ME 233 Spring 2008
Solution to Homework #3

1. In this problem, we will denote the three doors as x, y, and z. Without loss of generality, we will assume that the contestant originally picked door x. We now define C_i to be the event that the car is behind door i and H_j to be the event that the host opens door j. With this in mind, note that the mutually exclusive events C_x, $C_y \cap H_z$, and $C_z \cap H_y$ cover the sample space, i.e.

$$1 = P(C_x) + P(C_y \cap H_z) + P(C_z \cap H_y).$$

Given that the contestant switches her guess, the probability that she will win is given by $P((C_y \cap H_z) \cup (C_z \cap H_y))$. Since the event $C_y \cap H_z$ is disjoint from the event $C_z \cap H_y$, we can say that

$$\begin{aligned} P(\text{win}|\text{she switches}) &= P((C_y \cap H_z) \cup (C_z \cap H_y)) \\ &= P(C_y \cap H_z) + P(C_z \cap H_y) \\ &= 1 - P(C_x) = \frac{2}{3}. \end{aligned}$$

2. (a) First we will define

$$\begin{aligned} P_X(X = A) &\quad - \quad \text{probability that a randomly chosen item comes from factory A} \\ P_X(X = B) &\quad - \quad \text{probability that a randomly chosen item comes from factory B} \\ P_X(X = C) &\quad - \quad \text{probability that a randomly chosen item comes from factory C} \\ P_Y(Y = D) &\quad - \quad \text{probability that a randomly chosen item is defective} \\ P_Y(Y = N) &\quad - \quad \text{probability that a randomly chosen item is not defective} \end{aligned}$$

With these definitions, we can state our given information as

$$\begin{aligned} P_X(X = A) &= \frac{1}{2} \\ P_X(X = B) &= \frac{1}{4} \\ P_X(X = C) &= \frac{1}{4} \\ P_{Y|X}(Y = D|X = A) &= \frac{1}{100} \\ P_{Y|X}(Y = D|X = B) &= \frac{1}{100} \\ P_{Y|X}(Y = D|X = C) &= \frac{3}{100} \end{aligned}$$

Using Bayes' Rule, we can say

$$\begin{aligned} P_{X,Y}(X = A, Y = D) &= P_{Y|X}(Y = D|X = A)P_X(X = A) = \frac{1}{100}\frac{1}{2} \\ P_{X,Y}(X = B, Y = D) &= P_{Y|X}(Y = D|X = B)P_X(X = B) = \frac{1}{100}\frac{1}{4} \\ P_{X,Y}(X = C, Y = D) &= P_{Y|X}(Y = D|X = C)P_X(X = C) = \frac{3}{100}\frac{1}{4} \end{aligned}$$

With this data, we can now construct the array for the joint probability shown in Figure 1. To construct the last entry in the 'D' column, we add all of the elements above it. To construct the 'N' column, we subtract the 'D' column from the 'Marginal Probabilities' column.

	D	N	Marginal Probabilities
A	$\frac{1}{200}$	$\frac{1}{2}-\frac{1}{200}=\frac{99}{200}$	$\frac{1}{2}$
B	$\frac{1}{400}$	$\frac{1}{4}-\frac{1}{400}=\frac{99}{400}$	$\frac{1}{4}$
C	$\frac{3}{400}$	$\frac{1}{4}-\frac{3}{400}=\frac{97}{400}$	$\frac{1}{4}$
Marginal Probabilities	$\frac{1}{200}+\frac{1}{400}+\frac{3}{400}=\frac{3}{200}$	$1-\frac{3}{200}=\frac{197}{200}$	1

Figure 1: Array of joint probability

Using Bayes' Rule again, we see that our desired result is given by

$$P_{X|Y}(X=A|Y=D) \quad = \quad \frac{P_{X,Y}(X=A, Y=D)}{P_Y(Y=D)} = \frac{1}{3}$$

(b) Using Bayes' Rule, our desired result is given by

$$P_{X|Y}(X=C|Y=N) \quad = \quad \frac{P_{X,Y}(X=C, Y=N)}{P_Y(Y=N)} = \frac{97}{394}$$

3. (a) First we will define $Y = X_1 + X_2$. Now, since X_1 and X_2 are independent, we can apply the property that the PDF of Y is the convolution of the PDF of X_1 and the PDF of X_2:

$$p_Y(y) = \int_{-\infty}^{\infty} p_{X_1}(x_1)p_{X_2}(y-x_1)dx_1$$

Note that the PDF of X_1 only takes on the values of 1 and 0. Thus, we only need to integrate $p_{X_2}(y-x_1)$ over the regions where $p_{X_1}(x_1)$ has a value of 1. Thus,

$$p_Y(y) = \int_0^1 p_{X_2}(y-x_1)dx_1$$

Now note that the following conditions hold

$$\begin{aligned} & p_{X_2}(y-x_1) = 1 \\ \Leftrightarrow \quad & 0 \le y - x_1 \le 1 \\ \Leftrightarrow \quad & y-1 \le x_1 \le y \end{aligned}$$

Thus, we get

$$\begin{aligned} p_Y(y) \quad &= \quad \begin{cases} \int_0^y dx_1 & \text{for } 0 \le y \le 1 \\ \int_{y-1}^1 dx_1 & \text{for } 1 \le y \le 2 \\ 0 & \text{otherwise} \end{cases} \\ &= \quad \begin{cases} y & \text{for } 0 \le y \le 1 \\ 2-y & \text{for } 1 \le y \le 2 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(b) First we define $Z = X_1 + X_2 + X_3 = Y + X_3$. Following a similar procedure as before,

$$\begin{aligned} p_Z(z) &= \int_{-\infty}^{\infty} p_{X_3}(x_3) p_Y(z - x_3) dx_3 \\ &= \int_0^1 p_Y(z - x_3) dx_3 \\ p_Y(z - x_3) &= \begin{cases} z - x_3 & \text{for } z - 1 \leq x_3 \leq z \\ 2 - z + x_3 & \text{for } z - 2 \leq x_3 \leq z - 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Thus, we get that

$$\begin{aligned} p_Z(z) &= \begin{cases} \int_0^z (z - x_3) dx_3 & \text{for } 0 \leq z \leq 1 \\ \int_0^{z-1} (2 - z + x_3) dx_3 + \int_{z-1}^1 (z - x_3) dx_3 & \text{for } 1 \leq z \leq 2 \\ \int_{z-2}^1 (2 - z + x_3) dx_3 & \text{for } 2 \leq z \leq 3 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{2} z^2 & \text{for } 0 \leq z \leq 1 \\ -z^2 + 3z - \frac{3}{2} & \text{for } 1 \leq z \leq 2 \\ \frac{1}{2} z^2 - 3z + \frac{9}{2} & \text{for } 2 \leq z \leq 3 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Figure 2 shows the PDFs of X_1, Y, and Z. Notice that each time an extra variable is added onto the random variable being looked at, the mean moves to the right, the maximum value moves to the right, and, to compensate, the maximum value of the PDF starts to drop. Also, the plot of the PDF starts to look more and more like a Gaussian distribution.

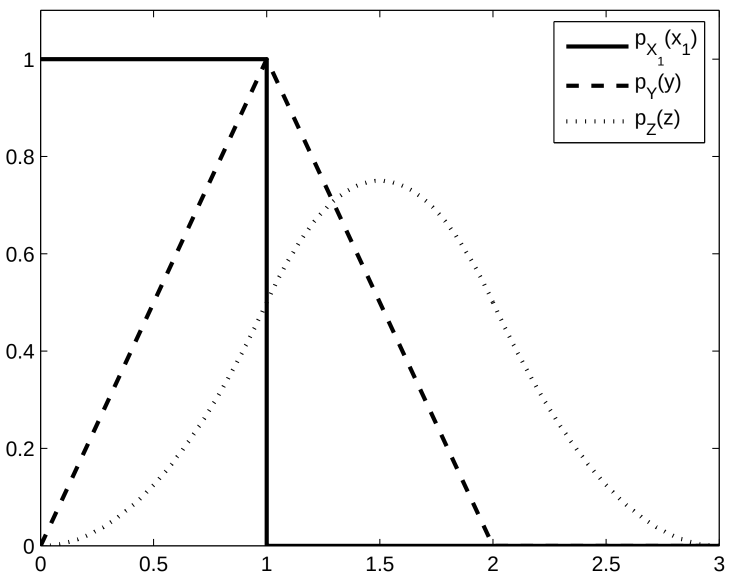

Figure 2: PDFs for X_1, Y, and Z

4. Since X and Y are Gaussian random variables, their moment generating functions are given by

$$P_X(j\omega) = \mathcal{F}\{p_X(\cdot)\} = \exp\left(j\omega m_X - \frac{\sigma_X^2 \omega^2}{2}\right)$$
$$P_Y(j\omega) = \mathcal{F}\{p_Y(\cdot)\} = \exp\left(j\omega m_Y - \frac{\sigma_Y^2 \omega^2}{2}\right)$$

Since $Z = X + Y$ is the sum of two independent random variables, we can say that

$$P_Z(j\omega) = \mathcal{F}\{p_Z(\cdot)\} = \mathcal{F}\{p_X(\cdot)\}\,\mathcal{F}\{p_Y(\cdot)\}$$

Substituting our expressions for the moment generating functions of X and Y then gives

$$\begin{aligned} P_Z(j\omega) &= \exp\left\{\left(j\omega m_X - \frac{\sigma_X^2\omega^2}{2}\right) + \left(j\omega m_Y - \frac{\sigma_Y^2\omega^2}{2}\right)\right\} \\ &= \exp\left\{j\omega\left(m_X + m_Y\right) - \frac{\left(\sigma_X^2 + \sigma_Y^2\right)\omega^2}{2}\right\} \end{aligned}$$

Note that this is the moment generating function of a Gaussian random variable with mean $m_X + m_Y$ and variance $\sigma_X^2 + \sigma_Y^2$. Therefore, $Z \sim N(m_X + m_Y, \sigma_X^2 + \sigma_Y^2)$.

5. (a) To begin, we find the conditional expectation of X given y:

$$m_{X|y} = m_X + \Lambda_{XY}\Lambda_{YY}^{-1}(y - m_Y)$$

Using the results of problem 4, we see that

$$\begin{aligned} \Lambda_{YY} &= \Lambda_{XX} + \Lambda_{V_1V_1} \\ m_Y &= m_X \end{aligned}$$

Noting that $X - m_X$ is independent of V_1, we calculate the cross-covariance of X and Y as

$$\begin{aligned} \Lambda_{XY} &= E\left[\left(X - m_X\right)\left(Y - m_Y\right)\right] \\ &= E\left[\left(X - m_X\right)\left(X + V_1 - m_X\right)\right] \\ &= E\left[\left(X - m_X\right)^2\right] + E\left[\left(X - m_X\right)V_1\right] \\ &= E\left[\left(X - m_X\right)^2\right] + E\left[X - m_X\right]E\left[V_1\right] \\ &= E\left[\left(X - m_X\right)^2\right] \\ &= \Lambda_{XX} \end{aligned}$$

Substituting the relevant values gives

$$m_{X|Y=11} = 10 + \frac{2(11-10)}{2+1} = 10\frac{2}{3}$$

(b) Using the same methodology as before, we see that

$$\begin{aligned} m_{X|z} &= m_X + \Lambda_{XZ}\Lambda_{ZZ}^{-1}(z - m_Z) \\ \Lambda_{ZZ} &= \Lambda_{XX} + \Lambda_{V_2V_2} \\ m_Z &= m_X \\ \Lambda_{XZ} &= \Lambda_{XX} \end{aligned}$$

Thus,

$$m_{X|Z=9} = 10 + \frac{2(9-10)}{2+2} = 9\frac{1}{2}$$

(c) First, we define the random vector W as

$$W = \begin{bmatrix} Y \\ Z \end{bmatrix}$$

The mean and covariance of this vector are given by

$$\begin{aligned} m_W &= \begin{bmatrix} m_Y \\ m_Z \end{bmatrix} \\ \Lambda_{WW} &= \begin{bmatrix} \Lambda_{YY} & \Lambda_{YZ} \\ \Lambda_{ZY} & \Lambda_{ZZ} \end{bmatrix} \end{aligned}$$

As before,

$$\begin{aligned} \Lambda_{YY} &= \Lambda_{XX} + \Lambda_{V_1V_1} \\ \Lambda_{ZZ} &= \Lambda_{XX} + \Lambda_{V_2V_2} \end{aligned}$$

The cross-covariance between Y and Z can be calculated as

$$\begin{aligned} \Lambda_{ZY} = \Lambda_{YZ} &= E\left[(X - m_X + V_1)(X - m_X + V_2)\right] \\ &= E\left[(X - m_X)^2\right] + E\left[(X - m_X)(V_1 + V_2)\right] + E\left[V_1 V_2\right] \\ &= E\left[(X - m_X)^2\right] \\ &= \Lambda_{XX} \end{aligned}$$

The cross-covariance between X and W can be expressed as

$$\Lambda_{XW} = \begin{bmatrix} \Lambda_{XY} & \Lambda_{XZ} \end{bmatrix} = \begin{bmatrix} \Lambda_{XX} & \Lambda_{XX} \end{bmatrix}$$

Thus,

$$\begin{aligned} m_{X|Y=11,Z=9} &= m_{X|W=[11\ 9]^T} \\ &= m_X + \Lambda_{XW}\Lambda_{WW}^{-1}(w - m_W) \\ &= 10 + \begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}^{-1} \left(\begin{bmatrix} 11 \\ 9 \end{bmatrix} - \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right) \\ &= 10\frac{1}{4} \end{aligned}$$

Note that the Y measurement has a greater impact on the conditional mean for X than the Z measurement. This means that our estimation is making use of the fact that Y is a more “reliable” measurement than Z, i.e. $\Lambda_{YY} < \Lambda_{ZZ}$.