Simple Examples of Random Variables and Random Processes:

1) Coin Toss- the outcome of flipping a coin can be considered as a Bernoulli random variable (*p* probability of being heads and *1-p* probability of being tails.)

   Flipping a coin multiple times in succession describes a random process. The process is considered "white" if the outcome of one particular coin flip does not affect the outcome of the other coin flips.

   Note: this does not mean the probability distribution of the coin flip have to remain the same for every coin flip, "whiteness" just implies that if the probability distribution were to change from one coin flip to the next, this change is uncorrelated with the output of the random process.

   The process is considered to be "colored" if the outcome of the *k-th* coin toss affects the probability distribution of other coin flips.

2) Die Roll- the outcome of rolling a die can be considered as a random variable.

   Rolling a die multiple times can be considered as a random process. The process is "white" if the outcome of each die roll is independent from each other.

Intuitive Interpretation of Least Square Properties:

Given two random variables *x* and *y* which are jointly Gaussian distributed, the conditional expectation $\hat{x} = \rho(x|y)$ is the "best" estimate of random variable *x* given we know the value of random variable *y*.

Note: By "best" estimate we mean that $\hat{x} = \rho(x|y)$ is the value that minimizes the 2-norm of the expected estimation error: $\tilde{x} = E[|x - \hat{x}|^2 |\, y]$ (Proof on pg. 18 in reader)

Recall that for the Gaussian case:

$$\hat{x} = E[x] + X_{xy} X_{yy}^{-1} (y - E[y])$$

$$E[(\hat{x} - E[\hat{x}])(\hat{x} - E[\hat{x}])^T] = X_{xy} X_{yy}^{-1} X_{yx}$$

Property i) Estimation error $\tilde{x} = x - \hat{x}$ is uncorrelated with *y*. Furthermore $\tilde{x}$ and $\hat{x}$ are orthogonal in the sense that:

$$E[\hat{x}^T \tilde{x}] = 0$$

The first part of Property i) is pretty intuitive. If $\tilde{x}$ is not uncorrelated with *y*, then that means there is more information we can learn about the error from the random variable *y*. The second part implies that our error should not be correlated with our estimation.

{this leads to Fig. 1 on pg. 31 of your reader}

Property ii) Let *y* and *z* be Gaussian and uncorrelated, then:

$$E[x|y,z] = E[x|y] + E[\tilde{x}_{|y}|z]$$

The first term is a direct consequence of Property i), the second term involves using information about *z* to further reduce $\tilde{x}$. Note that $\tilde{x}$ is "orthogonal" to *y*. It is important that the conditional expectation in the second term is taken with regards to an orthogonal component of $E[x|y]$ or else we may "double count" the information provided by the measurements.

{this leads to Fig. 2 on pg. 33 of the reader}

Property iii) If *y* and *z* are Gaussian and correlated, then:

$$E[x|y,z] = E[x|y] + E[\tilde{x}_{|y}|\tilde{z}_{|y}]$$

From Property i), $\tilde{z}_{|y}$ is the uncorrelated component of *z* to *y*. Hence at this point, this property is just an application of Property ii).

{this leads to Fig. 2 (b) on pg. 33 of the reader}