ME 233 Spring 2012
Solution to Homework #6

1. Choose the tuning parameters
$$\alpha_1 = 10^{-2},\ \alpha_2 = 1.0,\ \alpha_3 = 0.04$$
and design the controller using the following code

```
load me233hw6p1_model

alpha = [1e-2 1.0 0.04];
%lqg cost
Q = C1'*C1 + alpha(1)*C2'*C2;
R = diag(alpha(2:3));

%LQR design
sys_lqr = ss(A,B,[],[],-1);
K = lqr(sys_lqr,Q,R);

%Kalman filter design
sys_kf = ss(A,[B Bw],[C1; C2],zeros(2,4),-1);
Kest = kalman(sys_kf, eye(2), eye(2));

%optimal LQG controller
Klqg = lqgreg(Kest,K,'current');

%closed-loop system
Glft = ss(A, [Bw zeros(11,2) B], [C1; C2; zeros(2,11); C1; C2], ...
    [zeros(2,6); zeros(2,4) eye(2); zeros(2,2) eye(2) zeros(2,2)], -1);
Gcl = lft(Glft,Klqg);

%results
disp(['3 sigma PES: ' num2str(3*norm(Gcl(1,:)))])
disp(['3 sigma RPES: ' num2str(3*norm(Gcl(2,:)))])
disp(['3 sigma Uv: ' num2str(3*norm(Gcl(3,:)))])
disp(['3 sigma Um: ' num2str(3*norm(Gcl(4,:)))])
```

Then, we have

$$3\sqrt{E\{p_1^2(k)\}} = 21.75 \qquad 3\sqrt{E\{p_2^2(k)\}} = 87.48$$
$$3\sqrt{E\{u_1^2(k)\}} = 4.95 \qquad 3\sqrt{E\{u_2^2(k)\}} = 19.95$$

2. The proof is devided into two parts. First we are going to check that (A_e, B_e) is stabilizable, and then we are going to check the transmission zero condition.

(a) We first consider:

$$\begin{bmatrix} A^T - \lambda I & B_1^T & 0 \\ 0 & A_1^T - \lambda I & 0 \\ 0 & 0 & A_2^T - \lambda I \\ B^T & 0 & B_2^T \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = 0$$

where $|\lambda| \geq 1$. We then get:

$$(A_1^T - \lambda I)Y = 0$$
$$(A_2^T - \lambda I)Z = 0$$

Then by stability of A_1 and A_2, we know that λ is not an eigenvalue of A_1 or A_2, which implies that $Y = 0$ and $Z = 0$. After, we have:

$$\begin{bmatrix} A^T - \lambda I \\ B^T \end{bmatrix} X = 0$$

Since (A, B) is stabilizable, we obtain $X = 0$. So we deduce that:

$$\text{nullity} \begin{bmatrix} A_e^T - \lambda I \\ B_e^T \end{bmatrix} = \text{nullity} \begin{bmatrix} A^T - \lambda I & B_1^T & 0 \\ 0 & A_1^T - \lambda I & 0 \\ 0 & 0 & A_2^T - \lambda I \\ B^T & 0 & B_2^T \end{bmatrix} = 0$$

Since this equality holds $\forall \lambda$ such that $|\lambda| \geq 1$, we have that (A_e, B_e) is stabilizable.

(b) Now let consider:

$$\begin{bmatrix} A - \lambda I & 0 & 0 & B \\ B_1 & A_1 - \lambda I & 0 & 0 \\ 0 & 0 & A_2 - \lambda I & B_2 \\ D_1 & C_1 & 0 & 0 \\ 0 & 0 & C_2 & D_2 \end{bmatrix} \begin{bmatrix} W \\ X \\ Y \\ Z \end{bmatrix} = 0, \qquad |\lambda| = 1$$

We notice that:

$$\begin{bmatrix} A_2 - \lambda I & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} Y \\ Z \end{bmatrix} = 0$$

And because nullity $\begin{bmatrix} A_2 - \lambda I & B_2 \\ C_2 & D_2 \end{bmatrix} = 0$ whenever $|\lambda| = 1$, then we obtain $\begin{bmatrix} Y \\ Z \end{bmatrix} = 0$. Thanks to this result, we have:

$$\begin{bmatrix} A - \lambda I & 0 \\ B_1 & A_1 - \lambda I \\ D_1 & C_1 \end{bmatrix} \begin{bmatrix} W \\ X \end{bmatrix} = 0$$

We can consider two different cases.

i. First we consider that λ is an eigenvalue of A. Then we get:

$$\begin{bmatrix} A_1 - \lambda I & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} X \\ W \end{bmatrix} = 0$$

And because we have nullity $\begin{bmatrix} A_1 - \lambda I & B_1 \\ C_1 & D_1 \end{bmatrix} = 0$ whenever λ is an eigenvalue of A satisfying $|\lambda| = 1$, then we obtain $\begin{bmatrix} X \\ W \end{bmatrix} = 0$.

ii. Now let consider that λ is not an eigenvalue of A. So we have:

$$\begin{aligned}(A-\lambda I)W &= 0\\ \Rightarrow W &= 0\end{aligned}$$

This implies that:

$$(A_1-\lambda I)X = 0$$

Then by the stability condition, we obtain $X = 0$.

Combining all the results we obtained above, we deduce:

$$\text{nullity}\begin{bmatrix} A^T-\lambda I & 0 & 0 & B \\ B_1 & A_1^T-\lambda I & 0 & 0 \\ 0 & 0 & A_2^T-\lambda I & B_2 \\ D_1 & C_1 & 0 & 0 \\ 0 & 0 & C_2 & D_2 \end{bmatrix} = 0, \forall\lambda \text{ such that } |\lambda| = 1$$

Then we conclude that $C_e(zI-A_e)^{-1}B_e+D_e$ has no transmission zeros on the unit circle.