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ME 233 – Homework 9

1.a)

$$G_p(s)=\frac{\omega_b^2}{s^2+2\zeta_b\omega_b s+\omega_b^2},\ \zeta_b=0.707,\ \omega_b=10\,\text{rad/s}$$

$$G_{PA}(s)=G_p(s)\left(\frac{\omega_r(\zeta_r s+\omega_r)}{s^2+2\zeta_r\omega_r s+\omega_r^2}\right)\left(\frac{\omega_t^2(s^2+2\zeta_t\omega_n s+\omega_n^2)}{\omega_n^2(s^2+2\zeta_t\omega_t s+\omega_t^2)}\right)$$

$$\zeta_r=0.015,\ \omega_r=1000\,\text{rad/s},\ \zeta_t=0.015,\ \omega_t=1200\,\text{rad/s},\ \omega_n=0.9\,\omega_t$$

$$G_{PA}(s)=(1+\Delta(s))G_p(s)$$

$$\Delta(s)=\frac{G_{PA}(s)}{G_p(s)}-1=\left(\frac{\omega_r(\zeta_r s+\omega_r)}{s^2+2\zeta_r\omega_r s+\omega_r^2}\right)\left(\frac{\omega_t^2(s^2+2\zeta_t\omega_n s+\omega_n^2)}{\omega_n^2(s^2+2\zeta_t\omega_t s+\omega_t^2)}\right)-1$$

$$\Delta(s)=\frac{-100s^6-6162s^5-1.207\cdot10^8s^4-3.466\cdot10^9s^3-3.695\cdot10^{10}s^2-1.76\cdot10^{11}s-4}{100s^6+8014s^5+2.442\cdot10^8s^4+1.137\cdot10^{10}s^3+1.441\cdot10^{14}s^2+2.037\cdot10^{15}s+1.44\cdot10^{16}}$$

Noting the pole-zero cancellation at $-7.07\pm7.0721\,j$,

$$\text{minreal}(\Delta(s))=\frac{-s^4-47.48s^3-1.206\cdot10^6s^2-1.76\cdot10^7s}{s^4+66s^3+2.441\cdot10^6s^2+7.92\cdot10^7s+1.44\cdot10^{12}}$$

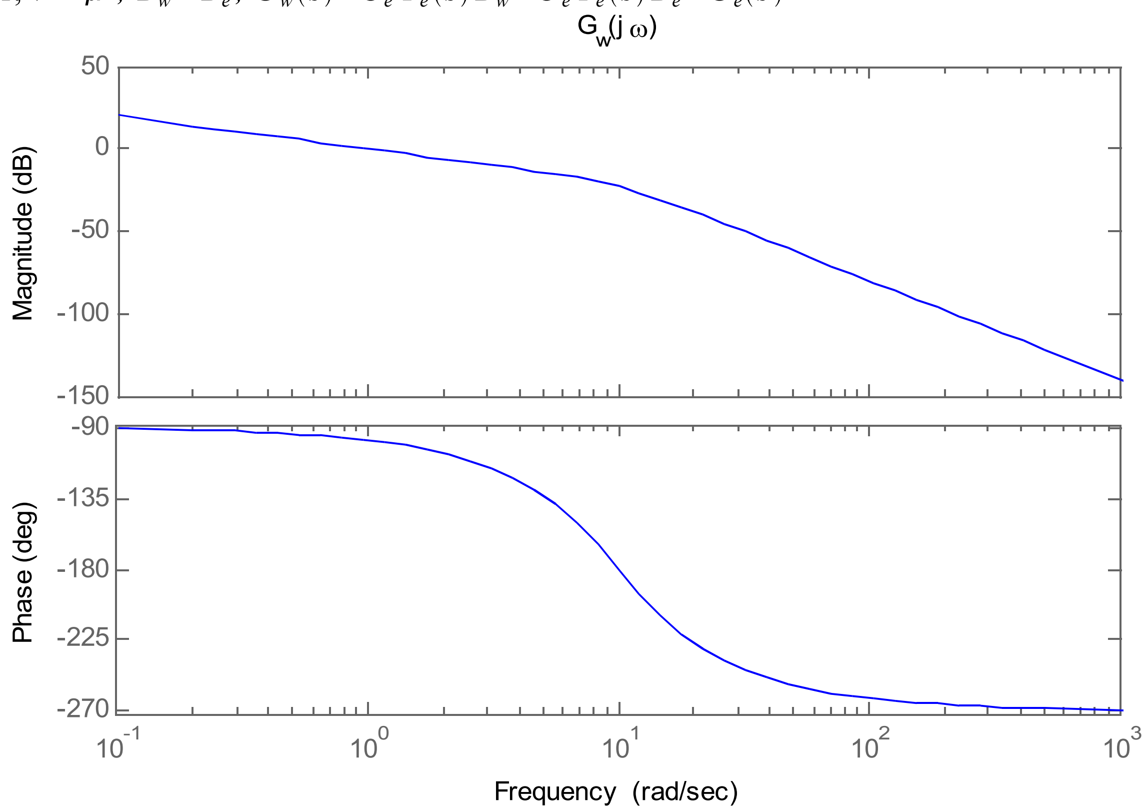
1.b)

$$G_a(s)=\frac{1}{s},\ G_e(s)=G_p(s)G_a(s)=\frac{\omega_b^2}{s^3+2\zeta_b\omega_b s^2+\omega_b^2 s}=\frac{100}{s^3+14.14s^2+100s}$$

$$G_{EA}(s)=G_{PA}(s)\,G_a(s)=\frac{G_p(s)}{s}\left(\frac{\omega_r(\zeta_r s+\omega_r)}{s^2+2\zeta_r\omega_r s+\omega_r^2}\right)\left(\frac{\omega_t^2(s^2+2\zeta_t\omega_n s+\omega_n^2)}{\omega_n^2(s^2+2\zeta_t\omega_t s+\omega_t^2)}\right)$$

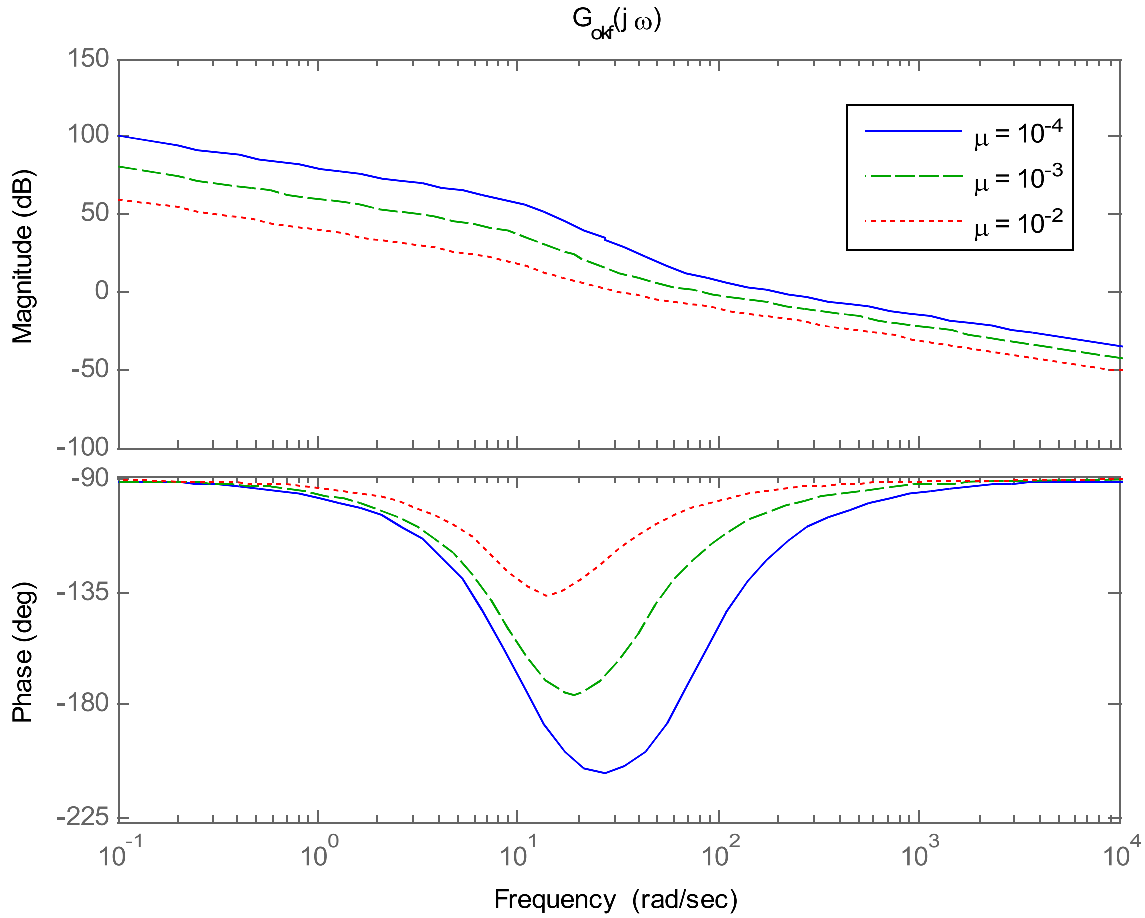
1.c)i.

$$W=1,\ V=\mu^2,\ B_w=B_e,\ G_w(s)=C_e\Phi_e(s)B_w=C_e\Phi_e(s)B_e=G_e(s)$$



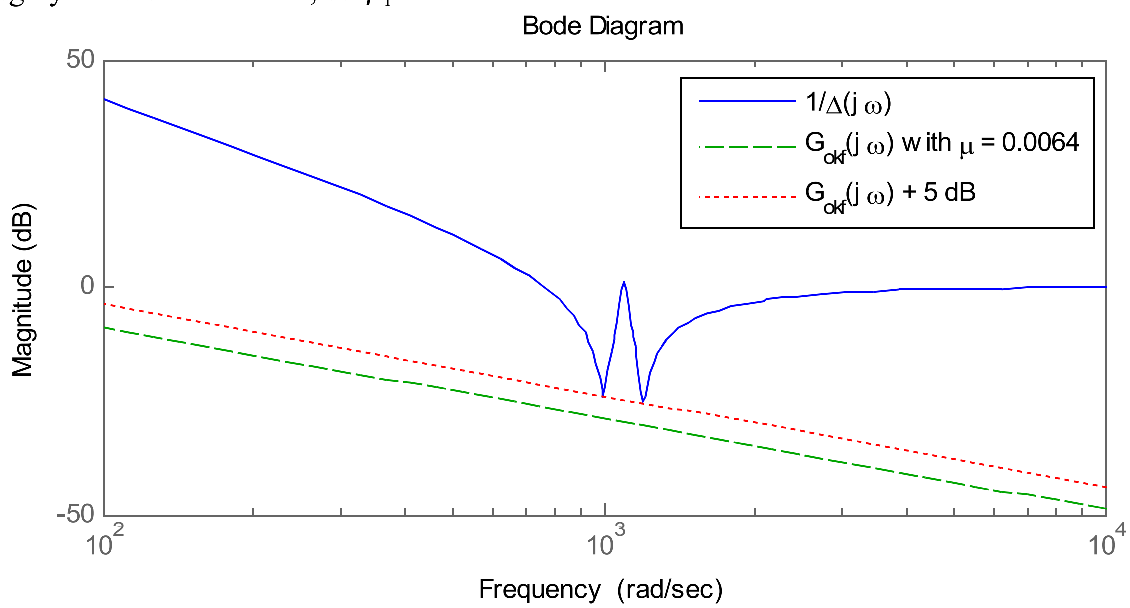
1.c)ii.

$L_e = M C_e^T V^{-1}$, $A_e M + M A_e^T + B_w B_w^T - M C_e^T V^{-1} C_e M = 0$



1.c)iii.

The most difficult part of $|1/\Delta(j\omega)|$ to remain below is the -24 dB point at 10^3 rad/s. That is a higher frequency than the slope changes in $G_{okf}(j\omega)$ so the simple inverse proportionality with μ doesn't really hold there. The magnitudes $|G_{okf}(j10^3)|$ were: -14.6 dB for $\mu = 10^{-4}$, -22.08 dB for $\mu = 10^{-3}$, and -30.7 dB for $\mu = 10^{-2}$. We want to be below -29 dB, so μ should be near 10^{-2} but towards 10^{-3} by roughly a factor of $10^{1.7/8.7}$, so $\mu_1 > 0.0064$



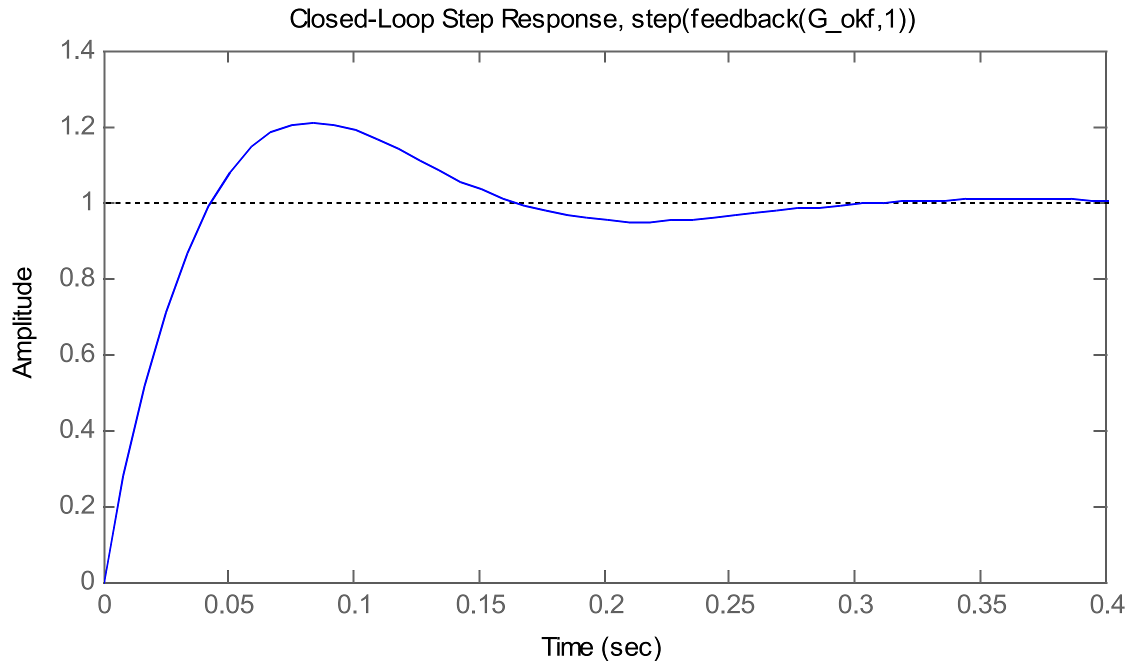
1.c)iv.

$T_{kf}(s)=\dfrac{G_{okf}(j\omega)}{1+G_{okf}(j\omega)}$, and at high frequencies we have $G_{okf}(j\omega)\ll 1$ so $T_{kf}(s)\approx G_{okf}(j\omega)$

$T_{kf}(s)\approx G_{okf}(j\omega)$ so $|T_{kf}(s)|\approx|G_{okf}(j\omega)|\leq|1/\Delta(j\omega)|\cdot 10^{(-5/20)}$

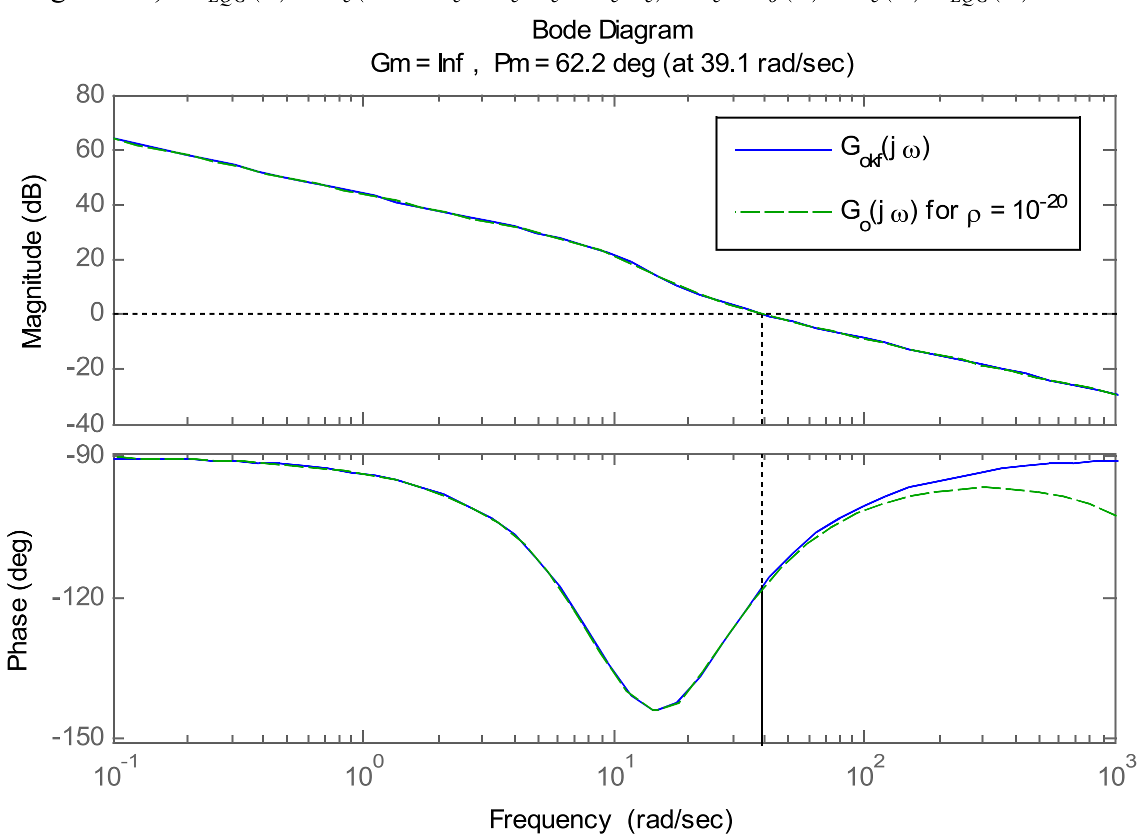
1.c)v.

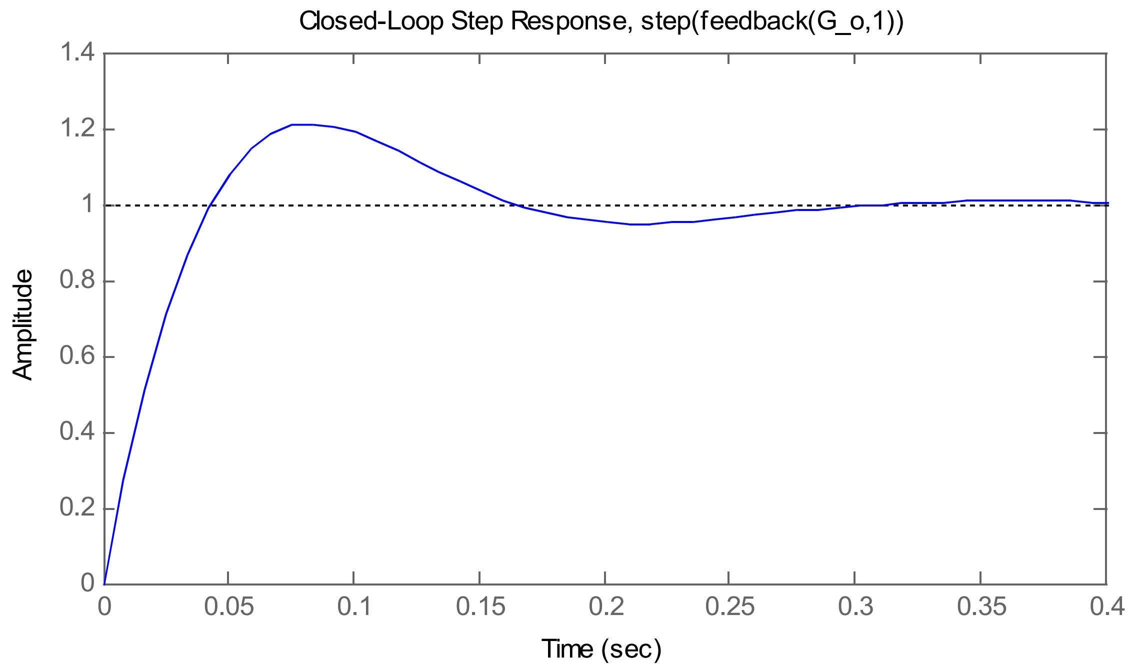
For $\mu_1=0.0064$, $L_e=[115.33 \quad 206.96 \quad 11.51]^T$



1.d)

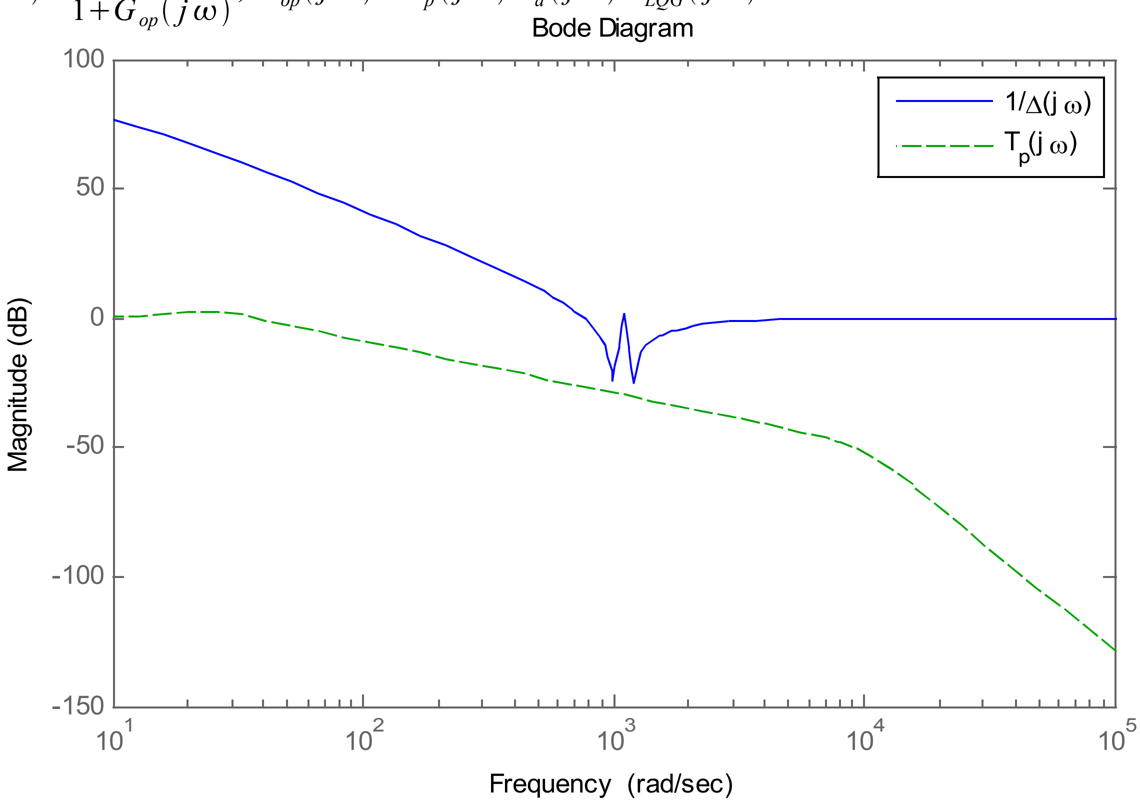
$A_e^T P+P A_e+C_e^T C_e-P B_e\rho^{-1}B_e^T P=0$, $K_e=\rho^{-1}N^{-1}B_e^T P=[5\cdot 10^3 \quad 6.25\cdot 10^6 \quad 3.125\cdot 10^{10}]$ for $\rho=10^{-20}$ (assuming $N=I$). $C_{LQG}(s)=K_e(sI-A_e+B_eK_e+C_eL_e)^{-1}L_e$, $G_o(s)=G_e(s)C_{LQG}(s)$





1.e)

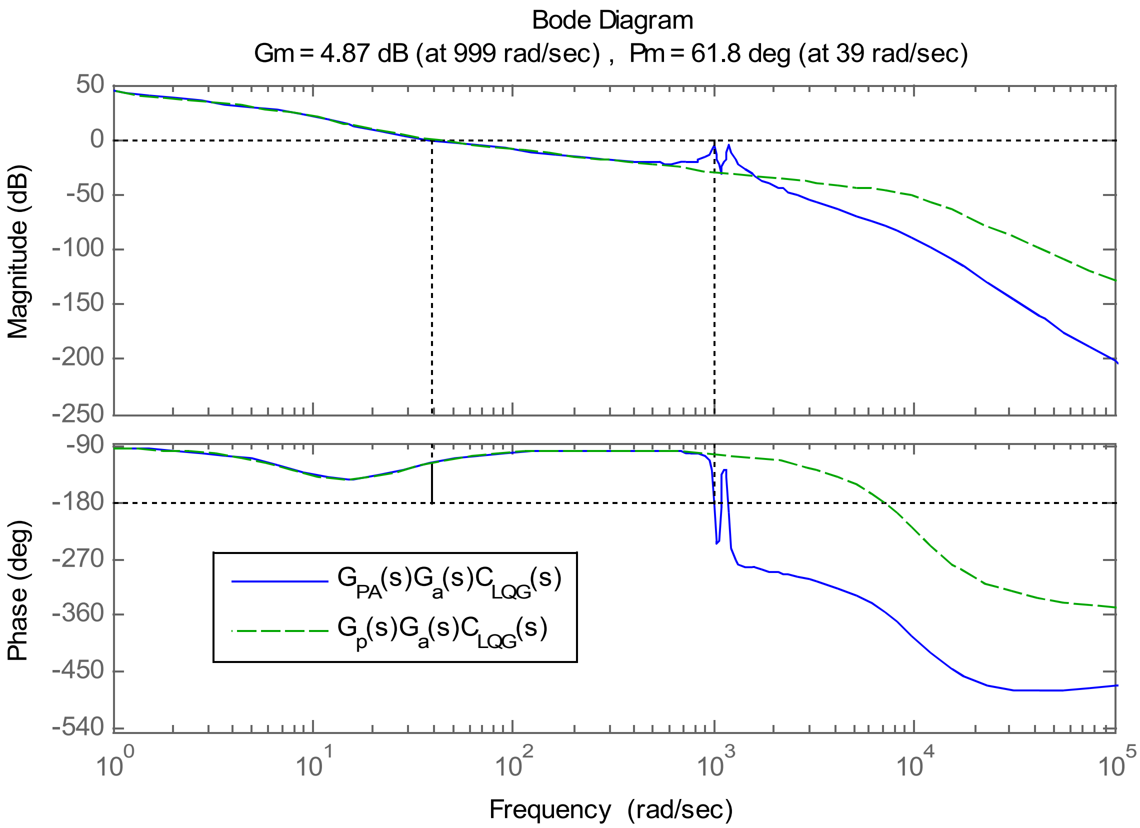
$$T_p(j\omega)=\frac{G_{op}(j\omega)}{1+G_{op}(j\omega)},\ G_{op}(j\omega)=G_p(j\omega)G_a(j\omega)C_{LQG}(j\omega)$$



We can see that yes, $|T_p(j\omega)|<|1/\Delta(j\omega)|$

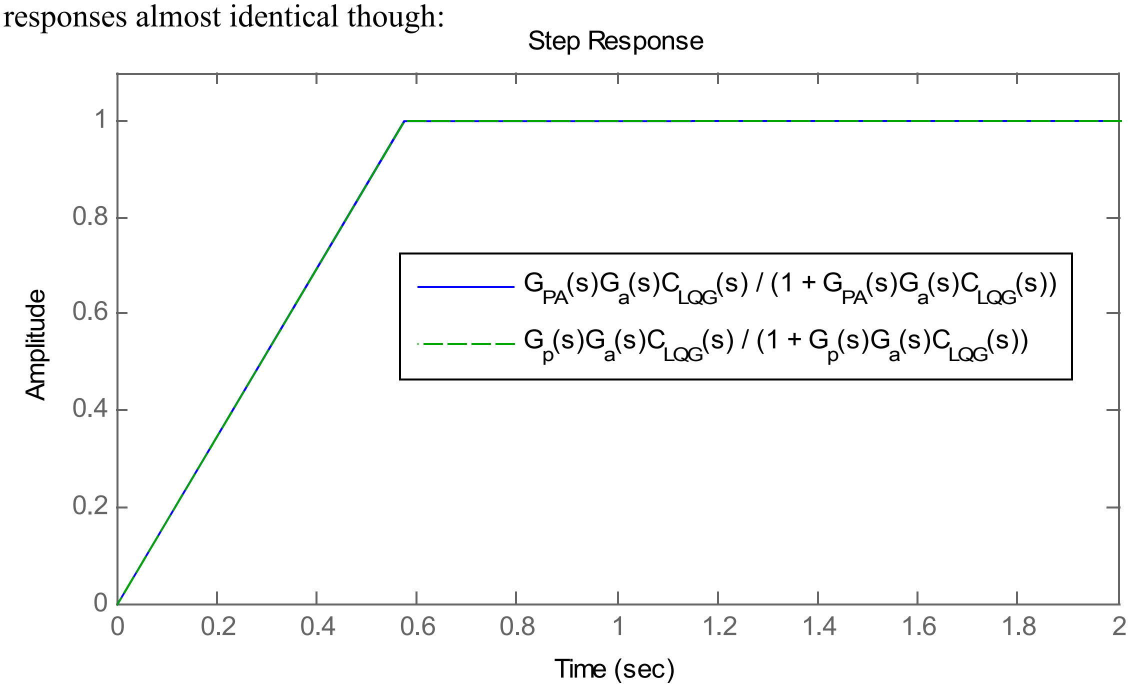
1.f)i.

For $G_p(s)G_a(s)C_{LQG}(s)$, gain margin=46.4 dB at $7.08\cdot10^3$ rad/s, phase margin=61.8 deg at 39 rad/s
For $G_{PA}(s)G_a(s)C_{LQG}(s)$, gain margin=4.87 dB at 999 rad/s, phase margin=61.8 deg at 39 rad/s
The unmodeled dynamics don't change anything at the low frequencies near the gain crossover, but they do introduce 2 additional high-frequency phase crossovers, with low gain margin (just less than 5 dB - this looks related to the earlier choice of μ, and my 0.0064 may have been a bit aggressive)



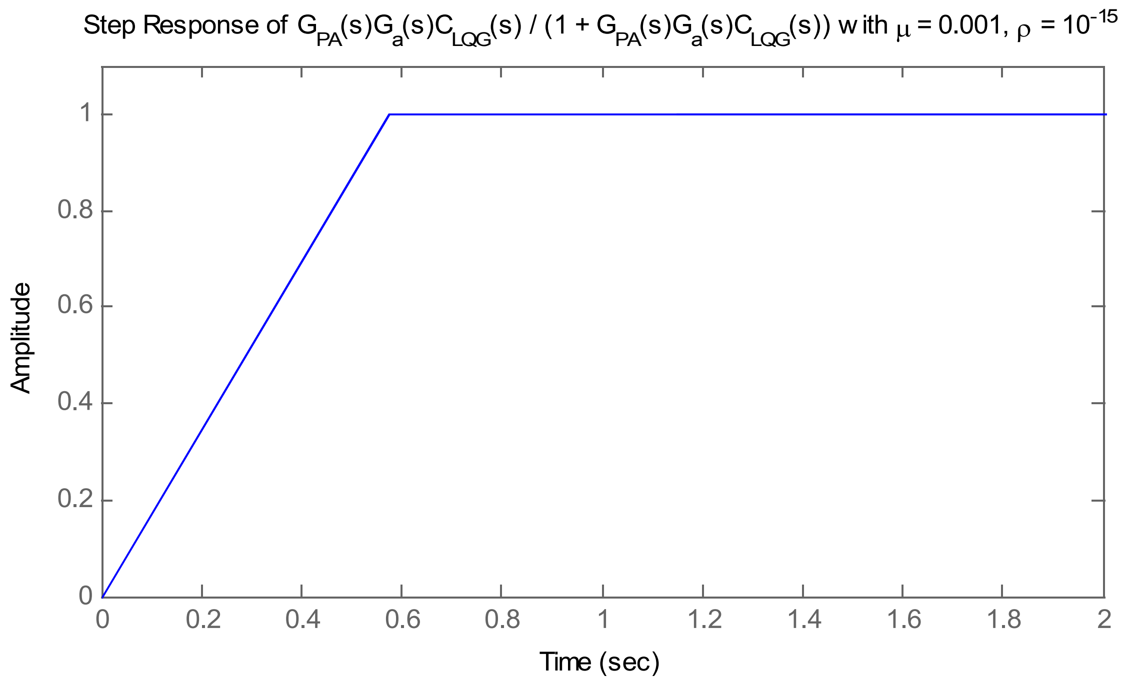
1.f)ii.

Step responses almost identical though:



1.g)i.

With $\mu=0.001$, $L_e=[1933.2 \quad 991.2 \quad 25.2]^T$. Interestingly, when we use $\rho=10^{-20}$ and the values for K_e from part d, $\frac{G_{PA}(s)G_a(s)C_{LQG}(s)}{1+G_{PA}(s)G_a(s)C_{LQG}(s)}$ is unstable. But it is stable if we take a less aggressive value of $\rho=10^{-15}$, for which $K_e=[730.3646 \quad 1.3465\cdot 10^5 \quad 9.882\cdot 10^7]$, so this case is borderline.



1.g)ii.

With $\mu=10^{-4}$, $L_e=[2.949\cdot 10^4 \quad 5527.1 \quad 59.476]^T$. This is now unstable with $\rho=10^{-15}$, but stable if $\rho=10^{-14}$, for which $K_e=[496.465 \quad 6.25\cdot 10^4 \quad 3.125\cdot 10^7]$. So we're seeing a trend where if we design the Kalman filter for less measurement noise (or less model uncertainty) we have to penalize control action more in the LQR cost function, so control is more expensive and gains are lower, if we want the closed-loop LQG system to be stable despite unmodeled dynamics.

2.

$$\dot{x}_e=A_e x_e+B_e u$$

$$J=\int_0^\infty \{x_e^T C_e^T C_e x_e+2x_e^T N_e u+u^T R_e u\}\,dt$$

$$A_e=\begin{bmatrix} A & 0 & 0 \\ B_1 & A_1 & 0 \\ 0 & 0 & A_2 \end{bmatrix},\ B_e=\begin{bmatrix} B \\ 0 \\ B_2 \end{bmatrix}$$

$$C_e=\begin{bmatrix} D_r C & C_r & 0 & 0 \\ D_1 & 0 & C_1 & 0 \\ 0 & 0 & 0 & \sqrt{\rho}C_2 \end{bmatrix}=\begin{bmatrix} & C_q & & \\ 0 & 0 & 0 & \sqrt{\rho}C_2 \end{bmatrix}$$

$$N_e^T=[0 \quad 0 \quad 0 \quad \rho D_2^T C_2],\ R_e=\rho D_2^T D_2>0$$

$$u=-Lx_e+v$$

$$J=\int_0^\infty \{x_e^T C_e^T C_e x_e+2x_e^T N_e(-Lx_e+v)+(-Lx_e+v)^T R_e(-Lx_e+v)\}\,dt$$

$$J=\int_0^\infty \{x_e^T(C_e^T C_e-2N_e L+L^T R_e L)x_e+2x_e^T(N_e-L^T R_e)v+v^T R_e v\}\,dt$$

We want the middle term $(N_e-L^T R_e)=0$, so $L=R_e^{-1}N_e^T$

$$J=\int_0^\infty \{x_e^T(C_e^T C_e-2N_e R_e^{-1}N_e^T+N_e R_e^{-1}R_e R_e^{-1}N_e^T)x_e+v^T R_e v\}\,dt$$

$$J=\int_0^\infty \{x_e^T(C_e^T C_e-N_e R_e^{-1}N_e^T)x_e+v^T R_e v\}\,dt$$

$$J=\int_0^\infty \{x_e^T \bar{Q}_e x_e+\rho v^T D_2^T D_2 v\}\,dt$$

$$\dot{x}_e=A_e x_e+B_e(-Lx_e+v)=(A_e-B_e L)x_e+B_e v=(A_e-B_e R_e^{-1}N_e^T)x_e+B_e v=\bar{A}_e x_e+B_e v$$

$$\bar{Q}_e = C_e^T C_e - N_e R_e^{-1} N_e^T$$

$$\bar{Q}_e = \begin{bmatrix} C^T D_r^T & D_1^T & 0 \\ C_r^T & 0 & 0 \\ 0 & C_1^T & 0 \\ 0 & 0 & \sqrt{\rho} C_2^T \end{bmatrix} \begin{bmatrix} D_r C & C_r & 0 & 0 \\ D_1 & 0 & C_1 & 0 \\ 0 & 0 & 0 & \sqrt{\rho} C_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ \rho C_2^T D_2 \end{bmatrix} (\rho D_2^T D_2)^{-1} [0 \quad 0 \quad 0 \quad \rho D_2^T C_2]$$

$$\bar{Q}_e = \begin{bmatrix} C^T D_r^T D_r C + D_1^T D_1 & C^T D_r^T C_r & D_1^T C_1 & 0 \\ C_r^T D_r C & C_r^T C_r & 0 & 0 \\ C_1^T D_1 & 0 & C_1^T C_1 & 0 \\ 0 & 0 & 0 & \rho C_2^T C_2 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho C_2^T D_2 (D_2^T D_2)^{-1} D_2^T C_2 \end{bmatrix}$$

$$\bar{Q}_e = \begin{bmatrix} C^T D_r^T D_r C + D_1^T D_1 & C^T D_r^T C_r & D_1^T C_1 & 0 \\ C_r^T D_r C & C_r^T C_r & 0 & 0 \\ C_1^T D_1 & 0 & C_1^T C_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho C_2^T (I - D_2 (D_2^T D_2)^{-1} D_2^T) C_2 \end{bmatrix}$$

Let $z = (I - D_2 (D_2^T D_2)^{-1} D_2^T)$, then we have $D_2^T z = D_2^T - D_2^T D_2 (D_2^T D_2)^{-1} D_2^T = D_2^T - D_2^T = 0$
assuming $\operatorname{rank}(D_2^T) \geq 1$, this can only be true if $z = (I - D_2 (D_2^T D_2)^{-1} D_2^T) = 0$

so $\bar{Q}_e = C_e^T C_e - N_e R_e^{-1} N_e^T = \begin{bmatrix} C^T D_r^T D_r C + D_1^T D_1 & C^T D_r^T C_r & D_1^T C_1 & 0 \\ C_r^T D_r C & C_r^T C_r & 0 & 0 \\ C_1^T D_1 & 0 & C_1^T C_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = C_q^T C_q$

Continuous infinite-horizon LQR solution: $v = -R_e^{-1} B_e^T P_e x_e$, $\bar{A}_e^{\,T} P_e + P_e \bar{A}_e - P_e B_e R_e^{-1} B_e^T P_e + \bar{Q}_e = 0$

$u = -L x_e + v = -R_e^{-1} N_e^T x_e - R_e^{-1} B_e^T P_e x_e = -R_e^{-1} (B_e^T P_e + N_e^T) x_e = -K_e x_e$

$(A_e - B_e R_e^{-1} N_e^T)^T P_e + P_e (A_e - B_e R_e^{-1} N_e^T) - P_e B_e R_e^{-1} B_e^T P_e + \bar{Q}_e = 0$

$A_e^T P_e - N_e R_e^{-1} B_e^T P_e + P_e A_e - P_e B_e R_e^{-1} N_e^T - P_e B_e R_e^{-1} B_e^T P_e + C_e^T C_e - N_e R_e^{-1} N_e^T = 0$

$A_e^T P_e - N_e R_e^{-1} (B_e^T P_e + N_e^T) + P_e A_e - P_e B_e R_e^{-1} (B_e^T P_e + N_e^T) + C_e^T C_e = 0$

$A_e^T P_e + P_e A_e - (B_e^T P_e + N_e^T)^T R_e^{-1} (B_e^T P_e + N_e^T) + C_e^T C_e = 0$

$[A_e, B_e]$ stabilizable implies $[A_e - B_e R_e^{-1} N_e, B_e]$ stabilizable and vice-versa because the system in terms of A_e and u is related to the system in terms of $\bar{A}_e$ and v by state feedback, and state feedback does not alter stabilizability. Controllable modes that might be moved to the right half plane by the feedback can just as easily be moved back to the left half plane, and uncontrollable modes can't be moved by the state feedback in the first place so they must be stable if either system is stabilizable.
So $[A_e, B_e]$ stabilizable and $[A_e - B_e R_e^{-1} N_e, C_q]$ detectable means we have satisfactory conditions for an exponentially stable LQR solution for the $\bar{A}_e$, v, $\bar{Q}_e$ system and hence also the above modified conditions for the A_e, u, $C_e^T C_e$ system.