Tony Kelman
ME 233 – Homework 1

1.

$x(k+1)=Ax(k)+Bu(k),\ y(k)=Cx(k)$

$y_{d[0,N]}=\{y_d(0),\ y_d(1),\ \ldots,\ y_d(N)\}$

$J=\frac{1}{2}[y_d(N)-y(N)]^T S[y_d(N)-y(N)]+\frac{1}{2}\sum_{k=0}^{N-1}\{[y_d(k)-y(k)]^T T[y_d(k)-y(k)]+u^T(k)Ru(k)\}$

$J^o[x(k),k]=\min_{u(k)}\{L[x(k),u(k),k]+J^o[x(k+1),k+1]\}$

$L[x(k),u(k),k]=\frac{1}{2}\{[y_d(k)-y(k)]^T T[y_d(k)-y(k)]+u^T(k)Ru(k)\}$

$2L[x(k),u(k),k]=[y_d^T(k)-y^T(k)]T\,y_d(k)-[y_d^T(k)-y^T(k)]T\,y(k)+u^T(k)Ru(k)$

$2L[x(k),u(k),k]=y_d^T(k)T\,y_d(k)-y^T(k)T\,y_d(k)-y_d^T(k)T\,y(k)+y^T(k)T\,y(k)+u^T(k)Ru(k)$

$y^T(k)T\,y_d(k)$ is a scalar so $y^T(k)T\,y_d(k)=[y^T(k)T\,y_d(k)]^T=y_d^T(k)T\,y(k)$

$2L[x(k),u(k),k]=y_d^T(k)T\,y_d(k)-2y^T(k)T\,y_d(k)+y^T(k)T\,y(k)+u^T(k)Ru(k)$

$2L[x(k),u(k),k]=y_d^T(k)T\,y_d(k)-2x^T(k)C^T T\,y_d(k)+x^T(k)C^T TC\,x(k)+u^T(k)Ru(k)$

$J^o[x(N),N]=\frac{1}{2}[y_d(N)-y(N)]^T S[y_d(N)-y(N)]$

$2J^o[x(N),N]=[y_d^T(N)-y^T(N)]S\,y_d(N)-[y_d^T(N)-y^T(N)]S\,y(N)$

$2J^o[x(N),N]=y_d^T(N)S\,y_d(N)-y^T(N)S\,y_d(N)-y_d^T(N)S\,y(N)+y^T(N)S\,y(N)$

$2J^o[x(N),N]=y_d^T(N)S\,y_d(N)-2y^T(N)S\,y_d(N)+y^T(N)S\,y(N)$

$2J^o[x(N),N]=y_d^T(N)S\,y_d(N)-2x^T(N)C^T S\,y_d(N)+x^T(N)C^T SC\,x(N)$

$J^o[x(N),N]=\frac{1}{2}y_d^T(N)S\,y_d(N)-x^T(N)C^T S\,y_d(N)+\frac{1}{2}x^T(N)C^T SC\,x(N)$

Using the form $J^o[x(k),k]=\frac{1}{2}x^T(k)P(k)x(k)+x^T(k)b(k)+c(k)$

We have final conditions $P(N)=C^T SC,\ b(N)=-C^T S\,y_d(N),\ c(N)=\frac{1}{2}y_d^T(N)S\,y_d(N)$

Using the same assumed form at $k+1$:

$$\begin{aligned}
2J^o[x(k+1),k+1]&=x^T(k+1)P(k+1)x(k+1)+2x^T(k+1)b(k+1)+2c(k+1)\\
&=[Ax(k)+Bu(k)]^T P(k+1)[Ax(k)+Bu(k)]+2[Ax(k)+Bu(k)]^T b(k+1)+2c(k+1)\\
&=[x^T(k)A^T+u^T(k)B^T]P(k+1)[Ax(k)+Bu(k)]+2[x^T(k)A^T+u^T(k)B^T]b(k+1)+2c(k+1)\\
&=x^T(k)A^T P(k+1)Ax(k)+2u^T(k)B^T P(k+1)Ax(k)+u^T(k)B^T P(k+1)Bu(k)\\
&\quad+2x^T(k)A^T b(k+1)+2u^T(k)B^T b(k+1)+2c(k+1)\\
&=x^T(k)A^T P(k+1)Ax(k)+u^T(k)B^T P(k+1)Bu(k)+2u^T(k)B^T[P(k+1)Ax(k)+b(k+1)]\\
&\quad+2x^T(k)A^T b(k+1)+2c(k+1)
\end{aligned}$$

$J^o[x(k),k]=\min_{u(k)}\{L[x(k),u(k),k]+J^o[x(k+1),k+1]\}$

$$\begin{aligned}
2J^o[x(k),k]=\min_{u(k)}\{&y_d^T(k)T\,y_d(k)-2x^T(k)C^T T\,y_d(k)+x^T(k)C^T TC\,x(k)+u^T(k)Ru(k)\\
&+x^T(k)A^T P(k+1)Ax(k)+u^T(k)B^T P(k+1)Bu(k)+2u^T(k)B^T[P(k+1)Ax(k)+b(k+1)]\\
&+2x^T(k)A^T b(k+1)+2c(k+1)\}\\
=\min_{u(k)}\{&y_d^T(k)T\,y_d(k)-2x^T(k)[C^T T\,y_d(k)-A^T b(k+1)]+x^T(k)[C^T TC+A^T P(k+1)A]x(k)\\
&+u^T(k)[R+B^T P(k+1)B]u(k)+2u^T(k)B^T[P(k+1)Ax(k)+b(k+1)]+2c(k+1)\}
\end{aligned}$$

Minimum achieved for $u^o(k)$ such that $\frac{\partial J[x(k),k]}{\partial u(k)}|_{u^o(k)} = 0$

$$\frac{\partial J[x(k),k]}{\partial u(k)}|_{u^o(k)} = [R+B^T P(k+1)B]u^o(k)+B^T[P(k+1)Ax(k)+b(k+1)]=0$$

$$u^o(k)=-[R+B^T P(k+1)B]^{-1} B^T[P(k+1)Ax(k)+b(k+1)]$$

Let $K_2(k+1)=[R+B^T P(k+1)B]^{-1} B^T$, and $K_1(k+1)=K_2(k+1)P(k+1)A$

$$u^o(k)=-K_1(k+1)x(k)-K_2(k+1)b(k+1)$$

$$\begin{aligned}2J^o[x(k),k]&=2c(k+1)-2x^T(k)[C^T T y_d(k)-A^T b(k+1)]+x^T(k)[C^T T C+A^T P(k+1)A]x(k)\\&\quad+u^o(k)^T[R+B^T P(k+1)B]u^o(k)+2u^o(k)^T B^T[P(k+1)Ax(k)+b(k+1)]+y_d^T(k)T y_d(k)\end{aligned}$$

$$\begin{aligned}2J^o[x(k),k]&=2c(k+1)-2x^T(k)[C^T T y_d(k)-A^T b(k+1)]+x^T(k)[C^T T C+A^T P(k+1)A]x(k)\\&\quad+[K_1(k+1)x(k)+K_2(k+1)b(k+1)]^T[R+B^T P(k+1)B][K_1(k+1)x(k)+K_2(k+1)b(k+1)]\\&\quad-2[K_1(k+1)x(k)+K_2(k+1)b(k+1)]^T B^T[P(k+1)Ax(k)+b(k+1)]+y_d^T(k)T y_d(k)\end{aligned}$$

$$\begin{aligned}2J^o[x(k),k]&=2c(k+1)-2x^T(k)[C^T T y_d(k)-A^T b(k+1)]+x^T(k)[C^T T C+A^T P(k+1)A]x(k)\\&\quad+x^T(k)K_1^T(k+1)[R+B^T P(k+1)B]K_1(k+1)x(k)\\&\quad+2x^T(k)K_1^T(k+1)[R+B^T P(k+1)B]K_2(k+1)b(k+1)\\&\quad+b^T(k+1)K_2^T(k+1)[R+B^T P(k+1)B]K_2(k+1)b(k+1)\\&\quad-2x^T(k)K_1^T(k+1)B^T P(k+1)Ax(k)-2x^T(k)K_1^T(k+1)B^T b(k+1)\\&\quad-2b^T(k+1)K_2^T(k+1)B^T P(k+1)Ax(k)-2b^T(k+1)K_2^T(k+1)B^T b(k+1)+y_d^T(k)T y_d(k)\end{aligned}$$

$$\begin{aligned}2J^o[x(k),k]&=2c(k+1)-2x^T(k)[C^T T y_d(k)-A^T b(k+1)]+x^T(k)[C^T T C+A^T P(k+1)A]x(k)\\&\quad+x^T(k)K_1^T(k+1)B^T P(k+1)Ax(k)\\&\quad+2x^T(k)A^T P^T(k+1)BK_2(k+1)b(k+1)\\&\quad+b^T(k+1)K_2^T(k+1)B^T b(k+1)\\&\quad-2x^T(k)K_1^T(k+1)B^T P(k+1)Ax(k)-2x^T(k)K_1^T(k+1)B^T b(k+1)\\&\quad-2b^T(k+1)K_2^T(k+1)B^T P(k+1)Ax(k)-2b^T(k+1)K_2^T(k+1)B^T b(k+1)+y_d^T(k)T y_d(k)\end{aligned}$$

$$\begin{aligned}2J^o[x(k),k]&=2c(k+1)-2x^T(k)[C^T T y_d(k)-A^T b(k+1)]+x^T(k)[C^T T C+A^T P(k+1)A]x(k)\\&\quad-x^T(k)K_1^T(k+1)B^T P(k+1)Ax(k)-b^T(k+1)K_2^T(k+1)B^T b(k+1)\\&\quad-2x^T(k)K_1^T(k+1)B^T b(k+1)+y_d^T(k)T y_d(k)\end{aligned}$$

$$\begin{aligned}2J^o[x(k),k]&=2c(k+1)-b^T(k+1)K_2^T(k+1)B^T b(k+1)+y_d^T(k)T y_d(k)\\&\quad+x^T(k)[C^T T C+A^T P(k+1)A-K_1^T(k+1)B^T P(k+1)A]x(k)\\&\quad+2x^T(k)[(A^T-K_1^T(k+1)B^T)b(k+1)-C^T T y_d(k)]\end{aligned}$$

Since $J^o[x(k),k]=\frac{1}{2}x^T(k)P(k)x(k)+x^T(k)b(k)+c(k)$ we can equate coefficients:

$$P(k)=C^T T C+A^T P(k+1)A-K_1^T(k+1)B^T P(k+1)A$$

$$P(k)=C^T T C+A^T P(k+1)A-A^T P^T(k+1)B([R+B^T P(k+1)B]^{-1})^T B^T P(k+1)A$$

$(M^{-1})^T=(M^T)^{-1}$ so:

$$P(k)=C^T T C+A^T P(k+1)A-A^T P^T(k+1)B[R+B^T P(k+1)B]^{-1} B^T P(k+1)A$$

$$b(k)=[A^T-K_1^T(k+1)B^T]b(k+1)-C^T T y_d(k)$$

$$b(k)=[A^T-A^T P^T(k+1)B[R+B^T P(k+1)B]^{-1} B^T]b(k+1)-C^T T y_d(k)$$

$$c(k)=c(k+1)-\frac{1}{2}b^T(k+1)K_2^T(k+1)B^T b(k+1)+\frac{1}{2}y_d^T(k)T y_d(k)$$

$$c(k)=c(k+1)-\frac{1}{2}b^T(k+1)B[R+B^T P(k+1)B]^{-1} B^T b(k+1)+\frac{1}{2}y_d^T(k)T y_d(k)$$

2.

$$x(k+1)=x(k)+u(k),\ x(0)=0$$
$$U_o^o=\{u^o(0),\ u^o(1),\ \ldots,\ u^o(N-1)\}$$
$$u^o(k)\geq 0,\ x(N)=L$$
$$J=\prod_{k=0}^{N-1} u(k)=u(0)u(1)\cdots u(N-1)$$
$$J^o[x(m)]=\max_{U_m}\prod_{k=m}^{N-1} u(k)$$
$$U_m=\{u(m),\ u(m+1),\ \ldots,\ u(N-1)\}$$
$$J^o[x(m)]=\max_{u(m)}\left\{u(m)\cdot\max_{U_{m+1}}\prod_{k=m+1}^{N-1} u(k)\right\}$$
$$J^o[x(m)]=\max_{u(m)}\{u(m)J^o[x(m+1)]\}$$
$$J^o[x(N-1)]=u^o(N-1)=L-x(N-1)$$
$$J^o[x(N-2)]=\max_{u(N-2)}\{u(N-2)J^o[x(N-1)]\}=\max_{u(N-2)}\{u(N-2)(L-x(N-1))\}$$
$x(N-1)=x(N-2)+u(N-2)$, so $J^o[x(N-2)]=\max\limits_{u(N-2)}\{u(N-2)(L-x(N-2)-u(N-2))\}$

Maximum achieved for $u^o(N-2)$ such that $\dfrac{\partial J[x(N-2)]}{\partial u(N-2)}\Big|_{u^o(N-2)}=0$
$$\frac{\partial J[x(N-2)]}{\partial u(N-2)}\Big|_{u^o(N-2)}=L-x(N-2)-2u^o(N-2)=0$$
$$u^o(N-2)=\frac{1}{2}(L-x(N-2)),\ J^o[x(N-2)]=\frac{1}{2^2}(L-x(N-2))^2$$
$$J^o[x(N-3)]=\max_{u(N-3)}\{u(N-3)J^o[x(N-2)]\}=\max_{u(N-3)}\left\{u(N-3)\frac{1}{2^2}(L-x(N-2))^2\right\}$$
$x(N-2)=x(N-3)+u(N-3)$, so $J^o[x(N-3)]=\max\limits_{u(N-3)}\left\{u(N-3)\dfrac{1}{2^2}(L-x(N-3)-u(N-3))^2\right\}$
$$\frac{\partial J[x(N-3)]}{\partial u(N-3)}\Big|_{u^o(N-3)}=\frac{1}{2^2}(L-x(N-3)-u^o(N-3))^2-u^o(N-3)\frac{1}{2}(L-x(N-3)-u^o(N-3))=0$$
$x(N-2)\neq L$ so $L-x(N-3)-u^o(N-3)\neq 0$
$$\frac{1}{2}(L-x(N-3)-u^o(N-3))-u^o(N-3)=0$$
$$u^o(N-3)=\frac{1}{3}(L-x(N-3)),\ J^o[x(N-3)]=\frac{1}{3^3}(L-x(N-3))^3$$
Propose general form: $u^o(N-k)=\dfrac{1}{k}(L-x(N-k)),\ J^o(x(N-k))=u^o(N-k)^k$
$$J^o[x(N-k-1)]=\max_{u(N-k-1)}\{u(N-k-1)J^o[x(N-k)]\}=\max_{u(N-k-1)}\left\{u(N-k-1)\frac{1}{k^k}(L-x(N-k))^k\right\}$$
$$J^o[x(N-k-1)]=\max_{u(N-k-1)}\left\{u(N-k-1)\frac{1}{k^k}(L-x(N-k-1)-u(N-k-1))^k\right\}$$
$$\frac{\partial J[x(N-k-1)]}{\partial u(N-k-1)}\Big|_{u^o(N-k-1)}=0=\frac{1}{k^k}(L-x(N-k-1)-u^o(N-k-1))^k$$
$$-u^o(N-k-1)\frac{1}{k^{k-1}}(L-x(N-k-1)-u^o(N-k-1))^{k-1}$$

For $k \neq 0$, $(L - x(N-k-1) - u^o(N-k-1))^{k-1} \neq 0$
$0 = (L - x(N-k-1) - u^o(N-k-1)) - k\,u^o(N-k-1)$
$u^o(N-k-1) = \frac{1}{k+1}(L - x(N-k-1))$ so proposed general form holds by induction.

3.
$x(k+1) = A\,x(k) + B\,u(k),\ x(0) \neq 0$
$J[x_m, m, S, N] = \frac{1}{2} x^T(N) S\,x(N) + \frac{1}{2} \sum_{k=m}^{N-1} \{x^T(k) Q\,x(k) + u^T(k) R\,u(k)\}$
$J^o[x_m, m, S, N] = \min_{U_m} J[x_m, m, S, N]$
$U_m = \{u(m),\ u(m+1),\ \ldots,\ u(N-1)\}$
$J[x_m, m, 0, N+1] = \frac{1}{2} \sum_{k=m}^{N} \{x^T(k) Q\,x(k) + u^T(k) R\,u(k)\}$
$J[x_m, m, 0, N+1] = \frac{1}{2}\{x^T(N) Q\,x(N) + u^T(N) R\,u(N)\} + J[x_m, m, 0, N]$
$J[x_m, m, 0, N] \geq J^o[x_m, m, 0, N]$, so:
$J[x_m, m, 0, N+1] \geq \frac{1}{2}\{x^T(N) Q\,x(N) + u^T(N) R\,u(N)\} + J^o[x_m, m, 0, N]$
$Q \geq 0$ and $R > 0$ so $J[x_m, m, 0, N+1] \geq J^o[x_m, m, 0, N]$
$J^o[x_m, m, 0, N+1] = \min_{U_m, u(N)} J[x_m, m, 0, N+1] \geq J^o[x_m, m, 0, N]$