ME 233 Advance Control II

Lecture 19

Stability Analysis of a discrete time Series Parallel Least Squares Parameter Identification Algorithm

ARMA Model

Consider the following system

$$A(q^{-1})y(k) = q^{-\mathrm{d}}\,B(q^{-1})u(k)$$

Where

- $u(k)$ known ***bounded*** input
- $y(k)$ measured output

ARMA Model

$$A(q^{-1})y(k) = q^{-\mathrm{d}}\,B(q^{-1})u(k)$$

$$A(q^{-1}) = 1 + a_1 q^{-1} + \cdots + a_n q^{-n}$$ **(Schur)**

$$B(q^{-1}) = b_o + b_1 q^{-1} + \cdots + b_m q^{-m}$$

- Orders n and m are ***known***
- a's and b's are ***unknown*** but ***constant*** coefficients

ARMA Model

ARMA model can be written as:

$$\begin{aligned} y(k+1) &= -\sum_{i=1}^{n} a_i\, y(k-i+1) + \sum_{i=0}^{m} b_i\, u(k-i-d+1) \\ &= \theta^T\,\phi(k) \end{aligned}$$

Where:

$$\theta = \left[\begin{array}{cccccc} a_1 & \cdots & a_n & b_o \cdots & b_m \end{array}\right]^T \in \mathcal{R}^{n+m+1}$$

$$\phi(k) = \left[\begin{array}{cccccc} -y(k) & \cdots & -y(k-n) & u(k-d) & \cdots & u(k-m-d) \end{array}\right]^T$$

Series-parallel estimation model

A-posteriori series-parallel estimation model

$$\hat{y}(k+1) = -\sum_{i=1}^{n} \hat{a}_i(k+1)\, y(k-i+1) + \sum_{i=0}^{m} \hat{b}_i(k+1)\, u(k-i-d+1)$$

Where

- $\hat{y}(k)$ a-posteriori estimate of $y(k)$
- $\hat{a}_i(k)$ estimate of a_i at sampling time k
- $\hat{b}_i(k)$ estimate of b_i at sampling time k

Series-parallel estimation model

A-posteriori series-parallel estimation model

$$\hat{y}(k+1) = \hat{\theta}^T(k+1)\,\phi(k)$$

Where

- $\hat{y}(k)$ a-posteriori estimate of $y(k)$

$$\hat{\theta}(k) = \left[\ \hat{a}_1(k) \ \cdots \ \hat{a}_n(k) \ \ \hat{b}_o(k) \cdots \ \hat{b}_m(k)\ \right]^T$$

$$\phi(k) = \left[\ -y(k) \ \cdots \ -y(k-n) \ \ u(k-d) \ \cdots \ u(k-m-d)\ \right]^T$$

Series-parallel estimation model

A-priori series-parallel estimation model

$$\hat{y}^o(k+1) = \hat{\theta}^T(k)\,\phi(k)$$

Where

- $\hat{y}^o(k)$ a-priori estimate of $y(k)$

$$\hat{\theta}(k) = \left[\ \hat{a}_1(k) \ \cdots \ \hat{a}_n(k) \ \ \hat{b}_o(k) \cdots \ \hat{b}_m(k)\ \right]^T$$

$$\phi(k) = \left[\ -y(k) \ \cdots \ -y(k-n) \ \ u(k-d) \ \cdots \ u(k-m-d)\ \right]^T$$

Additional Notation

- Unknown parameter vector:

$$\theta = \left[\ a_1 \ \cdots \ a_n \ \ b_o \cdots \ b_m\ \right]^T \in \mathcal{R}^{n+m+1}$$

- Parameter vector estimate:

$$\hat{\theta}(k) = \left[\ \hat{a}_1(k) \ \cdots \ \hat{a}_n(k) \ \ \hat{b}_o(k) \cdots \ \hat{b}_m(k)\ \right]^T$$

- **Parameter error estimate**:

$$\boxed{\tilde{\theta}(k) = \theta - \hat{\theta}(k)}$$

- Regressor vector:

$$\phi(k) = \left[\ -y(k) \ \cdots \ -y(k-n) \ \ u(k-d) \ \cdots \ u(k-m-d)\ \right]^T$$

Additional Notation

- ***A-posteriori*** output estimation error:

$$e(k) \;=\; y(k) - \hat{y}(k)$$

- ***A-priori*** output estimation error:

$$e^o(k) \;=\; y(k) - \hat{y}^o(k)$$

Parameter Adaptation Algorithm (PAA)

- **A-posteriori version**

$$\hat{\theta}(k+1) = \hat{\theta}(k) + F(k)\,\phi(k)\,e(k+1)$$

$$e(k+1) = \frac{e^o(k+1)}{1+\phi(k)^T F(k)\phi(k)}$$

Parameter Adaptation Algorithm (PAA)

- **Gain update**

$$F(k+1) = \frac{1}{\lambda_1(k)}\left[F(k) - \frac{F(k)\phi(k)\phi^T(k)F(k)}{\frac{\lambda_1(k)}{\lambda_2(k)} + \phi^T(k)F(k)\phi(k)}\right]$$

$$F^{-1}(k+1) = \lambda_1(k)\,F^{-1}(k) + \lambda_2(k)\,\phi(k)\phi^T(k)$$

$$0 \;<\; \lambda_1(k) \leq 1 \qquad 0 \leq \lambda_2(k) < 2$$

Parameter Adaptation Algorithm (PAA)

- **Gain update**

$$F(k+1) = \frac{1}{\lambda_1(k)}\left[F(k) - \frac{F(k)\phi(k)\phi^T(k)F(k)}{\frac{\lambda_1(k)}{\lambda_2(k)} + \phi^T(k)F(k)\phi(k)}\right]$$

$$F^{-1}(k+1) = \lambda_1(k)\,F^{-1}(k) + \lambda_2(k)\,\phi(k)\phi^T(k)$$

$$\lambda_1(k) = 1 \qquad \lambda_2(k) = 1$$

Least Squares PAA

Parameter Adaptation Algorithm (PAA)

- **Gain update**

$$F(k+1) = \frac{1}{\lambda_1(k)} \left[F(k) - \frac{F(k)\phi(k)\phi^T(k)F(k)}{\frac{\lambda_1(k)}{\lambda_2(k)} + \phi^T(k)F(k)\phi(k)} \right]$$

$$F^{-1}(k+1) = \lambda_1(k)\, F^{-1}(k) + \lambda_2(k)\, \phi(k)\phi^T(k)$$

$0 < \lambda_1(k) < 1$

$\lambda_2(k) = 1$

Least Squares with forgetting

Example

- Plant:

$$y(k) = \frac{q^{-1}\, 0.1(1+0.5q^{-1})}{(1+0.9q^{-1})(1+0.8q^{-1})} u(k)$$

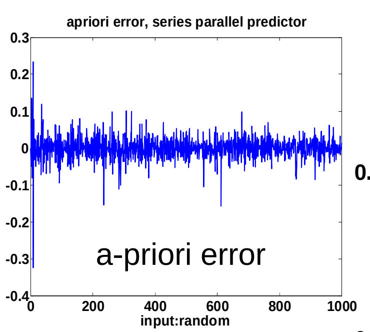
$$y(k+1) \; = \; \theta^T \phi(k)$$

$$\theta = \begin{bmatrix} 1.7 \\ 0.72 \\ 0.1 \\ 0.05 \end{bmatrix} \qquad \phi(k) = \begin{bmatrix} -y(k) \\ -y(k-1) \\ u(k) \\ u(k-1) \end{bmatrix}$$

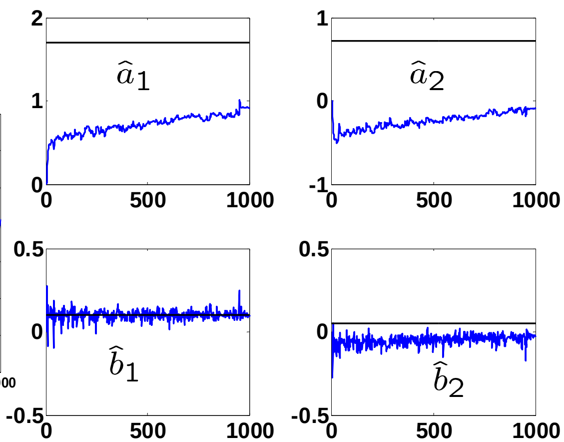
Example: Constant gain

u(k) : zero mean uniform white noise between [-1,1]

$F = 100 * I_4$



parameter convergence

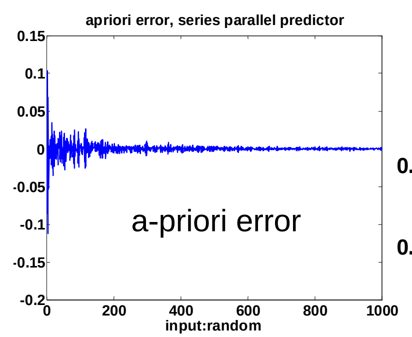


Example: Least Squares

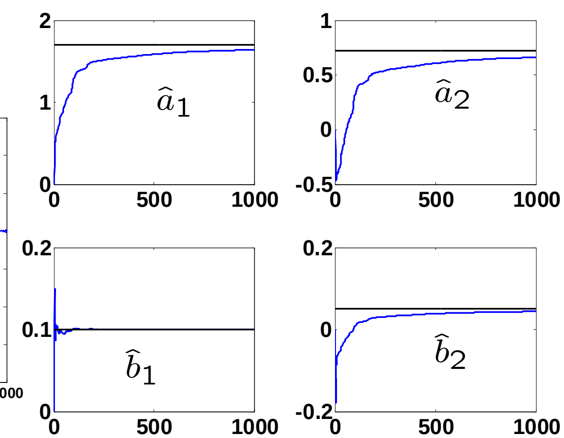
u(k) : zero mean uniform white noise between [-1,1]

$F(0) = 100 * I_4$

$\lambda_1 = 1 \quad \lambda_2 = 1$



parameter convergence



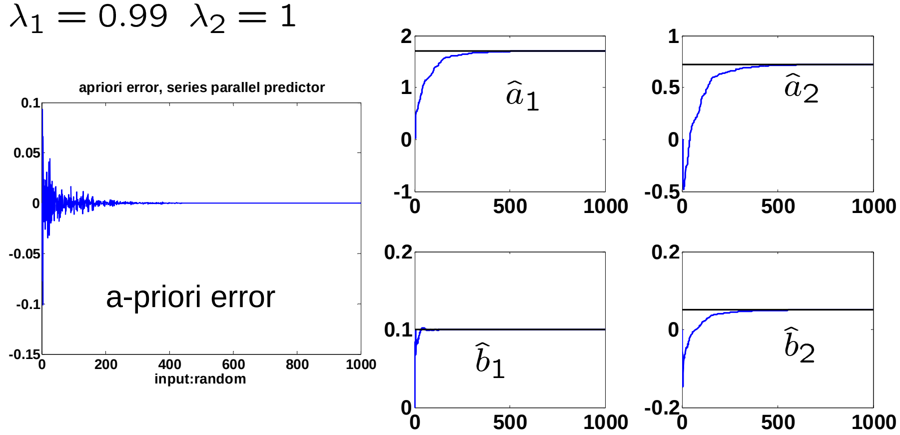
Example: Least Squares & forgetting factor

u(k) : zero mean uniform white noise between [-1,1]

$$F(0) = 100 * I_4$$

$$\lambda_1 = 0.99 \quad \lambda_2 = 1$$

parameter convergence



Theorem

Under the following conditions:

1. The input *u(k)* is bounded, I.e. $|u(k)| < \infty$
2. $A(q^{-1})$ is Schur
3. Maximum singular value of *F(k)* is uniformly bounded

$$0 < \lambda_{\max}\{F(k)\} < K_{max} < \infty \, .$$

$$\boxed{\lim_{k\to\infty} e(k) = 0 \quad \text{and} \quad \lim_{k\to\infty} e^o(k) = 0}$$

Parameter Adaptation Algorithm (PAA)

Since the unknown parameters are constant and

$$\tilde{\theta}(k) = \theta - \hat{\theta}(k)$$

Then, the PAA

$$\hat{\theta}(k+1) = \hat{\theta}(k) + F(k)\,\phi(k)\,e(k+1)$$

Implies that

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) - F(k)\,\phi(k)\,e(k+1)$$

A-posteriori error dynamics

- Plant

$$y(k+1) = \theta^T \phi(k)$$

- A-posteriori model

$$\hat{y}(k+1) \;=\; \hat{\theta}^T(k+1)\,\phi(k)$$

- A-posteriori output error

$$e(k+1) \;=\; \tilde{\theta}^T(k+1)\,\phi(k)$$

A-posteriori dynamics

- Error dynamics

$$e(k+1) \;=\; \tilde{\theta}^T(k+1)\phi(k)$$

- PAA

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) - F(k)\,\phi(k)\,e(k+1)$$

$$F(k+1) = \frac{1}{\lambda_1(k)}\left[F(k) - \frac{F(k)\phi(k)\phi^T(k)F(k)}{\frac{\lambda_1(k)}{\lambda_2(k)} + \phi^T(k)F(k)\phi(k)}\right]$$

A-posteriori dynamics

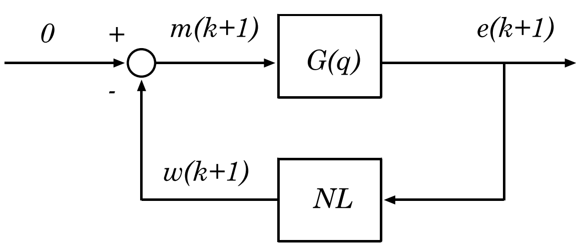
- Error dynamics

$$e(k+1) \;=\; \underbrace{\tilde{\theta}^T(k+1)\phi(k)}_{m(k+1)}$$

- Define:

$$w(k+1) \;=\; -m(k+1)$$

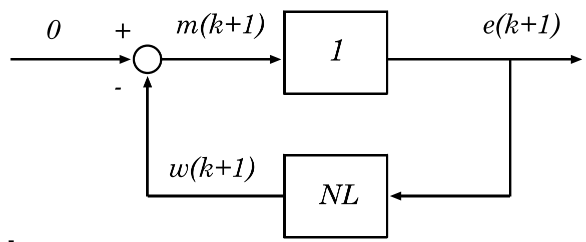
Equivalent Feedback Loop



$$m(k+1) \;=\; \tilde{\theta}^T(k+1)\phi(k)$$

$$w(k+1) \;=\; -m(k+1)$$

Equivalent Feedback Loop



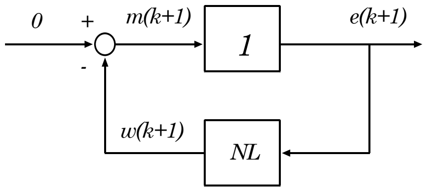
NL:

$$\tilde{\theta}(k+1) \;=\; \tilde{\theta}(k) - F(k)\,\phi(k)\,e(k+1)$$

$$F(k+1) = \frac{1}{\lambda_1(k)}\left[F(k) - \frac{F(k)\phi(k)\phi^T(k)F(k)}{\frac{\lambda_1(k)}{\lambda_2(k)} + \phi^T(k)F(k)\phi(k)}\right]$$

$$w(k+1) \;=\; -\phi(k)^T\tilde{\theta}(k+1)$$

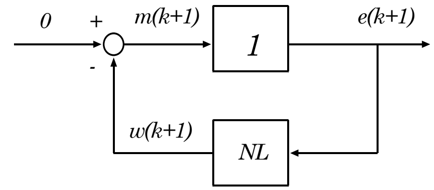
Stability analysis using Hyperstability



1. Verify that the LTI dynamics is SPR

2 Verify that the PAA dynamics is P-class

Good News: LTI “***very***” SPR



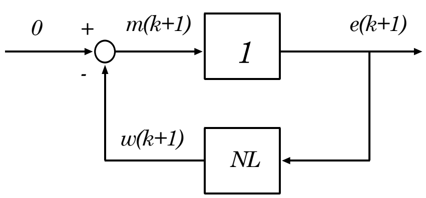
1. Verify that the LTI dynamics is SPR

$$e(k+1) \;=\; m(k+1)$$

$$\boxed{G(z) \;=\; 1}$$

Always SPR

Bad News: NL is ***not*** P-class



- Unfortunately the NL block is **not** P-class

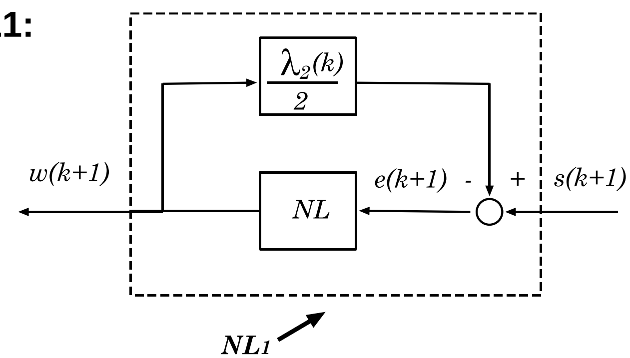
NL:

$$\left\{\begin{aligned} \tilde{\theta}(k+1) \;&=\; \tilde{\theta}(k) - F(k)\,\phi(k)\,e(k+1) \\ F(k+1) &= \frac{1}{\lambda_1(k)}\left[F(k) - \frac{F(k)\phi(k)\phi^T(k)F(k)}{\frac{\lambda_1(k)}{\lambda_2(k)} + \phi^T(k)F(k)\phi(k)}\right] \\ w(k+1) \;&=\; -\phi(k)^T\tilde{\theta}(k+1) \end{aligned}\right.$$

Solution: Modify the NL block

- **Add a feedback term to NL to make it P-class**

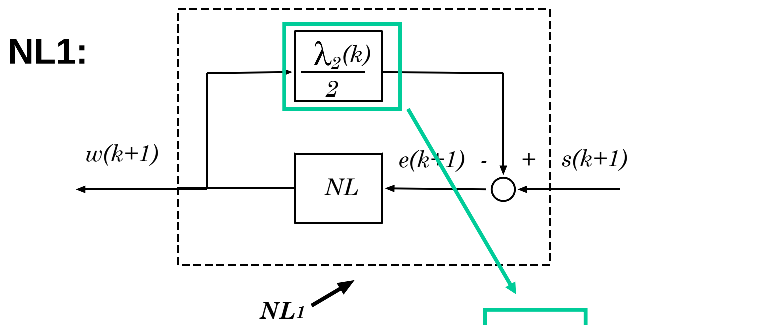
NL1:



$$\sum_{j=0}^{k} w(j)s(j) \geq -\gamma_o^2$$

Modifying the NL block

- **Add a feedback term to NL to make it P-class**

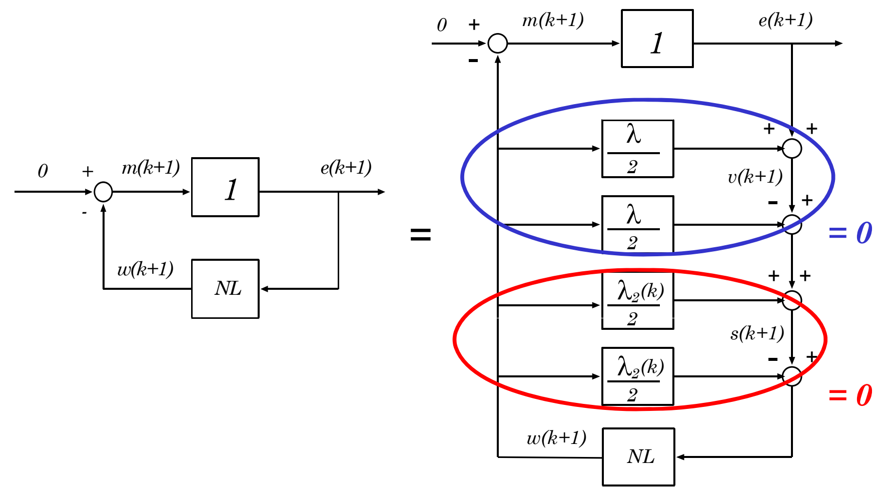


$$F^{-1}(k+1) = \lambda_1(k)\, F^{-1}(k) + \boxed{\lambda_2(k)}\, \phi(k)\phi^T(k)$$

Proof: Pages 27 and 28
ME233 Identification & Adaptive Control Notes

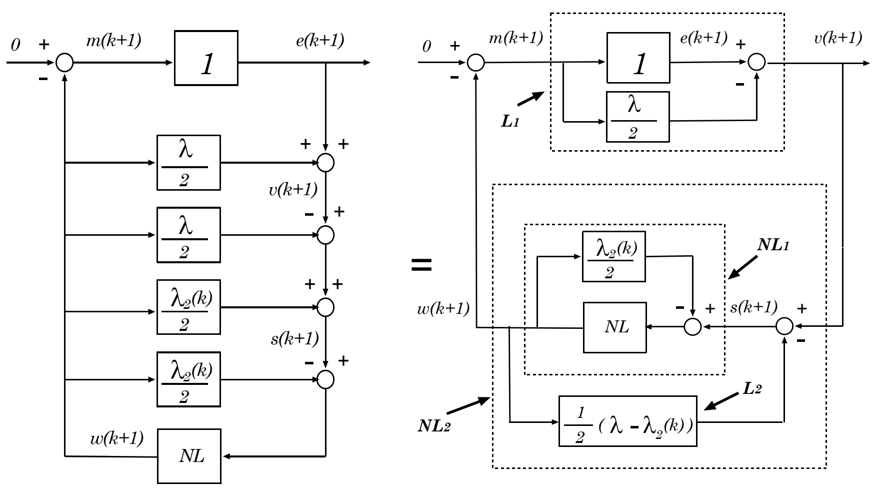
What happens to the feedback structure?

- **Add and subtract the same blocks:**



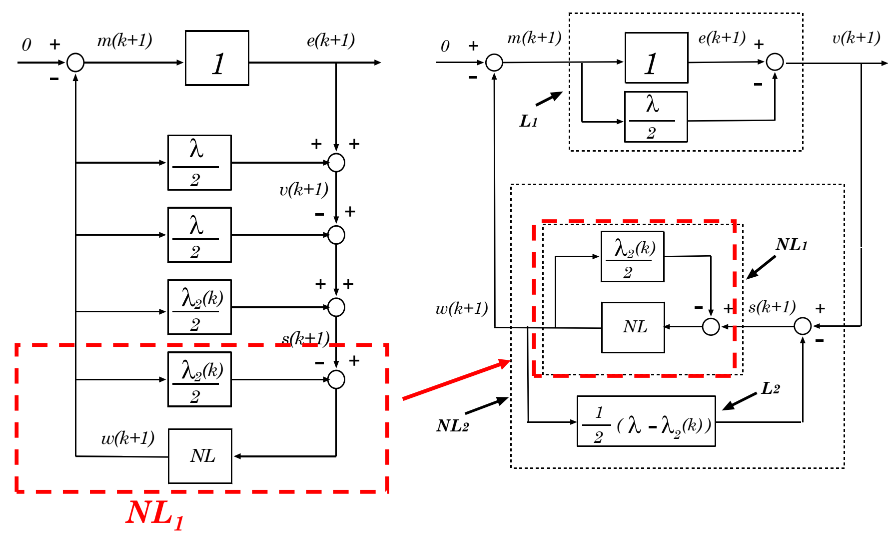
What happens to the feedback structure?

- **Rearranging blocks,**



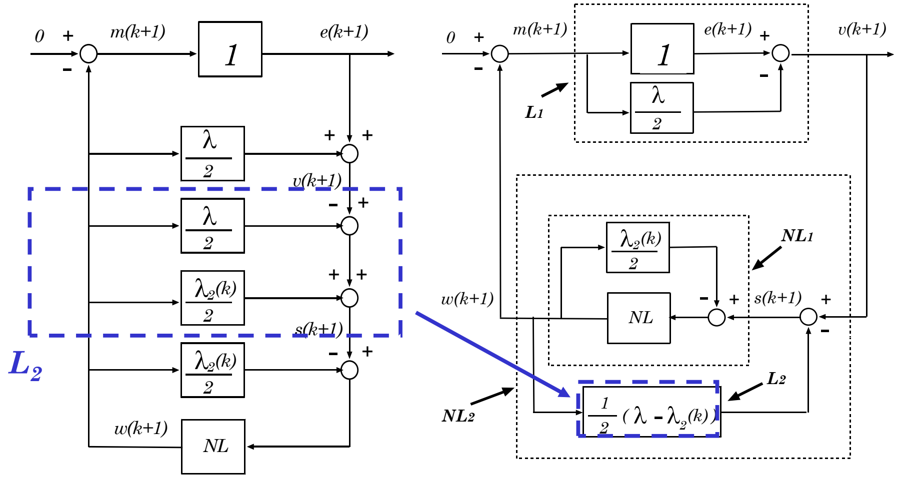
What happens to the feedback structure?

- **Rearranging blocks,**



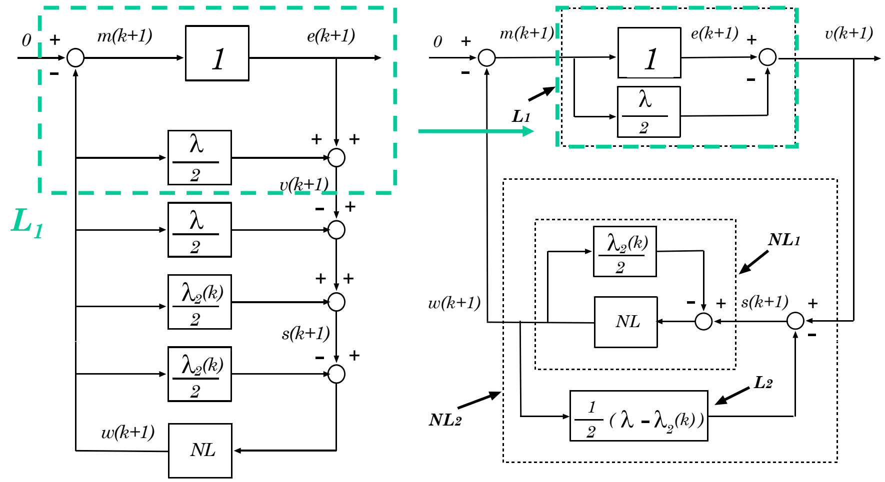
What happens to the feedback structure?

- **Rearranging blocks,**

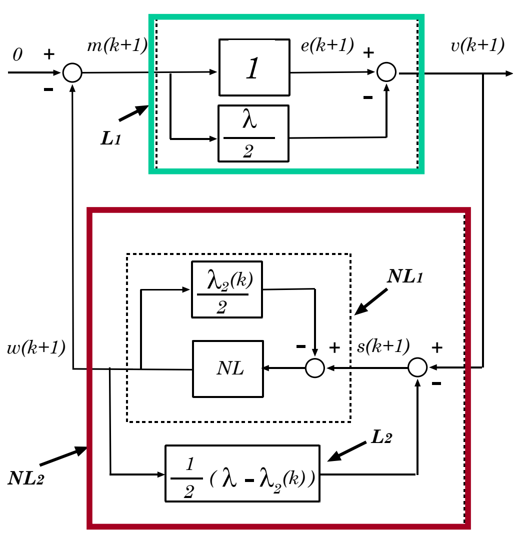


What happens to the feedback structure?

- **Rearranging blocks,**



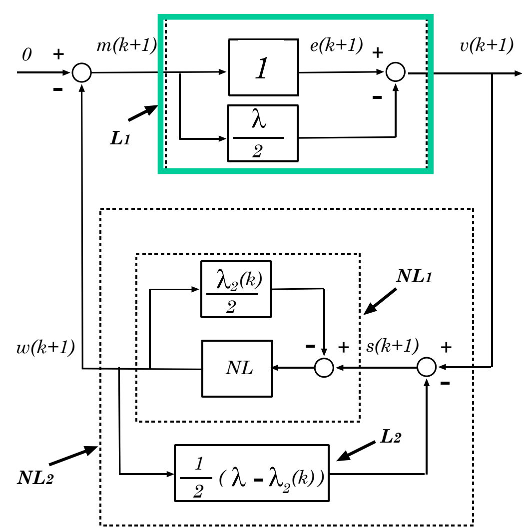
Can we now use Hyperstability Theory?



For Asymptotic Hyperstability:

1. L_1 **must be SPR**
2. NL_2 **must be P-class**

Linear Block L_1



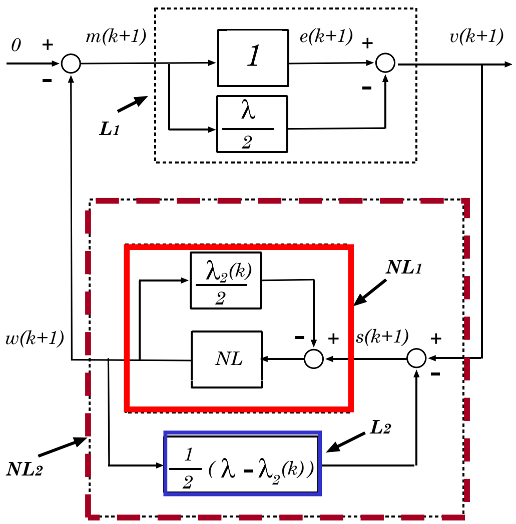
Since:

L_1: $1 - \frac{\lambda}{2}$

L_1 is SPR **iff**

$\lambda < 2$

Nonlinear Block NL_2



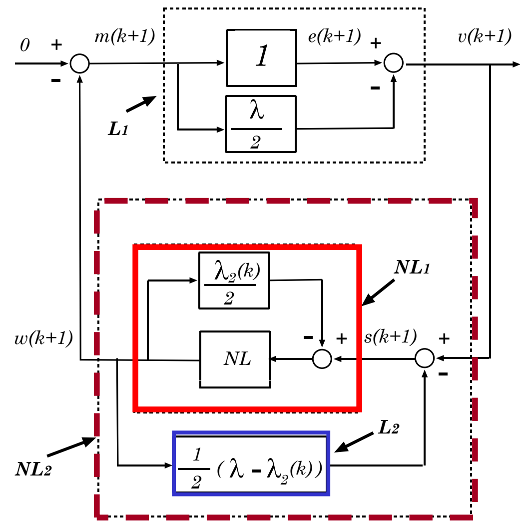
NL_2 :

Feedback combination of two blocks:

1. ***NL_1*** **: P-class**
2. ***L_2***

must be P-class

Nonlinear Block NL_2



NL_2 :

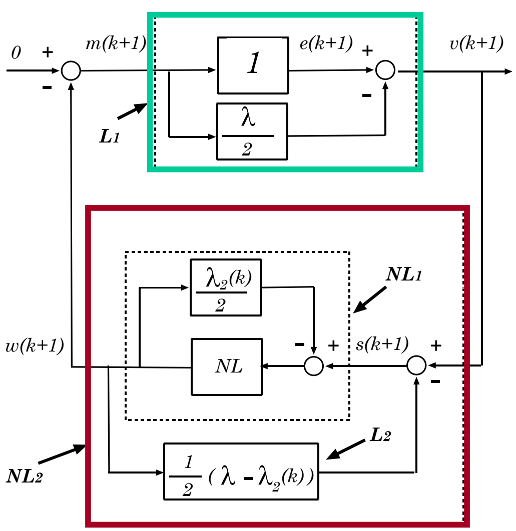
Feedback combination of two blocks:

1. ***NL_1*** **: P-class**
2. ***L_2*** **: must be P-class**

L_2: $\frac{1}{2}[\lambda - \lambda_2(k)]$

L_2 is P-class **iff** $\lambda_2(k) < 2$

Hyperstability Theorem



Iff

$$0 \leq \lambda_2(k) < 2$$

1. ***L_1*** **is SPR**
2. ***NL_2*** **is P-class**

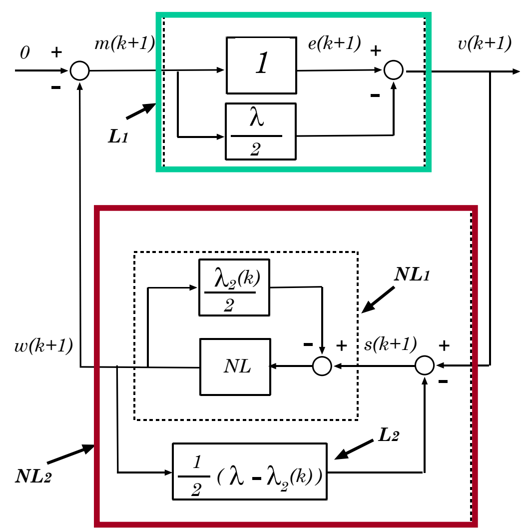
Therefore:

$$|v(k)| < \infty$$

$$|e(k)| < \infty$$

$$|m(k)| < \infty$$

Asymptotic Hyperstability Theorem



Since

1. ***L_1*** **is SPR**
2. ***NL_2*** **is P-class**

Therefore:

$$e(k) \rightarrow 0$$

$$\hat{y}(k) \rightarrow y(k)$$

A-posteriori error convergence

We have concluded that the a-posteriori output error converges to zero:

$$\lim_{k\to\infty} e(k) = 0$$

where

$$\begin{aligned} e(k) &= y(k) - \widehat{y}(k) \\ &= \tilde{\theta}(k)^T \phi(k) \end{aligned}$$

What about the a-priori output error?

A-posteriori error convergence

Notice that

$$e(k) = \frac{e^o(k)}{1 + \phi(k-1)^T F(k) \phi(k-1)}$$

Therefore, $\lim_{k\to\infty} e(k) = 0$ does not necessarily imply that $\lim_{k\to\infty} e^o(k) = 0$

To do so, we need to first show that

$$|\phi(k)| < \infty \qquad \forall\, k \geq 0$$

Bondedness of the regressor vector

Remember that:

$$\phi(k) = \left[\begin{array}{cccccc} -y(k) & \cdots & -y(k-n) & u(k-d) & \cdots & u(k-m-d) \end{array}\right]^T$$

Therefore,

$$|\phi(k)|^2 = \sum_{i=0}^{n} y^2(k-i) + \sum_{j=0}^{m} u^2(k-j-d)$$

By assumption,

$$|u(k)| < \infty \qquad \forall\, k \geq 0$$

Bondedness of the regressor vector

Since

$$|\phi(k)|^2 = \sum_{i=0}^{n} y^2(k-i) + \sum_{j=0}^{m} u^2(k-j)$$

and,

$$|u(k)| < \infty \qquad \forall\, k \geq 0$$

Therefore, we need to show that

$$|y(k)| < \infty \qquad \forall\, k \geq 0$$

Bondedness of the regressor vector

Remember that:

$$y(k) = \frac{B(q^{-1})}{A(q^{-1})}u(k)$$

and,

$A(q^{-1})$ is Schur,

Therefore LTI system is BIBO

$$G(q^{-1}) = \frac{B(q^{-1})}{A(q^{-1})}$$

Thus,

$$|u(k)| < \infty \Rightarrow |y(k)| < \infty$$

Q.E.D.