ME 233 Advance Control II

Lecture 12
Discrete Time
Linear Quadratic Gaussian (LQG)
Optimal Control

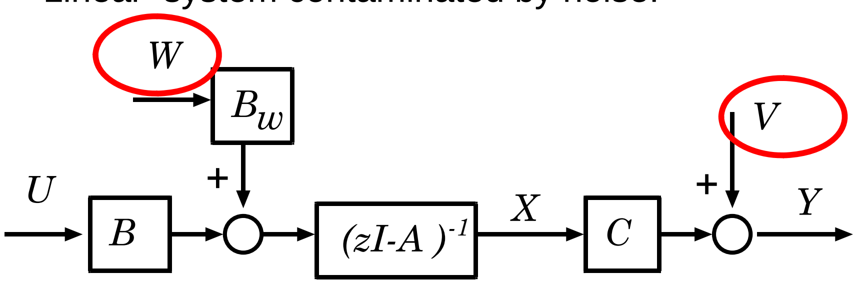
(ME233 Class Notes pp.LQG1-LQG7)

Outline

- Linear Quadratic Gaussian (LQG) regulator
- Finite horizon LQG
  - LQG under full state measurement
  - LQG under output measurement
- Stationary LQG

Stochastic Control

Linear system contaminated by noise:



Two random disturbances:

- Input noise ***w(k)*** - contaminates the state ***x(k)***
- Measurement noise ***v(k)*** - contaminates the output ***y(k)***

Stochastic state model

$$x(k+1) \;=\; A\,x(k) + B\,u(k) + B_w\,w(k)$$

$$y(k) \;=\; C\,x(k) + v(k)$$

Where:

- $y(k)$ available output
- $u(k)$ **control input**
- $w(k)$ Gaussian, white noise, zero mean, input noise
- $v(k)$ Gaussian, white noise, zero mean, meas. noise
- $x(0)$ Gaussian initial state

Assumptions – same as KF

- Initial conditions:

$$E\{x(0)\} = x_o \quad E\{\tilde{x}^o(0)\tilde{x}^{oT}(0)\} = X_o$$

- Noise properties:

$$\left.\begin{aligned} &E\{w(k+l)w^T(k)\} = W(k)\,\delta(l) \\ &E\{v(k+l)v^T(k)\} = V(k)\,\delta(l) \\ &E\{w(k+l)v^T(k)\} = 0 \end{aligned}\right\}$$

Zero-mean uncorrelated Gaussian white noises

$$E\{\tilde{x}^o(0)w^T(k)\} = 0 \qquad E\{\tilde{x}^o(0)v^T(k)\} = 0$$

Some notation – random variables

- The set of initial random conditions

$$\mathcal{X}_o \;=\; \{x(0)\}$$

- The set random input sequences **from k to N-1**

$$\mathcal{W}_k \;=\; \{w(k),\, w(k+1),\, \cdots,\, w(N-1)\}$$

- The set random measurement sequences **from k to N**

$$\mathcal{V}_k \;=\; \{v(k),\, v(k+1),\, \cdots,\, v(N)\}$$

Some notation- control and measurements

- The set of control sequences **from k to N-1**

$$U_k \;=\; \{u(k),\, u(k+1),\, \cdots,\, u(N-1)\}$$

The set of optimal control sequences **from k to N-1**

$$U_k^o \;=\; \{u^o(k),\, u^o(k+1),\, \cdots,\, u^o(N-1)\}$$

A set of available output measurements **up to k**

$$Y_k \;=\; \{y(0),\, y(1),\, \cdots,\, y(k)\}$$

Finite horizon LQG

For $N > 0$, find the optimal control sequence:

$$U_0^o \;=\; \{u^o(0),\, u^o(1),\, \cdots,\, u^o(N-1)\}$$

Which minimizes the cost functional:

$$J \;=\; \frac{1}{2}E\left\{x^T(N)\,S\,x(N)\right\} + \frac{1}{2}E\left\{\sum_{k=0}^{N-1} x^T(k)\,Q\,x(k) + u^T(k)\,R\,u(k)\right\}$$

where $u^o(k)$ can only be based on the observations:

$$Y_k = \{y(0),\, y(1),\, \cdots,\, y(k)\}$$

Finite horizon LQG

Notice that the expectation in $\boldsymbol{J}$ is taken as follows:

$$J = \frac{1}{2} E_{\substack{\mathcal{X}_o \\ \mathcal{W}_0 \\ \mathcal{V}_0}} \left\{ x^T(N)\, S\, x(N) + \sum_{k=0}^{N-1} \left[x^T(k)\, Q\, x(k) + u^T(k)\, R\, u(k) \right] \right\}$$

Over the combined PDF for

$$\mathcal{X}_o = \{x(0)\}$$
$$\mathcal{W}_0 = \{w(0), w(1), \cdots, w(N-1)\}$$
$$\mathcal{V}_0 = \{v(0), v(1), \cdots, v(N)\}$$

Separation Principle

Theorem:

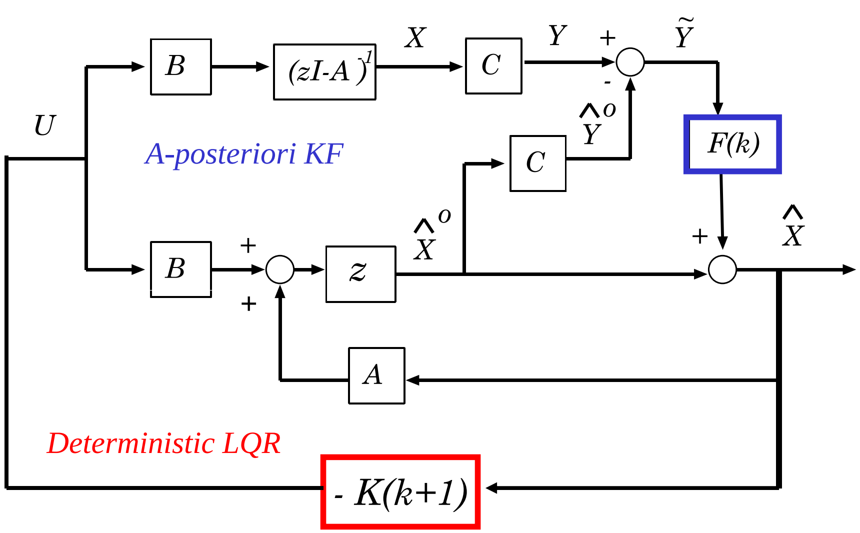
The optimal control is given by:

$$u^o(k) = -K(k+1)\,\hat{x}(k)$$

Where:

- The feedback gain $K(k)$ is obtained from the deterministic LQR solution.
- The state estimate $\hat{x}(k)$ is the **a-posteriori** Kalman Filter state estimate.

Separation Principle



Separation Principle Proof

The proof of the separation principle is conducted in two steps:

1. We will first assume that the state vector $x(k)$ is measurable and will solve the stochastic LQR problem.
2. We will then remove this assumption and show that the optimal solution is obtained by replacing $x(k)$ by the a-posteriori state estimated $\hat{x}(k)$

Finite horizon LQG with measured states

- Given

$$x(k+1) \;=\; A\,x(k) + B\,u(k) + B_w\,w(k)$$

Where,

$\mathcal{X}_o \;=\; \{x(0)\}$ *set of initial random conditions*

$\mathcal{W}_0 \;=\; \{w(0),\, w(1),\, \cdots,\, w(N-1)\}$ *set random input sequences*

The set of **measured state outcomes up to k**

$$X_k \;=\; \{x(0),\, x(1),\, \cdots,\, x(k)\}$$

Finite horizon LQG with measured states

Obtain the optimal control sequence which minimizes

$$J = \frac{1}{2} E_{\substack{\mathcal{X}_o \\ \mathcal{W}_0}} \left\{ x^T(N)\,S\,x(N) + \sum_{k=0}^{N-1} \left[x^T(k)\,Q\,x(k) + u^T(k)\,R\,u(k) \right] \right\}$$

Over the combined PDF for

$$\mathcal{X}_o \;=\; \{x(0)\}$$

$$\mathcal{W}_0 \;=\; \{w(0),\, w(1),\, \cdots,\, w(N-1)\}$$

Finite horizon LQG with measured states

This problem is similar to the standard deterministic finite horizon LQR

$$x(k+1) \;=\; A\,x(k) + B\,u(k) + B_w\,w(k)$$

Except that there is an additional input noise:

w(k) is Gaussian, white and zero mean

x(0) is also Gaussian, but not zero mean

Finite horizon LQG with measured states

Theorem 1:

a) The optimal control is given by

$$u^o(k) = -K(k+1)\,x(k)$$

$$K(k+1) = \left[R + B^T P(k+1) B\right]^{-1} B^T P(k+1) A$$

$$P(k-1) = Q + A^T P(k) A - A^T P(k) B \left[R + B^T P(k) B\right]^{-1} B^T P(k) A$$

Standard deterministic LQR solution! $P(N) = S$

Finite horizon LQG with <u>measured</u> states

Theorem 1:

b) The optimal cost $\boldsymbol{J^o}$ is given by

$$J^o \;=\; \frac{1}{2}x_o^T P(0)x_o + \frac{1}{2}\text{trace}\left[P(0)X_o\right] + b(0)$$

$$x_o = E\{x(0)\} \qquad\qquad X_o = E\{\tilde{x}^o(0)\tilde{x}^{oT}(0)\}$$

$$b(k) = b(k+1) + \text{trace}\left[B_w^T P(k+1) B_w W(k)\right]$$

$$b(N) = 0$$

Finite horizon LQG with <u>measured</u> states

Theorem 1:

b) The optimal cost is given by

$$J^o \;=\; \frac{1}{2}x_o^T P(0)x_o + \frac{1}{2}\text{trace}\left[P(0)X_o\right] + b(0)$$

b(k) *is a dynamic function of the noise intensity which is computed backwards in time with* ***b(N) = 0***

$$b(k) = b(k+1) + \text{trace}\left[B_w^T P(k+1) B_w W(k)\right]$$

This term reflects the harmful effect on the cost of ***w(k)***

Three useful Lemmas

- To prove Theorem 1, we need three Lemmas
- Lemma 1: - expectation and minimization commute
- Lemma 2: - Expectation of a quadratic function
- Lemma 3: - Stochastic Bellman Equation

Lemma 1

- Assume that the function $L(x,u)$ has a unique minimum $\boldsymbol{u^o(x)}$ for all $x \in \mathcal{X}$, i.e.
- $u^o(x) = \text{Arg}\left\{\min_u L(x,u)\right\}$ $\quad (\, L(x,u^o(x)) \leq L(x,u)\,)$

Let $x \in \mathcal{X}$ be random variable. Then,

$$\boxed{\min_{u(x)} E\left\{L(x,u)\right\} = E\left\{L(x,u^o(x))\right\} = E\left\{\min_u L(x,u)\right\}}$$

(i.e. expectation and minimization commute)

Proof of Lemma 1

For all admissible control strategies

$$L(x,u) \geq L(x,u^o(x)) = \min_u L(x,u)$$

Taking expectations

$$E\{L(x,u)\} \geq E\{L(x,u^o(x))\} = E\{\min_u L(x,u)\}$$

Minimizing the left hand side w/r all admissible strategies,

$$\min_{u(x)} E\{L(x,u))\} \geq E\{L(x,u^o(x)\} = E\{\min_u L(x,u)\}$$

Proof of Lemma 1

Therefore,

$$\min_{u(x)} E\{L(x,u)\} \geq E\{L(x,u^o(x))\} = E\{\min_u L(x,u)\}$$

Since $\boldsymbol{u^o(x)}$ is an admissible strategy,

$$E\{L(x,u^o(x))\} \geq \min_{u(x)} E\{L(x,u)\}$$

Thus,

$$\min_{u(x)} E\{L(x,u)\} = E\{L(x,u^o(x))\} = E\left\{\min_u L(x,u)\right\}$$

Q.E.D.

Lemma 2

Let $x \in \mathcal{X}$ be random variable, and

$$\hat{x} = E\{x\} \qquad \Lambda_{xx} = E\{(x-\hat{x})(x-\hat{x})^T\}$$

Then, for any symmetric matrix $\boldsymbol{P}$

$$\boxed{E\{x^T P\, x\} = E\{\hat{x}^T P\, \hat{x}\} + \mathrm{Tr}\,(P\Lambda_{xx})}$$

Proof of Lemma 2

Define $\tilde{x} = x - \hat{x}$ and remember that

$$E\{\hat{x}\tilde{x}^T\} = 0$$

$$E\{x^T P\, x\} = E\{(\underbrace{x - \overbrace{\hat{x} + \hat{x}}^{\boldsymbol{0}}}_{\tilde{x}})^T P(x-\hat{x}+\hat{x})\}$$

$$= E\{(\tilde{x}+\hat{x})^T P(\tilde{x}+\hat{x})\}$$

Proof of Lemma 2

$$E\{x^T P\, x\} = E\{(\tilde{x} + \hat{x})^T P(\tilde{x} + \hat{x})\}$$

$$= E\{\hat{x}^T P\hat{x})\} + E\{\tilde{x}^T P\tilde{x})\}$$

$$+2E\{\tilde{x}^T P\hat{x})\}$$

$$= E\{\hat{x}^T P\hat{x})\} + \mathrm{Tr}\left(P\underbrace{E\{\tilde{x}\tilde{x}^T\}}_{\Lambda_{xx}}\right)$$

$$+2\mathrm{Tr}\left(P\underbrace{E\{\hat{x}\tilde{x}^T\}}_{0}\right)$$

Q.E.D.

Finite horizon LQG with <u>measured</u> states

We use ***stochastic*** dynamic programming.

Define the optimal value function:

$$J_k^o[x(k)] = \frac{1}{2}\min_{U_k}\left[E_{\mathcal{W}_k}\left\{x^T(N)\, S\, x(N) + \sum_{j=k}^{N-1}\left[x^T(j)\, Q\, x(j) + u^T(j)\, R\, u(j)\right]\right\}\right]$$

function of the ***<u>variable</u>*** $x(k) \in \mathcal{X}(k)$

Finite horizon LQG with <u>measured</u> states

$$J_k^o[x(k)] = \frac{1}{2}\min_{U_k}\left[E_{\mathcal{W}_k}\left\{x^T(N)\, S\, x(N) + \sum_{j=k}^{N-1}\left[x^T(j)\, Q\, x(j) + u^T(j)\, R\, u(j)\right]\right\}\right]$$

minimization over

$$U_k = \{u(k),\, u(k+1),\, \cdots,\, u(N-1)\}$$

expectation over

$$\mathcal{W}_k = \{w(k),\, w(k+1),\, \cdots,\, w(N-1)\}$$

Notation...

Introduced some notation:

$$J_k^o[x(k)] = \frac{1}{2}\min_{U_k}\left[E_{\mathcal{W}_k}\left\{\underbrace{x^T(N)\, S\, x(N)}_{2\,S(x(N))} + \underbrace{\sum_{j=k}^{N-1}\left[x^T(j)\, Q\, x(j) + u^T(j)\, R\, u(j)\right]}_{2\,L(x(j),u(j))}\right\}\right]$$

$$J_k^o[x(k)] = \min_{U_k}\left[E_{\mathcal{W}_k}\left\{S(x(N)) + \sum_{j=k}^{N-1} L(x(j), u(j))\right\}\right]$$

Lemma 3

$$J_k^o[x(k)] \;=\; \min_{U_k}\left[E_{\mathcal{W}_k}\left\{S(x(N))+\sum_{j=k}^{N-1}L(x(j),u(j))\right\}\right]$$

satisfies the ***Stochastic Bellman equation***

$$\boxed{J_k^o[x(k)] \;=\; \min_{u(k)}\left[L(x(k),u(k))+E_{w(k)}\left\{J_{k+1}^o[x(k+1)]\right\}\right]}$$

with boundary condition $J_N^O[x(N)] \;=\; S(x(N))$

expectation $E_{w(k)}$ is taken only relative to PDF of $w(k)$

Proof of Lemma 3 (sketch)

$$J_k^o[x(k)] \;=\; \min_{U_k}\left[E_{\mathcal{W}_k}\left\{S(x(N))+\sum_{j=k}^{N-1}L(x(j),u(j))\right\}\right]$$

expand

$$J_k^o[x(k)] \;=\; \min_{U_k}\left[E_{\mathcal{W}_k}\left\{L(x(k),u(k))+S(x(N))+\sum_{j=k+1}^{N-1}L(x(j),u(j))\right\}\right]$$

$$J_k^o[x(k)] \;=\; \min_{U_k}\left[L(x(k),u(k))+E_{\mathcal{W}_k}\left\{S(x(N))+\sum_{j=k+1}^{N-1}L(x(j),u(j))\right\}\right]$$

because $w(k)$ is white and uncorrelated with $x(k)$ or $u(k)=f(x(k))$

$$E\{\tilde{x}(k)w^T(k+l)\}=0 \qquad l \geq 0$$
$$E\{\tilde{u}(k)w^T(k+l)\}=0$$

Proof of Lemma 3 (sketch)

$$J_k^o[x(k)] \;=\; \min_{U_k}\left[L(x(k),u(k))+E_{\mathcal{W}_k}\left\{S(x(N))+\sum_{j=k+1}^{N-1}L(x(j),u(j))\right\}\right]$$

$$J_k^o[x(k)] \;=\; \min_{\substack{u(k)\\U_{k+1}}}\left[L(x(k),u(k))+E_{\mathcal{W}_k}\left\{S(x(N))+\sum_{j=k+1}^{N-1}L(x(j),u(j))\right\}\right]$$

$$J_k^o[x(k)] \;=\; \min_{u_k}\left[L(x(k),u(k))+\min_{U_{k+1}}E_{\mathcal{W}_k}\left\{S(x(N))+\sum_{j=k+1}^{N-1}L(x(j),u(j))\right\}\right]$$

because $L(x(k),u(k))$ is not a function of $\{u(k+1) \ldots u(N-1)\}$,

$$J_k^o[x(k)] \;=\; \min_{u_k}\left[L(x(k),u(k))+\min_{U_{k+1}}E_{\substack{w(k)\\\mathcal{W}_{k+1}}}\left\{S(x(N))+\sum_{j=k+1}^{N-1}L(x(j),u(j))\right\}\right]$$

Proof of Lemma 3 (sketch)

$$J_k^o[x(k)] \;=\; \min_{u_k}\left[L(x(k),u(k))+\min_{U_{k+1}}E_{\substack{w(k)\\\mathcal{W}_{k+1}}}\left\{S(x(N))+\sum_{j=k+1}^{N-1}L(x(j),u(j))\right\}\right]$$

use Lemma 1 (commute expectation and minimization)

$$J_k^o[x(k)] \;=\; \min_{u_k}\left[L(x(k),u(k))+E_{w(k)}\left\{\underbrace{\min_{U_{k+1}}E_{\mathcal{W}_{k+1}}\left\{S(x(N))+\sum_{j=k+1}^{N-1}L(x(j),u(j))\right\}}_{J_{k+1}^O[x(k+1)]}\right\}\right]$$

$$J_k^o[x(k)] \;=\; \min_{u_k}\left[L(x(k),u(k))+E_{w(k)}\left\{J_{k+1}^o[x(k+1)]\right\}\right]$$

Q.E.D.

Finite horizon LQG with <u>measured</u> states

Theorem 1:

a) The optimal control is given by

$$u^o(k) = -K(k+1)\,x(k)$$

$$K(k+1) = \left[R + B^T P(k+1) B\right]^{-1} B^T P(k+1) A$$

$$P(k-1) = Q + A^T P(k) A - A^T P(k) B \left[R + B^T P(k) B\right]^{-1} B^T P(k) A$$

Standard deterministic LQR solution! $P(N) = S$

Finite horizon LQG with <u>measured</u> states

Theorem 1:

b) The optimal cost J^o is given by

$$J^o = \frac{1}{2} x_o^T P(0) x_o + \frac{1}{2}\text{trace}\,[P(0) X_o] + b(0)$$

$$x_o = E\{x(0)\} \qquad X_o = E\{\tilde{x}^o(0)\tilde{x}^{oT}(0)\}$$

$$b(k) = b(k+1) + \text{trace}\left[B_w^T P(k+1) B_w W(k)\right]$$

$$b(N) = 0$$

Proof of Theorem 1 (sketch)

We solve the Stochastic Bellman equation recursively

$$J_k^o[x(k)] = \frac{1}{2} \min_{u(k)} \Big[x^T(k)\,Q\,x(k) + u^T(k)\,R\,u(k) + E_{w(k)}\left\{J_{k+1}^o[x(k+1)]\right\}\Big]$$

State equation:

$$x(k+1) = A\,x(k) + B\,u(k) + B_w\,w(k)$$

Proof of Theorem 1 (by recursion)

Assume that $J_k^o[x(k)]$ has the following form:

$$J_k^o[x(k)] = \frac{1}{2} x^T(k)\,P(k)\,x(k) + b(k)$$

Additional term due to effect of $\boldsymbol{w(k)}$

With the boundary condition $b(N) = 0$

And substitute this expression into the Bellman equation

Proof of Theorem 1 (sketch)

$$
\begin{aligned}
J_k^o[x(k)] &= \frac{1}{2}\, x^T(k)\, P(k)\, x(k) + b(k) \\
&= \frac{1}{2} \min_{u(k)} \Big[x^T(k)\, Q\, x(k) + u^T(k)\, R\, u(k) \\
&\quad \underbrace{+ E_{w(k)} \left\{ x^T(k+1)\, P(k+1)\, x(k+1) + b(k+1) \right\}}_{J_{k+1}^o[x(k+1)]} \Big]
\end{aligned}
$$

$$x(k+1) = A\, x(k) + B\, u(k) + B_w\, w(k)$$

Proof of Theorem 1 (sketch)

Substituting

$$x(k+1) = A\, x(k) + B\, u(k) + B_w\, w(k)$$

Into,

$$E_{w(k)} \left\{ x^T(k+1)\, P(k+1)\, x(k+1) + b(k+1) \right\}$$

And using $E\{x(k)^T w(k)\} = 0$ we obtain,

$$
\begin{aligned}
&E_{w(k)} \left\{ x^T(k+1)\, P(k+1)\, x(k+1) + b(k+1) \right\} \\
&\quad = [Ax(k) + Bu(k)]^T P(k+1) [Ax(k) + Bu(k)] \\
&\quad + \mathrm{trace} \left[B_w^T P(k+1) B_w W(k) \right] + b(k+1)
\end{aligned}
$$

Proof of Theorem 1 (sketch)

We obtain,

$$\boxed{\frac{1}{2}\, x^T(k)\, P(k)\, x(k)} + \boxed{b(k)} =$$

Standard LQR solution

$$\boxed{\begin{aligned} = \frac{1}{2} \min_{u(k)} \Big[& x^T(k)\, Q\, x(k) + u^T(k)\, R\, u(k) \\ & + [Ax(k) + Bu(k)]^T P(k+1) [Ax(k) + Bu(k)] \Big] \end{aligned}}$$

$$\boxed{+ \,\mathrm{trace} \left[B_w^T P(k+1) B_w W(k) \right] + b(k+1)}$$

Additional stochastic component

Proof of Theorem 1 (sketch)

Thus,

Standard LQR solution

$$\boxed{\begin{aligned} \frac{1}{2}\, x^T(k)\, P(k)\, x(k) = \frac{1}{2} \min_{u(k)} \Big[& x^T(k)\, Q\, x(k) + u^T(k)\, R\, u(k) \\ & + [Ax(k) + Bu(k)]^T P(k+1) [Ax(k) + Bu(k)] \Big] \end{aligned}}$$

$$\boxed{b(k) = b(k+1) + \mathrm{trace} \left[B_w^T P(k+1) B_w W(k) \right]}$$

Recursive expression for the additional term in the value function $J_k^o[x(k)]$ due to $\boldsymbol{w(k)}$

Proof of Theorem 1 (sketch)

Solution:

Optimal control law:

$$u^o(k) = -K(k+1)\,x(k)$$

$$K(k+1) = \left[R + B^T P(k+1) B\right]^{-1} B^T P(k+1) A$$

$$P(k-1) = Q + A^T P(k) A - A^T P(k) B \left[R + B^T P(k) B\right]^{-1} B^T P(k) A$$

Standard deterministic LQR $P(N) = S$

Proof of Theorem 1 (sketch)

Solution: The optimal cost is obtained from

$$\begin{aligned} J^o &= J_0^o = E_{\mathcal{X}_o}\{J_0^o[x(0)]\} \\ &= \frac{1}{2} E\{x^T(0)\,P(0)\,x(0)\} + b(0) \end{aligned}$$

The Gaussian initial condition $x(0) \in \mathcal{X}_o$ satisfies

$$E\{x(0)\} = x_o \qquad X_o = E\{(x(0)-x_o)(x(0)-x_o)^T\}$$

Proof of Theorem 1 (sketch)

$$J^o = \frac{1}{2} E\{x^T(0)\,P(0)\,x(0)\} + b(0)$$

Using Lemma 2, we obtain

$$J^o = \frac{1}{2} x_o^T P(0) x_o + \frac{1}{2} \mathrm{Tr}\,[P(0) X_o] + b(0)$$

$$E\{x(0)\} = x_o \qquad X_o = E\{(x(0)-x_o)(x(0)-x_o)^T\}$$

Proof of Theorem 1 (sketch)

Thus,

$$J^o = \frac{1}{2} x_o^T P(0) x_o + \frac{1}{2} \mathrm{trace}\,[P(0) X_o] + b(0)$$

$$b(k) = b(k+1) + \mathrm{trace}\left[B_w^T P(k+1) B_w W(k)\right]$$

$$b(N) = 0$$

End of proof of Theorem 1

Finite horizon LQG

For $N > 0$, find the optimal control sequence:

$$U_0^o = \{u^o(0), u^o(1), \cdots, u^o(N-1)\}$$

Which minimizes the cost functional:

$$J = \frac{1}{2} E\left\{x^T(N)\, S\, x(N)\right\} + \frac{1}{2} E\left\{\sum_{k=0}^{N-1} x^T(k)\, Q\, x(k) + u^T(k)\, R\, u(k)\right\}$$

where $u^o(k)$ can only be based on the observations:

$$Y_k = \{y(0), y(1), \cdots, y(k)\}$$

Finite horizon LQG

Notice that the expectation in $\boldsymbol{J}$ is taken as follows:

$$J = \frac{1}{2} E_{\substack{\mathcal{X}_o \\ \mathcal{W}_0 \\ \mathcal{V}_0}} \left\{ x^T(N)\, S\, x(N) + \sum_{k=0}^{N-1} \left[x^T(k)\, Q\, x(k) + u^T(k)\, R\, u(k)\right]\right\}$$

Over the combined PDF for

$$\mathcal{X}_o = \{x(0)\}$$

$$\mathcal{W}_0 = \{w(0), w(1), \cdots, w(N-1)\}$$

$$\mathcal{V}_0 = \{v(0), v(1), \cdots, v(N)\}$$

State Estimate Error Covariances

A-priori estimation error covariance:

$$\hat{x}^o(k) = E\{x(k)|Y_{k-1}\} \qquad M(k) = E\{\tilde{x}^o(k)\tilde{x}^{oT}(k)\}$$

A-posteriori estimation error covariance:

$$\hat{x}(k) = E\{x(k)|Y_k\} \qquad Z(k) = E\{\tilde{x}(k)\tilde{x}^T(k)\}$$

satisfy

$$Z(k) = M(k) - M(k)C^T\left[CM(k)C^T + V(k)\right]^{-1} CM(k)$$

$$M(k+1) = A\,Z(k)A^T + B_w W(k) B_w^T$$

$$M(0) = X_o$$

Theorem 2: Separation Principle

1) The optimal control is given by the LQR replacing the state by the ***a-posteriori*** state estimate.

$$\boxed{u^o(k) = -K(k+1)\,\hat{x}(k)}$$

$$K(k+1) = \left[R + B^T P(k+1) B\right]^{-1} B^T P(k+1) A$$

$$P(k-1) = Q + A^T P(k) A - A^T P(k) B\left[R + B^T P(k) B\right]^{-1} B^T P(k) A$$

Standard deterministic LQR $\qquad P(N) = S$

Theorem 2: Separation Principle

2) The optimal cost J^o is given by

$$J^o = \hat{J}^o + \sum_{j=0}^{N-1} \mathrm{Tr}[QZ(j)] + \mathrm{Tr}[SZ(N)]$$

where

$$Z(k) \;=\; E\{\tilde{x}(k)\tilde{x}^T(k)\}$$

A-posteriori estimation error covariance:

$$Z(k) \;=\; M(k) - M(k)C^T\left[CM(k)C^T + V(k)\right]^{-1}CM(k)$$

$$M(k+1) \;=\; A\,Z(k)A^T + B_w W(k) B_w^T$$

$$M(0) \;=\; X_o$$

Theorem 2: Separation Principle

2) The optimal cost $\hat{J}^o$ is given by

$$J^o = \hat{J}^o + \sum_{j=0}^{N-1} \mathrm{Tr}[QZ(j)] + \mathrm{Tr}[SZ(N)]$$

$$\hat{J}^o \;=\; \frac{1}{2}x_o^T P(0)x_o + \frac{1}{2}\mathrm{trace}\,[P(0)X_o] + \hat{b}(0)$$

where

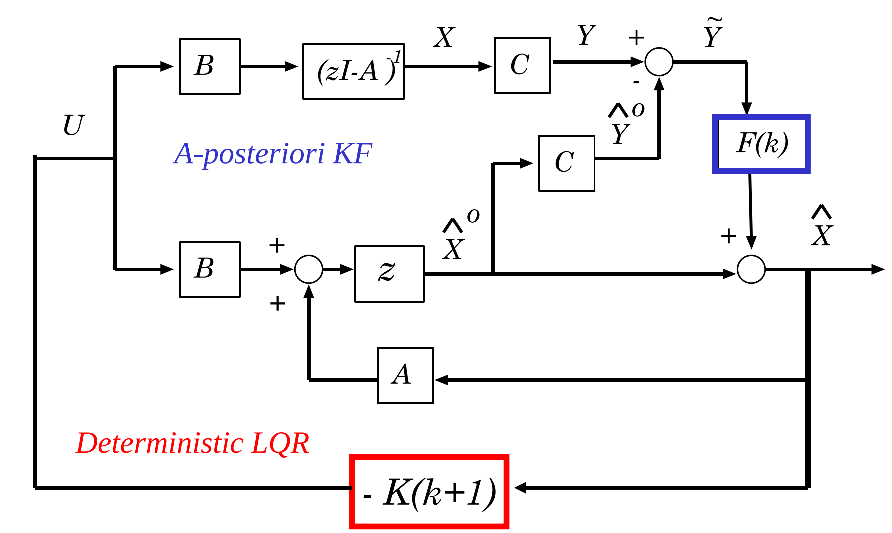
$$\hat{b}(k) = \hat{b}(k+1) + \mathrm{trace}\left[F^T(k+1)P(k+1)F(k+1)E(k+1)\right]$$

$$\hat{b}(N) = 0$$

$$F(k) \;=\; M(k)C^T\left[C\,M(k)C^T + V(k)\right]^{-1}$$

$$E(k) \;=\; E\{\tilde{y}^o(k)\tilde{y}^{oT}(k)\} = \left[C\,M(k)C^T + V(k)\right]$$

Separation Principle



Sketch of proof of Theorem 2

Remember some preliminary facts:

The a-posteriori state estimate $\hat{x}(k)$

$$\hat{x}(k) = E\{x(k)|Y_k\}$$

is the ***conditional*** expectation of ***x(k)*** given

$$Y_k = \{y(0), \cdots, y(k)\}$$

A-posteriori state estimate

Satisfies:

1) $$E\{x^T(k)\, g(Y_k)\} = E\{\hat{x}^T(k)\, g(Y_k)\}$$

For any function $g(\cdot)$ of the random variables $\boldsymbol{Y_k}$,

See Lemma in page 11, lecture notes 7
on Least Squares estimation

A-posteriori state estimate

Satisfies:

2) $$E\{\hat{x}(k)\, \tilde{x}^T(k)\} = 0$$

where $\tilde{x}(k) = x(k) - \hat{x}(k)$

See Property 1, page 35 , lecture notes 7
on Least Squares estimation

Sketch of proof of Theorem 2

We want to find the control that minimizes:

$$J = \frac{1}{2} E\left\{ x^T(N)\, S\, x(N) + \sum_{k=0}^{N-1} \left(x^T(k)\, Q\, x(k) + u^T(k)\, R\, u(k) \right) \right\}$$

lets focus on the quadratic terms of ***x(k)***

Sketch of proof of Theorem 2

We again use Lemma 2:

$$E\{x^T(k)\, Qx(k)\} = E\{\hat{x}^T(k)\, Q\hat{x}(k)\} + \text{trace}\,[QZ(k)\}]$$

$$E\{x^T(N)\, Sx(N)\} = E\{\hat{x}^T(N)\, S\hat{x}(N)\} + \text{trace}\,[SZ(N)\}]$$

where

$\hat{x}(k) = E\{x(k)|Y_k\}$ $\qquad Z(k) = E\{\tilde{x}(k)\tilde{x}^T(k)\}$

Proof of Lemma 2

$$E\{x^T(k)\,Qx(k)\} \;=$$

$$= \; E\{x^T(k)Q[\underbrace{x(k) - \hat{x}(k)}_{\tilde{x}(k)} + \hat{x}(k)]\}$$

$$\underbrace{-\hat{x}(k)+\hat{x}(k)}_{\boldsymbol{0}}$$

$$= \; E\{x^T(k)Q\tilde{x}(k)\} + \underbrace{E\{x^T(k)\,Q\hat{x}(k)\}}$$

By the LS lemma $\Rightarrow$ $E\{\hat{x}^T(k)\,Q\hat{x}(k)\}$

Proof of Lemma 2

$$E\{x^T(k)\,Qx(k)\} \;=$$

$$= \; \underbrace{E\{x^T(k)Q\tilde{x}(k)\}} + \; E\{\hat{x}^T(k)\,Q\hat{x}(k)\}$$

$$\text{trace}\left[QE\{\tilde{x}(k)x^T(k)\}\right] \; - \; \text{trace}\left[Q\underbrace{E\{\tilde{x}(k)\hat{x}^T(k)\}}\right]$$

By property 1 of LS estimation $\Rightarrow$ $\boldsymbol{0}$

Proof of Lemma 2

$$E\{x^T(k)\,Qx(k)\} \;= \quad E\{\hat{x}^T(k)\,Q\hat{x}(k)\}$$

$$\underbrace{+\text{trace}\left[QE\{\tilde{x}(k)x^T(k)\}\right] \quad - \; \text{trace}\left[QE\{\tilde{x}(k)\hat{x}^T(k)\}\right]}$$

$$\underbrace{+ \; \text{trace}\left[QE\{\tilde{x}(k)(x(k) - \hat{x}(k))^T\}\right]}$$

$$+ \; \text{trace}\left[Q\underbrace{E\{\tilde{x}(k)\tilde{x}(k)^T\}}_{Z(k)}\right]$$

Q.E.D.

Equivalent problems:

We want to find the control that minimizes:

$$J \;=\; \frac{1}{2}E\left\{ \; x^T(N)\,S\,x(N) + \sum_{k=0}^{N-1}\left(x^T(k)\,Q\,x(k) + u^T(k)\,R\,u(k)\right)\right\}$$

Equivalent problems:

We want to find the control that minimizes:

$$J = \underbrace{\frac{1}{2} E\left\{ \hat{x}^T(N)\, S\, \hat{x}(N) + \sum_{k=0}^{N-1} \left(\hat{x}^T(k)\, Q\, \hat{x}(k) + u^T(k)\, R\, u(k) \right) \right\}}_{\hat{J}}$$

$$+ \sum_{j=0}^{N-1} \text{trace}[QZ(j)] \;+ \text{trace}\,[SZ(N)\}]$$

Sketch of proof of Theorem 2

The cost J can be written as

$$J = \hat{J} \;+\; \underbrace{\sum_{j=0}^{N-1} \text{trace}[QZ(j)] + \text{trace}[SZ(N)]}$$

Minimized by Kalman filter!
These terms are not functions of the control

$$\mathcal{U}_0 \;=\; \{u(0),\, u(1),\, \cdots,\, u(N-1)\}$$

only the term $\hat{J}$ can be minimized w/r to the control

Sketch of proof of Theorem 2

Minimizing

$$J = \hat{J} \;+\; \sum_{j=0}^{N-1} \text{trace}[QZ(j)] + \text{trace}[SZ(N)]$$

is equivalent to minimizing

$$\hat{J} \;=\; \frac{1}{2} E\left\{ \hat{x}^T(N)\, S\, \hat{x}(N) + \sum_{k=0}^{N-1} \left(\hat{x}^T(k)\, Q\, \hat{x}(k) + u^T(k)\, R\, u(k) \right) \right\}$$

However, $\hat{x}(k)$ is measurable!

Sketch of proof of Theorem 2

Find the optimal control sequence that minimizes

$$\hat{J} = \frac{1}{2} E_{\substack{\mathcal{X}_o \\ \tilde{\mathcal{Y}}_0^o}} \left\{ \hat{x}^T(N)\, S\, \hat{x}(N) + \sum_{k=0}^{N-1} \left[\hat{x}^T(k)\, Q\, \hat{x}(k) + u^T(k)\, R\, u(k) \right] \right\}$$

Subject to

$$\hat{x}(k+1) = A\,\hat{x}(k) + B\,u(k) + F(k+1)\,\underbrace{\tilde{y}^o(k+1)}$$

Innovation

Stochastic disturbance which is uncorrelated

with all past a-posteriori <u>state estimates</u> $\hat{x}(k)$

Sketch of proof of Theorem 2

We can use Theorem 1:

a) The optimal control is given by

$$u^o(k) = -K(k+1)\,\hat{x}(k)$$

$$K(k+1) = \left[R + B^T P(k+1) B\right]^{-1} B^T P(k+1) A$$

$$P(k-1) = Q + A^T P(k) A - A^T P(k) B \left[R + B^T P(k) B\right]^{-1} B^T P(k) A$$

Standard deterministic LQR solution! $P(N) = S$

Sketch of proof of Theorem 2

A-posteriori state observer structure:

$$\begin{aligned}
\hat{x}(k) &= \hat{x}^o(k) + F(k)\,\tilde{y}^o(k) \\
\hat{x}^o(k+1) &= A\,\hat{x}(k) + B\,u(k) \\
\tilde{y}^o(k) &= y(k) - C\,\hat{x}^o(k)
\end{aligned}$$

$$\begin{aligned}
F(k) &= M(k) C^T \left[C\,M(k) C^T + V(k)\right]^{-1} \\
M(k+1) &= A M(k) A^T + B_w W(k) B_w^T \\
&\quad - A M(k) C^T \left[C M(k) C^T + V(k)\right]^{-1} C M(k) A^T
\end{aligned}$$

Sketch of proof of Theorem 2

We can use Theorem 1:

b) The optimal cost $\hat{J}^O$ is given by

$$\hat{J}^o = \frac{1}{2} x_o^T P(0) x_o + \frac{1}{2}\mathrm{trace}\left[P(0) X_o\right] + \hat{b}(0)$$

$$x_o = E\{x(0)\} \qquad X_o = E\{\tilde{x}^o(0)\tilde{x}^{oT}(0)\}$$

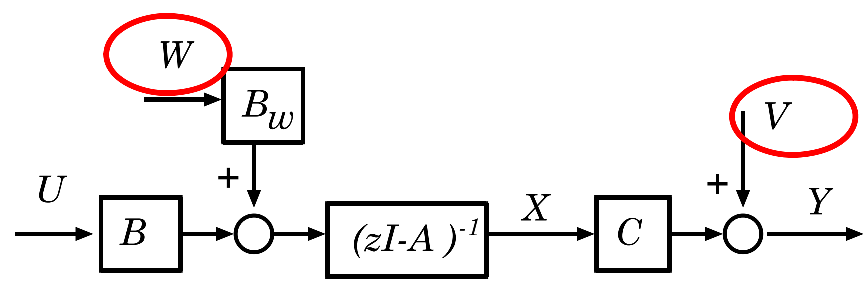
$$\hat{b}(k) = \hat{b}(k+1) + \mathrm{trace}\left[F^T(k+1) P(k+1) F(k+1) E(k+1)\right]$$

$$E(k) = \left[C\,M(k) C^T + V(k)\right] \qquad b(N) = 0$$

End of proof of Theorem 2

Stationary random inputs

Linear system contaminated by noise:



Assume now that both

- $w(k)$ and $v(k)$ are WSS

Stationary LQG

We want to regulate the state

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) + B_w\,w(k) \\ y(k) &= C\,x(k) + v(k) \end{aligned}$$

under

$$\left.\begin{aligned} E\{w(k+l)w^T(k)\} &= W\,\delta(l) \\ E\{v(k+l)v^T(k)\} &= V\,\delta(l) \\ E\{w(k+l)v^T(k)\} &= 0 \end{aligned}\right\}$$

WSS Gaussian Noise

Stationary LQG

Define the "incremental" cost

$$J' = \frac{1}{N}\,J$$

$$J = \frac{1}{2}E\left\{x^T(N)\,S\,x(N) + \sum_{k=0}^{N-1}\left[x^T(k)\,Q\,x(k) + u^T(k)\,R\,u(k)\right]\right\}$$

The control that minimizes J also minimizes J'

Stationary LQG

Define the "incremental" cost

$$J' = E\left\{\frac{1}{2N}\,x^T(N)\,S\,x(N) + \frac{1}{2N}\sum_{k=0}^{N-1}\left[x^T(k)\,Q\,x(k) + u^T(k)\,R\,u(k)\right]\right\}$$

Under the stationarity assumptions:

$$\lim_{N\to\infty} J' = J_s$$

$$J_s = \frac{1}{2}E\{x^T(k)Qx(k) + u^T(k)Ru(k)\}$$

Stationary LQG

Obtain the optimal control that minimizes:

$$J_s = \frac{1}{2}E\{x^T(k)Qx(k) + u^T(k)Ru(k)\}$$

under

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) + B_w\,w(k) \\ y(k) &= C\,x(k) + v(k) \end{aligned}$$

- $w(k)$ and $v(k)$ are WSS

Theorem 3: Stationary LQG

If, $[A,B]$ is controllable or stabilizable and

$[A, C_Q]$ is observable or detectable ($C_Q^T C_Q = Q$)

The optimal control is

$$u^o(k) = -K\,\widehat{x}(k)$$

where the gain $\boldsymbol{K}$ is obtained from the unique solution of the algebraic Riccati equation

$$A^T PA - P + Q - A^T PB\left[R + B^T PB\right]^{-1} B^T PA = 0$$

$$K = \left[R + B^T PB\right]^{-1} B^T PA$$

Theorem 3: Stationary LQG

If, $[A,C]$ is] is observable or detectable and

$[A, B'_w]$ is controllable or stabilizable ($B_w B'^T_w = B_w W B^T_w$)

The optimal control

$$u^o(k) = -K\,\widehat{x}(k)$$

- $\widehat{x}(k)$ is the steady state a-posteriori Kalman Filter estimator:

$$\begin{aligned} \widehat{x}(k) &= \widehat{x}^o(k) + F\,\tilde{y}^o(k) \\ \widehat{x}^o(k+1) &= A\,\widehat{x}(k) + B\,u(k) \\ \tilde{y}^o(k) &= y(k) - C\,\widehat{x}^o(k) \end{aligned}$$

Theorem 3: Stationary LQG

- **A-posteriori Kalman Filter estimator:**

$$\begin{aligned} \widehat{x}(k) &= \widehat{x}^o(k) + F\,\tilde{y}^o(k) \\ \widehat{x}^o(k+1) &= A\,\widehat{x}(k) + B\,u(k) \\ \tilde{y}^o(k) &= y(k) - C\,\widehat{x}^o(k) \end{aligned}$$

Where the gain $\boldsymbol{F}$ is the unique solution of

$$\begin{aligned} AMA^T - M &= -B_w W B_w^T \\ &\quad + AMC^T\left[CMC^T + V\right]^{-1} CMA^T \\ F &= MC^T\left[C\,MC^T + V\right]^{-1} \end{aligned}$$

Theorem 3: Stationary LQG

The Optimal cost is given by:

$$J_s^o = \text{trace}\left\{P\left[BKZA^T + B_w W B_w^T\right]\right\}$$

$$Z = E\{\tilde{x}(k)\tilde{x}^T(k)\}$$

(see the derivation of this result in the next slides)

Outline

- Linear Quadratic Gaussian (LQG) regulator
- Finite horizon LQG
  - LQG under full state measurement
  - LQG under output measurement
- Stationary LQG

- Additional material:

  - Derivation of optimal cost
  - Continuous time stationary LQG

Stationary LQG

Optimal cost (derivation)
The incremental optimal cost is

$$J_s^o = \lim_{N\to\infty} \frac{1}{N}\left\{\hat{J}^o + \sum_{j=0}^{N-1} \mathrm{Tr}[QZ(j)] + \mathrm{Tr}[SZ(N)]\right\}$$

$$\hat{J}^o = \frac{1}{2}x_o^T P(0) x_o + \frac{1}{2}\mathrm{trace}\,[P(0)X_o] + \hat{b}(0)$$

$$\hat{b}(k-1) = \hat{b}(k) + \mathrm{trace}\left[F^T(k)P(k)F(k)[CM(k)C^T + V]\right]$$

Thus

$$J_s^o = \mathrm{Tr}\left\{\left[QZ + F^T PF[CMC + V]\right]\right\}$$

Stationary LQG

Optimal cost (derivation)

$$J_s^o = \mathrm{Tr}\left\{\left[QZ + F^T PF[CMC + V]\right]\right\}$$

Note:

$$A^T PA - P = -Q + A^T PB\left[B^T PB + R\right]^{-1} B^T PA$$

$$F = MC^T\left[C\,MC^T + V\right]^{-1}$$

$$Z = M - MC^T\left[CMC^T + V\right]^{-1} CM \quad \text{(least squares)}$$

$$M = AZA^T + B_w W B_w^T$$

Stationary LQG

Optimal cost (derivation)

$$J_s^o = \mathrm{Tr}\left\{\left[QZ + F^T PF[CMC + V]\right]\right\}$$

last term:

$$\begin{aligned}\mathrm{Tr}\left\{F^T PF[CMC + V]\right\} &= \\ &= \mathrm{Tr}\{F^T PMC^T\} = \mathrm{Tr}\{PMC^T F^T\} \\ &= \mathrm{Tr}\{PMC^T[CMC^T + V]^{-1}CM\} \\ &= \mathrm{Tr}\{P(M - Z)\}\end{aligned}$$

first term:

$$\begin{aligned}\mathrm{Tr}\{QZ\} &= \\ &= \mathrm{Tr}\{\left[P - A^T PA + A^T PB[B^T PB + R]^{-1}B^T PA\right] Z\} \\ &= \mathrm{Tr}\{PZ + [-PA + PBK]\,ZA^T\}\end{aligned}$$

Stationary LQG

Optimal cost (derivation)

$$J_s^o = \text{Tr}\left\{QZ + F^T P F[CMC + V]\right\}$$

$$J_s^o = \text{Tr}\{PZ + [-PA + PBK]\, ZA^T + P(M - Z)\}$$

$$\begin{aligned} J_s^o &= \text{Tr}\{[-PA + PBK]ZA^T - P[AZA^T + B_w W B_w^T]\} \\ &= \text{Tr}\{PBKZA^T + PB_w W B_w^T\} \end{aligned}$$

Continuous time stationary LQG

Cost:

$$J_s = \frac{1}{2} E\{x^T(t) Q x(t) + u^T(t) R u(t)\}$$

- **Optimal control:** $u^o(t) = -K\,\widehat{x}(t)$

Where the gain is obtained from the solution of the steady state LQR

$$K = R^{-1} B^T P$$

$$A^T P + P A + Q - P B R^{-1} B^T P = 0$$

Stationary LQG

Solution:

- **Kalman Filter Estimator:**

$$\begin{aligned} \frac{d}{dt}\widehat{x}(t) &= A\,\widehat{x}(t) + B\,u(t) + L\,\tilde{y}(t) \\ \tilde{y}(t) &= y(t) - C\,\widehat{x}(t) \end{aligned}$$

$$L = M\,C^T\,V^{-1}$$

$$AM + MA^T = -B_w W B_w^T + MC^T V^{-1} CM$$

Stationary LQG

Solution:

- **Optimal cost:**

$$J_s^o = \text{Tr}\left\{P\left[BKM + B_w W B_w^T\right]\right\}$$

Stationary LQG

Optimal cost (derivation)

The incremental optimal cost is

$$J_s^o = \lim_{T\to\infty} \frac{1}{T}\left\{ \hat{J}^o + \int_0^T \mathrm{Tr}[QM(t)]dt + \mathrm{Tr}[SM(T)] \right\}$$

$$\begin{aligned} \hat{J}^o &= \frac{1}{2}x_o^T P(0) x_o + \frac{1}{2}\mathrm{trace}\,[P(0)X_o] \\ &+ \int_0^T \mathrm{trace}\{L^T(t)P(t)L(t)V(t)\}dt \end{aligned}$$

Thus

$$J_s^o = \mathrm{Tr}\left\{QM + L^T PLV\right\}$$

Stationary LQG

Optimal cost (derivation)

$$J_s^o = \mathrm{Tr}\left\{QM + L^T PLV\right\}$$

Note:

$$Q = -A^T P - P A + P B R^{-1} B^T P$$

$$K = R^{-1} B^T P$$

$$L = M C^T V^{-1}$$

$$AM + MA^T = -B_w W B_w^T + MC^T V^{-1} CM$$

Stationary LQG

Optimal cost (derivation)

$$J_s^o = \mathrm{Tr}\left\{QM + L^T PLV\right\}$$

last term:

$$\begin{aligned} &\mathrm{Tr}\left\{L^T PLV\right\} = \\ &= \mathrm{Tr}\{L^T PMC^T\} = \mathrm{Tr}\{PMC^T L^T\} \\ &= \mathrm{Tr}\{PMC^T V^{-1} CM\} \\ &= \mathrm{Tr}\{P[AM + MA^T + B_w W B_w^T]\} \end{aligned}$$

first term:

$$\begin{aligned} \mathrm{Tr}\{QM\} &= \mathrm{Tr}\{\left[-A^T P - P A + P B K\right] M\} \\ &= \mathrm{Tr}\{-PMA^T - P AM + P B KM\} \end{aligned}$$

Adding:

$$J_s^o = \mathrm{Tr}\left\{P\left[BKM + B_w W B_w^T\right]\right\}$$