Problem 1 Solution

1. Calculation of $\Lambda_{XX}(0)$: Utilizing the state equation, WSS, and the fact that $W(k)$ is white, zero-mean and $\Lambda_{WW}(j) = \delta(j)$,

$$E\{X(k+1)X(k+1)\} = a^2\, E\{X(k)X(k)\} + b^2\, E\{W(k)W(k)\}$$

$$(1-a^2)\,\Lambda_{XX}(0) = b^2\,\Lambda_{WW}(0) \quad \Rightarrow \Lambda_{XX}(0) = \frac{b^2}{(1-a^2)}$$

2. Calculation of $\Lambda_{XX}(1)$: Utilizing the state equation, WSS, and the fact that $W(k)$ is white, zero-mean and $\Lambda_{WW}(j) = \delta(j)$,

$$E\{X(k+1)X(k)\} = a\, E\{X(k)X(k)\} + b\, E\{W(k)X(k)\}$$

$$\Lambda_{XX}(1) = a\,\Lambda_{WW}(0) \quad \Rightarrow \Lambda_{XX}(1) = \frac{a\,b^2}{(1-a^2)}$$

3. Calculation of $\Lambda_{XW}(1)$: Utilizing the state equation, WSS, and the fact that $W(k)$ is white, zero-mean and $\Lambda_{WW}(j) = \delta(j)$,

$$E\{X(k+1)W(k)\} \;=\; a\, E\{X(k)W(k)\} + b\, E\{W(k)W(k)\}$$

$$\Lambda_{XW}(1) = b\,\Lambda_{WW}(0) \quad \Rightarrow \Lambda_{XW}(1) = b$$

4. Calculation of $\Lambda_{XW}(2)$: Utilizing the state equation, WSS, and the fact that $W(k)$ is white, zero-mean and $\Lambda_{WW}(j) = \delta(j)$,

$$\begin{aligned} E\{X(k+1)W(k-1)\} &= a\, E\{X(k)W(k-1)\} + b\, E\{W(k)W(k-1)\} \\ &= a\, \left[a\, E\{X(k-1)W(k-1)\} + b\, E\{W(k-1)W(k-1)\}\right] \end{aligned}$$

$$\Lambda_{XW}(2) = a\,b\,\Lambda_{WW}(0) \quad \Rightarrow \Lambda_{XW}(2) = a\,b$$

5. Minimum least square estimate of $X(k+1)$ given the measurements $x(k)$ and $w(k-1)$: We will use the least square estimation algorithm for jointly Gaussians. Defining,

$$Z(k) = \begin{bmatrix} X(k) & W(k-1) \end{bmatrix}^T$$

and remembering that all random sequences in this problem are zero mean,

$$\begin{aligned}\hat{x}(k+1)|_{Z(k)} &= E\{X(k+1)|X(k), W(k-1)\}\\ \Lambda_{XZ} &= \Lambda_{XZ}(0) = E\{X(k+1)Z(k)^T\}\\ \Lambda_{ZZ} &= \Lambda_{ZZ}(0) = E\{Z(k)Z(k)^T\},\end{aligned}$$

The optimal least square estimator is given by

$$\hat{x}(k+1)|_{Z(k)} = \Lambda_{XZ}\Lambda_{ZZ}^{-1}\begin{bmatrix} x(k) \\ w(k-1) \end{bmatrix}$$

We now need to calculate Λ_{XZ} and Λ_{ZZ}:

(a) Calculation of Λ_{XZ}

$$\begin{aligned}\Lambda_{XZ} &= \Lambda_{XZ}(1) = E\{X(k+1)Z(k)^T\} = \begin{bmatrix} E\{X(k+1)X(k)\} & E\{X(k+1)W(k-1)\} \end{bmatrix}\\ &= \begin{bmatrix} \Lambda_{XX}(1) & \Lambda_{XW}(2) \end{bmatrix}\end{aligned}$$

(b) Calculation of Λ_{ZZ}

$$\begin{aligned}\Lambda_{ZZ} &= \Lambda_{ZZ}(0) = E\{Z(k)Z(k)^T\} = \begin{bmatrix} E\{X(k)X(k)\} & E\{X(k)W(k-1)\} \\ E\{X(k)W(k-1)\} & E\{W(k-1)W(k-1) \end{bmatrix}\\ &= \begin{bmatrix} \Lambda_{XX}(0) & \Lambda_{XW}(1) \\ \Lambda_{XW}(1) & \Lambda_{WW}(0) \end{bmatrix}\end{aligned}$$

Therefore,

$$\begin{aligned}\hat{x}(k+1)|_{Z(k)} &= \begin{bmatrix} \Lambda_{XX}(1) & \Lambda_{XW}(2) \end{bmatrix}\begin{bmatrix} \Lambda_{XX}(0) & \Lambda_{XW}(1) \\ \Lambda_{XW}(1) & \Lambda_{WW}(0) \end{bmatrix}^{-1}\begin{bmatrix} x(k) \\ w(k-1) \end{bmatrix}\\ &= \cdots = ax(k)\end{aligned}$$

Alternate method using property 3 of least square estimation:

$$\begin{aligned}\hat{x}(k+1)|_{x(k),w(k-1)} &= \hat{x}(k+1)|_{x(k)} + E\left[\tilde{X}(k+1)|_{x(k)}|\tilde{w}(k-1)|_{x(k)}\right]\\ &= \Lambda_{XX}(1)\Lambda_{XX}^{-1}(0)x(k)\\ &\quad + \left[\Lambda_{XW}(2) - \Lambda_{XX}(1)\Lambda_{XX}^{-1}(0)\Lambda_{XW}(1)\right]\Lambda_{WW|_{x(k)}}^{-1}(0)\tilde{w}(k-1)|_{x(k)}\\ &= ax(k) + [ab - a*b]\,\Lambda_{WW|_{x(k)}}^{-1}(0)\tilde{w}(k-1)|_{x(k)}\\ &= ax(k)\end{aligned}$$

where in the last equality we use $\Lambda_{XX}(1) = a\Lambda_{XX}(0)$.

Both of these results confirm the fact that, since the state $x(k)$ already includes the information about $w(k-1)$, the least square estimate of $X(k+1)$ given $x(k)$ and $w(k-1)$ only depends on $x(k)$.

Problem 2 Solution

1. Obtain an expression for $\Lambda_{YW}(z)$: Since the system is WSS and $W(k)$ is white, zero-mean and $\Lambda_{WW}(j) = \delta(j)$,

$$\Lambda_{YW}(z) = G(z)\,\Lambda_{YW}(z) \quad \Rightarrow \Lambda_{YW}(z) = G(z) = b_o\,\frac{z-b}{z^2-a^2}$$

2. Obtain an expression for the output spectral density $\Phi_{YY}(\omega)$: Since the system is WSS and

$$E\left\{\begin{bmatrix} \tilde{W}(k+j) \\ \tilde{V}(k+j) \end{bmatrix}\begin{bmatrix} \tilde{W}(k+j) & \tilde{V}(k+j) \end{bmatrix}\right\} = \begin{bmatrix} 1 & 0 \\ 0 & \sigma_V^2 \end{bmatrix},$$

we obtain

$$\Phi_{YW}(\omega)) = G(e^{j\omega})\Phi_{WW}(\omega)G(e^{-j\omega})^T + \Phi_{VV}(\omega)$$

In this case,

$$\Phi_{YY}(\omega) = b_o^2\,\frac{1-2b\cos(\omega)+b^2}{[1+2a\cos(\omega)+a^2]\,[1-2a\cos(\omega)+a^2]} + \sigma_V^2$$

Problem 3 Solution

1. Obtain the matrices A, B in C in Eq. (4) in terms of the parameters a and b: Notice that

$$x(k+1) = ax(k) + bu(k) \quad \Rightarrow \quad \delta x(k+1) = a\delta x(k) + b\delta u(k)$$

and, since $e(k) = x(k) - \bar{x}$

$$e(k+1) - e(k) = (x(k+1) - \bar{x}) - (x(k) - \bar{x}) = \delta x(k+1)$$

which implies

$$e(k+1) = e(k) + \delta x(k+1) = e(k) + a\delta x(k) + b\delta u(k)\,.$$

Therefore,

$$\begin{aligned} \begin{bmatrix} e(k+1) \\ \delta x(k+1) \end{bmatrix} &= \begin{bmatrix} 1 & a \\ 0 & a \end{bmatrix} \begin{bmatrix} e(k) \\ \delta x(k) \end{bmatrix} + \begin{bmatrix} b \\ b \end{bmatrix} \delta u(k) \\ e(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} e(k) \\ \delta x(k) \end{bmatrix} \end{aligned} \tag{7}$$

2. Determine if there exist a unique stabilizing optimal incremental controller. lets check the controllability and observability matrices for (7):

$$Cnt = \begin{bmatrix} B & A\,B \end{bmatrix} = \begin{bmatrix} 1 & (1+a) \\ 1 & a \end{bmatrix} b \quad Obs = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & a \end{bmatrix}$$

Since the system is both controllable and observable, there exists a a unique stabilizing optimal incremental controller.

$$\delta u^o(k) = -K\,x_a(k)$$

3. Show that the optimal control action $u^o(k)$ is an integral action plus state feedback. Since,

$$\sum_{j=0}^{k} \delta u^o(k) = u^o(k) - u^o(-1) \quad \sum_{j=0}^{k} \delta x(k) = x(k) - x(-1)$$

and $u^o(-1) = x(-1) = 0$, we obtain

$$u^o(k) = -K_1 \sum_{j=0}^{k} e(k) - K_2\,x(k)$$

which is an I action with state feedback law.

4. We first show that

$$G(z) = \frac{E(z)}{\delta U(z)} = \frac{b\,z}{(z-a)(z-1)}, \qquad a < -1, \quad b > 0\,.$$

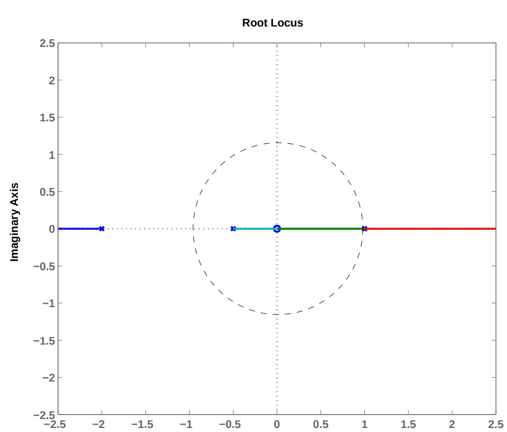
Plugging the values of A, B and C,

$$\begin{aligned} G(z) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} (z-1) & -a \\ 0 & (z-a) \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} b \\ &= \frac{b}{(z-1)(z-a)} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} (z-a) & a \\ 0 & (z-1) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{b}{(z-1)(z-a)} \begin{bmatrix} (z-a) & a \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{z\,b}{(z-1)(z-a)}\,. \end{aligned}$$

Using return difference equation results derived in class and remembering that $a < -1$ and $b > 0$, we obtain

$$\begin{aligned} \frac{A_c(z^{-1})A_c(z)}{A(z^{-1})A(z)} &= \gamma \left[1 + \frac{1}{R}\frac{b^2}{a}\frac{z^2}{(z-a)\,(z-\frac{1}{a})(z-1)^2} \right] \\ &= \gamma \left[1 - \frac{1}{R}\frac{b^2}{|a|}\frac{z^2}{(z-a)\,(z-\frac{1}{a})(z-1)^2} \right] \end{aligned}$$

(a) Plot the resulting reciprocal root locus for $R \in (0, \infty)$. Using the *positive feedback* root locus rules, we obtain the plot shown below



(b) Determine the closed loop poles as $R \rightarrow \infty$.

$$p_{c1} \longrightarrow 1 \quad p_{c2} \longrightarrow \frac{1}{a}$$

(c) Determine the closed loop poles as $R \rightarrow 0$.

$$p_{c1} \longrightarrow 0 \quad p_{c2} \longrightarrow 0$$