ME 233 Spring 2012
Final Exam Solutions

Problem 1

1. First note that the system is in controllable canonical form. Thus, the transfer function from $u(k)$ to $Cx(k)$ is

$$G(z) = \frac{z - 1.25}{z^2 - 0.3z - 0.4} = \frac{z - 1.25}{(z - 0.8)(z + 0.5)}$$

The reciprocal root locus plot for this LQR design is shown in Figure 1. Note that, regardless of how R is chosen, the closed-loop system always has a pole at $z = 0.8$.

2. Since the closed-loop poles converge to the open-loop zeros of G (or their inverses) as $R \to 0$, it is sufficient to choose the zeros so that they both have magnitude less than 0.5. Regardless of how C is chosen, $G(z)$ will always have relative degree of at least 1, so there will always be a zero at the origin. By choice of C, we can choose the location of the other zero. For instance, putting a zero at $z = 0.4$, which corresponds to

$$C = \begin{bmatrix} -0.4 & 1 \end{bmatrix}$$

results in the reciprocal root locus plot shown in Figure 2. Note that as $R \to 0$, the closed-loop poles converge to 0 and 0.4.

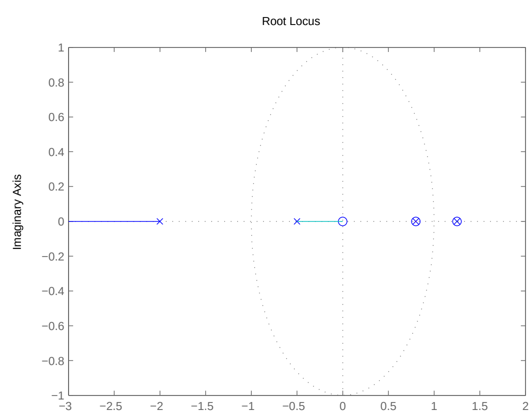

Figure 1: Reciprocal root locus plot for Problem 1, part 1

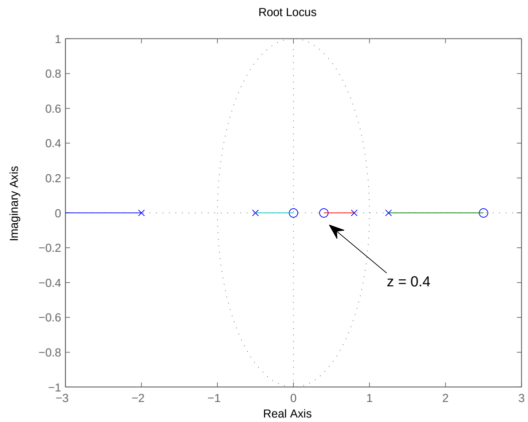

Figure 2: Reciprocal root locus plot for Problem 1, part 2

Problem 2

1. Suppose a causal output feedback controller achieves $E\{u^2(k)\} \leq \alpha$. Since the control law is suboptimal in terms of the cost function $J(\rho)$, we have

$$\begin{aligned} J(\rho) &\leq E\{x^T(k)Qx(k) + \rho u^T(k)u(k)\} \\ &= E\{x^T(k)Qx(k)\} + \rho E\{u^T(k)u(k)\} \\ &\leq E\{x^T(k)Qx(k)\} + \alpha\rho \end{aligned}$$

Rearranging terms, we have

$$E\{x^T(k)Qx(k)\} \geq J(\rho) - \alpha\rho$$

2. From the previous part, we know that any control law that satisfies $E\{u^2(k)\} \leq \alpha$ must also satisfy $E\{x^T(k)Qx(k)\} \geq J(\rho_*) - \alpha\rho_*$. This implies that $\bar{J} \geq J(\rho_*) - \alpha\rho_*$.

It now remains to show that the control law that optimizes $J(\rho_*)$ achieves this lower bound on $\bar{J}$. For the control law that optimizes $J(\rho_*)$, we have $E\{u^2(k)\} = \alpha$ and

$$\begin{aligned} J(\rho_*) &= E\{x^T(k)Qx(k) + \rho_* u^T(k)u(k)\} \\ &= E\{x^T(k)Qx(k)\} + \rho_* E\{u^T(k)u(k)\} \\ &= E\{x^T(k)Qx(k)\} + \alpha\rho_* \end{aligned}$$

which implies that

$$E\{x^T(k)Qx(k)\} = J(\rho_*) - \alpha\rho_* \qquad\qquad E\{u^2(k)\} \leq \alpha$$

Therefore, this control law satisfies the variance constraint on the control input and achieves the lower bound on $\bar{J}$ given by $J(\rho_*) - \alpha\rho_*$. Therefore, this control law optimizes the linear quadratic Gaussian control problem with a variance constraint and the corresponding cost is $\bar{J} = J(\rho_*) - \alpha\rho_*$.

Problem 3

1. Since $B(q^{-1})$ is a Schur polynomial of q^{-1}, none of the plant zeros should be canceled. Therefore, we use a standard zero phase error tracking compensator, which is given by

$$T(q, q^{-1}) = \frac{q^{+\mathrm{d}} A(q^{-1}) B(q)}{B^2(1)}$$

2. When the control policy $u(k) = T(q, q^{-1})u_d(k)$ is used, the system output is given by

$$\begin{aligned} y(k) &= \left(\frac{q^{-\mathrm{d}}B(q^{-1})}{A(q^{-1})}\right)\left(\frac{q^{+\mathrm{d}}A(q^{-1})B(q)}{B^2(1)}\right)u_d(k) \\ &= \frac{B(q^{-1})B(q)}{B^2(1)}u_d(k) \end{aligned}$$

Since the desired system output is expressed as a sum of discrete-time sinusoids, it is appropriate to use frequency domain analysis. The frequency response from $u_d(k)$ to $y(k)$ is

$$\frac{B(e^{-j\omega})B(e^{j\omega})}{B^2(1)} = \frac{|B(e^{j\omega})|^2}{B^2(1)}$$

Therefore, if we choose

$$u_d(k) = \sum_{i=1}^{r} \frac{B^2(1)}{|B(e^{j\omega})|^2} c_i \sin(\omega_i k + \phi_i)$$

the output $y(k)$ will perfectly track $y_d(k)$.

Problem 4

Since the system is stable, there is no need to do any pole placement. Moreover, note that $d(k)$ and $r(k)$ are both periodic with period $N = 35$. Therefore, to make the tracking error asymptotically converge to zero, we only need to use the repetitive controller

$$\begin{aligned} u(k) &= C_R(q^{-1})[r(k) - y(k)] \\ C_R(q^{-1}) &= \frac{k_r}{b}\frac{q^{-N}}{1 - q^{-N}}q^{+\mathrm{d}}A(q^{-1}) \end{aligned}$$

where

$$0 < k_r < 2 \qquad b \geq 1 \qquad \mathrm{d} = 2 \qquad A(q^{-1}) = 1 - 0.8q^{-1}$$

Problem 5

1. To find the $A(q^{-1})$ and $B(q^{-1})$ polynomials and the value of d, we find the transfer function from $u(k)$ to $y(k)$:

$$\begin{aligned} \frac{z^{-\mathrm{d}}B(z^{-1})}{A(z^{-1})} &= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}z-1 & -1 \\ 0 & z\end{bmatrix}^{-1}\begin{bmatrix}1 \\ -2\end{bmatrix} = \frac{1}{z^2 - z}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}z & 1 \\ 0 & z-1\end{bmatrix}^{-1}\begin{bmatrix}1 \\ -2\end{bmatrix} = \frac{z-2}{z^2-z} \\ &= \frac{z-2}{z^2-z}\frac{z^{-2}}{z^{-2}} = \frac{z^{-1}(1-2z^{-1})}{1-z^{-1}} \end{aligned}$$

Therefore, we have

$$A(q^{-1}) = 1 - q^{-1} \qquad B(q^{-1}) = 1 - 2q^{-1} \qquad \mathrm{d} = 1$$

To find the $C(q^{-1})$ polynomial, we first find the stationary Kalman filter for the system. The relevant DARE is

$$\begin{aligned} \begin{bmatrix}m_1 & m_2 \\ m_2 & m_3\end{bmatrix} &= \begin{bmatrix}1 & 1 \\ 0 & 0\end{bmatrix}\begin{bmatrix}m_1 & m_2 \\ m_2 & m_3\end{bmatrix}\begin{bmatrix}1 & 0 \\ 1 & 0\end{bmatrix} + \begin{bmatrix}1 \\ 0\end{bmatrix}1\begin{bmatrix}1 & 0\end{bmatrix} \\ &\quad - \begin{bmatrix}1 & 1 \\ 0 & 0\end{bmatrix}\begin{bmatrix}m_1 & m_2 \\ m_2 & m_3\end{bmatrix}\begin{bmatrix}1 \\ 0\end{bmatrix}\left(\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}m_1 & m_2 \\ m_2 & m_3\end{bmatrix}\begin{bmatrix}1 \\ 0\end{bmatrix} + 2\right)^{-1}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}m_1 & m_2 \\ m_2 & m_3\end{bmatrix}\begin{bmatrix}1 & 0 \\ 1 & 0\end{bmatrix} \end{aligned}$$

By examining the $(1,2)$ and $(2,2)$ blocks of the DARE, we immediately see that $m_2 = 0$ and $m_3 = 0$. Thus, it only remains to find a value of m_1 that satisfies the $(1,1)$ block of the DARE, i.e. we need to find m_1 such that

$$m_1 = m_1 + 1 - m_1(m_1+2)^{-1}m_1$$

This is equivalent to finding $m_1 \neq -2$ that satisfies

$$m_1 + 2 = m_1^2$$

The two solutions of this equation are $m_1 = 2$ and $m_1 = -1$. Since we are interested in the positive semi-definite solution of the DARE, we take $m_1 = 2$. The corresponding Kalman filter gain is

$$L = \begin{bmatrix} m_1 \\ 0 \end{bmatrix} (m_1+2)^{-1} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

Therefore, we have

$$\begin{aligned} C(z^{-1}) &= z^{-2} \det\left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right) = z^{-2} \det \begin{bmatrix} z - 0.5 & -1 \\ 0 & z \end{bmatrix} \\ &= z^{-2}(z^2 - 0.5z) \end{aligned}$$

This yields $C(q^{-1}) = 1 - 0.5z^{-1}$, which is anti-Schur as desired.

2. To solve the minimum variance regulator problem, we first factor $B(q^{-1}) = B^s(q^{-1})B^u(q^{-1})$ where $B^s(q^{-1})$ is anti-Schur, $B^u(q^{-1})$ is Schur, and $B^u(q^{-1})$ is monic. This corresponds to choosing

$$B^s(q^{-1}) = -2 \qquad\qquad B^u(q^{-1}) = -\frac{1}{2} + q^{-1}$$

We also define

$$\bar{B}^u(q) = qB^u(q^{-1}) = -\frac{1}{2}q + 1$$

and note that

$$\bar{B}^u(q^{-1}) = 1 - \frac{1}{2}q^{-1}$$

To design the minimum variance regulator, we must solve the Diophantine equation

$$C(q^{-1})\bar{B}^u(q^{-1}) = A(q^{-1})R(q^{-1}) + q^{-\mathrm{d}}B^u(q^{-1})S(q^{-1})$$

for $R(q^{-1})$ and $S(q^{-1})$. The orders of these polynomials are $n_r = 1$ and $n_s = 0$. Therefore, we need to solve for values of r_1 and s_0 that satisfy

$$(1 - 0.5q^{-1})(1 - 0.5q^{-1}) = (1 - q^{-1})(1 + r_1q^{-1}) + q^{-1}(-0.5 + q^{-1})s_0$$

Equating coefficients for the q^{-1} and q^{-2} terms on both sides of the equation respectively yields the equations

$$\begin{aligned} -1 &= r_1 - 1 - 0.5s_0 \\ 0.25 &= -r_1 + s_0 \end{aligned}$$

Solving these equations yields $s_0 = 0.5$ and $r_1 = 0.25$. The optimal controller is given by

$$B^s(q^{-1})R(q^{-1})u(k) = -S(q^{-1})y(k)$$

Plugging in all relevant values yields the optimal control law

$$-2(1 + 0.25q^{-1})u(k) = -0.5y(k)$$

Problem 6

1. We first define $d_k = d(k)$ for $k = 0, 1, 2$ and note that the sequence $d(k)$ is fully prescribed by d_0, d_1, and d_2. We now express the sequence $d(k)$ as

$$d(k) = d_0 f(k) + d_1 f(k-1) + d_2 f(k-2)$$

where $f(k)$ is the indicator function defined in the problem statement. We now express the system dynamics as

$$\begin{aligned}
&(1 + a_1 q^{-1} + a_2 q^{-2})y(k+1) = q^{-2}(b_0 + b_1 q^{-1})u(k+1) + q^{-1}d(k+1) \\
\Rightarrow \quad & y(k+1) = -a_1 y(k) - a_2 y(k-1) + b_0 u(k-1) + b_1 u(k-2) + d(k)
\end{aligned}$$

Using (1), we express

$$y(k+1) = -a_1 y(k) - a_2 y(k-1) + b_0 u(k-1) + b_1 u(k-2) + d_0 f(k) + d_1 f(k-1) + d_2 f(k-2)$$

Therefore, defining

$$\theta = \begin{bmatrix} a_1 & a_2 & b_0 & b_1 & d_0 & d_1 & d_2 \end{bmatrix}^T$$

we can write $y(k+1) = \phi^T(k)\theta$.

2. This is in the standard form for adaptive identification, so we use the general PAA presented in lecture

$$\begin{aligned}
e^o(k+1) &= y(k+1) - \phi^T(k)\hat{\theta}(k) \\
e(k+1) &= \frac{\lambda_1(k)}{\lambda_1(k) + \phi^T(k)F(k)\phi(k)} \\
\hat{\theta}(k+1) &= \hat{\theta}(k) + F(k)\phi(k)e(k+1) \\
F(k+1) &= \frac{1}{\lambda_1(k)}\left[F(k) - \lambda_2(k)\frac{F(k)\phi(k)\phi^T(k)F(k)}{\lambda_1(k) + \lambda_2(k)\phi^T(k)F(k)\phi(k)}\right]
\end{aligned}$$

where

$$F(0) \succ 0 \qquad\qquad 0 < \underline{\lambda}_1 \leq \lambda_1(k) \leq 2 \qquad\qquad 0 \leq \lambda_2(k) \leq \overline{\lambda}_2 < 2$$

and $\lambda_1(k)$ is the forgetting factor.

3. As in lecture, we can guarantee that $e^o(k)$ converges to zero if $F(k)$ and $\phi(k)$ remain bounded. We can guarantee this if:

   - $\lambda_{max}(F(k)) < K_{max} < \infty$
   - $u(k)$ is bounded
   - $A(q^{-1})$ is an anti-Schur polynomial of q^{-1}