[1]

This is a stationary stochastic control problem with exactly known state. The optimal feedback control law is the same as the deterministic case. So the closed loop system eigenvalues are the z's, which satisfy

$$1+\frac{1}{R}G(z^{-1})G(z)=0\,,$$

where

$$G(z)=C(zI-A)^{-1}B=\frac{0.2(z+1.6)}{z(z-0.9)}$$

and

$$G(z^{-1})=\frac{0.2(z^{-1}+1.6)}{z^{-1}(z^{-1}-0.9)}=-\frac{0.32}{0.9}\frac{z(z+1/1.6)}{z-1/0.9}\,.$$

Then

$$1+\frac{1}{R}G(z^{-1})G(z)=1-\frac{0.64/9}{R}\frac{z(z+1.6)(z+1/1.6)}{z(z-0.9)(z-1/0.9)}\,.$$

The symmetric root locus is shown in Fig. 1. The arrow is the direction of increasing *R*. One of the the optimal closed loop eigenvalues remains at the origin. The other one varies from –1/1.6 to 0.9 as *R* varies from 0 to infinity.

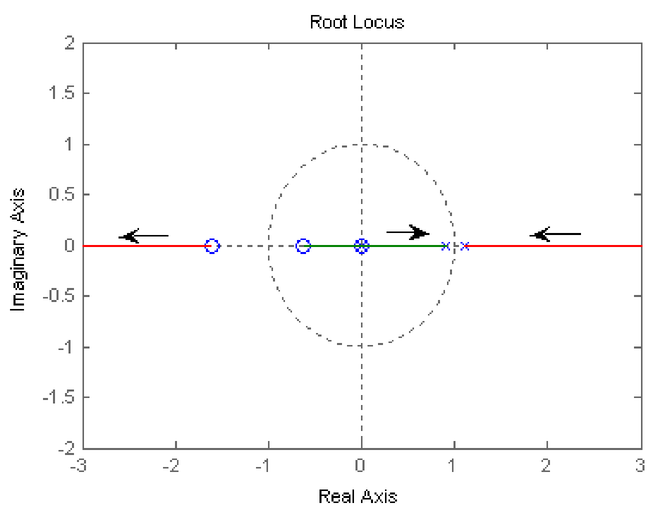

Fig. 1 Symmetric root locus.

[2]

The non-adaptive control law (when all parameters are known) to achieve the tracking objective is given by

$$u(k)=\frac{1}{S(z^{-1})}[-R(z^{-1})y(k)+D'(z^{-1})y_d(k+1)]-\frac{\alpha}{b_0}\sin(\omega k+\varsigma)\,,$$

where $D'(z^{-1})=1+d_1z^{-1}$ and $S'(z^{-1})$ ($S(z^{-1})=b_0S'(z^{-1})$) and $R(z^{-1})$ satisfy the following

Diophantine equation:

$$A(z^{-1})S'(z^{-1}) + z^{-1}R(z^{-1}) = D'(z^{-1}).$$

The solutions of this equation are

$$S'(z^{-1}) = 1 \text{ and } R(z^{-1}) = r_0 = d_1 - a_1.$$

The adaptive control law is base on the reparameterization of the plant:

$$\begin{aligned} D'(z^{-1})y(k) &= \left(A(z^{-1})S'(z^{-1}) + z^{-1}R(z^{-1})\right)y(k) \\ &= A(z^{-1})y(k) + R(z^{-1})y(k-1) \\ &= b_0 u(k-1) + \alpha\sin(\omega k + \varsigma) + r_0 y(k-1) \\ &= b_0 u(k-1) + \alpha\cos(\varsigma)\sin(\omega k) + \alpha\sin(\varsigma)\cos(\omega k) + r_0 y(k-1) \\ &= [b_0 \quad \alpha\cos(\varsigma) \quad \alpha\sin(\varsigma) \quad r_0]\begin{bmatrix} u(k-1) \\ \sin(\omega k) \\ \cos(\omega k) \\ y(k-1) \end{bmatrix} \end{aligned}$$

Define $c_1 = \alpha\cos(\varsigma)$, $c_2 = \alpha\sin(\varsigma)$, $\theta_{ce}^T = [b_0 \quad c_1 \quad c_2 \quad r_0]$, and $\phi_e^T(k-1) = [u(k-1) \quad \sin(\omega k) \quad \cos(\omega k) \quad y(k-1)]$. Then

$$D'(z^{-1})y(k) = \theta_{ce}^T \phi_e(k-1).$$

The controller parameter vector is updated by

$$\hat{\theta}_{ce}(k) = \hat{\theta}_{ce}(k-1) + \frac{F(k-1)\phi_e(k-1)}{1 + \phi_e^T(k-1)F(k-1)\phi_e(k-1)}[D'(z^{-1})y(k) - \hat{\theta}_{ce}^T(k-1)\phi_e(k-1)]$$

$$F(k) = \frac{1}{\lambda_1(k-1)}\left[F(k-1) - \frac{F(k-1)\phi_e(k-1)\phi_e^T(k-1)F(k-1)}{\lambda_1(k-1)/\lambda_2(k-1) + \phi_e^T(k-1)F(k-1)\phi_e(k-1)}\right]$$

with $0 < \lambda_1(k) \le 1$ and $0 \le \lambda_2(k) < 2$. The control signal is implicitly given by

$$D'(z^{-1})y_d(k+1) = \hat{\theta}_{ce}^T(k)\phi_e(k).$$

[3]

The controller transfer function is given by

$$G_c(z^{-1}) = -\frac{R(z^{-1})}{A_d(z^{-1})S(z^{-1})},$$

where $A_d(z^{-1}) = 1 - 0.98z^{-1}$ is the internal model of the disturbance and $S(z^{-1})$ and $R(z^{-1})$ satisfy the following Diophantine equation:

$$A(z^{-1})A_d(z^{-1})S(z^{-1}) + z^{-1}R(z^{-1}) = D(z^{-1}).$$

The degrees of $S(z^{-1})$ and $R(z^{-1})$ are both one and $S(z^{-1})$ is monic. Then we can obtain:

$$S(z^{-1}) = 1 - 0.5224z^{-1} \text{ and } R(z^{-1}) = 0.1024 - 0.0944z^{-1}.$$

Thus, the control law is given by

$$u(k) = -\frac{0.1024 - 0.0944z^{-1}}{(1-0.5224z^{-1})(1-0.98z^{-1})}.$$

Notice that the solution of the Diophantine equation is not unique. Another possible solution is

$$S(z^{-1}) = 1 \text{ and } R(z^{-1}) = -0.42 + 0.94z^{-1} - 0.512z^{-2}.$$

[4]

The spectral density of $w(t)$ can be decomposed as

$$\phi_{ww}(\omega) = \frac{1}{0.5j\omega+1} \cdot \frac{1}{-0.5j\omega+1},$$

which implies that $w(t)$ can be modeled as a white unit variance Gaussian noise, $n(t)$, filtered by $\frac{1}{0.5s+1}$. Then

$$Y(s) = \frac{1}{(Ts+1)(0.5s+1)} N(s) := G(s)N(s),$$

where $N(s)$ is the Laplace transform of $n(t)$. The transfer function, $G(s)$, can be minimally realized as

$$\begin{aligned} &\frac{d}{dt}x(t) = Ax(t) + Bn(t) \\ &y(t) = Cx(t) \end{aligned}$$

with

$$A = \begin{bmatrix} -1/T & 0 \\ 0 & -2 \end{bmatrix}, \; B = \begin{bmatrix} 2/(2T-1) \\ -2/(2T-1) \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

Then we can solve the Lyapunov function,

$$AX_{ss}(0) + X_{ss}(0)A^T = -BB^T,$$

to get the covariance matrix, $X_{ss}(0)$, of $x(t)$ at the steady state:

$$X_{ss}(0) = \begin{bmatrix} \frac{2T}{(2T-1)^2} & -\frac{4T}{(2T-1)^2(2T+1)} \\ -\frac{4T}{(2T-1)^2(2T+1)} & \frac{1}{(2T-1)^2} \end{bmatrix}.$$

The auto-covariance of $x(t)$ is

$$E[x(t)x^T(t+\tau)] = X_{ss}(0)\exp(A^T\tau) \text{ and } E[x(t)x^T(t-\tau)] = \exp(A\tau)X_{ss}(0),$$

for $\tau \geq 0$. Then the auto-covariance of $y(t)$ is

$$\begin{aligned} &E[y(t)y(t+\tau)] = E[y(t)y^T(t+\tau)] = E[Cx(t)x^T(t+\tau)C^T] = CE[x(t)x^T(t+\tau)]C^T \\ &= \begin{bmatrix} 1 & 1 \end{bmatrix} X_{ss}(0) \begin{bmatrix} e^{-\tau/T} & 0 \\ 0 & e^{-2\tau} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{2T}{(2T-1)(2T+1)} e^{-\tau/T} - \frac{1}{(2T-1)(2T+1)} e^{-2\tau} \end{aligned}$$

for $\tau \geq 0$ and $E[y(t)y(t+\tau)] = \dfrac{2T}{(2T-1)(2T+1)} e^{\tau/T} - \dfrac{1}{(2T-1)(2T+1)} e^{2\tau}$, for $\tau < 0$.

[5]

$y_f(t)$ and $u_f(t)$ satisfy the differential equation

$$\ddot{y}_f(t) + a_1\dot{y}_f(t) + a_0 y_f(t) = b_1\dot{u}_f(t) + b_0 u_f(t).$$

Define discrete time signals $y_2(k) = \ddot{y}_f(t)\big|_{t=kT}$, $y_1(k) = \dot{y}_f(t)\big|_{t=kT}$, $y_0(k) = y_f(t)\big|_{t=kT}$, $u_1(k) = \dot{u}_f(t)\big|_{t=kT}$, and $u_0(k) = u_f(t)\big|_{t=kT}$. These signals are known at time *k*. Then we have

$$y_2(k) = -a_1 y_1(k) - a_0 y_0(k) + b_1 u_1(k) + b_0 u_0(k) := \theta^T \phi(k),$$

where $\theta^T = \begin{bmatrix} a_1 & a_0 & b_1 & b_0 \end{bmatrix}$ and $\phi^T(k) = \begin{bmatrix} -y_1(k) & -y_0(k) & u_1(k) & u_0(k) \end{bmatrix}$. So we can use the following discrete-time series-parallel least squares PAA to estimate θ:

$$\hat{\theta}(k+1) = \hat{\theta}(k) + \frac{F(k)\phi(k)}{1+\phi^T(k)F(k)\phi(k)} \varepsilon^o(k+1)$$

$$F(k+1) = F(k) - \frac{F(k)\phi(k)\phi^T(k)F(k)}{1+\phi^T(k)F(k)\phi(k)}$$

$$\varepsilon^o(k+1) = y_2(k+1) - \hat{\theta}^T(k)\phi(k).$$

[6]

Define $\tilde{\theta}(k) = \hat{\theta}(k) - \theta$, $\theta^T = [a_1 \ \cdots \ a_n \ b_0 \ \cdots \ b_m]$, and $\varepsilon(k+1) = y(k+1) - \hat{\theta}^T(k+1)\phi(k)$.

The PAA can be represented by the equivalent feedback loop shown in Fig. 2, where the feedforward linear block is given by

$$G(z^{-1}) = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}}$$

and the feedback nonlinear/time-varying block is described by

$$\tilde{\theta}^T(k+1)\phi(k) = \tilde{\theta}^T(k)\phi(k) + \phi^T(k)F\phi(k)\varepsilon(k+1).$$

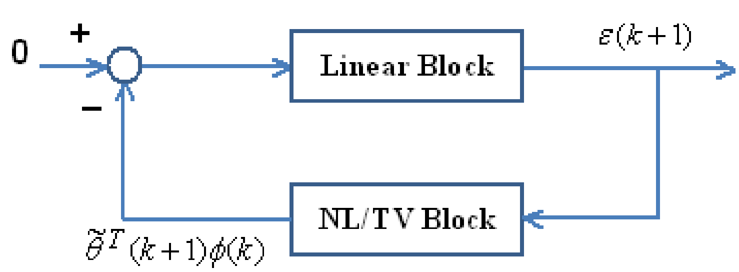

Fig. 2 Equivalent feedback system for the parallel MRAS.

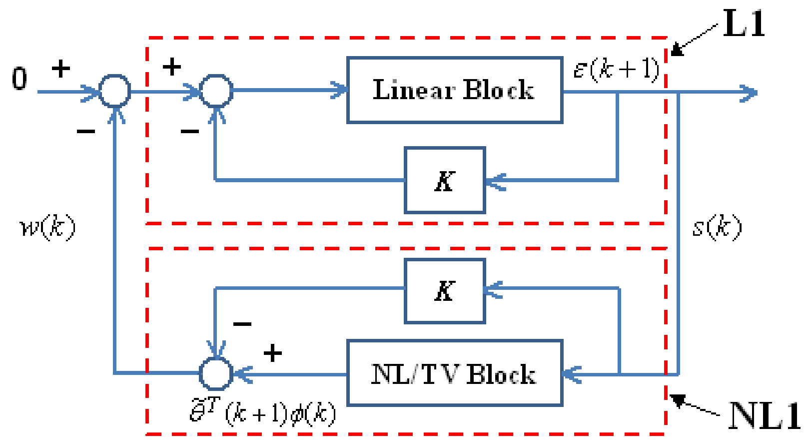

Fig. 3 Another equivalent feedback system for stability analysis

According to the stability proof for the parallel PAA with constant adaptation gain, this system is asymptotically hyperstable if $G(z^{-1})$ is SPR.

Applying "add and substract operations" as given by the hint on the equivalent feedback system, we can get another equivalent feedback system shown in Fig. 3. The transfer function of the feedforward linear block, L1, is given by

$$G_{L1}(z^{-1}) = \frac{\dfrac{1}{1+a_1z^{-1}+a_2z^{-2}+\cdots+a_nz^{-n}}}{1+\dfrac{K}{1+a_1z^{-1}+a_2z^{-2}+\cdots+a_nz^{-n}}} = \frac{1}{1+K+a_1z^{-1}+a_2z^{-2}+\cdots+a_nz^{-n}} = \frac{1/(1+K)}{1+(a_1z^{-1}+a_2z^{-2}+\cdots+a_nz^{-n})/(1+K)}.$$

From example 4 on page HS-3, we know that $G_{L1}(z^{-1})$ is SPR, if $\frac{1}{1+K}\sum_{i=1}^{n}|a_i|<1$. So choose a large $K>0$ that satisfies $\frac{1}{1+K}\sum_{i=1}^{n}|a_i|<1$ to make $G_{L1}(z^{-1})$ SPR. The feedback block, NL1, has input, $s(k)=\varepsilon(k+1)$, and output, $w(k)=\tilde{\theta}^T(k+1)\phi(k)-K\varepsilon(k+1)$. Let's verify that when $\frac{1}{2}\phi^T(k)F\phi(k)>K$, NL1 satisfies Popov inequality:

$$
\begin{aligned}
\sum_{k=0}^{k_1} w(k)s(k) &= \sum_{k=0}^{k_1} \varepsilon(k+1)[\phi^T(k)\tilde{\theta}(k+1) - K\varepsilon(k+1)] \\
&= \sum_{k=0}^{k_1} [\varepsilon(k+1)\phi^T(k)\tilde{\theta}(k+1)] - K\sum_{k=0}^{k_1} \varepsilon^2(k+1) \\
&= \frac{1}{2}\left[F\sum_{k=0}^{k_1}[\varepsilon(k+1)\phi(k)] + \tilde{\theta}(0)\right]^T F^{-1}\left[F\sum_{k=0}^{k_1}[\varepsilon(k+1)\phi(k)] + \tilde{\theta}(0)\right] \\
&\quad + \frac{1}{2}\sum_{k=0}^{k_1}[\phi^T(k)F\phi(k)\varepsilon^2(k+1)] - \frac{1}{2}\tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0) - K\sum_{k=0}^{k_1}\varepsilon^2(k+1) \\
&\ge \frac{1}{2}\sum_{k=0}^{k_1}[\phi^T(k)F\phi(k)\varepsilon^2(k+1)] - K\sum_{k=0}^{k_1}\varepsilon^2(k+1) - \frac{1}{2}\tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0) \\
&= \sum_{k=0}^{k_1}\{[\frac{1}{2}\phi^T(k)F\phi(k) - K]\varepsilon^2(k+1)\} - \frac{1}{2}\tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0) \\
&\ge -\frac{1}{2}\tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0) = -\gamma_0^2
\end{aligned}
$$

for any $k_1 > 0$ and some finite γ_0^2.

Therefore, the PAA is asymptotically hyperstable, when $\frac{1}{2}\phi^T(k)F\phi(k) > K$ and $\frac{1}{1+K}\sum_{i=1}^{n}|a_i| < 1$.