UNIVERSITY OF CALIFORNIA AT BERKELEY
Department of Mechanical Engineering
ME233 Advanced Control Systems II
Spring 2010

Homework #2

Assigned: Th., Feb. 4
Due: Th., Feb. 11

1. In this problem we will verify some results concerning the convergence of the LQR's discrete Riccati equation (DRE) to a steady state solution and the existence, uniqueness and closed loop stability of the discrete algebraic Riccati equation (DARE) solution.

   Consider the design of an optimal LQR for the LTI discrete time system

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\,x(k) \end{aligned} \tag{1}$$

   where $u(k) = -K(k+1)\,x(k)$ is the optimal control input that minimizes the following cost criteria

$$J[x_o, m, S, N] = \frac{1}{2} x^T(N)\,S\,x(N) + \frac{1}{2}\sum_{k=m}^{N-1}\left\{y^2(k) + R\,u^2(k)\right\} \quad x(m) = x_o\,,$$

   for $m = 0$, $S = S^T \succeq 0$ and $R = R^T \succ 0$, and any arbitrary initial condition $x_o \in \mathcal{R}^n$.

   Define the optimal value value function

$$J^o[x_o, m, S, N] = \min_{U_{[m,N-1]}} J[x_o, m, S, N]$$

   where $U_{[m,N-1]} = \{u(m), \cdots, u(N-1)\}$ is the set of all possible control actions from $k = m$ to $k = N-1$.

   (a) Assume:

$$A = \begin{bmatrix} 1.2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 10 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

   and verify that $[A, B]$ is stabilizable but $[A, C]$ is not detectable. Let $P(N) = S$. For the following three cases:

   (i) $S = 0$
   (ii) $S = \mathrm{Diag}(0,\ 0,\ 1)$,
   (iii) $S = \mathrm{Diag}(1,\ 1,\ 1)$,
   (iv) $S = \mathrm{Diag}(10,\ 1,\ 1)$,

   i. Compute the solution of the Riccati equation $P(k)$ backwards, for $N = 50$.
   ii. Compute $J^o[x_o, m, S, N]$ for $m = 0, \cdots, N$ and $x_o = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$.
   iii. Plot $J^o[x_o, N-m, S, N]$ vs m.
   iv. Compute the solution of the DARE using the matlab command `dare` and compare it with values of $P(0)$ and $P(1)$.

   Discuss your results.

(b) Assume:

$$A = \begin{bmatrix} 1.2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix}$$

and everything else is the same as a) and repeat a) for

(i) $S = 0$

(ii) $S = \mathrm{Diag}(0,\, 0,\, 1)$.

Discuss your results.

(c) Repeat part a) but in this case set

$$A = \begin{bmatrix} 0.8 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \qquad .B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \qquad R = 0.1$$

and consider

(i) $S = 0$

(ii) $S = \mathrm{Diag}(0,\, 0,\, 1)$.

Discuss your results.

2. Consider the discrete time system

$$x(k+1) \;=\; A\,x(k) + B\,u(k)\,, \qquad\qquad x(0) \neq 0$$

with $x(k) \in \mathcal{R}^n$ and $u(k) \in \mathcal{R}^m$.

We wish to find the optimal state feedback control law that minimizes the infinite horizon cost functional

$$J = \frac{1}{2} \sum_{k=0}^{\infty} \left\{ x^T(k) Q x(k) + 2\, x^T(k)\, S\, u(k) + u^T(k) R u(k) \right\}\,,$$

with $Q = Q^T \succeq 0$, $R = R^T \succ 0$ and $S \in \mathcal{R}^{n \times m}$ defined such that

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \succeq 0 \quad \Leftrightarrow Q - S R^{-1} S^T \succeq 0\,,$$

and to determine the necessary and sufficient conditions so that the infinite horizon optimal controller yields an asymptotically stable closed loop system. To solve this problem proceed as follows:

(a) Show that this problem can be transformed into a standard LQ problem by substituting the expression

$$u(k) = -K_1\, x(k) + u_1(k)\,,$$

where $u_1(k)$ is a new input, into the cost functional, and obtaining an expression for K_1 so that the resulting cost functional will not have a cross term between the state $x(k)$ and the input $u_1(k)$.

3. Consider the design of an optimal LQR for the SISO controllable and observable LTI discrete time system described by Eq. (1), where $u(k) = -K\,x(k)$ is the optimal control input that minimizes the following cost criteria

$$J = \sum_{k=0}^{\infty} \left\{ y^2(k) + R\,u^2(k) \right\} \ , \tag{2}$$

$R \in (0, \infty)$ is the input weight and

$$G(z) = C(zI - A)^{-1}B = \frac{z(z+2)}{(z-1)(z+0.5)(z-2)} \tag{3}$$

(a) Draw the locus of the eigenvalues of $A_c = A - B\,K$ and their respective reciprocals for $R \in (0, \infty)$.

(b) What are the eigenvalues of $A_c = A - B\,K$ for $R \to 0$?

(c) What are the eigenvalues of $A_c = A - B\,K$ for $R \to \infty$?

(d) Use the matlab function `rlocus` to verify your answers to parts (a)-(c).

(e) Use the matlab function `rlocfind` to determine the unique value of the input weight R_o for which all close loop eigenvalues are real and two eigenvalues are equal (i.e. double roots) and nonzero.

(f) Use the matlab function `canon` to obtain the controllable canonical realization for the transfer function $G(z)$ in Eq. (3).

(g) Using the canonical realization obtained in (f), compute, using the matlab function `dare`, the solution of the algebraic Riccati equation P_o, the optimal gain K_o and the location of the close loop eigenvalues for the LQR problem defined above with $R = R_o$, the value obtained in part (e).

(h) Using the matlab functions `bode` and `nyquist`, determine the gain and phase margins of the LQR open loop transfer function

$$G_o(z) = K_o(zI - A)^{-1}B$$

using the optimal gain K_o determined in part (g). Use the matlab functions `rlocus` and `rlocfind`, by plotting the root locus of $1 + \gamma G_o(z)$, for $\gamma \in (0, \infty)$, to verify your gain margin calculations.

(i) Compute the guaranteed LQR phase and gain margins of the open loop transfer function $G_o(z)$ in part (g) using the results based on the return difference inequality in slides in lecture 2, part II (also in the ME232 class notes pages 137-138).

(j) The guaranteed phase and gain margin results for a continuous time LQR in the ME232 class note pages 135-137 are independent of the input weight R and the state space realization of the transfer function $G(s) = C(sI - A)^{-1}B$. However, the corresponding results for discrete time systems depend on R. Show however that the guaranteed LQR phase and gain margin results are independent of the state space realization of the transfer function $G(z)$ in Eq. (3), when the realizations are related by a similarity transformation.

Hint: Let A, B, C and $\bar{A}$, $\bar{B}$, $\bar{C}$ be two state space realizations of $G(z)$, which are related by a similarity transformation. For a given value of R, let P and $\bar{P}$ be respectively the solutions of the discrete time algebraic Riccati equation for the two realizations. Show that $B^T P B = \bar{B}^T \bar{P} \bar{B}$.