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ME 233 – Homework 10

1.a)

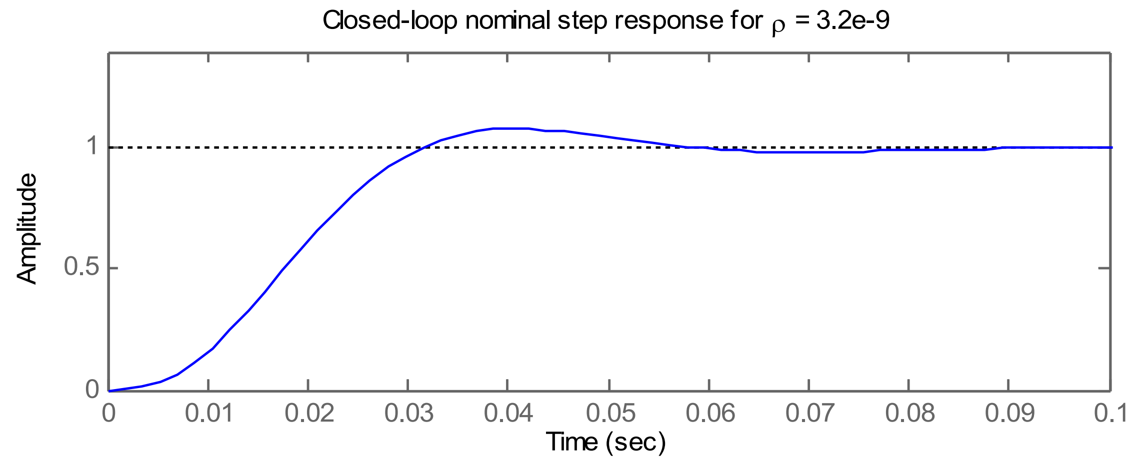
$$G_p(s)=\frac{\omega_b^2}{s^2+2\zeta_b\omega_b s+\omega_b^2},\ \zeta_b=0.707,\ \omega_b=10\,\text{rad/s}$$

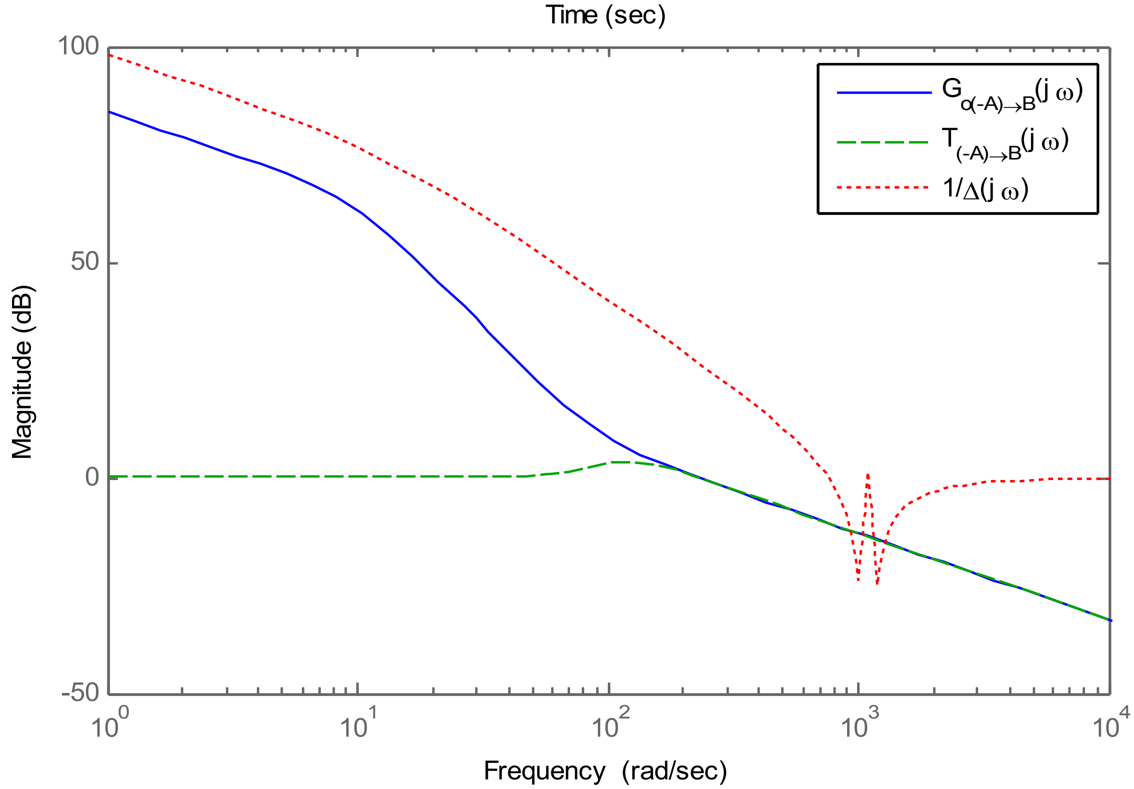
$$G_{PA}(s)=G_p(s)\left(\frac{\omega_r(\zeta_r s+\omega_r)}{s^2+2\zeta_r\omega_r s+\omega_r^2}\right)\left(\frac{\omega_t^2(s^2+2\zeta_t\omega_n s+\omega_n^2)}{\omega_n^2(s^2+2\zeta_t\omega_t s+\omega_t^2)}\right)$$

$$\zeta_r=0.015,\ \omega_r=1000\,\text{rad/s},\ \zeta_t=0.015,\ \omega_t=1200\,\text{rad/s},\ \omega_n=0.9\,\omega_t$$

$$Q_r(s)=\frac{1}{s},\ J=\frac{1}{2}\int_{-\infty}^{\infty}\{X^*(j\omega)C^T Q_r^*(j\omega)Q_r(j\omega)C\,X(j\omega)+\rho|U(j\omega)|^2\}\,d\omega$$

Using functions fslqr and fslqr_reg with $Q_f=\text{zeros(2)},\ R_f=1,\ \rho=3.2\cdot 10^{-9}$ gives a gain crossover frequency of 60 rad/sec for $G_{o\,E\to Y}(s)$





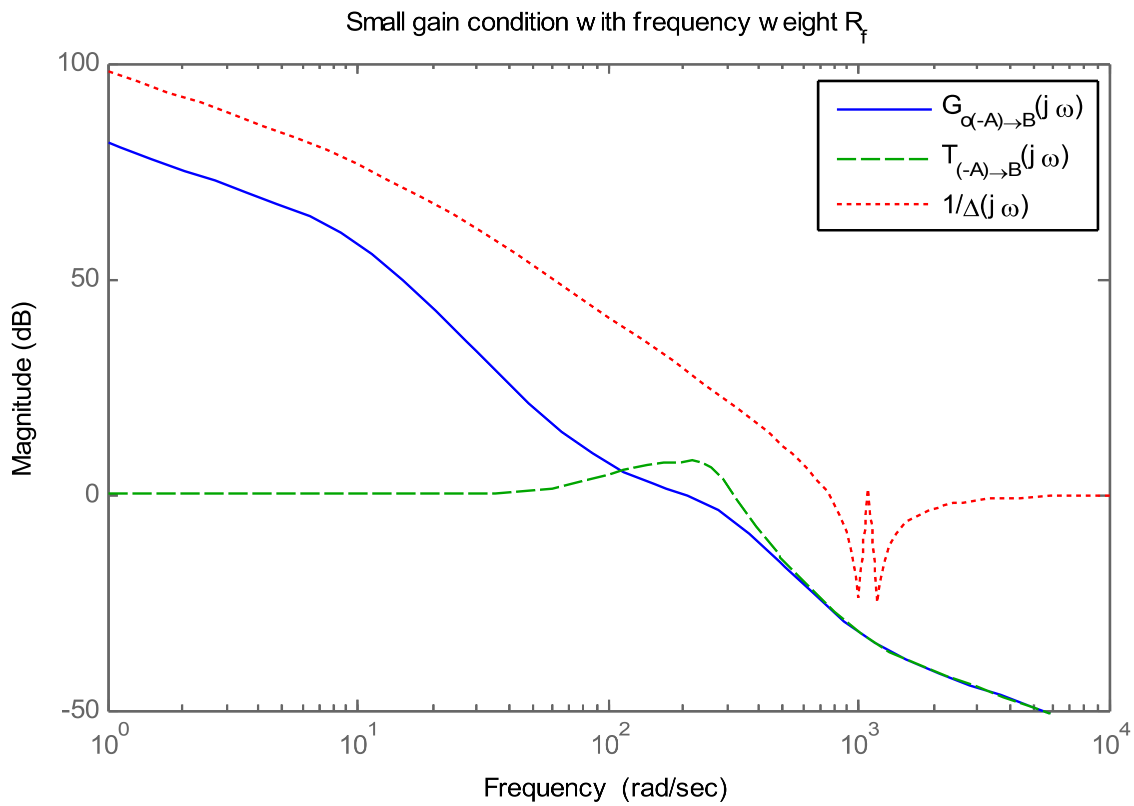
Small gain condition is not satisfied, $|T_{(-A)\to B}(j\omega)|>|1/\Delta(j\omega)|$ at some points around 10^3 rad/sec

Calculating this feedback design applied to the actual plant model using fslqr_reg_robust_test, the complementary transfer function $T_{E\to Y}(s)$ has unstable poles at $45.07\pm 1185\,j$ and $35.36\pm 1020.5\,j$ so the closed-loop feedback system with the actual plant model is not stable.

1.b)

$$J=\frac{1}{2}\int_{-\infty}^{\infty}\{X^*(j\omega)C^T Q_r^*(j\omega)Q_r(j\omega)C\,X(j\omega)+\rho\,R(j\omega)|U(j\omega)|^2\}d\omega$$

For $R_f(s)$ I will use a second-order lead filter with poles at 1000 rad/sec, zeros at 250 rad/sec, and damping ratios of 0.707 for the poles and 0.25 for the zeros. $R_f(s)=16\dfrac{s^2+125s+250^2}{s^2+1414s+1000^2}$



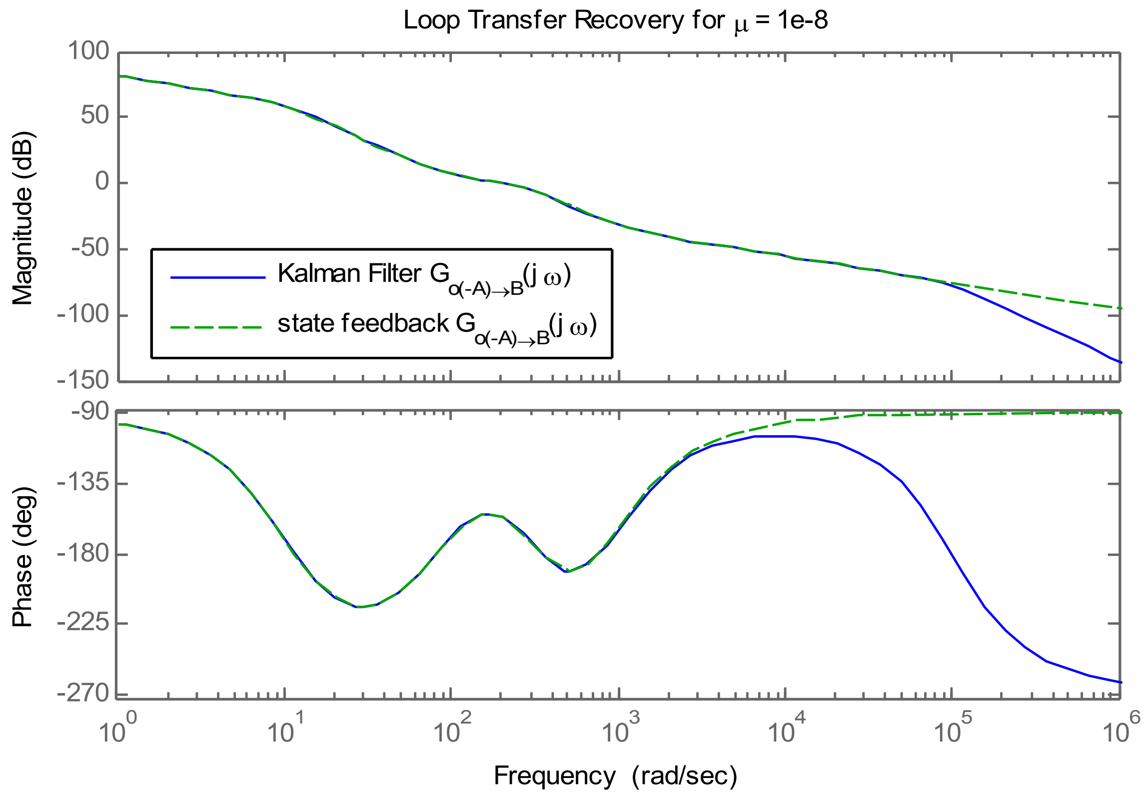
The gain crossover of $G_{oE\to Y}(s)$ in this design is now 55.3 rad/sec. Applying the same compensators to the actual model (figure 3 robustness test), the least stable closed-loop eigenvalue is $-13.5\pm 994\,j$ so now the closed-loop feedback system is stable.

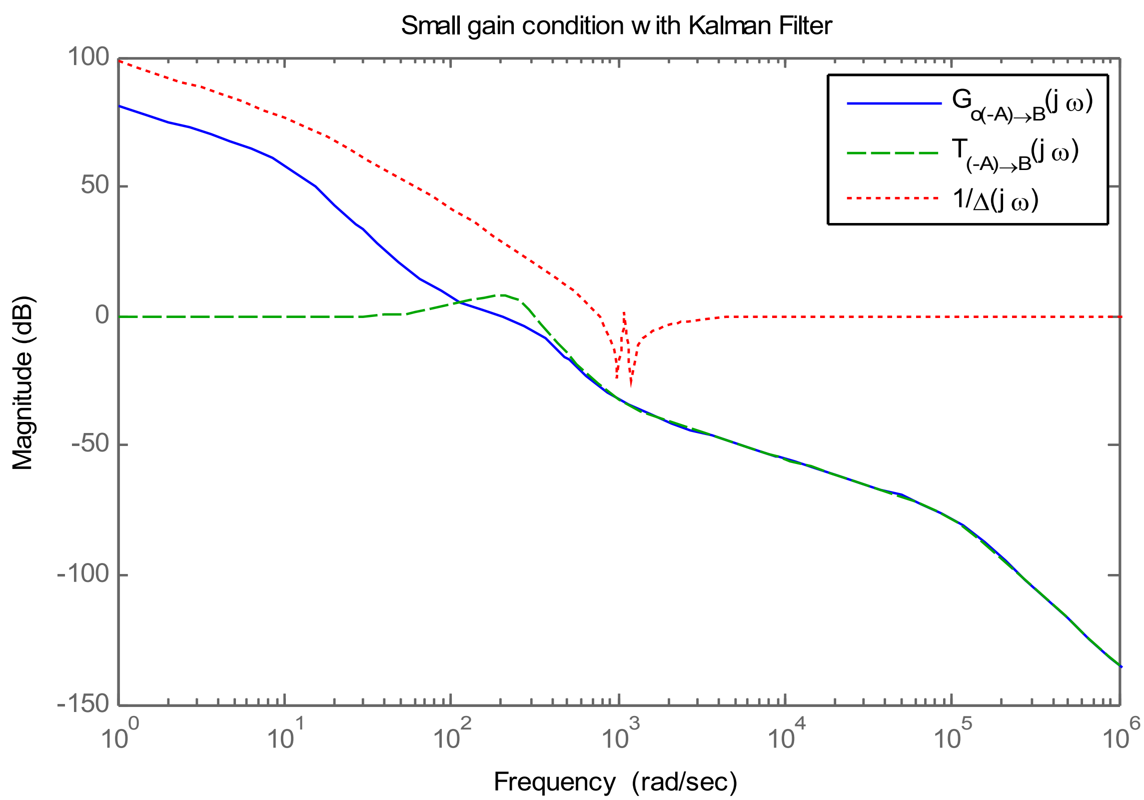
1.c)

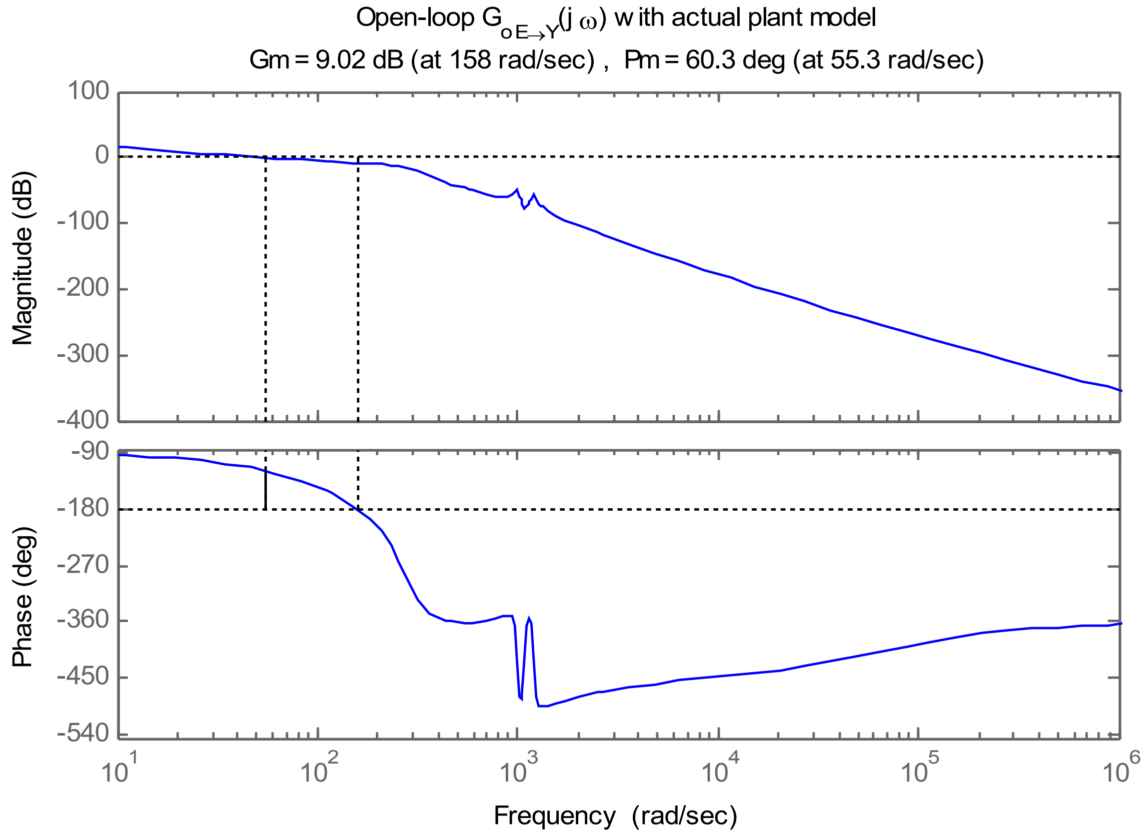
See plots on next page. The loop transfer recovery was pretty good for $\mu\leq 10^{-5}$, with better matching up to higher frequencies for smaller μ. $G_{oE\to Y}(s)$ appears to be unchanged by the Kalman filter so its gain crossover is again 55.3 rad/sec.

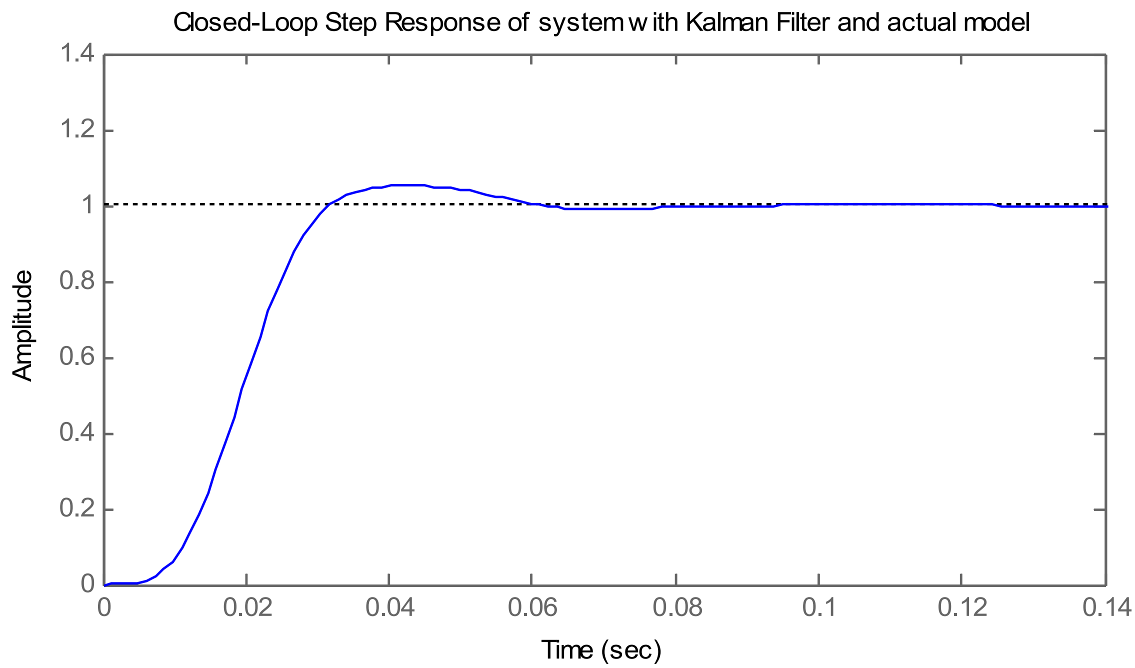
1.d)

The choice of $R_f(s)$ has a major effect on the closed-loop stability robustness. I explored ranges of coefficient values in $R_f(s)$ (poles, zeros, damping ratios) of second order, keeping the gain crossover where I wanted it using a nonlinear constraint in fmincon, and was a bit surprised by some results. Minimizing the largest real part of all closed-loop eigenvalues would occasionally drive one of the damping ratios to whatever lower limit I had set. Sometimes the best result had overdamped poles which I wasn't expecting, and the best frequency for the poles sometimes went even higher than the worst-case $\Delta(s)$ frequencies. The choice of $R_f(s)$ above was a hybrid of what I would have tried manually combined with the output trends that fmincon was giving me.


Loop Transfer Recovery for μ = 1e-8
Magnitude (dB)
Phase (deg)
Kalman Filter G_{α(-A)→B}(j ω)
state feedback G_{α(-A)→B}(j ω)
100
50
0
-50
-100
-150
-90
-135
-180
-225
-270
10^0
10^1
10^2
10^3
10^4
10^5
10^6
Frequency (rad/sec)


Small gain condition w ith Kalman Filter
Magnitude (dB)
G_{α(-A)→B}(j ω)
T_{(-A)→B}(j ω)
1/Δ(j ω)
100
50
0
-50
-100
-150
10^0
10^1
10^2
10^3
10^4
10^5
10^6
Frequency (rad/sec)





2.a)

$$G(z)=\frac{B^*(z)}{A^*(z)}=(1-z^{-1})\,\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{G(s)}{s}\right)\right\},\ G(s)=\frac{1}{s(s+1)}$$

$$G(z)=\mathrm{c2d}(G,0.5)=\frac{0.1065z+0.0902}{z^2-1.607z+0.6065}=\frac{q^{-1}(0.1065+0.0902q^{-1})}{1-1.607q^{-1}+0.6065q^{-2}}=\frac{q^{-d}B(q^{-1})}{A(q^{-1})}$$

2.b) $A_m(q^{-1})y_d(k)=q^{-d}B_m(q^{-1})u_d(k)$, we want natural frequency of 1 rad/sec and damping ratio of 0.707 so continuous poles $p_c=-0.707\pm 0.707j$, so $p_d=e^{p_c T}=0.659\pm 0.243j$

$A_m(q^{-1})=(1-(0.659+0.243j)q^{-1})(1-(0.659-0.243j)q^{-1})=1-1.318q^{-1}+0.493q^{-2}$

2.c) Static gain given by $q=1$, so for $B_m(q^{-1})=b_{mo}=A_m(1)=1-1.318+0.493=0.1756$

2.d) $A_c'(q^{-1})=1+a_{c1}'q^{-1}+a_{c2}'q^{-2}$, we want natural frequency of 2 rad/sec and damping ratio of 0.5 continuous characteristic equation of s^2+2s+2^2, so continuous poles $p_c=-1\pm 1.732j$

$p_d=e^{p_c T}=0.393\pm 0.462j$, $A_c'(q^{-1})=(1-(0.393+0.462j)q^{-1})(1-(0.393-0.462j)q^{-1})$

$A_c'(q^{-1})=1-0.786q^{-1}+0.368q^{-2}$

2.e) $A_c(q^{-1})=A_c'(q^{-1})B^s(q^{-1})$, $B^s(q^{-1})=\dfrac{1}{b_0}B(q^{-1})$, $B^u(q^{-1})=b_0$

From part a, leading coefficient of $B(q^{-1})$, $b_0=0.1065$

$d(k)=0$ so $A_d(q^{-1})=1$, $R(q^{-1})=R'(q^{-1})A_d(q^{-1})B^s(q^{-1})$

Perfect tracking feedforward $T(q^{-1},q)=\dfrac{q^d A_c'(q^{-1})}{b_0}=\dfrac{q(1-0.786q^{-1}+0.368q^{-2})}{0.1065}$

Diophantine equation $A_c(q^{-1})=A(q^{-1})R(q^{-1})+q^{-d}B(q^{-1})S(q^{-1})$

$A_c'(q^{-1})=A_d(q^{-1})A(q^{-1})R'(q^{-1})+q^{-d}B^u(q^{-1})S(q^{-1})$

$1-0.786q^{-1}+0.368q^{-2}=1(1-1.607q^{-1}+0.6065q^{-2})R'(q^{-1})+q^{-1}0.1065S(q^{-1})$

$m_u=0$ since $B_u(q^{-1})$ is a scalar, and $d=1$, so $n_r'=d+m_u-1=0$ therefore $R'(q^{-1})=1$

$n=2$, $n_d=0$, $n_c'=2$, so $n_s=\max(n+n_d-1,\ n_c'-d-m_u)=1$, $S(q^{-1})=s_0+s_1q^{-1}$

$1-0.786q^{-1}+0.368q^{-2}=(1-1.607q^{-1}+0.6065q^{-2})+q^{-1}0.1065(s_0+s_1q^{-1})$

$-0.786=-1.607+0.1065s_0$, $0.368=0.6065+0.1065s_1$

$s_0=(1.607-0.786)/0.1065=7.703$, $s_1=(0.368-0.6065)/0.1065=-2.24$

$T(q^{-1},q)=\dfrac{q-0.786+0.368q^{-1}}{0.1065}$, $r(k)=T(q^{-1},q)y_d(k)$, $S(q^{-1})=7.703-2.24q^{-1}$

$R(q^{-1})=R'(q^{-1})A_d(q^{-1})B^s(q^{-1})=\dfrac{0.1065+0.0902q^{-1}}{0.1065}$, $u(k)=\dfrac{1}{R(q^{-1})}[r(k)-S(q^{-1})y(k)]$

2.f) $y_d(-1)=y_d(0)=y(-1)=y(0)=0$, $u_d(k)=u_s(k)-2u_s(k-25)+2u_s(k-50)-2u_s(k-75)$

$d(k)=0.5u_s(k-40)$, $u_s(j)$ unit step: 0 for $j<1$, 1 for $j\geq 0$

See plots on next page, $y(k)$ perfectly tracks $y_d(k)$ when $d(k)=0$ but has an offset after 20 seconds

2.g) Assuming now that $d(k)=d(k-1)$, $A_d(q^{-1})=1-q^{-1}$

$A_c'(q^{-1})=A_d(q^{-1})A(q^{-1})R'(q^{-1})+q^{-d}B^u(q^{-1})S(q^{-1})$

$1-0.786q^{-1}+0.368q^{-2}=(1-q^{-1})(1-1.607q^{-1}+0.6065q^{-2})R'(q^{-1})+q^{-1}0.1065S(q^{-1})$

$m_u=0$ since $B_u(q^{-1})$ is a scalar, and $d=1$, so $n_r'=d+m_u-1=0$ therefore $R'(q^{-1})=1$

$n=2$, $n_d=1$, $n_c'=2$, so $n_s=\max(n+n_d-1,\ n_c'-d-m_u)=2$, $S(q^{-1})=s_0+s_1q^{-1}+s_2q^{-2}$

$1-0.786q^{-1}+0.368q^{-2}=(1-q^{-1})(1-1.607q^{-1}+0.6065q^{-2})+0.1065q^{-1}(s_0+s_1q^{-1}+s_2q^{-2})$

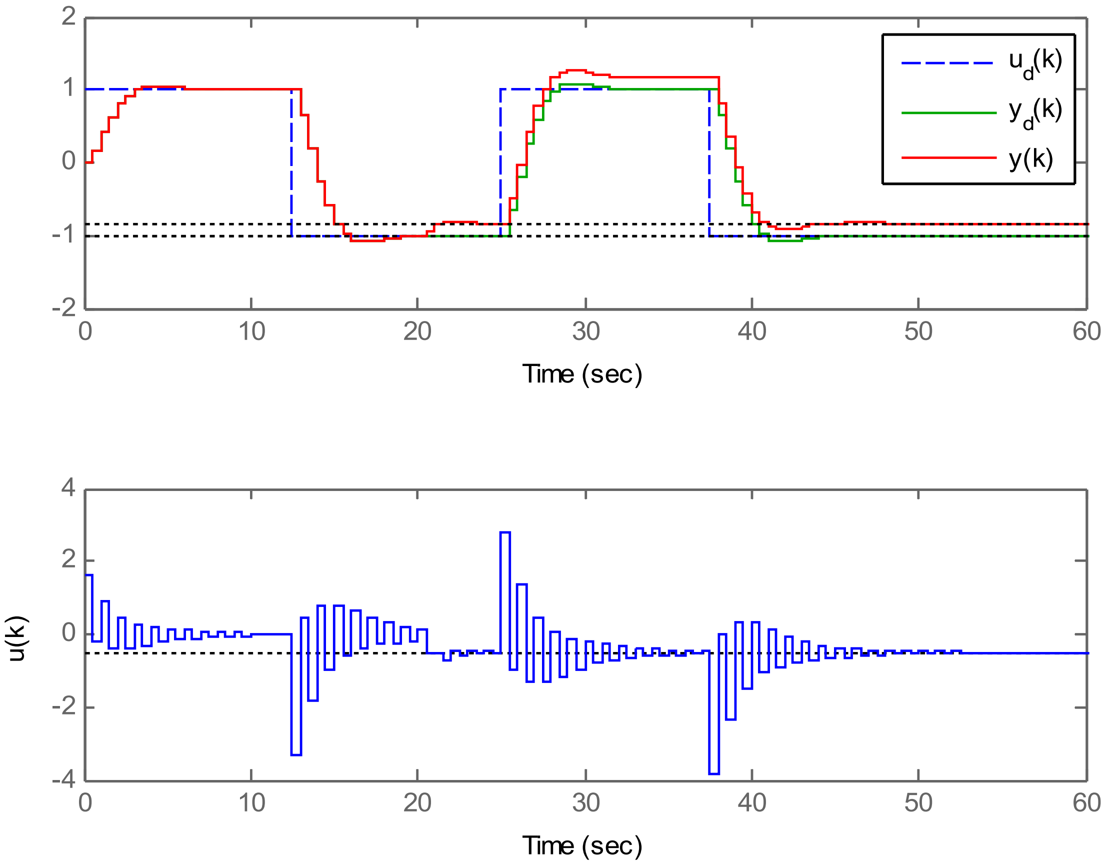
$1-0.786q^{-1}+0.368q^{-2}=1-2.607q^{-1}+2.213q^{-2}-0.6065q^{-3}+0.1065(s_0q^{-1}+s_1q^{-2}+s_2q^{-3})$

$s_0=(2.607-0.786)/0.1065=17.09$, $s_1=(0.368-2.213)/0.1065=-17.32$, $s_2=0.6065/0.1065=5.69$

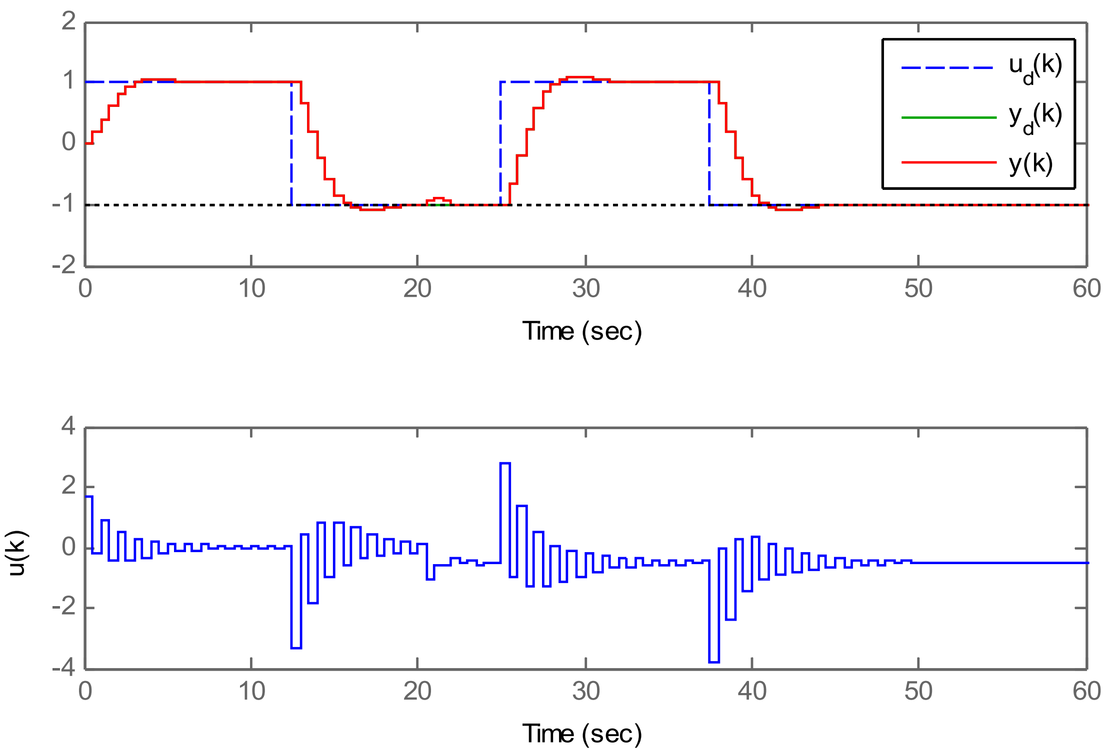
$S(q^{-1})=17.09-17.32q^{-1}+5.69q^{-2}$, $R(q^{-1})=R'(q^{-1})A_d(q^{-1})B^s(q^{-1})=(1-q^{-1})\dfrac{0.1065+0.0902q^{-1}}{0.1065}$

$R(q^{-1})=1-0.1533q^{-1}-0.8467q^{-2}$

2.f)



2.h)



2.i)

Now $B^s(q^{-1})=1,\ B^u(q^{-1})=B(q^{-1}),\ A_d(q^{-1})=1-q^{-1}$

$$A_c\,'(q^{-1})=A_d(q^{-1})\,A(q^{-1})\,R\,'(q^{-1})+q^{-d}\,B^u(q^{-1})\,S(q^{-1})$$

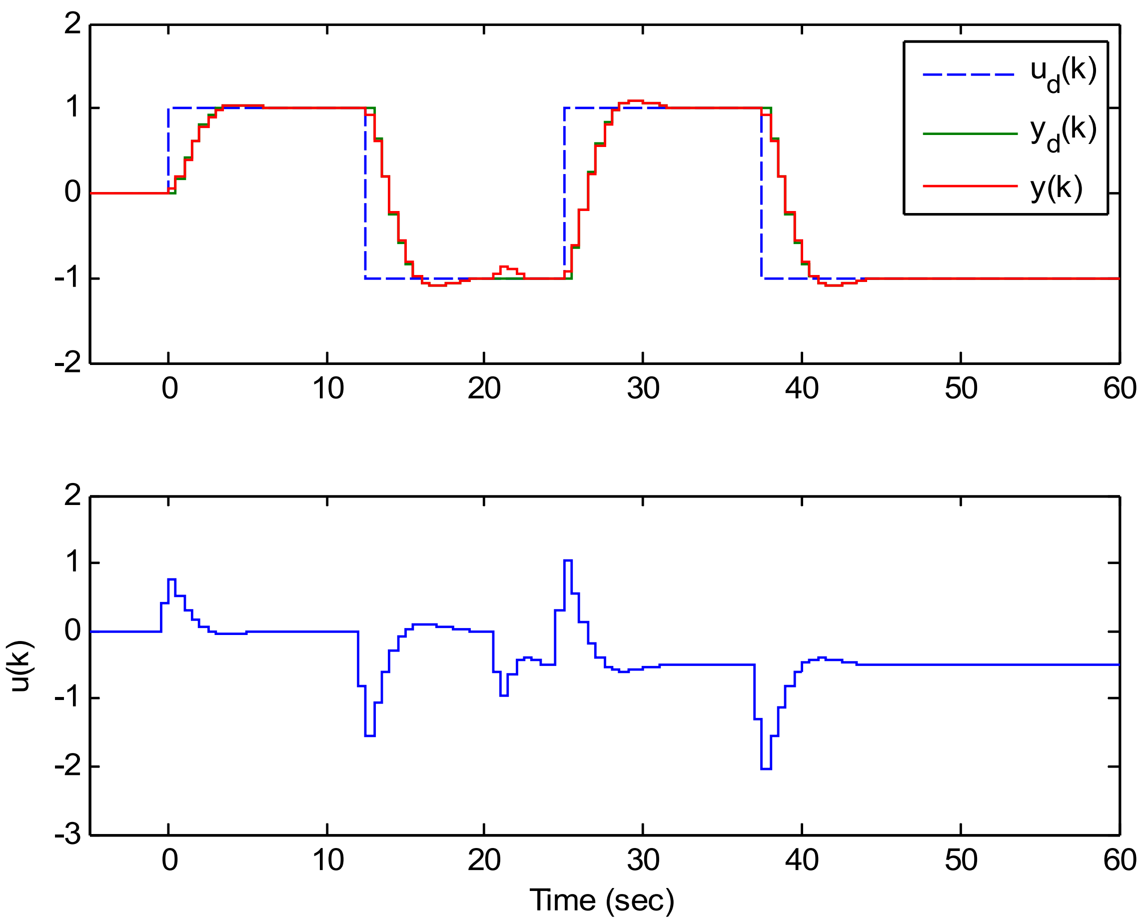
$$1-0.786\,q^{-1}+0.368\,q^{-2}=(1-q^{-1})(1-1.607\,q^{-1}+0.6065\,q^{-2})\,R\,'(q^{-1})+q^{-1}(0.1065+0.0902\,q^{-1})\,S(q^{-1})$$

$m_u = 1,\ d = 1,$ so $n_r{}' = d + m_u - 1 = 1$ therefore $R'(q^{-1}) = 1 + r_1{}' q^{-1}$
$n = 2,\ n_d = 1,\ n_c{}' = 2,$ so $n_s = \max(n + n_d - 1,\ n_c{}' - d - m_u) = 2,\ S(q^{-1}) = s_0 + s_1 q^{-1} + s_2 q^{-2}$
From bezout, $R'(q^{-1}) = 1 + 0.5935 q^{-1},\ S = 11.52 - 12.55 q^{-1} + 3.99 q^{-2}$
$R(q^{-1}) = R'(q^{-1}) A_d(q^{-1}) B^s(q^{-1}) = (1 + 0.5935 q^{-1})(1 - q^{-1}) = 1 - 0.4065 q^{-1} - 0.5935 q^{-2}$

Feedforward from zero-phase error tracking: $T(q^{-1}, q) = A_c{}'(q^{-1}) q^d \dfrac{B^u(q)}{[B^u(1)]^2}$

$$T(q^{-1}, q) = (1 - 0.786 q^{-1} + 0.368 q^{-2}) q \frac{0.1065 + 0.0902 q}{(0.1065 + 0.0902)^2} = 2.331 q^2 + 0.9208 q - 1.306 + 1.013 q^{-1}$$

2.j)
Transfer function from u_s to u is acausal, but we can simulate a delayed version then shift the result



2.k)
Other than the slight inconvenience of acausality, the results with the zero-phase feedforward were considerably cleaner. The control input was oscillatory when cancelling all the zeros, but well-behaved with the feed-forward and no cancelled zeros. That may be due to the fact that one of the plant zeros was -0.847, close to the unit circle.
We see that if disturbance is expected to be 0, then a non-zero disturbance introduces a constant offset between $y(k)$ and $y_d(k)$. If disturbance is expected to be constant, then a step causes a temporary mismatch but the controller compensates and returns after the disturbance is again constant. The value of the control input goes to a nonzero value which cancels the disturbance.