**UNIVERSITY OF CALIFORNIA AT BERKELEY**
**Department of Mechanical Engineering**
**ME233 Advanced Control Systems II**
Spring 2012

# Homework #4

Assigned: Feb. 16 (Th)
Due: Feb. 23 (Th)

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The first ME233 midterm will be held on Thursday, March 1st. The exam will be closed book and notes, but you are allowed to bring 2 double-sided sheets (i.e. 4 pages) of handwritten notes on $8.5'' \times 11''$ paper. You will not need a calculator. The midterm will cover material up to and including this homework assignment. Equivalently, the midterm will cover material up to and including Lecture 8 (LQG optimal control).

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1. **Finite-Horizon Optimal Tracking Problem:**

   Consider the discrete time system

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\,x(k) \end{aligned}$$

   where $x \in \mathbb{R}^n$ and $y,\ u \in \mathbb{R}^m$. Assume the existence of a *known* reference output sequence

$$\big(y_d(0),\, y_d(1),\, \ldots,\, y_d(N)\big)$$

   The optimal control is sought to minimize the finite-horizon quadratic performance index:

$$\begin{aligned} J = {} & [y_d(N) - y(N)]^T\,\bar{Q}_f\,[y_d(N) - y(N)] \\ & + \sum_{k=0}^{N-1} \left\{ [y_d(k) - y(k)]^T\,\bar{Q}\,[y_d(k) - y(k)] + u^T(k)\,Ru(k) \right\} \end{aligned}$$

   where $\bar{Q}_f$, $\bar{Q}$ and $R$ are symmetric and positive definite matrices of the appropriate dimensions. Find the optimal control law by applying dynamic programming and utilizing the Bellman equation

$$J_k^o[x(k)] = \min_{u(k)} \big( L[x(k), u(k)] + J_{k+1}^o[x(k+1)] \big)$$

   where

$$\begin{aligned} L[x(k), u(k)] &= [y_d(k) - y(k)]^T\,\bar{Q}\,[y_d(k) - y(k)] + u^T(k)Ru(k) \\ J_N^o[x(N)] &= [y_d(N) - y(N)]^T\,\bar{Q}_f\,[y_d(N) - y(N)]\,. \end{aligned}$$

**Hint:** Show that the optimal cost from state $x(k)$ to the final state can be expressed as

$$J_k^o[x(k)] = x^T(k)P(k)x(k) + x^T(k)b(k) + c(k)\ .$$

Obtain recursive expressions for $P(k)$, $b(k)$, and $c(k)$ (from $k = N$ to $k = 0$).

2. We wish to determine how to split a positive number $L$ into $N$ pieces, so that the product of the $N$ pieces is maximized. The problem can be solved using dynamic programming by formulating it as follows. Consider a first-order "pure integrator"

$$x(k+1) = x(k) + u(k) \qquad\qquad x(0) = 0,$$

we wish to determine the optimal control sequence

$$U_0^o = \{u^o(0),\, u^o(1),\, \cdots,\, u^o(N-1)\}$$

such that:

(a) $u^o(k) \geq 0$.

(b) $x(N) = L$.

(c) The following cost function is maximized:

$$J = \prod_{k=0}^{N-1} u(k) = u(0)\, u(1)\ \cdots\ u(N-1)$$

To use dynamic programming, it is convenient to define the following optimal value function

$$J_m^o[x(m)] = \max_{U_m} \prod_{k=m}^{N-1} u(k)$$

where $U_m = \{u(m),\, u(m+1),\, \cdots,\, u(N-1)\}$ is the set of all feasible control sequences from the instance $m$.

**Hint:** Notice that, because of the terminal condition $x(N) = L$, and the state equation, the optimal value function at $x(N-1)$ is given by

$$J^o[x(N-1)] = u^o(N-1) = L - x(N-1)$$

Use the Bellman equation starting from this boundary condition.

3. A LTI discrete-time system is described by

$$x(k+1) = A\,x(k) + B\,u(k) + B_w\,w(k)$$

where $x(k) \in \mathcal{R}^{p_1}$ is the state, $u(k)\mathcal{R}^{p_2}$ is the control input and $w(k) \in \mathcal{R}^{p_3}$ is stationary colored noise given by the output of the state space model

$$\begin{aligned} x_w(k+1) &= A_w\,x_w(k) + B_n\,\eta(k) \\ w(k) &= C_w\,x_w(k) \end{aligned}$$

where $n(k) \in \mathcal{R}^{p_5}$ is stationary and white and $x_w(k) \in \mathcal{R}^{p_4}$. Assume that $x(0)$, $x_w(0)$ and $\eta(k)$ are independent and normally distributed with

$$\begin{aligned} &E\{x(0)\} = x_o, \quad E\{(x(0)-x_o)(x(0)-x_o)^T\} = X_o \\ &E\{x_w(0)\} = x_{wo}, \quad E\{(x_w(0)-x_{wo})(x_w(0)-x_{wo})^T\} = X_{wo} \\ &E\{\eta(k)\} = 0, \quad E\{\eta(k)\eta(k+l)^T\} = \Gamma\delta(l) \end{aligned}$$

The performance index to be minimized is

$$J = E\left\{ x^T(N)\,Q_{_f}\,x(N) + \sum_{k=0}^{N-1}\left[x^T(k)\,Q\,x(k) + u^T(k)\,R\,u(k)\right]\right\}$$

where the expectation must be taken over all underlying random quantities in each of the following situations that will be described below. For each situation obtain the equations that must be solved to find the optimal control:

(a) $x(k)$ and $x_w(k)$ are measurable for all $k$.

(b) The only available measurement is $y(k) = C\,x(k) + v(k)$ where $v(k)$ is a zero-mean, white and Gaussian noise with covariance $E\{v(k)v(k+l)\} = V\,\delta(l)$ and is independent from $x(0)$, $x_w(0)$ and $\eta(k)$.