ME 233 Spring 2012
Solution to Homework #3

1. Write the measurement process as the standard state-space realization:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + B_w w(k) \\ y(k) &= Cx(k) + v(k) \\ x(0) &= X \end{aligned}$$

where $A = 1$, $B = 0$, $B_w = 0$, $C = 1$, and $x(k) = X$. Using a Kalman filter, we can get

$$\begin{aligned} \tilde{y}^o(k) &= y(k) - C\hat{x}^o(k) \\ \hat{x}(k) &= \hat{x}^o(k) + F(k)\tilde{y}^o(k) \\ \hat{x}^o(k+1) &= A\hat{x}(k) + Bu(k) \\ F(k) &= M(k)\left[M(k) + \Sigma_V\right]^{-1} \\ \hat{x}^o(k) &= E[X] = 0 \end{aligned}$$

Then, the above equations can be simplified as:

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + Bu(k) + F(k+1)\left(y(k+1) - CA\hat{x}(k) - CBu(k)\right) \\ &= \hat{x}(k) + F(k+1)\left(y(k+1) - \hat{x}(k)\right) \qquad (1) \end{aligned}$$

where $\hat{x}(0) = X_0\left[\Sigma_V + X_0\right]^{-1} y(0)$ and $F(k+1) = M(k+1)\left[M(k+1) + \Sigma_V\right]^{-1}$ and $M(k+1)$ is updated with the following recursive form:

$$\begin{aligned} M(k+1) &= Z(k) \qquad (2) \\ Z(k) &= M(k) - M(k)\left[M(k) + \Sigma_V\right]^{-1} M(k) \qquad (3) \\ M(0) &= X_0 \qquad (4) \end{aligned}$$

In the Kalman filter, $\hat{x}_{|\bar{y}(k)} = \hat{x}(k)$ and $\Lambda_{X_{|\bar{y}(k)}X_{|\bar{y}(k)}} = Z(k)$. Therefore, by using the recursive equations in (1)–(4), we can get $\hat{x}_{|\bar{y}(k)}$ and $\Lambda_{X_{|\bar{y}(k)}X_{|\bar{y}(k)}}$.

Actually, it is very easy to compute $\hat{x}(k)$ and $Z(k)$ with the recursive equations. From (2)–(4), we know

$$\begin{aligned} M(k+1) &= M(k) - \frac{M^2(k)}{M(k) + \Sigma_V} = \frac{M(k)\Sigma_V}{M(k) + \Sigma_V} \\ \Rightarrow \frac{1}{M(k+1)} &= \frac{1}{M(k)} + \frac{1}{\Sigma_V} = \frac{1}{M(k-1)} + \frac{2}{\Sigma_V} \\ &= \cdots = \frac{1}{M(0)} + \frac{k+1}{\Sigma_V} = \frac{1}{X_0} + \frac{k+1}{\Sigma_V} \\ \Rightarrow M(k+1) &= \frac{X_0\Sigma_V}{\Sigma_V + (k+1)X_0} \\ \Rightarrow \Lambda_{X_{|\bar{y}(k)}X_{|\bar{y}(k)}} &= Z(k) = \frac{X_0\Sigma_V}{\Sigma_V + (k+1)X_0}\,. \end{aligned}$$

Then, (1) changes to be

$$\begin{aligned} \hat{x}(k+1) &= \frac{\Sigma_V + (k+1)X_0}{\Sigma_V + (k+2)X_0}\hat{x}(k) + \frac{X_0}{\Sigma_V + (k+2)X_0}y(k+1) \\ \Rightarrow \left[\Sigma_V + (k+2)X_0\right]\hat{x}(k+1) &= \left[\Sigma_V + (k+1)X_0\right]\hat{x}(k) + X_0 y(k+1) \end{aligned}$$

Define $r(k) \triangleq \left[\Sigma_V + (k+1)X_0\right]\hat{x}(k)$, we get

$$\begin{aligned}
r(k+1) &= r(k) + X_0 y(k+1) = \cdots = r(0) + X_0 \sum_{i=1}^{k+1} y(i) \\
&= \left[\Sigma_V + X_0\right]\hat{x}(0) + X_0 \sum_{i=1}^{k+1} y(i) = X_0 y(0) + X_0 \sum_{i=1}^{k+1} y(i) \\
&= X_0 \sum_{i=0}^{k+1} y(i) \\
\Rightarrow \hat{x}(k) &= \frac{r(k)}{\Sigma_V + (k+1)X_0} = \frac{X_0}{\Sigma_V + (k+1)X_0} \sum_{i=0}^{k} y(i)
\end{aligned}$$

2. (a) We have $m_w \neq 0$, so:

$$\begin{aligned}
M(k) &= E\{\tilde{x}^o(k)\tilde{x}^{oT}(k)\} \\
\hat{x}^o(k) &= A\hat{x}^o(k-1) + Bu(0) + B_w m_w \\
x(k) &= Ax(k-1) + Bu(k-1) + B_w w(k-1) \\
\tilde{x}^o(k) &= A\tilde{x}(k-1) + B_w(w(k-1) - m_w)
\end{aligned}$$

Then we can deduce:

$$\begin{aligned}
M(k) &= E\{[A\tilde{x}^o(k-1) + B_w(w(k-1) - m_w)][A\tilde{x}^o(k-1) + B_w(w(k-1) - m_w)]^T\} \\
&= AE\{A\tilde{x}^o(k-1)\tilde{x}^{oT}(k-1)\}A^T + B_w E\{(w(k-1) - m_w)(w(k-1) - m_w)^T\}B_w \\
&= AZ(k-1)A + B_w W B_w
\end{aligned}$$

The following equation is still valid for $Z(k)$:

$$Z(k) = M(k) - M(k)C^T\left[CM(k)C^T + V\right]^{-1} CM(k)$$

And we have finally:

$$\Lambda_{\tilde{y}^o\tilde{y}^o}(k,0) = CM(k)C^T + V$$

The initial condition is:

$$M(0) = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

to iteratively find the estimation error covariances. To quantify when the estimation error covariance matrices are approaching their steady state value, use matrix norms. For example, one could say that they are approaching their steady state values when

$$\max\left\{\left\|M(k) - M(k-1)\right\|_{i2}, \left\|Z(k) - Z(k-1)\right\|_{i2}\right\} < 10^{-6}$$

Using these equations and this termination condition, the computed steady state estimation error covariances are

$$\begin{aligned}
M_{ss} &= \begin{bmatrix} 0.1608 & 0.0764 \\ 0.0764 & 0.1586 \end{bmatrix} \\
Z_{ss} &= \begin{bmatrix} 0.1335 & 0.0198 \\ 0.0198 & 0.0411 \end{bmatrix} \\
\left(\Lambda_{\tilde{y}^o\tilde{y}^o}\right)_{ss} &= 1.9275
\end{aligned}$$

It is easy to verify that

$$M_{ss} - Z_{ss} \quad > \quad 0$$

Thus, our a-posteriori estimates always do "better" than the a-priori estimates in the sense that they have a smaller covariance.
After we can deduce $F(k)$ and $L(k)$ thanks to the two following equations:

$$\begin{aligned} F(k) &= M(k)C^T[CM(k)C^T + V]^{-1} \\ L(k) &= AM(k)C^T[CM(k)C^T + V]^{-1} \end{aligned}$$

We obtain:

$$\begin{aligned} L &= \begin{bmatrix} -0.2564 \\ 0.1079 \end{bmatrix} \\ F &= \begin{bmatrix} 0.1189 \\ 0.2469 \end{bmatrix} \end{aligned}$$

(b) Figure 1 shows the a-priori output estimation error covariance as a function of time.

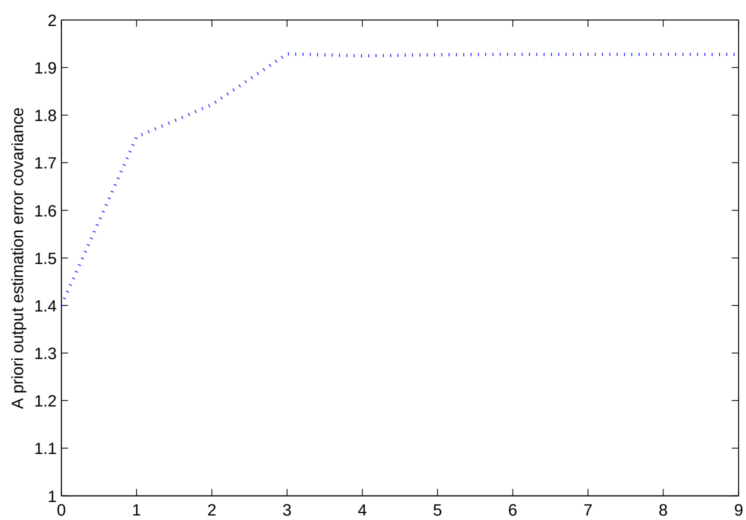

Figure 1: A priori output estimation error covariance vs. time

3. In this problem, we will derive the Kalman filter for a system in which $w(k)$ and $v(k)$ are correlated. As the first step, we will transform the coordinates of $w(k)$ so that its transformed coordinates are uncorrelated with $v(k)$. For simplicity, we will choose our transformation to be

$$w'(k) := w(k) + Tv(k)$$

With this choice of transformation, we get

$$\begin{aligned} \Lambda_{W'V}(0) &= E\left\{[w(k) + Tv(k)]\, v^T(k)\right\} \\ &= S + TV \end{aligned}$$

To make $w'(k)$ uncorrelated with $v(k)$, we will choose $T = -SV^{-1}$. The covariance of $w'(k)$ is given by

$$
\begin{aligned}
\Lambda_{W'W'}(0) &= E\left\{[w(k)+Tv(k)]\,[w(k)+Tv(k)]^T\right\} \\
&= W + ST^T + TS^T + TVT^T \\
&= W - SV^{-1}S^T
\end{aligned}
$$

Noting that

$$
\begin{aligned}
w(k) &= w'(k) + SV^{-1}v(k) \\
&= w'(k) + SV^{-1}\big[y(k) - Cx(k)\big]
\end{aligned}
$$

our governing equations become

$$
\begin{aligned}
x(k+1) &= \big[A - SV^{-1}C\big]x(k) + \begin{bmatrix} B & SV^{-1} \end{bmatrix}\begin{bmatrix} u(k) \\ y(k) \end{bmatrix} + w'(k) \\
y(k) &= Cx(k) + v(k)
\end{aligned}
$$

where $w'(k)$ and $v(k)$ are uncorrelated. Note that $\begin{bmatrix} u(k) & y(k) \end{bmatrix}^T$ is a deterministic quantity. Using the Kalman filter results derived in class, we get

$$
\begin{aligned}
\hat{x}(k) &= \hat{x}^o(k) + M(k)C^T\left[CM(k)C^T + V\right]^{-1}\tilde{y}^o(k) \\
\hat{x}^o(k+1) &= \big[A - SV^{-1}C\big]\hat{x}(k) + \begin{bmatrix} B & SV^{-1} \end{bmatrix}\begin{bmatrix} u(k) \\ y(k) \end{bmatrix} \\
Z(k) &= M(k) - M(k)C^T\left[CM(k)C^T + V\right]^{-1}CM(k) \\
M(k+1) &= \big[A - SV^{-1}C\big]Z(k)\big[A - SV^{-1}C\big]^T + \big[W - SV^{-1}S^T\big]
\end{aligned}
$$

To make the algebra easier, we will define

$$
\begin{aligned}
L(k) &:= \left[A - SV^{-1}C\right]\left[M(k)C^T\right]\left[CM(k)C^T + V(k)\right]^{-1} + SV^{-1} \\
&= \left[AM(k)C^T - SV^{-1}CM(k)C^T\right]\left[CM(k)C^T + V(k)\right]^{-1} \\
&\quad + \left[SV^{-1}CM(k)C^T + SV^{-1}V\right]\left[CM(k)C^T + V(k)\right]^{-1} \\
&= \left[AM(k)C^T + S\right]\left[CM(k)C^T + V(k)\right]^{-1}
\end{aligned}
$$

Simplifying the state estimation equations gives

$$
\begin{aligned}
\hat{x}^o(k+1) &= \left[A - SV^{-1}C\right]\Big(\hat{x}^o(k) + M(k)C^T\left[CM(k)C^T + V(k)\right]^{-1}\tilde{y}^o(k)\Big) \\
&\quad + Bu(k) + SV^{-1}y(k) \\
&= \left[A - SV^{-1}C\right]\hat{x}^o(k) + \left[L(k) - SV^{-1}\right]\tilde{y}^o(k) + Bu(k) + SV^{-1}y(k) \\
&= A\hat{x}^o(k) + \left[L(k) - SV^{-1}\right]\tilde{y}^o(k) + Bu(k) + SV^{-1}\tilde{y}^o(k) \\
&= A\hat{x}^o(k) + Bu(k) + L(k)\tilde{y}^o(k) \\
&= A\hat{x}^o(k) + Bu(k) + L(k)\left[y(k) - C\hat{x}^o(k)\right]
\end{aligned}
$$

Simplifying the state estimation covariance equations gives

$$
\begin{aligned}
M(k+1) &= \left[A - SV^{-1}C\right]M(k)\left[A - SV^{-1}C\right]^T + \left[W - SV^{-1}S^T\right] \\
&\quad - \left[A - SV^{-1}C\right]M(k)C^T\left[CM(k)C^T + V\right]^{-1}CM(k)\left[A - SV^{-1}C\right] \\
&= \left[A - SV^{-1}C\right]M(k)\left[A - SV^{-1}C\right]^T + \left[W - SV^{-1}S^T\right] \\
&\quad - \left[L(k) - SV^{-1}\right]\Big(\left[L(k) - SV^{-1}\right]\left[CM(k)C^T + V\right]\Big)^T \\
&= AM(k)A^T - SV^{-1}CM(k)A^T - \left[A - SV^{-1}C\right]M(k)C^TV^{-1}S^T \\
&\quad + W - SV^{-1}S^T - L(k)\left[CM(k)C^T + V\right]L^T(k) \\
&\quad + SV^{-1}\left[CM(k)C^T + V\right]L^T(k) + \left[L(k) - SV^{-1}\right]\left[CM(k)C^T + V\right]V^{-1}S^T
\end{aligned}
$$

Note that the third term in the last expression cancels with the last term in that expressions. Thus,

$$\begin{aligned}
M(k+1) &= AM(k)A^T - SV^{-1}CM(k)A^T + W - SV^{-1}S^T \\
&\quad -L(k)\left[CM(k)C^T + V\right]L^T(k) + SV^{-1}\left[CM(k)C^T + V\right]L^T(k) \\
&= AM(k)A^T + W - L(k)\left[CM(k)C^T + V\right]L^T(k) \\
&\quad -SV^{-1}\left[AM(k)C^T + S\right]^T + SV^{-1}\left[CM(k)C^T + V\right]L^T(k) \\
&= AM(k)A^T + W - \left[AM(k)C^T + S\right]\left[CM(k)C^T + V\right]^{-1}\left[CM(k)A^T + S^T\right]
\end{aligned}$$

Thus, in summary

$$\begin{aligned}
\hat{x}^o(k+1) &= A\hat{x}^o(k) + Bu(k) + L(k)\left[y(k) - C\hat{x}^o(k)\right] \\
L(k) &= \left[AM(k)C^T + S\right]\left[CM(k)C^T + V\right]^{-1} \\
M(k+1) &= AM(k)A^T + W - \left[AM(k)C^T + S\right]\left[CM(k)C^T + V(k)\right]^{-1}\left[CM(k)A^T + S^T\right]
\end{aligned}$$