UNIVERSITY OF CALIFORNIA AT BERKELEY
Department of Mechanical Engineering
ME233 Advanced Control Systems II
Spring 2016

Homework #1

Assigned:	Feb. 4 (Th)
Due:	Feb. 11 (Th)

1. We wish to determine how to split a positive number L into N pieces, so that the product of the N pieces is maximized. The problem can be solved using dynamic programming by formulating it as follows. Consider a first-order "pure integrator"

$$x(k+1) = x(k) + u(k), \qquad x(0) = 0.$$

We wish to determine the optimal control sequence

$$U_0^o = \{u^o(0),\, u^o(1),\, \cdots,\, u^o(N-1)\}$$

such that:

(a) $u^o(k) \geq 0$.

(b) $x(N) = L$.

(c) The following cost function is maximized:

$$J = \prod_{k=0}^{N-1} u(k) = u(0)\, u(1)\, \cdots\, u(N-1)$$

To use dynamic programming, it is convenient to define the following optimal value function

$$J_m^o[x(m)] = \max_{U_m} \prod_{k=m}^{N-1} u(k)$$

where $U_m = \{u(m),\, u(m+1),\, \cdots,\, u(N-1)\}$ is the set of all feasible control sequences from the instance m.

Hint: Notice that, because of the terminal condition $x(N) = L$, and the state equation, the optimal value function at $x(N-1)$ is given by

$$J^o[x(N-1)] = u^o(N-1) = L - x(N-1)$$

Use the Bellman equation starting from this boundary condition to determine an optimal control law for $u^o(k)$.

2. **Finite-Horizon Optimal Tracking Problem:**

   Consider the discrete time system

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\,x(k) \end{aligned}$$

   where $x \in \mathbb{R}^n$ and $y,\ u \in \mathbb{R}^m$. Assume the existence of a *known* reference output sequence

$$\big(y_d(0),\, y_d(1),\, \ldots,\, y_d(N)\big)$$

   The optimal control is sought to minimize the finite-horizon quadratic performance index:

$$\begin{aligned} J = &\left[y_d(N) - y(N)\right]^T \bar{Q}_f \left[y_d(N) - y(N)\right] \\ &+ \sum_{k=0}^{N-1} \left\{ \left[y_d(k) - y(k)\right]^T \bar{Q} \left[y_d(k) - y(k)\right] + u^T(k)\, R u(k) \right\} \end{aligned}$$

   where $\bar{Q}_f$, $\bar{Q}$ and R are symmetric and positive definite matrices of the appropriate dimensions. Find the optimal control law by applying dynamic programming and utilizing the Bellman equation

$$J_k^o[x(k)] = \min_{u(k)} \big(L[x(k), u(k)] + J_{k+1}^o[x(k+1)] \big)$$

   where

$$\begin{aligned} L[x(k), u(k)] &= \left[y_d(k) - y(k)\right]^T \bar{Q} \left[y_d(k) - y(k)\right] + u^T(k) R u(k) \\ J_N^o[x(N)] &= \left[y_d(N) - y(N)\right]^T \bar{Q}_f \left[y_d(N) - y(N)\right] . \end{aligned}$$

   Hint: Show that the optimal cost from state $x(k)$ to the final state can be expressed as

$$J_k^o[x(k)] = x^T(k) P(k) x(k) + x^T(k) b(k) + c(k) \; .$$

   Obtain recursive expressions for $P(k)$, $b(k)$, and $c(k)$ (from $k = N$ to $k = 0$).

3. A product is produced by three different factories: A, B, and C. Factories A, B, and C respectively produce 25%, 50%, and 25% of the total production. In factories A and B, 98% of the items produced are not defective, whereas in factory C, 99% are not defective. Calculate: (a) The probability that a randomly-chosen item is defective and (b) the probability that a (randomly-chosen) non-defective item comes from factory C.

4. In the "Monty Hall" three-door problem, a contestant is asked to choose one of three doors. One of the three doors conceals a prize while the other two do not. After the contestant chooses, Monty Hall (the master of ceremonies of the Let's Make a Deal television show) opens one of the two doors the player did not choose to reveal one door

that does not conceal the prize. The contestant is then permitted to either stay with their original choice or switch to the other unopened door. Determine the contestant's probability of getting the prize if she switches. Assume that before Monty Hall opens one of the doors, the prize is equally likely to be hidden behind each of the three doors.

Hint: Let the doors be called x, y, and z. Let C_x be the event that the prize is behind door x and so on. Let H_x be the event that the host opens door x and so on. Assuming that you choose door x, the probability that you win a car if you then switch your choice is given by

$$P((H_z \cap C_y) \cup (H_y \cap C_z)) \ .$$

Notice that, by Bayes' rule, $P(H_z \cap C_y) = P(H_z|C_y)P(C_y)$.

5. Consider a discrete random variable X with probability mass function $p_X(x_1) = a$, $p_X(x_n) = b$, with $p_X(x_i)$ varying linearly between a and b for $1 \leq i \leq n$.

   (a) What are the conditions on a and b to make the above a valid probability mass (density) function? How do these conditions vary with n and the range of possible values of x?

   (b) What is the expected value and variance of X?

   (c) What is the CDF $F_X(x)$?

   (d) If we take the sum of two independent random variables that have this same distribution, what is the probability mass (density) function of their sum?

   (e) Write and submit code (in Matlab, Julia, or Python) that implements a function to return a random variable with this distribution, with a, b, x_1 ($x_{\min}$), x_n ($x_{\max}$), and n as inputs. Verify the expected behavior by evaluating this function many times and plotting a histogram for several choices of parameter values - at least one case you plot should have $a \neq b$. (Hint: start with the output of the standard uniform `rand()` function.)

   Repeat the above steps for a continuous random variable with probability density function $p_X(x_{\min}) = a$, $p_X(x_{\max}) = b$, and $p_X(x)$ varying linearly between a and b for $x_{\min} \leq x \leq x_{\max}$. There is no n in the continuous case.