ME 233 Advance Control II

Lecture 23

Stability Analysis of a Direct Adaptive Control System

Deterministic SISO ARMA models

SISO ARMA model

$$A(q^{-1})\, y(k) = q^{-\mathrm{d}}\, B(q^{-1})\, u(k)$$

Where all inputs and outputs are scalars:

- $u(k)$ control input
- $y(k)$ output

d is the ***known*** pure time delay

Deterministic SISO ARMA models

$$A(q^{-1})\, y(k) = q^{-\mathrm{d}}\, B(q^{-1})\, u(k)$$

Where polynomials:

$$A(q^{-1}) \;=\; 1 + a_1 q^{-1} + \cdots + a_n q^{-n}$$

$$B(q^{-1}) \;=\; b_o + b_1 q^{-1} + \cdots + b_m q^{-m}$$

are co-prime

$\boxed{\text{and } B(q^{-1}) \text{ is Hurwitz}}$

Control Objectives

1. Pole Placement: The poles of the closed loop system must be placed at specific locations in the complex plane.

- **Closed loop pole polynomial:**

$$\boxed{A_c(q^{-1}) = B(q^{-1})\, A_c^{'}(q^{-1})}$$

Where:

- $B(q^{-1})$ cancelable plant zeros
- $A_c^{'}(q^{-1})$ monic Hurwitz polynomial chosen by the designer

$$A_c^{'}(q^{-1}) = 1 + a_{c1}^{'} q^{-1} + \cdots + a_{c_{n_c^{'}}}^{'} q^{-n_c^{'}}$$

Control Objectives

2. Tracking: The output sequence $y(k)$ must follow a ***reference*** sequence $y_d(k)$ which is known

- **Reference model:**

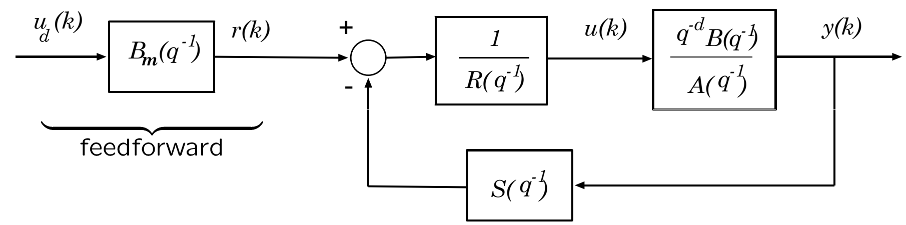
$$\boxed{A_c'(q^{-1})y_d(k) = q^{-d}\, B_m(q^{-1})\, u_d(k)}$$

Where:

- $u_d(k)$ **known** reference input control input sequence
- $A_c'(q^{-1})$ monic Hurwitz polynomial chosen by the designer
- $B_m(q^{-1})$ zero polynomial, chosen by the designer

Control Law

- Feedback and feedforward actions:



$$u(k) = \frac{1}{R(q^{-1})}\left[r(k) - S(q^{-1})y(k)\right]$$

$$r(k) = B_m(q^{-1})\, u_d(k)$$ *Feedforward (causal)*

Feedback Controller

Diophantine equation: Obtain polynomials $\underline{R'(q^{-1})}$, $\underline{S(q^{-1})}$ which satisfy:

$$\boxed{A_c'(q^{-1}) = A(q^{-1})\, \underline{R'(q^{-1})} + q^{-d}\, \underline{S(q^{-1})}}$$

Close loop poles *Plant poles* *Plant pure delays*

$$\boxed{\begin{aligned} R(q^{-1}) &= R'(q^{-1})\, B(q^{-1}) \\ A_c(q^{-1}) &= B(q^{-1})\, A_c'(q^{-1}) \end{aligned}}$$

Controller parameters

Start with the Diophantine equation

$$A_c'(q^{-1}) = A(q^{-1})\, R'(q^{-1}) + q^{-d}\, S(q^{-1})$$

Multiply both sides by *y(k)*

$$A_c'(q^{-1})\, y(k) = R'(q^{-1})\, A(q^{-1})\, y(k) + q^{-d}\, S(q^{-1})\, y(k)$$

Controller parameters

$$A_c'(q^{-1})\,y(k) = R'(q^{-1})\,\underline{A(q^{-1})\,y(k)} + q^{-\mathrm{d}}\,S(q^{-1})\,y(k)$$

Insert plant dynamics

$$A(q^{-1})\,y(k) = q^{-\mathrm{d}}\,B(q^{-1})\,u(k)$$

$$A_c'(q^{-1})\,y(k) = q^{-\mathrm{d}}\left[\underline{R'(q^{-1})\,B(q^{-1})}\,u(k) + S(q^{-1})\,y(k)\right]$$

$$\boxed{A_c'(q^{-1})\,y(k) = q^{-\mathrm{d}}\left[R(q^{-1})\,u(k) + S(q^{-1})\,y(k)\right]}$$

Pole placement

Key Idea: Parameterize plant close-loop dynamics in terms of the controller parameters

$$\underbrace{A_c'(q^{-1})}_{\text{Desired close loop polynomial (not including zeros)}}\,y(k) = q^{-\mathrm{d}}\left[R(q^{-1})\,u(k) + S(q^{-1})\,y(k)\right]$$

Desired close loop polynomial (not including zeros)

- Plant must have minimum phase zeros

$B(q^{-1})$ must be Hurwitz

Controller parameters and regressor

$$\underbrace{A_c'(q^{-1})\,y(k)}_{\eta(k)} = q^{-\mathrm{d}}\underbrace{\left[R(q^{-1})\,u(k) + S(q^{-1})\,y(k)\right]}$$

$$\underbrace{r_o u(k) + \cdots + r_{n_r} u(k-n_r) + s_o y(k) + \cdots + s_{n_s} y(k-n_s)}$$

$$\begin{bmatrix} s_o \cdots s_{n_s}\, r_o & \cdots & r_{n_r} \end{bmatrix}^T \begin{bmatrix} y(k) \\ \cdots \\ y(k-n_s) \\ u(k) \\ \cdots \\ u(k-n_r) \end{bmatrix}$$

Controller parameters and regressor

$$\underbrace{A_c'(q^{-1})\,y(k)}_{\eta(k)} = q^{-\mathrm{d}}\underbrace{\left[R(q^{-1})\,u(k) + S(q^{-1})\,y(k)\right]}$$

$$\underbrace{\begin{bmatrix} s_o \cdots s_{n_s}\, r_o & \cdots & r_{n_r} \end{bmatrix}^T}_{\theta_c^T} \underbrace{\begin{bmatrix} y(k) \\ \cdots \\ y(k-n_s) \\ u(k) \\ \cdots \\ u(k-n_r) \end{bmatrix}}_{\phi(k)}$$

$$\boxed{\eta(k) \;=\; \phi^T(k-d)\theta_c}$$

Control Objective

$$\underbrace{A_c'(q^{-1})\,y(k)}_{\eta(k)} = q^{-\mathrm{d}}\underbrace{\Big[R(q^{-1})\,u(k)+S(q^{-1})\,y(k)\Big]}_{\underbrace{A_c'(q^{-1})\,y_d(k)}_{\eta_d(k)}}$$

Such that

$$\boxed{\eta(k) = \eta_d(k)}$$

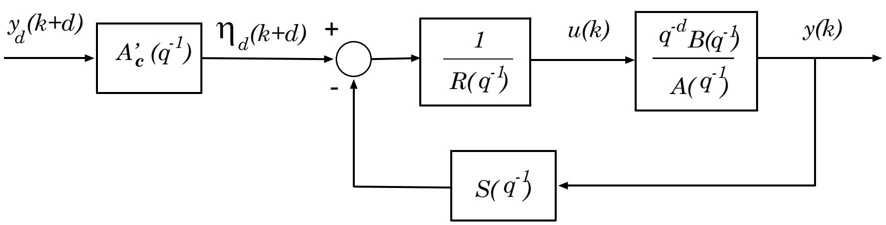
Control Law

$$A_c'(q^{-1})\,y(k) = q^{-\mathrm{d}}\underbrace{\Big[R(q^{-1})\,u(k)+S(q^{-1})\,y(k)\Big]}_{\underbrace{A_c'(q^{-1})\,y_d(k)}_{\eta_d(k)}}$$

Control law:

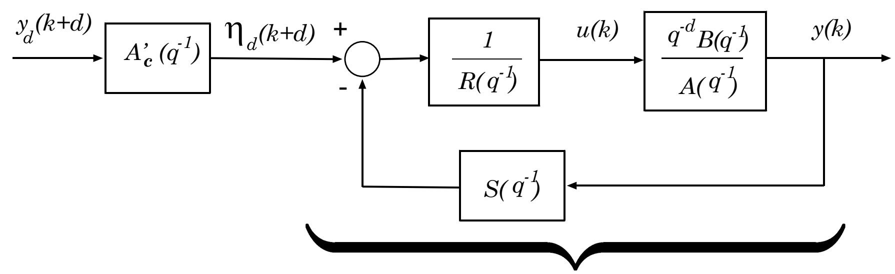
$$\Big[R(q^{-1})\,u(k-d)+S(q^{-1})\,y(k-d)\Big] = \eta_d(k)$$

Control Law



$$\underbrace{\Big[R(q^{-1})\,u(k-d)+S(q^{-1})\,y(k-d)\Big]}_{A_c'(q^{-1})y(k)=\eta(k)} = \eta_d(k)$$

Close loop dynamics



$$\frac{q^{-d}}{A_c(q^{-1})}$$

$$\underbrace{A_c'(q^{-1})\,y(k)}_{\eta(k)} = \underbrace{A_c'(q^{-1})\,y_d(k)}_{\eta_d(k)}$$

Control Law Implementation

$$A_c'(q^{-1})\,y(k) = \underbrace{q^{-\mathrm{d}}\left[R(q^{-1})\,u(k) + S(q^{-1})\,y(k)\right]}_{A_c'(q^{-1})\,y_d(k)}$$

Control law:

$$R(q^{-1})\,u(k) = \underbrace{A_c{}'(q^{-1})y_d(k+d)}_{B_m(q^{-1})\,u_d(k) \quad = r(k)} - S(q^{-1})\,y(k)$$

$$\boxed{R(q^{-1})\,u(k) = r(k) - S(q^{-1})\,y(k)}$$

PAA

$$\underbrace{A_c'(q^{-1})\,y(k)}_{\eta(k)} = \underbrace{q^{-\mathrm{d}}\left[R(q^{-1})\,u(k) + S(q^{-1})\,y(k)\right]}$$

$$\Downarrow$$

$$\underbrace{r_o u(k) + \cdots + r_{n_r} u(k-n_r) + s_o y(k) + \cdots + s_{n_s} y(k-n_s)}$$

$$\begin{bmatrix} s_o \cdots s_{n_s}\, r_o \;\cdots\; r_{n_r} \end{bmatrix}^T \begin{bmatrix} y(k) \\ \cdots \\ y(k-n_s) \\ u(k) \\ \cdots \\ u(k-n_r) \end{bmatrix}$$

PAA

$$\underbrace{A_c'(q^{-1})\,y(k)}_{\eta(k)} = \underbrace{q^{-\mathrm{d}}\left[R(q^{-1})\,u(k) + S(q^{-1})\,y(k)\right]}$$

$$\Downarrow$$

$$\underbrace{\begin{bmatrix} s_o \cdots s_{n_s}\, r_o \;\cdots\; r_{n_r} \end{bmatrix}^T}_{\theta_c^T} \underbrace{\begin{bmatrix} y(k) \\ \cdots \\ y(k-n_s) \\ u(k) \\ \cdots \\ u(k-n_r) \end{bmatrix}}_{\phi(k)}$$

$$\boxed{\eta(k) \;=\; \phi^T(k-d)\theta_c}$$

PAA

Plant dynamics:

$$\boxed{\eta(k) \;=\; \phi^T(k-d)\theta_c}$$

RLS PAA:

$$\boxed{e^o(k) \;=\; \eta(k) - \phi^T(k-d)\hat{\theta}_c(k-1)}$$

$$e(k+1) \;=\; \frac{e^o(k+1)}{1+\phi^T(k-d+1)F(k)\phi(k-d+1)}$$

$$\hat{\theta}_c^o(k+1) \;=\; \hat{\theta}_c(k) + F(k)\phi(k-d+1)\;e(k+1)$$

$$F(k+1) \;=\; \frac{1}{\lambda_1(k)}\left[F(k) - \frac{F(k)\phi(k-d+1)\phi^T(k-d+1)F(k)}{\frac{\lambda_1(k)}{\lambda_2(k)} + \phi^T(k-d+1)F(k)\phi(k-d+1)}\right]$$

PAA projection

PAA: Projection

$$\hat{\theta}_c(k) = \begin{cases} \hat{\theta}^o_c(k) & \text{if } \hat{r}^o_o(k) \geq b_{mino} \\ \left[\ \hat{s}^o_o(k) \cdots \hat{s}^o_{n_s}(k)\ b_{mino} \cdots \hat{r}^o_{n_r}(k)\ \right]^T & \text{if } \hat{r}^o_o(k) < b_{mino} \end{cases}$$

$\hat{r}_o(k)$

Replace $\hat{r}^o_o(k)$ by b_{mino} if it becomes too small.

Control law will divide by $\hat{r}_o(k)$. Thus, the projection algorithm prevents the control action from becoming too large.

PAA Gain matrix

Gain matrix:

$$F(k+1) = \frac{1}{\lambda_1(k)}\left[F(k) - \frac{F(k)\phi_f(k-d+1)\phi_f^T(k-d+1)F(k)}{\frac{\lambda_1(k)}{\lambda_2(k)} + \phi_f^T(k-d+1)F(k)\phi_f(k-d+1)}\right]$$

$$0 < \lambda_1(k) \leq 1$$
$$0 \leq \lambda_2(k) < 2$$

are adjusted so that the maximum singular value of *F(k)* is uniformly bounded, and

$$0 < K_{\min} \leq \lambda_{\min}\{F(k)\} \leq \lambda_{\max}\{F(k)\} < K_{max} < \infty\,.$$

A-priori and a-posteriori errors

Plant dynamics:

$$\boxed{\eta(k) = \phi^T(k-d)\theta_c}$$

A-priori estimation error

$$e^o(k) = \underbrace{\eta(k)}_{\phi^T(k-d)\theta_c} - \phi^T(k-d)\hat{\theta}_c(k-1)$$

$$\boxed{e^o(k) = \phi^T(k-d)\tilde{\theta}_c(k-1)}$$

A-priori and a-posteriori errors

Plant dynamics:

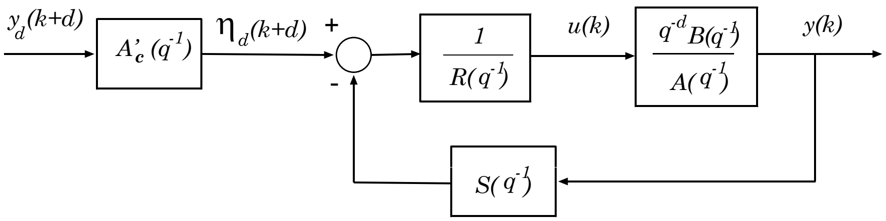
$$\boxed{\eta(k) = \phi^T(k-d)\theta_c}$$

A-posteriori estimation error

$$e(k) = \underbrace{\eta(k)}_{\phi^T(k-d)\theta_c} - \phi^T(k-d)\hat{\theta}_c(k)$$

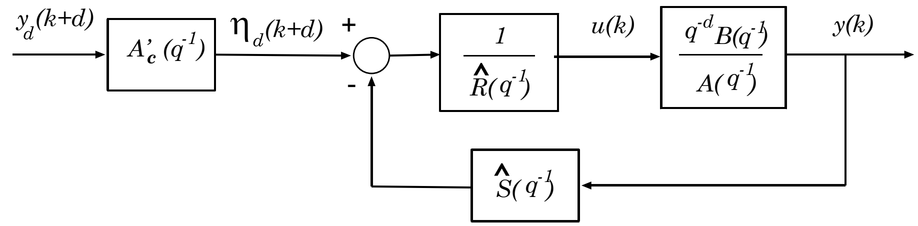
$$\boxed{e(k) = \phi^T(k-d)\tilde{\theta}_c(k)}$$

Control Law – Known Parameters



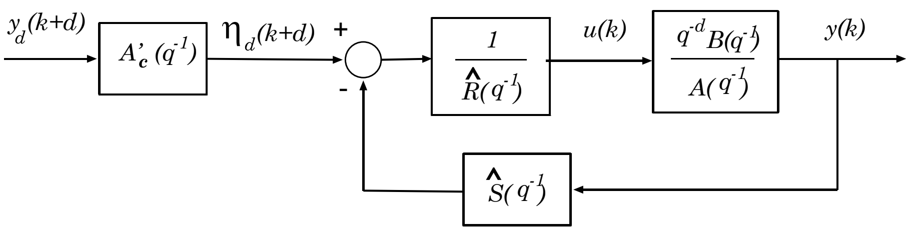
$$\eta_d(k+d) = \underbrace{\left[R(q^{-1})\,u(k) + S(q^{-1})\,y(k)\right]}_{\phi^T(k)\theta_c \;=\; \eta(k+d)}$$

Adaptive Control Law



$$\eta_d(k+d) = \underbrace{\left[\widehat{R}(q^{-1},k)\,u(k) + \widehat{S}(q^{-1},k)\,y(k)\right]}_{\phi^T(k)\widehat{\theta}_c(k)}$$

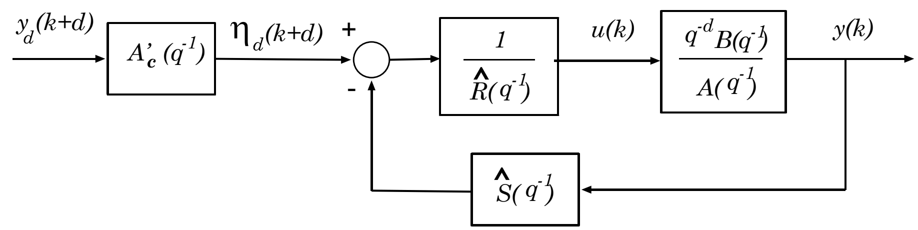
Adaptive Control Law



$$\eta_d(k+d) = \underbrace{\phi^T(k)\widehat{\theta}_c(k)}$$

not necessarily $= \underbrace{\eta(k+d)}_{\phi^T(k)\theta_c}$ ***why?***

Adaptive Control Objective



Filter tracking error:

$$\epsilon(k) = \eta(k) - \eta_d(k)$$

$$\boxed{\lim_{k\to\infty} \epsilon(k) = 0}$$

Filter error Dynamics

$$\epsilon(k) = \underbrace{\underbrace{\eta(k)}_{\phi^T(k-d)\theta_c} \quad \underbrace{-\eta_d(k)}_{-\phi^T(k-d)\hat{\theta}_c(k-d)}}_{\phi^T(k-d)\underbrace{\left[\theta_c - \hat{\theta}_c(k-d)\right]}_{\tilde{\theta}_c(k-d)}}$$

$$\boxed{\epsilon(k) = \phi^T(k-d)\tilde{\theta}_c(k-d)}$$

Adaptive Control with **Constant Gain**

Adaptive control algorithm:

1. $\eta(k) = A_c^{'}(q^{-1})\, y(k)$ **filtered output signal**

2. $e^o(k) \;=\; \eta(k) - \phi^T(k-d)\hat{\theta}_c(k-1)$ **a-priori error**

3. $$\left.\begin{aligned} e(k) &= \frac{e^o(k)}{1+\phi^T(k-d)F\phi(k-d)} \\ \hat{\theta}_c(k) &= \hat{\theta}_c(k-1) + F\phi(k-d)\; e(k) \end{aligned}\right\}\textbf{PAA}$$

4. $\phi^T(k)\hat{\theta}_c(k) = \eta_d(k+d)$ **control action**

Notes on error terms

- **Parameter** error vector: $\tilde{\theta}_c(k) = \theta_c - \hat{\theta}_c(k)$
- **A-posteriori** output error:

$$e(k) \;=\; \phi(k-d)^T\,\tilde{\theta}_c(k)$$

- **A-priori** output error:

$$e^o(k) \;=\; \phi(k-d)^T\,\tilde{\theta}_c(k-1)$$

- **Tracking** filter error:

$$\epsilon(k) \;=\; \phi(k-d)^T\tilde{\theta}_c(k-d)$$

Adaptive Control with **RLS**

Adaptive control algorithm:

1. $\eta(k) = A_c^{'}(q^{-1})\, y(k)$ **filtered output signal**

2. $e^o(k) \;=\; \eta(k) - \phi^T(k-d)\hat{\theta}_c(k-1)$ **a-priori error**

3. **PAA**

$$\begin{aligned} e(k) &= \frac{e^o(k)}{1+\phi^T(k-d)F(k-1)\phi(k-d)} \\ \hat{\theta}_c(k) &= \hat{\theta}_c(k-1) + F(k-1)\phi(k-d)\; e(k) \\ F(k+1) &= \frac{1}{\lambda_1(k)}\left[F(k) - \frac{F(k)\phi(k-d+1)\phi^T(k-d+1)F(k)}{\frac{\lambda_1(k)}{\lambda_2(k)} + \phi^T(k-d+1)F(k)\phi(k-d+1)}\right] \end{aligned}$$

4. $\phi^T(k)\hat{\theta}_c(k) = \eta_d(k+d)$ **control action**

Stability Theorem

Under the conditions:

1. Model orders and delay: n, m and d are known
2. $B(q^{-1})$ is Hurwitz
3. Projections are used: $\hat{r}_o(k) \geq b_{mino}$

$$0 < K_{\min} \leq \lambda_{\min}\{F(k)\} \leq \lambda_{\max}\{F(k)\} < K_{max} < \infty\,.$$

The tracking error converges to zero.

$$\boxed{\lim_{k\to\infty} \epsilon(k) = 0}$$

Stability Theorem

Notice that the theorem **does not require**:

- control input be a-priori bounded

$$|u(k)| < \infty$$

- $A(q^{-1})$ be Hurwitz
- Persistence of excitation or parameter convergence $\tilde{\theta}_c(k) \to 0$

Stability Analysis Step 1

Prove that the a-posteriori error converges to zero using Hyperstability theory.

$$\lim_{k\to\infty} e(k) = 0$$

where,

$$e(k) \;=\; \phi^T(k-d)\,\tilde{\theta}_c(k)$$

$$e(k) = \frac{e^o(k)}{1 + \phi(k-d)^T F(k-1)\phi(k-d)}$$

Stability Analysis Step 2

Prove that the parameter error stops changing:

$$\lim_{k\to\infty} |\Delta\tilde{\theta}_c(k)| = \lim_{k\to\infty} |\tilde{\theta}_c(k) - \tilde{\theta}_c(k-1)| = 0$$

Note: $\Delta\tilde{\theta}_c(k) \to 0 \quad \not\Rightarrow \quad \tilde{\theta}_c(k) \to 0$

Stability Analysis Step 3

Prove that:

$$\lim_{k\to\infty} \frac{(e^o(k))^2}{1+\phi(k-d)^T F(k-1)\phi(k-d)} = 0$$

and

$$\lim_{k\to\infty} \frac{\epsilon^2(k)}{1+\phi(k-d)^T F(k-1)\phi(k-d)} = 0$$

Stability Analysis Step 3

- Note that the result in step 3:

$$\lim_{k\to\infty} \frac{(e^o(k))^2}{1+\phi(k-d)^T F(k-1)\phi(k-d)} = 0$$

is different (**stronger**) than the one in step 1:

$$\lim_{k\to\infty} e(k) = \lim_{k\to\infty} \frac{e^o(k)}{1+\phi(k-d)^T F(k-1)\phi(k-d)} = 0$$

Stability Analysis Step 4

Prove that the regressor vector is an **affine function** of the truncated infinity norm of the filtered tracking error

$$|\phi(k-d)| \le C_1 + C_2 \max_{j\in[0,k]} |\epsilon(j)|\,.$$

For some finite **non-negative** constants C_1 and C_2

$$0 \le C_1 < \infty \qquad 0 \le C_2 < \infty$$

Goodwin's Key technical Lemma

We will use **Goodwin's Lemma** to prove that the filtered tracking error converges to zero

$$\boxed{\lim_{k\to\infty} \epsilon(k) = 0}$$

Therefore, since $A_c'(q^{-1})$ is Hurwitz

$$\boxed{\lim_{k\to\infty} y(k) = y_d(k)}$$

Goodwin's technical lemma

Given sequences $\epsilon(k) \in \mathcal{R}$ and $\phi(k) \in \mathcal{R}^n$

Under the conditions:

1. $$\lim_{k\to\infty} \frac{\epsilon^2(k)}{1 + b(k)\,|\phi(k-d)|^2} = 0 \qquad 0 \le b(k) < B < \infty$$

2. $$|\phi(k-d)| \le C_1 + C_2 \max_{j\in[0,k]} |\epsilon(j)|\,. \qquad \begin{array}{l} 0 \le C_1 < \infty \\ 0 \le C_2 < \infty \end{array}$$

Then:

$$\boxed{\lim_{k\to\infty} \epsilon(k) = 0 \qquad \text{and} \qquad |\phi(k)| < \infty}$$

Goodwin's technical lemma

Proof:

Assume first that $|\epsilon(k)| < \infty$ $\qquad |\phi(k)| < \infty$

Then, $\qquad 1 + b(k)\,|\phi(k-d)|^2 < \infty$

numerator must be going to zero

$$\lim_{k\to\infty} \frac{\epsilon^2(k)}{1 + b(k)\,|\phi(k-d)|^2} = 0 \quad \Longrightarrow \quad \lim_{k\to\infty} \epsilon(k) = 0$$

denominator is not going to infinity

Goodwin's technical lemma

We will now that

$$\lim_{k\to\infty} |\epsilon(k)| = \infty \quad \Longrightarrow \quad \lim_{k\to\infty} \frac{\epsilon^2(k)}{1 + b(k)\,|\phi(k-d)|^2} > 0$$

Goodwin's technical lemma

Assume that $\qquad \lim_{k\to\infty} |\epsilon(k)| = \infty$

there exists a subsequence $\{k_n\}$ of the sampling sequence $\{k\}$ such that

$$|\epsilon(k)| \le |\epsilon(k_n)| \qquad \text{for } k \le k_n$$

Goodwin's technical lemma

Assume: $\lim_{k\to\infty} |\epsilon(k)| = \infty$

Then, since $|\phi(k-d)| \le C_1 + C_2 \max_{j\in[0,k]} |\epsilon(j)|\,.$

along $\{k_n\}$

$$|\phi(k_n - d)| \le C_1 + C_2\,|\epsilon(k_n)|\,.$$

Goodwin's technical lemma

Assume: $\lim_{k\to\infty} |\epsilon(k)| = \infty$

Then, since $|\phi(k-d)| \le C_1 + C_2 \max_{j\in[0,k]} |\epsilon(j)|\,.$

along $\{k_n\}$

$$\frac{\epsilon^2(k_n)}{1+b(k_n)\,|\phi(k_n-d)|^2} \ge \frac{\epsilon^2(k_n)}{1+b(k_n)\,[C_1+C_2|\epsilon(k_n)|]^2}$$

smaller

Goodwin's technical lemma

$$\frac{\epsilon^2(k_n)}{1+b(k_n)\,|\phi(k_n-d)|^2} \ge \frac{\epsilon^2(k_n)}{1+b(k_n)\,[C_1+C_2|\epsilon(k_n)|]^2}$$

since $\lim_{k_n\to\infty} |\epsilon(k_n)| = \infty$ $\qquad 0 \le b(k) < B < \infty$

$$\lim_{k_n\to\infty} \frac{\epsilon^2(k_n)}{1+b(k_n)\,|\phi(k_n-d)|^2} \ge \lim_{k_n\to\infty} \frac{\epsilon^2(k_n)}{1+b(k_n)\,[C_1+C_2|\epsilon(k_n)|]^2}$$

$$\ge \lim_{\epsilon\to\infty} \frac{\epsilon^2}{1+B\,[C_1+C_2|\epsilon|]^2}$$

$0 \le C_1 < \infty$

$0 \le C_2 < \infty$

$$\ge \frac{1}{BC_2^2} > 0$$

Goodwin's technical lemma

Thus, if $\lim_{k\to\infty} |\epsilon(k)| = \infty$

Then, $\lim_{k\to\infty} \frac{\epsilon^2(k)}{1+b(k)\,|\phi(k-d)|^2} > 0$

Therefore,

$$\lim_{k\to\infty} \frac{\epsilon^2(k)}{1+b(k)\,|\phi(k-d)|^2} = 0 \quad \Rightarrow \quad \begin{array}{l} |\epsilon(k)| < \infty \\ |\phi(k)| < \infty \end{array}$$

$$\Rightarrow \boxed{\lim_{k\to\infty} \epsilon(k) = 0}$$

Q.E.D

Stability Analysis

I will do the stability analysis using the constant gain PAA.

$$
\begin{aligned}
e(k) &= \frac{e^o(k)}{1+\phi^T(k-d)F\phi(k-d)} \\
\hat{\theta}_c(k) &= \hat{\theta}_c(k-1) + F\phi(k-d)\, e(k)
\end{aligned}
$$

$$F = F^T \quad \text{and} \quad F > 0$$

The proof for the RLS PAA can be found in pages 52-65 of the ME233 part II class notes.

Stability Analysis Step 1

Prove that the a-posteriori error converges to zero using Hyperstability theory.

$$\lim_{k\to\infty} e(k) = 0$$

where,

$$e(k) = \phi^T(k-d)\,\tilde{\theta}_c(k)$$

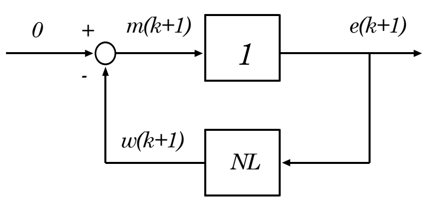
$$e(k) = \frac{e^o(k)}{1+\phi(k-d)^T F(k-1)\phi(k-d)}$$

Stability Analysis Step 1

Proof: The proof is similar to the one that we did for the parallel identification system with constant gain:

- **Linear block** ***1*****:**

$$
\begin{aligned}
e(k) &= \phi(k-d)\tilde{\theta}_c(k) \\
&= m(k)
\end{aligned}
$$



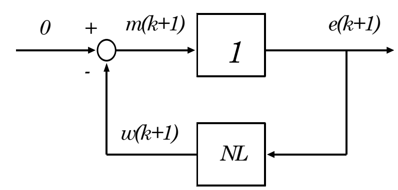
- **Nonlinear block** ***NL*****:**

$$
\begin{aligned}
\tilde{\theta}_c(k) &= \tilde{\theta}_c(k-1) - F\phi(k-d)\, e(k) \\
w(k) &= -m(k) = -\tilde{\theta}_c(k)^T\, \phi(k-d)
\end{aligned}
$$

Stability Analysis Step 1

We need to show that $\boldsymbol{NL}$ is a P-class

$$\sum_{j=1}^{k} w(j)e(j) \geq -\gamma_o^2$$



where

$$
\begin{aligned}
\tilde{\theta}_c(k) &= \tilde{\theta}_c(k-1) - F\phi(k-d)\, e(k) \\
w(k) &= -\tilde{\theta}_c(k)^T\, \phi(k-d)
\end{aligned}
$$

Stability Analysis Step 1

- Doing a bit of algebra we obtain:

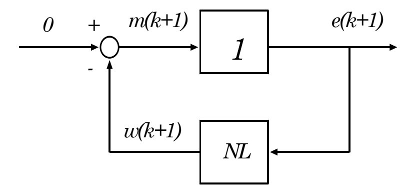
$$w(k)e(k) = \tilde{\theta}_c(k)^T F^{-1} \underbrace{\left[\tilde{\theta}_c(k) - \tilde{\theta}_c(k-1)\right]}_{\Delta\tilde{\theta}_c(k)}$$

- Completing the squares,
  (see the class notes on Lecture 20 for details)

$$\begin{aligned} w(k)e(k) &= \frac{1}{2}\tilde{\theta}_c(k)^T F^{-1}\tilde{\theta}_c(k) - \frac{1}{2}\tilde{\theta}_c(k-1)^T F^{-1}\tilde{\theta}_c(k-1) \\ &\quad + \frac{1}{2}\Delta\tilde{\theta}_c(k)^T F^{-1}\Delta\tilde{\theta}_c(k) \end{aligned}$$

Stability Analysis Step 1

$$\sum_{j=1}^{k} w(j)e(j) \geq -\gamma_o^2$$



Using the expression in the previous slide,

$$\begin{aligned} 2\sum_{j=1}^{k} w(j)e(j) &= \underbrace{\sum_{j=1}^{k}\left\{\tilde{\theta}_c(j)^T F^{-1}\tilde{\theta}_c(j) - \tilde{\theta}_c(j-1)^T F^{-1}\tilde{\theta}_c(j-1)\right\}}_{\tilde{\theta}_c(k)^T F^{-1}\tilde{\theta}_c(k) - \tilde{\theta}_c(0)^T F^{-1}\tilde{\theta}_c(0)} \\ &\quad + \sum_{j=1}^{k}\Delta\tilde{\theta}_c(j)^T F^{-1}\Delta\tilde{\theta}_c(j) \end{aligned}$$

Stability Analysis Step 1

Therefore,

$$\boxed{\begin{aligned} \sum_{j=1}^{k} w(j)e(j) &= \frac{1}{2}\tilde{\theta}_c(k)^T F^{-1}\tilde{\theta}_c(k) - \frac{1}{2}\tilde{\theta}_c(0)^T F^{-1}\tilde{\theta}_c(0) \\ &\quad + \frac{1}{2}\sum_{j=1}^{k}\Delta\tilde{\theta}_c(j)^T F^{-1}\Delta\tilde{\theta}_c(j) \end{aligned}}$$

Since every term is a positive definite function,

$$\sum_{j=1}^{k} w(j)e(j) \geq -\gamma_o^2 \qquad \gamma_o^2 = \frac{1}{2}\tilde{\theta}_c^T(0)F^{-1}\tilde{\theta}_c(0)$$

Stability Analysis Step 1

By the asymptotic Hyperstability theorem,

$$\lim_{k\to\infty} e(k) = 0$$

where

$$e(k) = \phi^T(k-d)\,\tilde{\theta}_c(k)$$

$$e(k) = \frac{e^o(k)}{1 + \phi(k-d)^T F(k-1)\phi(k-d)}$$

Q.E.D

Stability Analysis Step 2

Prove that the parameter error stops changing:

$$\lim_{k\to\infty} |\Delta\tilde{\theta}_c(k)| = \lim_{k\to\infty} |\tilde{\theta}_c(k) - \tilde{\theta}_c(k-1)| = 0$$

Note: $\Delta\tilde{\theta}_c(k) \to 0 \quad \not\Rightarrow \quad \tilde{\theta}_c(k) \to 0$

Stability Analysis Step 2

Prove that the parameter error stops changing.

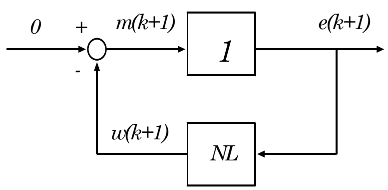
Lets remember, from the analysis done to show that $\boldsymbol{NL}$ is P-class,

$$\begin{aligned}\sum_{j=1}^{k} w(j)e(j) &= \frac{1}{2}\tilde{\theta}_c(k)^T F^{-1}\tilde{\theta}_c(k) - \frac{1}{2}\tilde{\theta}_c(0)^T F^{-1}\tilde{\theta}_c(0) \\ &+ \frac{1}{2}\sum_{j=1}^{k}\Delta\tilde{\theta}_c(j)^T F^{-1}\Delta\tilde{\theta}_c(j)\end{aligned}$$

Stability Analysis Step 2

On the other hand, from the feedback loop and the linear block, we have

$$e(k) = -w(k)$$



and

$$\sum_{j=1}^{k} w(j)e(j) = -\sum_{j=1}^{k} e(j)e(j)$$

Stability Analysis Step 2

Therefore,

$$\begin{aligned}\underbrace{\sum_{j=1}^{k} w(j)e(j)}_{-\sum_{j=1}^{k} e(j)e(j)} &= \frac{1}{2}\tilde{\theta}_c(k)^T F^{-1}\tilde{\theta}_c(k) - \frac{1}{2}\tilde{\theta}_c(0)^T F^{-1}\tilde{\theta}_c(0) \\ &+ \frac{1}{2}\sum_{j=1}^{k}\Delta\tilde{\theta}_c(j)^T F^{-1}\Delta\tilde{\theta}_c(j)\end{aligned}$$

Stability Analysis Step 2

Combining,

$$-\sum_{j=1}^{k} e^2(j) = \frac{1}{2}\tilde{\theta}_c(k)^T F^{-1}\tilde{\theta}_c(k) - \frac{1}{2}\tilde{\theta}_c(0)^T F^{-1}\tilde{\theta}_c(0) + \frac{1}{2}\sum_{j=1}^{k} \Delta\tilde{\theta}_c(j)^T F^{-1}\Delta\tilde{\theta}_c(j)$$

Rearranging terms,

$$\frac{1}{2}\sum_{j=1}^{k} \Delta\tilde{\theta}_c(j)^T F^{-1}\Delta\tilde{\theta}_c(j) + \sum_{j=1}^{k} e^2(j) + \frac{1}{2}\tilde{\theta}_c(k)^T F^{-1}\tilde{\theta}_c(k) = \gamma_o^2$$

$$\gamma_o^2 = \frac{1}{2}\tilde{\theta}_c^T(0) F^{-1}\tilde{\theta}_c(0)$$

Stability Analysis Step 2

Notice that the expression below applies for all k

$$\frac{1}{2}\sum_{j=1}^{k} \underbrace{\Delta\tilde{\theta}_c(j)^T F^{-1}\Delta\tilde{\theta}_c(j)}_{\geq 0} + \sum_{j=1}^{k} \underbrace{e^2(j)}_{\geq 0} + \frac{1}{2}\underbrace{\tilde{\theta}_c(k)^T F^{-1}\tilde{\theta}_c(k)}_{\geq 0} = \gamma_o^2$$

bounded

Therefore, since all terms are non-negative,

$$\sum_{j=1}^{\infty} \Delta\tilde{\theta}_c(j)^T F^{-1}\Delta\tilde{\theta}_c(j) \leq \gamma_0^2 < \infty$$

$$\sum_{j=1}^{\infty} e^2(j) \leq \gamma_o^2 < \infty$$

Stability Analysis Step 2

$$\sum_{j=1}^{\infty} \underbrace{\Delta\tilde{\theta}_c(j)^T F^{-1}\Delta\tilde{\theta}_c(j)}_{\geq 0} \leq \gamma_0^2 < \infty$$

$$\Longrightarrow \quad \lim_{k\to\infty} \Delta\tilde{\theta}_c(k)^T F^{-1}\Delta\tilde{\theta}_c(k) = 0$$

Since, $F = F^T$, $F > 0$ and $F^{-1} > 0$

$$\Longrightarrow \quad \boxed{\lim_{k\to\infty} |\Delta\tilde{\theta}_c(k)| = 0}$$

Q.E.D

Stability Analysis Step 2

Similarly, since

$$\sum_{j=1}^{\infty} e^2(j) \leq \gamma_o^2 < \infty$$

$$\Longrightarrow \quad \boxed{\lim_{k\to\infty} e(k) = 0}$$

*We have proven the **sufficiency** portion of the **Asymptotic Hyperstability Theorem**, when the linear block is an identity.*

So far we have:

1. $$e(k) \rightarrow 0 \quad \not\Rightarrow \quad \begin{cases} e^o(k) \rightarrow 0 \\ \epsilon(k) \rightarrow 0 \end{cases}$$

$$\Downarrow$$

$$\frac{e^o(k)}{1+\phi(k-d)^T F(k-1)\phi(k-d)} \rightarrow 0$$

2. $$|\Delta\tilde{\theta}_c(k)| \rightarrow 0$$

Stability Analysis Step 3

Prove that:

$$\lim_{k\rightarrow\infty} \frac{(e^o(k))^2}{1+\phi(k-d)^T F(k-1)\phi(k-d)} = 0$$

and

$$\lim_{k\rightarrow\infty} \frac{\epsilon^2(k)}{1+\phi(k-d)^T F(k-1)\phi(k-d)} = 0$$

Stability Analysis Step 3

- Note that this result:

$$\lim_{k\rightarrow\infty} \frac{(e^o(k))^2}{1+\phi(k-d)^T F(k-1)\phi(k-d)} = 0$$

is stronger than the one in step 1:

$$\lim_{k\rightarrow\infty} e(k) = \lim_{k\rightarrow\infty} \frac{e^o(k)}{1+\phi(k-d)^T F(k-1)\phi(k-d)} = 0$$

Stability Analysis Step 3

From the PAA we have

$$\hat{\theta}_c(k) = \hat{\theta}_c(k-1) + F\phi(k-d)\, e(k)$$

Which implies

$$\begin{aligned} \hat{\theta}_c(k) - \hat{\theta}_c(k-1) &= F\phi(k-d)\, e(k) \\ \Delta\hat{\theta}_c(k) &= F\phi(k-d)\, e(k) \end{aligned}$$

$$\Delta\tilde{\theta}_c(k) = -F\phi(k-d)\, e(k)$$

Stability Analysis Step 3

Inserting
$$\Delta\tilde{\theta}_c(k) \;=\; -F\phi(k-d)\,e(k)$$

Into
$$\lim_{k\to\infty} \Delta\tilde{\theta}_c(k)^T F^{-1} \Delta\tilde{\theta}_c(k) \;=\; 0$$

We obtain
$$\lim_{k\to\infty} \phi^T(k-d)F(k-1)\phi(k-d)e^2(k) = 0$$

Adding $\lim_{k\to\infty} e(k) \;=\; 0$ we obtain
$$\lim_{k\to\infty} \left[1+\phi^T(k-d)F(k-1)\phi(k-d)\right] e^2(k) = 0$$

Stability Analysis Step 3

From
$$\lim_{k\to\infty} \left[1+\phi^T(k-d)F(k-1)\phi(k-d)\right] e^2(k) = 0$$

And the fact that
$$e(k) = \frac{e^o(k)}{1+\phi(k-d)^T F(k-1)\phi(k-d)}$$

we obtain our first result
$$\boxed{\lim_{k\to\infty} \frac{(e^o(k))^2}{1+\phi(k-d)^T F(k-1)\phi(k-d)} = 0}$$

Stability Analysis Step 3

To prove the second result
$$\lim_{k\to\infty} \frac{\epsilon^2(k)}{1+\phi(k-d)^T F(k-1)\phi(k-d)} = 0$$

We use
$$\epsilon(k) \;=\; \phi(k-d)^T \tilde{\theta}_c(k-d)$$

and add and subtract:
$$e^o(k) \;=\; \phi(k-d)^T\, \tilde{\theta}_c(k-1)$$

$$\epsilon(k) \;=\; e^o(k) - \phi(k-d)^T \left[\tilde{\theta}_c(k-1) - \tilde{\theta}_c(k-d)\right]$$

Stability Analysis Step 3

Notice that,
$$|\tilde{\theta}_c(k-1) - \tilde{\theta}_c(k-d)| = |\Delta\tilde{\theta}_c(k-1) + \cdots + \Delta\tilde{\theta}_c(k-d+1)|$$
$$\le |\Delta\tilde{\theta}_c(k-1)| + \cdots + |\Delta\tilde{\theta}_c(k-d+1)|$$

Thus, for a finite d, since
$$\lim_{k\to\infty} |\Delta\tilde{\theta}_c(k)| \;=\; 0$$

$$\Longrightarrow \qquad \lim_{k\to\infty} |\tilde{\theta}_c(k-1) - \tilde{\theta}_c(k-d)| = 0$$

Stability Analysis Step 3

From

$$\epsilon(k) \;=\; e^o(k) - \phi(k-d)^T\left[\tilde{\theta}_c(k-1) - \tilde{\theta}_c(k-d)\right]$$

we obtain

$$\frac{\epsilon(k)}{(1+\phi^T(k-d)F\phi(k-d))^{\frac{1}{2}}} = \frac{e^o(k)}{(1+\phi^T(k-d)F\phi(k-d))^{\frac{1}{2}}} - \frac{\phi(k-d)^T\left[\hat{\theta}_c(k-1) - \hat{\theta}_c(k-d)\right]}{(1+\phi^T(k-d)F\phi(k-d))^{\frac{1}{2}}}$$

Stability Analysis Step 3

Also, using Schwartz inequality

$$\left|\frac{\phi(k-d)^T\left[\hat{\theta}_c(k-1) - \hat{\theta}_c(k-d)\right]}{(1+\phi^T(k-d)F\phi(k-d))^{\frac{1}{2}}}\right| \le \underbrace{\frac{|\phi(k-d)|}{(1+\phi^T(k-d)F\phi(k-d))^{\frac{1}{2}}}}_{\le \frac{1}{\lambda_{\min}^{1/2}(F)} < \infty} \underbrace{|\hat{\theta}_c(k-1) - \hat{\theta}_c(k-d)|}_{\to 0}$$

therefore

$$\lim_{k\to\infty}\left|\frac{\phi(k-d)^T\left[\hat{\theta}_c(k-1) - \hat{\theta}_c(k-d)\right]}{(1+\phi^T(k-d)F\phi(k-d))^{\frac{1}{2}}}\right| = 0$$

Stability Analysis Step 3

Since,

$$\frac{\epsilon(k)}{(1+\phi^T(k-d)F\phi(k-d))^{\frac{1}{2}}} = \underbrace{\frac{e^o(k)}{(1+\phi^T(k-d)F\phi(k-d))^{\frac{1}{2}}}}_{\to 0} - \underbrace{\frac{\phi(k-d)^T\left[\hat{\theta}_c(k-1) - \hat{\theta}_c(k-d)\right]}{(1+\phi^T(k-d)F\phi(k-d))^{\frac{1}{2}}}}_{\to 0}$$

Thus

$$\boxed{\lim_{k\to\infty}\frac{\epsilon^2(k)}{1+\phi^T(k-d)F(k-1)\phi(k-d)} = 0}$$

Q.E.D

Stability Analysis Step 4

Prove that the regressor vector is an **affine function** of the truncated infinity norm of the filtered tracking error

$$|\phi(k-d)| \le C_1 + C_2 \max_{j\in[0,k]} |\epsilon(j)|\,.$$

For some finite **non-negative** constants C_1 and C_2

$$0 \le C_1 < \infty \qquad 0 \le C_2 < \infty$$

Stability Analysis Step 4

The full proof is in pages 61-65 of ME233 class notes part II.

The proof utilizes:

1) Triangle inequality:

$$\boxed{a = b + c \Longrightarrow |a| \leq |b| + |c|}$$

Stability Analysis Step 4

The full proof is in pages 61-65 of ME233 class notes part II.

The proof utilizes:

2) BIBO stability of linear asymptotically stable systems

$$u(k-d) = \frac{A(q^{-1})}{B(q^{-1})} y(k)$$

Where $B(q^{-1})$ is **Hurwitz**

$$\boxed{|u(k-d)| \leq K_1 + K_2 \max_{j \in [0,k]} |y(j)|}$$

For some bounded non-negative constants

$$0 \leq K_1 < \infty$$

$$0 \leq K_2 < \infty$$

Stability Analysis Step 4

We want to show that: $|\phi(k-d)| \leq C_1 + C_2 \max_{j \in [0,k]} |\epsilon(j)|$.

For simplicity, lets assume that

$$\phi(k-d) = \left[\; y(k-d) \quad u(k-d) \; \right]^T$$

Then, using the Euclidean norm,

$$|\phi(k-d)|^2 = |y(k-d)|^2 + |u(k-d)|^2$$

$$\boxed{|\phi(k-d)| \leq |y(k-d)| + |u(k-d)|}$$

Stability Analysis Step 4

We have $|\phi(k-d)| \leq |y(k-d)| + |u(k-d)|$

- Now we use the fact that $B(q^{-1})$ is **Hurwitz**

$$|u(k-d)| \leq K_1 + K_2 \max_{j \in [0,k]} |y(j)|$$

Therefore

$$|\phi(k-d)| \leq |y(k-d)| + K_1 + K_2 \max_{j \in [0,k]} |y(j)|$$

Notice that $|y(k-d)| \leq \max_{j \in [0,k]} |y(j)|$

Stability Analysis Step 4

Therefore, setting $K_3 = K_2 + 1$

$$|\phi(k-d)| \le K_1 + K_3 \max_{j\in[0,k]} |y(j)|$$

- Now we use the fact that $A_c'(q^{-1})$ is **Hurwitz**

$$y(k) = \frac{1}{A_c'(q^{-1})}\eta(k)$$

Therefore,

$$|y(k)| \le L_1 + L_2 \max_{j\in[0,k]} |\eta(j)|$$

For some bounded non-negative constants L_1 L_2

Stability Analysis Step 4

Therefore, setting $J_1 = K_1 + K_3 L_1$ and $J_2 = K_3 L_2$

$$|\phi(k-d)| \le J_1 + J_2 \max_{j\in[0,k]} |\eta(j)|$$

- Now we use the triangle inequality.

$$\epsilon(k) = \eta(k) - \eta_d(k) \implies |\eta(k)| \le |\epsilon(k)| + |\eta_d(k)|$$

Since the desired filter trajectory, $\eta_d(k)$, is bounded,

Define: $$J_4 = \max_{j\in[0,\infty)} |\eta_d(k)|$$

Stability Analysis Step 4

Therefore,

$$|\phi(k-d)| \le J_1 + J_2 \max_{j\in[0,k]} |\eta(j)|$$

$$\max_{j\in[0,k]} |\eta(j)| \le \max_{j\in[0,k]} |\epsilon(j)| + J_4$$

Setting, $C_1 = J_1 + J_2 J_4$ and $C_2 = J_2$ we obtain:

$$\boxed{|\phi(k-d)| \le C_1 + C_2 \max_{j\in[0,k]} |\epsilon(j)|\,.}$$

Q.E.D

What do we have so far?

1. $$\frac{\epsilon^2(k)}{1+\phi^T(k-d)F(k-1)\phi(k-d)} \to 0$$

$$\epsilon(k) = \eta(k) - \eta_d(k) \quad = A_c'(q^{-1})(y(k) - y_d(k))$$

2. $$|\phi(k-d)| \le C_1 + C_2 \max_{j\in[0,k]} |\epsilon(j)|\,.$$

$$0 \le C_1 < \infty$$
$$0 \le C_2 < \infty$$

Notice that

$$\phi^T(k-d)F(k-1)\phi(k-d) \leq \lambda_{\max}(F)\,|\phi(k-d)|^2$$

Therefore, defining $\quad b = \lambda_{\max}(F) < \infty$

$$\underbrace{\frac{\epsilon^2(k)}{1+\phi^T(k-d)F(k-1)\phi(k-d)}}_{\to 0} \geq \underbrace{\frac{\epsilon^2(k)}{1+b\,|\phi(k-d)|^2}}_{\to 0}$$

Thus we have shown

1. $$\frac{\epsilon^2(k)}{1+b\,|\phi(k-d)|^2} \to 0 \qquad 0 < b < \infty$$

2. $$|\phi(k-d)| \leq C_1 + C_2 \max_{j\in[0,k]} |\epsilon(j)|\,.$$

$$0 \leq C_1 < \infty$$
$$0 \leq C_2 < \infty$$

Stability Analysis Step 5

Finally, we will use **Goodwin's Lemma**

$$\boxed{\lim_{k\to\infty} \epsilon(k) = 0}$$

Therefore, since $\quad A_c'(q^{-1}) \quad$ is Hurwitz

$$\boxed{\lim_{k\to\infty} y(k) = y_d(k)}$$

Goodwin's technical lemma

Given sequences $\epsilon(k) \in \mathcal{R}$ and $\phi(k) \in \mathcal{R}^n$

Under the conditions:

1. $$\lim_{k\to\infty} \frac{\epsilon^2(k)}{1+b(k)\,|\phi(k-d)|^2} = 0 \qquad 0 \leq b(k) < B < \infty$$

2. $$|\phi(k-d)| \leq C_1 + C_2 \max_{j\in[0,k]} |\epsilon(j)|\,. \qquad \begin{array}{l} 0 \leq C_1 < \infty \\ 0 \leq C_2 < \infty \end{array}$$

Then:

$$\boxed{\lim_{k\to\infty} \epsilon(k) = 0 \qquad \text{and} \qquad |\phi(k)| < \infty}$$