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ME 233 – Homework 6

1.1.

$x(k+1)=A\,x(k)+B\,u(k)+B_w w(k),\ E\{x(0)\}=x_o,\ E\{(x(0)-x_o)(x(0)-x_o)^T\}=X_o$

$y(k)=C\,x(k)+v(k),\ E\{w(k)w(k+l)^T\}=W(k)\delta(l),\ E\{v(k)v(k+l)^T\}=V(k)\delta(l)$

$\hat{x}(k)=\hat{x}^o(k)+F(k)\tilde{y}^o(k),\ F(k)=M(k)C^T[C\,M(k)C^T+V(k)]^{-1}$

$\tilde{y}^o(k)=y(k)-C\,\hat{x}^o(k),\ \Lambda_{\tilde{y}^o\tilde{y}^o}(k,0)=E\{\tilde{y}^o(k)(\tilde{y}^o(k))^T\}=C\,M(k)C^T+V(k)$

$\tilde{y}(k)=y(k)-C\,\hat{x}(k)=y(k)-C\,\hat{x}^o(k)-C\,F(k)\tilde{y}^o(k)=\tilde{y}^o(k)-C\,F(k)\tilde{y}^o(k)=[I-C\,F(k)]\tilde{y}^o(k)$

$\Lambda_{\tilde{y}\tilde{y}}(k,0)=E\{\tilde{y}(k)(\tilde{y}(k))^T\}=E\{[I-C\,F(k)]\tilde{y}^o(k)(\tilde{y}^o(k))^T[I-C\,F(k)]^T\}$

$\Lambda_{\tilde{y}\tilde{y}}(k,0)=[I-C\,F(k)]E\{\tilde{y}^o(k)(\tilde{y}^o(k))^T\}[I-C\,F(k)]^T=[I-C\,F(k)]\Lambda_{\tilde{y}^o\tilde{y}^o}(k,0)[I-C\,F(k)]^T$

$\Lambda_{\tilde{y}\tilde{y}}(k,0)=[I-C\,F(k)][C\,M(k)C^T+V(k)][I-C\,F(k)]^T$

$\Lambda_{\tilde{y}\tilde{y}}(k,0)=[I-C\,F(k)][C\,M(k)C^T+V(k)][C\,M(k)C^T+V(k)]^{-1}[C\,M(k)C^T+V(k)][I-C\,F(k)]^T$

$[I-C\,F(k)][C\,M(k)C^T+V(k)]=C\,M(k)C^T+V(k)-C\,F(k)[C\,M(k)C^T+V(k)]$

$C\,F(k)[C\,M(k)C^T+V(k)]=C\,M(k)C^T[C\,M(k)C^T+V(k)]^{-1}[C\,M(k)C^T+V(k)]=C\,M(k)C^T$

So $[I-C\,F(k)][C\,M(k)C^T+V(k)]=C\,M(k)C^T+V(k)-C\,M(k)C^T=V(k)$

$V(k)$ and $M(k)$ are symmetric so $[C\,M(k)C^T+V(k)]=[C\,M(k)C^T+V(k)]^T$

and $V(k)=V(k)^T=[C\,M(k)C^T+V(k)]^T[I-C\,F(k)]^T=[C\,M(k)C^T+V(k)][I-C\,F(k)]^T$

$\Lambda_{\tilde{y}\tilde{y}}(k,0)=[I-C\,F(k)][C\,M(k)C^T+V(k)][C\,M(k)C^T+V(k)]^{-1}[C\,M(k)C^T+V(k)][I-C\,F(k)]^T$

$\Lambda_{\tilde{y}\tilde{y}}(k,0)=V(k)[C\,M(k)C^T+V(k)]^{-1}V(k)$

1.2.

$\hat{x}^o(k+1)=A\,\hat{x}(k)+B\,u(k),\ \hat{x}^o(0)=x_0,\ M(0)=X_o$

$\hat{x}^o(k+1)=A[\hat{x}^o(k)+F(k)\tilde{y}^o(k)]+B\,u(k)=A\,\hat{x}^o(k)+B\,u(k)+A\,F(k)\tilde{y}^o(k)$

So $L(k)=A\,F(k)=A\,M(k)C^T[C\,M(k)C^T+V(k)]^{-1}$

1.3.

$Z(k)=M(k)-M(k)C^T[C\,M(k)C^T+V(k)]^{-1}C\,M(k),\ M(k+1)=A\,Z(k)A^T+B_w W(k)B_w^T$

$M(k+1)=A[M(k)-M(k)C^T[C\,M(k)C^T+V(k)]^{-1}C\,M(k)]A^T+B_w W(k)B_w^T$

$M(k+1)=[A\,M(k)-A\,M(k)C^T[C\,M(k)C^T+V(k)]^{-1}C\,M(k)]A^T+B_w W(k)B_w^T$

$M(k+1)=A\,M(k)A^T+B_w W(k)B_w^T-A\,M(k)C^T[C\,M(k)C^T+V(k)]^{-1}C\,M(k)A^T$

2.a)

$\begin{bmatrix}x_1(k+1)\\x_2(k+1)\end{bmatrix}=\begin{bmatrix}-0.08 & -1\\0.7 & 0.1\end{bmatrix}\begin{bmatrix}x_1(k)\\x_2(k)\end{bmatrix}+\begin{bmatrix}0.34\\0.3\end{bmatrix}(u(k)+w(k))$, and $y(k)=[0\quad 3]x(k)+v(k)$

$x(0)\sim\mathcal{N}\left(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0.1 & 0\\0 & 0.1\end{bmatrix}\right),\ \begin{bmatrix}w(k)\\v(k)\end{bmatrix}\sim\mathcal{N}\left(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1 & 0\\0 & V\end{bmatrix}\right),\ E\{x(0)[w(k)\quad v(k)]\}=0$

By $k=10$, the Riccati equation for $M(k)$ has more or less converged to steady-state.

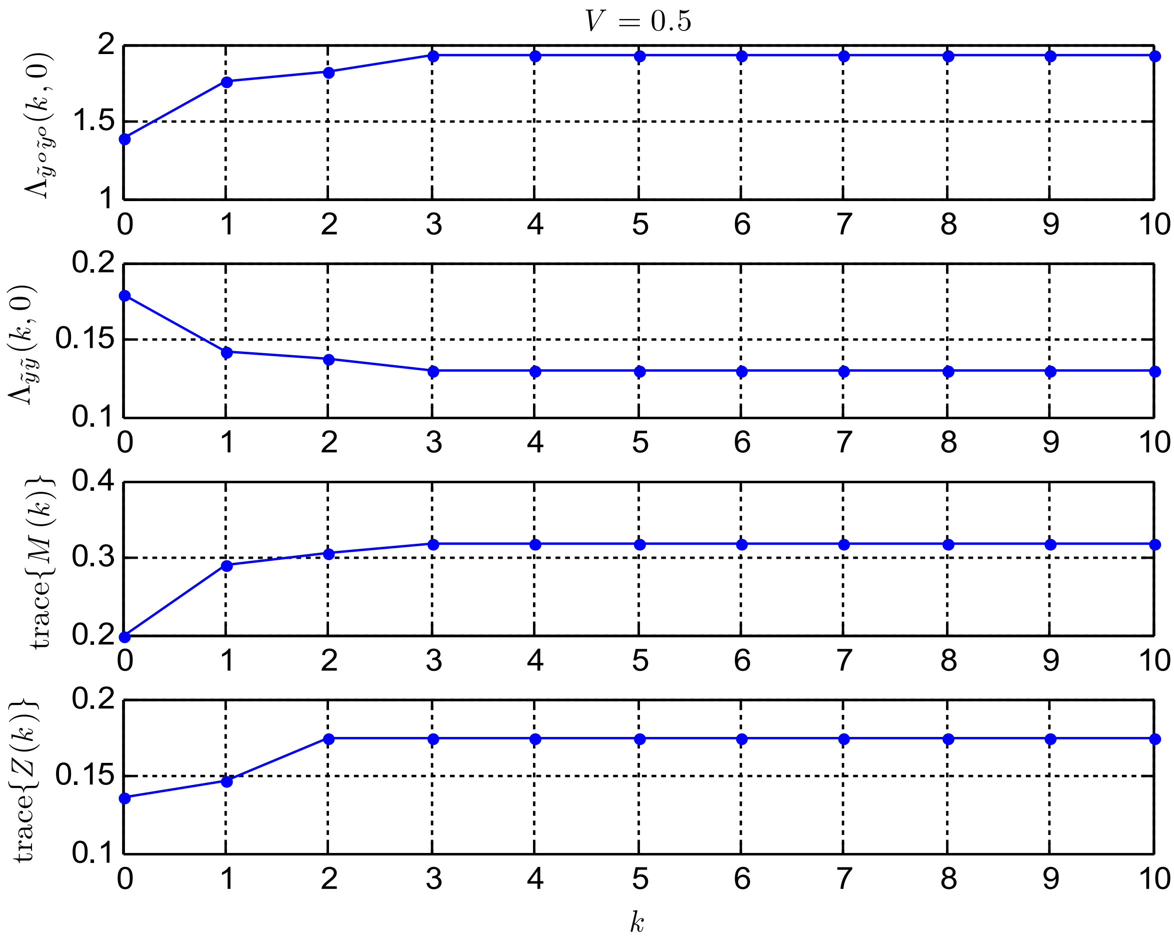
for $V=0.5$, $\text{trace}\{M(10)\}=0.3194$, $\text{trace}\{Z(10)\}=0.1747$, $\Lambda_{\tilde{y}^o\tilde{y}^o}(10,0)=1.9275$, $\Lambda_{\tilde{y}\tilde{y}}(10,0)=0.1297$

2.b) see next page for plot

2.c)

$\bar{M}=\text{dare}(A^T,C^T,B\,W\,B^T,V)=\begin{bmatrix}0.1608 & 0.0764\\0.0764 & 0.1586\end{bmatrix}$

$\bar{Z}=\bar{M}-\bar{M}\,C^T[C\,\bar{M}\,C^T+V]^{-1}C\,\bar{M}=\begin{bmatrix}0.1335 & 0.0198\\0.0198 & 0.0411\end{bmatrix}$



$$\bar{\Lambda}_{\tilde{y}^o\tilde{y}^o} = C\bar{M}C^T + V = 1.9275,\ \bar{\Lambda}_{\tilde{y}\tilde{y}} = V[C\bar{M}C^T + V]^{-1}V = 0.1297$$

$$\bar{F} = \bar{M}C^T[C\bar{M}C^T + V]^{-1} = \begin{bmatrix} 0.1189 \\ 0.2469 \end{bmatrix},\ \bar{L} = A\bar{F} = \begin{bmatrix} -0.2564 \\ 0.1079 \end{bmatrix},\ \text{eig}(A - \bar{L}C) = -0.1519 \pm 0.3955\,j$$

2.d)

$x_0 = \sqrt{0.1}\,\text{randn}(2,1),\ w = \sqrt{W}\,\text{randn}(N,1),\ v = \sqrt{V}\,\text{randn}(N,1)$ for some large N

2.e)

$$\bar{M}_{\text{sim}} = \text{cov}\left(\begin{bmatrix} \tilde{x}_1^o(k) & \tilde{x}_2^o(k) \\ \vdots & \vdots \\ \tilde{x}_1^o(N) & \tilde{x}_2^o(N) \end{bmatrix}\right) = \begin{bmatrix} 0.1542 & 0.0734 \\ 0.0734 & 0.1565 \end{bmatrix},\ \bar{Z}_{\text{sim}} = \text{cov}\left(\begin{bmatrix} \tilde{x}_1(k) & \tilde{x}_2(k) \\ \vdots & \vdots \\ \tilde{x}_1(N) & \tilde{x}_2(N) \end{bmatrix}\right) = \begin{bmatrix} 0.1293 & 0.0199 \\ 0.0199 & 0.0416 \end{bmatrix}$$

where $\tilde{x}^o(k) = x(k) - \hat{x}^o(k)$, $\tilde{x}(k) = x(k) - \hat{x}(k)$, and $k < N$ is large enough to reach steady state but enough smaller than N to give a large sample size. For the above I used $k = 1000$, $N = 10000$

$\bar{\Lambda}_{\tilde{y}^o\tilde{y}^o,\text{sim}} = \text{cov}(\tilde{y}^o(k:N)) = 1.9144$, $\bar{\Lambda}_{\tilde{y}\tilde{y},\text{sim}} = \text{cov}(\tilde{y}(k:N)) = 0.1288$

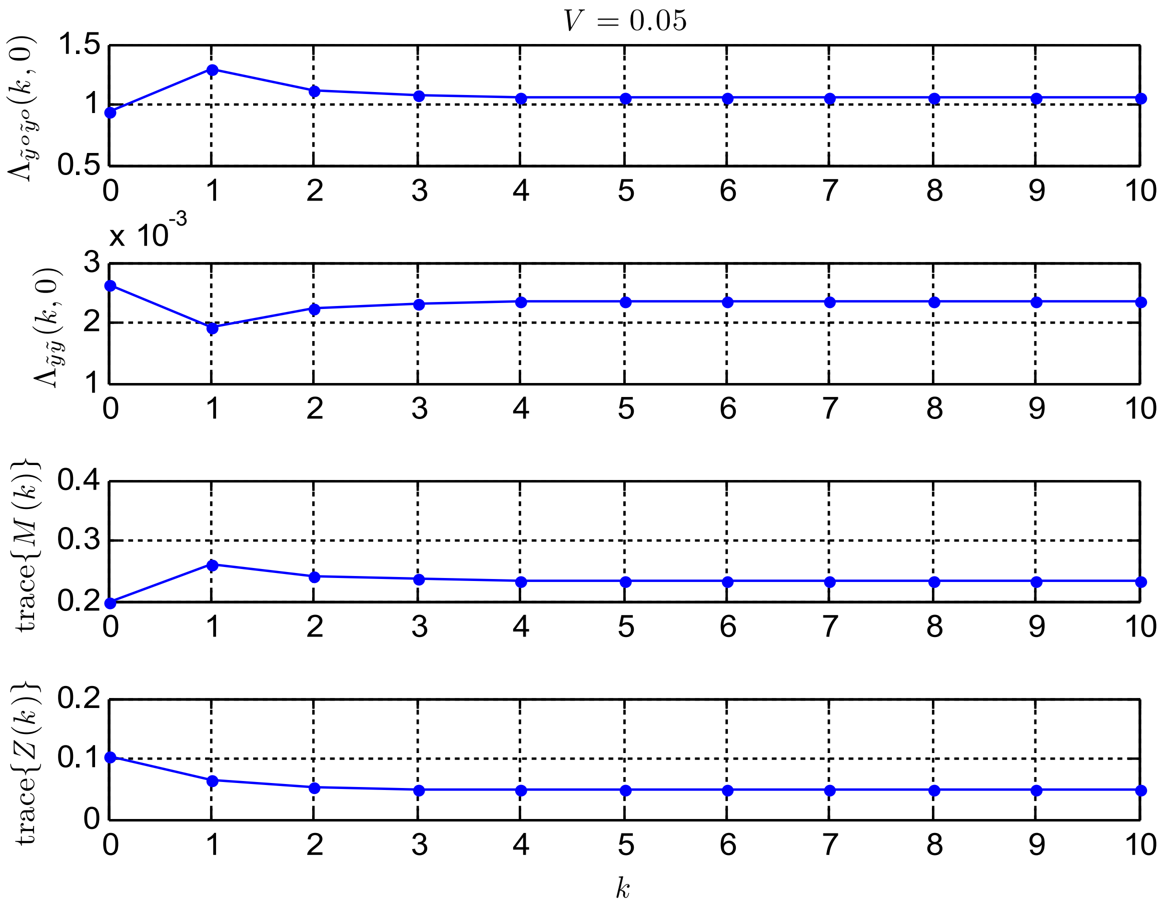
All simulation approximate values are close to the previously calculated actual values.

2.f.i)

$V = 0.05$: trace$\{M(10)\} = 0.2341$, trace$\{Z(10)\} = 0.0493$, $\Lambda_{\tilde{y}^o\tilde{y}^o}(10,0) = 1.0601$, $\Lambda_{\tilde{y}\tilde{y}}(10,0) = 0.0024$

$$\bar{M} = \begin{bmatrix} 0.1219 & 0.0958 \\ 0.0958 & 0.1122 \end{bmatrix},\ \bar{Z} = \begin{bmatrix} 0.044 & 0.0045 \\ 0.0045 & 0.0053 \end{bmatrix},\ \bar{F} = \begin{bmatrix} 0.2711 \\ 0.3176 \end{bmatrix},\ \bar{L} = \begin{bmatrix} -0.3393 \\ 0.2216 \end{bmatrix}$$

$\bar{\Lambda}_{\tilde{y}^o\tilde{y}^o} = 1.06$, $\bar{\Lambda}_{\tilde{y}\tilde{y}} = 0.0024$, $\text{eig}(A - \bar{L}C) = -0.0554$ and -0.5893



$\bar{M}_{\text{sim}} = \begin{bmatrix} 0.1228 & 0.0956 \\ 0.0956 & 0.1117 \end{bmatrix}$, $\bar{Z}_{\text{sim}} = \begin{bmatrix} 0.0449 & 0.0045 \\ 0.0045 & 0.0053 \end{bmatrix}$, $\bar{\Lambda}_{\tilde{y}^o\tilde{y}^o,\text{sim}} = 1.0623$, $\bar{\Lambda}_{\tilde{y}\tilde{y},\text{sim}} = 0.0024$

Again, all simulation results are consistent with the calculations. With less measurement noise the a-priori output and state and output estimation covariances are improved a bit. The a-posteriori output estimate covariance is 2 orders of magnitude smaller (for one order of magnitude smaller measurement noise variance), and the expected norm of the a-posteriori state estimate is more than 3 times smaller. The closed-loop Kalman filter does not have complex eigenvalues with the smaller variance measurement noise, but it does have a slower-converging component.

2.f.ii)

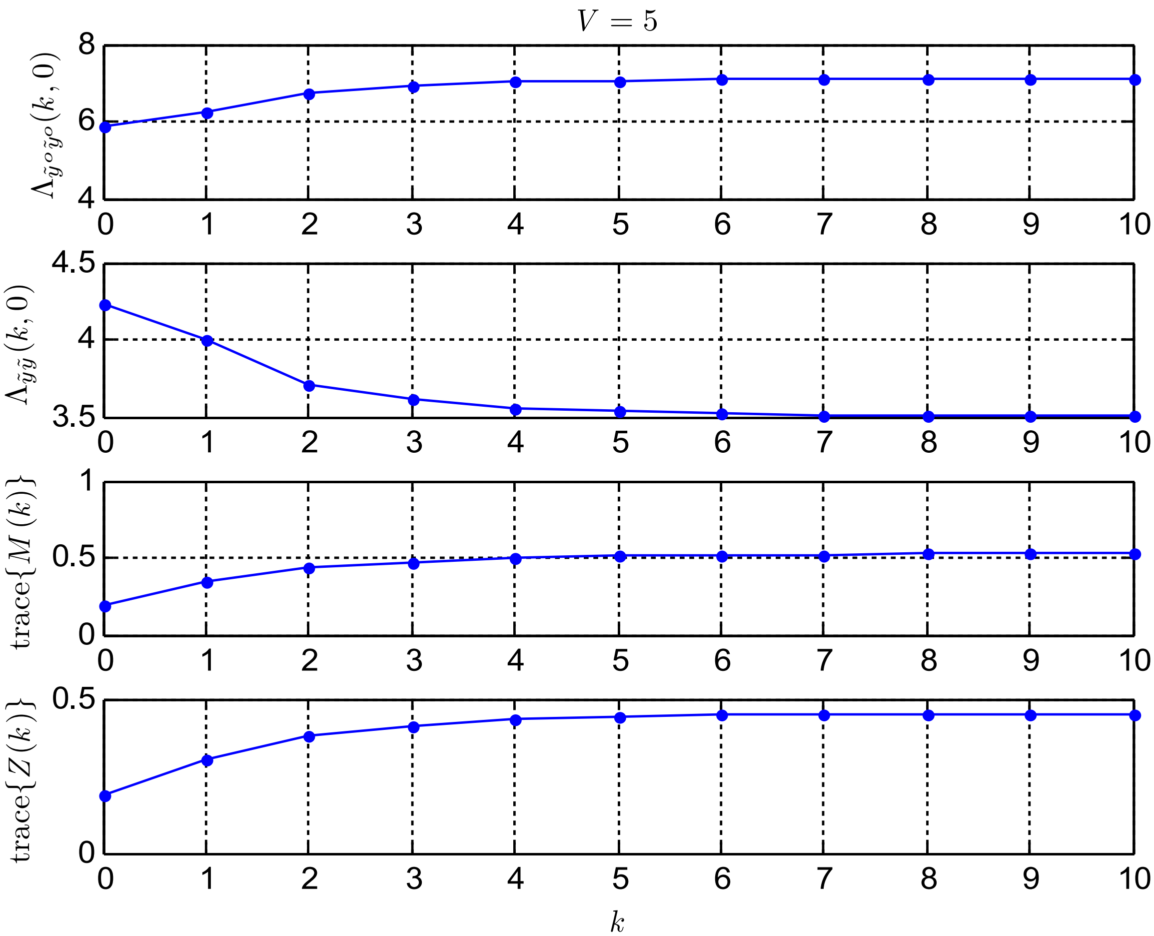
For $V=5$, trace$\{M(10)\}=0.5242$, trace$\{Z(10)\}=0.4512$, $\Lambda_{\tilde{y}^o\tilde{y}^o}(10,0)=7.124$, $\Lambda_{\tilde{y}\tilde{y}}(10,0)=3.5093$

$\bar{M} = \begin{bmatrix} 0.2884 & 0.0464 \\ 0.0464 & 0.2362 \end{bmatrix}$, $\bar{Z} = \begin{bmatrix} 0.2856 & 0.0325 \\ 0.0325 & 0.1657 \end{bmatrix}$, $\bar{F} = \begin{bmatrix} 0.0195 \\ 0.0994 \end{bmatrix}$, $\bar{L} = \begin{bmatrix} -0.101 \\ 0.0236 \end{bmatrix}$

$\bar{\Lambda}_{\tilde{y}^o\tilde{y}^o} = 7.1256$, $\bar{\Lambda}_{\tilde{y}\tilde{y}} = 3.5085$, $\text{eig}(A - \bar{L}C) = -0.0254 \pm 0.6964\,j$

$\bar{M}_{\text{sim}} = \begin{bmatrix} 0.2872 & 0.0439 \\ 0.0439 & 0.2321 \end{bmatrix}$, $\bar{Z}_{\text{sim}} = \begin{bmatrix} 0.2854 & 0.0328 \\ 0.0328 & 0.1648 \end{bmatrix}$, $\bar{\Lambda}_{\tilde{y}^o\tilde{y}^o,\text{sim}} = 7.0221$, $\bar{\Lambda}_{\tilde{y}\tilde{y},\text{sim}} = 3.4575$

With more measurement noise the Kalman filter converges more slowly - here there are slight differences between the estimates after 10 steps and the eventual steady-state estimates. As expected the estimate error variances are all higher, with the most pronounced difference in the a-posteriori output estimate covariance.



3.

$x(k+1)=A\,x(k)+B\,u(k)+w(k),\;\; y(k)=C\,x(k)+v(k)$

$E\{x(0)\}=x_o,\;\; E\{(x(0)-x_o)(x(0)-x_o)^T\}=X_o,\;\; E\left\{\begin{bmatrix} w(k) \\ v(k)\end{bmatrix}\begin{bmatrix} w(j)^T & v(j)^T\end{bmatrix}\right\}=\begin{bmatrix} W & S \\ S^T & V\end{bmatrix}\delta(k-j)$

$x(k+1)=A\,x(k)+B\,u(k)+w(k)-T\,y(k)+T\,y(k)=(A-T\,C)\,x(k)+B\,u(k)+w(k)-T\,v(k)+T\,y(k)$

$E\{(w(k)-T\,v(k))\,v(j)^T\}=E\{w(k)\,v(j)^T\}-T\,E\{v(k)\,v(j)^T\}=(S-T\,V)\,\delta(k-j)$

So for $T=S\,V^{-1}$ (invertible by positive definiteness), $E\{(w(k)-T\,v(k))\,v(j)^T\}=0$

$E\{(w(k)-T\,v(k))(w(k)-T\,v(k))^T\}=W-S\,T^T-T\,S^T+T\,V\,T^T$

$E\{(w(k)-T\,v(k))(w(k)-T\,v(k))^T\}=W-S\,(V^{-1})^T\,S^T-S\,V^{-1}\,S^T+S\,V^{-1}\,V\,(V^{-1})^T\,S^T=W-S\,V^{-1}\,S^T$

So for $w'(k)=w(k)-S\,V^{-1}\,v(k),\;\; E\left\{\begin{bmatrix} w'(k) \\ v(k)\end{bmatrix}\begin{bmatrix} w'(j)^T & v(j)^T\end{bmatrix}\right\}=\begin{bmatrix} W-S\,V^{-1}\,S^T & 0 \\ 0 & V\end{bmatrix}\delta(k-j)$

$x(k+1)=(A-S\,V^{-1}\,C)\,x(k)+B\,u(k)+w'(k)+S\,V^{-1}\,y(k)=A'\,x(k)+B'\,u'(k)+w'(k)$

Where $A'=A-S\,V^{-1}\,C,\;\; B'=[B\quad S\,V^{-1}],\;\; u'(k)=\begin{bmatrix} u(k) \\ y(k)\end{bmatrix}$

This is now of the same form as in problem 1, with $B_w=I,\; V(k)=V,\; W'(k)=W'=W-S\,V^{-1}\,S^T$

$\hat{x}^o(k+1)=A'\,\hat{x}^o(k)+B'\,u'(k)+A'\,F(k)\,\tilde{y}^o(k)=A'\,\hat{x}^o(k)+B'\,u'(k)+A'\,F(k)[\,y(k)-C\,\hat{x}^o(k)]$

$\hat{x}^o(k+1)=(A-S\,V^{-1}\,C)\,\hat{x}^o(k)+B\,u(k)+S\,V^{-1}\,y(k)+(A-S\,V^{-1}\,C)\,F(k)[\,y(k)-C\,\hat{x}^o(k)]$

$\hat{x}^o(k+1)=A\,\hat{x}^o(k)+B\,u(k)+[S\,V^{-1}+(A-S\,V^{-1}\,C)\,F(k)][\,y(k)-C\,\hat{x}^o(k)]$

$$\hat{x}^o(k+1) = A\hat{x}^o(k) + Bu(k) + [SV^{-1}(I - CF(k)) + AF(k)][y(k) - C\hat{x}^o(k)]$$
$$F(k) = M(k)C^T[CM(k)C^T + V]^{-1}$$
$$I - CF(k) = [CM(k)C^T + V][CM(k)C^T + V]^{-1} - CM(k)C^T[CM(k)C^T + V]^{-1}$$
$$I - CF(k) = V[CM(k)C^T + V]^{-1}$$
$$L(k) = SV^{-1}(I - CF(k)) + AF(k) = SV^{-1}V[CM(k)C^T + V]^{-1} + AM(k)C^T[CM(k)C^T + V]^{-1}$$
$$L(k) = [AM(k)C^T + S][CM(k)C^T + V]^{-1}$$
$$\hat{x}^o(k+1) = A\hat{x}^o(k) + Bu(k) + L(k)[y(k) - C\hat{x}^o(k)]$$
Riccati $M(k+1) = A'M(k)(A')^T + W' - A'M(k)C^T[CM(k)C^T + V]^{-1}CM(k)(A')^T$
$$M(k+1) = A'M(k)(A')^T + W' - (A')F(k)CM(k)(A')^T = A'[I - F(k)C]M(k)(A')^T + W'$$
$$M(k+1) = (A - SV^{-1}C)[I - F(k)C]M(k)(A - SV^{-1}C)^T + W - SV^{-1}S^T$$
$$M(k+1) = [A - AF(k)C - SV^{-1}C + SV^{-1}CF(k)C]M(k)(A - SV^{-1}C)^T + W - SV^{-1}S^T$$
$$M(k+1) = [A - (AF(k) + SV^{-1}(I - CF(k)))C]M(k)(A - SV^{-1}C)^T + W - SV^{-1}S^T$$
$$M(k+1) = [A - L(k)C]M(k)(A - SV^{-1}C)^T + W - SV^{-1}S^T$$
V is symmetric so $(A - SV^{-1}C)^T = A^T - C^T(V^{-1})^T S^T = A^T - C^T V^{-1} S^T$
$$M(k+1) = AM(k)A^T - AM(k)C^T V^{-1} S^T - L(k)CM(k)[A^T - C^T V^{-1} S^T] + W - SV^{-1}S^T$$
$$AM(k)C^T V^{-1} S^T = AM(k)C^T[CM(k)C^T + V]^{-1}[CM(k)C^T + V]V^{-1}S^T$$
$$AM(k)C^T V^{-1} S^T = (L(k) - S[CM(k)C^T + V]^{-1})[CM(k)C^T + V]V^{-1}S^T$$
$$AM(k)C^T V^{-1} S^T = L(k)[CM(k)C^T + V]V^{-1}S^T - SV^{-1}S^T$$
$$M(k+1) = AM(k)A^T - L(k)[CM(k)C^T + V]V^{-1}S^T - L(k)CM(k)[A^T - C^T V^{-1} S^T] + W$$
$$M(k+1) = AM(k)A^T - L(k)[CM(k)C^T V^{-1} S^T + S^T + CM(k)A^T - CM(k)C^T V^{-1} S^T] + W$$
$$M(k+1) = AM(k)A^T - L(k)[S^T + CM(k)A^T] + W$$
$M(k)$ is symmetric so $S^T + CM(k)A^T = (AM(k)C^T + S)^T = (L(k)[CM(k)C^T + V])^T$
$$M(k+1) = AM(k)A^T - L(k)[CM(k)C^T + V]^T L(k)^T + W$$
By symmetry of $M(k)$ and V, we have $M(k+1) = AM(k)A^T - L(k)[CM(k)C^T + V]L(k)^T + W$

4.a)

$y(k) = x + v(k)$, $E\{x\} = 0$, $E\{x^2\} = X_0$, $E\{v(k)\} = 0$, $E\{v(k)v(k+j)\} = V\delta(j)$, $E\{xv(k)\} = 0$

Let $Y = [y(0) \ \cdots \ y(k)]^T = [x + v(0) \ \cdots \ x + v(k)]^T$

least squares estimate $\hat{x}(k) = E\{x | y(0) \cdots y(k)\} = E\{x|Y\} = E\{x\} + \Lambda_{xY}\Lambda_{YY}^{-1}(Y - E\{Y\})$

$E\{y(k)\} = E\{x\} + E\{v(k)\} = 0$ so $\hat{x}(k) = \Lambda_{xY}\Lambda_{YY}^{-1} Y$

$$\Lambda_{xY} = E\{xY^T\} = E\{x[x + v(0) \ \cdots \ x + v(k)]\} = E\{[x^2 + xv(0) \ \cdots \ x^2 + xv(k)]\} = X_0[1 \ \cdots \ 1]$$
$$\Lambda_{YY} = E\{YY^T\} = E\{[x + v(0) \ \cdots \ x + v(k)]^T[x + v(0) \ \cdots \ x + v(k)]\}$$
$$\Lambda_{YY} = E\{[x \ \cdots \ x]^T[x \ \cdots \ x]\} + E\{[x \ \cdots \ x]^T[v(0) \ \cdots \ v(k)]\}$$
$$\quad + E\{[v(0) \ \cdots \ v(k)]^T[x \ \cdots \ x]\} + E\{[v(0) \ \cdots \ v(k)]^T[v(0) \ \cdots \ v(k)]\}$$
$$\Lambda_{YY} = X_0[1 \ \cdots \ 1]^T[1 \ \cdots \ 1] + 0 + 0 + VI$$
By the matrix inversion lemma (Sherman-Morrison-Woodbury formula),
$$\Lambda_{YY}^{-1} = V^{-1}I - V^{-1}I[1 \ \cdots \ 1]^T(X_0^{-1} + [1 \ \cdots \ 1]V^{-1}[1 \ \cdots \ 1]^T)^{-1}[1 \ \cdots \ 1]V^{-1}I$$
$$\Lambda_{YY}^{-1} = V^{-1}I - V^{-2}[1 \ \cdots \ 1]^T(X_0^{-1} + (k+1)V^{-1})^{-1}[1 \ \cdots \ 1]$$
$$\Lambda_{YY}^{-1} = V^{-1}I - \frac{V^{-2}}{X_0^{-1} + (k+1)V^{-1}}[1 \ \cdots \ 1]^T[1 \ \cdots \ 1]$$
$$\Lambda_{xY}\Lambda_{YY}^{-1} = V^{-1}X_0[1 \ \cdots \ 1] - \frac{V^{-2}X_0}{X_0^{-1} + (k+1)V^{-1}}[1 \ \cdots \ 1][1 \ \cdots \ 1]^T[1 \ \cdots \ 1]$$

$$\Lambda_{xY}\Lambda_{YY}^{-1}=V^{-1}X_0[1 \quad \cdots \quad 1]-\frac{(k+1)V^{-2}X_0}{X_0^{-1}+(k+1)V^{-1}}[1 \quad \cdots \quad 1]$$

$$\Lambda_{xY}\Lambda_{YY}^{-1}=\frac{V^{-1}+(k+1)V^{-2}X_0}{X_0^{-1}+(k+1)V^{-1}}[1 \quad \cdots \quad 1]-\frac{(k+1)V^{-2}X_0}{X_0^{-1}+(k+1)V^{-1}}[1 \quad \cdots \quad 1]$$

$$\Lambda_{xY}\Lambda_{YY}^{-1}=\frac{V^{-1}}{X_0^{-1}+(k+1)V^{-1}}[1 \quad \cdots \quad 1]=\frac{1}{VX_0^{-1}+k+1}[1 \quad \cdots \quad 1]$$

$$\hat{x}(k)=\Lambda_{xY}\Lambda_{YY}^{-1}Y=\frac{1}{VX_0^{-1}+k+1}[1 \quad \cdots \quad 1][y(0) \quad \cdots \quad y(k)]^T=\frac{1}{VX_0^{-1}+k+1}\sum_{i=0}^{k}y(i)$$

$$\hat{x}(k)=\frac{1}{VX_0^{-1}+k+1}\sum_{i=0}^{k}(x+v(i))=\frac{(k+1)x}{VX_0^{-1}+k+1}+\frac{1}{VX_0^{-1}+k+1}\sum_{i=0}^{k}v(i)$$

$$x-\hat{x}(k)=\frac{1}{VX_0^{-1}+k+1}\left(VX_0^{-1}x-\sum_{i=0}^{k}v(i)\right)$$

All cross terms go away when squaring and taking expectations, only squared terms remain

$$E\{(x-\hat{x}(k))^2\}=\frac{1}{(VX_0^{-1}+k+1)^2}\left(V^2X_0^{-1}+(k+1)V\right)=\frac{V}{VX_0^{-1}+k+1}$$

4.b)

$$\lim_{X_0\to\infty}\hat{x}(k)=\lim_{X_0\to\infty}\left(\frac{1}{VX_0^{-1}+k+1}\sum_{i=0}^{k}y(i)\right)=\frac{1}{k+1}\sum_{i=0}^{k}y(i)$$

$$\lim_{X_0\to\infty}E\{(x-\hat{x}(k))^2\}=\lim_{X_0\to\infty}\left(\frac{V}{VX_0^{-1}+k+1}\right)=\frac{V}{k+1}$$