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ME 233 – Homework 8

1.a)

$$x(k+1)=Ax(k)+Bu(k)+B_w w(k)$$
$$w(k)=C_w x_w(k),\ x_w(k+1)=A_w x_w(k)+B_n \eta(k)$$
$$E\{x(0)\}=x_o,\ E\{(x(0)-x_o)(x(0)-x_o)^T\}=X_o$$
$$E\{x_w(0)\}=x_{wo},\ E\{(x_w(0)-x_{wo})(x_w(0)-x_{wo})^T\}=X_{wo}$$
$$E\{\eta(k)\}=0,\ E\{\eta(k)\eta(k+l)^T\}=\Gamma\delta(l)$$
$$J=\frac{1}{2}E\left\{x^T(N)Sx(N)+\sum_{k=0}^{N-1}[x^T(k)Qx(k)+u^T(k)Ru(k)]\right\}$$

If $x(k)$ and $x_w(k)$ are both measurable for all k, let extended state $x_e(k)=\begin{bmatrix} x(k) \\ x_w(k)\end{bmatrix}$

$$J=\frac{1}{2}E\left\{\begin{bmatrix} x(N) \\ x_w(N)\end{bmatrix}^T\begin{bmatrix} S & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix} x(N) \\ x_w(N)\end{bmatrix}+\sum_{k=0}^{N-1}\left(\begin{bmatrix} x(k) \\ x_w(k)\end{bmatrix}^T\begin{bmatrix} Q & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix} x(k) \\ x_w(k)\end{bmatrix}+u^T(k)Ru(k)\right)\right\}$$

$$x_e(k+1)=\begin{bmatrix} x(k+1) \\ x_w(k+1)\end{bmatrix}=\begin{bmatrix} A & B_w C_w \\ 0 & A_w\end{bmatrix}\begin{bmatrix} x(k) \\ x_w(k)\end{bmatrix}+\begin{bmatrix} B \\ 0\end{bmatrix}u(k)+\begin{bmatrix} 0 \\ B_n\end{bmatrix}\eta(k)$$

Optimal control solution is as in deterministic LQR, $u^o(k)=-K(k+1)x_e(k)=-K(k+1)\begin{bmatrix} x(k) \\ x_w(k)\end{bmatrix}$

$$K(k+1)=\left(R+\begin{bmatrix} B \\ 0\end{bmatrix}^T P(k+1)\begin{bmatrix} B \\ 0\end{bmatrix}\right)^{-1}\begin{bmatrix} B \\ 0\end{bmatrix}^T P(k+1)\begin{bmatrix} A & B_w C_w \\ 0 & A_w\end{bmatrix}$$

$$P(k-1)=\begin{bmatrix} Q & 0 \\ 0 & 0\end{bmatrix}+\begin{bmatrix} A & B_w C_w \\ 0 & A_w\end{bmatrix}^T P(k)\begin{bmatrix} A & B_w C_w \\ 0 & A_w\end{bmatrix}-\begin{bmatrix} A & B_w C_w \\ 0 & A_w\end{bmatrix}^T P(k)\begin{bmatrix} B \\ 0\end{bmatrix}K(k)$$

$P(N)=\begin{bmatrix} S & 0 \\ 0 & 0\end{bmatrix}$, and assuming initial values $x(0)$ and $x_w(0)$ are uncorrelated, the optimal cost is:

$$J^o=\frac{1}{2}\begin{bmatrix} x_o \\ x_{wo}\end{bmatrix}^T P(0)\begin{bmatrix} x_o \\ x_{wo}\end{bmatrix}+\frac{1}{2}\operatorname{trace}\left(P(0)\begin{bmatrix} X_o & 0 \\ 0 & X_{wo}\end{bmatrix}\right)+b(0)$$

$$b(k)=b(k+1)+\operatorname{trace}\left(\begin{bmatrix} 0 \\ B_n\end{bmatrix}^T P(k+1)\begin{bmatrix} 0 \\ B_n\end{bmatrix}\Gamma\right),\ b(N)=0$$

1.b)

$$y(k)=Cx(k)+v(k),\ E\{v(k)\}=0,\ E\{v(k)v(k+l)^T\}=V\delta(l)$$

$v(k)$ independent from $x(0)$, $x_w(0)$, and $\eta(k)$

$$y(k)=[C \quad 0]\begin{bmatrix} x(k) \\ x_w(k)\end{bmatrix}+v(k)$$

Construct Kalman filter observer $\hat{x}_e(k)=\begin{bmatrix} \hat{x}(k) \\ \hat{x}_w(k)\end{bmatrix}=\hat{x}_e^o(k)+F(k)\tilde{y}^o(k)$

$$\hat{x}_e^o(k+1)=\begin{bmatrix} \hat{x}^o(k+1) \\ \hat{x}_w^o(k+1)\end{bmatrix}=\begin{bmatrix} A & B_w C_w \\ 0 & A_w\end{bmatrix}\hat{x}_e(k)+\begin{bmatrix} B \\ 0\end{bmatrix}u(k)$$

$$\tilde{y}^o(k)=y(k)-[C \quad 0]\hat{x}_e^o(k)$$

$E(k)=[C \quad 0]M(k)[C \quad 0]^T+V$
$F(k)=M(k)[C \quad 0]^T E(k)^{-1}$
$Z(k)=M(k)-F(k)[C \quad 0]M(k)$
$M(k+1)=\begin{bmatrix} A & B_w C_w \\ 0 & A_w \end{bmatrix} Z(k) \begin{bmatrix} A & B_w C_w \\ 0 & A_w \end{bmatrix}^T + \begin{bmatrix} 0 \\ B_n \end{bmatrix} \Gamma \begin{bmatrix} 0 \\ B_n \end{bmatrix}^T$
$M(0)=\begin{bmatrix} X_o & 0 \\ 0 & X_{wo} \end{bmatrix}$, optimal control based on a-posteriori state estimate: $u^o(k)=-K(k+1)\hat{x}_e(k)$
where $K(k+1)$ is defined exactly as in part a.
optimal cost $J^o=\hat{J}^o+\sum_{j=0}^{N-1} \text{trace}\left(\begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} Z(j)\right)+\text{trace}\left(\begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} Z(N)\right)$
$\hat{J}^o=\frac{1}{2}\begin{bmatrix} x_o \\ x_{wo} \end{bmatrix}^T P(0) \begin{bmatrix} x_o \\ x_{wo} \end{bmatrix}+\frac{1}{2}\text{trace}\left(P(0)\begin{bmatrix} X_o & 0 \\ 0 & X_{wo} \end{bmatrix}\right)+\hat{b}(0)$
$\hat{b}(k)=\hat{b}(k+1)+\text{trace}[F(k+1)^T P(k+1) F(k+1) E(k+1)]$, $\hat{b}(N)=0$
$\hat{b}(k)=\hat{b}(k+1)+\text{trace}[F(k+1)^T P(k+1) M(k+1)[C \quad 0]^T]$

1.c)

If $x(k)$ is measurable for all k, we have $x(k)-Ax(k-1)-Bu(k-1)=B_w C_w x_w(k-1)$
Let $y_w(k-1)=B_w C_w x_w(k-1)$, so at step k we know $y_w(k-1)$ with zero measurement noise
$x_w(k)=A_w x_w(k-1)+B_n \eta(k-1)$
Construct Kalman filter observer just for $\hat{x}_w(k-1\,|\,k)=\hat{x}_w^o(k-1)+F_w(k-1)\tilde{y}_w^o(k-1)$
$\hat{x}_w^o(k)=A_w \hat{x}_w(k-1\,|\,k)$, $\tilde{y}_w^o(k-1)=y_w(k-1)-B_w C_w \hat{x}_w^o(k-1)$
$E_w(k-1)=B_w C_w M_w(k-1) C_w^T B_w^T+0$
$F_w(k-1)=M_w(k-1) C_w^T B_w^T E_w(k-1)^{\#}$, using the pseudoinverse since $E_w(k-1)$ may be singular
$Z_w(k-1)=M_w(k-1)-F_w(k-1) B_w C_w M_w(k-1)$
$M_w(k)=A_w Z_w(k-1) A_w^T+B_n \Gamma B_n^T$, initial condition $M_w(0)=X_{wo}$
Let $Y_w(k-1)=\{y_w(0), \dots, y_w(k-1)\}$
$E\{x_w(k)\,|\,Y_w(k-1)\}=E\{A_w x_w(k-1)+B_n \eta(k-1)\,|\,Y_w(k-1)\}$
$E\{x_w(k)\,|\,Y_w(k-1)\}=A_w E\{x_w(k-1)\,|\,Y_w(k-1)\}+B_n E\{\eta(k-1)\,|\,Y_w(k-1)\}$
$y_w(k-1)=B_w C_w x_w(k-1)=B_w C_w A_w x_w(k-2)+B_w C_w B_n \eta(k-2)$
Therefore $y(k-1)$ and earlier don't contain any information about $\eta(k-1)$
so $E\{\eta(k-1)\,|\,Y_w(k-1)\}=E\{\eta(k-1)\}=0$
$\hat{x}_w(k\,|\,k)=E\{x_w(k)\,|\,Y_w(k-1)\}=A_w E\{x_w(k-1)\,|\,Y_w(k-1)\}=A_w \hat{x}_w(k-1\,|\,k)=\hat{x}_w^o(k)$
LQR control based on measured $x(k)$ and estimated $\hat{x}_w^o(k)$, so $u^o(k)=-K(k+1)\begin{bmatrix} x(k) \\ \hat{x}_w^o(k) \end{bmatrix}$
where again $K(k+1)$ is defined exactly as in part a. In the special circumstances of this problem with the one-step observation delay, the value used by the controller happens to be the a-priori estimate instead of the usual a-posteriori estimate. (I wonder if this also holds in more general cases?)
Luckily the cost $J=\frac{1}{2}E\left\{x^T(N) S x(N)+\sum_{k=0}^{N-1}[x^T(k) Q x(k)+u^T(k) R u(k)]\right\}$ is equivalent to:
$$J=\frac{1}{2}E\left\{\begin{bmatrix} x(N) \\ \hat{x}_w^o(N) \end{bmatrix}^T \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(N) \\ \hat{x}_w^o(N) \end{bmatrix}+\sum_{k=0}^{N-1}\left(\begin{bmatrix} x(k) \\ \hat{x}_w^o(k) \end{bmatrix}^T \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ \hat{x}_w^o(k) \end{bmatrix}+u^T(k) R u(k)\right)\right\}$$
$\hat{x}_w^o(k)=A_w \hat{x}_w(k-1\,|\,k)=A_w \hat{x}_w^o(k-1)+A_w F_w(k-1)\tilde{y}_w^o(k-1)$

$\tilde{y}_w^o(k-1)=y_w(k-1)-B_w C_w \hat{x}_w^o(k-1)=B_w C_w x_w(k-1)-B_w C_w \hat{x}_w^o(k-1)$

$x(k)=A x(k-1)+B u(k-1)+B_w C_w x_w(k-1)=A x(k-1)+B u(k-1)+B_w C_w \hat{x}_w^o(k-1)+\tilde{y}_w^o(k-1)$

$$\begin{bmatrix} x(k) \\ \hat{x}_w^o(k) \end{bmatrix}=\begin{bmatrix} A & B_w C_w \\ 0 & A_w \end{bmatrix}\begin{bmatrix} x(k-1) \\ \hat{x}_w^o(k-1) \end{bmatrix}+\begin{bmatrix} B \\ 0 \end{bmatrix}u(k-1)+\begin{bmatrix} I \\ A_w F_w(k-1) \end{bmatrix}\tilde{y}_w^o(k-1)$$

$$\begin{bmatrix} x(k+1) \\ \hat{x}_w^o(k+1) \end{bmatrix}=\begin{bmatrix} A & B_w C_w \\ 0 & A_w \end{bmatrix}\begin{bmatrix} x(k) \\ \hat{x}_w^o(k) \end{bmatrix}+\begin{bmatrix} B \\ 0 \end{bmatrix}u(k)+\begin{bmatrix} I \\ A_w F_w(k) \end{bmatrix}\tilde{y}_w^o(k)$$

$E\{\tilde{y}_w^o(k)\,\tilde{y}_w^o(k)^T\}=E_w(k)=B_w C_w M_w(k) C_w^T B_w^T$

So using this form, the optimal cost is $J^o=\frac{1}{2}\begin{bmatrix} x_o \\ x_{wo} \end{bmatrix}^T P(0)\begin{bmatrix} x_o \\ x_{wo} \end{bmatrix}+\frac{1}{2}\text{trace}\left(P(0)\begin{bmatrix} X_o & 0 \\ 0 & X_{wo} \end{bmatrix}\right)+b(0)$

$$b(k)=b(k+1)+\text{trace}\left(\begin{bmatrix} I \\ A_w F_w(k) \end{bmatrix}^T P(k+1)\begin{bmatrix} I \\ A_w F_w(k) \end{bmatrix}E_w(k)\right),\ b(N)=0$$

2.a)

$$Z_1(s)=\frac{1.367\cdot 10^4}{s^2+134 s+4.52\cdot 10^9}W(s)$$

$s^2 Z_1(s)+134 s Z_1(s)+4.52\cdot 10^9\cdot Z_1(s)=1.367\cdot 10^4\cdot W(s)$

Inverse Laplace transform, $\ddot{z}_1(t)+134\dot{z}_1(t)+4.52\cdot 10^9\cdot z_1(t)=1.367\cdot 10^4\cdot w(t)$

Assuming the given transfer function was for the suspension by itself, since we know $w(t)$ is in units of force we have $m_1=1/(1.367\cdot 10^4),\ b_1=134/(1.367\cdot 10^4),\ k_1=4.52\cdot 10^9/(1.367\cdot 10^4)$

For the actuator, $m\ddot{z}(t)+b(\dot{z}(t)-\dot{z}_1(t))+k(z(t)-z_1(t))=K_e u(t)$

$z(t)=z_1(t)+z_2(t)$, so $m(\ddot{z}_1(t)+\ddot{z}_2(t))+b\dot{z}_2(t)+k z_2(t)=K_e u(t)$

$2\cdot 10^{-6}\ddot{z}_1(t)+2\cdot 10^{-6}\ddot{z}_2(t)+5\cdot 10^{-3}\dot{z}_2(t)+400 z_2(t)=2.5\cdot 10^{-5}u(t)$

Taking into account the reaction forces on the suspension from the actuator spring, damper, and input,

$\ddot{z}_1(t)+134\dot{z}_1(t)+4.52\cdot 10^9\cdot z_1(t)=1.367\cdot 10^4\cdot[w(t)+5\cdot 10^{-3}\dot{z}_2(t)+400 z_2(t)-2.5\cdot 10^{-5}u(t)]$

$\ddot{z}_1(t)=-134\dot{z}_1(t)-4.52\cdot 10^9\cdot z_1(t)+1.367\cdot 10^4\cdot w(t)+68.35\dot{z}_2(t)+5.468\cdot 10^6 z_2(t)-0.34175u(t)$

$\ddot{z}_2(t)=-\ddot{z}_1(t)-2500\dot{z}_2(t)-2\cdot 10^8 z_2(t)+12.5u(t)$

$\ddot{z}_2(t)=134\dot{z}_1(t)+4.52\cdot 10^9\cdot z_1(t)-1.367\cdot 10^4\cdot w(t)-2568.35\dot{z}_2(t)-2.05468\cdot 10^8 z_2(t)+12.84175u(t)$

Let $x(t)=[z_1(t)\quad z_2(t)\quad \dot{z}_1(t)\quad \dot{z}_2(t)]^T$, then $\dot{x}(t)=[\dot{z}_1(t)\quad \dot{z}_2(t)\quad \ddot{z}_1(t)\quad \ddot{z}_2(t)]^T$

$$\dot{x}(t)=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4.52\cdot 10^9 & 5.468\cdot 10^6 & -134 & 68.35 \\ 4.52\cdot 10^9 & -2.05468\cdot 10^8 & 134 & -2568.35 \end{bmatrix}\begin{bmatrix} z_1(t) \\ z_2(t) \\ \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix}+\begin{bmatrix} 0 \\ 0 \\ -0.34175 \\ 12.84175 \end{bmatrix}u(t)+\begin{bmatrix} 0 \\ 0 \\ 1.367\cdot 10^4 \\ -1.367\cdot 10^4 \end{bmatrix}w(t)$$

$y(t)=z(t)+v(t)=z_1(t)+z_2(t)+v(t)=[1\quad 1\quad 0\quad 0]x(t)+v(t)$

2.b)

TF from u to x is $G(s)=C(sI-A)^{-1}B=\dfrac{12.5s^2+1675s+5.65\cdot 10^{10}}{s^4+2702s^3+4.726\cdot 10^9 s^2+1.133\cdot 10^{13}s+9.04\cdot 10^{17}}$

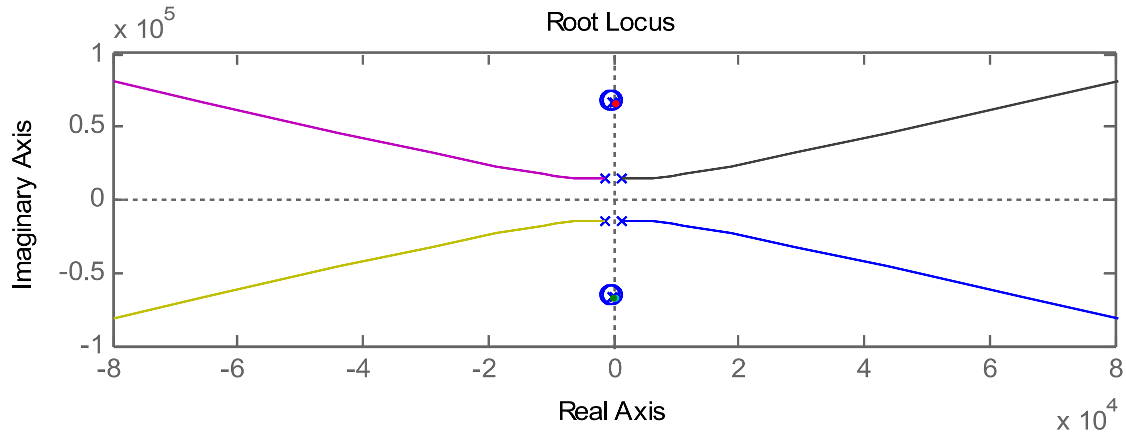
Symmetric root locus of $G(-s)G(s)$ shown on next page

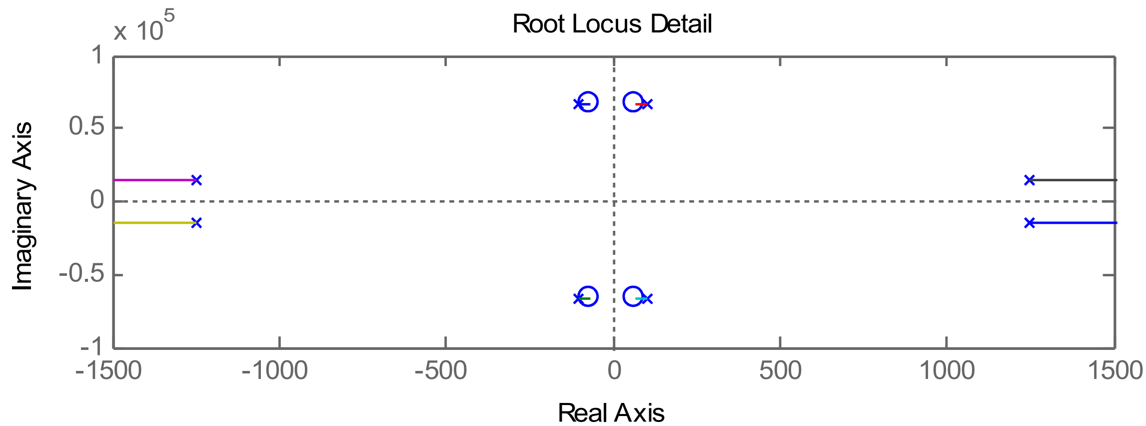
2.c)

stationary LQG regulator minimizing cost $J=E\{z^2(t)+Ru^2(t)\}/2=E\{x^T(t)C^T C x(t)+Ru^2(t)\}/2$

optimal controller $u^o(t)=-K\hat{x}(t),\ K=R^{-1}B^T P,\ A^T P+PA+C^T C-PBR^{-1}B^T P=0$

KF estimator $\dot{\hat{x}}(t)=A\hat{x}(t)+Bu(t)+L(y(t)-C\hat{x}(t)),\ L=MC^T V^{-1}$





$A\,M+M\,A^T=-B_w\,W\,B_w^T+M\,C^T\,V^{-1}\,C\,M\ ,\ W=E\{w(t)^2\}=1.01\cdot10^{-7},\ V=E\{v(t)^2\}=1.66\cdot10^{-14}$

Matlab says $M=\begin{bmatrix} 9.9388\cdot10^{-12} & -1.0381\cdot10^{-11} & 5.895\cdot10^{-12} & 2.7248\cdot10^{-8} \\ -1.0381\cdot10^{-11} & 1.0902\cdot10^{-11} & -2.7262\cdot10^{-8} & 8.1553\cdot10^{-12} \\ 5.895\cdot10^{-12} & -2.7262\cdot10^{-8} & 4.4979\cdot10^{-2} & -4.6987\cdot10^{-2} \\ 2.7248\cdot10^{-8} & 8.1553\cdot10^{-12} & -4.6987\cdot10^{-2} & 4.9168\cdot10^{-2} \end{bmatrix},\ L=\begin{bmatrix} -26.65 \\ 31.346 \\ -1.642\cdot10^{6} \\ 1.642\cdot10^{6} \end{bmatrix}$

from $\mathrm{care}(A^T, C^T, B_w W\,B_w^T, V)$

For $R=1$, Matlab gives $P=\begin{bmatrix} 2.288\cdot10^{-4} & 2.098\cdot10^{-4} & 6.7825\cdot10^{-12} & -1.0384\cdot10^{-10} \\ 2.098\cdot10^{-4} & 2.0674\cdot10^{-4} & 2.7321\cdot10^{-9} & 2.5062\cdot10^{-9} \\ 6.7825\cdot10^{-12} & 2.7321\cdot10^{-9} & 1.1008\cdot10^{-12} & 1.0502\cdot10^{-12} \\ -1.0384\cdot10^{-10} & 2.5062\cdot10^{-9} & 1.0502\cdot10^{-12} & 1.0037\cdot10^{-12} \end{bmatrix}$

and $K=[-1.3358\cdot10^{-9}\quad 3.125\cdot10^{-8}\quad 1.311\cdot10^{-11}\quad 1.2531\cdot10^{-11}]$, both from $\mathrm{care}(A, B, C^T C, R)$

optimal cost $J^o=\mathrm{trace}\{P[B\,K\,M+B_w W\,B_w^T]\}$, for $R=1$ we have $J^o=7.8156\cdot10^{-14}$

For $R=10^{-10}$, LQR result is $P=\begin{bmatrix} 2.2873\cdot10^{-4} & 2.0973\cdot10^{-4} & 7.6393\cdot10^{-12} & -1.0298\cdot10^{-10} \\ 2.0973\cdot10^{-4} & 2.0668\cdot10^{-4} & 2.7321\cdot10^{-9} & 2.5062\cdot10^{-9} \\ 7.6393\cdot10^{-12} & 2.7321\cdot10^{-9} & 1.1005\cdot10^{-12} & 1.0499\cdot10^{-12} \\ -1.0298\cdot10^{-10} & 2.5062\cdot10^{-9} & 1.0499\cdot10^{-12} & 1.0034\cdot10^{-12} \end{bmatrix}$

and $K=[-13.251\quad 312.5\quad 0.13106\quad 0.12527]$, with optimal cost $J^o=7.8156\cdot10^{-14}$

2.d)

$J^o = \text{trace}\{C_Q^T C_Q M + L^T P L V\}$

First let $Q = C_Q^T C_Q$. Next, note that the matrices Q, M, and V are positive semidefinite.

$Q \geq 0$ by construction from C_Q, $M \geq 0$ since it is the solution of a Kalman filter Riccati equation, and $V \geq 0$ since it is a covariance matrix. These positive semidefinite matrices are symmetric and real-valued so they are each diagonalizable. Let $M = T_M \Lambda_M T_M^{-1}$ and $V = T_V \Lambda_V T_V^{-1}$

The transformations T_M and T_V of the symmetric real matrices are orthogonal: $T_M^{-1} = T_M^T$, $T_V^{-1} = T_V^T$

$C_Q^T C_Q M = Q T_M \Lambda_M T_M^T$

The trace of a matrix is the sum of its eigenvalues, so the trace is invariant to a similarity transform.

$\text{trace}\{C_Q^T C_Q M\} = \text{trace}\{T_M^T Q T_M \Lambda_M\}$

Since Λ_M is diagonal, the columns with zero eigenvalues will result in columns of all zeros in the product matrix $T_M^T Q T_M \Lambda_M$, and those columns will not contribute to the trace.

So defining $\Lambda_M^{1/2}$ as the matrix with positive square roots of the eigenvalues of M on its diagonal, and $(\Lambda_M^{1/2})^{\#}$ as the pseudoinverse matrix with reciprocal values at the nonzero diagonals of $\Lambda_M^{1/2}$, but zeros everywhere else, we can take a pseudo-similarity transform without altering the trace.

$\text{trace}\{C_Q^T C_Q M\} = \text{trace}\{\Lambda_M^{1/2} T_M^T Q T_M \Lambda_M (\Lambda_M^{1/2})^{\#}\} = \text{trace}\{\Lambda_M^{1/2} T_M^T Q T_M \Lambda_M^{1/2}\}$

For any vector x, let $z = T_M \Lambda_M^{1/2} x$. Then $x^T \Lambda_M^{1/2} T_M^T Q T_M \Lambda_M^{1/2} x = z^T Q z$

By positive semidefiniteness $z^T Q z \geq 0$, so that implies $\Lambda_M^{1/2} T_M^T Q T_M \Lambda_M^{1/2}$ is positive semidefinite, and therefore has nonnegative eigenvalues and a nonnegative trace.

$\text{trace}\{C_Q^T C_Q M\} = \text{trace}\{\Lambda_M^{1/2} T_M^T Q T_M \Lambda_M^{1/2}\} \geq 0$

Likewise $\text{trace}\{L^T P L V\} = \text{trace}\{L^T P L T_V \Lambda_V T_V^T\} = \text{trace}\{T_V^T L^T P L T_V \Lambda_V\}$

Using corresponding definitions of $\Lambda_V^{1/2}$ and $(\Lambda_V^{1/2})^{\#}$ and the same zero-eigenvalue argument,

$\text{trace}\{L^T P L V\} = \text{trace}\{\Lambda_V^{1/2} T_V^T L^T P L T_V \Lambda_V (\Lambda_V^{1/2})^{\#}\} = \text{trace}\{\Lambda_V^{1/2} T_V^T L^T P L T_V \Lambda_V^{1/2}\}$

For any vector x, let $z = L T_V \Lambda_V^{1/2} x$. Then $x^T \Lambda_V^{1/2} T_V^T L^T P L T_V \Lambda_V^{1/2} x = z^T P z$

$z^T P z \geq 0$ by positive semidefiniteness of P, so $\Lambda_V^{1/2} T_V^T L^T P L T_V \Lambda_V^{1/2}$ is positive semidefinite

$\text{trace}\{L^T P L V\} = \text{trace}\{\Lambda_V^{1/2} T_V^T L^T P L T_V \Lambda_V^{1/2}\} \geq 0$

For this example $\text{trace}\{C_Q^T C_Q M\} = 7.7945 \cdot 10^{-14}$

With $R = 1$, $\text{trace}\{L^T P L V\} = 2.1091 \cdot 10^{-16}$

With $R = 10^{-10}$, $\text{trace}\{L^T P L V\} = 2.1090 \cdot 10^{-16}$

This is essentially at roundoff error, definitely two orders of magnitude smaller than the other term.

Difficult to tell for sure with only this data, but it appears as $R \to 0$, $J^o \to \text{trace}\{C_Q^T C_Q M\}$

2.e)

Now z_1 does not depend on z_2 or $u(t)$, so $\ddot{z}_1(t) + 134\,\dot{z}_1(t) + 4.52 \cdot 10^9 \cdot z_1(t) = 1.367 \cdot 10^4 \cdot w(t)$

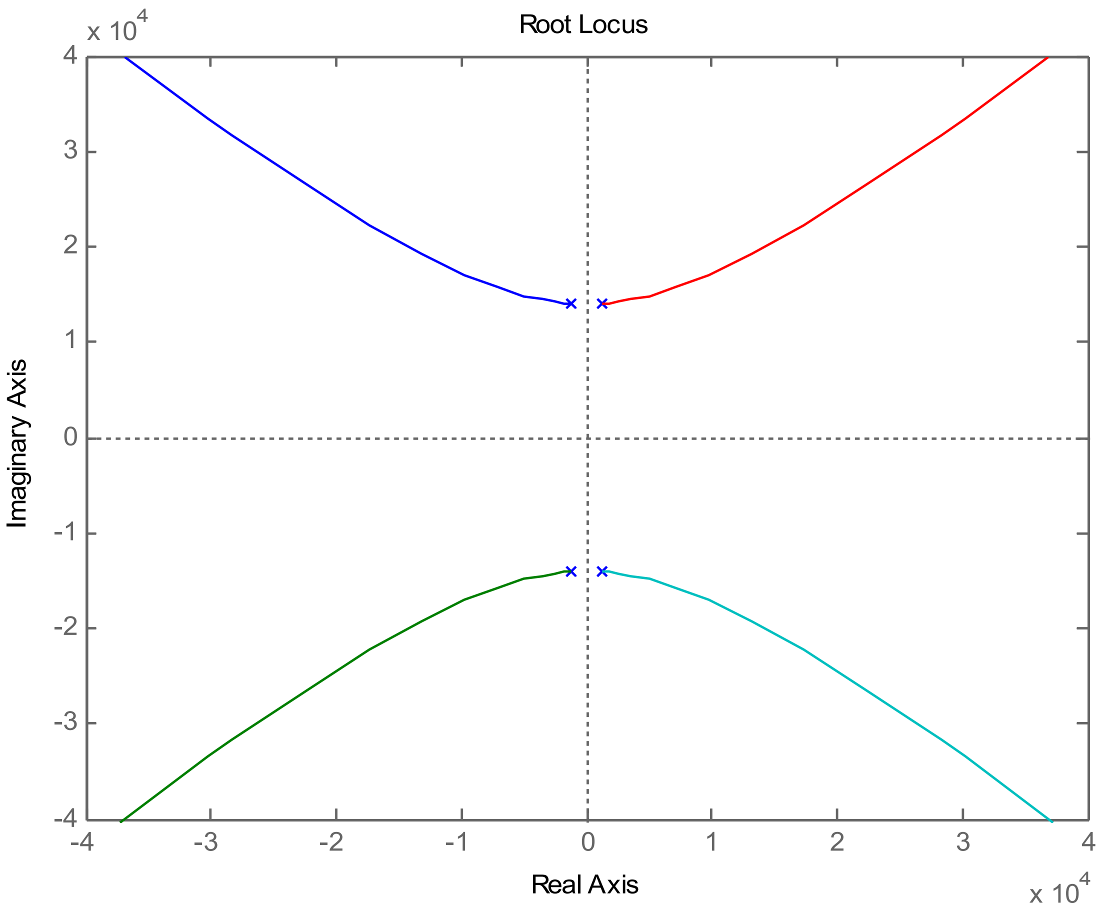
As before, $\ddot{z}_2(t) = -\ddot{z}_1(t) - 2500\,\dot{z}_2(t) - 2 \cdot 10^8 z_2(t) + 12.5\,u(t)$

$\ddot{z}_2(t) = 134\,\dot{z}_1(t) + 4.52 \cdot 10^9 z_1(t) - 1.367 \cdot 10^4 w(t) - 2500\,\dot{z}_2(t) - 2 \cdot 10^8 z_2(t) + 12.5\,u(t)$

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4.52 \cdot 10^9 & 0 & -134 & 0 \\ 4.52 \cdot 10^9 & -2 \cdot 10^8 & 134 & -2500 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \\ \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 12.5 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 1.367 \cdot 10^4 \\ -1.367 \cdot 10^4 \end{bmatrix} w(t)$$

TF from u to x is $G(s) = C(sI - A)^{-1} B = \dfrac{12.5}{s^2 + 2500\,s + 2 \cdot 10^8}$

Symmetric root locus of $G(-s)G(s)$ shown on next page



Now $M=\begin{bmatrix} 1.5386\cdot10^{-11} & -1.6073\cdot10^{-11} & 1.4198\cdot10^{-11} & 4.2129\cdot10^{-8} \\ -1.6073\cdot10^{-11} & 1.6857\cdot10^{-11} & -4.2162\cdot10^{-8} & 1.8534\cdot10^{-11} \\ 1.4198\cdot10^{-11} & -4.2162\cdot10^{-8} & 6.9546\cdot10^{-2} & -7.2655\cdot10^{-2} \\ 4.2129\cdot10^{-8} & 1.8534\cdot10^{-11} & -7.2655\cdot10^{-2} & 7.6027\cdot10^{-2} \end{bmatrix}$, $L=\begin{bmatrix} -41.359 \\ 47.255 \\ -2.539\cdot10^{6} \\ 2.539\cdot10^{6} \end{bmatrix}$

For $R=1$, $P=\begin{bmatrix} 2.3319\cdot10^{-4} & 2.093\cdot10^{-4} & 6.9127\cdot10^{-12} & -1.0371\cdot10^{-10} \\ 2.093\cdot10^{-4} & 2.0625\cdot10^{-4} & 2.7258\cdot10^{-9} & 2.5\cdot10^{-9} \\ 6.9127\cdot10^{-12} & 2.7258\cdot10^{-9} & 1.098\cdot10^{-12} & 1.0464\cdot10^{-12} \\ -1.0371\cdot10^{-10} & 2.5\cdot10^{-9} & 1.0464\cdot10^{-12} & 1\cdot10^{-12} \end{bmatrix}$

and $K=[-1.2963\cdot10^{-9} \quad 3.125\cdot10^{-8} \quad 1.308\cdot10^{-11} \quad 1.25\cdot10^{-11}]$, $J^{o}=9.8598\cdot10^{-14}$

For $R=10^{-10}$, LQR result is $P=\begin{bmatrix} 2.3312\cdot10^{-4} & 2.0924\cdot10^{-4} & 7.7655\cdot10^{-12} & -1.0285\cdot10^{-10} \\ 2.0924\cdot10^{-4} & 2.0619\cdot10^{-4} & 2.7258\cdot10^{-9} & 2.5\cdot10^{-9} \\ 7.7655\cdot10^{-12} & 2.7258\cdot10^{-9} & 1.0976\cdot10^{-12} & 1.046\cdot10^{-12} \\ -1.0285\cdot10^{-10} & 2.5\cdot10^{-9} & 1.046\cdot10^{-12} & 9.9968\cdot10^{-13} \end{bmatrix}$

and $K=[-12.857 \quad 312.5 \quad 0.13075 \quad 0.12496]$, with optimal cost $J^{o}=9.8598\cdot10^{-14}$

LQG results quite similar between the two models. Biggest difference was in the transfer function and therefore root locus as well. There was a pole/zero cancellation and a reduction of order.

3.a)

$$\frac{d}{dt}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}=\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}+\begin{bmatrix}0\\m\end{bmatrix}u(t)+\begin{bmatrix}1\\1\end{bmatrix}w(t),\ y(t)=\begin{bmatrix}1&0\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}+v(t)$$

$$J=E\{x^T(\tau)Q\,x(\tau)+u^2(\tau)\}=\lim_{T\to\infty}E\left\{\frac{1}{T}\int_0^T[x^T(\tau)Q\,x(\tau)+u^2(\tau)]\,d\tau\right\}$$

$$Q=\rho\begin{bmatrix}1\\1\end{bmatrix}[1\quad 0]=\rho\begin{bmatrix}1&1\\1&1\end{bmatrix},\ R=1,\ \text{nominal } m_o=1$$

optimal controller $u(t)=-K_{LQ}x(t)$, $K_{LQ}=R^{-1}B^TP$, $A^TP+PA+Q-PBR^{-1}B^TP=0$

$$0=\begin{bmatrix}1&0\\1&1\end{bmatrix}\begin{bmatrix}p_{11}&p_{12}\\p_{12}&p_{22}\end{bmatrix}+\begin{bmatrix}p_{11}&p_{12}\\p_{12}&p_{22}\end{bmatrix}\begin{bmatrix}1&1\\0&1\end{bmatrix}+\rho\begin{bmatrix}1&1\\1&1\end{bmatrix}-\begin{bmatrix}p_{11}&p_{12}\\p_{12}&p_{22}\end{bmatrix}\begin{bmatrix}0\\m_o\end{bmatrix}[0\quad m_o]\begin{bmatrix}p_{11}&p_{12}\\p_{12}&p_{22}\end{bmatrix}$$

$$0=\begin{bmatrix}p_{11}&p_{12}\\p_{11}+p_{12}&p_{12}+p_{22}\end{bmatrix}+\begin{bmatrix}p_{11}&p_{11}+p_{12}\\p_{12}&p_{12}+p_{22}\end{bmatrix}+\rho\begin{bmatrix}1&1\\1&1\end{bmatrix}-\begin{bmatrix}p_{12}m_o\\p_{22}m_o\end{bmatrix}[m_op_{12}\quad m_op_{22}]$$

$$0=\begin{bmatrix}2p_{11}+\rho&p_{11}+2p_{12}+\rho\\p_{11}+2p_{12}+\rho&2p_{12}+2p_{22}+\rho\end{bmatrix}-\begin{bmatrix}p_{12}^2&p_{12}p_{22}\\p_{12}p_{22}&p_{22}^2\end{bmatrix}$$

$p_{11}=(p_{12}^2-\rho)/2$, $p_{11}+2p_{12}+\rho-p_{12}p_{22}=0$, $p_{12}=-p_{22}+(p_{22}^2-\rho)/2$

$p_{11}=(p_{22}^2-p_{22}(p_{22}^2-\rho)+(p_{22}^2-\rho)^2/4-\rho)/2$

$p_{22}^2/2-p_{22}(p_{22}^2-\rho)/2+(p_{22}^2-\rho)^2/8-\rho/2+(-p_{22}+(p_{22}^2-\rho)/2)(2-p_{22})+\rho=0$

$p_{22}^2/2-p_{22}^3/2+\rho\,p_{22}/2+p_{22}^4/8-\rho\,p_{22}^2/4+\rho^2/8-\rho/2-2p_{22}+p_{22}^2-\rho+p_{22}^2-p_{22}^3/2+\rho\,p_{22}/2+\rho=0$

$p_{22}^4/8-p_{22}^3+(5/2-\rho/4)p_{22}^2+(\rho-2)p_{22}+\rho^2/8-\rho/2=0$

$p_{22}^4-8p_{22}^3+(20-2\rho)p_{22}^2+(8\rho-16)p_{22}+\rho^2-4\rho=0$

$(p_{22}^2-4p_{22}-\rho)(p_{22}^2-4p_{22}+4-\rho)=0$

$p_{22}=2\pm\sqrt{4+\rho}$ or $p_{22}=2\pm\sqrt{\rho}$

$p_{12}=-p_{22}+(p_{22}^2-\rho)/2$, for the possible solutions $p_{22}=2\pm\sqrt{\rho}$ we have $p_{12}=\pm\sqrt{\rho}$

$p_{11}=(p_{12}^2-\rho)/2$, for the possible solutions $p_{22}=2\pm\sqrt{\rho}$ we have $p_{11}=0$

$\det(sI-P)=(s-p_{11})(s-p_{22})-p_{12}^2=s^2-(2\pm\sqrt{\rho})s-\rho$

$\lambda=\left(2+\sqrt{\rho}\pm\sqrt{(2+\sqrt{\rho})^2+4\rho}\right)/2$ or $\lambda=\left(2-\sqrt{\rho}\pm\sqrt{(2-\sqrt{\rho})^2+4\rho}\right)/2$

and both $p_{22}=2\pm\sqrt{\rho}$ options have 2 eigenvalues of different signs, not positive definite

$p_{12}=-p_{22}+(p_{22}^2-\rho)/2$, for the possible solutions $p_{22}=2\pm\sqrt{4+\rho}$ we have $p_{12}=p_{22}=2\pm\sqrt{4+\rho}$

$p_{11}=(p_{12}^2-\rho)/2$, for the possible solutions $p_{22}=2\pm\sqrt{4+\rho}$ we have $p_{11}=2p_{22}=4\pm2\sqrt{4+\rho}$

$\det(sI-P)=(s-p_{11})(s-p_{22})-p_{12}^2=s^2-3p_{22}s+p_{22}^2$

$\lambda=\left(3p_{22}\pm\sqrt{5p_{22}^2}\right)/2$, both eigenvalues same sign as p_{22}

$p_{22}=2+\sqrt{4+\rho}$ is therefore the positive definite solution, $P=\begin{bmatrix}4+2\sqrt{4+\rho}&2+\sqrt{4+\rho}\\2+\sqrt{4+\rho}&2+\sqrt{4+\rho}\end{bmatrix}$

$$K_{LQ}=R^{-1}B^TP=[0\quad m_o]\begin{bmatrix}4+2\sqrt{4+\rho}&2+\sqrt{4+\rho}\\2+\sqrt{4+\rho}&2+\sqrt{4+\rho}\end{bmatrix}=m_o[2+\sqrt{4+\rho}\quad 2+\sqrt{4+\rho}]$$

$K_{LQ}=(2+\sqrt{4+\rho})[1\quad 1]=\alpha(\rho)[1\quad 1]$

3.b)

$$\frac{d}{dt}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}=\left(\begin{bmatrix}1&1\\0&1\end{bmatrix}-\begin{bmatrix}0\\m\end{bmatrix}(2+\sqrt{4+\rho})[1\quad 1]\right)\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}=\begin{bmatrix}1&1\\-2m-m\sqrt{4+\rho}&1-2m-m\sqrt{4+\rho}\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}$$

Eigenvalues given by $(s-1)(s-1+2m+m\sqrt{4+\rho})+2m+m\sqrt{4+\rho}=0$

$s^2+\left(-2+2m+m\sqrt{4+\rho}\right)s+1=0$

$\sqrt{4+\rho}\geq 2$ so for $m>0.5$ the middle coefficient is positive, and the real parts of the eigenvalues will be negative, giving a stable closed-loop system

For $\rho=5$ and $m=1$, $s^2+\sqrt{9}s+1=0$

$\lambda=\left(-3\pm\sqrt{9-4}\right)/2=\left(-3\pm\sqrt{5}\right)/2<0$

3.c)

$$\frac{d}{dt}\begin{bmatrix}\hat{x}_1(t)\\ \hat{x}_2(t)\end{bmatrix}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}\hat{x}_1(t)\\ \hat{x}_2(t)\end{bmatrix}+\begin{bmatrix}0\\ m\end{bmatrix}u(t)+\begin{bmatrix}L_1\\ L_2\end{bmatrix}\tilde{y}(t),\ \tilde{y}(t)=y(t)-\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\hat{x}_1(t)\\ \hat{x}_2(t)\end{bmatrix}$$

$AM+MA^T=-B_w W B_w^T+MC^TV^{-1}CM$, $W=1,\ V=\sigma$

Note that $B_w W B_w^T=\begin{bmatrix}1\\ 1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$, and let $M=\begin{bmatrix}M_{11} & M_{12}\\ M_{12}^T & M_{22}\end{bmatrix}$

$$\frac{d}{dt}\begin{bmatrix}\hat{x}_2(t)\\ \hat{x}_1(t)\end{bmatrix}=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}\hat{x}_2(t)\\ \hat{x}_1(t)\end{bmatrix}+\begin{bmatrix}m\\ 0\end{bmatrix}u(t)+\begin{bmatrix}L_2\\ L_1\end{bmatrix}\tilde{y}(t),\ \tilde{y}(t)=y(t)-\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\hat{x}_2(t)\\ \hat{x}_1(t)\end{bmatrix}$$

Let $M_{\text{flip}}=\begin{bmatrix}M_{22} & M_{12}^T\\ M_{12} & M_{11}\end{bmatrix}$, $C_{\text{flip}}=\begin{bmatrix}0 & 1\end{bmatrix}$

$A^TM_{\text{flip}}+M_{\text{flip}}A=-B_w W B_w^T+M_{\text{flip}}C_{\text{flip}}^TV^{-1}C_{\text{flip}}M_{\text{flip}}$

recall the LQR Riccati equation from part a was: $A^TP+PA+Q-PBR^{-1}B^TP=0$

$B=C_{\text{flip}}^T$ for $m=1$, so if we multiply every term by σ^{-1} we get the same Riccati equation:

$A^T(\sigma^{-1}M_{\text{flip}})+(\sigma^{-1}M_{\text{flip}})A+B_w\sigma^{-1}B_w^T-(\sigma^{-1}M_{\text{flip}})C_{\text{flip}}^TC_{\text{flip}}(\sigma^{-1}M_{\text{flip}})=0$

Where $Q=B_w\sigma^{-1}B_w^T$ for $\rho=\sigma^{-1}$

$$\sigma^{-1}M_{\text{flip}}=\sigma^{-1}\begin{bmatrix}M_{22} & M_{12}^T\\ M_{12} & M_{11}\end{bmatrix}=P=\begin{bmatrix}4+2\sqrt{4+\sigma^{-1}} & 2+\sqrt{4+\sigma^{-1}}\\ 2+\sqrt{4+\sigma^{-1}} & 2+\sqrt{4+\sigma^{-1}}\end{bmatrix}$$

$$M=\begin{bmatrix}M_{11} & M_{12}\\ M_{12}^T & M_{11}\end{bmatrix}=\sigma\begin{bmatrix}2+\sqrt{4+\sigma^{-1}} & 2+\sqrt{4+\sigma^{-1}}\\ 2+\sqrt{4+\sigma^{-1}} & 4+2\sqrt{4+\sigma^{-1}}\end{bmatrix}$$

$$K_{KF}=MC^TV^{-1}=\sigma\begin{bmatrix}2+\sqrt{4+\sigma^{-1}} & 2+\sqrt{4+\sigma^{-1}}\\ 2+\sqrt{4+\sigma^{-1}} & 4+2\sqrt{4+\sigma^{-1}}\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}\sigma^{-1}=\begin{bmatrix}2+\sqrt{4+\sigma^{-1}}\\ 2+\sqrt{4+\sigma^{-1}}\end{bmatrix}=\beta\begin{bmatrix}1\\ 1\end{bmatrix}$$

3.d)

Assuming expectations over all noise variables in the following:

$$\frac{d}{dt}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+\begin{bmatrix}0\\ m\end{bmatrix}u(t)=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}-\begin{bmatrix}0\\ m\end{bmatrix}\alpha\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}\hat{x}_1(t)\\ \hat{x}_2(t)\end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix}\hat{x}_1(t)\\ \hat{x}_2(t)\end{bmatrix}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}\hat{x}_1(t)\\ \hat{x}_2(t)\end{bmatrix}+\begin{bmatrix}0\\ m_o\end{bmatrix}u(t)+\begin{bmatrix}L_1\\ L_2\end{bmatrix}\tilde{y}(t),\ \tilde{y}(t)=y(t)-\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\hat{x}_1(t)\\ \hat{x}_2(t)\end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix}\hat{x}_1(t)\\ \hat{x}_2(t)\end{bmatrix}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}\hat{x}_1(t)\\ \hat{x}_2(t)\end{bmatrix}-\begin{bmatrix}0\\ m_o\end{bmatrix}\alpha\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}\hat{x}_1(t)\\ \hat{x}_2(t)\end{bmatrix}+\beta\begin{bmatrix}1\\ 1\end{bmatrix}\left(\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}-\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\hat{x}_1(t)\\ \hat{x}_2(t)\end{bmatrix}\right)$$

$$\frac{d}{dt}\begin{bmatrix}x_1(t)\\ x_2(t)\\ \hat{x}_1(t)\\ \hat{x}_2(t)\end{bmatrix}=\begin{bmatrix}1 & 1 & 0 & 0\\ 0 & 1 & -m\alpha & -m\alpha\\ \beta & 0 & 1-\beta & 1\\ \beta & 0 & -\beta-m_o\alpha & 1-m_o\alpha\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ \hat{x}_1(t)\\ \hat{x}_2(t)\end{bmatrix}=\begin{bmatrix}1 & 1 & 0 & 0\\ 0 & 1 & -m\alpha & -m\alpha\\ \beta & 0 & 1-\beta & 1\\ \beta & 0 & -\beta-\alpha & 1-\alpha\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ \hat{x}_1(t)\\ \hat{x}_2(t)\end{bmatrix}$$

3.e)

For $\rho = \sigma = 5,\ m = 1,\ \alpha = 2 + \sqrt{4+5} = 5,\ \beta = 2 + \sqrt{4+5^{-1}} = 4.0494$
the eigenvalues of the part d matrix in this case are $-2.618,\ -1.2483,\ -0.80109,$ and -0.38197.
the eigenvalues of full-state LQR from part b were -2.618 and -0.38197 so the separation principle holds in this $m=1$ example

3.f)

With $m=1.1$, the eigenvalues of the above matrix are $-3.4846,\ -0.83368 \pm 1.4736,$ and 0.10258. The eigenvalues are not where we expected them to be from the LQR and KF results, especially since we now have a complex pair, and even worse one of the closed-loop eigenvalues is unstable.