ME233 Advance Control II
Lecture 2

Review of
ME 232 Lectures 25 & 26

Linear Quadratic Regulators (LQR)
PART I

(ME232 Class Notes pp. 135-137)

Outline

Previous lecture :

- Dynamic programming
- Solution of finite-horizon LQR

This Lecture: Review ME232 results on

- Infinite horizon LQR (steady state)

LTI Optimal regulators

- State space description of a discrete time LTI

$$x(k+1) \;=\; A\,x(k) + B\,u(k) \qquad x(0) = x_o$$

- Find optimal control $u^0(k), \;\; k = 0,\, 1,\, 2 \cdots$

- That drives the state to the origin

$$x \rightarrow 0$$

Finite Horizon LQ optimal regulator

LTI system:

$$x(k+1) \;=\; A\,x(k) + B\,u(k) \qquad x(0) = x_o$$

We want to find the optimal control sequence:

$$U_0^o \;=\; \{u^o(0),\, u^o(1),\, \cdots,\, u^o(N-1)\}$$

which minimizes the cost functional:

$$J[x_o] = \frac{1}{2}x^T(N)\,S\,x(N) + \frac{1}{2}\sum_{k=0}^{N-1}\left\{x^T(k)\,Q\,x(k) + u^T(k)\,R\,u(k)\right\}$$

LQ Cost Functional:

$$J = \frac{1}{2} x^T(N)\, S\, x(N) + \frac{1}{2} \sum_{k=0}^{N-1} \left\{ x^T(k)\, Q\, x(k) + u^T(k)\, R\, u(k) \right\}$$

- N is the total number of steps
- $\frac{1}{2} x^T(N)\, S\, x(N)$ penalizes the final state deviation from the origin
- $\frac{1}{2} x^T(k)\, Q\, x(k)$ penalizes the transient state deviation from the origin
- $\frac{1}{2} u^T(k)\, R\, u(k)$ penalizes the control effort

$$S = S^T \succeq 0 \qquad Q = Q^T \succeq 0 \qquad R = R^T \succ 0$$

Finite Horizon LQR Solution:

For ***k = 0, … N-1*** we have:

$$u^o(k) = -K(k+1)\, x(k)$$

$$K(k+1) = \left[R + B^T P(k+1) B\right]^{-1} B^T P(k+1) A$$

Riccati difference equation (computed backwards):

$$P(k-1) = Q + A^T P(k) A - A^T P(k) B \left[R + B^T P(k) B\right]^{-1} B^T P(k) A$$

$$P(N) = S$$

The optimal cost function $J^o[x(k)]$

$$J^o[x(k)] \;=\; \frac{1}{2} x^T(k)\, P(k) x(k)$$

$$P(k-1) \;=\; Q + A^T P(k) A$$
$$-A^T P(k) B \left[R + B^T P(k) B\right]^{-1} B^T P(k) A$$

$P(N) = S$ *boundary condition*

Computation of P(k) entirely recursive !!
(starting from N and going backwards)

Properties of Matrix *P(k)*

Assume that $P(N) = S$ $\qquad S = S^T \succeq 0$, $Q = Q^T \succeq 0$, $R = R^T \succ 0$

Then:

1) $P(k) = P^T(k)$ (symmetric)

2) $P(k) \succeq 0$ (positive semi-definite)

These two properties are easy to proof

Proof that $P(k) = P^T(k)$

- $P(N) = S$ where $S = S^T \succ 0$, $Q = Q^T \succeq 0$

Use induction and assume $P(k) = P^T(k)$

$$\underbrace{P(k-1)}_{symmetric} = Q + \overbrace{A^T P(k) A}^{symmetric} \underbrace{- A^T P(k) B \left[R + B^T P(k) B \right]^{-1} B^T P(k) A}_{symmetric}$$

symmetric

- Then, $P(k-1) = P^T(k-1)$

Proof that $P(k) \succeq 0$

Riccati Equation (RE)

$$\begin{aligned} P(k-1) &= Q + A^T P(k) A \\ &\quad - A^T P(k) B \left[R + B^T P(k) B \right]^{-1} B^T P(k) A \end{aligned}$$

Can be written as the ***Joseph stabilized RE***

$$\begin{aligned} P(k-1) &= Q + K^T(k)\, R\, K(k) \\ &\quad + (A - BK(k))^T P(k) (A - BK(k)) \end{aligned}$$

- where ***K(k)*** is the optimal feedback gain

$$K(k) = \left[R + B^T P(k) B \right]^{-1} B^T P(k) A$$

Proof that $P(k) \succeq 0$

$P(N) = S$ where $S = S^T \succeq 0$, $Q = Q^T \succeq 0$, $R = R^T \succ 0$

Use induction and assume $P(k) \succeq 0$

$$\underbrace{P(k-1)}_{\succeq 0} = Q + \overbrace{K^T(k)\, R\, K(k)}^{\succeq 0} + \underbrace{(A - BK(k))^T P(k) (A - BK(k))}_{\succeq 0}$$

- Then, $P(k-1) \succeq 0$

Derivation of the Joseph stabilized RE

$$\begin{aligned} P(k-1) &= Q + A^T P(k) A \\ &\quad - A^T P(k) B \underbrace{\left[R + B^T P(k) B \right]^{-1} B^T P(k) A}_{K(k)} \end{aligned}$$

$$= Q + A^T P(k) A - A^T P(k) B K(k)$$

(eliminate (k))

$$= Q + A^T P A - 2 A^T P B K + A^T P B \overbrace{\left[R + B^T P B \right]^{-1} \left[R + B^T P B \right]}\, K$$

$$= Q + A^T P A - 2 A^T P B K + K^T [R + B^T P B] K$$

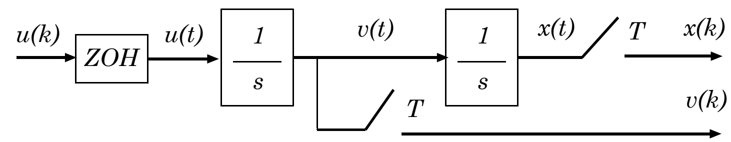
Derivation of the Joseph stabilized RE

$$\begin{aligned} P(k-1) &= Q + A^T P A \\ &\quad -2A^T PBK + K^T[R + B^T PB]K \\ &= Q + K^T RK \\ &\quad + A^T PA - 2A^T PBK + K^T B^T PBK \end{aligned}$$

$$\boxed{\begin{aligned} P(k-1) &= Q + K^T(k) R K(k) \\ &\quad + [A - BK(k)]^T P(k) [A - BK(k)] \\ K(k) &= \left[R + B^T P(k) B\right]^{-1} B^T P(k) A \end{aligned}}$$

Example – Double Integrator

Double integrator with ZOH and sampling time T=1:



$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} u(k)$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u(k)$$

Example – Double Integrator

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u(k)$$

LQR cost:

$$J_N[x_o] = \frac{1}{2} x^T(N)\, S\, x(N) + \frac{1}{2} \sum_{k=0}^{N-1} \left\{ x^T(k)\, Q\, x(k) + R\, u^2(k) \right\}$$

Choose: $R > 0$ $\quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ only penalize position x_1

$S = S^T \succeq 0$ (remember that) $S = P(N)$

Example – Double Integrator (DI)

Compute ***P(k)*** for an arbitrary $P(N) = S$ and N.

Computing backwards: $P(N) = S$

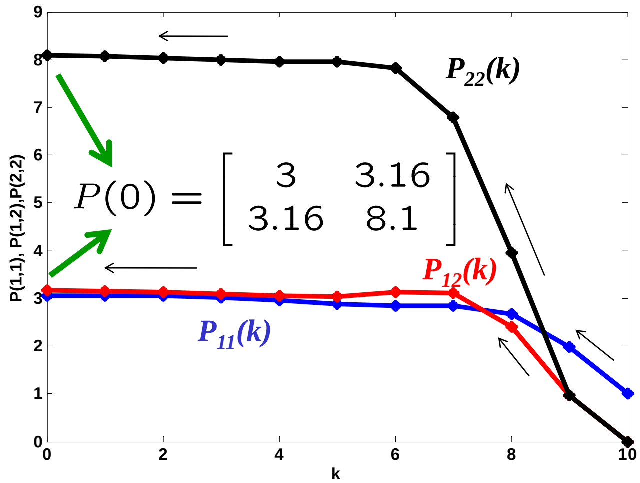
$$\begin{aligned} P(k-1) &= Q + A^T P(k) A \\ &\quad - A^T P(k) B \left[R + B^T P(k) B\right]^{-1} B^T P(k) A \end{aligned}$$

$R > 0$

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$ $\quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

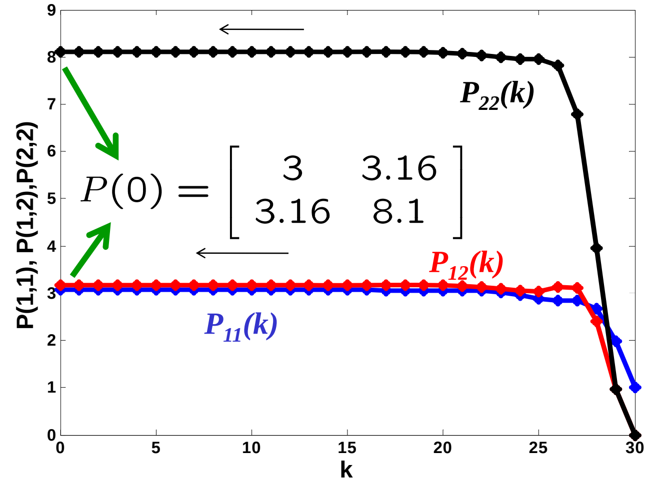
Example – DI Finite Horizon Case 1

- $N = 10$, $R = 10$, $P(10) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$



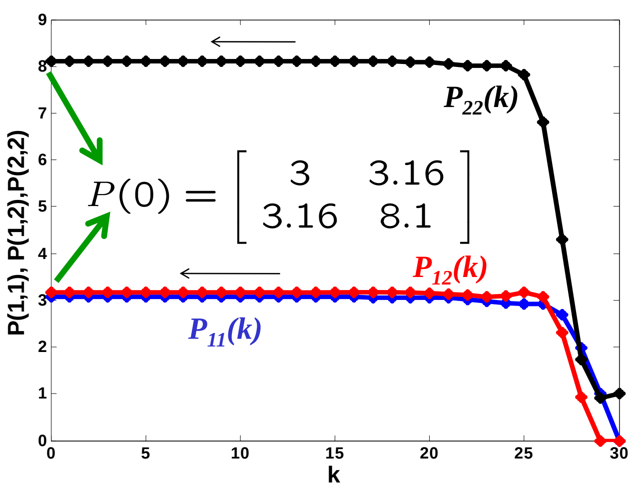
Example – DI Finite Horizon Case 2

- $N = 30$, $R = 10$, $P(30) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$



Example – DI Finite Horizon Case 3

- $N = 30$, $R = 10$, $P(30) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$



Example – DI Finite Horizon

Observation:

In all cases, regardless of the choice of $P(N) = S$

when the finite horizon index, N, is sufficiently large

- the backwards computation of the Riccati Eq. always converges to the same solution:

$$P(0) = \begin{bmatrix} 3 & 3.16 \\ 3.16 & 8.1 \end{bmatrix}$$

Infinite Horizon LQ regulator

LTI system:

$$x(k+1) \;=\; A\,x(k) + B\,u(k) \qquad x(0) = x_o$$

LQR that minimizes the cost:

$$J_N[x_o] = \frac{1}{2}x^T(N)\,S\,x(N) + \frac{1}{2}\sum_{k=0}^{N-1}\left\{x^T(k)\,Q\,x(k) + u^T(k)\,R\,u(k)\right\}$$

- We now consider the limiting behavior when

$$N \longrightarrow \infty$$

Infinite Horizon (IH) LQ regulator

Consider the limiting behavior when $N \longrightarrow \infty$

LTI system:

$$x(k+1) \;=\; A\,x(k) + B\,u^o(k) \qquad x(0) = x_o$$

$$u^o(k) = -K(k+1)\,x(k)$$

Riccati equation:

$$P(N) = S$$

$$P(k-1) = Q + A^T P(k) A - A^T P(k) B\left[R + B^T P(k) B\right]^{-1} B^T P(k) A$$

$$K(k+1) = \left[R + B^T P(k+1) B\right]^{-1} B^T P(k+1) A$$

Infinite Horizon LQ regulator property 1

Consider the limiting behavior when $N \longrightarrow \infty$

1) When does there exist a **<u>BOUNDED limiting</u>** solution

$$P(0) = P_\infty$$

to the Riccati Eq.

$$P(k-1) = Q + A^T P(k) A - A^T P(k) B\left[R + B^T P(k) B\right]^{-1} B^T P(k) A$$

<u>for all</u> choices $P(N) = S = S^T \succeq 0$?

Infinite Horizon LQ regulator property 2

Consider the limiting behavior when $N \longrightarrow \infty$

2) When does there exist a **<u>UNIQUE limiting</u>** solution

$$P(0) = P_\infty$$

to the Riccati Eq.

$$P(k-1) = Q + A^T P(k) A - A^T P(k) B\left[R + B^T P(k) B\right]^{-1} B^T P(k) A$$

<u>regardless</u> of the choice $P(N) = S = S^T \succeq 0$?

Infinite Horizon LQ regulator property 3

Consider the limiting behavior when $N \longrightarrow \infty$

3) When does the **limiting** solution

$$P(0) = P_\infty$$

to the Riccati Eq.

yield an **asymptotically stable** closed loop system?

$$A_c = A - BK_\infty$$

is Schur
(all eigenvalues inside unit circle)

$$K_\infty = \left[R + B^T P_\infty B\right]^{-1} B^T P_\infty A$$

LQ regulator Cost

$$J_N[x_o] = \frac{1}{2} x^T(N)\, S\, x(N) + \frac{1}{2} \sum_{k=0}^{N-1} \left\{ \underline{x^T(k)\, Q\, x(k)} + u^T(k)\, R\, u(k) \right\}$$

$$\frac{1}{2} x^T(k)\, Q\, x(k)$$

penalizes the state deviation from the origin

Define the square root matrix of ***Q***, i.e.

Define the matrix ***C*** such that $C^T C = Q$

$$J_N[x_o] = \frac{1}{2} x^T(N)\, S\, x(N) + \frac{1}{2} \sum_{k=0}^{N-1} \left\{ \underline{x^T(k)\, C^T\, C\, x(k)} + u^T(k)\, R\, u(k) \right\}$$

LQ regulator Cost

$$J_N[x_o] = \frac{1}{2} x^T(N)\, S\, x(N) + \frac{1}{2} \sum_{k=0}^{N-1} \left\{ \underline{x^T(k)\, Q\, x(k)} + u^T(k)\, R\, u(k) \right\}$$

- Define the matrix ***C*** such that $C^T C = Q$
- Define the fictitious output ***y(k)*** such that

$$y(k) = C\, x(k)$$

$$J_N[x_o] = \frac{1}{2} x^T(N)\, S\, x(N) + \frac{1}{2} \sum_{k=0}^{N-1} \left\{ \underline{y^T(k)\, y(k)} + u^T(k)\, R\, u(k) \right\}$$

Infinite Horizon LQ optimal regulator

LTI system:

$$x(k+1) = A\, x(k) + B\, u(k) \qquad x(0) = x_o$$
$$y(k) = C\, x(k)$$

Find optimal control which minimizes the cost functional:

$$J_N[x_o] = \frac{1}{2} x^T(N)\, S\, x(N) + \frac{1}{2} \sum_{k=0}^{N-1} \left\{ y^T(k)\, y(k) + u^T(k)\, R\, u(k) \right\}$$

Theorem-1 : Existence of a bounded $\boldsymbol{P}_\infty$

Let $[A\ B]$ be stabilizable
(uncontrollable modes are asymptotically stable)

Then, for $P(N) = S = 0$, as $N \to \infty$

the “backwards” solution of the Riccati Eq.

$$P(k-1) = Q + A^T P(k) A - A^T P(k) B \left[R + B^T P(k) B\right]^{-1} B^T P(k) A$$

converges to a **<u>BOUNDED limiting</u>** solution $\boldsymbol{P}_\infty$ that satisfies the algebraic Riccati equation (DARE):

$$P_\infty = Q + A^T P_\infty A - A^T P_\infty B \left[R + B^T P_\infty B\right]^{-1} B^T P_\infty A$$

Theorem-1 : Notes

1. Theorem-1 only guarantees the existence of a bounded solution $\boldsymbol{P}_\infty$ to the algebraic Riccati Equation

$$P_\infty = Q + A^T P_\infty A - A^T P_\infty B \left[R + B^T P_\infty B\right]^{-1} B^T P_\infty A$$

the solution may not be unique.

Different final conditions $P(N) = S = S^T$
may result in different limiting solutions $\boldsymbol{P}_\infty$ or may return no solution at all!

Theorem 2 – Existence and uniqueness of a positive definite asymptotic stabilizing solution

Let $[A\ C]$ be <u>observable</u> where $C^T C = Q$

Then, $[A\ B]$ is stabilizable <u>if and only if</u>

1) There exists a **<u>unique</u>**, bounded $P_\infty \succ 0$ solution to the ARE:

$$P_\infty = Q + A^T P_\infty A - A^T P_\infty B \left[R + B^T P_\infty B\right]^{-1} B^T P_\infty A$$

Theorem 2 – Existence and uniqueness of a positive definite asymptotic stabilizing solution

2) The close loop plant

$$x(k+1) = [A - B\,K_\infty]\; x(k)$$

is **<u>asymptotically stable</u>**

$$K_\infty = \left[R + B^T P_\infty B\right]^{-1} B^T P_\infty A$$

$$P_\infty = Q + A^T P_\infty A - A^T P_\infty B \left[R + B^T P_\infty B\right]^{-1} B^T P_\infty A$$

Theorem-3 : existence of a stabilizing solution

If ***[A,B]*** stabilizable and ***[A,C]*** is detectable,

1) There exists a unique, bounded $P_\infty \succeq 0$ solution to the ARE.

2) The close loop plant $x(k+1) = [A - B\,K_\infty]\;x(k)$ is **asymptotically stable**

[A,C] is detectable if the unobservable modes are asymptotically stable.

Notes

When ***[A,B]*** stabilizable and ***[A,C]*** observable or detectable, the infinite horizon cost (N →∞) becomes

$$J[x_o] = \frac{1}{2}\sum_{k=0}^{\infty}\left\{x^T(k)\,Q\,x(k) + u^T(k)\,R\,u(k)\right\}$$

- The close loop plant is asymptotically stable

  ⇒ $\lim_{N\to\infty} x(N) = 0$

- Solution of the ARE is unique ⇒ independent of ***P(N)***

Explanation: why is stabilizability needed

$[A\;B]$ not stabilizable ⇒ there are unstable uncontrollable modes

⇒ there are some initial conditions, such that

$$\lim_{N\to\infty} J_N[x_o] = \lim_{N\to\infty}\left\{\frac{1}{2}x^T(N)\,S\,x(N) + \frac{1}{2}\sum_{k=0}^{N-1}\left\{y^T(k)\,y(k) + u^T(k)\,R\,u(k)\right\}\right\} = \infty$$

since the optimal cost is given by

$$J_N^o[x_o] = \frac{1}{2}x_o^T\,P(0)x_0$$

⇒ $\lim_{N\to\infty} ||P(0)|| = \infty$

Explanation: why is detectability is needed

$[A\;C]$ not detectable ⇒ there are unstable unobservable modes

⇒ these modes do not affect the optimal cost

$$J^o[x_o] = \frac{1}{2}\sum_{k=0}^{\infty}\left\{x^T(k)\,Q\,x(k) + u^T(k)\,R\,u(k)\right\}$$

⇒ no need to stabilize these modes

Explanation: why is observability needed

The ARE can be written in the **Joseph stabilized** form:

$$A_c^T P_\infty A_c - P_\infty = -C^T C - K_\infty^T R K_\infty$$

$A_c = [A - B K_\infty]$ (close loop matrix)

$Q = C^T C \succeq 0$

$R = D^T D \succ 0$

$\xrightarrow{\text{define}}$ $\bar{C} = \begin{bmatrix} C \\ D K_\infty \end{bmatrix}$

Explanation: why is observability needed

Joseph stabilized ARE:

$$A_c^T P_\infty A_c - P_\infty = -\bar{C}^T \bar{C}$$

Looks like a Lyapunov equation and, in fact, it is the Lyapunov equation for the **observability Grammian** of the pair

$A_c = [A - B K_\infty]$ $\bar{C} = \begin{bmatrix} C \\ D K_\infty \end{bmatrix}$

Explanation: why is observability needed

Joseph stabilized ARE:

$$A_c^T P_\infty A_c - P_\infty = -\bar{C}^T \bar{C}$$

It can be shown that:

$[A\ C]$ observable $\Longleftrightarrow$ $[A_c\ \bar{C}]$ observable

$[A\ B]$ stabilizable $\Longleftrightarrow$ $P_\infty \succ 0$ and

$A_c = [A - B K_\infty]$ asymptotically stable (Schur)

Example – Double Integrator

LQR

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)$$

$$J = \frac{1}{2} \sum_{k=0}^{\infty} \left\{ y^2(k) + R\, u^2(k) \right\} \qquad R > 0$$

Example – Double Integrator

Penalize position in the infinite horizon cost functional:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)$$

$[A\ C]$ Observable

$[A\ B]$ Controllable

$$\begin{bmatrix} C \\ C\,A \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} B & A\,B \end{bmatrix} = \begin{bmatrix} 0.5 & 1.5 \\ 1 & 1 \end{bmatrix}$$

Example - Steady State Solution

- The steady state solution of the DARE:

$$A^T P A - P + C^T C - A^T P B \left[R + B^T P B\right]^{-1} B^T P A = 0$$

- Use matlab function dare

$$P = \mathtt{dare}(A,\ B,\ C' * C,\ R)$$

- Get steady state answer: $P = \begin{bmatrix} 3 & 3.16 \\ 3.16 & 8.1 \end{bmatrix}$

Example - Infinite Horizon LQ Regulator

- The control law is given by:

$$u(k) = -K\,x(k) \qquad K = \left[R + B^T P B\right]^{-1} B^T P A$$

Answer ➔ $K = \begin{bmatrix} 0.21 & 0.65 \end{bmatrix}$

- Close loop poles are the eigenvalues of

$$A_c = A - B\,K$$

$$= \begin{bmatrix} 0.9 & 0.67 \\ -0.21 & 0.345 \end{bmatrix}$$

is Schur

- Use matlab command:

```
>> abs(eig(Ac))
ans =

    0.6736
    0.6736
```


Summary

- Convergence of LQR as horizon $N \to \infty$
  - $[A\ B]$ stabilizable
  - $[A\ C]$ detectable
- Infinite horizon LQR
- Unique, positive definite solution of algebraic Riccati equation
- Close loop system is asymptotically stable

Additional Material

- Solutions of Infinite Horizon LQR using the Hamiltonian Matrix
  - (see ME232 class notes by M. Tomizuka)
- Strong and stabilizing solutions of the discrete time algebraic Riccati equation (DARE)
- Some additional results on the asymptotic convergence of the discrete time Riccati equation (DRE)

Infinite Horizon LQ optimal regulator

Consider the nth order discrete time LTI system:

$$x(k+1) = A\,x(k) + B\,u(k) \qquad x(0) = x_o$$

We want to find the optimal control which minimizes the cost functional :

$$J = \frac{1}{2}\sum_{k=0}^{\infty}\left\{x^T(k)\,\underbrace{C^T C}_{Q}\,x(k) + u^T(k)\,R\,u(k)\right\}$$

Assume:

- $\{A,\, B\}$ is controllable or asymptotically stabilizable
- $\{A,\, C\}$ is observable or asymptotically detectable

Infinite Horizon LQR Solution:

$$\boxed{\begin{gathered} J^o[x(0)] = \frac{1}{2}\,x^T(0)\,P\,x(0) \\ u^o(k) = -K\,x(k) \\ K = \left[R + B^T P B\right]^{-1} B^T P A \end{gathered}}$$

Discrete time Algebraic Riccati (DARE) equation:

$$\boxed{A^T P A - P + Q - A^T P B\left[R + B^T P B\right]^{-1} B^T P A = 0}$$

Solution of the DARE

DARE:

$$A^T P A - P + Q - A^T P B\left[R + B^T P B\right]^{-1} B^T P A = 0$$

1) Assume that A is nonsingular and define the 2n x 2n ***Backwards*** Hamiltonian matrix:

$$H_b = \left[\begin{array}{c|c} A^{-1} & A^{-1} B\,R^{-1}\,B^T \\ -C^T C\,A^{-1} & A^T + C^T C\,A^{-1} B\,R^{-1}\,B^T \end{array}\right]$$

2) Compute its first ***n*** eigenvalues ($|\lambda_i| < 1$):

$$\{\lambda_1,\, \lambda_2,\, \cdots,\, \lambda_n \,|\, \lambda_{n+1},\, \cdots,\, \lambda_{2n}\}$$

Solution of the DARE

- The first n eigenvalues of $\boldsymbol{H}$ are the eigenvalues of

$$A_c = A - B\,K \quad \text{where} \quad K = \left[R + B^T P B\right]^{-1} B^T P A$$

and are all inside the unit circle, $|\lambda_i| < 1$
(I.e. asymptotically stable)

- The remaining eigenvalues of $\boldsymbol{H}$ satisfy:

$$\lambda_{n+i} = \frac{1}{\lambda_i} \qquad i = 1, \cdots, n$$

Solution of the DARE

3) For each ***unstable*** eigenvalue of $\boldsymbol{H}$ **(*outside the unit circle*)**, compute its associated eigenvector :

$$H_b \underbrace{\begin{bmatrix} f_{n+i} \\ g_{n+i} \end{bmatrix}}_{v_{n+i}} = \lambda_{n+i} \underbrace{\begin{bmatrix} f_{n+i} \\ g_{n+i} \end{bmatrix}}_{v_{n+i}} \qquad \begin{array}{l} |\lambda_{n+i}| > 1 \\ i = 1, \cdots, n \\ f_{n+i},\, g_{n+i} \in \mathcal{C}^n \end{array}$$

4) Define the $n\ x\ n$ matrices:

$$X_1 = \begin{bmatrix} f_{n+1} & f_{n+2} & \cdots & f_{2n} \end{bmatrix}$$

$$X_2 = \begin{bmatrix} g_{n+1} & g_{n+2} & \cdots & g_{2n} \end{bmatrix}$$

Solution of the ARE

5) Finally, $\boldsymbol{P}$ is computed as follows:

$$\boxed{P = X_2\, X_1^{-1}}$$

- Matlab command **`Dare`**: (Discrete time ARE)

$$\boxed{[\,P,\, \Lambda,\, K\,, rr\,] = \texttt{dare}(A,\, B,\, C^T C,\, R)}$$

$$P = X_2\, X_1^{-1} \qquad \Lambda = \mathrm{Diag}(\lambda_1, \cdots, \lambda_n)$$

$$K \;=\; R^{-1}\, B^T\, P \qquad |\lambda_i| < 1$$

Strong Solution of the DARE

A solution $P = P^T \succeq 0$ of the DARE

$$A^T P A - P + Q - A^T P B \left[R + B^T P B\right]^{-1} B^T P A = 0$$

is said to be a **strong solution**
if the corresponding closed loop matrix $\boldsymbol{A_c}$

$$A_c = A - BK \qquad K = \left[R + B^T P B\right]^{-1} B^T P A$$

has all its eigenvalues on or inside the unit circle.

$$|\lambda_i(A_c)| \leq 1;\; i = 1 \cdots n$$

Stabilizing Solution of the DARE

A strong solution $P = P^T \succeq 0$ of the DARE

$$A^T PA - P + Q - A^T PB \left[R + B^T PB\right]^{-1} B^T PA = 0$$

is said to be **stabilizing**
if the corresponding closed loop matrix $\boldsymbol{A_c}$

$$A_c = A - BK \qquad K = \left[R + B^T PB\right]^{-1} B^T PA$$

is Schur, i.e. it has all its eigenvalues inside the unit circle.

$$|\lambda_i(A_c)| < 1;\ i = 1 \cdots n$$

Theorem – Solutions to the DARE

Provided that ***[A,B]*** is stabilizable, then

i. the strong solution of the DARE exists and is unique.
ii. if ***[A,C]*** is detectable, the strong solution is the only nonnegative definite solution of the DARE.
iii. if ***[A,C]*** is has no unobservable modes on the unit circle, then the strong solution coincides with the stabilizing solution.
iv. if ***[A,C]*** has an unobservable mode on the unit circle, then there is no stabilizing solution.

Theorem – Solution to the DARE

Provided that ***[A,B]*** is stabilizable, then

v. if ***[A,C]*** has an unobservable mode inside or on the unit circle, then the strong solution is not positive definite.
vi. if ***[A,C]*** has an unobservable mode outside the unit circle, then as well as the the strong solution, there is at least one nonnegative definite solution of the DARE

S. W. Chan, G.C. Goodwin and K.S. Sin, "Convergence properties of the Riccati difference equation in optimal filtering of nonstabilizable systems, "*IEEE Trans. of Automatic Control* AC-29 (1984) pp 110-118.

Theorems - convergence of the DRE

Consider the "backwards" solution of the discrete time Riccati Equation

$$P(k-1) = C^T C + A^T P(k)A - A^T P(k)B \left[R + B^T P(k)B\right]^{-1} B^T P(k)A$$

$$P(N) = S = S^T$$

1) Subject to

i. ***[A,B]*** is stabilizable and ***[A,C]*** is detectable,

ii. $S \succeq 0$

then, as $N \to \infty$ ***P(k)*** converges exponentially to a unique **stabilizing** solution $\boldsymbol{P_\infty}$ of the DARE

$$P_\infty = Q + A^T P_\infty A - A^T P_\infty B \left[R + B^T P_\infty B\right]^{-1} B^T P_\infty A$$

Theorems - convergence of the DRE

Consider the "backwards" solution of the discrete time Riccati Equation

2) Subject to

i. ***[A,B]*** is stabilizable

ii. ***[A,C]*** is has no unobservable modes on the unit circle

iii. $S \succ 0$

then, as $N \to \infty$ ***P(k)*** converges exponentially to a unique **stabilizing** solution $\boldsymbol{P_\infty}$ of the DARE

$$P_\infty = Q + A^T P_\infty A - A^T P_\infty B \left[R + B^T P_\infty B\right]^{-1} B^T P_\infty A$$

Theorems - convergence of the DRE

Consider the "backwards" solution of the discrete time Riccati Equation

3) Subject to

i. ***[A,B]*** is controllable

ii. $S - P_\infty \succ 0$ or $S = P_\infty$

then, as $N \to \infty$ ***P(k)*** converges to a unique **strong** solution $\boldsymbol{P_\infty}$ of the DARE

$$P_\infty = Q + A^T P_\infty A - A^T P_\infty B \left[R + B^T P_\infty B\right]^{-1} B^T P_\infty A$$

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