1. Solution:
(1) To obtain $\Phi_{xx}(\omega)$, we first compute the spectral density of $w(k)$. We give two ways to do this:
Method 1: By definition

$$
\begin{aligned}
\Phi_{ww}(\omega) &= \sum_{l=-\infty}^{\infty} 0.2^{|l|}\frac{3}{8}e^{-j\omega l} = \sum_{l=0}^{\infty} 0.2^{l}\frac{3}{8}e^{-j\omega l} + \sum_{l=-\infty}^{0} 0.2^{-l}\frac{3}{8}e^{-j\omega l} - 0.2^{0}\frac{3}{8}e^{-j\omega\times 0} \\
&= \frac{3}{8}\left[\frac{1}{1-0.2e^{-j\omega}} + \frac{1}{1-0.2e^{j\omega}} - 1\right] \\
&= \left(\frac{3}{5}\right)^2 \frac{1}{1-0.2e^{-j\omega}}\frac{1}{1-0.2e^{j\omega}} \qquad (1)
\end{aligned}
$$

Method 2: Notice that $X_{ww}(l) = 0.2^{|l|}\frac{3}{8}$ defines a first-order dynamics that comes from

$$w(k+1) = 0.2w(k) + n(k) \qquad (2)$$

which gives $X_{ww}(l) = 0.2^{|l|}X_{ww}(0)$ under the whiteness assumption on $n(k)$.[1] From (2), the steady-state variance $X_{ww}(0)$ can be obtained from

$$X_{ww}(0) = a^2 X_{ww}(0) + W_{nn} \Rightarrow X_{ww}(0) = \frac{W_{nn}}{1-a^2}$$

In our problem, $X_{ww}(0) = 3/8$. Setting $\frac{W_{nn}}{1-a^2} = 3/8$ gives $W_{nn} = (3/5)^2$. The relationship between $n(k)$ and $w(k)$ can now be summarized by

$$n(k) \longrightarrow \boxed{G_{wn}(z)} \longrightarrow w(k)$$

where $n(k)$ is a zero-mean, white, Gaussian random process with $E[n(k)n(k+l)] = (3/5)^2\delta_l$; and the transfer function $G_{wn}(z^{-1}) = \frac{1}{z-a}$. The spectral density of $w(k)$ is thus

$$\Phi_{ww}(\omega) = \left.\frac{1}{z-a}\frac{1}{z^{-1}-a}\right|_{z=e^{j\omega}} W_{nn} = \left(\frac{3}{5}\right)^2 \frac{1}{1-0.2e^{-j\omega}}\frac{1}{1-0.2e^{j\omega}}$$

+4 points

The transfer function from $w(k)$ to $x(k)$ is given by

$$G_{xw}(z) = \frac{1}{z-0.8} = \frac{z^{-1}}{1-0.8z^{-1}}$$

+1 points

Hence the spectral density of $x(k)$ is

$$
\begin{aligned}
\Phi_{xx}(\omega) &= \left.G_{xw}(z)G_{xw}(z^{-1})\right|_{z=e^{j\omega}}\Phi_{ww}(w) \\
&= \frac{e^{-j\omega}}{1-0.8e^{-j\omega}}\frac{e^{j\omega}}{1-0.8e^{j\omega}}\left(\frac{3}{5}\right)^2 \frac{1}{1-0.2e^{-j\omega}}\frac{1}{1-0.2e^{j\omega}} \\
&= \left(\frac{3}{5}\right)^2 \frac{1}{1.64-1.6\cos\omega}\frac{1}{1.04-0.4\cos\omega}
\end{aligned}
$$

+3 points

(2) For simplicity, from now on we will use a normalized version of (2)

$$w(k+1) = 0.2w(k) + \frac{3}{5}w_o(k) \qquad (3)$$

where $w_o(k)$ has zero mean and *unit* variance. We then have the following picture

$$w_o(k) \longrightarrow \boxed{G_{ww_o}(z)} \longrightarrow w(k)$$

[1] The general case is discussed on page PR-10 in the reader.

where

$$G_{ww_o}(z) = \frac{3}{5}\frac{z^{-1}}{1-0.2z^{-1}}$$

Augmenting the original system with (3) yields

$$\begin{bmatrix} x(k+1) \\ w(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 0.8 & 1 \\ 0 & 0.2 \end{bmatrix}}_{A_e} \underbrace{\begin{bmatrix} x(k) \\ w(k) \end{bmatrix}}_{x_e(k)} + \underbrace{\frac{3}{5}\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B_{w_o}} w_o(k) \tag{4}$$

$$y(k) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C_e} x_e(k) + v(k) \tag{5}$$

$w(k)$ and $w_o(k)$ are zero mean, and the variance of $w_o(k)$ is 1. If $x(0)$ is Gaussian with $E[x(0)] = x_o$ and variance X_o, and $x(0)$ is independent from $w(0)$, then

$$E[x_e(0)] = \begin{bmatrix} x_o \\ 0 \end{bmatrix},\ E\left[(x_e(0) - E[x_e(0)])(x_e(0) - E[x_e(0)])^T\right] = \begin{bmatrix} X_o & 0 \\ 0 & 1 \end{bmatrix},\ W_{w_o w_o} = 1$$

+6 points

Now the standard assumptions for Kalman filter hold for the augmented system. The steady-state Kalman filter is given by

$$\begin{aligned}
\hat{x}_e(k+1|k+1) &= \hat{x}_e(k+1|k) + F_s\left(y(k+1) - C_e\hat{x}_e(k+1|k)\right) \\
\hat{x}_e(k+1|k) &= A_e\hat{x}_e(k|k),\ \hat{x}_e(0|-1) = \begin{bmatrix} x_o \\ 0 \end{bmatrix} \\
F_s &= M_s C_e^T\left[C_e M_s C_e^T + V\right]^{-1} \\
M_s &= A_e Z_s A_e^T + B_{w_o}B_{w_o}^T \\
Z_s &= M_s - M_s C_e^T\left[C_e M_s C_e^T + V\right]^{-1} C_e M_s
\end{aligned}$$

with the ARE

$$M_s = A_e M_s A_e^T - A_e M_s C_e^T\left[C_e M_s C_e^T + V\right]^{-1} C_e M_s A_e^T + B_{w_o}B_{w_o}^T$$

+2 points

(3) The transfer function from $w(k)$ to $y(k)$ is

$$G_{yw}(z) = G_{xw}(z) = \frac{1}{z-0.8} = \frac{z^{-1}}{1-0.8z^{-1}}$$

Hence the transfer function from $w_o(k)$ to $y(k)$ is

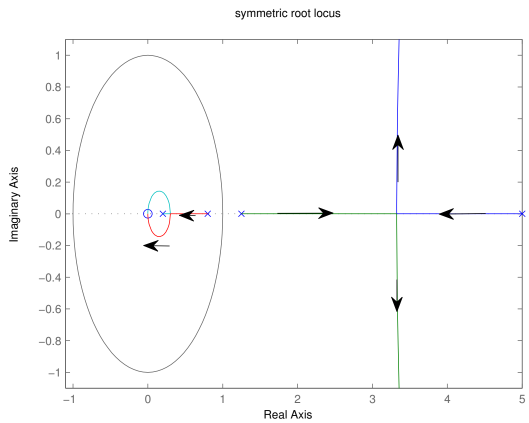
$$G_{yw_o}(z) = G_{yw}(z)G_{ww_o}(z) = \frac{3}{5}\frac{z^{-1}}{1-0.2z^{-1}}\frac{z^{-1}}{1-0.8z^{-1}}$$

The symmetric root locus for the Kalman filter is determined by

$$1 + \frac{W_{w_o w_o}}{V}G_{yw_o}(z)G_{yw_o}(z^{-1}) = 0$$

$$\begin{aligned}
G_{yw_o}(z)G_{yw_o}(z^{-1}) &= \left(\frac{3}{5}\right)^2 \frac{z^{-1}}{1-0.2z^{-1}}\frac{z^{-1}}{1-0.8z^{-1}}\frac{z}{1-0.2z}\frac{z}{1-0.8z} \\
&= \left(\frac{3}{5}\right)^2 z^2 \frac{1}{z-0.2}\frac{1}{z-0.8}\frac{1}{1-0.2z}\frac{1}{1-0.8z}
\end{aligned} \tag{6}$$

We thus have



The arrows indicate the direction of increasing $W_{w_o w_o}/V = 1/V$.
+4 points

2. Solution:
We can use the standard way of dynamic programming, by considering first $J(N)$, $J(N-1)$, and generalizing the results.

$$\begin{aligned}
J_N^o &= J_N = \frac{1}{2}x^T(N)\,Sx(N)\\
J_{N-1}^o &= \min_{u(N-1)} J_{N-1}\\
&= \min_{u(N-1)} \{\frac{1}{2}x^T(N)\,Sx(N)\\
&+ \frac{1}{2}\left[x^T(N-1)\,Qx(N-1) + 2u^T(N-1)\,Mx(N-1) + u^T(N-1)\,Ru(N-1)\right]\}\\
&= \min_{u(N-1)} \{\frac{1}{2}\left[Ax(N-1) + Bu(N-1)\right]^T S\left[Ax(N-1) + Bu(N-1)\right]\\
&+ \frac{1}{2}\left[x^T(N-1)\,Qx(N-1) + 2u^T(N-1)\,Mx(N-1) + u^T(N-1)\,Ru(N-1)\right]\}
\end{aligned}$$

Taking the partial derivative w.r.t. $u(N-1)$ gives

$$\frac{\partial J_{N-1}}{\partial u(N-1)} = B^T S\left[Ax(N-1) + Bu(N-1)\right] + Mx(N-1) + Ru(N-1)$$

and

$$\frac{\partial J_{N-1}}{\partial u(N-1)} = 0 \Rightarrow u^o(N-1) = -\left[R + B^T SB\right]^{-1}\left[B^T SA + M\right]x(N-1)$$

+8 points

Let $P(N) = S$. After simplification, the optimal cost under $u^o(N-1)$ is,

$$J_{N-1}^o = \frac{1}{2}x^T(N-1)\underbrace{\left\{Q + A^T P(N)A - \left[A^T P(N)\,B + M^T\right]\left[R + B^T P(N)B\right]^{-1}\left[B^T P(N)\,A + M\right]\right\}}_{P(N-1)} x(N-1)$$

+2 points

which is a quadratic function of the state $x(N-1)$. Generalizing the result and considering

$$J_{k+1}^o\left(x(k+1)\right) = \frac{1}{2}x^T(k+1)\,P(k+1)\,x(k+1)$$

we have

$$\begin{aligned}J_k^o = \min_{u(k)} J_k &= \min_{u(k)}\{\frac{1}{2}x^T(k)\,Qx(k) + u^T(k)\,Mx(k) + \frac{1}{2}u^T(k)\,Ru(k) + J_{k+1}^o(k+1)\}\\ &= \min_{u(k)}\{\frac{1}{2}\left[x^T(k)\,Qx(k) + 2u^T(k)\,Mx(k) + u^T(k)\,Ru(k)\right]\\ &+ \frac{1}{2}\left[Ax(k)+Bu(k)\right]^T P(k+1)\left[Ax(k)+Bu(k)\right]\}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial J_k}{\partial u(k)} &= Mx(k) + Ru(k) + B^T P(k+1)\left[Ax(k)+Bu(k)\right]\\ \frac{\partial J_k}{\partial u(k)} = 0 &\Rightarrow u^o(k) = \left[R + B^T P(k+1)B\right]^{-1}\left[B^T P(k+1)A + M\right]x(k)\end{aligned}$$

+2 points

Substituting in $u^o(k)$, and after simplification, we have

$$J_k^o(x(k)) = \frac{1}{2}x^T(k)\underbrace{\left\{Q + A^T P(k+1)A - \left[A^T P(k+1)B + M^T\right]\left[R + B^T P(k+1)B\right]^{-1}\left[B^T P(k+1)A + M\right]\right\}}_{P(k)}x(k)$$

hence the Riccati equation

$$P(k) = Q + A^T P(k+1)A - \left[A^T P(k+1)B + M^T\right]\left[R + B^T P(k+1)B\right]^{-1}\left[B^T P(k+1)A + M\right],\ P(N) = S$$

+2 points

To conclude the positive semi definiteness of $Q - M^T R^{-1} M$, we notice that

$$\begin{aligned}J &= \frac{1}{2}x^T(N)Sx(N) + \frac{1}{2}\sum_{j=0}^{N-1}\left\{x^T(j)Qx(j) + 2u^T(j)Mx(j) + u^T(j)Ru(j)\right\}\\ &= \frac{1}{2}x^T(N)Sx(N) + \frac{1}{2}\sum_{j=0}^{N-1}\left\{x^T(j)Qx(j) + 2u^T(j)\underbrace{R^{1/2}R^{-1/2}}_{\text{identity matrix}}Mx(j) + u^T(j)Ru(j)\right\}\\ &= \frac{1}{2}x^T(N)Sx(N) + \frac{1}{2}\sum_{j=0}^{N-1}\left\{x^T(j)Qx(j) + 2u^T(j)R^{1/2}R^{-1/2}Mx(j) + u^T(j)Ru(j) \pm x^T(j)M^T R^{-1/2}R^{-1/2}Mx(j)\right\}\\ &= \frac{1}{2}x^T(N)Sx(N) + \frac{1}{2}\sum_{j=0}^{N-1}\left\{x^T(j)\left(Q - M^T R^{-1}M\right)x(j) + \left(R^{1/2}u(j) + R^{-1/2}Mx(j)\right)^T\left(R^{1/2}u(j) + R^{-1/2}Mx(j)\right)\right\}\\ &= \frac{1}{2}x^T(N)Sx(N) + \frac{1}{2}\sum_{j=0}^{N-1}\left\{x^T(j)\underbrace{\left(Q - M^T R^{-1}M\right)}_{\bar{Q}}x(j) + \left(u(j) + R^{-1}Mx(j)\right)^T R\underbrace{\left(u(j) + R^{-1}Mx(j)\right)}_{\bar{u}(j)}\right\} \qquad (7)\end{aligned}$$

We have transformed (7) to be in the standard LQ form. Clearly we need $\bar{Q} = Q - M^T R^{-1} M$ to be positive semidefinite from standard requirements in LQ.

+6 points

Actually if we started with (7), the problem can be solved in an alternative (and simpler) way: By the change of input

$$\bar{u}(k) = u(k) + R^{-1}Mx(k)$$

the system becomes

$$\begin{aligned}x(k+1) = Ax(k) + Bu(k) &= Ax(k) + B\left(\bar{u}(k) - R^{-1}Mx(k)\right)\\ &= \underbrace{\left(A - BR^{-1}M\right)}_{\bar{A}}x(k) + B\bar{u}(k)\end{aligned}$$

We have the system equation and the standard performance index. The remaining steps about dynamic programming are the same as those in standard discrete-time LQ. The solution is given by

$$\bar{u}^o(k) = u^o(k) + R^{-1}Mx(k) = -\left[R + B^T P(k+1)B\right]^{-1} B^T P(k+1)A^* x(k)$$

which gives

$$\begin{aligned}
& u^o(k) \\
&= -\left[R + B^T P(k+1)B\right]^{-1} B^T P(k+1)A^* x(k) - R^{-1}Mx(k) \\
&= -\left\{R^{-1}M + \left[R + B^T P(k+1)B\right]^{-1} B^T P(k+1)\left(A - BR^{-1}M\right)\right\} x(k) \\
&= -\left\{R^{-1}M + \left[R + B^T P(k+1)B\right]^{-1} B^T P(k+1)A - \left[R + B^T P(k+1)B\right]^{-1} B^T P(k+1)BR^{-1}M\right\} x(k) \\
&= -\left\{\left[R + B^T P(k+1)B\right]^{-1} B^T P(k+1)A + \left[I - \left[R + B^T P(k+1)B\right]^{-1} B^T P(k+1)B\right] R^{-1}M\right\} x(k) \\
&= -\left\{\left[R + B^T P(k+1)B\right]^{-1} B^T P(k+1)A + \left[I - \left[R + B^T P(k+1)B\right]^{-1} \left[R + B^T P(k+1)B - R\right]\right] R^{-1}M\right\} x(k) \\
&= -\left\{\left[R + B^T P(k+1)B\right]^{-1} B^T P(k+1)A + \left[R + B^T P(k+1)B\right]^{-1} RR^{-1}M\right\} x(k) \\
&= -\left[R + B^T P(k+1)B\right]^{-1} \left[B^T P(k+1)A + M\right] x(k)
\end{aligned}$$

where $P(k)$ is the positive definite solution of the following Riccati equation

$$P(k) = \bar{A}^T P(k+1)\bar{A} - \bar{A}^T P(k+1)B\left[R + B^T P(k+1)B\right]^{-1} B^T P(k+1)\bar{A} + \bar{Q}$$

i.e.

$$\begin{aligned}
P(k) = &\left(A - BR^{-1}M\right)^T P(k+1)\left(A - BR^{-1}M\right) + Q - M^T R^{-1}M \\
&- \left(A - BR^{-1}M\right)^T P(k+1)B\left[R + B^T P(k+1)B\right]^{-1} B^T P(k+1)\left(A - BR^{-1}M\right)
\end{aligned}$$

which is equivalent to

$$P(k) = Q + A^T P(k+1)A - \left[A^T P(k+1)B + M^T\right]\left[R + B^T P(k+1)B\right]^{-1}\left[B^T P(k+1)A + M\right]$$

with the boundary condition

$$P(N) = S$$

3. Solution: From the separation theorem, the closed-loop eigenvalues are composed of two parts: one from LQ and the other from Kalman filter. Hence we know that the eigenvalues of $A - BK_s$ and $A - F_sC$ are $-\sqrt{2}$ and $-\sqrt{3}$.
+2 points
Here K_s and F_s are

$$\begin{aligned}
K_s &= R^{-1}B^T P_s \\
F_s &= M_s C^T V^{-1}
\end{aligned}$$

where P_s and M_s are the positive definite solutions of the following Riccati equations:

$$\begin{aligned}
A^T P_s + P_s A + Q - P_s BR^{-1}B^T P_s &= 0 \\
M_s A^T + AM_s + B_w W B_w - M_s C^T V^{-1} CM_s &= 0
\end{aligned}$$

+2 points
Before computing the detailed algebra, we know that the optimal control law is the same for $J = E\left[Qx^2(t) + Ru^2(t)\right]$ and $\bar{J} = E\left[kQx^2(t) + kRu^2(t)\right]$, $\forall k > 0$. Hence, it is expected that the exact values for Q and R are not unique. The same applies to W and V in Kalman filter. Keeping these in mind, we notice that in the considered problem we have

$$A = -1,\ B = 1,\ B_w = 1,\ C = 1$$

yielding

$$K_s = \frac{P_s}{R}$$
$$F_s = \frac{M_s}{V}$$

and

$$-2P_s + Q - P_s \frac{P_s}{R} = 0 \tag{8}$$
$$-2M_s + W - M_s \frac{M_s}{V} = 0 \tag{9}$$

+2 points
Consider two cases for the closed-loop eigenvalues:
Case (i):

$$A - BK_s = -1 - \frac{P_s}{R} = -\sqrt{2} \Rightarrow \frac{P_s}{R} = -1 + \sqrt{2} \tag{10}$$
$$A - F_s C = -1 - \frac{M_s}{V} = -\sqrt{3} \Rightarrow \frac{M_s}{V} = -1 + \sqrt{3} \tag{11}$$

yielding the following simplified versions of (8) and (9):

$$-2P_s + Q + P_s(1 - \sqrt{2}) = 0 \Rightarrow Q = \left(1 + \sqrt{2}\right) P_s \tag{12}$$
$$-2M_s + W + M_s\left(1 - \sqrt{3}\right) = 0 \Rightarrow W = \left(1 + \sqrt{3}\right) M_s \tag{13}$$

(10) - (13) give us

$$\frac{Q}{R} = 1$$
$$\frac{W}{V} = 2$$

Case (ii): analogous derivations yield that, if $A - BK_s = -\sqrt{3}$ and $A - F_s C = -\sqrt{2}$, we get

$$\frac{Q}{R} = 2$$
$$\frac{W}{V} = 1$$

+4 points
Remark: an alternative approach is to use the Return Difference Equality. The symmetric root locus is derived based on the Return Difference Equality, which is a result of the ARE. Hence, in this problem, symmetric root locus holds independently for the LQ and Kalman filter. For LQ, the symmetric root locus is determined by

$$1 + \frac{Q}{R} G(s) G(-s) = 0$$
$$\Leftrightarrow 1 + \frac{Q}{R} \frac{1}{(s+1)(-s+1)} = 0$$
$$1 - s^2 + \frac{Q}{R} = 0$$

For Case (i), we have the LQ closed-loop eigenvalue is $-1 - \sqrt{2}$. Hence

$$1 - s^2 + \frac{Q}{R} = \left(s + \sqrt{2}\right)\left(-s + \sqrt{2}\right) = -s^2 + 2$$
$$\Rightarrow \frac{Q}{R} = 1$$

Analogous analysis gives that, for the Kalman filter we have

$$
\begin{aligned}
1 + \frac{W}{V} G_w(s) G_w(-s) &= 0 \\
\Rightarrow 1 - s^2 + \frac{W}{V} &= \left(s + \sqrt{3}\right)\left(-s + \sqrt{3}\right) = 0 \\
\Rightarrow \frac{W}{V} &= 2
\end{aligned}
$$

Case (ii) can be similarly derived. The results are the same as those using the first approach.