ME 233 Spring 2010
Solution to Homework #8

1. For this problem, it is useful to combine the plant dynamics with the noise dynamics. To do this, note that

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + B_w w(k) \\ &= Ax(k) + Bu(k) + B_w C_w x_w(k). \end{aligned}$$

Thus, if we define

$$\bar{x}(k) := \begin{bmatrix} x(k) \\ x_w(k) \end{bmatrix}, \qquad \bar{A} := \begin{bmatrix} A & B_w C_w \\ 0 & A_w \end{bmatrix}, \qquad \bar{B} := \begin{bmatrix} B \\ 0 \end{bmatrix}, \qquad \bar{B}_w := \begin{bmatrix} 0 \\ B_n \end{bmatrix}$$

we can write the augmented system dynamics as

$$\begin{aligned} \bar{x}(k+1) &= \bar{A}\bar{x}(k) + \bar{B}u(k) + \bar{B}_w \eta(k) \\ E\{\bar{x}(0)\} &= \bar{x}_o := \begin{bmatrix} x_o \\ x_{wo} \end{bmatrix} \\ E\left\{(\bar{x}(0) - \bar{x}_o)(\bar{x}(0) - \bar{x}_o)^T\right\} &= \bar{X}_o := \begin{bmatrix} X_o & 0 \\ 0 & X_{wo} \end{bmatrix}. \end{aligned}$$

To finish the redefinition of the problem, we need to reformulate the LQG cost. Note that if we define

$$\bar{Q} := \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}, \qquad \bar{S} := \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}$$

we can rewrite the LQG cost as

$$J = \frac{1}{2} E\left\{ \bar{x}^T(N)\bar{S}\bar{x}(N) + \sum_{k=0}^{N-1} \bar{x}^T(k)\bar{Q}\bar{x}(k) + u^T(k)Ru(k) \right\}$$

(a) Since assuming that $x(k)$ and $x_w(k)$ are measurable for all k is equivalent to assuming that $\bar{x}(k)$ is measurable for all k, this problem is simply an LQG problem with exactly known state. The optimal control is thus given by

$$u(k) = -K(k+1)\bar{x}(k)$$

where

$$\begin{aligned} K(k) &= \left(\bar{B}^T P(k)\bar{B} + R\right)^{-1} \bar{B}^T P(k)\bar{A} \\ P(k-1) &= \bar{A}^T P(k)\bar{A} + \bar{Q} - \bar{A}^T P(k)\bar{B}\left(\bar{B}^T P(k)\bar{B} + R\right)^{-1} \bar{B}^T P(k)\bar{A} \\ P(N) &= \bar{S}. \end{aligned}$$

The optimal cost is

$$\begin{aligned} J^o &= \frac{1}{2}\bar{x}_o^T P(0)\bar{x}_o + \frac{1}{2}\,\mathrm{trace}\left[P(0)\bar{X}_o\right] + b(0) \\ b(k) &= b(k+1) + \mathrm{trace}\left[\bar{B}_w^T P(k+1)\bar{B}_w \Gamma\right] \\ b(N) &= 0. \end{aligned}$$

(b) In this part, we assume that we only have access to the measurements

$$y(k) = Cx(k) + v(k).$$

If we define

$$\bar{C} := \begin{bmatrix} C & 0 \end{bmatrix}$$

the measurements can be expressed

$$y(k) = \bar{C}\bar{x}(k) + v(k).$$

Thus, this problem is simply an LQG problem. The Kalman filter for this system is given by

$$\begin{aligned} \hat{\bar{x}}^o(k+1) &= \bar{A}\hat{\bar{x}}(k) + \bar{B}u(k) \\ \hat{\bar{x}}(k) &= \hat{\bar{x}}^o(k) + F(k)\left(y(k) - \bar{C}\hat{\bar{x}}^o(k)\right) \end{aligned}$$

where

$$\begin{aligned} F(k) &= M(k)\bar{C}^T\left[\bar{C}M(k)\bar{C}^T + V\right]^{-1} \\ M(k+1) &= \bar{A}Z(k)\bar{A}^T + \bar{B}_w\Gamma\bar{B}_w^T \\ Z(k) &= M(k) - M(k)\bar{C}^T\left[\bar{C}M(k)\bar{C}^T + V\right]^{-1}\bar{C}M(k) \\ M(0) &= \bar{X}_o. \end{aligned}$$

The optimal control is thus given by

$$u(k) = -K(k+1)\hat{\bar{x}}(k)$$

where $K(k)$ is the same as in part (a). The optimal cost is

$$\begin{aligned} J^o = \frac{1}{2}\bar{x}_o^T P(0)\bar{x}_o + \frac{1}{2}\,\mathrm{trace}\left[P(0)\bar{X}_o\right] + \mathrm{trace}\left[SZ(N)\right] + \sum_{j=0}^{N-1}\mathrm{trace}\left[QZ(j)\right] + \hat{b}(0) \\ \hat{b}(k) = \hat{b}(k+1) + \mathrm{trace}\left[F^T(k+1)P(k+1)F(k+1)\left(\bar{C}M(k+1)\bar{C}^T + V\right)\right] \\ \hat{b}(N) = 0 \end{aligned}$$

where $P(k)$ is the same as in part (a).

(c) Since $x(k)$ is measurable whereas $x_w(k)$ is not, we just need to use a Kalman filter to estimate $x_w(k)$ and then we can construct the LQG control using the estimated $\hat{x}_w(k|k)$:

$$u(k) = -K(k+1)\begin{bmatrix} x(k) \\ \hat{x}_w(k) \end{bmatrix}$$

where the optimal LQG gain $K(k)$ is the same as in part (a) and $\hat{x}_w(k) = \hat{x}_w(k|k)$ is the conditional estimate of $x_w(k)$ given the available measurable data up to the instant k. In what follows, we will use three different methods to determine $\hat{x}_w(k)$.

i. **First Method**

Let $y_{we}(k) = x(k) - Ax(k-1) - Bu(k-1) = B_w C_w x_w(k-1)$ which is measurable. Then we have:

$$\begin{aligned} x_w(k+1) &= A_w x_w(k) + B_n\eta(k) \\ y_w(k) &= B_w C_w x_w(k-1). \end{aligned}$$

Here, we will develop a Kalman filter to estimate $\hat{x}_w(k|k)$ based on the output $y_{we}(k)$.

Define $Y_w(k) = \{y_w(k), \cdots, y_w(1)\}$ to be the set of all available measurable data up to instant k. Then we have:

$$\begin{aligned}
\hat{x}_w(k|k) &= E\left\{x_w(k)|Y_w(k)\right\} = E\left\{A_w x_w(k-1) + B_n \eta(k-1)|Y_w(k)\right\} \\
&= E\left\{A_w x_w(k-1) + B_n \eta(k-1)|\{B_w C_w x_w(k-1), \cdots, B_w C_w x_w(0)\}\right\} \\
&= A_w \hat{x}_w(k-1|k)
\end{aligned}$$

where we define

$$\hat{x}_w(k-1|k) = E\left\{x_w(k-1)|Y_w(k)\right\} .$$

Note that for the last equality, we used the fact that $\eta(k-1)$ is uncorrelated with $\{x_w(k-1), \cdots, x_w(0)\}$ and $E(\eta(k-1) = 0$.

We can now use the following Kalman filter to recursively update $\hat{x}_w(k-1|k) = E\left\{x_w(k-1)|Y_w(k)\right\}$ and obtain $\hat{x}_w(k|k) = E\left\{x_w(k)|Y_w(k)\right\}$.

$$\begin{aligned}
\hat{x}_w(k-1|k) &= A_w \hat{x}_w(k-2|k-1) + L(k)\left[y_w(k) - B_w C_w A_w \hat{x}_w(k-1|k-2)\right] \\
\hat{x}_w(k|k) &= A_w \hat{x}_w(k-1|k)
\end{aligned}$$

where

$$\begin{aligned}
L(k) &= M_1(k) C_w^T B_w^T \left[B_w C_w M_1(k) C_w^T B_w^T\right]^{\#} \\
M_1(k+1) &= A_w Z_1(k) A_w^T + B_n \Gamma B_n^T \\
Z_1(k) &= M_1(k) - M_1(k) C_w^T B_w^T \left[B_w C_w M_1(k) C_w^T B_w^T\right]^{\#} C_w B_w M_1(k)
\end{aligned}$$

The initial conditions are

$$\begin{aligned}
\hat{x}_w(0|1) &= x_{w0} + X_{w0} C_w^T B_w^T \left[B_w C_w X_{w0} C_w^T B_w^T\right]^{\#} C_w B_w \left(y_w(1) - B_w C_w x_{w0}\right) \\
Z_1(1) &= X_{w0} C_w^T B_w^T \left[B_w C_w X_{w0} C_w^T B_w^T\right]^{\#} C_w B_w X_{w0}
\end{aligned}$$

Notice that, since the matrix $\left[B_w C_w M_1(k) C_w^T B_w^T\right]$ may be nonsingular it may be necessary to compute its pseudo-inverse $\left[B_w C_w M_1(k) C_w^T B_w^T\right]^{\#}$, as will be explained in the section *Conditional Estimation with singular covariance matrices* below.

We now need to calculate the optimal cost for the designed Kalman filter. From the Kalman filter, we have the following update equation for $\hat{x}_w(k|k)$:

$$\hat{x}_w(k+1|k+1) = A_w \hat{x}_w(k|k) + A_w L(k+1)\left\{y_w(k+1) - B_w C_w \hat{x}_w(k|k)\right\}$$

Then, we have the following update equation for the state $\begin{bmatrix} x(k) \\ \hat{x}_w(k|k) \end{bmatrix}$ used in the LQR control.

$$\begin{aligned}
\begin{bmatrix} x(k+1) \\ \hat{x}_w(k+1|k+1) \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 0 & A_w \end{bmatrix} \begin{bmatrix} x(k) \\ \hat{x}_w(k|k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} y_w(k+1) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ A_w L(k+1) \end{bmatrix} r(k+1) \\
\Rightarrow \begin{bmatrix} x(k+1) \\ \hat{x}_w(k+1|k+1) \end{bmatrix} &= \begin{bmatrix} A & B_w C_w \\ 0 & A_w \end{bmatrix} \begin{bmatrix} x(k) \\ \hat{x}_w(k|k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} I \\ A_w L(k+1) \end{bmatrix} r(k+1) \\
\text{where } r(k+1) &= y_w(k+1) - B_w C_w \hat{x}_w(k|k)
\end{aligned}$$

Thus, the optimal cost is

$$J^o = \frac{1}{2}\bar{x}_o^T P(0)\bar{x}_o + \frac{1}{2}\,\text{trace}\left[P(0)\bar{X}_o\right] + \text{trace}\left[\bar{S}\bar{Z}(N)\right] + \sum_{j=0}^{N-1}\text{trace}\left[\bar{Q}\bar{Z}(j)\right] + \hat{b}(0)$$

$$\hat{b}(k) = \hat{b}(k+1) + \text{trace}\left[\bar{F}^T(k+1)P(k+1)\bar{F}(k+1)\left(B_w C_w M_1(k+1)C_w^T B_w^T\right)\right]$$

$$\hat{b}(N) = 0,\ \bar{Z}(k) = \begin{bmatrix} 0 & 0 \\ 0 & M_1(k+1)\end{bmatrix},\ \bar{F}(k+1) = \begin{bmatrix} I \\ A_w L(k+1)\end{bmatrix}$$

where P_k, $\bar{S}$, $\bar{Q}$ and $\bar{X}_o$ are the same as in part (a).

Conditional Estimation with singular covariance matrices

Here we derive a result for conditioning a Gaussian vector on another Gaussian vector whose covariance is singular (i.e. positive semi-definite). Let X_1 and X_2 be random vectors and assume that $\Lambda_{X_2X_2}$ is singular and has rank p. The eigenvalue decomposition of $\Lambda_{X_2X_2}$ can then be expressed as

$$\Lambda_{X_2X_2} = \begin{bmatrix} U_3 & U_4 \end{bmatrix}\begin{bmatrix} \Sigma & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix} U_3^T \\ U_4^T\end{bmatrix} = U_3\Sigma U_3^T$$

where

$$I = \begin{bmatrix} U_3 & U_4 \end{bmatrix}\begin{bmatrix} U_3^T \\ U_4^T\end{bmatrix} = \begin{bmatrix} U_3^T \\ U_4^T\end{bmatrix}\begin{bmatrix} U_3 & U_4 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I\end{bmatrix} \tag{1}$$

and $\Sigma \in \mathbb{R}^{p\times p}$ is a diagonal matrix which is positive definite (i.e. invertible). We now define

$$\begin{bmatrix} X_3 \\ X_4\end{bmatrix} := \begin{bmatrix} U_3^T \\ U_4^T\end{bmatrix} X_2$$

and note that, because $[U_3\ U_4]$ is invertible, no information is lost when we transform X_2 into X_3 and X_4. In particular, this means that conditioning with respect to the outcome x_2 is equivalent to conditioning with respect to the outcomes x_3 and x_4. To perform this conditioning, we first note that using Eq. (1) gives

$$E\left\{\tilde{X}_4\tilde{X}_4^T\right\} = U_4^T E\left\{\tilde{X}_2\tilde{X}_2^T\right\} U_4 = U_4^T U_3\Sigma U_3 U_4^T = 0$$

which implies that X_4 **is deterministic**. As a result

$$E\left\{\begin{bmatrix}\tilde{X}_1 \\ \tilde{X}_3\end{bmatrix}\tilde{X}_4^T\right\} = E\left\{\begin{bmatrix}\tilde{X}_1 \\ \tilde{X}_3\end{bmatrix}\right\}\times 0 = 0.$$

This means that X_4 is uncorrelated with X_1 and X_3, since it contains no information which is useful in estimating either X_1 or X_3. Thus, we can see that conditioning X_1 on the outcomes x_3 and x_4 is equivalent to conditioning X_1 on the outcome x_3 only.

Pseudo-inverses

Before proceeding further we review some basic concepts of pseudo-inverses. Give the matrix $A \in R^{n\times m}$, the pseudoinverse of A, denoted by $A^{\#}$, satisfies the following properties:

$$A\,A^{\#}\,A = A \quad \text{and} \quad A^{\#}\,AA^{\#} = A^{\#}$$
$$(A\,A^{\#})^T = (A\,A^{\#}) \quad \text{and} \quad (A^{\#}\,A)^T = (A^{\#}\,A)$$

The following two formulas for computing pseudo-inverses are useful

A. Given the singular value decomposition of $A \in R^{n\times m}$,

$$A = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \qquad \Sigma = \mathrm{diag}(\sigma_1, \cdots \sigma_p) \succ 0$$

then

$$A^{\#} = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix}^T \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} \tag{2}$$

B. Given the matrix $A = B\,C$ of rank p, where $B \in R^{n\times p}$ and $C \in R^{p\times m}$,

$$A^{\#} = C^T\,(C\,C^T)^{-1}\,(B^T\,B)^{-1}\,B^T$$

We can now use Eq. (2) to compute the pseudo-inverse of the covariance matrix $\Lambda_{X_2X_2}$. Noting that the pseudoinverse of $\Lambda_{X_2X_2}$ is given by

$$\Lambda_{X_2X_2}^{\#} = \begin{bmatrix} U_3 & U_4 \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_3^T \\ U_4^T \end{bmatrix}$$

and

$$\Lambda_{X_1X_3} = E\left\{\tilde{X}_1\tilde{X}_3^T\right\} = E\left\{\tilde{X}_1\tilde{X}_2^T\right\}U_3 = \Lambda_{X_1X_2}U_3$$
$$\Lambda_{X_3X_3} = E\left\{\tilde{X}_3\tilde{X}_3^T\right\} = U_3^T E\left\{\tilde{X}_2\tilde{X}_2^T\right\}U_3 = U_3^T\Lambda_{X_2X_2}U_3 = \Sigma$$
$$\begin{bmatrix} U_3 & U_4 \end{bmatrix} \begin{bmatrix} x_3 - m_{X_3} \\ x_4 - m_{X_4} \end{bmatrix} = \begin{bmatrix} U_3 & U_4 \end{bmatrix} \begin{bmatrix} U_3^T \\ U_4^T \end{bmatrix} (x_2 - m_{X_2}) = x_2 - m_{X_2}$$

we arrive at the conclusion that

$$\begin{aligned}
(\hat{x}_1)_{|x_2} &= (\hat{x}_1)_{|x_3,x_4} = (\hat{x})_{|x_3} \\
&= m_{X_1} + (\Lambda_{X_1X_2}U_3)\,\Sigma^{-1}\,(x_3 - m_{X_3}) \\
&= m_{X_1} + \Lambda_{X_1X_2}\begin{bmatrix} U_3 & U_4 \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} U_3^T \\ U_4^T \end{bmatrix} \begin{bmatrix} U_3 & U_4 \end{bmatrix} \right) \begin{bmatrix} x_3 - m_{X_3} \\ x_4 - m_{X_4} \end{bmatrix} \\
&= m_{X_1} + \Lambda_{X_1X_2}\Lambda_{X_2X_2}^{\#}\,(x_2 - m_{X_2}) \\
\Lambda_{(X_{1|x_2},X_{1|x_2})} &= \Lambda_{(X_{1|x_3,x_4},X_{1|x_3,x_4})} = \Lambda_{(X_{1|x_3},X_{1|x_3})} \\
&= \Lambda_{X_1X_1} - \Lambda_{X_1X_2}U_3\Sigma^{-1}U_3^T\Lambda_{X_2X_1} \\
&= \Lambda_{X_1X_1} - \Lambda_{X_1X_2}\begin{bmatrix} U_3 & U_4 \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_3^T \\ U_4^T \end{bmatrix} \Lambda_{X_2X_1} \\
&= \Lambda_{X_1X_1} - \Lambda_{X_1X_2}\Lambda_{X_2X_2}^{\#}\Lambda_{X_2X_1}.
\end{aligned}$$

Thus, we can see that we just need to replace inverses with pseudoinverses in order for the conditioning formulas to work when X_2 has a singular covariance matrix.

ii. **Second Method**

Let $y_{we}(k) = x(k) - Ax(k-1) - Bu(k-1) = B_wC_wx_w(k-1)$ and $x_{we} = \begin{bmatrix} x_w(k) \\ x_w(k-1) \end{bmatrix}$. Then we have:

$$\begin{aligned}
x_{we}(k+1) &= A_{we}x_{we}(k) + B_{we}\eta(k) \\
y_{we}(k) &= C_{we}x_{we}(k).
\end{aligned}$$

where

$$A_{we} := \begin{bmatrix} A_w & 0 \\ I & 0 \end{bmatrix}, \qquad B_{we} := \begin{bmatrix} B_n \\ 0 \end{bmatrix}, \qquad C_{we} := \begin{bmatrix} 0 & B_w C_w \end{bmatrix}$$

Then we can use Kalman filter to estimate $\hat{x}_w(k|k)$:

$$\begin{aligned} \hat{x}^o_{we}(k+1) &= A_{we}\hat{x}_{we}(k) \\ \hat{x}_{we}(k) &= \hat{x}^o_{we}(k) + F(k)\left(y_{we}(k) - C_{we}\hat{x}^o_{we}(k)\right) \end{aligned}$$

where

$$\begin{aligned} F(k) &= M(k)C_{we}^T\left[C_{we}M(k)C_{we}^T\right]^{\#} \\ M(k+1) &= A_{we}Z(k)A_{we}^T + B_{we}\Gamma B_{we}^T \\ Z(k) &= M(k) - M(k)C_{we}^T\left[C_{we}M(k)C_{we}^T\right]^{\#} C_{we}M(k) \end{aligned}$$

The initial conditions are

$$\begin{aligned} \hat{x}_{we}(1) &= \begin{bmatrix} \hat{x}_w(1|1) \\ \hat{x}_w(0|1) \end{bmatrix} = \begin{bmatrix} A_w \\ I \end{bmatrix}\left(x_{w0} + X_{w0}C_w^T B_w^T\left[B_w C_w X_{w0} C_w^T B_w^T\right]^{\#} C_w B_w\left(y_w(1) - B_w C_w x_{w0}\right)\right) \\ Z(1) &= \begin{bmatrix} A_w \Lambda A_w^T & A_w\Lambda \\ \Lambda A_w^T & \Lambda \end{bmatrix} \\ \Lambda &= X_{w0}C_w^T B_w^T\left[B_w C_w X_{w0} C_w^T B_w^T\right]^{\#} C_w B_w X_{w0} \end{aligned}$$

Thus $\hat{x}_{we}(k) = \begin{bmatrix} \hat{x}_w(k|k) \\ \hat{x}_w(k-1|k) \end{bmatrix}$, which provides the estimation of x_w for the LQG control.
Notice that with some linear algebra computation, we can show that the second method is equivalent to the first method.
In addition, for the above methods, we use $\hat{x}_w(0) = x_{w0}$.

iii. **Third Method**

In this part, we assume that $x(k)$ is measurable whereas $x_w(k)$ is not. One way to utilize this information is to assume that we have the measurements

$$\begin{aligned} y_2(k) &= \begin{bmatrix} I & 0 \end{bmatrix}\bar{x} + v_2(k) \\ v_2(k) &= 0 \end{aligned}$$

i.e. we can measure $x(k)$ with no measurement noise. Defining

$$\bar{C}_2 := \begin{bmatrix} I & 0 \end{bmatrix}$$

so that

$$y_2(k) = \bar{C}_2\bar{x}(k) + v_2(k)$$

the Kalman filter for this system is given by

$$\hat{\bar{x}}^o(k+1) = \bar{A}\hat{\bar{x}}(k) + \bar{B}u(k) \tag{3a}$$

$$\hat{\bar{x}}(k) = \hat{\bar{x}}^o(k) + F(k)\left(y_2(k) - \bar{C}_2\hat{\bar{x}}^o(k)\right) \tag{3b}$$

where

$$F(k) = M(k)\bar{C}_2^T\left[\bar{C}_2 M(k)\bar{C}_2^T\right]^{-1} \tag{4a}$$

$$M(k+1) = \bar{A}Z(k)\bar{A}^T + \bar{B}_w\Gamma\bar{B}_w^T \tag{4b}$$

$$Z(k) = M(k) - M(k)\bar{C}_2^T\left[\bar{C}_2 M(k)\bar{C}_2^T\right]^{-1}\bar{C}_2 M(k) \tag{4c}$$

$$M(0) = \bar{X}_o. \tag{4d}$$

and we have assumed that $\bar{C}_2 M(k)\bar{C}_2^T$ is invertible for all k. (Note that the values of $F(k)$, $M(k)$, and $Z(k)$ are different than in the previous part.) Under this invertibility assumption, the optimal control is thus given by

$$u(k) = -K(k+1)\hat{\bar{x}}(k)$$

where $K(k)$ is the same as in part (a). The optimal cost is

$$\begin{aligned}
J^o &= \frac{1}{2}\bar{x}_o^T P(0)\bar{x}_o + \frac{1}{2}\,\text{trace}\left[P(0)\bar{X}_o\right] + \text{trace}\left[\bar{S}Z(N)\right] + \sum_{j=0}^{N-1} \text{trace}\left[\bar{Q}Z(j)\right] + \hat{b}(0) \\
\hat{b}(k) &= \hat{b}(k+1) + \text{trace}\left[F^T(k+1)P(k+1)F(k+1)\left(\bar{C}_2 M(k+1)\bar{C}_2^T\right)\right] \\
\hat{b}(N) &= 0
\end{aligned}$$

where P_k is the same as in part (a).
To gain intuition about what the Kalman filter is doing in this case, first partition

$$M(k) = \begin{bmatrix} M_{11}(k) & M_{12}(k) \\ M_{12}^T(k) & M_{22}(k) \end{bmatrix}$$

and assume that $M_{11}(k)$ is invertible. This allows us to write Eq. (4a) as

$$F(k) = \begin{bmatrix} M_{11}(k) \\ M_{12}^T(k) \end{bmatrix} M_{11}^{-1}(k) = \begin{bmatrix} I \\ M_{12}^T M_{11}^{-1} \end{bmatrix}$$

which in turn allows us to write Eq. (3b) as

$$\begin{aligned}
\begin{bmatrix} \hat{x}(k) \\ \hat{x}_w(k) \end{bmatrix} &= \begin{bmatrix} \hat{x}^o(k) \\ \hat{x}_w^o(k) \end{bmatrix} + \begin{bmatrix} I \\ M_{12}^T M_{11}^{-1} \end{bmatrix} (x(k) - \hat{x}^o(k)) \\
&= \begin{bmatrix} x(k) \\ \hat{x}_w^o(k) + M_{12}^T M_{11}^{-1}(x(k) - \hat{x}^o(k)) \end{bmatrix}.
\end{aligned}$$

In particular, note that $\hat{x}(k) = x(k)$. We therefore expect the a posteriori estimation error of $x(k)$ to have zero covariance. This is verified by writing Eq. (4c) as

$$\begin{aligned}
Z(k) &= \begin{bmatrix} M_{11}(k) & M_{12}(k) \\ M_{12}^T(k) & M_{22}(k) \end{bmatrix} - \begin{bmatrix} M_{11}(k) \\ M_{12}^T(k) \end{bmatrix} M_{11}^{-1}(k) \begin{bmatrix} M_{11}(k) & M_{12}(k) \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & M_{22}(k) - M_{12}^T(k)M_{11}^{-1}(k)M_{12}(k) \end{bmatrix}.
\end{aligned}$$

Using this fact, it is possible to simplify the LQG cost formula by noting that

$$\begin{aligned}
\bar{Q}Z(j) &= 0, \quad \forall j \\
\bar{S}Z(N) &= 0.
\end{aligned}$$

If the covariance matrices $\bar{C}_2 M(k)\bar{C}_2^T$ and $M(k)$ are not invertible we need to replace their inverses in the above expression by their respective pseudo-inverses. With this in mind, the Kalman filter for this system can be expressed as

$$\begin{aligned}
\hat{\bar{x}}^o(k+1) &= \bar{A}\hat{\bar{x}}(k) + \bar{B}u(k) \\
\hat{\bar{x}}(k) &= \hat{\bar{x}}^o(k) + F(k)\left(y_2(k) - \bar{C}_2\hat{\bar{x}}^o(k)\right)
\end{aligned}$$

where

$$\begin{aligned}
F(k) &= M(k)\bar{C}_2^T\left[\bar{C}_2 M(k)\bar{C}_2^T\right]^{\#} \\
M(k+1) &= \bar{A}Z(k)\bar{A}^T + \bar{B}_w \Gamma \bar{B}_w^T \\
Z(k) &= M(k) - M(k)\bar{C}_2^T\left[\bar{C}_2 M(k)\bar{C}_2^T\right]^{\#}\bar{C}_2 M(k) \\
M(0) &= \bar{X}_o.
\end{aligned}$$

As before, we will now manipulate these equations to give intuition into what the Kalman filter is doing in this case. First note that if we partition

$$Z(k) = \begin{bmatrix} Z_{11}(k) & Z_{12}(k) \\ Z_{12}(k) & Z_{22}(k) \end{bmatrix}$$

we see that

$$\begin{bmatrix} Z_{11}(k) & Z_{12}(k) \\ Z_{12}^T(k) & Z_{22}(k) \end{bmatrix} = \begin{bmatrix} M_{11}(k) & M_{12}(k) \\ M_{12}^T(k) & M_{22}(k) \end{bmatrix} - \begin{bmatrix} M_{11}(k) \\ M_{12}^T(k) \end{bmatrix} M_{11}^{\#}(k) \begin{bmatrix} M_{11}(k) & M_{12}(k) \end{bmatrix}.$$

Since $M_{11}(k) = M_{11}(k)M_{11}^{\#}(k)M_{11}(k)$ by the properties of the pseudoinverse, $Z_{11}(k) = 0$. This means that the a posteriori estimation error of $x(k)$ has zero covariance, i.e. it is deterministic. Thus, $Z_{12}(k)$ must also be 0, which implies that

$$\begin{bmatrix} Z_{11}(k) & Z_{12}(k) \\ Z_{12}^T(k) & Z_{22}(k) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & M_{22}(k) - M_{12}^T(k)M_{11}^{\#}(k)M_{12}(k) \end{bmatrix}.$$

Because $\tilde{x}(k)$ is deterministic and estimation errors in Kalman filtering are unbiased (i.e. their mean is zero), this implies that $\tilde{x}(k) = 0$. Therefore

$$\tilde{x}(k) = x(k) - \hat{x}(k) \quad \Rightarrow \quad \hat{x}(k) = x(k).$$

This means that the a posteriori estimate of $x(k)$ always takes the value $x(k)$.

2. (a) Using Newtonian dynamics, a model of the system is given by

$$\begin{aligned} m_s\ddot{z}_1 &= w - k_s z_1 - b_s\dot{z}_1 + kz_2 + b\dot{z}_2 - K_e u \\ &= w + (-k_s - k)z_1 + (-b_s - b)\dot{z}_1 + kz + b\dot{z} - K_e u \\ m\ddot{z} &= K_e u - kz_2 - b\dot{z}_2 \\ &= K_e u + kz_1 + b\dot{z}_1 - kz - b\dot{z} \end{aligned} \tag{5}$$

where the subscript 's' refers to parameters for the suspension model. For convenience, we will define

$$\begin{bmatrix} a_{s1} \\ a_{s2} \\ n_s \end{bmatrix} := \frac{1}{m_s}\begin{bmatrix} k_s \\ b_s \\ 1 \end{bmatrix} = \begin{bmatrix} 4.52 \times 10^9 & N/(m \cdot kg) \\ 134 & N/(m \cdot s \cdot kg) \\ 1.367 \times 10^4 & kg^{-1} \end{bmatrix}$$

$$\begin{bmatrix} a_{m1} \\ a_{m2} \\ n \end{bmatrix} := \frac{1}{m}\begin{bmatrix} k \\ b \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \times 10^8 & N/(m \cdot kg) \\ 2500 & N/(m \cdot s \cdot kg) \\ 5 \times 10^5 & kg^{-1} \end{bmatrix}.$$

Now note that Eq. (5) can be expressed

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ \dot{z}_1 \\ z \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -a_{s1} - n_s k & -a_{s2} - n_s b & n_s k & n_s b \\ 0 & 0 & 0 & 1 \\ a_{m1} & a_{m2} & -a_{m1} & -a_{m2} \end{bmatrix}\begin{bmatrix} z_1 \\ \dot{z}_1 \\ z \\ \dot{z} \end{bmatrix} + \begin{bmatrix} 0 \\ -n_s K_e \\ 0 \\ nK_e \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ n_s \\ 0 \\ 0 \end{bmatrix} w(t)$$

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ \dot{z}_1 \\ z \\ \dot{z} \end{bmatrix} + v.$$

Thus, if we define

$$\bar{x} := \begin{bmatrix} z_1 \\ \dot{z}_1 \\ z \\ \dot{z} \end{bmatrix} \quad \bar{A} := \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4.5255 \times 10^9 & -202.35 & 5.468 \times 10^6 & 68.35 \\ 0 & 0 & 0 & 1 \\ 2 \times 10^8 & 2500 & -2 \times 10^8 & -2500 \end{bmatrix} \quad \bar{B} := \begin{bmatrix} 0 \\ -0.34175 \\ 0 \\ 12.5 \end{bmatrix}$$

$$\bar{B}_w := \begin{bmatrix} 0 \\ 1.367 \times 10^4 \\ 0 \\ 0 \end{bmatrix} \qquad \bar{C} := \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$

the system can be realized as

$$\frac{d}{dt}\bar{x} = \bar{A}\bar{x} + \bar{B}u + \bar{B}_w w$$
$$y = \bar{C}\bar{x} + v.$$

(b) Since $C_q = \bar{C}$ in this case, we draw the root locus of $G(-s)G(s)$, where

$$G(s) = \bar{C}\left(sI - \bar{A}\right)^{-1}\bar{B}_u.$$

This is shown in Figure 1. From examining this plot, it is evident that in order to affect the closed-loop system poles, the gain on the root locus plot needs to be on the order of at least 10^{11}, which means that R needs to be smaller than 10^{-11}. However, even if we do make R that small, it will not appreciably affect the two closed-loop poles that are nearly cancelled with zeros. Since these are the slowest poles, they are the factor that limits performance. Since R has almost no effect on these poles, we could make R anything we want without significantly changing the closed loop LQR performance.

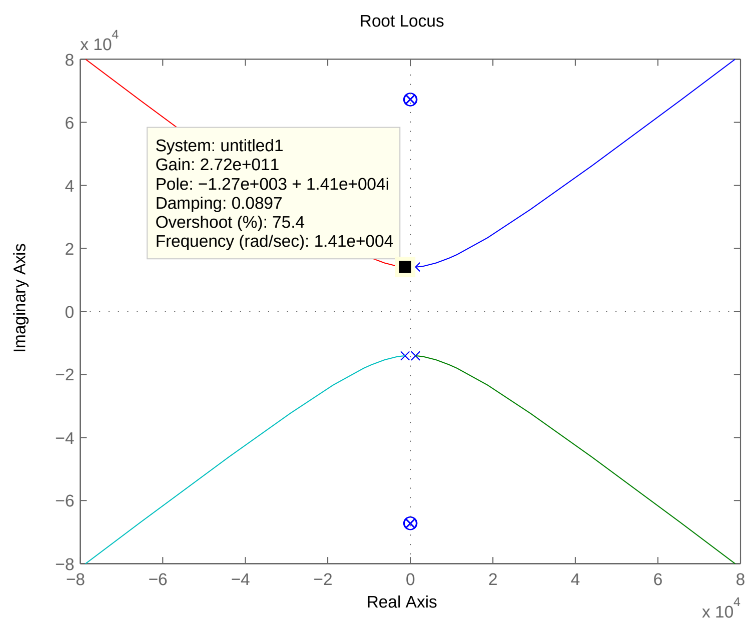

Figure 1: Locus of the closed-loop LQR system eigenvalues

(c) To solve this problem, we will use the MATLAB commands `kalman` and `lqr`. We will not use `lqg` because it does not return the Riccati equation solutions. Thus, we use the following code:

```
>> sys_KF = ss(Abar, [Bubar Bwbar], Cbar, [0 0]);
>> sys_LQR = ss(Abar, Bubar, Cbar, 0);
>> [Kest,L,M] = kalman(sys_KF, W, V);
>> [K,P,E] = lqr(sys_LQR, Cbar'*Cbar, R);
```

Thus, since the optimal stationary LQG cost is given by $\mathrm{Tr}\{P[\bar{B}KM + \bar{B}_w W \bar{B}_w^T]\}$, we compute the optimal cost using the code

```
>> Js0 = trace(P * (Bubar*K*M + Bwbar*W*Bwbar') )
```

which gives

$$J_s^o = 7.8156267 \times 10^{-14} \text{ for } R = 1.$$
$$J_s^o = 7.8156222 \times 10^{-14} \text{ for } R = 10^{-10}.$$

(d) **Proof** Note that P is the only matrix in this expression that varies as R is decreased. First of all, note that $P \succeq 0$ for any value of R. Thus, for any R, we can decompose $P = S_P^T S_P$, $M = S_M^T S_M$, and $V = S_V^T S_V$. Using this, we see that

$$\begin{aligned} S_M C_Q^T \left(S_M C_Q^T\right)^T &\succeq 0 \\ \Rightarrow \text{trace}\left\{S_M C_Q^T C_Q S_M^T\right\} &\geq 0 \\ \Rightarrow \text{trace}\left\{S_M C_Q^T C_Q S_M^T\right\} &= \text{trace}\left\{C_Q^T C_Q S_M^T S_M\right\} = \text{trace}\left\{C_Q^T C_Q M\right\} \geq 0 \end{aligned}$$

and

$$\begin{gathered} \left\|S_P L S_V^T x\right\|^2 \geq 0, \quad \forall x \\ \Leftrightarrow x^T S_V L^T S_P^T S_P L S_V^T x \geq 0, \quad \forall x \\ \Leftrightarrow S_V L^T P L S_V^T \succeq 0 \\ \Leftrightarrow \text{trace}\left\{S_V L^T P L S_V^T\right\} \geq 0 \\ \Leftrightarrow \text{trace}\left\{P L V L^T\right\} \geq 0. \end{gathered}$$

We thus conclude that for any positive R

$$J_s^o \geq \text{trace}\left\{\bar{C}^T \bar{C} M\right\} = 7.7945318 \times 10^{-14}.$$

Note that this lower bound on the cost is very close to the value obtained by setting $R = 1$ and $R = 10^{-10}$.

Now we examine the properties of P as R is varied. Consider the system

$$\begin{gathered} x(k+1) = Ax(k) + Bu(k) \\ x(0) = x_o \end{gathered}$$

and the two cost functions

$$\begin{aligned} J_1 &= \frac{1}{2} \lim_{N\to\infty} \sum_{k=0}^{N-1} \left[x^T(k) Q x(k) + u^T(k) R_1 u(k)\right] \\ J_2 &= \frac{1}{2} \lim_{N\to\infty} \sum_{k=0}^{N-1} \left[x^T(k) Q x(k) + u^T(k) R_2 u(k)\right] \end{aligned}$$

where $u(k)$ and x_o are deterministic and $R_1 > R_2$. Recall that finding the control laws that minimize these costs are standard LQR problems with solutions

$$\begin{gathered} u_i(k) = -K_i x(k) \\ K_i = \left[B^T P_i B + R_i\right]^{-1} B^T P_i A \\ P_i = A^T P_i A + Q - A^T P_i B \left[B^T P_i B + R_i\right]^{-1} B^T P_i A \\ J_i^o = x_0^T P_i x_0 \end{gathered}$$

for $i = 1, 2$. Now note that for any given control sequence and initial condition

$$J_1 - J_2 = \frac{1}{2} \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \left[u^T(k) \left(R_1 - R_2\right) u(k)\right] \geq 0.$$

If we now consider the control sequence that minimizes J_1 for the initial condition x_o and define its cost with respect to J_2 as $\bar{J}_2$, we see that $J_1^o - \bar{J}_2 \geq 0$. Thus,

$$\begin{gathered} 0 \leq J_2^o \leq \bar{J}_2 \leq J_1^o \\ \Rightarrow 0 \leq x_o^T P_2 x_o \leq x_o^T P_1 x_o. \end{gathered}$$

Since x_o is arbitrary, this implies that

$$\begin{aligned} x_o^T (P_1 - P_2) x_o \geq 0, \quad \forall x_o \\ \Rightarrow P_1 - P_2 \succeq 0 \\ \Rightarrow \text{trace}\left\{(P_1 - P_2) LVL^T\right\} \geq 0 \\ \Rightarrow \text{trace}\left\{P_1 LVL^T\right\} \geq \text{trace}\left\{P_2 LVL^T\right\}. \end{aligned}$$

Since these P's come from the same Riccati equation used in LQG controller design, this means that as R is decreased, the LQG cost can not increase. This justifies estimating J_s^o as $R \to 0$ by the lower bound on J_s^o. Now we examine the limiting behavior of P as $R \to 0$. For a general system, since the function $x_o^T P x_o$ is a lower bounded and is monotonically nonincreasing as R is decreased, it must have a limit as $R \to 0$. Since x_o is arbitrary, this implies that P must have a limit as $R \to 0$. For a general system $P \nrightarrow 0$ as $R \to 0$. However, since this system is a minimum phase SISO system, it can be shown that $P \to 0$, which implies that the LQG cost converges to the lower bound (which corresponds to setting $P = 0$) as $R \to 0$. For a detailed proof that $P \to 0$ as $R \to 0$, refer to:

Kwakernaak, H.; Sivan, R., "The maximally achievable accuracy of linear optimal regulators and linear optimal filters," *Automatic Control, IEEE Transactions on*, vol.17, no.1, pp. 79-86, Feb 1972.

(e) For the simplified model, we ignore the effect of the microacuator control effort $u(t)$ and motion on the suspension. Thus, the dynamics of the suspension in Eq. (5) changes to be:

$$m_s \ddot{z}_1 = w - k_s z_1 - b_s \dot{z}_1 \tag{6}$$

Then we need to change the $\bar{A}$ matrix and the $\bar{B}_u$ matrix for the simplified model:

$$\begin{aligned} \bar{A} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -a_{s1} & -a_{s2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ a_{m1} & a_{m2} & -a_{m1} & -a_{m2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4.52 \times 10^9 & -134 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 2 \times 10^8 & 2500 & -2 \times 10^8 & -2500 \end{bmatrix} \\ \bar{B}_u &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ nK_e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 12.5 \end{bmatrix} \end{aligned}$$

Note the first two states related to the suspension are uncontrollable but stable. Thus there is pole-zero cancelation for the system $G(s) = \bar{C}\left(sI - \bar{A}\right)^{-1} \bar{B}_u$. The corresponding root locus is shown in Figure 2.

The LQG design results for both original model and simplified model are listed in Table 1. In the table, E represents the eigenvalues of the closed-loop A matrix with the designed LQG controllers. From the results we can see the two different values of R almost have no effect on the closed loop poles, which is expected from the root locus plots in Figure 1 and Figure 2. However, with the smaller R, we have the higher LQR control gain K. For the same value of R, the original model and the simplified model achieve the similar closed-loop eigenvalues. For the original model, the first four closed-loop poles are very closed to the open loop zeros. For the simplified model, the first four closed-loop poles are exactly canceled by the zeros.

In addition, the closed-loop poles for applying the control designed using the simplified to the original model are also listed in the last two columns of Table 1. Obviously, the closed-loop poles are almost the same, which implies that it is reasonable to design the control by using the simplified model.

The corresponding optimal cost function changes to be which gives

$$\begin{aligned} J_s^o &= 9.8598205 \times 10^{-14} \text{ for } R = 1. \\ J_s^o &= 9.8598199 \times 10^{-14} \text{ for } R = 10^{-10}. \end{aligned}$$

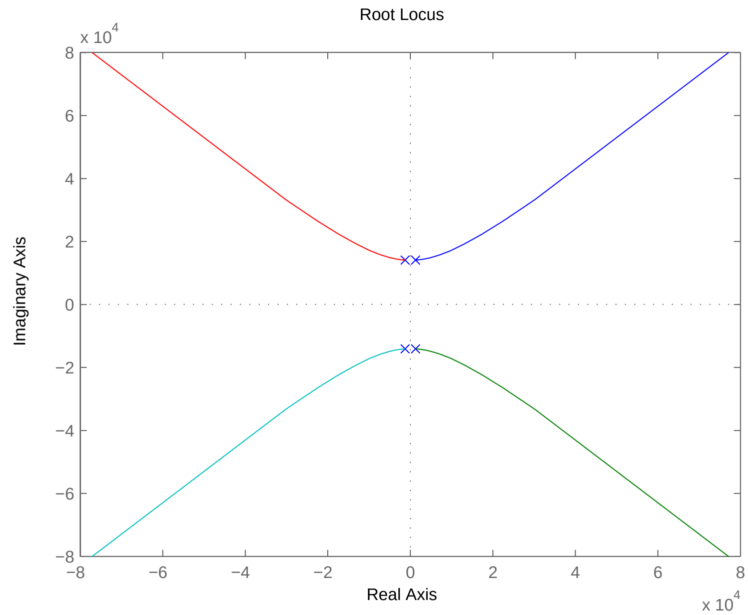

Figure 2: Locus of the closed-loop LQR system eigenvalues for the simplified model

And

$$J_s^o \to \text{trace}\left\{\bar{C}^T\bar{C}M\right\} = 9.7867158 \times 10^{-14} \text{ as } R \to 0.$$

3. (a) To begin, we verify the relevant stabilizability and detectability conditions. For the existence of the optimal stationary LQR controller, we need $[A, B]$ to be stabilizable and $[A, C_Q]$ to be detectable where C_Q for this cost function is $\sqrt{\rho}[1\ 1]$. The controllability matrix for $[A, B]$ and the observability matrix for $[A, C_Q]$ are respectively given by

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = m\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad \mathcal{O} = \begin{bmatrix} C_Q \\ C_Q A \end{bmatrix} = \sqrt{\rho}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Thus, for all $\rho, m > 0$, these matrices are full rank, which implies that the optimal LQR controller exists. The optimal controller is given by

$$u(t) = -K_{LQ}x(t)$$
$$K_{LQ} = R^{-1}B^T P$$

where P is the unique positive semi-definite solution to

$$0 = A^T P + PA + Q - PBR^{-1}B^T P.$$

In this problem, we are asked to verify that

$$K_{LQ} = \alpha \begin{bmatrix} 1 & 1 \end{bmatrix} \tag{7}$$

and find the value of α. Rather than trying to solve the Riccati equation for P and show that the solution gives the desired value of K_{LQ}, we will instead show that K_{LQ} is the optimal gain in another way.

Suppose that the optimal controller gain is given by Eq. (7). This implies that the Riccati equation can be written

$$\begin{aligned} 0 &= A^T P + PA + Q - \left(R^{-1}B^T P\right)^T R\left(R^{-1}B^T P\right) \\ &= A^T P + PA + Q - K_{LQ}^T R K_{LQ} \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \rho\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \alpha^2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \end{aligned}$$

	Original model		Simplified model		Original model	
R	1	10^{-10}	1	10^{-10}	1	10^{-10}
L	$\begin{bmatrix} -26.6504 \\ -1.642 \times 10^6 \\ 4.6955 \\ 11.0239 \end{bmatrix}$	$\begin{bmatrix} -26.6504 \\ -1.642 \times 10^6 \\ 4.6955 \\ 11.0239 \end{bmatrix}$	$\begin{bmatrix} -41.3595 \\ -2.5390 \times 10^6 \\ 5.8956 \\ 17.3791 \end{bmatrix}$	$\begin{bmatrix} -41.3595 \\ -2.5390 \times 10^6 \\ 5.8956 \\ 17.3791 \end{bmatrix}$	$\begin{bmatrix} -41.3595 \\ -2.5390 \times 10^6 \\ 5.8956 \\ 17.3791 \end{bmatrix}$	$\begin{bmatrix} -41.3595 \\ -2.5390 \times 10^6 \\ 5.8956 \\ 17.3791 \end{bmatrix}$
K^T	$\begin{bmatrix} -3.2586 \times 10^{-8} \\ 5.7951 \times 10^{-13} \\ 3.1250 \times 10^{-8} \\ 1.2531 \times 10^{-11} \end{bmatrix}$	$\begin{bmatrix} -325.7475 \\ 0.0058 \\ 312.4969 \\ 0.1253 \end{bmatrix}$	$\begin{bmatrix} -3.2546 \times 10^{-8} \\ 5.7954 \times 10^{-13} \\ 3.1250 \times 10^{-8} \\ 1.2500 \times 10^{-11} \end{bmatrix}$	$\begin{bmatrix} -325.3537 \\ 0.0058 \\ 312.4969 \\ 0.1250 \end{bmatrix}$	$\begin{bmatrix} -3.2546 \times 10^{-8} \\ 5.7954 \times 10^{-13} \\ 3.1250 \times 10^{-8} \\ 1.2500 \times 10^{-11} \end{bmatrix}$	$\begin{bmatrix} -325.3537 \\ 0.0058 \\ 312.4969 \\ 0.1250 \end{bmatrix}$
E	$\begin{bmatrix} -104 \pm 67272i \\ -105 \pm 67272i \\ -1247 \pm 14078i \\ -1248 \pm 14078i \end{bmatrix}$	$\begin{bmatrix} -104 \pm 67272i \\ -105 \pm 67272i \\ -1247 \pm 14078i \\ -1248 \pm 14078i \end{bmatrix}$	$\begin{bmatrix} -67 \pm 67231i \\ -69 \pm 67272i \\ -1250 \pm 14087i \\ -1251 \pm 14087i \end{bmatrix}$	$\begin{bmatrix} -67 \pm 67231i \\ -69 \pm 67272i \\ -1250 \pm 14087i \\ -1251 \pm 14087i \end{bmatrix}$	$\begin{bmatrix} -104 \pm 67272i \\ -105 \pm 67272i \\ -1247 \pm 14078i \\ -1248 \pm 14078i \end{bmatrix}$	$\begin{bmatrix} -104 \pm 67272i \\ -105 \pm 67272i \\ -1247 \pm 14078i \\ -1248 \pm 14078i \end{bmatrix}$

Table 1: Summary of LQG design for two models with different values of R

Note that we have exploited that P should be symmetric. Using this matrix equation, we can extract scalar equations for each matrix coefficient:

$$\begin{aligned} 0 &= p_1 + p_1 + \left(\rho - \alpha^2\right) \\ 0 &= p_2 + (p_1 + p_2) + \left(\rho - \alpha^2\right) \\ 0 &= (p_2 + p_3) + (p_2 + p_3) + \left(\rho - \alpha^2\right). \end{aligned}$$

(The second equation listed above arises for both the upper right and lower left matrix coefficients.) Solving these equations gives

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \frac{\alpha^2 - \rho}{4} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Note that this value of P is positive definite if and only if $\alpha^2 - \rho > 0$. Now it remains to show that the given structure of K_{LQ} corresponds to the optimal controller. If the optimal controller has the structure of K_{LQ}, then

$$\alpha \begin{bmatrix} 1 & 1 \end{bmatrix} = K_{LQ} = R^{-1} B^T P = m \frac{\alpha^2 - \rho}{4} \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

Thus, this can only be satisfied if

$$\begin{aligned} 4\alpha &= m \left(\alpha^2 - \rho\right) \\ \Rightarrow \alpha &= \frac{2 \pm \sqrt{4 + m^2 \rho}}{m}. \end{aligned}$$

Note that this gives a positive and a negative solution for α. Since the sign of $\left(\alpha^2 - \rho\right)$ is the same as the sign of $4\alpha/m$, we are interested in the positive solution for α so that P will be positive definite. Thus the optimal controller gain is given by

$$\begin{aligned} K_{LQ} &= \alpha \begin{bmatrix} 1 & 1 \end{bmatrix} \\ \alpha &= \frac{2 + \sqrt{4 + m^2 \rho}}{m} \end{aligned}$$

and the positive definite solution of the Riccati equation is given by

$$P = \frac{\alpha^2 - \rho}{4} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

(b) If we set $m = m_o = 1$, $\rho = 5$ we get the LQR gain

$$K_{LQ} = \begin{bmatrix} 5 & 5 \end{bmatrix}.$$

¿From known LQR robustness results, the feedback system in Figure 3 is guaranteed to be stable for $\gamma \in (0.5, \infty)$. If we lump γ with B, we see that the closed loop system is guaranteed to be

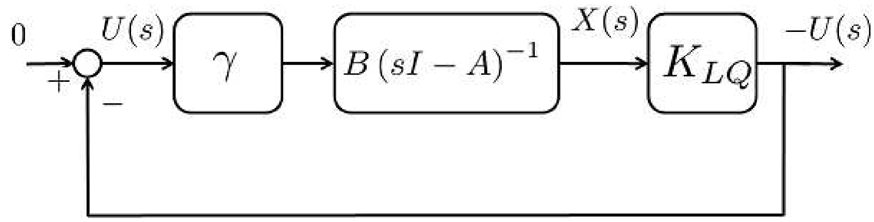

Figure 3: LQR Closed Loop System

stable if the actual B matrix has the form γB where $\gamma \in (0.5, \infty)$. Equivalently, the closed loop system is guaranteed to be stable if the actual B matrix is equal to $[0\ m]^T$, where $m \in (0.5, \infty)$. For this system, the closed loop eigenvalues are the eigenvalues of A_{cl}, which is defined as

$$A_{cl} := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 \\ m \end{bmatrix} K_{LQ} = \begin{bmatrix} 1 & 1 \\ -5m & 1-5m \end{bmatrix}$$

Thus, the characteristic polynomial for this system is

$$\begin{aligned} 0 &= (s-1)(s+5m-1) + 5m \\ &= s^2 + (5m-2)s + 1. \end{aligned}$$

We can obtain two pieces of information from this characteristic polynomial. First, we can deduce that the closed loop eigenvalues for $m = 1$ are given by

$$\lambda = \frac{-3 \pm \sqrt{5}}{2} = \{-2.6180, -0.3820\}.$$

Second, using the Routh-Hurwitz stability criterion, the system is stable if and only if $5m - 2 > 0$, which is equivalent to $m \in (0.4, \infty)$. Thus, we can see that the guaranteed robustness results are only slightly conservative in this case.

(c) First note that if we exchange the order of the states, the state update and output equations for the plant can be written

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_2(t) \\ x_1(t) \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_2(t) \\ x_1(t) \end{bmatrix} + \begin{bmatrix} m \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t) \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x_2(t) \\ x_1(t) \end{bmatrix} + v(t). \end{aligned}$$

Thus, if we define $\bar{x} := [x_2(t)\ x_1(t)]^T$, we see that the plant has the realization

$$\begin{aligned} \frac{d}{dt}\bar{x}(t) &= A^T\bar{x}(t) + mC^Tu(t) + B_w w(t) \\ y(t) &= \frac{1}{m}B^T\bar{x}(t) + v(t). \end{aligned}$$

The Riccati equation to find the Kalman Filter gain matrix, K_{KF}, for this new realization is

$$\begin{aligned} 0 &= \left(A^T\right)M + M\left(A^T\right)^T + B_wWB_w^T - M\left(\frac{1}{m}B^T\right)^T V^{-1}\left(\frac{1}{m}B^T\right)M \\ &= A^TM + MA + B_wB_w^T - \frac{1}{\sigma m^2}MBB^TM \\ &= A^TM + MA + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - M\begin{bmatrix} 0 & 0 \\ 0 & \sigma \end{bmatrix}M. \end{aligned}$$

Compare this to the Riccati equation used to find the LQR:

$$0 = A^T P + PA + \rho \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - P \begin{bmatrix} 0 & 0 \\ 0 & m^2 \end{bmatrix} P.$$

If we choose $\rho = 1$, $m = \sqrt{\sigma}$, we see that the LQR Riccati equation is the same as the Kalman Filter Riccati equation, which implies that they have the same solution. Thus, we see that

$$\begin{aligned} M &= \frac{\beta^2 - 1}{4} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \\ \beta &= \frac{2 + \sqrt{4 + \sigma}}{\sqrt{\sigma}} \\ \bar{K}_{KF} &= \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned}$$

Note that $\bar{K}_{KF}$ is the optimal filtering gain for estimating $\bar{x}(t)$. The Kalman Filter dynamics are given by

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} \hat{x}_2(t) \\ \hat{x}_1(t) \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_2(t) \\ \hat{x}_1(t) \end{bmatrix} + \begin{bmatrix} m \\ 0 \end{bmatrix} u(t) + \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(y(t) - \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_2(t) \\ \hat{x}_1(t) \end{bmatrix} \right) \\ \Rightarrow \frac{d}{dt} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ m \end{bmatrix} u(t) + \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(y(t) - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} \right) \\ \Rightarrow \frac{d}{dt} \hat{x}(t) &= A\hat{x}(t) + Bu(t) + K_{KF} \left(y(t) - C\hat{x}(t) \right) \end{aligned}$$

where K_{KF} is the filtering gain, which is given by

$$K_{KF} = \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(d) To avoid confusion, we first define

$$B_o := \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

This is the nominal value of the B matrix in the realization; the actual value of the B matrix is given by mB_o. With this in mind, we write down the equations which govern the closed loop system:

$$\begin{aligned} \frac{d}{dt} x(t) &= Ax(t) + mB_o u(t) + B_w w(t) \\ \frac{d}{dt} \hat{x}(t) &= A\hat{x}(t) + B_o u(t) + K_{KF} \left(y(t) - C\hat{x}(t) \right) \\ u(t) &= -K_{LQ} \hat{x}(t). \end{aligned}$$

Note that the Kalman Filter, because it was designed for the value $m = 1$, uses the nominal value of B. Closing the control loop gives the equations

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} = \begin{bmatrix} A & -mB_o K_{LQ} \\ K_{KF} C & A - K_{KF} C - B_o K_{LQ} \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} + \begin{bmatrix} B_w w(t) \\ K_{KF} \; v(t) \end{bmatrix}.$$

Thus, the eigenvalues of the closed loop system are the eigenvalues of the matrix

$$\bar{A}_c = \begin{bmatrix} A & -mB_o K_{LQ} \\ K_{KF} C & A - K_{KF} C - B_o K_{LQ} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -m\alpha & -m\alpha \\ \beta & 0 & 1 - \beta & 1 \\ \beta & 0 & -\beta - \alpha & 1 - \alpha \end{bmatrix}.$$

(e) For $\sigma = 5$, $\beta = \sqrt{5}$. The eigenvalues of the Kalman Filter are the eigenvalues of $A - K_{KF}C$, which are $-0.1180 \pm 0.9930i$. The eigenvalues of the actual closed loop LQG system with $m = 1$ are, as computed by MATLAB,

$$\lambda = \begin{bmatrix} -2.6180 \\ -0.3820 \\ -0.1180 \pm 0.9930i \end{bmatrix}.$$

Note that the first two eigenvalues are the LQR eigenvalues and the last two eigenvalues are the Kalman Filter eigenvalues, i.e. the separation principle holds.

(f) The eigenvalues of the actual closed loop LQG system with $m = 1.1$ are, as computed by MATLAB,

$$\lambda = \begin{bmatrix} -2.9513 \\ -0.1531 \pm 1.3604i \\ 0.0213 \end{bmatrix}.$$

Since one of these eigenvalues is in the closed right-half plane, the closed loop system is unstable. This reflects the fact that even though LQRs and Kalman Filters have guaranteed robustness margins, LQG has no guaranteed robustness margins.