ME 233 Spring 2012
Solution to Homework #2

1. (a) We have the two-sided infinite sequence:

$$h(k) = f(k) + f(-k) + c\delta(k)$$

where $c \in \mathbb{R}$ and $\delta(k)$ is the Kronecker delta. We have after the $\mathbb{Z}$ - transform of $f(k)$ such that:

$$F(z) = \sum_{n=-\infty}^{+\infty} f(n) z^{-n}$$

Then we deduce the $\mathbb{Z}$ - transform of $f(-k)$ such that:

$$\begin{aligned} F_1(z) &= \sum_{n=-\infty}^{+\infty} f(-n) z^{-n} \\ &= \sum_{n=-\infty}^{+\infty} f(n) z^{n} \\ &= \sum_{n=-\infty}^{+\infty} f(n) (z^{-1})^{-n} \\ &= F(z^{-1}) \end{aligned}$$

And we know that the $\mathbb{Z}$ - transform of the Kronecker delta is 1. Then we deduce $H(z)$:

$$\begin{aligned} H(z) &= F(z) + F_1(z) + c \\ &= F(z) + F(z^{-1}) + c \end{aligned}$$

(b) $f(k)$ is defined as:

$$f(k) = \begin{cases} ba^k, & k \geq 1 \\ 0, & k \leq 0 \end{cases}$$

where $a, b \in \mathbb{R}$ and $|a| < 1$. Then we deduce:

$$\begin{aligned} F(z) &= \sum_{n=-\infty}^{+\infty} f(n) z^{-n} \\ &= \sum_{n=1}^{+\infty} ba^n z^{-n} \\ &= b\frac{a}{z} \sum_{n=0}^{+\infty} (\frac{z}{a})^{-n} \\ &= \frac{ba}{z} \frac{1}{1 - (z/a)^{-1}} \\ &= \frac{ba}{z - a} \end{aligned}$$

Then:

$$\begin{aligned} H(z) &= F(z) + F(z^{-1}) + c \\ &= \frac{ba}{z-a} + \frac{ba}{z^{-1}-a} + c \\ &= \frac{a(b-c)(z+z^{-1}) + a^2c + c - 2ba^2}{(z-a)(z^{-1}-a)} \end{aligned}$$

So we get:

$$\begin{aligned} \alpha &= a(b-c) \\ \beta &= a^2c + c - 2ba^2 \end{aligned}$$

2. (a) Figure 1 shows the MATLAB estimates of the auto-covariances and cross-covariances of W and Y. As we would expect, $\Lambda_{WW}(j)$ is approximately a unit pulse and $\Lambda_{YY}(j)$ is approximately symmetric. Also, $\Lambda_{YW}(-j) \approx \Lambda_{WY}(j)$ is approximately 0 for positive j, as causality dictates.

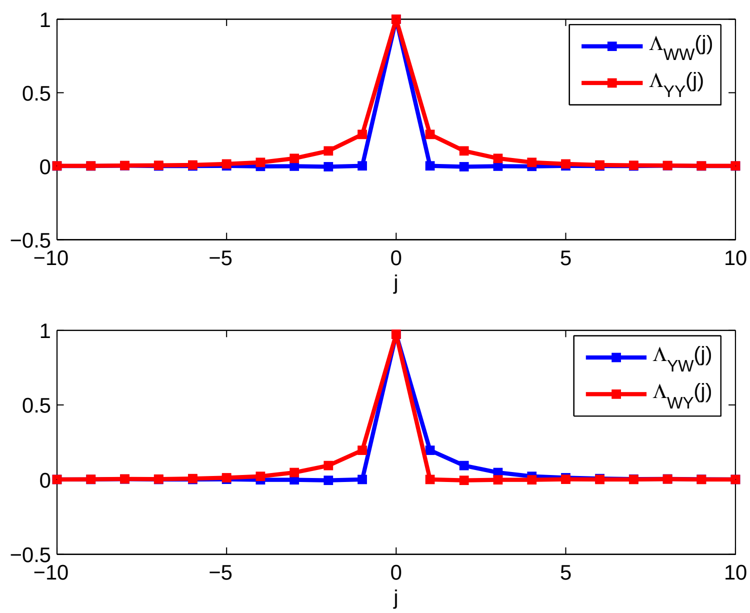

Figure 1: MATLAB estimates of auto-covariances and cross-covariances

(b) To find $\Lambda_{YW}(l)$, it is easiest to first find $\hat{\Lambda}_{YW}(z)$. Thus, we first note that

$$\begin{aligned} \hat{\Lambda}_{YW}(z) &= G(z)\hat{\Lambda}_{WW}(z) \\ G(z) &= \frac{z-0.3}{z-0.5} \\ \hat{\Lambda}_{WW}(z) &= \mathcal{Z}\{\delta(l)\} = 1 \\ \Rightarrow \hat{\Lambda}_{YW}(z) &= \frac{z-0.3}{z-0.5}. \end{aligned}$$

Now, with the aid of inverse Z-transform tables, we get that

$$\begin{aligned} \Lambda_{YW}(l) &= \mathcal{Z}^{-1}\left\{\frac{0.4z}{z-0.5} + 0.6\right\} \\ &= \left\{ \begin{array}{cc} 0.4(0.5)^l + 0.6\delta(l) & l \geq 0 \\ 0 & l < 0 \end{array} \right. . \end{aligned}$$

Figure 2 shows that the values of $\Lambda_{YW}(l)$ determined through MATLAB simulation match up well with the values determined above.

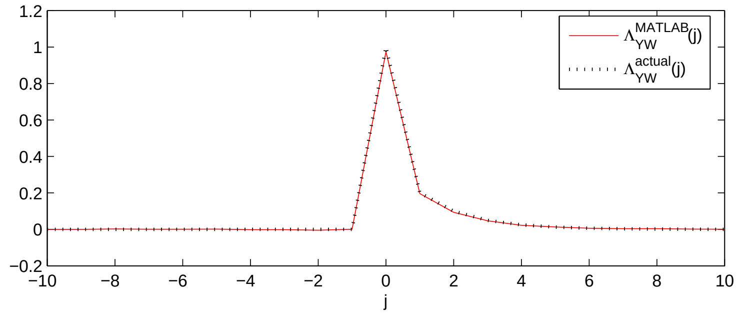

Figure 2: Comparison of MATLAB-determined cross-covariance to actual values

(c) Now that we have $\Lambda_{YW}(l)$, finding $\Lambda_{WY}(l)$ is a trivial matter. Using the property that $\Lambda_{YW}(l) = \Lambda_{WY}(-l)$, we see that

$$\Lambda_{WY}(l) = \begin{cases} 0.4(0.5)^{-l} + 0.6\delta(l) & l \leq 0 \\ 0 & l > 0 \end{cases}.$$

Figure 3 shows that the values of $\Lambda_{WY}(l)$ determined through MATLAB simulation match up well with the values determined above.

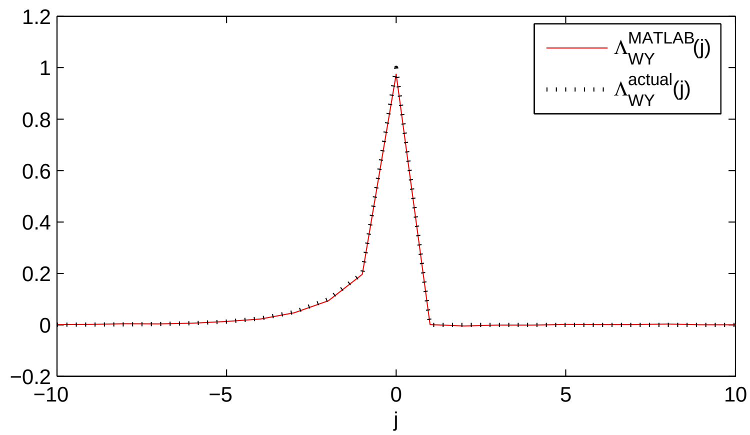

Figure 3: Comparison of MATLAB-determined cross-covariance to actual values

To find $\hat{\Lambda}_{WY}(z)$, it is easiest to recognize that the following general property applies to any random variables X and U:

$$\begin{aligned} \hat{\Lambda}_{XU}(z) &= \sum_{l=-\infty}^{\infty} z^{-l}\Lambda_{XU}(l) \\ &= \sum_{l=-\infty}^{\infty} (z^{-1})^{l}\Lambda_{UX}(-l) \\ &= \sum_{l=-\infty}^{\infty} (z^{-1})^{-l}\Lambda_{UX}(l) \\ &= \hat{\Lambda}_{UX}(z^{-1}). \end{aligned}$$

Applying this property to our system here gives

$$\hat{\Lambda}_{WY}(z) = \hat{\Lambda}_{YW}(z^{-1}) = \frac{z^{-1} - 0.3}{z^{-1} - 0.5} = \frac{0.3z - 1}{0.5z - 1}.$$

(d) We have the following:

$$\begin{aligned}\hat{\Lambda}_{YY}(z) &= \left(\frac{z-0.3}{z-0.5}\right)\left(\frac{z^{-1}-0.3}{z^{-1}-0.5}\right) \\ &= \frac{-0.3(z+z^{-1})+1.09}{(z-0.5)(z^{-1}-0.5)}.\end{aligned}$$

Using the results obtain in problem 1, we obtain:

$$\begin{aligned} a &= 0.5 \\ \alpha &= -0.3 \\ \beta &= 1.09. \end{aligned}$$

Then we can deduce b and c:

$$\begin{aligned} b &= 0.4533 \\ c &= 1.0533. \end{aligned}$$

So we obtain:

$$\hat{\Lambda}_{YY}(l) = f(l) + f(-l) + c\delta(l)$$

Where $c = 1.0533$ and where $f(l)$ is defined as:

$$f(l) = \begin{cases} 0.4533(0.5)^l, & l \geq 1 \\ 0, & l \leq 0 \end{cases}.$$

Figure 4 shows that the values of $\Lambda_{YY}(l)$ determined through MATLAB simulation match up well with the values determined above. (Note that the auto-covariance was normalized in this figure, i.e. $\Lambda_{YY}(l)$ was scaled so that its maximum value was 1.)

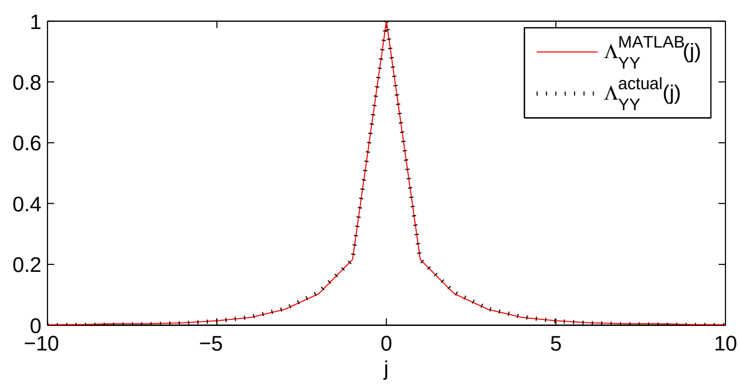

Figure 4: Comparison of MATLAB-determined auto-covariance to actual values

(e) Here, we want to compute covariances using the orginal series equation and compare our results to those obtained using transforms. To start, note that

$$\begin{aligned}\Lambda_{YW}(0) &= E\{Y(k)W(k)\} \\ &= E\Big\{\big[0.5Y(k-1) + W(k) - 0.3W(k-1)\big]W(k)\Big\} \\ &= E\left\{W^2(k)\right\} + 0.5E\{Y(k-1)W(k)\} - 0.3E\{W(k-1)W(k)\}.\end{aligned}$$

Since the system is causal we know that the system's output should not depend on future inputs. Thus, the system's output should be independent of future inputs. Also, since W is white, its value should be independent of its value at any other timestep. Using these two facts gives

$$\begin{aligned}\Lambda_{YW}(0) &= E\left\{W^2(k)\right\} + E\{W(k)\}\left[0.5E\{Y(k-1)\} - 0.3E\{W(k-1)\}\right] \\ &= E\left\{W^2(k)\right\} = 1\end{aligned}$$

where we have used the fact that W is zero-mean. Note that this result agrees with the result found in part (b).

(f) Using the wide-sense stationarity of the signals and the results from the previous part,

$$\begin{aligned}
\lambda_{YW}(1) &= E\{Y(k+1)W(k)\} \\
&= E\{Y(k)W(k-1)\} \\
&= -0.3E\left\{W^2(k-1)\right\} + 0.5E\{Y(k-1)W(k-1)\} + E\{W(k)W(k-1)\} \\
&= -0.3E\left\{W^2(k-1)\right\} + 0.5E\{Y(k-1)W(k-1)\} \\
&= -0.3E\left\{W^2(k)\right\} + 0.5E\{Y(k)W(k)\} \\
&= -0.3 + 0.5\Lambda_{YW}(0) = 0.2.
\end{aligned}$$

Note that this result agrees with the result found in part (b).

(g) To solve this problem, we will first note that

$$Y^2(k) = [0.5Y(k-1) + W(k) - 0.3W(k-1)]^2 .$$

Taking the expected value of both sides gives

$$\begin{aligned}
\Lambda_{YY}(0) &= 0.25E\left\{Y^2(K-1)\right\} + E\left\{W^2(k)\right\} + 0.09E\left\{W^2(k-1)\right\} \\
&\quad + E\{Y(k-1)W(k)\} - 0.3E\{Y(k-1)W(k-1)\} - 0.6E\{W(k)W(k-1)\} \\
&= 0.25\Lambda_{YY}(0) + 1 + 0.09 + 0 - 0.3\Lambda_{YW}(0) + 0 \\
&= \frac{0.79}{0.75} = 1.0533.
\end{aligned}$$

Note that this result agrees with the result found in part (e).

3. (a) First, we express our system as

$$\begin{aligned}
X(k+1) &= AX(k) + BW(k) \\
Y(k) &= CX(k) + V(k).
\end{aligned}$$

Taking expectation of our system equations gives

$$\begin{aligned}
m_x(k+1) &= Am_x(k) + Bm_w(k) \\
m_y(k) &= Cm_x(k).
\end{aligned}$$

Thus, finding $m_y(k)$ is equivalent to finding a step response of this system with magnitude 10. Figure 5 shows a plot of $m_y(k)$ versus the time step. Note that because $W(k)$ is not a zero-mean sequence, $Y(k)$ does not settle out to 0; the steady state value of m_y is given by

$$\overline{m}_y = 10.084.$$

(b) As discussed in lecture, the covariance of X propagates in the following way:

$$\Lambda_{XX}(k+1,0) = A\Lambda_{XX}(k,0)A^T + B\Sigma_{WW}(k)B^T.$$

Since we know the initial condition $\Lambda_{XX}(0,0)$, we can find $\Lambda_{XX}(k,0)$ iteratively using this Lyapunov equation. To find the covariance of Y, note that

$$\begin{aligned}
\Lambda_{YY}(k,0) &= E\left\{\tilde{Y}^2(k)\right\} \\
&= E\left\{\left[C\tilde{X}(k) + V(k)\right]\left[C\tilde{X}(k) + V(k)\right]^T\right\} \\
&= C\Lambda_{XX}(k,0)C^T + \Sigma_{vv}
\end{aligned}$$

where we made use of the fact that $X(k)$ and $V(k)$ are uncorrelated. Thus, we can use our simulation results for $\Lambda_{XX}(k,0)$ to find $\Lambda_{YY}(k,0)$. Figure 6 shows the results of simulating the

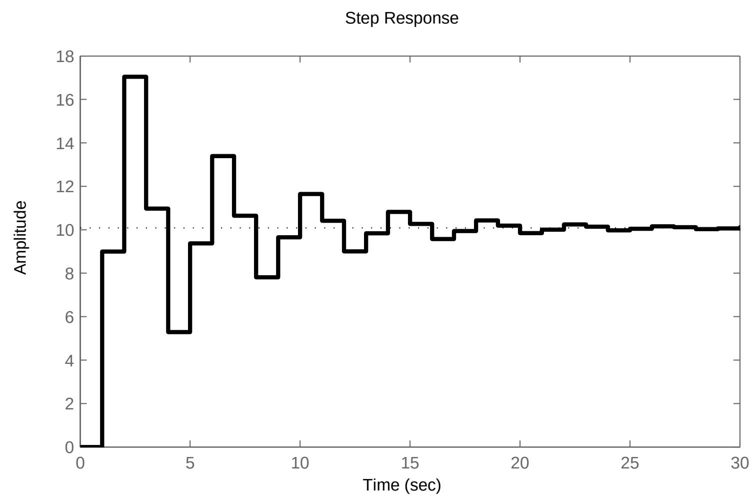

Figure 5: Evolution of $m_y(k)$ with time

evolution of $\Lambda_{XX}(k,0)$ and then using it to find $\Lambda_{YY}(k,0)$. This set of simulations terminated when

$$\left\|\Lambda_{XX}(k,0) - \Lambda_{XX}(k-1,0)\right\|_{i2} \leq 10^{-5}.$$

Note that we could have used any matrix norm in this termination condition (Frobenius norm, i1 norm, i2 norm, i∞ norm, etc). The steady state covariance of y was found to be

$$\Lambda_{YY}(0) = 3.27.$$

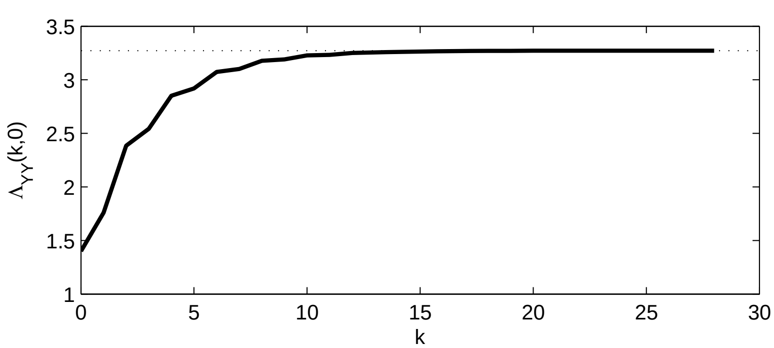

Figure 6: Evolution of $\Lambda_{YY}(k,0)$ with time

(c) To find $\Lambda_{XX}(5)$, recall that

$$\begin{aligned} \Lambda_{XX}(k,l) &= A^l \Lambda_{XX}(k,0) \\ \Rightarrow \Lambda_{XX}(k,5) &= A^5 \Lambda_{XX}(k,0). \end{aligned}$$

To find $\Lambda_{YY}(5)$, note that

$$\begin{aligned} \Lambda_{YY}(k,5) &= E\left\{\left[C\tilde{X}(k+5) + V(k+5)\right]\left[C\tilde{X}(k) + V(k)\right]^T\right\} \\ &= C\Lambda_{XX}(k,5)C^T \end{aligned}$$

where we have used that the measurement noise is white and uncorrelated with the state. Figure 7 shows the simulation results. At steady state,

$$\Lambda_{YY}(5) = 0.27.$$

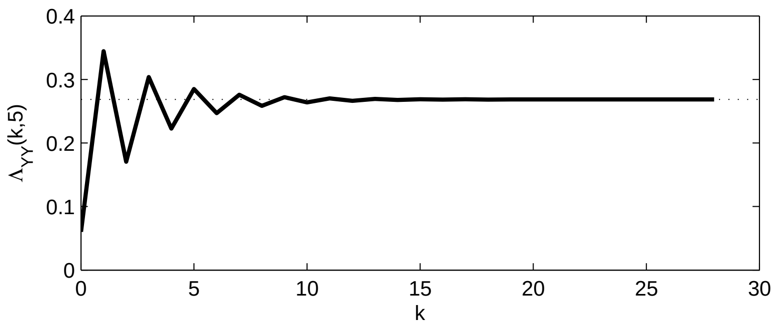

Figure 7: Evolution of $\Lambda_{YY}(k,5)$ with time

(d) At steady state,

$$A\Lambda_{XX}(0)A^T - \Lambda_{XX}(0) = -B\Sigma_{ww}B^T.$$

A call to `dlyap(A,B*Sigma_ww*B')` gives

$$\Lambda_{XX}(0) = \begin{bmatrix} 0.4308 & 0.0276 \\ 0.0276 & 0.3080 \end{bmatrix}.$$

At steady state, the stationary covariances of x and y are given by

$$\begin{aligned} \Lambda_{XX}(l) &= \begin{cases} \Lambda_{XX}(0)\left(A^{-l}\right)^T & l < 0 \\ \Lambda_{XX}(0) & l = 0 \\ A^l \Lambda_{XX}(0) & l > 0 \end{cases} \\ \Lambda_{YY}(l) &= E\left\{\left[C\tilde{X}(k+l) + V(k+l)\right]\left[C\tilde{X}(k) + V(k)\right]^T\right\} \\ &= C\Lambda_{XX}(l)C^T + \Sigma_{vv}\delta(l). \end{aligned}$$

Figure 8 shows the computed stationary covariance of Y. As expected, the plot is symmetric and the largest value occurs at $j = 0$. Note that the values of $\Lambda_{YY}(0)$ and $\Lambda_{YY}(5)$ are the same as the steady state covariances found in the two previous parts.

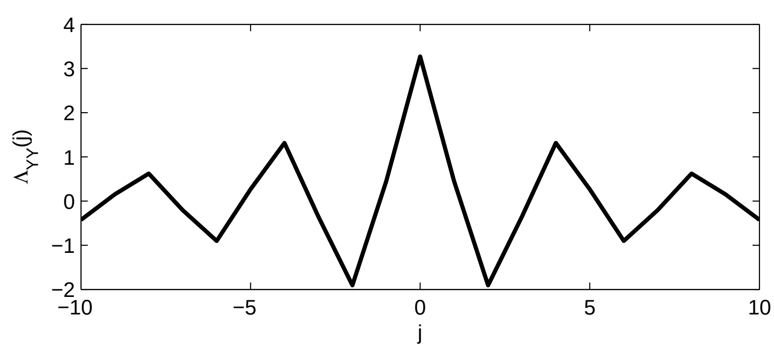

Figure 8: Stationary covariance of Y

(e) First, we define

$$\begin{aligned} \overline{W}(k) &= \begin{bmatrix} W(k) \\ V(k) \end{bmatrix} \\ \overline{G}(z) &= \begin{bmatrix} G(z) & 1 \end{bmatrix} \end{aligned}$$

so that our governing equations in the Z domain become

$$Y(z) = \overline{G}(z)\overline{W}(z).$$

Thus, the output spectral density is given by

$$\begin{aligned}
\Phi_{YY}(\omega) &= \overline{G}(\omega)\Phi_{\overline{WW}}(\omega)\overline{G}^T(-\omega) \\
&= \begin{bmatrix} G(\omega) & 1 \end{bmatrix} \begin{bmatrix} \Sigma_{ww} & 0 \\ 0 & \Sigma_{vv} \end{bmatrix} \begin{bmatrix} G(-\omega) \\ 1 \end{bmatrix} \\
&= |G(\omega)|^2 \Sigma_{ww} + \Sigma_{vv}.
\end{aligned}$$

(f) Figure 9 shows the spectral density of Y. As expected this graph is symmetric. Notice, however, that Φ_{YY} is not a maximum when $\omega = 0$. Unlike auto-covariances, spectral densities do not have to be a maximum when the argument is zero.

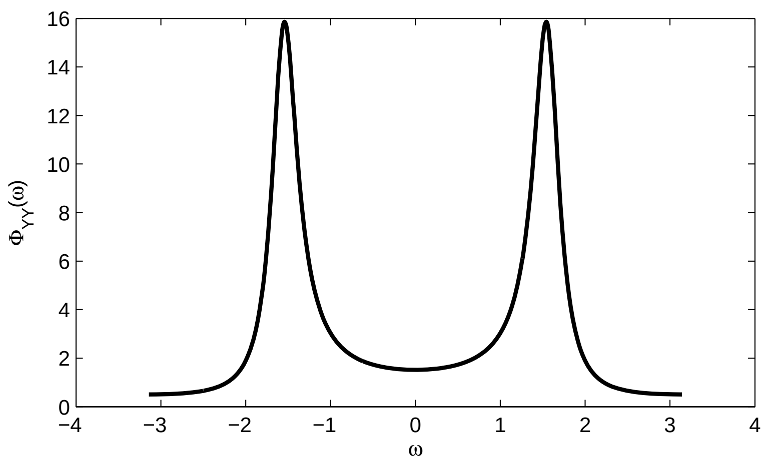

Figure 9: Spectral density of Y

To see where these peaks come from, we will find the equivalent damping and natural frequency of this system. Recall from classical controls that for a continuous time second-order underdamped system with damping ζ and natural frequency ω_n, the poles are given by

$$\begin{aligned}
q_1 &= \sigma + j\omega_d \\
q_2 &= \sigma - j\omega_d
\end{aligned}$$

where

$$\begin{aligned}
\sigma &= -\zeta\omega_n \\
\omega_d &= \omega_n\sqrt{1-\zeta^2} \\
&= \sqrt{\omega_n^2 - \sigma^2}.
\end{aligned}$$

Now recall from ME232 that if we have poles $\lambda_1, \ldots, \lambda_n$ in continuous time, the poles of the discrete time system obtained using a zero-order hold are given by $e^{\lambda_1 T}, \ldots, e^{\lambda_n T}$, where T is the sampling time. (Refer to page `ME232-36` of the ME232 class notes.) Thus, letting $T = 1$, our discrete time poles are given by

$$\begin{aligned}
p_1 &= e^{q_1} = e^{\sigma}e^{j\omega_d} = e^{\sigma}\left[cos(\omega_d) + jsin(\omega_d)\right] \\
p_2 &= e^{q_2} = e^{\sigma}e^{-j\omega_d} = e^{\sigma}\left[cos(\omega_d) - jsin(\omega_d)\right].
\end{aligned}$$

Thus, we can solve for

$$\begin{aligned}
\sigma &= \ln\{\,|p_1|\,\} \\
\omega_d &= \tan^{-1}\left\{\left|\frac{\mathrm{Im}(p_1)}{\mathrm{Re}(p_1)}\right|\right\} \\
\omega_n &= \sqrt{\sigma^2+\omega_d^2} \\
\zeta &= \frac{-\sigma}{\omega_n}.
\end{aligned}$$

In our system here, these values are

$$\begin{aligned}
\sigma &= -0.1841 \\
\omega_d &= 1.5588 \\
\omega_n &= 1.5696 \\
\zeta &= 0.1173.
\end{aligned}$$

Because we have a small value of ζ, our system is lightly damped, resulting in a relatively high peak gain at ω_d. Thus, we can expect a lot of the system output to be at the frequency ω_d. This explains our peaks in our output spectral density. This analysis can be verified by looking at the Bode plot for $G(z)$, shown in Figure 10. (Note that you would have to square the peak gain in the Bode plot and then add Σ_{vv} to get the peak in the output spectral density.)

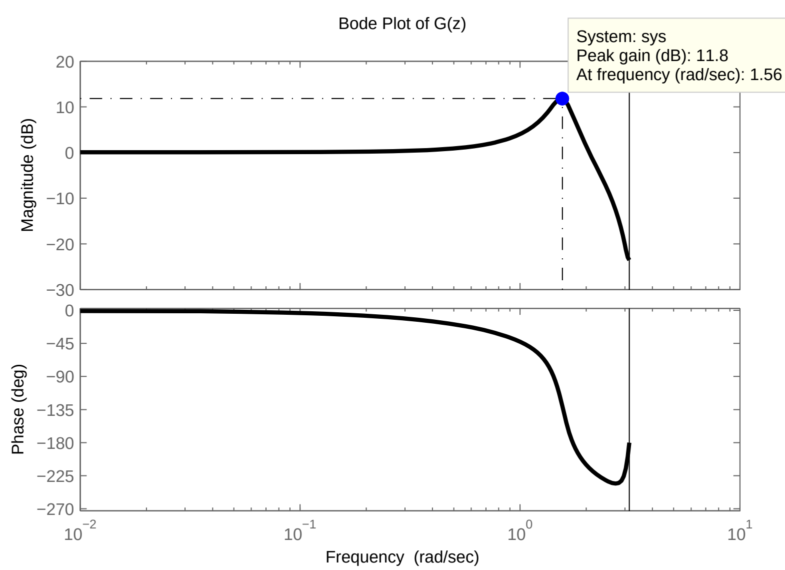

Figure 10: Bode plot of $G(z)$

Also note that $\Phi_{YY}(\omega)$ is large when $\omega \sim \pi/2$, i.e. half of our sampling frequency. This corresponds to the relative extrema of Figure 8, which occur at even correlation indices.

4. (a) To begin, we find the conditional expectation of X given y:

$$m_{X|y} = m_X + \Lambda_{XY}\Lambda_{YY}^{-1}(y-m_Y)$$

Since X and V_1 are two independent normal distributed random variables, with the results from Problem 5 in HW#1 we see that

$$\begin{aligned}
\Lambda_{YY} &= \Lambda_{XX}+\Lambda_{V_1V_1} \\
m_Y &= m_X
\end{aligned}$$

Noting that $X - m_X$ is independent of V_1, we calculate the cross-covariance of X and Y as

$$\begin{aligned}
\Lambda_{XY} &= E\left[(X - m_X)(Y - m_Y)\right] \\
&= E\left[(X - m_X)(X + V_1 - m_X)\right] \\
&= E\left[(X - m_X)^2\right] + E\left[(X - m_X) V_1\right] \\
&= E\left[(X - m_X)^2\right] + E\left[X - m_X\right] E\left[V_1\right] \\
&= E\left[(X - m_X)^2\right] \\
&= \Lambda_{XX}
\end{aligned}$$

Substituting the relevant values gives

$$m_{X|Y=9} = 10 + \frac{2(9-10)}{2+1} = 9\frac{1}{3}$$

(b) Using the same methodology as before, we see that

$$\begin{aligned}
m_{X|z} &= m_X + \Lambda_{XZ}\Lambda_{ZZ}^{-1}(z - m_Z) \\
\Lambda_{ZZ} &= \Lambda_{XX} + \Lambda_{V_2V_2} \\
m_Z &= m_X \\
\Lambda_{XZ} &= \Lambda_{XX}
\end{aligned}$$

Thus,

$$m_{X|Z=11} = 10 + \frac{2(11-10)}{2+2} = 10\frac{1}{2}$$

(c) First, we define the random vector W as

$$W = \begin{bmatrix} Y \\ Z \end{bmatrix}$$

The mean and covariance of this vector are given by

$$\begin{aligned}
m_W &= \begin{bmatrix} m_Y \\ m_Z \end{bmatrix} \\
\Lambda_{WW} &= \begin{bmatrix} \Lambda_{YY} & \Lambda_{YZ} \\ \Lambda_{ZY} & \Lambda_{ZZ} \end{bmatrix}
\end{aligned}$$

As before,

$$\begin{aligned}
\Lambda_{YY} &= \Lambda_{XX} + \Lambda_{V_1V_1} \\
\Lambda_{ZZ} &= \Lambda_{XX} + \Lambda_{V_2V_2}
\end{aligned}$$

The cross-covariance between Y and Z can be calculated as

$$\begin{aligned}
\Lambda_{ZY} = \Lambda_{YZ} &= E\left[(X - m_X + V_1)(X - m_X + V_2)\right] \\
&= E\left[(X - m_X)^2\right] + E\left[(X - m_X)(V_1 + V_2)\right] + E\left[V_1 V_2\right] \\
&= E\left[(X - m_X)^2\right] \\
&= \Lambda_{XX}
\end{aligned}$$

The cross-covariance between X and W can be expressed as

$$\Lambda_{XW} = \begin{bmatrix} \Lambda_{XY} & \Lambda_{XZ} \end{bmatrix} = \begin{bmatrix} \Lambda_{XX} & \Lambda_{XX} \end{bmatrix}$$

Thus,

$$\begin{aligned}
m_{X|Y=9,Z=11} &= m_{X|W=[9\ 11]^T} \\
&= m_X + \Lambda_{XW}\Lambda_{WW}^{-1}(w - m_W) \\
&= 10 + \begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}^{-1} \left(\begin{bmatrix} 9 \\ 11 \end{bmatrix} - \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right) \\
&= 9\frac{3}{4}
\end{aligned}$$

Note that the Y measurement has a greater impact on the conditional mean for X than the Z measurement. This means that our estimation is making use of the fact that Y is a more "reliable" measurement than Z, i.e. $\Lambda_{YY} < \Lambda_{ZZ}$.

5. (a) First, we define

$$Z := \begin{bmatrix} Y(0) & Y(1) & \cdots & Y(k) \end{bmatrix}^T.$$

And Z takes the outcome of $\bar{y}(k) = \begin{bmatrix} y(0) & \cdots & y(k) \end{bmatrix}^T$.
With this notation in mind, we are interested in finding $\hat{x}_{|z}$. Recall that

$$\begin{aligned}
\hat{x}_{|\bar{y}(k)} &= E\{X\} + \Lambda_{XZ}\Lambda_{ZZ}^{-1}\left(\bar{y}(k) - E\{Z\}\right) \\
&= \Lambda_{XZ}\Lambda_{ZZ}^{-1}\bar{y}(k).
\end{aligned}$$

Note that we used that X and Z are zero mean. In order to find this quantity, we need to find expressions for Λ_{XZ} and Λ_{ZZ}^{-1}. First, we will start by finding Λ_{XZ}. Note that

$$\begin{aligned}
E\left\{XY(j)\right\} &= E\left\{X^2\right\} + E\left\{XV(j)\right\} \\
&= X_0.
\end{aligned}$$

Thus, if we define

$$w = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T \in \mathbb{R}^{k+1}$$

we can express

$$\Lambda_{XZ} = X_0 w^T$$

Now we turn our attention to finding Λ_{ZZ}^{-1}. Note that

$$\begin{aligned}
E\left\{Y(k+j)Y(k)\right\} &= E\left\{\left(X + V(k+j)\right)\left(X + V(k)\right)\right\} \\
&= E\left\{X^2\right\} + E\left\{XV(k)\right\} + E\left\{XV(k+j)\right\} + E\left\{V(k+j)V(k)\right\} \\
&= X_0 + \Sigma_V \delta(j).
\end{aligned}$$

Thus, we can express

$$\begin{aligned}
\Lambda_{ZZ} &= \Sigma_V I + X_0 w w^T \\
&= \Sigma_V \left(I + \frac{X_0}{\Sigma_V} w w^T \right).
\end{aligned}$$

In order to invert this matrix, we must use the matrix inversion lemma, which states that

$$(I + AB)^{-1} = I - A\left(I + BA\right)^{-1} B.$$

Using this, we can say that

$$\begin{aligned}
\Lambda_{ZZ}^{-1} &= \frac{1}{\Sigma_V}\left(I + \frac{X_0}{\Sigma_V} w w^T \right)^{-1} \\
&= \frac{1}{\Sigma_V}\left[I - \frac{X_0}{\Sigma_V} w \left(1 + \frac{X_0}{\Sigma_V} w^T w \right)^{-1} w^T \right] \\
&= \frac{1}{\Sigma_V}\left[I - \frac{X_0}{\Sigma_V} \cdot \frac{\Sigma_V}{\Sigma_V + (k+1)X_0} w w^T \right] \\
&= \frac{1}{\Sigma_V}\left[I - \frac{X_0}{\Sigma_V + (k+1)X_0} w w^T \right].
\end{aligned}$$

Thus the estimate of X is given by

$$\begin{aligned}
\hat{x}(k) = \hat{x}_{|\bar{y}(k)} &= \frac{X_0}{\Sigma_V} w^T \left[I - \frac{X_0}{\Sigma_V + (k+1)X_0} w w^T \right] \bar{y}(k) \\
&= \frac{X_0}{\Sigma_V} \left[1 - \frac{X_0}{\Sigma_V + (k+1)X_0} w^T w \right] w^T \bar{y}(k) \\
&= \frac{X_0}{\Sigma_V + (k+1)X_0} w^T \bar{y}(k) \\
&= \frac{X_0}{\Sigma_V + (k+1)X_0} \sum_{i=0}^{k} y(i).
\end{aligned}$$

The covariance of the estimate is given by

$$\begin{aligned}
\Lambda_{\tilde{X}\tilde{X}}(k,0) &= \Lambda_{XX} - \Lambda_{XZ}\Lambda_{ZZ}^{-1}\Lambda_{ZX} \\
&= X_0 - \left(\frac{X_0}{\Sigma_V + (k+1)X_0} w^T \right) \left(X_0 w \right) \\
&= \frac{X_0 \Sigma_V}{\Sigma_V + (k+1)X_0}.
\end{aligned}$$

(b) Using the results of the previous part, it is trivial to see that

$$\begin{aligned}
\lim_{X_0 \to \infty} \hat{x}(k) &= \frac{1}{k+1} \sum_{i=0}^{k} y(k) \\
\lim_{X_0 \to \infty} \Lambda_{\tilde{X}\tilde{X}}(k,0) &= \frac{\Sigma_V}{k+1}.
\end{aligned}$$