ME 233 Advance Control II

Lecture 10

Kalman Filter Reciprocal Root Locus and ARMAX models

(ME233 Class Notes pp.KF1-KF6)

Outline

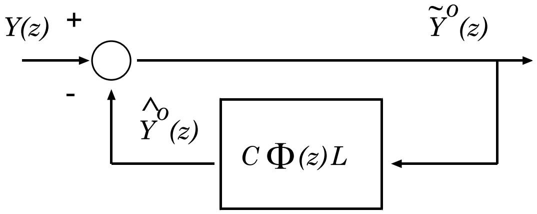
- KF return difference equality
- KF reciprocal root locus
- Stochastic Auto Regressive Moving Average eXtra (ARMAX) SISO models

Kalman filter close loop eigenvalues

- A-priori KF

$$\hat{x}^o(k+1) = A\,\hat{x}^o(k) + L\,\tilde{y}^o(k)$$

$$\tilde{y}^o(k) = y(k) - C\,\hat{x}^o(k)$$



$$\tilde{Y}^o(z) = [I + C\Phi(z)L]^{-1}\,Y(z)$$

Kalman filter close loop eigenvalues

$$\hat{x}^o(k+1) = \underbrace{(A - LC)}_{A_c}\,\hat{x}^o(k) + L\,y(k)$$

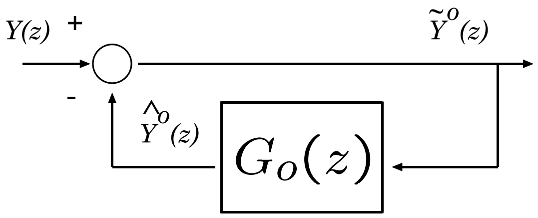
$$\tilde{y}^o(k) = y(k) - C\,\hat{x}^o(k)$$

- KF close loop eigenvalues

$$\begin{aligned} C(z) &= \det\{(zI - A_c)\} = 0 \\ &= \det\{(zI - A + LC)\} = 0 \end{aligned}$$

Kalman filter close loop eigenvalues

- A-priori KF



$$\tilde{Y}^o(z) \; = \; [I + C\Phi(z)L]^{-1}\, Y(z)$$

$$\tilde{Y}^o(z) \; = \; [I + G_o(z)]^{-1}\, Y(z)$$

Kalman filter close loop eigenvalues

$$\det\{[I + \underbrace{C\Phi(z)L}_{G_o(z)}]\} = \frac{C(z)}{A(z)}$$

Close loop eigenvalues

$$C(z) \; = \; \det\{(zI - A + LC)\} = 0$$

Open loop eigenvalues

$$A(z) \; = \; \det\{(zI - A)\} = 0$$

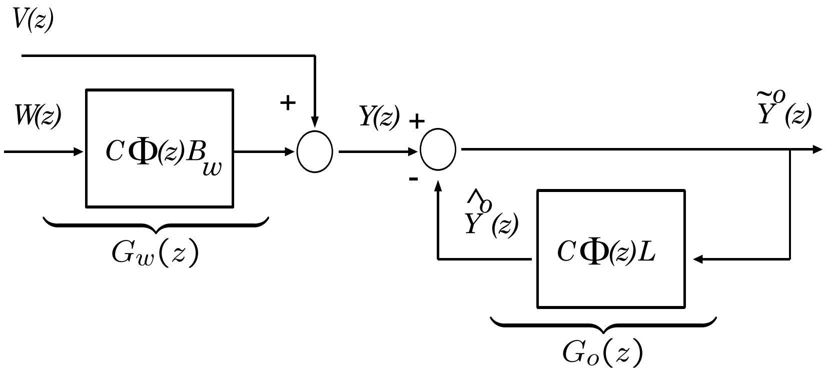
Kalman filter close loop eigenvalues

- Return difference

$$\det\{[I + C\Phi(z)L]\} = \frac{C(z)}{A(z)}$$

$$\begin{aligned}\det\{[I + C\Phi(z)L]\} &= \det\{[I + LC\Phi(z)]\} \\ &= \det\{[(\phi^{-1}(z) + LC)\,\Phi(z)]\} \\ &= \det\{[zI - A + LC]\,(zI - A)^{-1}\} \\ &= \frac{\det\{(zI - A + LC)\}}{\det\{(zI - A)\}}\end{aligned}$$

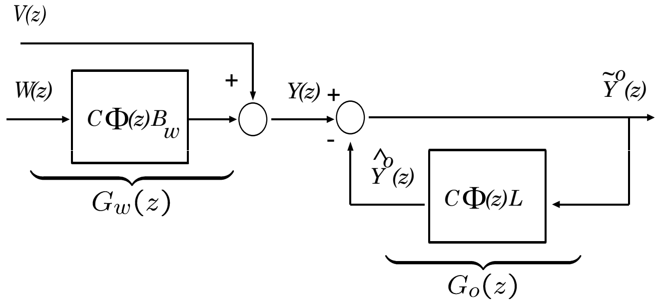
KF return difference equality



$$[I + G_o(z)]\, \underbrace{\left[C\,MC^T + V\right]}_{\Lambda_{\tilde{y}^o\tilde{y}^o}(0)} \left[I + G_o(z^{-1})\right]^T = V + G_w(z)\,WG_w^T(z^{-1})$$

KF return difference equality (SISO)

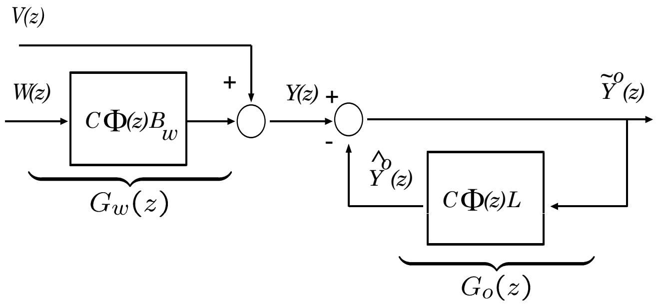
Assume that both, $w(k) \in \mathcal{R}$ and $y(k),\, v(k) \in \mathcal{R}$



$$\boxed{[1+G_o(z)][1+G_o(z^{-1})] = \gamma\,[1+\frac{W}{V}G_w(z)G_w(z^{-1})]}$$

$$\gamma = \frac{V}{V + C\,MC^T}$$

KF root locus for SISO Systems



$$[1+G_o(z)] = [1+C\Phi(z)L] = \frac{C(z)}{A(z)}$$

c.l. poles

o.l. poles

$$G_w(z) = C\Phi(z)B_w = \frac{B_w(z)}{A(z)}$$

o.l. zeros

o.l. poles

KF root locus for SISO Systems

$$\boxed{\frac{C(z^{-1})C(z)}{A(z^{-1})A(z)} = \gamma\left[1+\rho\,\frac{B_w(z^{-1})B_w(z)}{A(z^{-1})A(z)}\right]}$$

$$\boxed{\rho = \frac{W}{V} \geq 0}$$

input noise intensity / *output noise intensity*

$$\gamma = \frac{V}{V + C\,MC^T} > 0, \quad \text{for} \quad V \in (0,\infty)$$

Example in HW5 and HW6

- Stationary system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.08 & -1 \\ 0.7 & 0.1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.34 \\ 0.3 \end{bmatrix} (u(k)+w(k))$$

white noises

$$y(k) = \begin{bmatrix} 0 & 3 \end{bmatrix} x(k) + v(k)$$

Example in HW5 and HW6

- Open loop KF zeros and poles

$$G_w(z) = C[zI - A]^{-1} B_w = \frac{B_w(z)}{A(z)}$$

$$A \ = \ \begin{bmatrix} -0.08 & -1 \\ 0.7 & 0.1 \end{bmatrix} \qquad B_w = \begin{bmatrix} 0.34 \\ 0.3 \end{bmatrix}$$

$$C \ = \ \begin{bmatrix} 0 & 3 \end{bmatrix}$$

Example in HW5 and HW6

- Open loop KF zeros and poles

$$G_w(z) = C[zI - A]^{-1} B_w = \frac{B_w(z)}{A(z)}$$

$$\frac{B_w(z)}{A(z)} = \frac{0.9\ z + 0.786}{z^2 - 0.02\ z + 0.692}$$

$$\frac{B_w(z)}{A(z)} = \frac{0.9(z + 0.87)}{(z - (0.01 + 0.83j))(z - (0.01 - 0.83j))}$$

Example in HW5 and HW6

- Open loop KF zeros and poles

$$\frac{B_w(z)}{A(z)} = \frac{0.9(z + 0.87)}{z^2 - 0.02\ z + 0.69}$$

- Open loop KF zeros and poles reciprocals

$$\frac{B_w(z^{-1})}{A(z^{-1})} = \left(\frac{0.87}{0.69}\right) \ \frac{0.9z(z + 1.15)}{z^2 - 0.029\ z + 1.45}$$

Example in HW5 and HW6

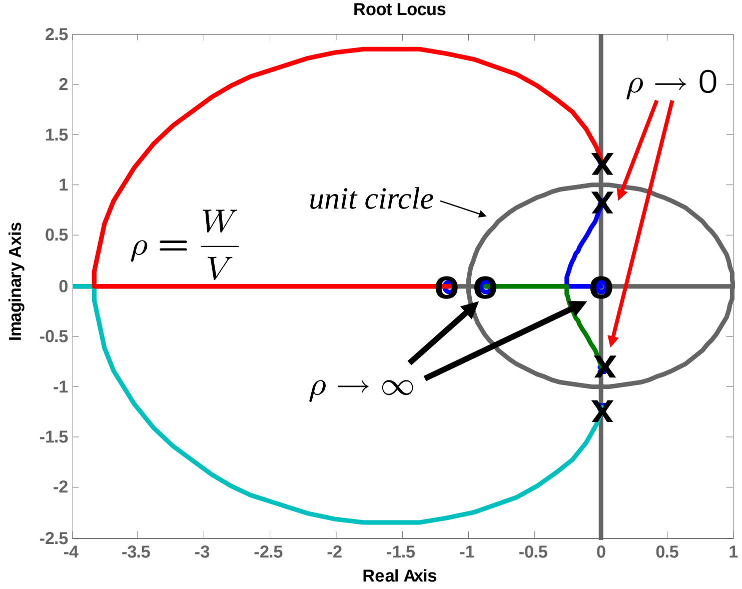
- Reciprocal root locus:

$$\frac{C(z^{-1})C(z)}{A(z^{-1})A(z)} = \gamma \left[1 + \rho \frac{B_w(z^{-1})B_w(z)}{A(z^{-1})A(z)} \right]$$

$$\begin{aligned} \frac{C(z^{-1})C(z)}{A(z^{-1})A(z)} \ &= \ \gamma \Bigg[1 + \ \rho\ 0.9^2 \left(\frac{0.87}{0.69}\right) \\ & \qquad \frac{(z + 0.87)}{z^2 - 0.02\ z + 0.69} \ \frac{0.9z(z + 1.15)}{z^2 - 0.029\ z + 1.45} \Bigg] \end{aligned}$$

$$\rho = \frac{W}{V} \qquad \rho \in (0, \infty)$$

Example in HW5 and HW6



ARMAX stochastic models

- State space stochastic model

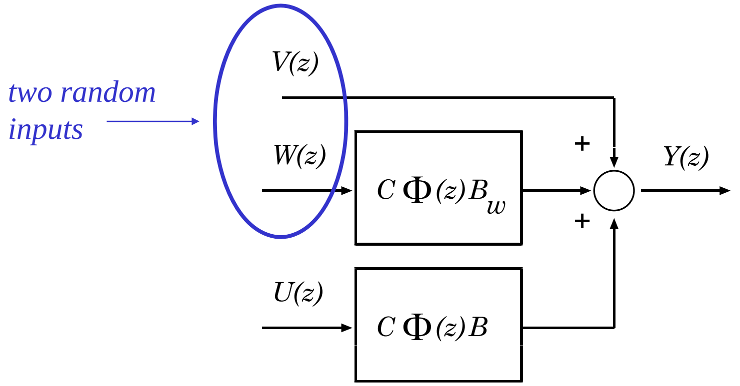
$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) + B_w\,w(k) \\ y(k) &= C\,x(k) + v(k) \end{aligned}$$

Where:

- $u(k)$ **deterministic input**
- $w(k)$ Gaussian, white noise, zero mean, input noise
- $v(k)$ Gaussian, white noise, zero mean, meas. noise

Transfer Function:

$$Y(z) = [C\Phi(z)B]\,U(z) + [C\Phi(z)B_w]\,\underline{W(z)} + \underline{V(z)}$$



Innovations-driven model

- A-priori Kalman filter

$$\begin{aligned} \hat{x}^o(k+1) &= A\,\hat{x}^o(k) + B\,u(k) + L\,\tilde{y}^o(k) \\ y(k) &= C\,\hat{x}^o(k) + \tilde{y}^o(k) \end{aligned}$$

Where:

- $u(k)$ **deterministic input**
- $\tilde{y}^o(k)$ innovations sequence, Gaussian, white, zero mean
- L optimal KF gain $\Lambda_{\tilde{y}^o\tilde{y}^o}(0) = CMC^T + V$

ARMAX stochastic models

- From

$$\begin{aligned}\hat{x}^o(k+1) &= A\,\hat{x}^o(k) + B\,u(k) + L\,\tilde{y}^o(k)\\ y(k) &= C\,\hat{x}^o(k) + \tilde{y}^o(k)\end{aligned}$$

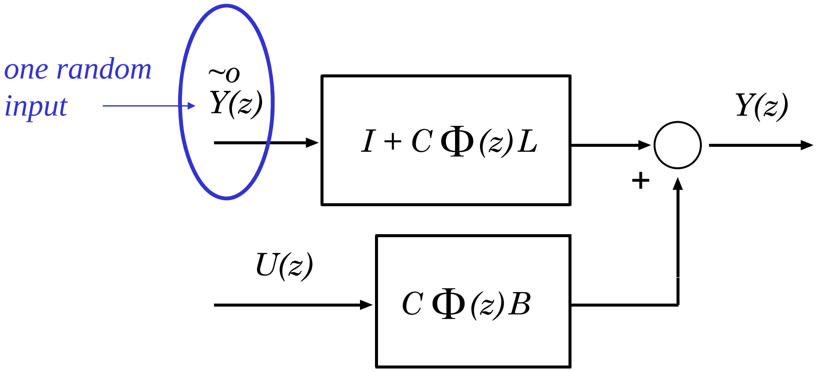
We obtain the ARMAX input/output representation:

$$\begin{aligned}Y(z) &= [C\Phi(z)B]\,U(z)\\ &+ [I + C\Phi(z)L]\,\tilde{Y}^o(z)\end{aligned}$$

$$\Phi(z) = (zI - A)^{-1}$$

ARMAX stochastic models

$$Y(z) = [C\Phi(z)B]\,U(z) + [I + C\Phi(z)L]\,\underline{\tilde{Y}^o(z)}$$



Random input noise intensity: $\Lambda_{\tilde{y}^o\tilde{y}^o}(0) = CMC^T + V$

SISO ARMAX stochastic models

$$Y(z) = \underbrace{[C\Phi(z)B]}\,U(z) + \underbrace{[1 + C\Phi(z)L]}\,\tilde{Y}^o(z)$$

$$\boxed{Y(z) = \frac{B(z)}{A(z)}U(z) + \frac{C(z)}{A(z)}\tilde{Y}^o(z)}$$

SISO ARMAX stochastic models

$$Y(z) = \frac{B(z)}{A(z)}U(z) + \frac{C(z)}{A(z)}\tilde{Y}^o(z)$$

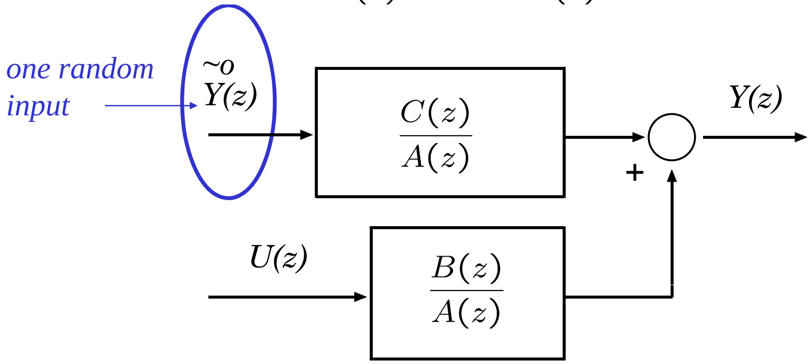
$$C\Phi(z)B = \frac{B(z)}{A(z)} \qquad [1 + C\Phi(z)L] = \frac{C(z)}{A(z)}$$

$$\begin{aligned}C(z) &= \det\{(zI - A + LC)\}\\ A(z) &= \det\{(zI - A)\}\end{aligned}$$

(Schur)

SISO ARMAX stochastic models

$$Y(z) = \frac{B(z)}{A(z)}U(z) + \frac{C(z)}{A(z)}\tilde{Y}^o(z)$$



Random input noise intensity: $\Lambda_{\tilde{y}^o\tilde{y}^o}(0) = CMC^T + V$

Example in HW5 and HW6

- Stationary system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.08 & -1 \\ 0.7 & 0.1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.34 \\ 0.3 \end{bmatrix} (u(k) + w(k))$$

white noises

$$y(k) = \begin{bmatrix} 0 & 3 \end{bmatrix} x(k) + v(k)$$

Example in HW5 and HW6

- Plant parameters:

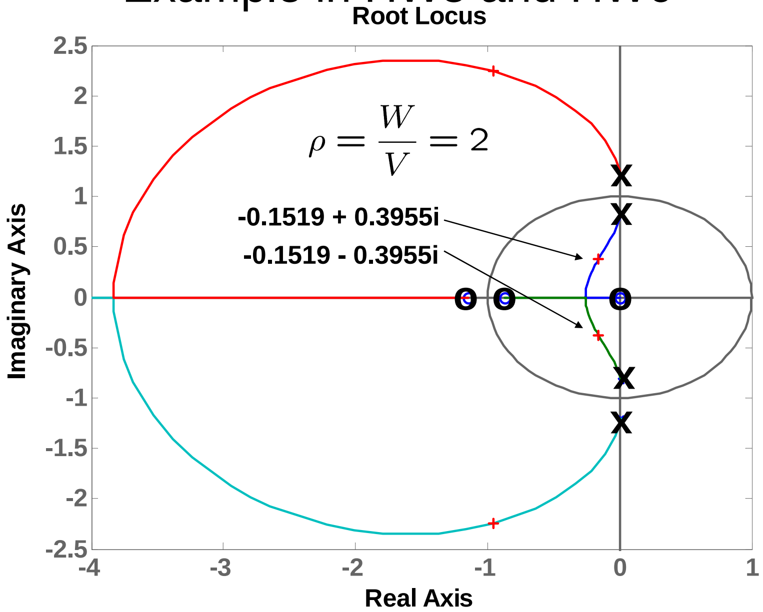
$$A = \begin{bmatrix} -0.08 & -1 \\ 0.7 & 0.1 \end{bmatrix} \qquad B = \begin{bmatrix} 0.34 \\ 0.3 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 3 \end{bmatrix} \qquad B_w = \begin{bmatrix} 0.34 \\ 0.3 \end{bmatrix}$$

- Noise intensities:

$$W = 1 \qquad V = 0.5$$

Example in HW5 and HW6



Example in HW5 and HW6

ARMAX:

$$Y(z) = \frac{B(z)}{A(z)}U(z) + \frac{C(z)}{A(z)}\tilde{Y}^o(z)$$

- Deterministic polynomials:

$$\frac{B(z)}{A(z)} = C[zI - A]^{-1}B \quad = \frac{0.9(z + 0.87)}{z^2 - 0.02\ z + 0.69}$$

Example in HW5 and HW6

A-priori covariance $\boldsymbol{M}$, KF feedback gain $\boldsymbol{L}$ and close loop eigenvalues (Eig):

- Use matlab function **`dare`** and LQ-KF duality:

```
>>[M,Eig,Lt] = dare(A',C',Bw*W*Bw',V)
```

```
M =
   0.1608    0.0764
   0.0764    0.1586
```

```
Lt =
   -0.2564    0.1079

Eig =
  -0.1519 + 0.3955i
  -0.1519 - 0.3955i
```

Example in HW5 and HW6

ARMAX:

$$Y(z) = \frac{B(z)}{A(z)}U(z) + \frac{C(z)}{A(z)}\tilde{Y}^o(z)$$

Stochastic parameters:

$$C(z) = \det[zI - A + LC] \ = z^2 + 0.3\ z + 0.18$$

$$\Lambda_{\tilde{y}^o\tilde{y}^o}(0) = CMC^T + V \quad = 1.93$$

Outline

- KF return difference equality
- KF reciprocal root locus
- Stochastic ARMAX SISO models

KF root locus for SISO Systems

$$\frac{C(z^{-1})C(z)}{A(z^{-1})A(z)} = \gamma \left[1 + \rho \frac{B_w(z^{-1})B_w(z)}{A(z^{-1})A(z)}\right]$$

$$\rho = \frac{W}{V} \geq 0$$

Open loop poles:

$$A(z) = \prod_{i=1}^{n} (z - p_{oi})$$

Closed loop poles:

$$C(z) = \prod_{i=1}^{n} (z - p_{ci})$$

Zeros: roots of $B_w(z)$

$$B_w(z) = b_{wm} \prod_{i=1}^{m} (z - z_{oi})$$

KF Root Locus with open loop eigenvalues and zeros at the origin

Since the KF gain is given by:

$$L = AMC\left[V + CMC^T\right]^{-1} = A\,F$$

Then, if A is singular, the close loop matrix A_c will also be singular:

$$A_c = A - LC = A\,(I - FC)$$

KF Root Locus with open loop eigenvalues and zeros at the origin

Therefore if,

$$A(z) = \det(zI - A) = z^r\,A^{'}(z)$$

$$A^{'}(z) = z^{n-r} + \cdots + a^{'}_o, \quad a^{'}_o \neq 0$$

then:

$$C(z) = \det(zI - A + LC) = z^{r_c}\,C^{'}(z)$$

$$C^{'}(z) = z^{n-r_c} + \cdots + c^{'}_o, \qquad r_c \geq 1$$

Lets also assume that there are p zeros at the origin:

$$B_w(z) = z^p\,B^{'}_w(z)$$

$$B^{'}_w(z) = b_{wm}\,(z^{m-p} + \cdots + b^{'}_{wo}), \quad b^{'}_{wo} \neq 0$$

KF Root Locus with open loop eigenvalues and zeros at the origin

Thus, from

$$C(z^{-1})\,C(z) = \gamma\left[A(z^{-1})\,A(z) + \rho\,B_w(z^{-1})\;B_w(z)\right]$$

we obtain

$$z^{-(n-r_c)} \quad \prod_{i=1}^{n-r_c}(z - p_{ci})(z - \frac{1}{p_{ci}}) = \beta\left[z^{-(n-r)}\prod_{i=1}^{n-r}(z - p_{oi})(z - \frac{1}{p_{oi}})\right.$$

$$\left. + \rho\,b^2_{wm}\frac{b^{'}_{wo}}{a^{'}_o}\,z^{-(m-p)}\prod_{i=1}^{m-p}(z - z_{oi})(z - \frac{1}{z_{oi}})\right]$$

where

$$r_c = n - \max\left[(n-r),\,(m-p)\right]$$

KF Root Locus with open loop eigenvalues and zeros at the origin

Case 1: $(n-r) \geq (m-p) \;\Rightarrow\; r_c = r$

There are r close loop eigenvalues at the origin and

The remaining n-r close loop eigenvalues are plotted using:

$$\frac{\prod_{i=1}^{n-r}(z-p_{ci})(z-\frac{1}{p_{ci}})}{\prod_{i=1}^{n-r}(z-p_{oi})(z-\frac{1}{p_{oi}})} =$$

$$\beta\left[1+\;\rho\,\frac{b'_{wo}}{a'_o}\,b^2_{wm}\,\frac{z^{[(n-r)-(m-p)]}\prod_{i=1}^{m-p}(z-z_{oi})(z-\frac{1}{z_{oi}})}{\prod_{i=1}^{n-r}(z-p_{oi})(z-\frac{1}{p_{oi}})}\right]$$

$$\beta = \left(\frac{a'_o}{c'_o}\right)\frac{V}{V+CMC^T} \qquad\qquad \rho = \frac{W}{V} \geq 0,$$

KF Root Locus with open loop eigenvalues and zeros at the origin

Case 2: $(m-p) > (n-r) \;\Rightarrow r_c = n-(m-p)$

There are $r_c < r$ close loop eigenvalues at the origin and

the remaining m - p close loop eigenvalues are plotted using:

$$\frac{\prod_{i=1}^{m-p}(z-p_{ci})(z-\frac{1}{p_{ci}})}{\prod_{i=1}^{m-p}(z-z_{oi})(z-\frac{1}{z_{oi}})} =$$

$$\alpha\left[1+\frac{b^2_{wm}}{\rho}\,\frac{b'_{wo}}{a'_o}\,\frac{z^{[(m-p)-(n-r)]}\prod_{i=1}^{n-r}(z-p_{oi})(z-\frac{1}{p_{oi}})}{\prod_{i=1}^{m-p}(z-z_{oi})(z-\frac{1}{z_{oi}})}\right]$$

$$\alpha = \left(\frac{b'_{wo}}{c'_o}\frac{\rho}{b^2_{wm}}\right)\frac{V}{V+CMC^T}$$

Notice that ρ is in the denominator and the zeros are in the denominator