1. (15 pints: 5pts for (a), 10pts for (b)) The first order system is defined by:

$$x(k+1) \quad = \quad ax(k) + w(k) + c$$

$x(0), c$ are RV's and $w(k)$ is a white random process. Also, c remains constant once the experiments starts.

There are two methods to solve this problem,

- Method 1: Start with the definition of the variance:

$$\begin{aligned}
m_x(k+1) &= E[x(k+1)] = E[ax(k)] + E[w(k)] + E[c] = aE[x(k)] = am_x(k) \\
\Longrightarrow m_x(k+1) &= a^{k+1}m_x(0) = 0; \\
X(k) &= E[x^2(k)] = E[(ax(k-1) + w(k-1) + c)^2] \\
&= a^2X(k-1) + 2aE[x(k-1)w(k-1)] + 2aE[x(k-1)c] \\
&\quad +E[w^2(k-1)] + E[c^2] \\
E[x(k-1)w(k-1)] &= aE[x(k-2)w(k-1)] + E[w(k-2)w(k-1)] + E[cw(k-1)] \\
&= aE[x(k-2)w(k-1)] = a^{k-1}E[x(0)w(k-1)] = 0 \\
E[x(k-1)c] &= aE[x(k-2)c] + E[w(k-2)c] + E[c^2] = aE[x(k-2)c] + C \\
\Longrightarrow E[x(k-1)c] &= a^{k-1}E[x(0)c] + \sum_{i=1}^{k-1} a^{k-1-i}C = \frac{1-a^{k-1}}{1-a}C \\
E[w^2(k-1)] &= W \\
E[c^2] &= C \\
\Longrightarrow X(k) &= a^2X(k-1) + (2a\frac{1-a^{k-1}}{1-a} + 1)C + W, \; X(0) = 4
\end{aligned}$$

  Assume the system is aymptotically stable, i.e. $|a| < 1$, the steady state is when $k \to \infty$:

$$X_{ss} = a^2X_{ss} + (2a\frac{1}{1-a} + 1)C + W$$
$$\Longrightarrow X_{ss} = \frac{1}{(1-a)^2}C + \frac{1}{1-a^2}W$$

- Method 2: Since c is a constant once the experiment starts, we can rewrite the system equations as:

$$\begin{bmatrix} x(k+1) \\ c(k+1) \end{bmatrix} \quad = \quad \begin{bmatrix} a & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(k) \\ c(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(k)$$

  Define $x_a(k) := \begin{bmatrix} x(k) & c(k) \end{bmatrix}^T$, $A := \begin{bmatrix} a & 1 \\ 0 & 1 \end{bmatrix}$, $B := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then,

$$\begin{aligned}
E[x_a(k+1)] &= AE[x_a(k)] + BE[w(k)] \\
&= AE[x_a(k)]
\end{aligned}$$

  Define $E[x_a(k)] := m_{x_a}(k)$. Then, from the definition of $x_a(k)$,

$$\begin{aligned}
X_{x_ax_a}(k) &= E[\begin{bmatrix} x(k) - m_x(k) \\ c(k) - m_c(k) \end{bmatrix} \begin{bmatrix} x(k) - m_x(k) & c(k) - m_c(k) \end{bmatrix}] \\
&= \begin{bmatrix} X_{xx}(k) & COV(x(k), c(k)) \\ COV(c(k), x(k)) & X_{cc}(k) \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
X_{x_a x_a}(k+1) = &= E[(x_a(k+1) - m_{x_a}(k+1))(x_a(k+1) - m_{x_a}(k+1))^T] \\
&= E[(Ax_a(k) + Bw(k) - Am_{x_a}(k))(Ax_a(k) + Bw(k) - Am_{x_a}(k))^T] \\
&= E[(A(x_a(k) - m_{x_a}(k)) + Bw(k))(A(x_a(k) - m_{x_a}(k)) + Bw(k))^T] \\
&= AX_{x_a x_a}(k)A^T + BWB^T
\end{aligned}
$$

with the initial condition $X_{x_a x_a}(0) = \begin{bmatrix} 4 & 0 \\ 0 & C \end{bmatrix}$

Note: We have used the fact that $x_a(k)$ is uncorrelated with $w(k)$ since $x_a(k)$ is a fn of $x(0), w(1), w(2), \cdots, w(k-1)$ all of which are uncorrelated with $w(k)$

If the system is asymptoticall stable, then A is marginally stable since A has eigenvalues at a and 1. Howver, c remains constant. Therefore, $x_a(k)$ reaches a steady state value as $k \longrightarrow \infty$. $X_{ss} := \lim_{k\to\infty} X_{x_a x_a}(k)$ can be obtained by solving the Lyapunov equation:

$$
X_{ss} = AX_{ss}A^T + BWB^T
$$

Solution of this Lyapunov equation yields: $X_{ss} = \begin{bmatrix} \left(\frac{C}{(1-a)^2} + \frac{W}{1-a^2}\right) & \frac{C}{1-a} \\ \frac{C}{1-a} & C \end{bmatrix}$

Therefore, the steady state variance of $x(k)$ is given by: $\lim_{k\to\infty} X_{xx}(k) = \frac{C}{(1-a)^2} + \frac{W}{1-a^2}$

2. (20 points: 10pts for each approach) Repeated Measurements
   There are also two methods to solve this problem: using Kalman Filter and using least square estimation directly.

   - We do the least-square approach first. Since x & v are Gaussian distributed, y (sum of two Gaussian RV) will be Gaussian. Define $Y_{vec} := \begin{bmatrix} y(0) & y(1) & \cdots & y(k) \end{bmatrix}^T, V_{vec} := \begin{bmatrix} v(0) & v(1) & \cdots & v(k) \end{bmatrix}^T$ and $X_{vec}^2 := \begin{bmatrix} x^2 & x^2 & \cdots & x^2 \end{bmatrix}$. Then,

$$
\begin{aligned}
\hat{x}(k) &= E[x|y(0), y(1), \cdots, y(k)] \\
&= E[x] + X_{xY_{vec}} X_{Y_{vec}Y_{vec}}^{-1} (Y_{vec} - E[Y_{vec}])) \\
X_{\tilde{x}|Y_{vec}\tilde{x}|Y_{vec}} &= X_{xx} - X_{xY_{vec}} X_{Y_{vec}\,Y_{vec}}^{-1} X_{Y_{vec}x}
\end{aligned}
$$

We will now compute the quantities in the above expression.

$$
\begin{aligned}
E[Y_{vec}] &= E[\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(k) \end{bmatrix}] = \mathbf{0} \\
X_{xY_{vec}} : &= E[(x - E[x])(Y_{vec} - E[Y_{vec}])^T] \\
&= E[x\, Y_{vec}] = E[X_{vec}^2 + x\, V_{vec}] = E[X_{vec}^2] + E[x\, V_{vec}] = \begin{bmatrix} X_0 & X_0 & \cdots & X_0 \end{bmatrix} \\
X_{Y_{vec}Y_{vec}} : &= E[(Y_{vec} - E[Y_{vec}])(Y_{vec} - E[Y_{vec}])^T] \\
&= E[Y_{vec} Y_{vec}^T] \\
&= E[\begin{bmatrix} x + v(0) \\ x + v(1) \\ \vdots \\ x + v(k) \end{bmatrix} \begin{bmatrix} x + v(0) & x + v(1) & \cdots & x + v(k) \end{bmatrix}] \\
&= \begin{bmatrix} X_0 + V & X_0 & \cdots & X_0 \\ X_0 & X_0 + V & \cdots & X_0 \\ \vdots & \vdots & \ddots & \vdots \\ X_0 & \cdots & X_0 & X_0 + V \end{bmatrix} \qquad (1)
\end{aligned}
$$

We provide two methods to do the remaining steps:

- Evaluating, $X^{-1}_{Y_{vec}Y_{vec}}$ for the $k \times k$ matrix case seems cumbersome. So we evaluate $\hat{x}$ for simple cases.
  In the 2×2 case,

$$
\begin{aligned}
X^{-1}_{Y_2Y_2} &= \begin{bmatrix} X_0 + V & X_0 \\ X_0 & X_0 + V \end{bmatrix}^{-1} \\
&= \frac{1}{V(2X_0 + V)} \begin{bmatrix} X_0 + V & -X_0 \\ -X_0 & X_0 + V \end{bmatrix} \\
\hat{x}|_{y(0),y(1)} &= \frac{1}{V(2X_0 + V)} \begin{bmatrix} X_0 & X_0 \end{bmatrix} \begin{bmatrix} X_0 + V & -X_0 \\ -X_0 & X_0 + V \end{bmatrix} \begin{bmatrix} y(0) \\ y(1) \end{bmatrix} \\
&= \frac{X_0}{2X_0 + V} y(0) + \frac{V X_0}{2X_0 + V} y(1)
\end{aligned}
$$

$$
\begin{aligned}
X_{\tilde{x}_2\tilde{x}_2} &= X_0 - \frac{1}{V(2X_0 + V)} \begin{bmatrix} X_0 & X_0 \end{bmatrix} \begin{bmatrix} X_0 + V & -X_0 \\ -X_0 & X_0 + V \end{bmatrix} \begin{bmatrix} X_0 \\ X_0 \end{bmatrix} \\
&= X_0 - \frac{2X_0^2}{2X_0 + V} \\
&= \frac{V X_0}{2X_0 + V}
\end{aligned}
$$

In the 3×3 case,

$$\begin{aligned}
X_{Y_3Y_3}^{-1} &= \begin{bmatrix} X_0+V & X_0 & X_0 \\ X_0 & X_0+V & X_0 \\ X_0 & X_0 & X_0+V \end{bmatrix}^{-1} \\
&= \frac{1}{V(3X_0+V)} \begin{bmatrix} 2X_0V+V^2 & -VX_0 & -VX_0 \\ -VX_0 & 2X_0V+V^2 & -VX_0 \\ -VX_0 & -VX_0 & 2X_0V+V^2 \end{bmatrix} \\
\hat{x}|_{y(0),y(1),y(2)} &= \frac{\begin{bmatrix} X_0 & X_0 & X_0 \end{bmatrix}}{V(3X_0+V)} \begin{bmatrix} 2X_0V+V^2 & -VX_0 & -VX_0 \\ -VX_0 & 2X_0V+V^2 & -VX_0 \\ -VX_0 & -VX_0 & 2X_0V+V^2 \end{bmatrix} \begin{bmatrix} y(0) \\ y(1) \\ y(2) \end{bmatrix} \\
&= \frac{X_0}{3X_0+V} y(0) + \frac{X_0}{3X_0+V} y(1) + \frac{X_0}{3X_0+V} y(2) \\
X_{\tilde{x}_3\tilde{x}_3} &= X_0 - \frac{\begin{bmatrix} X_0 & X_0 & X_0 \end{bmatrix}}{V(3X_0+V)} \begin{bmatrix} 2X_0V+V^2 & -VX_0 & -VX_0 \\ -VX_0 & 2X_0V+V^2 & -VX_0 \\ -VX_0 & -VX_0 & 2X_0V+V^2 \end{bmatrix} \begin{bmatrix} X_0 \\ X_0 \\ X_0 \end{bmatrix} \\
&= X_0 - \frac{3X_0^2}{3X_0+V} \\
&= \frac{VX_0}{3X_0+V}
\end{aligned}$$

Therefore, we can expect that (You can also prove this by induction):

$$\begin{aligned}
\hat{x}|_{y(0),y(1),\cdots,y(k)} &= \frac{X_0}{(k+1)X_0+V} y(0) + \frac{X_0}{(k+1)X_0+V} y(1) + \cdots + \frac{X_0}{(k+1)X_0+V} y(k) \\
X_{\tilde{x}|_{Y_{vec}}\tilde{x}|_{Y_{vec}}} &= \frac{VX_0}{(k+1)X_0+V}
\end{aligned}$$

Also, from the above equations,

$$\begin{aligned}
\lim_{X_0\to\infty} \hat{x}|_{y(0),y(1),\cdots,y(k)} &= \frac{1}{k+1} \left(y(0) + y(1) + \cdots + y(k) \right) \\
\lim_{X_0\to\infty} X_{\tilde{x}|_{Y_{vec}}\tilde{x}|_{Y_{vec}}} &= \frac{V}{k+1}
\end{aligned}$$

– An alternative approach is as follows: with some manipulations, we can directly compute the inverse of (1). To do this, we need the matrix inversion lemma:

$$(A+BDC)^{-1} = A^{-1} + A^{-1}B\left(-D^{-1} - CA^{-1}B\right)^{-1} CA^{-1}$$

Now notice that (1) is equivalent to

$$VI_{(k+1)\times(k+1)} + \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{(k+1)\times 1} X_o \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{(k+1)\times 1}^T$$

Hence

$$\begin{bmatrix} X_0+V & X_0 & \cdots & X_0 \\ X_0 & X_0+V & \cdots & X_0 \\ \vdots & \vdots & \ddots & \vdots \\ X_0 & \cdots & X_0 & X_0+V \end{bmatrix}^{-1}_{(k+1)\times(k+1)}$$

$$= \frac{1}{V} I_{k+1} + \frac{1}{V}\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \left(-\frac{1}{X_o} - \begin{bmatrix} 1 & \dots & 1 \end{bmatrix} \frac{1}{V} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & \dots & 1 \end{bmatrix} \frac{1}{V}$$

$$= \frac{1}{V} I_{k+1} + \frac{1}{V^2} \frac{1}{-\frac{1}{X_o} - \frac{1}{V}(k+1)} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}$$

The remaining steps are similar to the previous discussions and ommited here.

- The Kalman-Filter approach: to use the results of Kalman Filter, we need to write the given process in the standard state space form:

$$\begin{aligned} x(k+1) &= x(k), \qquad x(0) = x \\ y(k) &= x(k) + v(k) \end{aligned}$$

Kalman Filter gives:

$$\begin{aligned} \hat{x}(k|k-1) &= \hat{x}(k-1|k-1) \\ \hat{x}(k|k) &= \hat{x}(k|k-1) + F(k)(y(k) - \hat{x}(k|k-1)) \\ \Longrightarrow \hat{x}(k|k) &= (1-F(k))\hat{x}(k-1|k-1) + F(k)y(k) \end{aligned}$$

And:

$$\begin{aligned} M(k+1) &= Z(k),\ M(0) = X_0 \\ Z(k+1) &= M(k+1) - M(k+1)(M(k+1) + V(k+1))^{-1}M(k+1) \end{aligned}$$

$$\begin{aligned} \Longrightarrow Z(k+1) &= Z(k) - \frac{Z^2(k)}{Z(k) + V(k+1)} = \frac{Z(k)V}{Z(k)+V},\ Z(-1) = X_0 \\ \Longrightarrow Z(k) &= \frac{X_0 V}{(k+1)X_0 + V} \end{aligned}$$

$$F(k) = M(k)(M(k)+V(k))^{-1} = \frac{Z(k-1)}{Z(k-1)+V} = Z(k) = \frac{X_0}{(k+1)X_0+V}$$

$$\begin{aligned} 1 - F(k) &= \frac{kX_0 + V}{(k+1)X_0 + V} \\ \Longrightarrow \hat{x}(k|k) &= \frac{kX_0+V}{(k+1)X_0+V}\hat{x}(k-1|k-1) + \frac{X_0}{(k+1)X_0+V}y(k) \\ &= \frac{(k-1)X_0+V}{(k+1)X_0+V}\hat{x}(k-2|k-2) + \frac{X_0}{(k+1)X_0+V}(y(k-1)+y(k)) \\ &\cdots \\ &= \frac{X_0}{(k+1)X_0+V}(y(0)+y(1)+\cdots+y(k)) \end{aligned}$$

When $X_0 \to \infty$,

$$\begin{aligned} Z(k) &= \frac{X_0 V}{(k+1)X_0 + V} \to \frac{V}{k+1} \\ \hat{x}(k) &= \frac{X_0}{(k+1)X_0 + V}(y(0) + y(1) + \cdots + y(k)) \to \frac{1}{k+1}(y(0) + y(1) + \cdots + y(k)) \end{aligned}$$

3. (20 points) Simulating Kalman Filter

The discrete time system is described by:

$$\begin{aligned} \begin{bmatrix} x_(k+1) \\ x_2(k+1) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 0.7114 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.0384 \\ 0.0722 \end{bmatrix} w(k) \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + v(k) \end{aligned}$$

Define $X(k) := E[x(k)x(k)^T]$ where $x(k) := \begin{bmatrix} x_1(k) & x_2(k) \end{bmatrix}^T$. Then $X(k)$ satisfies the Lyapunov difference equation below:

$$X(k+1) = AX(k)A^T + B_w\,.1\,.B_w^T$$

where A and B_w are obvious.

(a) (5pts) Since A, B_w are constant matrices, $X(k)$ reaches a steady state. The steady-state solution therefore satisfies the algebraic Lyapunov equation:

$$X_{ss} = AX_{ss}A^T + B_w B_w^T$$

Solving the above using the matlab command: $\mathtt{X}_{ss} = \mathtt{dlyap}(\mathtt{A}, \mathtt{B}_w * \mathtt{B}_w^T)$, we get,

$$\begin{aligned} X_{ss} &= \begin{bmatrix} 0.01203 & 0.0103 \\ 0.0103 & 0.0106 \end{bmatrix} \\ \Longrightarrow X_{11} &= 0.01203 \end{aligned}$$

(b) (5pts) To get the steady state Kalman filter gain, first solve the Ricatti Equation to get M_{ss}, and then use $F_{ss} = M_{ss}C^T(CM_{ss}C^T + V)^{-1}$ for F_{ss}. The results are:

$$\begin{aligned} r &= 0.05, \quad M_{ss} = \begin{bmatrix} 0.0022 & 0.0033 \\ 0.0033 & 0.0056 \end{bmatrix}, \quad F_{ss} = \begin{bmatrix} 0.9866 \\ 1.4705 \end{bmatrix} \\ r &= 0.5, \quad M_{ss} = \begin{bmatrix} 0.0050 & 0.0053 \\ 0.0053 & 0.0070 \end{bmatrix}, \quad F_{ss} = \begin{bmatrix} 0.6239 \\ 0.6594 \end{bmatrix} \end{aligned}$$

(c) (5pts) Use **kalman** for Kalman filter design, and use either *Simulink* or *Matlab* command **dlsim** for simulation. The results are shown in Figure 3c and 3c.

Error covariance Z and the time average of the error covariance Z_t are:

$$\begin{aligned} r &= 0.05, \quad Z = \begin{bmatrix} 0.0297 & 0.0442 \\ 0.0442 & 0.7376 \end{bmatrix} \times 10^{-3}, \quad Z_t = \begin{bmatrix} 0.02965 & 0.04381 \\ 0.04381 & 0.7317 \end{bmatrix} \times 10^{-3} \\ r &= 0.5, \quad Z = \begin{bmatrix} 0.0019 & 0.0020 \\ 0.0020 & 0.0035 \end{bmatrix}, \quad Z_t = \begin{bmatrix} 0.00181 & 0.00196 \\ 0.00196 & 0.00353 \end{bmatrix} \end{aligned}$$

(d) (5pts) From the Return Difference Equation, we have

$$\left[I + C(zI - A)^{-1}F_{ss}\right](CM_{ss}C^T + V)\left[I + C(z^{-1}I - A)^{-1}F_{ss}\right]^T = V + G(z)WG(z^{-1})^T$$

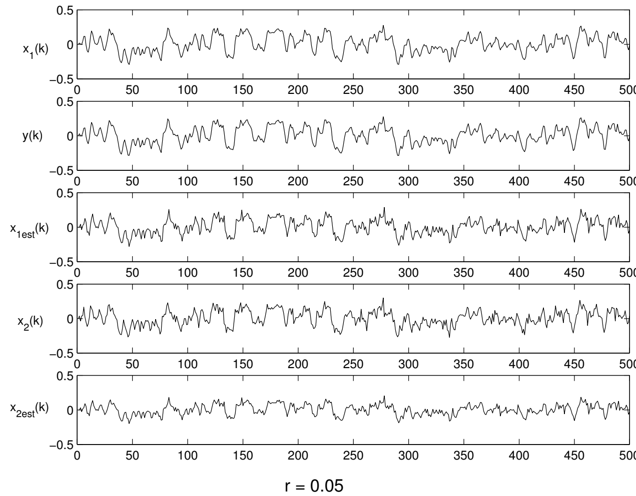

Figure 1: States and their estimation by Kalman filter (r=0.05)

$$\begin{aligned}\beta(z)\beta(z^{-1}) &= \phi(z)\phi(z^{-1})\frac{V}{V + CM_{ss}C^T + V}\left(1 + G(z)\frac{W}{V}G\left(z^{-1}\right)\right)\\ &= \frac{V}{V + CM_{ss}C^T + V}\left(\phi(z)\phi(z^{-1}) + \frac{W}{V}\psi(z)\psi\left(z^{-1}\right)\right)\end{aligned}$$

with $\beta(z)$ the closed-loop characteristic equation, $\phi(z)$ the open-loop characteristic equation, and $G(z) = \frac{0.0384(z+1.169)}{z(z-0.7114)} = \frac{\psi(z)}{\phi(z)}$. The closed-loop poles satisfy $\phi(z)\phi(z^{-1}) + \frac{W}{V}\psi(z)\psi\left(z^{-1}\right) = 0$. The root locus plot should look like Figure 3. Keep in mind, however, that we always have a closed-loop pole at the origin.

4. (15 points) The innovation process is a white random sequence!
(This fact is exploited in the solution of the LQG problem)
Two methods are given below, but only the first method is a complete answer. The second method, which is based on Return Difference Equation, only proves that the innovation process is a white random sequence in the steady-state case.

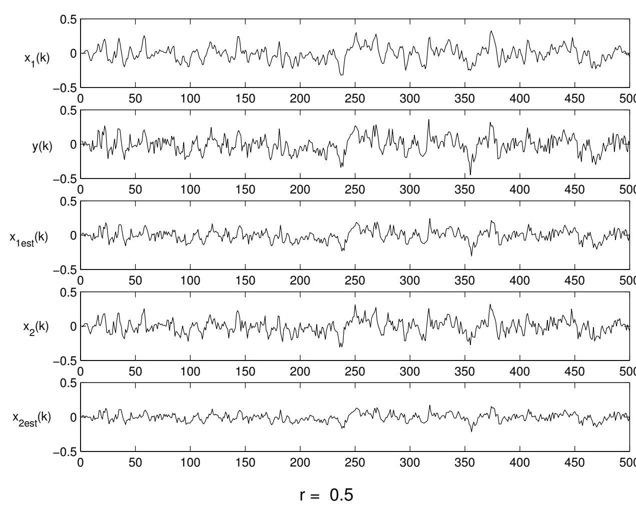

Figure 2: States and their estimation by Kalman filter (r=0.5)

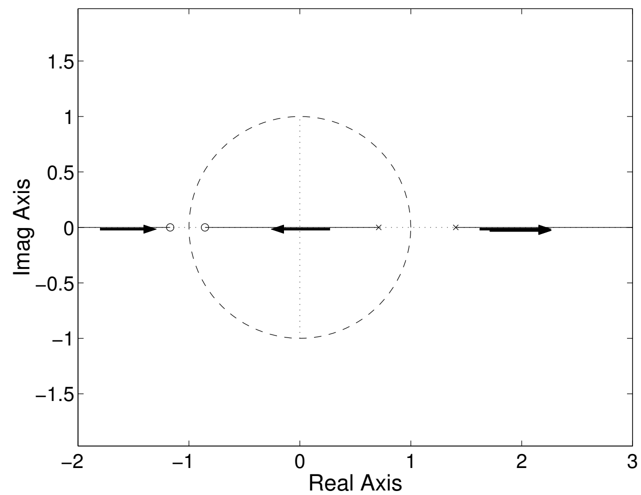

Figure 3: Root Locus plot, Arrows point in the direction of decreasing r

(a) Method 1. The innovation process $e_y(k) = y(k) - C\hat{x}(k|k-1) = v(k) + C\tilde{x}(k|k-1)$, where $\tilde{x}(k|k-1) = x(k) - \hat{x}(k|k-1)$.

$$\begin{aligned} E[e_y(k)] &= E[v(k)] + CE[\tilde{x}(k|k-1)] = 0 \\ E[e_y(k)e_y^T(k)] &= E[(v(k) + C\tilde{x}(k|k-1))(v(k) + C\tilde{x}(k|k-1))^T] \\ &= E[v(k)v^T(k)] + CE[\tilde{x}(k|k-1)v^T(k)] + E[v(k)\tilde{x}^T(k|k-1)]C^T \\ &\quad + CE[\tilde{x}(k|k-1)\tilde{x}^T(k|k-1)]C^T \\ &= V + CM(k)C^T \end{aligned}$$

$e_y(j)$, $j > k$ is a linear function of $\tilde{x}(k|k) = x(k) - \hat{x}(k|k)$, $v(k+1)$, $v(k+2)$, $\cdots$, $v(j)$, $w(k)$, $\cdots$, $w(j-1)$, which are all independent of $e_y(k)$. $v(k+1)$, $\cdots$, $v(j)$, $w(k)$, $\cdots$, $w(j-1)$ are obviously independent of $e_y(k)$. $\tilde{x}(k|k)$ and $e_y(k)$ are independent[1] because:

$$\begin{aligned} E[e_y(k)(x(k) - \hat{x}(k|k))^T] &= E[(v(k) + c\tilde{x}(k|k-1))(\tilde{x}(k|k-1) - F(k)e_y(k))^T] \\ &= CE[\tilde{x}(k|k-1)\tilde{x}^T(k|k-1)] - E[e_y(k)e_y^T(k)F^T(k)] \\ &= CM(k) - (V + CM(k)C^T)F^T(k) \\ &= CM(k) - (V + CM(k)C^T)(V + CM(k)C^T)^{-1}CM(k) \\ &= 0 \end{aligned}$$

Hence, we have $E[e_y(k)e_y^T(j)] = 0$, and $E[e_y(j)e_y^T(k)] = 0$ for $j \neq k$, i.e. $e_y(k)$ is white noise.

(b) Method 2 (You get at most B if you did this way). The innovation sequence $e_y(k) := y(k) - C\hat{x}(k|k-1)$ (in the steady state) is Wide Sense Stationary .

To show that a WSS random process is white, it is enough if we show that the power spectral density of the process is a constant (over all frequencies). This is because the autocorrelation function of a WSS random process is a delta-function.

Since we are looking at the difference between y and $\hat{x}$ both of which have the Bu term, we may

[1] Extended concept: the same conclusion can be obtained by using the properties of the least square estimation, since $\tilde{x}(k|k)$ is the residual of estimation and $e_y(k)$ is the residual of the projection of $y(k)$ onto $y(0), \ldots, y(k-1)$ (refer to the discussion note 4). Please come to see the GSI if you want to know more about this.

assume that $u = 0$. Then $Y(s)$ may be written as:

$$\begin{aligned} y(z) &= C(zI - A)^{-1} B_w \, w(z) + v(z) \\ \Longrightarrow \Phi_{yy}(z) &= C(zI - A)^{-1} B_w \, W \, B_w^T (z^{-1} I - A)^{-T} C^T + V \\ \texttt{i.e.}, \Phi_{yy}(z) &= G(z) \, W \, G^T(z^{-1}) + V \end{aligned}$$

where $G(s) := C(zI - A)^{-1} B_w$
From the block diagram in page $KF - 14$, we get,

$$\begin{aligned} e_y(z) &= [I + CA(zI - A)^{-1} F_s] y(z) \\ \Longrightarrow \Phi_{e_y e_y}(z) &= \left(I + CA(zI - A)^{-1} F_s\right) \Phi_{yy}(z) \left(I + F_s^T (z^{-1} I - A) - A^T C^T\right) \\ &= C \, M \, C^T + V \end{aligned}$$

The last equality stems from the Return Difference equality $(KF - 53)$. Therefore, the innovation process e_y is white.