# ME 233 Advanced Control II

Course Review



## Course Outline

- Unit 0: Probability
- Unit 1: State-space control, estimation
- Unit 2: Input/output control
- Unit 3: Adaptive control



## Course Outline

- Unit 0: Probability
- Unit 1: State-space control, estimation
- Unit 2: Input/output control
- Unit 3: Adaptive control



## Unit 0: Intro to probability (L1-L3)

- Random vector (RV)
- Cumulative distribution function (CDF)
- Probability density function (PDF)
  - Joint PDF
  - Marginal PDF
  - Conditional PDF

## Unit 0: Intro to probability (L1-L3)

5

- Expected value
- Mean, covariance, correlation
- Uncorrelated RVs, orthogonal RVs
- Independence

## Unit 0: Intro to probability (L1-L3)

6

- Gaussian RVs
  - Independent if and only if uncorrelated
  - If $X$ Gaussian, then $AX + b$ is Gaussian
  - If $X$ and $Y$ are jointly Gaussian, then $X_{|Y=y}$ is Gaussian: $X_{|Y} \sim \mathcal{N}(m_{X|y},\ \Lambda_{X|yX|y})$

$$m_{X|y} = m_X + \Lambda_{XY}\Lambda_{YY}^{-1}(y - m_Y)$$
$$\Lambda_{X|yX|y} = \Lambda_{XX} - \Lambda_{XY}\Lambda_{YY}^{-1}\Lambda_{YX}$$

*independent of outcome, y*

## Unit 0: Random vector sequences (L4)

7

- Mean: $m_X(k) = E\{X(k)\}$
- Auto-covariance:
  $\Lambda_{XX}(k,j) = E\{(X(k+j) - m_X(k+j))(X(k) - m_X(k))^T\}$
- Wide sense stationary (WSS)
  - Mean, auto-covariance are time-invariant
    $m_X = E\{X(k)\}$
    $\Lambda_{XX}(j) = E\{(X(k+j) - m_X(k+j))(X(k) - m_X(k))^T\}$
- Ergodic

## Unit 0: Random vector sequences (L4)

8

- Power spectral density

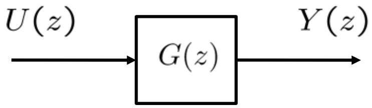
$$\Phi_{XX}(\omega) = \mathcal{F}\{\Lambda_{XX}(j)\}$$
$$\Lambda_{XX}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{XX}(\omega)d\omega$$

*covariance*

- White random sequences

9

## Unit 0: Random vector sequences (L4)




- If *U(k)* is WSS and *G(z)* is asymptotically stable, then *Y(k)* is WSS

$$\Phi_{YY}(\omega) = G(e^{j\omega})\, \Phi_{UU}(\omega)\, G^*(e^{j\omega})$$

$$\hat{\Lambda}_{YY}(z) = G(z)\, \hat{\Lambda}_{UU}(z)\, G^T(z^{-1})$$

10

## Unit 0: Random vector sequences (L4)

Consider the state-space system

$$\begin{aligned} X(k+1) &= AX(k) + BW(k) \\ Y(k) &= CX(k) \end{aligned}$$

where *W(k)* is an uncorrelated RVS

- The mean propagates as

$$\begin{aligned} m_X(k+1) &= Am_X(k) + Bm_W(k) \\ m_Y(k) &= Cm_X(k) \end{aligned}$$

11

## Unit 0: Random vector sequences (L4)

Consider the state-space system

$$\begin{aligned} X(k+1) &= AX(k) + BW(k) \\ Y(k) &= CX(k) \end{aligned}$$

where *W(k)* is an uncorrelated RVS

- The auto-covariance propagates as

$$\Lambda_{XX}(k+1,0) = A\Lambda_{XX}(k,0)A^T + B_w\Lambda_{WW}(k,0)B_w^T$$

$$\Lambda_{XX}(k,l) \;=\; A^l\,\Lambda_{XX}(k,0) \qquad\qquad l \geq 0$$

12

## Unit 0: Random vector sequences (L4)

Consider the state-space system

$$\begin{aligned} X(k+1) &= AX(k) + BW(k) \\ Y(k) &= CX(k) \end{aligned}$$

where *W(k)* is white (i.e uncorrelated and WSS)

- The steady-state means are

$$\begin{aligned} m_X &= (I - A)^{-1}Bm_W \\ m_Y &= Cm_X \end{aligned}$$

## Unit 0: Random vector sequences (L4)

Consider the state-space system

$$X(k+1) = AX(k) + BW(k)$$
$$Y(k) = CX(k)$$

where *W(k)* is white (i.e uncorrelated and WSS)

- The steady state auto-covariance is given by

$$\Lambda_{XX}(0) = A\Lambda_{XX}(0)A^T + B_w\Lambda_{WW}(0)B_w^T$$

$$\Lambda_{XX}(l) = A^l\Lambda_{XX}(0) \qquad l \geq 0$$

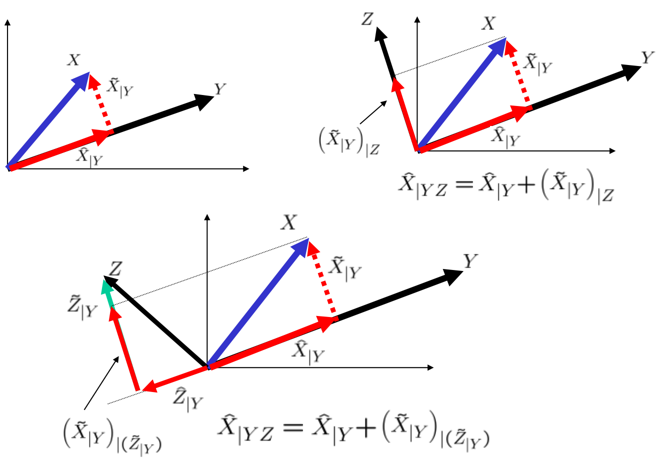
## Unit 0: Least squares (L5)

$\hat{X}|_Y$ is the least squares minimum conditional estimator of ***X*** given ***Y***, *i.e.*

$$E\{\|X - \hat{X}|_Y\|^2\} \leq E\{\|X - f(Y)\|^2\}$$

*for all functions* $f(\cdot)$ *of* $Y$

## Unit 0: Least squares (L5)






## Course Outline

- Unit 0: Probability
- Unit 1: State-space control, estimation
- Unit 2: Input/output control
- Unit 3: Adaptive control

# What we covered in Unit 1

**Finite-horizon results**

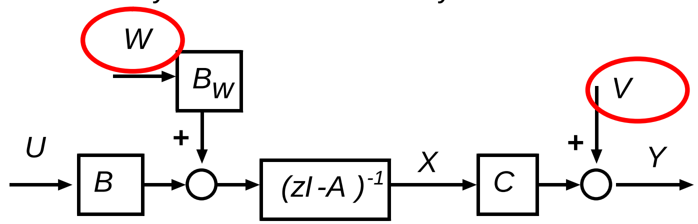
- Kalman filter
- Optimal LQR
- Optimal LQG
  - state feedback
  - output feedback

**Infinite-horizon results**

- Kalman filter
- Optimal LQR
- Optimal LQG
  - output feedback
- Frequency-shaped LQR

# Unit 1: Kalman filter (L6)

Linear system contaminated by noise:




*Two random disturbances:*

- *Input noise* $w(k)$ *- contaminates the state* $x(k)$
- *Measurement noise* $v(k)$ *- contaminates the output* $y(k)$

# Unit 1: Kalman filter (L6)

***Key Ideas:***

**Kalman Filter**

- Optimal state estimator (in the least squares sense) for uncorrelated Gaussian measurement and input noises
- Is expressed in a recursive form

# Unit 1: Kalman filter (L6)

- Kalman Filter – predictor/corrector

$$\hat{x}^o(k+1) = A\hat{x}(k) + Bu(k) \qquad \hat{x}^o(0) = E\{x(0)\}$$

$$\hat{x}(k+1) = \hat{x}^o(k+1) + F(k+1)\left[y(k+1) - \hat{x}^o(k+1)\right]$$

$$F(k+1) = M(k+1)C^T[C^T M(k+1)C + V(k+1)]^{-1}$$

$$M(k+1) = AZ(k)A^T + B_w W(k) B_w^T \qquad M(0) = X(0)$$

$$Z(k+1) = M(k+1) - M(k+1)C^T[C^T M(k+1)C + V(k+1)]^{-1} CM(k+1)$$

## Unit 1: Kalman filter (L6)

- State Space Kalman Filter

$$\hat{x}^o(k+1) = [A - L(k)C]\hat{x}^o(k) + \begin{bmatrix} B & L(k) \end{bmatrix} \begin{bmatrix} u(k) \\ y(k) \end{bmatrix} \qquad \hat{x}^o(0) = E\{x(0)\}$$

$$\hat{x}(k) = [I - F(k)C]\hat{x}^o(k) + \begin{bmatrix} 0 & F(k) \end{bmatrix} \begin{bmatrix} u(k) \\ y(k) \end{bmatrix}$$

$$F(k) = M(k)C^T \left[C\,M(k)C^T + V(k)\right]^{-1}$$

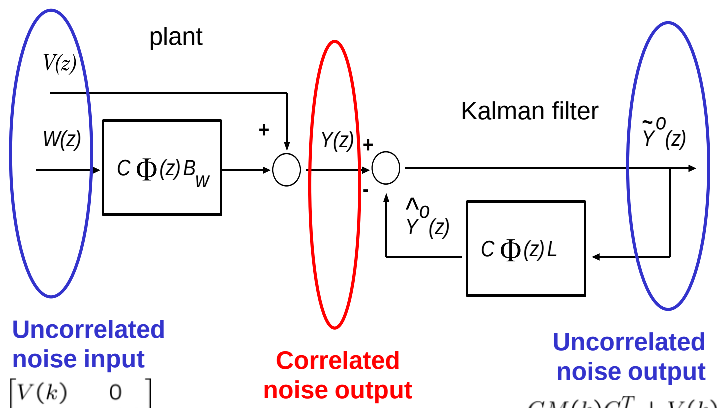
$$L(k) = AM(k)C^T \left[C\,M(k)C^T + V(k)\right]^{-1}$$

$$M(k+1) = AM(k)A^T + B_w W(k) B_w^T - AM(k)C^T \left[CM(k)C^T + V(k)\right]^{-1} CM(k)A^T \qquad M(0) = X_o$$

23

## Unit 1: Kalman filter (L6)

$$\Phi(z) = (zI - A)^{-1}$$




$$\begin{bmatrix} V(k) & 0 \\ 0 & W(k) \end{bmatrix} \qquad CM(k)C^T + V(k)$$

## Unit 1: Optimal LQR, LQG (L7-L8)

***Key Important Ideas:***

**Bellman's Dynamic Programming**

- used to derive LQR

**Stochastic Dynamic Programming**

- used to derive LQG

## Unit 1: Optimal LQR (L7)

$$x(k+1) = A\,x(k) + B\,u(k) \qquad x(0) = x_o$$

Cost functional:

$$J[x_o] = x^T(N)Q_f X(N) + \sum_{k=0}^{N-1} \left\{ \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \right\}$$

The optimal control is:

$$u^o(k) = -K(k+1)x(k)$$

$$K(k) = [B^T P(k) B + R]^{-1}[B^T P(k) A + S^T]$$

$$P(k-1) = A^T P(k) A + Q - [A^T P(k) B + S][B^T P(k) B + R]^{-1}[B^T P(k) A + S^T]$$

$$P(N) = Q_f$$

## Unit 1: Optimal LQG (L8)

$$x(k+1) = Ax(k) + Bu(k) + B_w w(k) \qquad E\{x(0)\} = x_0$$
$$y(k) = Cx(k) + v(k) \qquad E\{\tilde{x}(0)\tilde{x}^T(0)\} = X_0$$

Optimal output feedback control that minimizes the cost functional:

$$J[x_o] = E\left\{x^T(N)Q_f x(N) + \sum_{k=0}^{N-1}\left[x^T(k)Qx(k) + u^T(k)Ru(k)\right]\right\}$$

The optimal control is:

$$u^o(k) = -K(k+1)\ \hat{x}(k)$$

Where:

- The feedback gain $K(k+1)$ is obtained from the deterministic LQR solution.
- The state estimate $\hat{x}(k)$ is the a-posteriori Kalman filter state estimate.

## Unit 1: Infinite-horizon LQR (L10-L11)

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \qquad x(0) = x_o \\ y(k) &= C\,x(k) \end{aligned}$$

Cost functional:

$$J[x_o] = \sum_{k=0}^{\infty}\left\{y^T(k)y(k) + u^T(k)Ru(k)\right\}$$

The optimal control is:

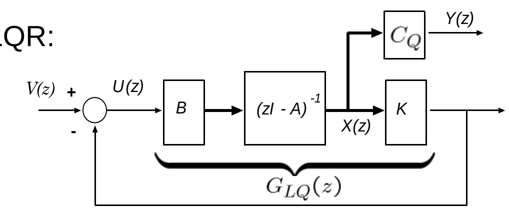
$$\left.\begin{aligned} u(k) &= -K\,x(k) \\ K &= \left[R + B^T P B\right]^{-1} B^T P A \\ P &= Q + A^T P A - A^T P B\left[R + B^T P B\right]^{-1} B^T P A \end{aligned}\right\} \begin{array}{l} (A, B) \\ \textit{stabilizable} \\ (C, A) \\ \textit{detectable} \end{array}$$

such that $A - BK$ is Schur

*Uniqueness and exponential closed-loop system stability*

## Unit 1: Infinite-horizon LQR (L10-L11)

Steady State LQR:




$$J = \sum_{k=0}^{\infty}\left\{y^2(k) + r\,u^2(k)\right\}$$

$$G_y(z) = C_Q\,(zI - A)^{-1}\,B$$

•*Return difference:* $$1 + \underbrace{K\,(zI - A)^{-1}\,B}_{G_{LQ}(z)}$$

## Unit 1: Infinite-horizon LQR (L10-L11)

Return Difference Equality (single input systems)

$$(1 + G_{LQ}(z^{-1}))(1 + G_{LQ}(z)) = \frac{R}{R + B^T P B}\left[1 + \frac{1}{R} G_y(z^{-1})^T G_y(z)\right]$$

- *LQR guaranteed robustness margins*
- *Reciprocal root locus*

30

## Unit 1: Stationary Kalman filter (L12)

**A-posteriori state observer structure:**

$$\begin{aligned}\hat{x}(k) &= \hat{x}^o(k) + F\tilde{y}^o(k)\\ \hat{x}^o(k+1) &= A\,\hat{x}(k) + B\,u(k)\\ \tilde{y}^o(k) &= y(k) - C\,\hat{x}^o(k)\end{aligned}$$

$$\begin{aligned}F &= MC^T\left[C\,MC^T + V\right]^{-1}\\ M &= AMA^T + B_wWB_w^T\\ &\quad - AMC^T(CMC^T+V)^{-1}CMA^T\end{aligned}$$

$A - AFC$ is Schur

31

## Unit 1: Stationary Kalman filter (L12)

**State space form:**

$$\hat{x}^o(k+1) = [A - LC]\hat{x}^o(k) + \begin{bmatrix} B & L\end{bmatrix}\begin{bmatrix}u(k)\\ y(k)\end{bmatrix}$$

$$\hat{x}(k) = [I - FC]\hat{x}^o(k) + \begin{bmatrix} 0 & F\end{bmatrix}\begin{bmatrix}u(k)\\ y(k)\end{bmatrix}$$

$$\begin{aligned}F &= MC^T\left[C\,MC^T + V\right]^{-1}\\ L &= AMC^T\left[CMC^T + V\right]^{-1}\\ M &= AMA^T + B_wWB_w^T\\ &\quad - AMC^T(CMC^T+V)^{-1}CMA^T\end{aligned}$$

$A - LC$ is Schur

## Unit 1: Stationary Kalman filter (L12)

Comparing ARE's and feedback gains, we obtain the following duality

*duality* →

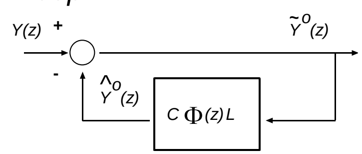
| **LQR** | **KF** |
|---|---|
| | |
| $P$ | $M$ |
| $A$ | $A^T$ |
| $B$ | $C^T$ |
| $R$ | $V$ |
| $C_Q^T$ | $B'_w = B_w W^{1/2}$ |
| $K$ | $L^T$ |
| $(A-BK)$ | $(A-LC)^T$ |

## Unit 1: Stationary Kalman filter (L12)

Return Difference Equality (SISO)

$$(1 + G_{KF}(z))(1 + G_{KF}(z^{-1})) = \gamma\,(1 + \frac{W}{V}G_w(z)G_w(z^{-1}))$$

- *Guaranteed robustness margins for the Kalman Filter feedback loop*




- *Reciprocal root locus*

36

## Unit 1: Stationary LQG (L13)

We want to regulate the state

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) + B_w\,w(k) \\ y(k) &= C\,x(k) + v(k) \end{aligned}$$

under

$$\left.\begin{aligned} E\{w(k)\} &= 0 \\ E\{v(k)\} &= 0 \\ E\{w(k+l)w^T(k)\} &= W\,\delta(l) \\ E\{v(k+l)v^T(k)\} &= V\,\delta(l) \\ E\{w(k+l)v^T(k)\} &= 0 \end{aligned}\right\}$$

***WSS zero-mean white Gaussian Noise***

37

## Unit 1: Stationary LQG (L13)

"Incremental" cost:

$$J' = E\left\{\frac{1}{N} x^T(N)\,Q_f\,x(N) + \frac{1}{N}\sum_{k=0}^{N-1}\left[x^T(k)\,C_Q^T C_Q\,x(k) + u^T(k)\,R\,u(k)\right]\right\}$$

Under the stationarity assumptions:

$$\lim_{N\to\infty} J' = J_s$$

$$J_s = E\left\{x^T(k) C_Q^T C_Q x(k) + u^T(k) R u(k)\right\}$$

## Unit 1: Stationary LQG (L13)

**Theorem:**

*The optimal <u>output feedback</u> control is given by:*

$$u^o(k) = -K\;\hat{x}(k)$$

*Where:*

- *The feedback gain* $K$ *is obtained from the deterministic infinite-horizon LQR solution.*
- *The state estimate* $\hat{x}(k)$ *is the a-posteriori Kalman filter state estimate.*

39

## Unit 1: Frequency-shaped LQR (L14)

**Key idea:** Make matrices $Q$ and $R$ functions of frequency

$$J = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{X^*(e^{j\omega})\underline{Q(e^{j\omega})}X(e^{j\omega}) + U^*(e^{j\omega})\underline{R(e^{j\omega})}U(e^{j\omega})\right\}d\omega$$

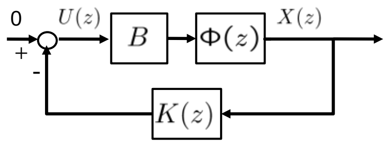
where

$$\underline{Q(e^{j\omega})} = Q_f^*(e^{j\omega})Q_f(e^{j\omega}) \succeq 0$$

$$\underline{R(e^{j\omega})} = R_f^*(e^{j\omega})R_f(e^{j\omega}) \succ 0$$

40

## Unit 1: Frequency-shaped LQR (L14)




$K(z)$ has the state space realization:

$$\begin{bmatrix} z_1(k+1) \\ z_2(k+1) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ -B_2K_1 & A_2 - B_2K_2 \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ -B_2K_x \end{bmatrix} x(k)$$

$$-u(k) = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} + K_x x(k)$$

41

## What we <u>skipped</u> in Unit 1

- Continuous-time versions of:
  - Kalman filter
  - Optimal LQG
  - Frequency-shaped LQR
- Loop transfer recovery

*Slides are posted on bSpace*

42

## Course Outline

- Unit 0: Probability
- Unit 1: State-space control, estimation
- Unit 2: Input/output control
- Unit 3: Adaptive control

43

## What we covered in Unit 2

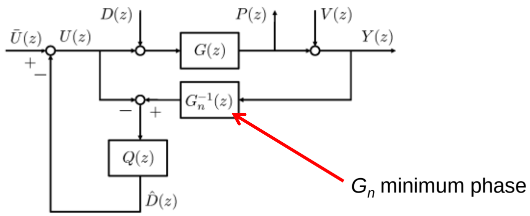
A collection of 4 SISO input/output control design techniques:

- Disturbance observer
- Pole placement, disturbance rejection, and tracking control
- Repetitive control
- Minimum variance regulators

# Unit 2: Disturbance observer (L15)

## Disturbance Observer

The following control structure is referred to as a disturbance observer:




The signals are:

| | |
|---|---|
| $U(z)$ : control input | $V(z)$ : measurement noise |
| $D(z)$ : disturbance | $\hat{D}(z)$ : estimate of $D(z)$ |
| $Y(z)$ : measured output | $P(z)$ : performance output |



# Unit 2: Disturbance observer (L15)

## Choosing $Q(z)$

Closed-loop dynamics:

$$\boxed{P = \frac{G_n(1+\Delta)(1-Q)}{1+Q\Delta}D + \frac{G_n(1+\Delta)}{1+Q\Delta}\bar{U} - \frac{Q(1+\Delta)}{1+Q\Delta}V}$$

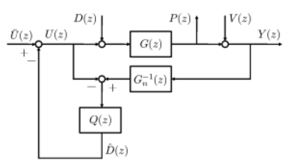
Concerns when choosing $Q(z)$:

1. **Robust disturbance rejection:** Choose $Q(e^{j\omega}) \approx 1$ at frequencies for which disturbance rejection is important
2. **Sensor noise insensitivity:** Choose $|Q(e^{j\omega})|$ to be small at frequencies for which sensor noise is large
3. **Robustness:** Choose $|Q(e^{j\omega})|$ to be small at frequencies for which $|\Delta(e^{j\omega})|$ is large



# Unit 2: Disturbance observer (L15)

## Choosing $Q(z)$




Concerns when choosing $Q(z)$:

4. **Realizability:** Choose $Q(z)$ so that $\hat{D}(z) = Q(z)[G_n^{-1}(z)Y(z) - U(z)]$ is realizable

   $\Rightarrow$ Choose $Q(z)$ realizable so that $\dfrac{Q(z)}{G_n(z)}$ is also realizable

   This is a constraint on the relative degree of $Q(z)$

# Unit 2: Pole placement, disturbance rejection, and tracking control (L16)

## Control objectives

1. **Pole Placement:** Closed-loop pole polynomial:

$$A_c(q^{-1}) = B^s(q^{-1})\, A_c'(q^{-1})$$

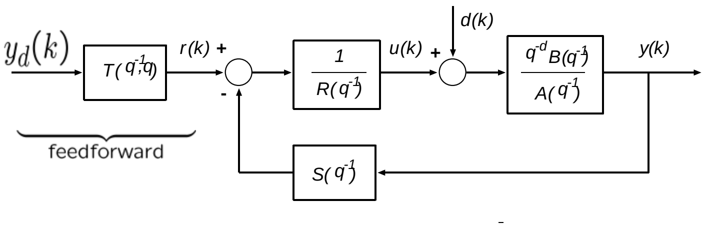
2. **Tracking:**

$$y(k) - y_d(k) \quad \text{small}$$

3. **Disturbance rejection:** Disturbance model:

$$A_d(q^{-1})d(k) = 0$$

## Unit 2: Pole placement, disturbance rejection, and tracking control (L16)




$$u(k) = \frac{1}{R(q^{-1})}\left[r(k) - S(q^{-1})y(k)\right]$$

$$r(k) = T(q^{-1},q)\, y_d(k)$$ *Feedforward (a-causal)*

## Unit 2: Pole placement, disturbance rejection, and tracking control (L16)

Diophantine equation: Obtain polynomials $R'(q^{-1}),\ S(q^{-1})$ which satisfy:

$$\boxed{A'_c(q^{-1}) = A_d(q^{-1})\, A(q^{-1})\, R'(q^{-1}) + q^{-\mathrm{d}} B^u(q^{-1})\, S(q^{-1})}$$

*Closed-loop poles without cancelled zeros*

*Disturbance annihilating polynomial*

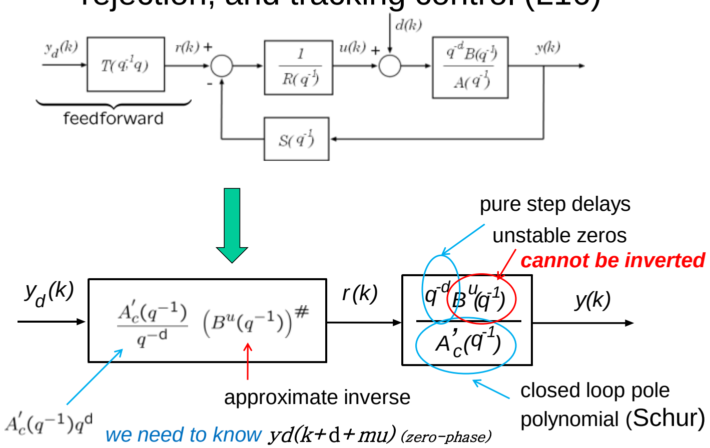
*Plant poles*

*Unstable plant zeros*

$$\boxed{\begin{aligned} R(q^{-1}) &= R'(q^{-1})\, A_d(q^{-1})\, B^s(q^{-1}) \\ A_c(q^{-1}) &= B^s(q^{-1})\, A'_c(q^{-1}) \end{aligned}}$$

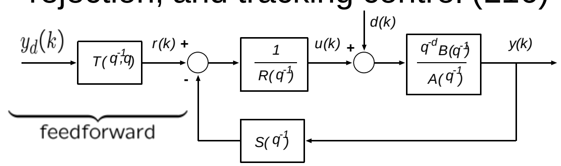
## Unit 2: Pole placement, disturbance rejection, and tracking control (L16)

50




we need to know *yd(k+d+mu)* *(zero-phase)*

## Unit 2: Pole placement, disturbance rejection, and tracking control (L16)

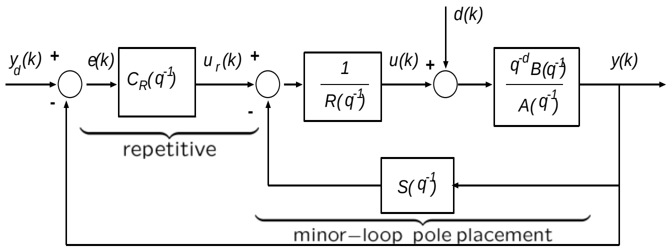



$$T(q^{-1},q) = A'_c(q^{-1})q^{\mathrm{d}}\,\frac{B^u(q)}{[B^u(1)]^2}$$

*Tracking error dynamics*

$$A'_c(q^{-1})\left\{\underbrace{G_{zp}(q^{-1},q)}_{\text{zero-phase transfer function}}y_d(k) - y(k)\right\} = 0$$

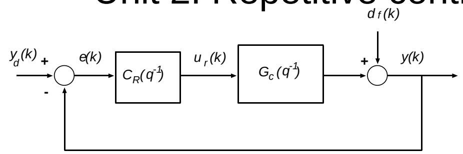
## Unit 2: Repetitive control (L17)




***Control strategy:*** *We design the controller in two stages*

1. ***Minor-loop pole placement:*** *Place minor-loop poles, (that will be cancelled later)*
2. ***Repetitive compensator:*** *Reject periodic disturbance*
   *Follow periodic reference*

## Unit 2: Repetitive control (L17)

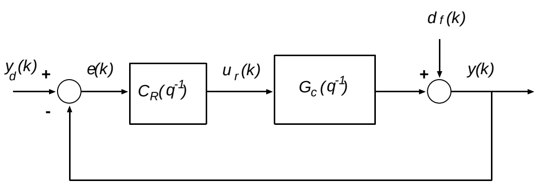



$$G_c(q^{-1}) = \frac{q^{-\mathrm{d}}B^u(q^{-1})}{A_c'(q^{-1})}$$

Repetitive control strategy:

1. Cancel stable poles $A_c'(q^{-1}) = 0$
2. Zero-phase error compensation $B^u(q^{-1})$
3. Include annihilating polynomial $1 - q^{-N} = 0$
   $q^N - 1 = 0$

## Unit 2: Repetitive control (L17)



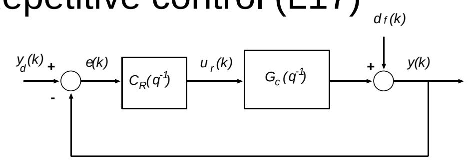

Repetitive controller:

$$\boxed{C_R(q^{-1}) = \frac{k_r}{b}\left[\frac{q^{-N}}{1-q^{-N}}\right]\left[q^{\mathrm{d}}A_c'(q^{-1})\,B^u(q)\right]}$$

$$(N \geq \mathrm{d} + m_u)$$

## Unit 2: Repetitive control (L17)

*Add Q-filter for robustness*




$$\boxed{C_R(q^{-1}) = \frac{k_r}{b}q^{-(N-\mathrm{d})}\frac{A_c'(q^{-1})B^u(q)}{1-\underline{Q(q,q^{-1})}q^{-N}}}$$

$Q(q,q^{-1})$ *moving average filter with zero-phase shift characteristics*

*Controller's* $N$ *open-loop poles are no longer on the unit circle*

## Unit 2: Minimum variance regulator (L18)

- Plant (contains no zeros on unit circle):

$$A(q^{-1})\,y(k) = q^{-\mathrm{d}}\,B(q^{-1})\,u(k) + C(q^{-1})\,\epsilon(k)$$

- MVR feedback law:

$$u(k) \;=\; \frac{-S(q^{-1})}{B^s(q^{-1})R(q^{-1})}\,y(k)$$

- Diophantine equation:

$$\boxed{\underbrace{C(q^{-1})\bar{B}^u(q^{-1})}_{\text{Since } C \text{ and } \bar{B}^u \text{ are anti-Schur, this product is anti-Schur}} = A(q^{-1})\,R(q^{-1}) + q^{-\mathrm{d}}B^u(q^{-1})\,S(q^{-1})}$$

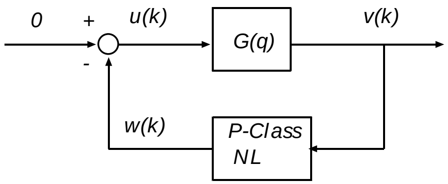


## Course Outline

- Unit 0: Probability
- Unit 1: State-space control, estimation
- Unit 2: Input/output control
- Unit 3: Adaptive control



## Unit 3: Hyperstability (L19)




**Asymptotical Hyperstability Theorem:** The above feedback system is asymptotically hyperstable **iff** the transfer function $G(z)$ of the LTI block is **Strictly Positive Real.**

Under this condition, the state of *G(q), u(k),* and *v(k)* all converge to 0



## Unit 3: Hyperstability (L19)

$$G(z) = C(zI - A)^{-1}B + D$$

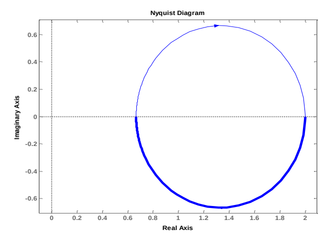
Is **Strictly Positive Real (SPR)** iff:

1. All poles of $G(z)$ are asymptotically stable.
2. $G(e^{j\omega}) + G^T(e^{-j\omega}) \succ 0$

   for all $0 \leq \omega \leq \pi$

Example:

$$G(z) = \frac{z}{z + 0.5}$$



## Unit 3: Least squares parameter estimation (L20)

Parameter estimate after $k$ observations: $\hat{\theta}(k)$

$$V(\hat{\theta}(k)) = \frac{1}{2}\sum_{j=1}^{k}\left[y(j) - \phi^T(j-1)\,\hat{\theta}(k)\right]^2$$

*Optimal $\hat{\theta}(k)$ satisfies the* ***Normal Equation***

$$\left[\sum_{i=1}^{k}\phi(i-1)\phi^T(i-1)\right]\hat{\theta}(k) = \sum_{i=1}^{k}\phi(i-1)y(i)$$

## Unit 3: Least squares parameter estimation (L20)

Recursive implementation of general PAA:

$$e^o(k+1) \;=\; y(k+1) - \phi^T(k)\hat{\theta}(k)$$

$$e(k+1) = \frac{\lambda_1(k)}{\lambda_1(k) + \phi^T(k)F(k)\phi(k)}e^o(k+1)$$

$$\hat{\theta}(k+1) = \hat{\theta}(k) + \frac{1}{\lambda_1(k)}F(k)\phi(k)e(k+1)$$

$$F(k+1) = \frac{1}{\lambda_1(k)}\left[F(k) - \lambda_2(k)\frac{F(k)\phi(k)\phi^T(k)F(k)}{\lambda_1(k) + \lambda_2(k)\phi^T(k)F(k)\phi(k)}\right]$$

*Initial conditions:* $F(0) = F^T(0) > 0 \qquad \hat{\theta}(0)$

## Unit 3: Series-parallel least squares convergence (L21-L22)

Two-step approach:

1. Prove output estimation error convergence

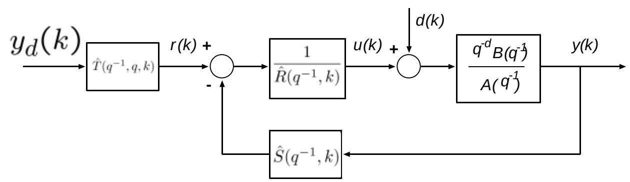
- Asymptotic hyperstability

2. Prove parameter convergence

*Not covered this semester*

- Persistence of excitation

## Unit 3: Indirect Adaptive Control (L23)

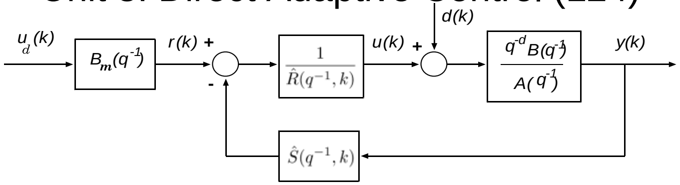



$$\hat{R}(q^{-1},k)\,u(k) = \hat{T}(q^{-1},q,k)\,y_d(k) - \hat{S}(q^{-1},k)y(k)$$

1. Identify plant parameters using RLS
   - Prefiltering
   - Parameter projection
2. Compute controller parameters at each *k*, e.g.

$$A_c'(q^{-1}) \;=\; A_d(q^{-1})\,\hat{A}(q^{-1},k)\,\hat{R}'(q^{-1},k) + q^{-d}\,\hat{B}^u(q^{-1},k)\hat{S}(q^{-1},k)$$

## Unit 3: Direct Adaptive Control (L24)




$$\widehat{R}(q^{-1},k)\,u(k) = B_m(q^{-1})\,u_d(k) - \widehat{S}(q^{-1},k)y(k)$$

Controller parameters updated directly using RLS

- Prefiltering
- Parameter projection