ME232 Lecture #23

Linear Quadratic (LQ) Optimal Control Problem

We have found that the state feedback allows us to assign any closed loop system eigenvalues if the system is controllable. For single input systems, we have given a procedure for obtaining the gain given a desired set of eigenvalues. The eigenvalues must be selected by the designer. Another way to determine feedback control gains is to solve the linear quadratic (LQ) optimal control problem. The LQ problem is one of the most frequently appearing optimal control problems. It is the basis of many modern robust control system design methods.

Continuous Time Case

Problem Statement

The controlled system is described by

$$\frac{dx(t)}{dt} = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0 \tag{23-1}$$

The optimal control is sought to minimize the quadratic performance index,

$$J = \frac{1}{2}x^T(t_f)Sx(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\{x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\}dt \tag{23-2}$$

where S and $Q(t) = C^T(t)C(t)$ are positive semidefinite matrices and R(t) is positive definite.

Note 1. The LQ problem as formulated above is concerned with the regulation of the system around the origin of the state space. The resulting system is called the Linear Quadratic Regulator (LQR).

Note 2. In the performance index, the first term penalizes the deviation of **x** from the origin at the final time. Inside the integral, the first term penalizes the transient deviation of **x** from the origin and the second the control energy. The first term in the integral can be interpreted as the deviation of the output vector defined by **y** = C**x**.

Note 3. The problem statement is given for general cases where A, B, Q and R are time varying. In most cases, A, B, Q and R are time invariant.

Derivation of Solution

The LQ optimal control can be found in various ways. In the present notes, we will derive the solution in an elementary way. To simplify writing, we write A, B, Q and R in place of A(t), B(t), Q(t) and R(t).

Let P(t) be an $n \times n$ symmetric matrix and x(t) satisfy the state equation (23-1). Then

$$
\begin{aligned}
x^T(t_f)P(t_f)x(t_f)-x^T(t_0)P(t_0)x(t_0) &= \int_{t_0}^{t_f}\frac{d}{dt}\left[x^T(t)P(t)x(t)\right]dt \\
&= \int_{t_0}^{t_f}\left[\left(\frac{dx}{dt}\right)^T Px+x^T\frac{dP}{dt}x+x^TP\frac{dx}{dt}\right]dt \\
&= \int_{t_0}^{t_f}\left[(Ax+Bu)^TPx+x^T\frac{dP}{dt}x+x^TP(Ax+Bu)\right]dt
\end{aligned} \tag{23-3}
$$

Siince Eq. (23-3) is true for any nxn matrix, select P(t) so that it satisfies

$$
\frac{dP(t)}{dt}+A^TP(t)+P(t)A = P(t)BR^{-1}B^TP(t)-Q, \quad P(t_f) = S \tag{23-4}
$$

From Eqs. (23-3) and (23-4)

$$
0 = -\frac{1}{2}x^T(t_f)Sx(t_f)+\frac{1}{2}x^T(t_0)P(t_0)x(t_0)+\frac{1}{2}\int_{t_0}^{t_f}\left[x^T(PBR^{-1}B^TP-Q)x+u^TB^TPx+x^TPBu\right]dt \tag{23-5}
$$

Adding this net zero quantity to the cost functional, i.e. (23-2)+(23-5),

$$
\begin{aligned}
J &= \frac{1}{2}x^T(t_0)P(t_0)x(t_0)+\frac{1}{2}\int_{t_0}^{t_f}\left[x^TPBR^{-1}B^TPx+u^TB^TPx+x^TPBu+u^TRu\right]dt \\
&= \frac{1}{2}x^T(t_0)P(t_0)x(t_0)+\frac{1}{2}\int_{t_0}^{t_f}(R^{-1}B^TPx+u)^TR(R^{-1}B^TPx+u)dt
\end{aligned} \tag{23-6}
$$

Since R is positive definite, in order for J to be minimum

$$
u(t) = -R^{-1}(t)B^T(t)P(t)x(t)
$$

and the minimum value of J is

$$
J^o(x(t_0)) = \frac{1}{2}x^T(t_0)P(t_0)x(t_0)
$$

Now, the result is summarized in the following theorem.

Theorem LQ-1(**Solution of the Continuous Time LQ problem in State Feedback Form**): The solution of the continuous time LQ problem is given by the state feedback control law,

$$
u(t) = -R^{-1}(t)B^T(t)P(t)x(t) \tag{23-7}
$$

where P(t) is the positive semidefinite solution of the Riccati differential equation

$$-\frac{dP(t)}{dt} = A^T(t)P(t)+P(t)A(t)-P(t)B(t)R^{-1}(t)B^T(t)P(t)+C^T(t)C(t), \quad P(t_f) = S \tag{23-8}$$

and the minimum value of the quadratic performance index is given by

(End of Theorem)

$$J^o(x(t_0)) = \frac{1}{2}x^T(t_0)P(t_0)x(t_0) \tag{23-9}$$

Note 4. The boundary condition of the matrix Riccati differential equation is given at the final time. Therefore, the equation must be integrated backward in time.

Note 5. Since the performance index is nonnegative, Eq. (23-9) implies that $P(t_0)$ is at least positive semidefinite. In fact, any t ($< t_f$) is a valid initial time of an LQ problem. Therefore, P(t) is at least positive semidefinite for any t.

Note 6. The state feedback gain in (23-7) is time varying. Even if the matrices, A, B, R and Q ($=C^T C$) are time invariant, the feedback gain is time varying because of its dependence on P(t). A practically useful case is obtained by letting $t_f \to \infty$, which is discussed below.

Stationary Case

We will let $t_f \to \infty$. In this case, we introduce the following two key assumptions.

Assumption 1. The system is controllable or stabilizable. This assumption is necessary to make it sure that the performance index remains bounded. In view of (23-9), this assumption assures that P(t) converges to a stationary solution of the Riccati differential equation when integrated backward in time. Namely, if the final time is infinity, the cost should not depend on when the control action is initiated.

Assumption 2. (A, C) is observable or detectable. This assumption assures that the optimal state feedback control system is asymptotically stable. If (A, C) is observable, it can be further said that the stationary solution of the Riccati differential equation is positive definite. If (A, C) is only detectable, the stationary solution is assured to be positive semidefinite. The stationary solution is obtained by solving the algebraic Riccati equation (ARE)

$$A^TP+PA-PBR^{-1}B^TP+C^TC = 0 \tag{23-10}$$

Theorem LQ-2(**Stationary LQ Problem and Solution**): Consider the time invariant controllable (or stabilizable) system

$$\frac{dx(t)}{dt} = Ax(t)+Bu(t), \qquad x(t_0) = x_0 \tag{23-11}$$

and the quadratic performance index

$$J = \frac{1}{2}\int_{t_0}^{\infty} x^T(t)Qx(t)+u^T(t)Ru(t)dt, \quad Q = C^TC \tag{23-12}$$

where Q is positive semidefinite, R is positive definite and (A, C) is observable (or *detectable*). Then, the optimal control input is given by the stationary state feedback control law

$$u(t) = -R^{-1}B^TP_+x(t) \tag{23-13}$$

where P_+ is the positive definite (or *positive semidefinite*) solution of the algebraic Riccati equation (23-10), and the optimal closed loop system

$$\frac{dx(t)}{dt} = (A-BR^{-1}B^TP_+)x(t) \tag{23-14}$$

is asymptotically stable. The minimum value of the performance index is given by

$$J^o(x(t_0)) = \frac{1}{2}x^T(t_0)P_+x(t_0) \tag{23-15}$$

(End of Theorem)

Note 7. When the final time is infinity, the term penalizing the final state does not have to be included and has been dropped in (23-12).

Note 8. When P_+ is positive definite, which is assured if A and C are observable, (23-15) is a Lyapunov function for the closed loop system.

Note 9. The asymptotic stability of the closed loop system is very attractive from a practical point of view. The LQ theory is a practical approach for finding stabilizing feedback control gains.

Note 10. The optimality of the state feedback control gain is in the sense of minimizing the performance index given Q and R. Whether it defines a good system in any engineering sense depends of the choice of Q and R. It is not possible to discuss this point fully in ME232. Further analysis will be continued in ME233. At this stage, we state rather intuitive rules of thumb.

1. If there is not a good idea for the structure for Q and R, start with diagonal Q and R.
2. Obtain some idea how large the magnitude of each of the state variables and input variables can be. Call them $x_{i,\,max}$ (i = 1,...,n) and $u_{i,\,max}$ (i=1,...,r). Make the diagonal elements of Q and R inversely proportional to $\|x_{i,max}\|^2$ and $\|u_{i,max}\|^2$, respectively. This approach has been known for a long time, and is given in Bryson and Ho, Applied Optimal Control, 1969.

Obtaining P_+

The positive solution of the algebraic Riccati equation can be obtained in several ways. The most straightforward method is to integrate the differential Riccati equation (23-8) backward in time until the steady state solution is reached. The boundary condition S can be any positive semidefinite matrix. However, the backward integration is not necessarily the best approach. "lqr" in MATLAB is based on the properties of the Hamiltonian matrix summarized below.

Another method is based on the Hamiltonian matrix given below.

$$H = \begin{bmatrix} A & -BR^{-1}B^T \\ -C^TC & -A^T \end{bmatrix} \tag{23-16}$$

is called the Hamiltonian matrix.

Theorem LQ-3(Hamiltonian Matrix): Under the conditions as stated in the previous theorem, the Hamiltonian matrix H possesses the following properties.

i. n eigenvalues of H are those of the optimal closed loop matrix, $A - BR^{-1}B^TP_+$. Call them λ_i, $i = 1, \ldots, n$.
ii. remaining n eigenvalues of H are given by $\lambda_{(n+i)} = -\lambda_i$.
iii. i. and ii. imply that H has no eigenvalues on the imaginary axis.
iv. For each eigenvalue in the left half side of the complex plane (i.e. those from i), find the eigenvector: i.e. find f_i and g_i in

$$H\begin{bmatrix} f_i \\ g_i \end{bmatrix} = \lambda_i \begin{bmatrix} f_i \\ g_i \end{bmatrix}, \quad i = 1,\ldots,n \tag{23-17}$$

Form two $n \times n$ matrices by

$$X_1 = [f_1, f_2, \ldots, f_n], \qquad X_2 = [g_1, g_2, \ldots, g_n] \tag{23-18}$$

Then,

$$P_+ = X_2 X_1^{-1} \tag{23-19}$$

(End of Theorem)

Note 11. The eigenvalues of H are in general complex. This implies that X_1 and X_2 defined about are in general complex. However, the operation in (23-19) provides a real P_+. Proof of this theorem will be in Lecture #24.

Example (A pure inertia system): A pure inertia system is described by

$$dx_1(t)/dt = x_2(t), \quad dx_2(t)/dt = u(t)$$

where the mass m is assumed to be unity. The performance index is

$$J = \frac{1}{2}\int_0^\infty x_1^2(t) + Ru^2(t)dt$$

Notice that C is [1, 0]. It can be easily checked that (A, B) is controllable and (A,C) is observable.

The algebraic Riccati equation for this problem is

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} P + P \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - P \begin{bmatrix} 0 \\ 1 \end{bmatrix} (1/R) \begin{bmatrix} 0 & 1 \end{bmatrix} P + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

This second order nonlinear equation can be solved analytically, and the positive definite solution is

$$P_+ = \begin{bmatrix} \sqrt{2}R^{1/4} & R^{1/2} \\ R^{1/2} & \sqrt{2}R^{3/4} \end{bmatrix}$$

The closed loop system matrix is

$$A_c = A - BR^{-1}B^T P_+ = \begin{bmatrix} 0 & 1 \\ R^{-1/2} & \sqrt{2}R^{-1/4} \end{bmatrix}$$

Solving the closed loop characteristic equation, the closed loop eigenvalues are

$$\lambda_{1,2} = \frac{-\sqrt{2}R^{-1/4} \pm \sqrt{2}R^{-1/4} j}{2}$$

Notice that the closed loop system has a 0.707 damping and the magnitude of the closed loop eigenvalues is inversely proportional to R: i.e. the smaller the value of R, the wider the closed loop bandwidth.

ME232 Lecture #24

Proof of Theorem LQ-3 on page 125:

i-iii. The Hamiltonian matrix is

$$H = \begin{bmatrix} A & -BR^{-1}B^T \\ -C^TC & -A^T \end{bmatrix}$$

The eigenvalues of H are determined from $\det[\lambda I_{2n} - H] = 0$. Let P_+ be the positive definite solution of the algebraic Riccati equation (23-10): i.e.

$$A^TP_+ + P_+A - P_+BR^{-1}B^TP_+ + C^TC = 0$$

Define a nonsingular matrix T and its inverse by

$$T = \begin{bmatrix} I_n & 0 \\ P_+ & I_n \end{bmatrix} \quad T^{-1} = \begin{bmatrix} I_n & 0 \\ -P_+ & I_n \end{bmatrix}$$

Then,

$$T^{-1}HT = \begin{bmatrix} A-BR^{-1}B^TP_+ & -BR^{-1}B^T \\ 0 & -(A-BR^{-1}B^TP_+)^T \end{bmatrix}$$

where the 2-1 block becomes zero because P_+ satisfies the algebraic Riccati equation. Recall that the eigenvalues of H do not change under the similarity transformation T: i..e eig{H} = eig($T^{-1}HT$}. Since $T^{-1}HT$ is block diagonal, its eigenvalues consist of those of the diagonal blocks: $A-BR^{-1}B^TP_+$ and $-(A-BR^{-1}B^TP_+)^T$. This gives us i. and ii. Note that the eigenvalues of a matrix are invariant under matrix transposition. Furthermore $A-BR^{-1}B^TP_+$ defines the optimal closed loop system and is asymptotically stable under the controllability and observability (or stabilizability and detectability) assumptions. This gives us iii.

iv. From the algebraic Riccati equation

$$A^TP_+ + P_+A - P_+BR^{-1}B^TP_+ + C^TC + \lambda P_+ - \lambda P_+ = 0$$
$$\Rightarrow \ (\lambda I_n + A^T)P_+ - P_+(\lambda I_n - A) - P_+BR^{-1}B^TP_+ + C^TC = 0$$
$$\Rightarrow \ P_+[\lambda I_n - (A - BR^{-1}B^TP_+] + (\lambda I_n + A^T)P_+ + C^TC = 0$$

Let f_i be the eigenvector of $A-BR^{-1}B^TP_+$ associated with the eigenvalue λ_i. Set $\lambda = \lambda_i$ in the last equality and multiply f_i from the right. Then, the first term vanishes after multiplication: i.e.

$$(\lambda_i I_n + A^T)P_+f_i + C^TCf_i = 0$$

Then,

$$H\begin{bmatrix} f_i \\ P f_i \end{bmatrix} = \begin{bmatrix} Af_i - BR^{-1}B^T P f_i \\ -C^T C f_i - A^T P f_i \end{bmatrix} = \lambda_i \begin{bmatrix} f_i \\ P f_i \end{bmatrix}$$

This implies that

$$\begin{bmatrix} f_i \\ P f_i \end{bmatrix}$$

is the eigenvector of H associated with a stable eigenvalue λ_i. iv. follows from this fact. (End)

Discrete Time Case

The formulation and solution of the discrete time LQ optimal control problem is analogous to the continuous time case.

Discrete Time LQ Problem

The controlled system is described by

$$x(k+1) = A(k)x(k) + B(k)u(k), \quad x(k_0) = x_0 \tag{24-1}$$

The optimal control is sought to minimize the quadratic performance index,

$$J = \frac{1}{2}x^T(k_f)Sx(k_f) + \frac{1}{2}\sum_{k=k_0}^{k_f-1}\{x^T(k)Q(k)x(k) + u^T(k)R(k)u(k)\}, \quad Q(k) = C^T(k)C(k) \tag{24-2}$$

Theorem LQ-4 (**Solution of the Discrete Time LQ Problem**): The solution of the discrete time LQ problem is given by the state feedback control law,

$$u(k) = -[R(k) + B^T(k)P(k+1)B(k)]^{-1}B^T(k)P(k+1)A(k)x(k) \tag{24-3}$$

where P(k) is the positive semidefinite solution of the matrix Riccati difference equation

$$\begin{gathered} P(k) = A^T(k)P(k+1)A(k) + C^T(k)C(k) \\ -A^T(k)P(k+1)B(k)[R(k) + B^T(k)P(k+1)B(k)]^{-1}B^T(k)P(k+1)A(k), \\ P(k_f) = S \end{gathered} \tag{24-4}$$

The minimum value of the quadratic performance index is given by

$$J^o(x(k_0)) = \frac{1}{2}x^T(k_0)P(k_0)x(k_0) \tag{24-5}$$

(End of Theorem)

Verification of the results stated in the Theorem

We will provide below a verification of the results.

Let P(k) be any nxn symmetric matrix and x(k) satisfy the state equation (24-1). Then

$$\sum_{k=k_0}^{k_f-1}[x^T(k+1)P(k+1)x(k+1)-x^T(k)P(k)x(k)] = x^T(k_f)P(k_f)x(k_f)-x^T(k_0)P(k_0)x(k_0)$$

$$= \sum_{k=k_0}^{k_f-1}[(A(k)x(k)+B(k)u(k))^TP(k+1)(A(k)x(k)+B(k)u(k))-x^T(k)P(k)x(k)] \qquad (24\text{-}6)$$

From (24-6),

$$x^T(k_f)P(k_f)x(k_f)-x^T(k_0)P(k_0)x(k_0)$$

$$= \sum_{k=k_0}^{k_f-1}\{x^T(k)[A^T(k)P(k+1)A(k)-P(k)]x(k)+x^T(k)A^T(k)P(k+1)B(k)u(k) \qquad (24\text{-}7)$$

$$+u^T(k)B^T(k)P(k+1)A(k)x(k)+u^T(k)B^T(k)P(k+1)B(k)u(k)\}$$

Since (24-7) is true for any nxn matrix, select P(k) that satisfies the Riccati equation (24-4). Then, from (24-4) and (24-7) (note $P(k_f) = S$), we obtain

$$0 = -\frac{1}{2}x^T(k_f)Sx(k_f)+\frac{1}{2}x^T(k_0)P(k_0)x(k_0)$$

$$\frac{1}{2}\sum_{k=k_0}^{k_f-1}\{x^T(k)\big[-C^T(k)C(k)+A^T(k)P(k+1)B(k)[R(k)+B^T(k)P(k+1)B(k)]^{-1}B^TP(k+1)A(k)\big]x(\qquad (24\text{-}8)$$

$$+x^T(k)A^T(k)P(k+1)B(k)u(k)+u^T(k)B^T(k)P(k+1)A(k)x(k)+u^T(k)B^T(k)P(k+1)B(k)u(k)\}$$

Adding this net zero quantity to the performance index (i.e. (24-2) + (24-8)), we obtain

$$J = \frac{1}{2}x^T(k_0)P(k_0)x(k_0)$$

$$+\frac{1}{2}\sum_{k=k_0}^{k_f-1}\{x^T(k)A^T(k)P(k+1)B(k)[R(k)+B^T(k)P(k+1)B(k)]^{-1}B^T(k)P(k+1)A(k)x(k)$$

$$+x^T(k)A^T(k)P(k+1)B(k)u(k)+u^T(k)B^T(k)P(k+1)A(k)x(k)+u^T(k)[R(k)+B^T(k)P(k+1)B(k)]u(k)\} \qquad (24\text{-}9)$$

$$= \frac{1}{2}x^T(k_0)P(k_0)x(k_0)+\frac{1}{2}\sum_{k=k_0}^{k_f-1}\big\{[[R(k)+B^T(k)P(k+1)B(k)]^{-1}B^T(k)P(k+1)A(k)x(k)+u(k)\big]^T$$

$$\cdot[R(k)+B^T(k)P(k+1)B(k)]\big[[R(k)+B^T(k)P(k+1)B(k)]^{-1}B^T(k)P(k+1)A(k)x(k)+u(k)\big]\big\}$$

Since $R(k) + B^T(k)P(k)B(k)$ is positive definite, in order for J to be minimum, u(k) must be given by (24-3) and the minimum value of the performance index is given by (24-5). (End of Verification)

Stationary Discrete-Time LQ Problem and Solution

Theorem LQ-5: Consider the time invariant discrete time controllable (or stabilizable) system

$$x(k+1) = Ax(k)+Bu(k), \quad x(k_0) = x_0 \tag{24-10}$$

and the quadratic performance index

$$J = \frac{1}{2}\sum_{k_0}^{\infty} \{x^T(k)Qx(k)+u^T(k)Ru(k)\}, \quad Q = C^TC \tag{24-11}$$

where Q is positive semidefinite, R is positive definite and (A, C) is observable (or *detectable*). Then the optimal control input is given by the stationary feedback control law,

$$u(k) = -[R+B^TP_+B]^{-1}B^TP_+Ax(k) \tag{24-12}$$

where P_+ is the positive definite (or *positive semidefinite*) solution of the algebraic Riccati equation

$$A^TPA-P-A^TPB[R+B^TPB]^{-1}B^TPA+C^TC=0 \tag{24-13}$$

Furthermore, the optimal closed loop system

$$x(k+1) = [A-B(R+B^TP_+B)^{-1}B^TP_+A]x(k) \tag{24-14}$$

is asymptotically stable. The minimum value of the performance index is given by

$$J^o(x(k_0)) = \frac{1}{2}x^T(k_0)P_+x(k_0) \tag{24-15}$$

(End of Theorem)

Obtaining P_+

The positive (semidefinite) solution can be obtained by backwards recursion of the Riccati difference equation. Another way, which is utilized in MATLAB, is based on properties of the backwards Hamiltonian,

$$H_b = \begin{bmatrix} A^{-1} & A^{-1}BR^{-1}B^T \\ C^TCA^{-1} & A^T+C^TCA^{-1}BR^{-1}B^T \end{bmatrix} \tag{24-16}$$

where det A is assumed to be nonzero.

Theorem LQ-6(**Properties of the backwards Hamiltonian**): Assume that the pair (A, B) is controllable (or stabilizable) and the pair (A, C) is observable (or detectable). Then, the backwards Hamiltonian possesses the following properties.

i. n eigenvalues of H_b are those of the optimal closed loop matrix, $A - B[R+B^TP_+B]^{-1}B^TP_+A$. Call them λ_i, $i = 1, \ldots, n$.

ii. remaining n eigenvalues of H_b are given by $\lambda_{(n+i)} = \lambda_i^{-1}$.
iii. for each eigenvalue outside the unit circle (i.e. those from ii), find the eigenvector: i.e. find f_{n+i} and g_{n+i} in

$$H_b \begin{bmatrix} f_{n+i} \\ g_{n+i} \end{bmatrix} = \lambda_{n+i} \begin{bmatrix} f_{n+i} \\ g_{n+i} \end{bmatrix}, \quad i = 1, \ldots, n \tag{24-17}$$

Form two $n \times n$ matrices by

$$X_1 = [f_{n+1}, f_{n+2}, \ldots, f_{2n}], \quad X_2 = [g_{n+1}, g_{n+2}, \ldots, g_{2n}] \tag{24-18}$$

Then,

$$P_+ = X_2 X_1^{-1} \tag{24-19}$$

ME232 Lecture #24-Part 2

Conversion of Continuous Time LQ Problem to Discrete Time LQ Problem

Consider a continuous time LQ problem defined by the following plant equation and quadratic performance index.

$$\frac{dx(t)}{dt}=Ax(t)+Bu(t) \tag{24-20}$$

$$J=\frac{1}{2}\int_0^{\infty}\{x^T(\tau)Qx(\tau)+u^T(\tau)Ru(\tau)\}d\tau \tag{24-21}$$

Digital control law may be obtained as an approximation of the continuous time LQ control law. Alternatively, let us reformulate the problem as a discrete time LQ problem. For a sampling time of Δt, the discrete time state equation of the plant becomes

$$x(k+1)=A_d x(k)+B_d u(k), \quad A_d=e^{A\Delta t},\ B_d=\int_0^{\Delta t}e^{A\sigma}Bd\sigma \tag{24-22}$$

where k denotes the k-th sampling instance and $x(k) = x(k\Delta t)$. To find a discrete time quadratic performance index equivalent to the one in the original continuous time LQ problem, we first note

$$x(t)=e^{A(t-k\Delta t)}x(k\Delta t)+\int_{k\Delta t}^{t}e^{A(t-\tau)}Bd\tau u(k),\ k\Delta t\le t<(k+1)\Delta t \tag{24-23}$$

x(t) can also be expressed as

$$x(k\Delta t+\tau)=e^{A\tau}x(k)+\int_0^{\tau}e^{A\sigma}Bd\tau u(k), \quad 0\le\tau<\Delta t \tag{24-24}$$

We utilize Eq. (24-24) to obtain

$$\int_{k\Delta t}^{(k+1)\Delta t} x^T(\tau)Qx(\tau)+u^T(\tau)Ru(\tau)d\tau = x^T(k)\int_0^{\Delta t}[e^{A\tau}]^T Q e^{A\tau}d\tau x(k)$$
$$+2x^T(k)\int_0^{\Delta}[e^{A\tau}]^T Q\int_0^{\tau} e^{A\sigma}Bd\sigma d\tau u(k) + u^T(k)\int_0^{\Delta t}\{[\int_0^{\tau} e^{A\sigma}Bd\sigma]^T Q\int_0^{\tau} e^{A\sigma}Bd\sigma + R\}d\tau u(k) \tag{24-25}$$
$$= x^T(k)Q_d x(k) + 2x^T(k)S_d u(k) + u^T(k)R_d u(k)$$

where

$$Q_d = \int_0^{\Delta t}[e^{A\tau}]^T Q e^{A\tau}d\tau,\ S_d = \int_0^{\Delta t}[e^{A\tau}]^T Q\int_0^{\tau} e^{A\sigma}Bd\sigma d\tau$$
$$\text{and}\quad R_d = \int_0^{\Delta t}\{[\int_0^{\tau} e^{A\sigma}Bd\sigma]^T Q\int_0^{\tau} e^{A\sigma}Bd\sigma + R\}d\tau \tag{24-26}$$

Noting Eq. (24-25), the quadratic performance index (24-21) can be written as

$$J = \frac{1}{2}\sum_{k=0}^{\infty}\{x^T(k)Q_d x(k) + 2x^T(k)S_d u(k) + u^T(k)R_d u(k)\} \tag{24-27}$$

Equations (24-23) and (24-27) define a discrete time LQ problem. When the continuous time plant (24-21) is under digital control, the optimal discrete time control law obtained from this LQ problem minimizes the performance index (24-22).

LQ Problem with Generalized Quadratic Performance Index

Notice the presence of the cross product term(k) $x^T(k)S_d u(k)$ in the performance index (24-27). This LQ problem can be transformed to a standard LQ problem without a cross product term and the solution can be obtained. The transformation can be accomplished by writing

$$u(k) = Fx(k) + u_d(k)$$

The discrete state equation (24-23) may be written as

$$x(k+1) = (A_d + B_d F)x(k) + B_d u_d(k) \tag{24-28}$$

The right hand side quantity of (24-27) is

$$x^T(k)Q_d x(k)+2x^T S_d u(k)+u^T(k)R_d u(k)$$
$$=x^T(k)\{Q_d+S_d F+F^T S_d^T+F^T R_d F\}x(k)+2x^T(k)\{S_d+F^T R_d\}u_d(k)+u_d^T(k)R_d u_d(k)$$

Thus by letting $F^T = -S_d R_d^{-1}$, the performance index (24-27) is

$$J=\frac{1}{2}\sum_{k=0}^{\infty}\{x^T(k)[Q_d-S_d R_d^{-1}S_d^T]x(k)+u_d^T R_d u_d(k)\} \tag{24-29}$$

Notice that Eqs. (24-28) and (24-29) define a standard LQ problem. As is evident from Eq. (24-29), Q_d, S_d and R_d must satisfy $R_d > 0$ and $Q_d - S_d R_d^{-1} S_d^T \geq 0$. The solution is

$$u_d(k)=-[R_d+B_d^T P_{d+}B_d]^{-1}B_d^T P_{d+}(A_d-B_d R_d^{-1}S_d^T)x(k) \tag{24-30}$$

where P_{d+} is the positive definite solution of the algebraic Riccati equation

$$\begin{aligned}&(A_d-B_d R_d^{-1}S_d^T)^T P_d(A_d-B_d R_d^{-1}S_d^T)-P_d\\ &-(A_d-B_d R_d^{-1}S_d^T)^T P_d B_d[R_d+B_d^T P_d B_d]^{-1}B_d^T P_d(A_d-B_d R_d^{-1}S_d^T)+\{Q_d-S_d R_d^{-1}S_d^T\}=0\end{aligned} \tag{24-31}$$

Noting $u(k) = -R_d^{-1}S_d x(k) + u_d(k)$, the optimal input u(k) is

$$u(k)=-[R_d+B_d^T P_{d+}B_d]^{-1}[B_d^T P_{d+}A_d+S_d^T]x(k) \tag{24-32}$$

The Riccati Equation (24-31) can be rearranged as

$$\begin{aligned}&A_d^T P_d A_d-P_d+Q_d\\ &-[B_d^T P_d A_d+S_d^T]^T[R_d+B_d^T P_d B_d]^{-1}[B_d^T P_d A_d+S_d^T]=0\end{aligned} \tag{24-33}$$

Equations (24-32) and (24-33) give the solution of the discrete time LQ problem defined by Eqs. (24-22) and (24-27) where as stated already $R_d > 0$ and $Q_d - S_d R_d^{-1} S_d^T \geq 0$.