(f) Test the response of the LQG-LTR feedback system in Fig. 3 with the LQG compensators that were designed in the previous section, but use the actual plant $G_{PA}(s)$ in Eq. (2) in place of the simplified plant $G_p(s)$ in Eq. (1):

i. Compare the bode plots of $G_p(s)G_a(s)C_{_{LQG}}(s)$ with that of $G_{PA}(s)G_a(s)C_{_{LQG}}(s)$ and find their respective stability (gain and phase) margins.

ii. Compare the step responses of

$$\frac{G_p(s)G_a(s)C_{_{LQG}}(s)}{1+G_p(s)G_a(s)C_{_{LQG}}(s)} \qquad \text{with} \qquad \frac{G_{PA}(s)G_a(s)C_{_{LQG}}(s)}{1+G_{PA}(s)G_a(s)C_{_{LQG}}(s)}$$

for the LQG-LTR compensators $C_{_{LQG}}(s)$ that were designed in sections (c) and (d).

(g) We will now explore the consequences of violating the the small gain constraint (5). Remember that this constraint is both necessary and sufficient for the closed loop system to remain stable *for all* possible uncertainties $\bar{\Delta}(s)$ with frequency response magnitude $|\bar{\Delta}(j\omega)| \leq |\Delta(j\omega)|$. However, in this case, we are considering a *specific* uncertainty $\Delta(s)$. Therefore, the robustness constraint (5) will be only a sufficient condition.

i. Consider the case when $\mu = 0.001$, for which the target "fictitious" Kalman filter complementary sensitivity transfer function, $|T_{_{kf}}(j\omega)|$ is slightly larger than $|\frac{1}{\Delta(j\omega)}|$ for some frequencies. Obtain, the LQG compensator through the loop transfer recovery process and test whether it will stabilize the actual plant $G_{PA}(s)$.

ii. Consider now the case when $\mu_2 = 10^{-4}$. In this case, the target "fictitious" Kalman filter complementary sensitivity transfer function, $|T_{_{kf}}(j\omega)|$ is significantly larger than $|\frac{1}{\Delta(j\omega)}|$ for some frequencies. Obtain, the LQG compensator through the loop transfer recovery process and test whether it will stabilize the actual plant $G_{PA}(s)$.

2. Consider the FS-LTR extended dynamics and cost function:

$$\dot{x}_e = A_e\, x_e + B_e\, u \tag{6}$$

$$J = \int_0^\infty \left\{ x_e^T C_e^T C_e x_e + 2 x_e^T N_e u + u^T R_e u \right\} dt \tag{7}$$

where

$$A_e = \begin{bmatrix} A & 0 & 0 \\ B_1 & A_1 & 0 \\ 0 & 0 & A_2 \end{bmatrix} \qquad B_e = \begin{bmatrix} B \\ 0 \\ B_2 \end{bmatrix}$$

and

$$C_e = \begin{bmatrix} D_r C & C_r & 0 & 0 \\ D_1 & 0 & C_1 & 0 \\ 0 & 0 & 0 & \sqrt{\rho} C_2 \end{bmatrix} = \begin{bmatrix} & C_q & & \\ 0 & 0 & 0 & \sqrt{\rho} C_2 \end{bmatrix}$$

$$N_e^T = \begin{bmatrix} 0 & 0 & 0 & \rho D_2^T C_2 \end{bmatrix} \qquad R_e = \rho D_2^T D_2 \succ 0$$

We will prove that when

- the pair $[A_e, B_e]$ is stabilizable and
- the pair $[A_e - B_e R_e^{-1} N_e^T, C_q]$ is detectable

the optimal control

$$
\begin{aligned}
&u = -K_e x_e \\
&K_e = R_e^{-1} \left[B_e^T P_e + N_e^T \right] \\
&P_e A_e + A_e^T P_e + Q_e - \left[B_e^T P_e + N_e^T \right]^T R_e^{-1} \left[B_e^T P_e + N_e^T \right] = 0
\end{aligned}
$$

yields an exponentially stable close loop system.

Step 1: Define the control law

$$
u = -L x_e + v \,, \tag{8}
$$

where L is a gain to be determine and v is the new control input. Insert the control law (8) into Eqs. (6) and (7).

Step 2: Determine the required value of L so that we now obtain

$$
\begin{aligned}
\dot{x}_e &= \bar{A}_e \, x_e + B_e \, v & (9) \\
J &= \int_0^\infty \left\{ x_e^T \bar{Q}_e x_e + \rho v^T D_2^T D_2 v \right\} dt & (10)
\end{aligned}
$$

and show that $\bar{A}_e = A_e - B_e R_e^{-1} N_e^T$ and $\bar{Q}_e = C_q^T C_q$. Finally, remember that $[A_e, B_e]$ is stabilizable iff $[A_e - B_e R_e^{-1} N_e^T, B_e]$ is stabilizable (why?).