[1]

When we take the measurement noise into consideration, the steady state Kalman filter is given by

$$\hat{x}(k+1|k+1) = \hat{x}(k+1|k) + F_s[y(k+1) - C\hat{x}(k+1|k)] \tag{1}$$

$$\hat{x}(k+1|k) = A\hat{x}(k|k) + Bu(k) \tag{2}$$

where $F_s = M_s C^T [CM_s C^T + V_a]^{-1}$ and M_s is the positive definite solution of the following algebraic Riccati equation

$$M_s = AM_s A^T + B_w W B_w^T - AM_s C^T [CM_s C^T + V]^{-1} CM_s A^T . \tag{3}$$

From (1) and (2), we can get

$$\hat{x}(k+1|k) = (A - AF_s C)\hat{x}(k|k-1) + AF_s y(k) + Bu(k) . \tag{4}$$

Then

$$\begin{aligned} x(k+1) - \hat{x}(k+1|k) &= Ax(k) + Bu(k) + B_w w(k) - (A - AF_s C)\hat{x}(k|k-1) - AF_s y(k) - Bu(k) \\ &= Ax(k) - (A - AF_s C)\hat{x}(k|k-1) - AF_s Cx(k) + B_w w(k) \\ &= (A - AF_s C)[x(k) - \hat{x}(k|k-1)] + B_w w(k) \end{aligned} \tag{5}$$

So

$$\begin{aligned} &E\{[x(k+1) - \hat{x}(k+1|k)][x(k+1) - \hat{x}(k+1|k)]^T\} \\ &= (A - AF_s C)E\{[x(k) - \hat{x}(k|k-1)][x(k) - \hat{x}(k|k-1)]^T\}(A - AF_s C)^T + B_w E[w(k)w^T(k)]B_w^T \end{aligned} \tag{6}$$

(Notice that we have made use of $E\{[x(k) - \hat{x}(k|k-1)]w^T(k)\} = 0$ to get eq. (6))

Define $X_{ss} = E\{[x(k+1) - \hat{x}(k+1|k)][x(k+1) - \hat{x}(k+1|k)]^T\}$. At the steady state, we have $E\{[x(k) - \hat{x}(k|k-1)][x(k) - \hat{x}(k|k-1)]^T\} = X_{ss}$. Then eq. (6) becomes

$$X_{ss} = (A - AF_s C)X_{ss}(A - AF_s C)^T + B_w W B_w^T . \tag{7}$$

The solution, X_{ss}, of this Lyapunov equation is $E\{[x(k+1) - \hat{x}(k+1|k)][x(k+1) - \hat{x}(k+1|k)]^T\}$.

Moreover,

$$\begin{aligned} x(k+1) - \hat{x}(k+1|k+1) &= x(k+1) - \hat{x}(k+1|k) - F_s[y(k+1) - C\hat{x}(k+1|k)] \\ &= (I - F_s C)[x(k+1) - \hat{x}(k+1|k)] \end{aligned} . \tag{8}$$

So

$$\begin{aligned} &E\{[x(k+1) - \hat{x}(k+1|k+1)][x(k+1) - \hat{x}(k+1|k+1)]^T\} \\ &= (I - F_s C)E\{[x(k+1) - \hat{x}(k+1|k)][x(k+1) - \hat{x}(k+1|k)]^T\}(I - F_s C)^T \\ &= (I - F_s C)X_{ss}(I - F_s C)^T \end{aligned} \tag{9}$$

[2]

The performance index can be written as

$$J = \int_{-\infty}^{\infty} \left\{ \left[\frac{1}{-j\omega} CX(-j\omega) \right]^T \left[\frac{1}{j\omega} CX(j\omega) \right] + RU(-j\omega)U(j\omega) \right\} d\omega$$
$$= 2\pi \int_0^{\infty} \left\{ x_f^T(t) x_f(t) + Ru^T(t)u(t) \right\} dt \tag{10}$$

where $x_f(t)$ is the output of the following state space model:

$$\begin{aligned} \dot{z}_1(t) &= 0 \cdot z_1(t) + Cx(t) \\ x_f(t) &= z_1(t) \end{aligned} \tag{11}$$

Combine eq. (11) and the plant equations to get the extended system:

$$\dot{x}_e(t) = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -2 & -3 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{A_e} x_e(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{B_e} u(t) \tag{12}$$

and rewrite the performance index in terms of $x_e(t) = \begin{bmatrix} x(t) & z_1(t) \end{bmatrix}^T$:

$$J = 2\pi \int_0^{\infty} \left\{ x_e^T(t) C_e^T C_e x_e(t) + Ru^T(t)u(t) \right\} dt, \tag{13}$$

where $C_e = [0 \quad 0 \quad 1]$. Then the symmetric root locus for the optimal closed loop system is determined by

$$1 + \frac{1}{R} G_e(-s) G_e(s) = 0, \tag{14}$$

where

$$G_e(s) = C_e (sI - A_e)^{-1} B_e = \frac{1}{s(s+1)(s+2)}. \tag{15}$$

The symmetric root locus is shown in Fig. 1. The branches in the left half plane are the optimal closed loop poles.

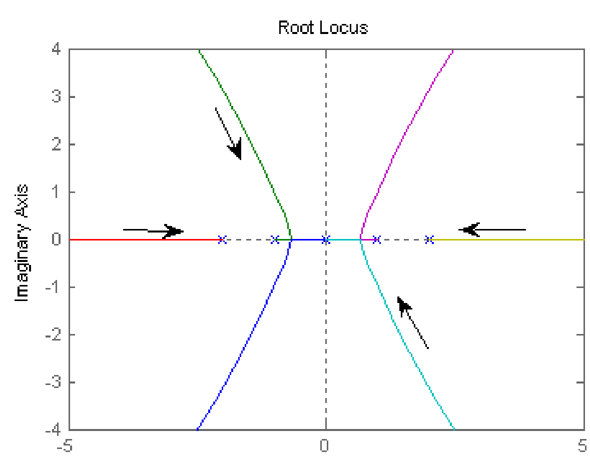

Fig. 1 Symmetric root locus. The arrows are the directions of increasing R.

[3]

According to the internal model principle, $S(z^{-1})$ must incorporate the internal model of the disturbance, i.e.,

$$S(z^{-1}) = S'(z^{-1})A_d(z^{-1}), \tag{16}$$

where $A_d(z^{-1}) = (1 - z^{-1})(1 - (\cos\omega)z^{-1} + z^{-2})$. Then the output of the closed loop system is

$$y(k) = z^{-1}d(k) - \frac{z^{-1}R(z^{-1})}{S'(z^{-1})A_d(z^{-1})}y(k), \tag{17}$$

which implies

$$y(k) = \frac{z^{-1}S'(z^{-1})A_d(z^{-1})}{S'(z^{-1})A_d(z^{-1}) + z^{-1}R(z^{-1})}d(k). \tag{18}$$

To assign all closed loop poles at 0.9, we must have the following Diophantine equation for some integer n:

$$S'(z^{-1})A_d(z^{-1}) + z^{-1}R(z^{-1}) = (1 - 0.9z^{-1})^n. \tag{19}$$

The minimal order solutions are

$$S'(z^{-1}) = 1 \text{ and } R(z^{-1}) = (-1.7 + 2\cos\omega) + (1.43 - 2\cos\omega)z^{-1} + 0.271z^{-2}.$$

Thus, the internal model controller for asymptotic regulation is given by

$$\frac{(-1.7 + 2\cos\omega) + (1.43 - 2\cos\omega)z^{-1} + 0.271z^{-2}}{(1 - z^{-1})(1 - (\cos\omega)z^{-1} + z^{-2})}. \tag{20}$$

[4]

See the solution to problem [3] of homework set #7.

[5]

a. The output of this system satisfies the following difference equation:

$$y(k+1) = ay(k) + b[u_f(k) + k_c e(k)]. \tag{21}$$

The feedforward controller gives

$$u_f(k) = w_0 y_d(k) + w_1 y_d(k+1). \tag{22}$$

Then

$$\begin{aligned} y(k+1) &= ay(k) + b[w_0 y_d(k) + w_1 y_d(k+1) + k_c e(k)] \\ &= ay(k) + bw_0 y_d(k) + bw_1 y_d(k+1) + bk_c e(k) \\ &= ay(k) - ay_d(k) + y_d(k+1) + bk_c e(k) \\ &= y_d(k+1) - (a - bk_c)e(k) \end{aligned} \tag{23}$$

So we get

$$e(k+1) = y_d(k+1) - y(k+1) = (a - bk_c)e(k). \tag{24}$$

Since $0 < a - bk_c < a < 1$, $e(k)$ approaches to zero.

b. For the error equation to be represented by the loop in Fig. 5-2, we must have

$$e(k+1) = (a - bk_c)e(k) - b\tilde{\theta}^T(k)\phi_d(k) = (a - bk_c)e(k) - b\hat{\theta}^T(k)\phi_d(k) + b\theta^T\phi_d(k), \qquad (25)$$

where $\tilde{\theta}(k) = \hat{\theta}(k) - \theta$.

From the definition, $u_f(k) = \hat{\theta}^T(k)\phi_d(k)$, and eq. (21), we get

$$\begin{aligned}
e(k+1) &= y_d(k+1) - y(k+1) \\
&= y_d(k+1) - ay(k) - b[u_f(k) + k_c e(k)] \\
&= y_d(k+1) - ay(k) - b\hat{\theta}^T(k)\phi_d(k) - bk_c e(k) \\
&= b \cdot \frac{1}{b} y_d(k+1) - b \cdot \frac{a}{b} y_d(k) + ay_d(k) - ay(k) - b\hat{\theta}^T(k)\phi_d(k) - bk_c e(k) \\
&= b\begin{bmatrix} w_0 & w_1 \end{bmatrix}\begin{bmatrix} y_d(k) \\ y_d(k+1) \end{bmatrix} - b\hat{\theta}^T(k)\phi_d(k) + (a - bk_c)e(k) \\
&= (a - bk_c)e(k) - b\hat{\theta}^T(k)\phi_d(k) + b\theta^T\phi_d(k)
\end{aligned}$$

which verifies eq. (25). So the feedforward block in Fig. 5-2 agrees with the error equation. The feedback nonlinear block is determined by the parameter adaptation algorithm.

c. The transfer function of the feedforward linear block is given by

$$G(z^{-1}) = \frac{\dfrac{b}{1 - (a - bk_c)z^{-1}}}{1 - K \cdot \dfrac{b}{1 - (a - bk_c)z^{-1}}} = \frac{b}{1 - Kb - (a - bk_c)z^{-1}} = \frac{b/(1 - Kb)}{1 - \dfrac{a - bk_c}{1 - Kb} z^{-1}}. \qquad (26)$$

When $K < \frac{1}{b}[1 - (a - bk_c)]$, we have $0 < a - bk_c < 1 - Kb$. Then $\frac{a - bk_c}{1 - Kb} < 1$. From the example 4 on page HS-3 of the reader or problem [1] c of homework #7, we can conclude that $G(z^{-1})$ is SPR.

Define the output of the modified nonlinear block (the NL block connected with the constant gain K as shown in Fig. 5-3) as $w(k)$. Notice that $w(k) = \tilde{\theta}^T(k-1)\phi_d(k-1) + Ke(k)$. Let's now verify that when $\frac{\gamma}{2}\phi_d^T(k-1)\phi_d(k-1) < K$, the modified nonlinear block satisfies Popov inequality. We start with the left hand side of the Popov inequality (suppose our intial estimate is given at time –1):

$$\begin{aligned}
\sum_{k=0}^{k_1} w(k)e(k) &= \sum_{k=0}^{k_1} e(k)[\phi_d^T(k-1)\tilde{\theta}(k-1) + Ke(k)] \\
&= \sum_{k=0}^{k_1}[e(k)\phi_d^T(k-1)\tilde{\theta}(k-1)] + \sum_{k=0}^{k_1}[Ke^2(k)]
\end{aligned} \qquad (27)$$

From the parameter adaptation algorithm, we can get

$$\begin{aligned}\tilde{\theta}(k-1) &= \tilde{\theta}(k-2) + \gamma\phi_d(k-2)e(k-1) \\ &= \tilde{\theta}(k-3) + \gamma\phi_d(k-3)e(k-2) + \gamma\phi_d(k-2)e(k-1) \\ &\vdots \\ &= \gamma\sum_{i=0}^{k-1}\{\phi_d(i-1)e(i)\} + \tilde{\theta}(-1)\end{aligned}$$

Then

$$\begin{aligned}\sum_{k=0}^{k_1}[e(k)\phi_d^T(k-1)\tilde{\theta}(k-1)] &= \sum_{k=0}^{k_1}\left\{e(k)\phi_d^T(k-1)\left[\gamma\sum_{i=0}^{k-1}\{\phi_d(i-1)e(i)\} + \tilde{\theta}(-1)\right]\right\} \\ &= \sum_{k=0}^{k_1}\left\{e(k)\phi_d^T(k-1)\left[\gamma\sum_{i=0}^{k}\{\phi_d(i-1)e(i)\} - \gamma\phi_d(k-1)e(k) + \tilde{\theta}(-1)\right]\right\} \\ &= \sum_{k=0}^{k_1}\left\{e(k)\phi_d^T(k-1)\left[\gamma\sum_{i=0}^{k}\{\phi_d(i-1)e(i)\} + \tilde{\theta}(-1)\right]\right\} \\ &\quad -\gamma\sum_{k=0}^{k_1}\left\{\phi_d^T(k-1)\phi_d(k-1)e^2(k)\right\}\end{aligned} \tag{28}$$

Applying eq. (PIAC-49) on page PIAC-12 of the reader with $x(i) = \phi_d(i-1)e(i)$ and $F = \gamma$, we have

$$\begin{aligned}&\sum_{k=0}^{k_1}\left\{e(k)\phi_d^T(k-1)\left[\gamma\sum_{i=0}^{k}\{\phi_d(i-1)e(i)\} + \tilde{\theta}(-1)\right]\right\} \\ &= \frac{1}{2}\left[\gamma\sum_{k=0}^{k_1}[e(k)\phi_d(k-1)] + \tilde{\theta}(-1)\right]^T \gamma^{-1}\left[\gamma\sum_{k=0}^{k_1}[e(k)\phi_d(k-1)] + \tilde{\theta}(-1)\right] \\ &\quad + \frac{\gamma}{2}\sum_{k=0}^{k_1}[\phi_d^T(k-1)\phi_d(k-1)e^2(k)] - \frac{1}{2}\tilde{\theta}^T(-1)\gamma^{-1}\tilde{\theta}(-1)\end{aligned} \tag{29}$$

Plugging eq. (29) into eq. (28), we can get

$$\begin{aligned}\sum_{k=0}^{k_1}[e(k)\phi_d^T(k-1)\tilde{\theta}(k-1)] &= \frac{1}{2}\left[\gamma\sum_{k=0}^{k_1}[e(k)\phi_d(k-1)] + \tilde{\theta}(-1)\right]^T \gamma^{-1}\left[\gamma\sum_{k=0}^{k_1}[e(k)\phi_d(k-1)] + \tilde{\theta}(-1)\right] \\ &\quad + \frac{\gamma}{2}\sum_{k=0}^{k_1}[\phi_d^T(k-1)\phi_d(k-1)e^2(k)] - \frac{1}{2}\tilde{\theta}^T(-1)\gamma^{-1}\tilde{\theta}(-1) - \gamma\sum_{k=0}^{k_1}\left\{\phi_d^T(k-1)\phi_d(k-1)e^2(k)\right\} \\ &= \frac{1}{2}\left[\gamma\sum_{k=0}^{k_1}[e(k)\phi_d(k-1)] + \tilde{\theta}(-1)\right]^T \gamma^{-1}\left[\gamma\sum_{k=0}^{k_1}[e(k)\phi_d(k-1)] + \tilde{\theta}(-1)\right] \\ &\quad - \frac{\gamma}{2}\sum_{k=0}^{k_1}[\phi_d^T(k-1)\phi_d(k-1)e^2(k)] - \frac{1}{2}\tilde{\theta}^T(-1)\gamma^{-1}\tilde{\theta}(-1) \\ &\geq -\frac{\gamma}{2}\sum_{k=0}^{k_1}[\phi_d^T(k-1)\phi_d(k-1)e^2(k)] - \frac{1}{2}\tilde{\theta}^T(-1)\gamma^{-1}\tilde{\theta}(-1)\end{aligned} \tag{30}$$

Plugging inequality (30) into eq. (27), we have

$$\begin{aligned}
\sum_{k=0}^{k_1} w(k)e(k) &\geq -\frac{\gamma}{2}\sum_{k=0}^{k_1}[\phi_d^T(k-1)\phi_d(k-1)e^2(k)] - \frac{1}{2}\tilde{\theta}^T(-1)\gamma^{-1}\tilde{\theta}(-1) + \sum_{k=0}^{k_1}[Ke^2(k)] \\
&= \sum_{k=0}^{k_1}\left\{\left[K - \frac{\gamma}{2}\phi_d^T(k-1)\phi_d(k-1)\right]e^2(k)\right\} - \frac{1}{2}\tilde{\theta}^T(-1)\gamma^{-1}\tilde{\theta}(-1) \\
&\geq -\frac{1}{2}\tilde{\theta}^T(-1)\gamma^{-1}\tilde{\theta}(-1)
\end{aligned}$$

This inequality holds for any $k_1 > 0$. So the feedback nonlinear block satisfies Popov inequality. Therefore, the PAA is asymptotically hyperstable and $e(k)$ converges to zero.