No.

Date

Least square

$$\hat{x} = \arg\min_{z} E[\|x - z\|_2^2 \mid y]$$

Given two random vectors x and y, what is the best estimation of x given y?

For any distribution : $\hat{x} = E(x|y)$

For Gaussian distribution :

- Estimation error of $\hat{x} = E(x|y)$ $\tilde{x} = x - \hat{x}$ is uncorrelated with y.
- $E(x|y,z) = E(x|y) + E(\tilde{x}_{|y} | z)$ when y, z uncorrelated
- $E(x|y,z) = E(x|y) + E(\tilde{x}|y \,|\, \tilde{z}|y)$ when y, z are correlated
- $m_{x|y} = m_x + \Sigma_{xy}\Sigma_{yy}^{-1}(y - m_y)$
- $\Sigma_{x|y} = \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx} = Var(\tilde{x})$.
- $Var(\hat{x}) = \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx}$

No.
Date

Example: $T_A = T + v_A$ $\quad T \sim N(T_0, X_0)$

$T_B = T + v_B$ $\quad v_A \sim N(0, V_A)$

$\quad v_B \sim N(0, v_B)$

T is uncorrelated with A & B, A & B uncorrelated.

Calculate $\hat{T} = E(T \mid T_A, T_B)$

① Recursive Approach

$$\hat{T} = E(T \mid T_A, T_B) = E(T \mid T_A) + E(\tilde{T}|_{T_A} \mid \tilde{T}_B|_{T_A})$$

$$E(T \mid T_A) = E(T) + \Sigma_{TT_A} \Sigma_{T_A T_A}^{-1} (T_A - m_{T_A})$$

$$\boxed{\begin{array}{l} m_{T_A} = E[(T + v_A)] = T_0 \\ \Sigma_{T_A T_A} = X_0 + V_A \qquad \Sigma_{TT_A} = X_0 \end{array}}$$

$$E(T \mid T_A) = T_0 + \frac{X_0}{X_0 + V_A}(T_A - T_0)$$

$$E(\tilde{T}|_{T_A} \mid \tilde{T}_B|_{T_A}) = \cancel{E(\tilde{T}|_{T_A})} + \Sigma_{\tilde{T}|_{T_A} \tilde{T}_B|_{T_A}} \Sigma^{-1}_{\tilde{T}_B|_{T_A} \tilde{T}_B|_{T_A}} (\tilde{T}_B|_{T_A} - \cancel{E(\tilde{T}_B|_{T_A})})$$

$$= \Sigma_{\tilde{T}|_{T_A} \tilde{T}_B|_{T_A}} \Sigma_{\tilde{T}_B|_{T_A} \tilde{T}_B|_{T_A}} (T_B - E(T_B \mid T_A))$$

$$\tilde{T}_{B|A} = T_B - (E(T_B) - \Sigma_{T_B T_A} \Sigma^{-1}_{T_A T_A} (T_A - m_{T_A}))$$

$$= T_B - (E(T_B) - X_0/(X_0 + V_A) \cdot (T_A - T_0))$$

$$\Sigma_{\tilde{T}|_{T_A} \tilde{T}_B|_{T_A}} = E[(\tilde{T}|_{T_A} \cdot \tilde{T}_B|_{T_A})]$$

$$= E\left[\left((T - T_0) + \frac{X_0}{X_0 + V_A}(T_A - T_0)\right)\left((T_B - T_0) + \frac{X_0}{X_0 + V_A}(T_A - T_0)\right)\right]$$

$$= \frac{X_0 V_A}{X_0 + V_A}.$$

No.

Date

$$\Sigma_{\tilde{T}_{B|A}\tilde{T}_{B|T_A}}$$

$$= \Sigma_{T_B T_B} - \Sigma_{T_B T_A} \Sigma_{T_A T_A}^{-1} \Sigma_{T_B T_A}$$

$$= (X_0 + V_B) - X_0 \frac{1}{X_0 + V_A} X_0 = \frac{\cancel{X_0^2} + (V_A + V_B) X_0 + V_B V_A - \cancel{X_0^2}}{X_0 + V_A}$$

$$\hat{T} = T_0 + \frac{X_0}{X_0(V_A+V_B) + V_A V_B} [V_B (T_A - T_0) + V_A (T_B - T_0)]$$

Batch Approach

$$T_t = \begin{bmatrix} T_A \\ T_B \end{bmatrix} \qquad T_t \sim N\left(\begin{bmatrix} T_0 \\ T_0 \end{bmatrix}, \begin{bmatrix} X_0 + V_A & X_0 \\ X_0 & X_0 + V_B \end{bmatrix} \right)$$

$$E[(T \mid \begin{bmatrix} T_A \\ T_B \end{bmatrix})] = E[T] + \Sigma_{T\begin{bmatrix} T_A \\ T_B \end{bmatrix}} \Sigma^{-1}_{\begin{bmatrix} T_A \\ T_B \end{bmatrix}\begin{bmatrix} T_A \\ T_B \end{bmatrix}} \left(\begin{bmatrix} T_A \\ T_B \end{bmatrix} - \begin{bmatrix} T_0 \\ T_0 \end{bmatrix} \right)$$

$$= T_0 + [X_0 \ X_0] \begin{bmatrix} X_0 + V_A & X_0 \\ X_0 & X_0 + V_B \end{bmatrix}^{-1} \begin{bmatrix} T_A - T_0 \\ T_B - T_0 \end{bmatrix}$$