ME 233 Advanced Control II

Lecture 20

Least Squares Parameter Estimation

Least Squares Estimation

Model

$$y(k) = \sum_{i=1}^{n} \phi_i(k-1)\,\theta_i$$

Where

- $y(k)$ observed output
- $\phi_i(k)$ **known** and measurable function
- θ_i **unknown** but constant parameter

Least Squares Estimation

Model

$$y(k) = \phi^T(k-1)\,\theta$$

Where

$y(k)$ measured output

$$\underbrace{\phi(k) = \begin{bmatrix} \phi_1(k) \\ \vdots \\ \phi_n(k) \end{bmatrix}}_{n\times 1 \text{ regressor}} \qquad \underbrace{\theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_n \end{bmatrix}}_{\text{unknown vector}}$$

Batch Least Squares Estimation

Assume that we have collected k data sets:

$$\left.\begin{array}{l} y(1), \cdots, y(k) \\ \phi(0), \cdots, \phi(k-1) \end{array}\right\} \textit{collected data}$$

We want to find the parameter estimate at instant k: $\hat{\theta}(k)$

that best fits **<u>all collected</u>** data in the **<u>least squares</u>** sense:

$$\min_{\hat{\theta}(k)} \left\{ \frac{1}{2} \sum_{j=1}^{k} \left[y(j) - \phi^T(j-1)\,\hat{\theta}(k) \right]^2 \right\}$$

kept constant in the summation

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Batch Least Squares Estimation

Defining the cost functional

$$V(\hat{\theta}(k)) = \frac{1}{2} \sum_{j=1}^{k} \left[y(j) - \phi^T(j-1)\,\hat{\theta}(k) \right]^2$$

$\hat{\theta}(k)$ is obtained by solving

$$\frac{dV(\hat{\theta}(k))}{d\hat{\theta}(k)} = 0$$

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Batch Least Squares Solution

The least squares parameter estimate $\hat{\theta}(k)$ which solves

$$\frac{dV(\hat{\theta}(k))}{d\hat{\theta}(k)} = 0$$

Satisfies the **normal equation:**

$$\underbrace{\left[\sum_{i=1}^{k} \phi(i-1)\phi^T(i-1) \right]}_{n \times n \text{ matrix}} \hat{\theta}(k) = \underbrace{\sum_{i=1}^{k} \phi(i-1)\, y(i)}_{n \times 1 \text{ vector}}$$

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Normal Equation Derivation

$$V(\hat{\theta}(k)) = \frac{1}{2} \sum_{j=1}^{k} \left[y(j) - \phi^T(j-1)\,\hat{\theta}(k) \right]^2$$

$$= \frac{1}{2} \left\| \begin{bmatrix} y(1) - \phi^T(0)\hat{\theta}(k) \\ \vdots \\ y(k) - \phi^T(k-1)\hat{\theta}(k) \end{bmatrix} \right\|^2$$

$$= \frac{1}{2} \left\| \underbrace{\begin{bmatrix} y(1) \\ \vdots \\ y(k) \end{bmatrix}}_{Y(k)} - \underbrace{\begin{bmatrix} \phi^T(0) \\ \vdots \\ \phi^T(k-1) \end{bmatrix}}_{\Phi^T(k-1)} \hat{\theta}(k) \right\|^2$$

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Normal Equation Derivation

$$V(\hat{\theta}(k)) = \frac{1}{2} \left\| Y(k) - \Phi^T(k-1)\hat{\theta}(k) \right\|^2$$

$$= \frac{1}{2} \left[Y^T(k)Y(k) + \hat{\theta}^T(k)\Phi(k-1)\Phi^T(k-1)\hat{\theta}(k) - 2\hat{\theta}^T(k)\Phi(k-1)Y(k) \right]$$

Taking the partial derivative with respect to $\hat{\theta}(k)$

$$\frac{\partial V(\hat{\theta}(k))}{\hat{\theta}(k)} = \Phi(k-1)\Phi^T(k-1)\hat{\theta}(k) - \Phi(k-1)Y(k)$$

For optimality, we therefore need

$$\Phi(k-1)\Phi^T(k-1)\hat{\theta}(k) = \Phi(k-1)Y(k)$$

Normal Equation Derivation

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$$\Phi(k-1) = \begin{bmatrix} \phi(0) & \cdots & \phi(k-1) \end{bmatrix}$$

$$Y(k) = \begin{bmatrix} y(1) & \cdots & y(k) \end{bmatrix}^T$$

For optimality, we need

$$\underbrace{\Phi(k-1)\Phi^T(k-1)}_{\sum_{i=1}^{k} \phi(i-1)\phi^T(i-1)}\hat{\theta}(k) = \underbrace{\Phi(k-1)Y(k)}_{\sum_{i=1}^{k} \phi(i-1)y(i)}$$

Therefore, we need

$$\left[\sum_{i=1}^{k} \phi(i-1)\phi^T(i-1)\right] \hat{\theta}(k) = \sum_{i=1}^{k} \phi(i-1)\, y(i)$$

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Batch Least Squares Estimation

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The solution of the normal equation

$$\left[\sum_{i=1}^{k} \phi(i-1)\phi^T(i-1)\right] \hat{\theta}(k) = \sum_{i=1}^{k} \phi(i-1)\, y(i)$$

Is given by:

$$\hat{\theta}(k) = \left[\sum_{i=1}^{k} \phi(i-1)\phi^T(i-1)\right]^{\#} \sum_{i=1}^{k} \phi(i-1)\, y(i)$$

Pseudoinverse

Moore-Penrose pseudoinverse

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- Let *A* have the singular value decomposition

orthogonal matrices

$$A = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$$

$$\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_r) \qquad \sigma_1 \geq \cdots \geq \sigma_r > 0$$

- Then the Moore-Penrose pseudoinverse of *A* is

$$A^\sharp = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix}$$

In MATLAB: `pinv(A)`

Moore-Penrose pseudoinverse

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Let $A \in \mathcal{R}^{n \times m}$ and $A^\sharp$ be its Moore-Penrose pseudoinverse

Then $A^\sharp$ has the dimension of A^T and satisfies:

- $A\, A^\sharp\, A = A$
- $A^\sharp\, A\, A^\sharp = A^\sharp$
- $A^\sharp A$ and $A A^\sharp$ are Hermitian

In this case, since $A = \Phi\Phi^T$

$\Phi = \begin{bmatrix} \phi(0) \cdots \phi(k-1) \end{bmatrix}$

$$\boxed{A\, A^\sharp\, \Phi = \Phi}$$

Batch Least Squares Estimation

Assume that we have collected sufficient data and the data has sufficient richness so that

$$\left[\sum_{i=1}^{k} \phi(i-1)\phi^T(i-1)\right] = \phi(0)\phi^T(0) + \phi(1)\phi^T(1) + \cdots + \phi(k-1)\phi^T(k-1)$$

has full rank*.*

Then,

$$\hat{\theta}(k) = \underbrace{\left[\sum_{i=1}^{k} \phi(i-1)\phi^T(i-1)\right]^{-1}}_{F(k)} \sum_{i=1}^{k} \phi(i-1)\, y(i)$$

Recursive Least Squares (RLS)

Assume that we have collected $k-1$ sets of data and have computed $\hat{\theta}(k-1)$ using

$$\hat{\theta}(k-1) = \underbrace{\left[\sum_{i=1}^{k-1} \phi(i-1)\phi^T(i-1)\right]^{-1}}_{F(k-1)} \sum_{i=1}^{k-1} \phi(i-1)y(i)$$

Then, given a new set of data: $y(k) \quad \phi(k-1)$

We want to find $\hat{\theta}(k)$ in a recursive fashion:

$$\hat{\theta}(k) = \hat{\theta}(k-1) + [\, correction\ \ term\,]$$

Recursive Least Squares Algorithm

Define the ***a-priori*** output estimate:

$$\hat{y}^o(k) = \phi^T(k-1)\hat{\theta}(k-1)$$

and the ***a-priori*** output estimation error:

$$e^o(k) = y(k) - \phi^T(k-1)\hat{\theta}(k-1)$$

The RLS algorithm is given by:

$$\boxed{\hat{\theta}(k) \;=\; \hat{\theta}(k-1) + F(k)\phi(k-1)\, e^o(k)}$$

where *F(k)* has the recursive relationship on the next slide

Recursive Least Squares Gain

The RLS gain $F(k)$ is defined by

$$F^{-1}(k) \;=\; \sum_{i=1}^{k} \phi(i-1)\phi^T(i-1)$$

Therefore,

$$F^{-1}(k) \;=\; F^{-1}(k-1) + \phi(k-1)\phi^T(k-1)$$

Using the matrix inversion lemma, we obtain

$$\boxed{F(k) \;=\; F(k-1) - \frac{F(k-1)\,\phi(k-1)\,\phi(k-1)^T F(k-1)}{1 + \phi(k-1)^T F(k-1)\,\phi(k-1)}}$$

Recursive Least Squares Derivation

Notice that

$$\hat{\theta}(k) = F(k) \sum_{i=1}^{k} \phi(i-1)y(i)$$

$$= F(k) \left[\phi(k-1)y(k) + \underbrace{\sum_{i=1}^{k-1} \phi(i-1)y(i)}_{F^{-1}(k-1)\hat{\theta}(k-1)} \right]$$

$$F^{-1}(k-1) \;=\; F^{-1}(k) - \phi(k-1)\phi^T(k-1)$$

Recursive Least Squares Derivation

Therefore plugging the previous two results,

$$\begin{aligned} \hat{\theta}(k) \;=\; & F(k) \left[\left(F(k)^{-1} - \phi(k-1)\,\phi^T(k-1) \right) \hat{\theta}(k-1) \right. \\ & \left. + \phi(k-1)\, y(k) \right] \end{aligned}$$

And rearranging terms, we obtain

$$\begin{aligned} \hat{\theta}(k) \;=\; & \hat{\theta}(k-1) \\ & + F(k)\phi(k-1) \underbrace{\left[y(k) - \phi^T(k-1)\hat{\theta}(k-1) \right]}_{e^o(k)} \end{aligned}$$

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RLS Estimation Algorithm

A-priori version:

$$e^o(k+1) = y(k+1) - \phi^T(k)\hat{\theta}(k)$$

$$\hat{\theta}(k+1) = \hat{\theta}(k) + F(k+1)\phi(k)e^o(k+1)$$

$$F(k+1) = F(k) - \frac{F(k)\phi(k)\phi^T(k)F(k)}{1 + \phi^T(k)F(k)\phi(k)}$$

Initial conditions:

$$F(0) = F^T(0) > 0 \qquad \hat{\theta}(0)$$

RLS Estimation Algorithm

A-posteriori version (used to prove that $e(k) \longrightarrow 0$):

$$e^o(k+1) = y(k+1) - \phi^T(k)\hat{\theta}(k)$$

$$e(k+1) = \frac{e^o(k+1)}{1 + \phi^T(k)F(k)\phi(k)}$$

$$\hat{\theta}(k+1) = \hat{\theta}(k) + F(k)\phi(k)e(k+1)$$

$$F(k+1) = F(k) - \frac{F(k)\phi(k)\phi^T(k)F(k)}{1 + \phi^T(k)F(k)\phi(k)}$$

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RLS with forgetting factor

The inverse of the gain matrix in the RLS algorithm is given by:

$$F^{-1}(k) = F^{-1}(k-1) + \phi(k-1)\phi^T(k-1)$$

Its trace is given by:

$$\text{tr}\left[F^{-1}(k)\right] = \text{tr}\left[F^{-1}(k-1)\right] + \|\phi(k-1)\|^2$$

which always increases when $\|\phi(k-1)\| \neq 0$

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RLS with forgetting factor

Similarly, the trace of the gain matrix is given by

$$\text{tr}\,[F(k)] = \text{tr}\,[F(k-1)] - \frac{\|F(k-1)\phi(k-1)\|^2}{1+\phi^T(k-1)F(k-1)\phi(k-1)}$$

always decreases when $\|F(k-1)\phi(k-1)\| \neq 0$

Problem: RLS eventually stops updating

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RLS with forgetting factor

We can modify cost function to “forget” old data

$$V(\hat{\theta}(k)) = \frac{1}{2}\sum_{j=1}^{k} \lambda^{(k-j)} \left[y(j) - \phi^T(j-1)\,\hat{\theta}(k)\right]^2$$

$$0 < \lambda \leq 1$$

Key idea: Discount old data, e.g. the term

$$\lambda^{(k-1)} \left[y(1) - \phi^T(0)\,\hat{\theta}(k)\right]^2$$

is small when k is large since $\lim_{m\to\infty} \lambda^m = 0$

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RLS with forgetting factor

A-priori version:

$$\left.\begin{aligned} e^o(k+1) &= y(k+1) - \phi^T(k)\hat{\theta}(k) \\ \hat{\theta}(k+1) &= \hat{\theta}(k) + F(k+1)\phi(k)e^o(k+1) \end{aligned}\right\}$$

Same as RLS without forgetting factor

$$F(k+1) = \frac{1}{\lambda}\left[F(k) - \frac{F(k)\,\phi(k)\,\phi(k)^T\,F(k)}{\lambda + \phi(k)^T\,F(k)\,\phi(k)}\right]$$

$$F^{-1}(k+1) = \lambda F^{-1}(k) + \phi(k)\phi^T(k)$$

RLS with forgetting factor

A-posteriori version (used to prove that $e(k) \longrightarrow 0$):

$$e^o(k+1) = y(k+1) - \phi^T(k)\hat{\theta}(k)$$

$$e(k+1) = \frac{\lambda e^o(k+1)}{\lambda + \phi^T(k)F(k)\phi(k)}$$

$$\hat{\theta}(k+1) = \hat{\theta}(k) + \frac{1}{\lambda}F(k)\phi(k)e(k+1)$$

$$F(k+1) = \frac{1}{\lambda}\left[F(k) - \frac{F(k)\,\phi(k)\,\phi(k)^T\,F(k)}{\lambda + \phi(k)^T\,F(k)\,\phi(k)}\right]$$

RLS with forgetting factor

The gain of the RLS with FF may blow up

$$\mathrm{tr}\,[F(k)] = \frac{1}{\lambda}\mathrm{tr}\,[F(k-1)] - \frac{\|F(k-1)\phi(k-1)\|^2}{\lambda^2 + \lambda\phi^T(k-1)F(k-1)\phi(k-1)}$$

if $\phi(k)$ is not persistently exciting
(more on this later)

General PAA gain formula

$$F^{-1}(k+1) = \lambda_1(k)\,F^{-1}(k) + \lambda_2(k)\,\phi(k)\phi^T(k)$$

$$0 < \lambda_1(k) \leq 1 \qquad 0 \leq \lambda_2(k) < 2$$

- Constant adaptation gain: $\lambda_1(k) = 1,\ \lambda_2(k) = 0$
  (We talked about this case in the previous lecture)
- RLS: $\lambda_1(k) = 1,\ \lambda_2(k) = 1$
- RLS with forgetting factor: $\lambda_1(k) < 1,\ \lambda_2(k) = 1$

General PAA gain formula

$$F^{-1}(k+1) = \lambda_1(k)\,F^{-1}(k) + \lambda_2(k)\,\phi(k)\phi^T(k)$$

$$0 < \lambda_1(k) \leq 1 \qquad 0 \leq \lambda_2(k) < 2$$

$$F(k+1) = \frac{1}{\lambda_1(k)}\left[F(k) - \lambda_2(k)\frac{F(k)\phi(k)\phi^T(k)F(k)}{\lambda_1(k) + \lambda_2(k)\phi^T(k)F(k)\phi(k)}\right]$$

$$F(0) = F^T(0) > 0$$

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General PAA

A-priori version:

$$e^o(k+1) = y(k+1) - \phi^T(k)\hat{\theta}(k)$$

$$\hat{\theta}(k+1) = \hat{\theta}(k) + \frac{1}{\lambda_1(k) + \phi^T(k)F(k)\phi(k)} F(k)\phi(k)e^o(k+1)$$

$$F(k+1) = \frac{1}{\lambda_1(k)}\left[F(k) - \lambda_2(k)\frac{F(k)\phi(k)\phi^T(k)F(k)}{\lambda_1(k) + \lambda_2(k)\phi^T(k)F(k)\phi(k)}\right]$$

When $\lambda_2(k) = 1$, the parameter estimate equation simplifies to

$$\hat{\theta}(k+1) = \hat{\theta}(k) + F(k+1)\phi(k)e^o(k+1)$$

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General PAA

A-posteriori version (used to prove that $e(k) \longrightarrow 0$):

$$e^o(k+1) = y(k+1) - \phi^T(k)\hat{\theta}(k)$$

$$e(k+1) = \frac{\lambda_1(k)e^o(k+1)}{\lambda_1(k) + \phi^T(k)F(k)\phi(k)}$$

$$\hat{\theta}(k+1) = \hat{\theta}(k) + \frac{1}{\lambda_1(k)}F(k)\phi(k)e(k+1)$$

$$F(k+1) = \frac{1}{\lambda_1(k)}\left[F(k) - \lambda_2(k)\frac{F(k)\phi(k)\phi^T(k)F(k)}{\lambda_1(k) + \lambda_2(k)\phi^T(k)F(k)\phi(k)}\right]$$

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Additional Material
(you are not responsible for this)

- The Matrix Inversion Lemma
- Relationships for the General PAA

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Matrix Inversion Lemma
(simplified version)

- Since $\det(I + RL) = \det(I + LR)$, we know that

$$I + RL \text{ is invertible}$$
$$\Updownarrow$$
$$I + LR \text{ is invertible}$$

- The matrix inversion lemma (simplified version) states that

$$(I + RL)^{-1} = I - R(I + LR)^{-1}L$$

Matrix Inversion Lemma (simplified version)

$$(I+RL)^{-1} = I - R(I+LR)^{-1}L$$

Proof:

Define $\Phi = I - R(I+LR)^{-1}L$

We want to show that $(I+RL)\Phi = I$

$$(I+RL)\Phi = (I+RL) - \underbrace{(I+RL)R}_{R+RLR=R(I+LR)}(I+LR)^{-1}L$$

$$\begin{aligned}(I+RL)\Phi &= I + RL - R(I+LR)(I+LR)^{-1}L \\ &= I + RL - RL\end{aligned}$$

$\blacksquare$

Matrix Inversion Lemma

If A, C, and $(A+UCV)$ are invertible, then

$$(A+UCV)^{-1} = A^{-1} - A^{-1}U\left(C^{-1} + VA^{-1}U\right)^{-1}VA^{-1}$$

Proof:

$$\begin{aligned}(A+UCV)^{-1} &= \left[\left(I + UCVA^{-1}\right)A\right]^{-1} \\ &= A^{-1}\left(I + UCVA^{-1}\right)^{-1} \\ &= A^{-1}\left[I - UC\left(I + VA^{-1}UC\right)^{-1}VA^{-1}\right] \\ &= A^{-1}\left[I - U\left[\left(I + VA^{-1}UC\right)C^{-1}\right]^{-1}VA^{-1}\right] \\ &= A^{-1} - A^{-1}U\left(C^{-1} + VA^{-1}U\right)^{-1}VA^{-1}\end{aligned}$$

$\blacksquare$

Relationships for General PAA

$$F(k+1) = \frac{1}{\lambda_1(k)}\left[F(k) - \lambda_2(k)\frac{F(k)\phi(k)\phi^T(k)F(k)}{\lambda_1(k) + \lambda_2(k)\phi^T(k)F(k)\phi(k)}\right]$$

Proof: We know that

$$F^{-1}(k+1) = \lambda_1(k)F^{-1}(k) + [\lambda_2(k)\phi(k)]\,\phi^T(k)$$

By the Matrix Inversion Lemma

$$F(k+1) = \frac{1}{\lambda_1(k)}F(k)$$
$$-\left[\frac{1}{\lambda_1(k)}F(k)\right][\lambda_2(k)\phi(k)]\left[\frac{1}{1+\phi^T(k)\left[\frac{1}{\lambda_1(k)}F(k)\right][\lambda_2(k)\phi(k)]}\right]\phi^T(k)\left[\frac{1}{\lambda_1(k)}F(k)\right]$$

This simplifies to the stated expression for $F(k+1)$

$\blacksquare$

Relationships for General PAA

$$F(k+1)\phi(k) = \frac{1}{\lambda_1(k) + \lambda_2(k)\phi^T(k)F(k)\phi(k)}F(k)\phi(k)$$

Proof:

$$F^{-1}(k+1) = \lambda_1(k)F^{-1}(k) + \lambda_2(k)\phi(k)\phi^T(k)$$
$$\Downarrow$$
$$\begin{aligned}&F(k+1)\left[F^{-1}(k+1)\right]F(k)\phi(k) \\ &\quad = F(k+1)\left[\lambda_1(k)F^{-1}(k) + \lambda_2(k)\phi(k)\phi^T(k)\right]F(k)\phi(k)\end{aligned}$$
$$\Downarrow$$
$$\begin{aligned}F(k)\phi(k) &= \lambda_1(k)F(k+1)\phi(k) \\ &\quad + \lambda_2(k)F(k+1)\phi(k)\phi^T(k)F(k)\phi(k) \\ &= F(k+1)\phi(k)\left[\lambda_1(k) + \lambda_2(k)\phi^T(k)F(k)\phi(k)\right]\end{aligned}$$

$\blacksquare$

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Relationships for General PAA

$$\boxed{e(k+1) = \frac{\lambda_1(k)}{\lambda_1(k) + \phi^T(k)F(k)\phi(k)} e^o(k+1)}$$

Proof:

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) - \frac{1}{\lambda_1(k)} F(k)\phi(k)e(k+1)$$

$$\Downarrow$$

$$\underbrace{\phi^T(k)\tilde{\theta}(k+1)}_{e(k+1)} = \phi^T(k)\left[\tilde{\theta}(k) - \frac{1}{\lambda_1(k)} F(k)\phi(k)e(k+1)\right]$$

$$= \underbrace{\phi^T(k)\tilde{\theta}(k)}_{e^o(k+1)} - \frac{1}{\lambda_1(k)}\phi^T(k)F(k)\phi(k)e(k+1)$$

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Relationships for General PAA

$$\boxed{e(k+1) = \frac{\lambda_1(k)}{\lambda_1(k) + \phi^T(k)F(k)\phi(k)} e^o(k+1)}$$

Proof (continued):

From the previous slide,

$$e(k+1) = e^o(k+1) - \frac{1}{\lambda_1(k)}\phi^T(k)F(k)\phi(k)e(k+1)$$

$$\Downarrow$$

$$\left[\lambda_1(k) + \phi^T(k)F(k)\phi(k)\right] e(k+1) = \lambda_1(k)e^o(k+1)$$

$\blacksquare$