1}\right)$, the control law is



i.e.
$$u\left(k\right)=\frac{1}{\hat{S}(z^{-1})}\left[D^{'}(z^{-1})y_d\left(k+d\right)-\hat{R}(z^{-1})y\left(k\right)\right]$$

yielding

$$\underline{D^{'}(z^{-1})y_d\left(k+d\right)}=\hat{S}(z^{-1})u\left(k\right)+\hat{R}(z^{-1})y\left(k\right)=\underline{\hat{\theta}_c^T\phi\left(k\right)} \qquad (3)$$

This is a direct adaptive control: no explicit $B(z^{-1})$ and $A(z^{-1})$ in $\hat{\theta}_c$

Pole placement for plants with stable zeros

Hence we can define

$$\varepsilon(k+d) = D^{'}(z^{-1})y(k+d) - \hat{\theta}_c^T(k+d)\phi(k)$$

or equivalently

$$\text{a posteriori:} \qquad \varepsilon(k) = D^{'}(z^{-1})y(k) - \hat{\theta}_c^T(k)\phi(k-d)$$

$$\text{a priori:} \qquad \varepsilon^o(k) = D^{'}(z^{-1})y(k) - \hat{\theta}_c^T(k-1)\phi(k-d)$$

and apply parameter adaptation for θ_c, e.g., using series-parallel predictors

$$\hat{\theta}_c(k) = \hat{\theta}_c(k-1) + \frac{F(k-1)\phi(k-d)}{1+\phi(k-d)^T F(k-1)\phi(k-d)}\varepsilon^o(k)$$

$$F^{-1}(k) = \lambda_1(k)F^{-1}(k-1) + \lambda_2(k)\phi(k-d)\phi^T(k-d)$$

Comparison with system identification

Comparison:

standard system identification:

$$
\begin{aligned}
y(k+1) &= \theta^T \phi(k) \\
\hat{\theta}(k+1) &= \hat{\theta}(k) + F(k)\phi(k)\varepsilon(k+1) \\
\varepsilon(k+1) &= \frac{\varepsilon^o(k+1)}{1+\phi^T(k)F(k)\phi(k)}
\end{aligned}
$$

adaptive pole placement:

$$
\begin{aligned}
D'\left(z^{-1}\right) y(k) &= \theta_c^T \phi(k-d) \\
\hat{\theta}_c(k) &= \hat{\theta}_c(k-1) + F(k-1)\phi(k-d)\varepsilon(k) \\
\varepsilon(k) &= \frac{\varepsilon^o(k)}{1+\phi^T(k-d)F(k-1)\phi(k-d)}
\end{aligned}
$$

Pole placement for plants with stable zeros

PAA Stability

First obtain the *a posteriori* dynamics of the parameter error:

$$\hat{\theta}_c(k) = \hat{\theta}_c(k-1) + F(k-1)\phi(k-d)\varepsilon(k)$$
$$\Rightarrow \tilde{\theta}_c(k) = \tilde{\theta}_c(k-1) + F(k-1)\phi(k-d)\varepsilon(k)$$

In the mean time

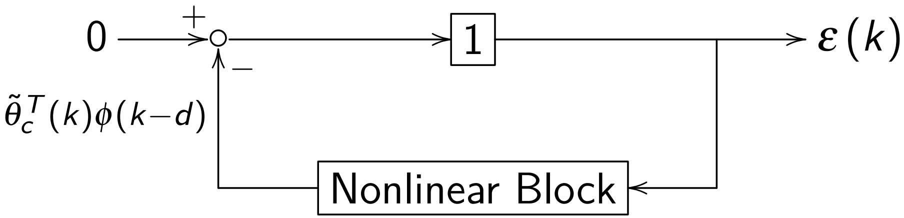
$$\begin{aligned}
\varepsilon(k) &= D^{'}(z^{-1})y(k) - \hat{\theta}_c^T(k)\phi(k-d) \\
&\Downarrow \text{recall } D^{'}\left(z^{-1}\right)y(k+d) = \theta_c^T\phi(k) \\
&= \theta_c^T\phi(k-d) - \hat{\theta}_c^T(k)\phi(k-d) \\
&= -\tilde{\theta}_c(k)^T\phi(k-d)
\end{aligned}$$

Pole placement for plants with stable zeros

PAA Stability

$$\varepsilon(k) = -\tilde{\theta}_c(k)^T \phi(k-d)$$

$$\tilde{\theta}_c(k) = \tilde{\theta}_c(k-1) + F(k-1)\phi(k-d)\varepsilon(k)$$



The PAA thus is in a standard series-parallel structure with the LTI block being 1. Hyperstability easily follows, which gives

$$\lim_{k\to\infty} \varepsilon(k) = \frac{D'(z^{-1})y(k) - \hat{\theta}_c^T(k-1)\phi(k-d)}{1+\phi^T(k-d)F(k-1)\phi(k-d)} \to 0$$

Similar as before, to prove $\varepsilon^o(k) = D'(z^{-1})(y(k) - y_d(k)) \to 0$, we need to show that $\phi(k-d)$ is bounded, which can be shown to be true (see ME233 reader).

Pole placement for plants with stable zeros

Design procedure:

Step 1: choose desired $D^{'}(z^{-1})$ ($\deg D^{'}(z^{-1}) \leq n+d-1$). The overall closed-loop characteristic polynomial is $D^{'}(z^{-1})B(z^{-1})$.

Step 2: determine orders in the Diophantine equation $S^{'}(z^{-1})$ ($\deg S^{'}(z^{-1}) = d-1$) and $R(z^{-1})$ ($\deg R(z^{-1}) = n-1$).

Step 3: at each time index, do the following:

- apply an appropriate PAA to estimate the coefficients of $S(z^{-1}) = S^{'}(z^{-1})B(z^{-1})$ and $R(z^{-1})$, based on the reparameterized plant model

$$D^{'}(z^{-1})y(k) = \theta_c^T \phi(k-d)$$

- use the estimated parameter vector, $\hat{\theta}_c(k)$, to compute the control signal $u(k)$ according to

$$u(k) = \frac{1}{\hat{S}(z^{-1})}\left[D^{'}(z^{-1})y_d(k+d) - \hat{R}(z^{-1})y(k)\right]$$

Example

Consider a plant (discrete-time model of $1/(ms+b)$ with an extra delay)

$$G_p\left(z^{-1}\right) = \frac{z^{-2}b_0}{1+a_1z^{-1}}$$

We have $B\left(z^{-1}\right) = b_0$ ($m=0$ here); $A\left(z^{-1}\right) = 1+a_1z^{-1}$ ($n=1$ here); $d=2$. The pole placement equation is

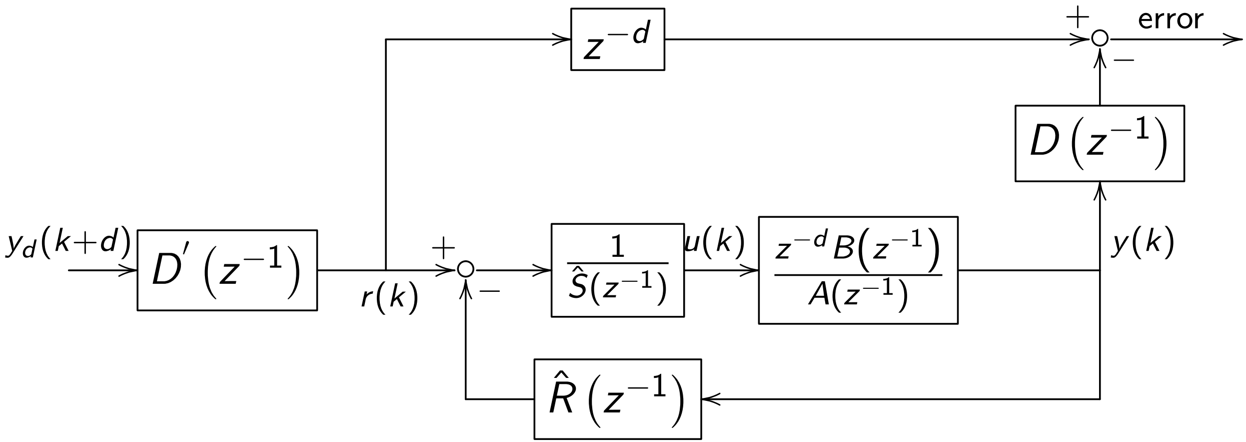
$$(1+a_1z^{-1})(1+s_1^{'}z^{-1})+z^{-2}r_0 = 1+d_1^{'}z^{-1}+d_2^{'}z^{-2}$$

$$\Rightarrow s_1^{'} = d_1^{'}-a_1, r_0 = d_2^{'}-a_1(d_1^{'}-a_1)$$

and $S(z^{-1}) = S^{'}(z^{-1})B(z^{-1}) = s_0+s_1z^{-1}$; $R(z^{-1}) = r_o$

$$\begin{aligned} u(k) &= \frac{1}{\hat{S}(z^{-1})}\left[D^{'}(z^{-1})y_d(k+d)-\hat{R}(z^{-1})y(k)\right] \\ &= \frac{1}{\hat{s}_0(k)}\left[D^{'}(z^{-1})y_d(k+2)-\hat{r}_0(k)y(k)-\hat{s}_1(k)u(k-1)\right] \end{aligned}$$

Remark



Parameter convergence is achieved if the excitation y_d is rich in frequency (which may not be assured in practice). Yet the performance goal of making $D'\left(z^{-1}\right)[y(k) - y_d(k)]$ small can still be achieved even if y_d is not rich in frequency.

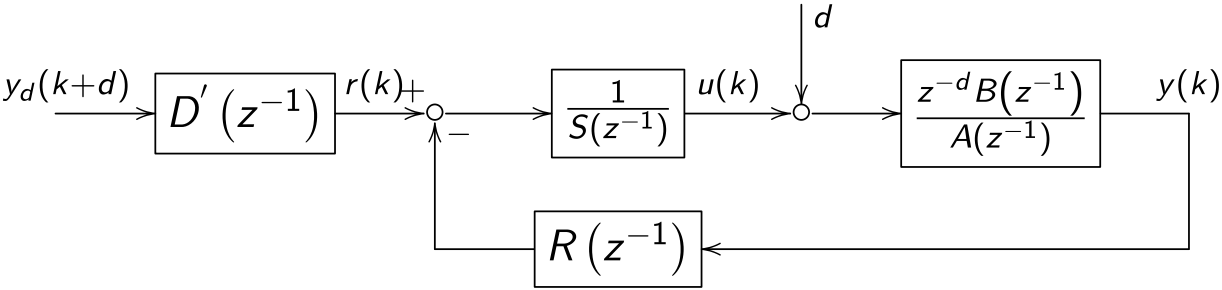
Add now disturbance cancellation

If the disturbance structure is known, we can estimate its parameters for disturbance cancellation. Consider, e.g.,

$$y\left(k\right)=\frac{z^{-d}B\left(z^{-1}\right)}{A\left(z^{-1}\right)}\left[u\left(k\right)+d\left(k\right)\right]$$

where $B\left(z^{-1}\right)$ is cancallable and the disturbance satisfies

$$W\left(z^{-1}\right)d\left(k\right)=\left(1-z^{-1}\right)d\left(k\right)=0$$



the deterministic control law should be:

$$u\left(k\right)=\frac{1}{S\left(z^{-1}\right)}\left[-R\left(z^{-1}\right)y\left(k\right)+D'(z^{-1})y_d\left(k+d\right)\right]-d$$

Disturbance cancellation

$$u(k) = \frac{1}{S(z^{-1})}\left[-R\left(z^{-1}\right)y(k) + D^{'}(z^{-1})y_d(k+d)\right] - d$$

can be equivalently represented as

$$\begin{aligned} D^{'}\left(z^{-1}\right)y_d(k+d) &= \theta_c^T\phi(k) + d^*,\ d^* = S\left(z^{-1}\right)d \\ &= \theta_{ce}^T\phi_e(k),\ \theta_{ce} = \left[\theta_c^T, d^*\right]^T,\ \phi_e(k) = \left[\phi^T(k), 1\right] \end{aligned}$$

In the adaptive case:

$$D^{'}\left(z^{-1}\right)y_d(k+d) = \hat{\theta}_{ce}^T(k+d)\phi_e(k)$$

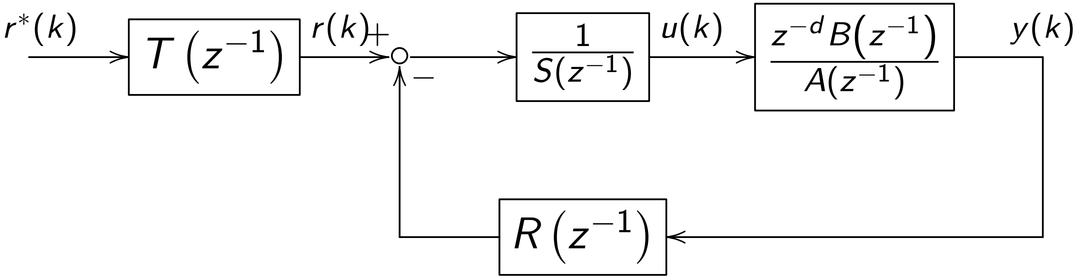
where $\hat{\theta}_{ce}(k)$ is updated via a PAA, e.g.

$$\hat{\theta}_{ce}(k) = \hat{\theta}_{ce}(k-1) + \frac{F(k-1)\phi_e(k-d)\left[D^{'}(z^{-1})y(k) - \hat{\theta}_{ce}^T(k-1)\phi_e(k-d)\right]}{1+\phi_e^T(k-d)F(k-1)\phi_e(k-d)}$$

Outline

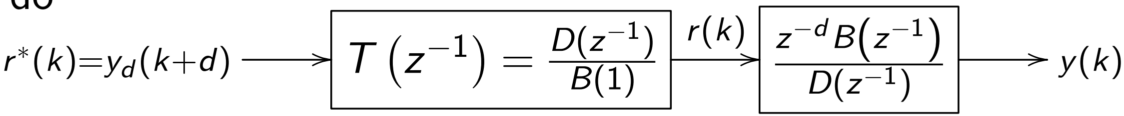
1. Big picture
2. Adaptive pole placement
   Cancellable $B\left(z^{-1}\right)$
   Remark
3. Extension: adaptive pole placement with disturbance cancellation
4. Pole placement with no cancellation of $B\left(z^{-1}\right)$
5. Indirect adaptive pole placement

Uncancellable $B\left(z^{-1}\right)$



$$\frac{Y\left(z^{-1}\right)}{R\left(z^{-1}\right)} = \frac{z^{-d}B\left(z^{-1}\right)}{A(z^{-1})\,S\left(z^{-1}\right) + z^{-d}B\left(z^{-1}\right)R\left(z^{-1}\right)} = \frac{z^{-d}B\left(z^{-1}\right)}{D\left(z^{-1}\right)}$$

If $B\left(z^{-1}\right)$ contains unstable roots or if we don't want to cancel it, we can do



$$\Rightarrow D\left(z^{-1}\right)\left[y\left(k+d\right) - \frac{B\left(z^{-1}\right)}{B\left(1\right)} y_d\left(k+d\right)\right] = 0$$

Uncancellable $B\left(z^{-1}\right)$

or

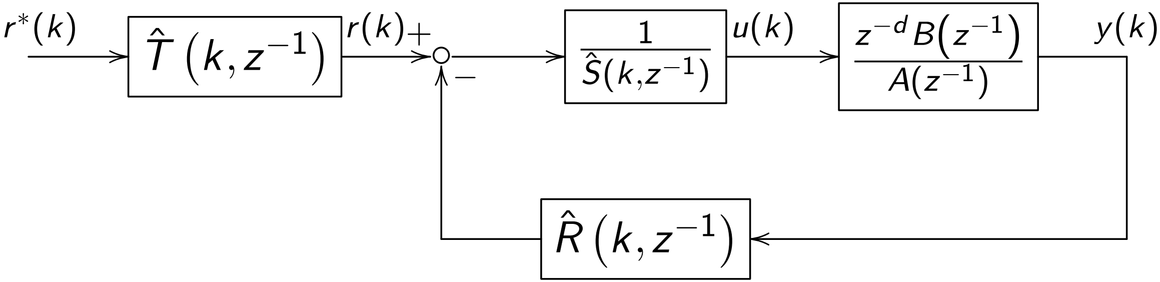
$$r^*(k) = y_d(k+d) \longrightarrow \boxed{T\left(z^{-1}\right) = \frac{D(z^{-1})B(z)}{[B(1)]^2}} \xrightarrow{r(k)} \boxed{\frac{z^{-d}B\left(z^{-1}\right)}{D(z^{-1})}} \longrightarrow y(k)$$

$$\Rightarrow D\left(z^{-1}\right)\left[y(k+d) - \frac{B\left(z^{-1}\right)B(z)}{[B(1)]^2} y_d(k+d)\right] = 0$$

which gives zero phase error tracking.
Remark: can also partially cancel the stable parts of $B\left(z^{-1}\right)$
Note: now we explicitly need $B(1)$ and/or $B(z)$ in $T\left(z^{-1}\right) \Rightarrow$ need adaptation to find the plant parameters $\Rightarrow$ indirect adaptive control

Indirect adaptive pole placement: big picture

Consider the plant $z^{-d}B\left(z^{-1}\right)/A\left(z^{-1}\right)$.



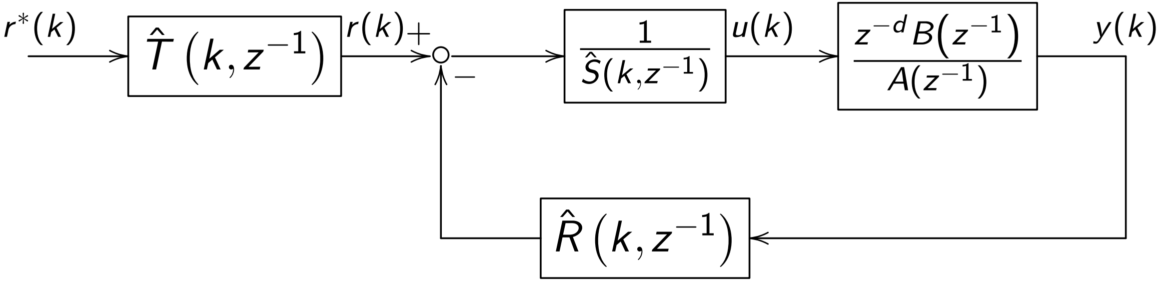
Pole placement with known plant parameters:

$$A\left(z^{-1}\right)S\left(z^{-1}\right)+z^{-d}B\left(z^{-1}\right)R\left(z^{-1}\right)=D\left(z^{-1}\right)$$

Assumptions:

- we know n, m, and d;
- the plant is irreducible.

Indirect adaptive pole placement: big picture



- At time k, identify $\hat{B}\left(k, z^{-1}\right)$ and $\hat{A}\left(k, z^{-1}\right)$ (using a suitable PAA); design $\hat{T}\left(k, z^{-1}\right)$ based on methods previously discussed.
- Solve Diophantine equation

$$\hat{A}\left(k, z^{-1}\right)\hat{S}\left(k, z^{-1}\right) + z^{-1}\hat{B}\left(k, z^{-1}\right)\hat{R}\left(k, z^{-1}\right) = D\left(z^{-1}\right)$$

for $\hat{S}\left(k, z^{-1}\right)$ and $\hat{R}\left(k, z^{-1}\right)$.

Indirect adaptive pole placement: details

- Controller order:

$$\underbrace{\hat{A}\left(k,z^{-1}\right)}_{\text{order: } n}\ \underbrace{\hat{S}\left(k,z^{-1}\right)}_{\text{order: } d+m-1}+z^{-d}\underbrace{\hat{B}\left(k,z^{-1}\right)}_{\text{order: } d+m}\underbrace{\hat{R}\left(k,z^{-1}\right)}_{\text{order: } n-1}=\underbrace{D\left(z^{-1}\right)}_{\text{order}\le n+m+d-1}$$

- Controller parameters:

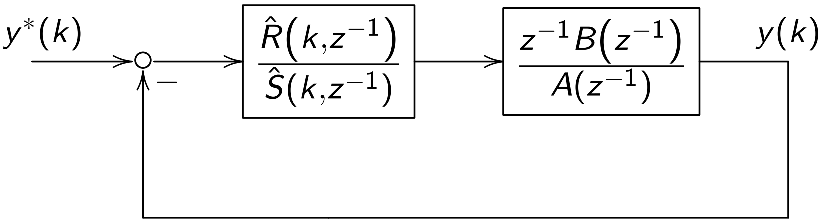
$$\hat{S}\left(k,z^{-1}\right)=\hat{s}_0(k)+\hat{s}_1(k)z^{-1}+\cdots+\hat{s}_{r-1}(k)z^{-d-m+1}$$
$$\hat{R}\left(k,z^{-1}\right)=\hat{r}_0(k)+\hat{r}_1(k)z^{-1}+\cdots+\hat{r}_{r-1}(k)z^{-n+1}$$

- Solvability of the Diophantine equation: $\hat{A}\left(k,z^{-1}\right)$ and $\hat{B}\left(k,z^{-1}\right)$ need to be coprime. If not, use the previous estimation.
- Control law:

$$u(k)=\frac{1}{\hat{S}\left(k,z^{-1}\right)}\left[\hat{T}\left(k,z^{-1}\right)r^*(k)-\hat{R}\left(k,z^{-1}\right)y(k)\right]$$

Indirect adaptive pole placement: extension

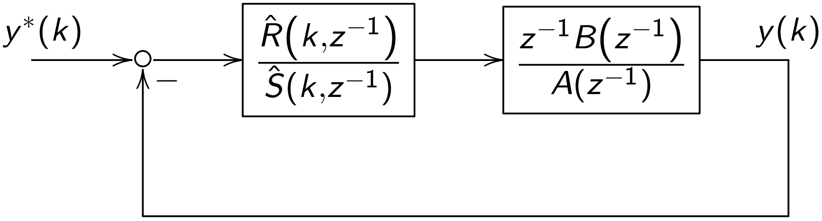
Consider the plant $z^{-1}B\left(z^{-1}\right)/A\left(z^{-1}\right)$ with the general feedback design



Similar as before, but assume we **know only the order of the plant**: $r = \max\left(n, m+1\right)$.
Pole placement with known plant parameters:

$$A\left(z^{-1}\right)S\left(z^{-1}\right) + z^{-1}B\left(z^{-1}\right)R\left(z^{-1}\right) = D\left(z^{-1}\right)$$

Indirect adaptive pole placement: extension



- Can write $B\left(z^{-1}\right) = b_0 + b_1 z^{-1} + \cdots + b_{r-1} z^{-r+1}$ and $A\left(z^{-1}\right) = 1 + a_1 z^{-1} + \cdots + a_r z^{-r}$
- At time k, identify $\hat{B}\left(k, z^{-1}\right)$ and $\hat{A}\left(k, z^{-1}\right)$
- Solve Diophantine equation

$$\hat{A}\left(k, z^{-1}\right)\hat{S}\left(k, z^{-1}\right) + z^{-1}\hat{B}\left(k, z^{-1}\right)\hat{R}\left(k, z^{-1}\right) = D\left(z^{-1}\right)$$

for $\hat{S}\left(k, z^{-1}\right)$ and $\hat{R}\left(k, z^{-1}\right)$

Indirect adaptive pole placement: extension

- Controller order:

$$\underbrace{\hat{A}\left(k,z^{-1}\right)}_{\text{order: } r}\underbrace{\hat{S}\left(k,z^{-1}\right)}_{\text{order: } r-1}+\underbrace{z^{-1}\hat{B}\left(k,z^{-1}\right)}_{\text{order: } r}\underbrace{\hat{R}\left(k,z^{-1}\right)}_{\text{order: } r-1}=\underbrace{D\left(z^{-1}\right)}_{\text{order}\leq 2r-1}$$

- Controller parameters:

$$\hat{S}\left(k,z^{-1}\right)=\hat{s}_0\left(k\right)+\hat{s}_1\left(k\right)z^{-1}+\cdots+\hat{s}_{r-1}\left(k\right)z^{-r+1}$$
$$\hat{R}\left(k,z^{-1}\right)=\hat{r}_0\left(k\right)+\hat{r}_1\left(k\right)z^{-1}+\cdots+\hat{r}_{r-1}\left(k\right)z^{-r+1}$$

- Control law:

$$u\left(k\right)=\frac{\hat{R}\left(k,z^{-1}\right)}{\hat{S}\left(k,z^{-1}\right)}\left[y^*\left(k\right)-y\left(k\right)\right]$$
$$=\frac{1}{\hat{s}_0\left(k\right)}\left\{-\hat{s}_1\left(k\right)u\left(k-1\right)-\cdots-\hat{s}_{r-1}u\left(k-r+1\right)\right.$$
$$\left.+\hat{r}_0\left(k\right)\left[y^*\left(k\right)-y\left(k\right)\right]+\cdots+\hat{r}_{r-1}\left(k\right)\left[y^*\left(k-r+1\right)-y\left(k-r+1\right)\right]\right)$$

Summary

1. Big picture

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   Remark

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References

Goodwin and Sin, "Adaptive Filtering, Prediction and Control," Prentice Hall.

Exercises

- We mentioned that direct adaptive control requires no identification of the plant. In direct adaptive pole placement, the closed loop characteristic polynomial is

$$D\left(z^{-1}\right) = D^{'}\left(z^{-1}\right) B\left(z^{-1}\right)$$

which depends on $B\left(z^{-1}\right)$. So the closed-loop design directly depends on the plant zeros. Why is it still direct adaptive control?