ME 233 Advance Control II

Lecture 15

Deterministic Input/Output Approach to SISO Discrete Time Systems

Pole Placement, Disturbance Rejection and Tracking Control

SISO ARMA models

- SISO State space model

$$x(k+1) = A\,x(k) + B\,u(k)$$
$$y(k) = C\,x(k) + D\,u(k)$$

Where all inputs and outputs are scalars:

- $u(k) \in \mathcal{R}$ control input
- $y(k) \in \mathcal{R}$ output
- $x(k) \in \mathcal{R}^n$ state

SISO transfer function

$$Y(z) = \left[C(sI-A)^{-1}B + D\right] U(z) \quad = \frac{B^*(z)}{A^*(z)} U(z)$$

$$A^*(z) = \det\{(zI-A)\} = z^n + a_1 z^{n-1} + \cdots + a_n$$

$$B^*(z) = C Adj\{(sI-A)\}B + D$$

$$= b_o\, z^m + b_1\, z^{n-1} + \cdots + b_m$$

$d = n - m \geq 0$ relative degree

ARMA Models

- Define the ***back-step*** operator q^{-1} such that

$$y(k-1) = q^{-1}\, y(k)$$

- the polynomials

$$A(q^{-1}) = q^{-n} A^*(q) = 1 + a_1 q^{-1} + \cdots + a_n q^{-n}$$

$$B(q^{-1}) = q^{-m} B^*(q) = b_o + b_1 q^{-1} + \cdots + b_m q^{-m}$$

- relative degree (pure time delay)

$$d = n - m$$

SISO ARMA models

$$A(q^{-1})\, y(k) = q^{-\mathsf{d}}\, B(q^{-1})\, u(k)$$

$$y(k) \;=\; b_o u(k-d) + \cdots + b_m u(k-n)$$

$$-a_1 y(k-1) - \cdots - a_n y(k-n)$$

SISO ARMA models with persistent disturbances

SISO ARMA model with disturbance

$$A(q^{-1})\, y(k) = q^{-\mathsf{d}}\, B(q^{-1})\, [\, u(k) + d(k)]$$

Where all inputs and outputs are scalars:

- $u(k)$ control input
- $d(k)$ persistent (deterministic) but unknown disturbance
- $y(k)$ output

Deterministic SISO ARMA models

$$A(q^{-1})\, y(k) = q^{-\mathsf{d}}\, B(q^{-1})\, [\, u(k) + d(k)]$$

Where polynomials:

$$A(q^{-1}) \;=\; 1 + a_1 q^{-1} + \cdots + a_n q^{-n}$$

$$B(q^{-1}) \;=\; b_o + b_1 q^{-1} + \cdots + b_m q^{-m}$$

are co-prime and d is the ***known*** pure time delay

Polynomials in *q-1*

- Monic polynomial

$$A(q^{-1}) \;=\; 1 + a_1 q^{-1} + \cdots + a_n q^{-n}$$

non-delay coefficient is 1

$$A^*(q) = q^n A(q^{-1})$$

$$A^*(q) \;=\; q^n + a_1 q^{n-1} + \cdots + a_n$$

leading coefficient is 1

Polynomials in q^{-1}

- Co-prime polynomials have no common roots

$$A(q^{-1}) \;=\; 1 + a_1 q^{-1} + \cdots + a_n q^{-n}$$

$$B(q^{-1}) \;=\; b_o + b_1 q^{-1} + \cdots + b_m q^{-m}$$

are co-prime if and only if

for all $p \in \mathcal{C}$ such that $A(p) \;=\; 0$

$$B(p) \;\neq\; 0$$

Polynomials in q^{-1}

- Schur polynomials in q^{-1} have all roots outside the unit circle

$$A(q^{-1}) \;=\; 1 + a_1 q^{-1} + \cdots + a_n q^{-n}$$

is Schur if and only if, for all sequences $\{y(k)\} \in \mathcal{R}$

such that $A(q^{-1})\, y(k) \;=\; 0$

$$\lim_{k \to \infty} y(k) = 0$$

Polynomials in q^{-1}

- Schur polynomials in q^{-1} have all roots outside the unit circle

$$A(q^{-1}) \;=\; 1 + a_1 q^{-1} + \cdots + a_n q^{-n}$$

is Schur if and only if,

for all $p \in \mathcal{C}$ such that $A(p^{-1}) \;=\; 0\,, \qquad |p| < 1$

$$A^*(p) = 0 \qquad\qquad A^*(q) \;=\; q^n A(q^{-1})$$

Factorization of the zero polynomial $B(q^{-1})$

The m order zero polynomial:

$$B^*(q) = q^m\, B(q^{-1}) = 0$$

has

- m_u zeros which **<u>we do not wish to cancel</u>**.
- m_s zeros inside the unite circle (asymptotically stable) which we wish to cancel.

$$B(q^{-1}) = B^s(q^{-1})\, B^u(q^{-1})$$

$B^s(q^{-1})$ is Schur

$B^{u*}(q) = q^{m_u} B^u(q^{-1})$ has zeros which we **<u>do not wish to cancel</u>**

Example

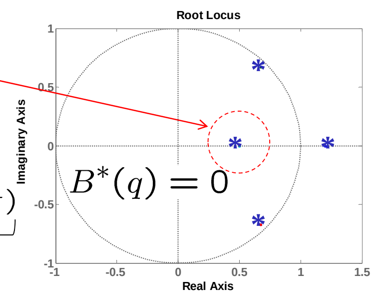
$$B^*(q) = 1.2(q - 0.5)(q - 1.2)(q - 0.95e^{j\frac{\pi}{4}})(q - 0.95e^{-j\frac{\pi}{4}})$$

$$B(q^{-1}) = 1.2(1 - 0.5q^{-1})(1 - 1.2q^{-1})(1 - 0.95e^{j\frac{\pi}{4}}q^{-1})(1 - 0.95e^{-j\frac{\pi}{4}}q^{-1})$$
$$= B^s(q^{-1})\, B^u(q^{-1})$$

<u>Example</u>

$$B^s(q) = (1 - 0.5q^{-1})$$

$$B^u(q^{-1}) = 1.2(1 - 1.2q^{-1})\underbrace{(1 - 0.95e^{j\frac{\pi}{4}}q^{-1})(1 - 0.95e^{-j\frac{\pi}{4}}q^{-1})}_{\textit{we could have chosen to cancel these}}$$



Deterministic SISO ARMA models

The zero polynomial:

$$B(q^{-1}) = B^s(q^{-1})\, B^u(q^{-1})$$

Without loss of generality, we will assume that

$$B^s(q^{-1}) = 1 + \cdots + b^s_{m_s}\, q^{-m_s}$$

$$B^u(q^{-1}) = b_o + \cdots + b^u_{m_u}\, q^{-m_u}$$

i.e. the Schur polynomial $B^s(q^{-1})$ is monic

Control Objectives

1. **<u>Pole Placement</u>**: The poles of the closed loop system must be placed at specific locations in the complex plane.

- **Closed loop pole polynomial:**

$$\boxed{A_c(q^{-1}) = B^s(q^{-1})\, A_c'(q^{-1})}$$

Where:

- $B^s(q^{-1})$ cancelable plant zeros
- $A_c'(q^{-1})$ monic Schur polynomial chosen by the designer

$$A_c'(q^{-1}) = 1 + a_{c1}'q^{-1} + \cdots + a_{c n_c'}'q^{-n_c'}$$

Control Objectives

2. **<u>Tracking</u>**: The output sequence $y(k)$ must follow a ***reference*** sequence $y_d(k)$ which is known

- **Reference model:**

$$A_m(q^{-1})y_d(k) = q^{-\mathrm{d}}\, B_m(q^{-1})\, u_d(k)$$

Where:

- $y_d(k)$ **reference output sequence,** which is known in advance (i.e. $y_d(k+L)$ is available at instance k for some $L>0$).
- $A_m(q^{-1})$ monic Schur polynomial chosen by the designer
- $B_m(q^{-1})$ polynomial chosen by the designer

Control Objectives

3. <u>Disturbance rejection</u>: The closed loop system must reject a class of **<u>persistent</u>** disturbances $d(k)$

- **Disturbance model:**

$$\boxed{A_d(q^{-1})d(k) = 0}$$

Where

- $A_d(q^{-1})$ is a ***known*** annihilating polynomial <u>with roots on the unit circle</u>

- $A_d(q^{-1}),\ B(q^{-1})$ are co-prime

Deterministic disturbance examples

a) Constant disturbance:

$$d(k) = d(k-1)$$

Then,

$$\boxed{A_d(q^{-1}) = 1 - q^{-1}}$$

b) Sinusoidal disturbance of ***<u>known</u>*** frequency:

$$d(k) = D\ \sin(\omega\, k + \phi)$$

Then,

$$\boxed{A_d(q^{-1}) = 1 - 2\cos(\omega)\, q^{-1} + q^{-2}}$$

Deterministic disturbance examples

c) Periodic disturbance of ***<u>known</u>*** period N

$$d(k) = d(k-N)$$

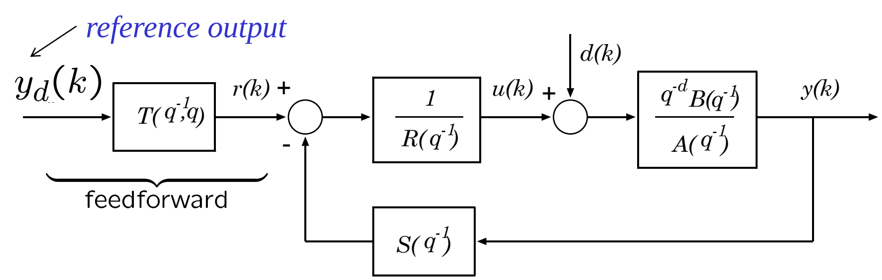
Then,

$$\boxed{A_d(q^{-1}) = 1 - q^{-N}}$$

In all of these three examples, the polynomial $A_d(q^{-1})$ has its roots <u>on the unit circle</u>.

Control Law

- Feedback and feedforward actions:

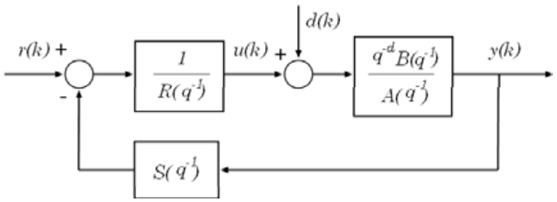


$$u(k) = \frac{1}{R(q^{-1})}\left[r(k) - S(q^{-1})y(k)\right]$$

$$r(k) = T(q^{-1},q)\, y_d(k)$$

Feedforward action (a-causal)

Close Loop TF from *r(k)* to *y(k)*

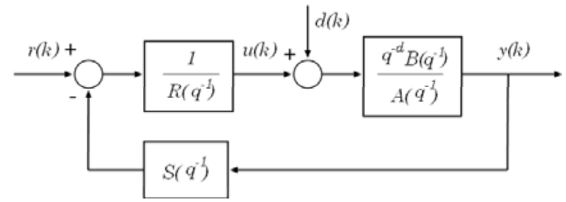


$$\frac{y(k)}{r(k)} = \frac{\frac{1}{R(q^{-1})}\,\frac{q^{-\mathrm{d}}B(q^{-1})}{A(q^{-1}}}{1+\frac{S(q^{-1})}{R(q^{-1})}\,\frac{q^{-\mathrm{d}}B(q^{-1})}{A(q^{-1})}}$$

$$= \frac{q^{-\mathrm{d}}B(q^{-1})}{\underbrace{A(q^{-1})R(q^{-1}) + q^{-\mathrm{d}}B(q^{-1})S(q^{-1})}_{A_c(q^{-1})}}$$

Desired closed loop characteristic polynomial

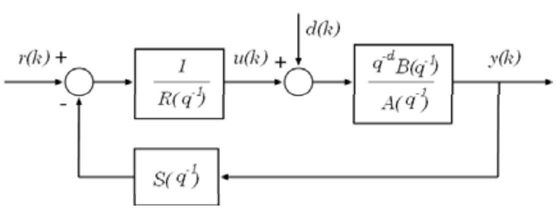
Close Loop TF from *r(k)* to *y(k)*



$$\frac{y(k)}{r(k)} = \frac{q^{-\mathrm{d}}B(q^{-1})}{A_c(q^{-1})} = \cancel{\frac{B^s(q^{-1})}{B^s(q^{-1})}}\,\frac{q^{-\mathrm{d}}B^u(q^{-1})}{A_c'(q^{-1})}$$

Since $\begin{cases} B(q^{-1}) = B^s(q^{-1})\,B^u(q^{-1}) \\ A_c(q^{-1}) = B^s(q^{-1})\,A_c'(q^{-1}) \end{cases}$

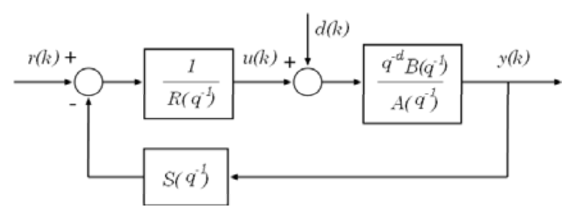
Close Loop TF from *r(k)* to *y(k)*



$$\Big[\underbrace{A(q^{-1})R(q^{-1}) + q^{-\mathrm{d}}B(q^{-1})S(q^{-1})}_{A_c(q^{-1})}\Big]\,y(k) = q^{-\mathrm{d}}B(q^{-1})\,r(k)$$

Desired closed loop polynomial

Close Loop TF from *r(k)* to *y(k)*



$$\Big[A(q^{-1})R(q^{-1}) + q^{-\mathrm{d}}B(q^{-1})S(q^{-1})\Big]\,y(k) = q^{-\mathrm{d}}B(q^{-1})\,r(k)$$

We need to find polynomials $R(q^{-1})$ and $S(q^{-1})$ so that

$$A(q^{-1})R(q^{-1}) + q^{-\mathrm{d}}B(q^{-1})S(q^{-1}) = A_c(q^{-1})$$

The Diophantine (Bezout) equation

- Given the ***co-prime*** polynomials

$$\mathcal{A}(q^{-1}) \;=\; 1 + a_1 q^{-1} + \cdots + a_n q^{-n}$$

$$\mathcal{B}(q^{-1}) \;=\; b_o + b_1 q^{-1} + \cdots + b_m q^{-m}$$

  - $\mathcal{A}(q^{-1})$ is monic and order n
  - $\mathcal{B}(q^{-1})$ is order m

- and a monic polynomial of order n_c

$$\mathcal{C}(q^{-1}) \;=\; 1 + c_1 q^{-1} + \cdots + c_{n_c} q^{-n_c}$$

The Diophantine (Bezout) equation

We wish to find the polynomials

same order as $\beta(q^{-1})$

$$\mathcal{R}(q^{-1}) \;=\; 1 + r_1 q^{-1} + \cdots + r_m\, q^{-m}$$

$$\mathcal{S}(q^{-1}) \;=\; s_o + \cdots + s_{n_s}\, q^{-n_s}$$

which satisfy the Diophantine equation:

$$\boxed{\mathcal{C}(q^{-1}) = \mathcal{A}(q^{-1})\,\mathcal{R}(q^{-1}) + q^{-1}\,\mathcal{B}(q^{-1})\mathcal{S}(q^{-1})}$$

given

The Diophantine (Bezout) equation

Expanding in terms of q^{-1} coefficients:

$$\underbrace{\mathcal{C}(q^{-1})}_{\text{order } n_c} = \underbrace{\mathcal{A}(q^{-1})\,\mathcal{R}(q^{-1})}_{\text{order } n+m} + \underbrace{q^{-1}\,\mathcal{B}(q^{-1})\mathcal{S}(q^{-1})}_{\text{order } m+n_s+1}$$

$$\boxed{n_s = \max\{n-1, n_c - m - 1\}}$$

The Diophantine (Bezout) equation

Expanding in terms of q^{-1} coefficients:

$$\mathcal{C}(q^{-1}) = \mathcal{A}(q^{-1})\,\mathcal{R}(q^{-1}) + q^{-1}\,\mathcal{B}(q^{-1})\mathcal{S}(q^{-1})$$

We obtain:

$$\begin{array}{c} q^{-1}\\ q^{-2}\\ \vdots\\ \vdots\\ \vdots\\ \vdots\\ q^{-(n_s+m+1)} \end{array} \qquad D\begin{bmatrix} r_1\\ r_2\\ \vdots\\ r_m\\ s_o\\ \vdots\\ s_{n_s}\end{bmatrix} = \begin{bmatrix} c_1\\ c_2\\ \vdots\\ c_{nc}\\ 0\\ \vdots\\ 0\end{bmatrix} - \begin{bmatrix} a_1\\ a_2\\ \vdots\\ a_n\\ 0\\ \vdots\\ 0\end{bmatrix}$$

The Diophantine (Bezout) equation

Where the matrix $D \in \mathcal{R}^{(n_s+1+m)\times(n_s+1+m)}$ is given by:

$$D = \left[\begin{array}{ccccc|ccccc}
1 & 0 & 0 & \cdots & 0 & b_0 & 0 & 0 & \cdots & 0 \\
a_1 & 1 & 0 & \cdots & 0 & b_1 & b_0 & 0 & \cdots & 0 \\
a_2 & a_1 & 1 & \cdots & 0 & b_2 & b_1 & b_0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n-1} & a_{n-2} & a_{n-3} & \cdots & 1 & b_m & b_{m-1} & b_{m-2} & \cdots & b_o \\
a_n & a_{n-1} & a_{n-2} & \cdots & a_1 & 0 & b_m & b_{m-1} & \cdots & b_1 \\
0 & a_n & a_{n-1} & \cdots & a_2 & 0 & 0 & b_m & \cdots & b_2 \\
0 & 0 & a_n & \cdots & a_3 & 0 & 0 & 0 & \cdots & b_3 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & a_n & 0 & 0 & 0 & \cdots & b_m
\end{array}\right]$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{m} \qquad \underbrace{\qquad\qquad\qquad\qquad}_{n_s+1}$$

The Diophantine (Bezout) equation

Theorem: D is nonsingular iff the polynomials $\mathcal{A}(q^{-1})$ and $q^{-1}\mathcal{B}(q^{-1})$ are co-prime.

The solution to the Diophantine equation is:

$$\boxed{\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \\ s_o \\ \vdots \\ s_{n_s} \end{bmatrix} = D^{-1}\left\{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{nc-1} \\ c_{nc} \\ \vdots \\ 0 \end{bmatrix} - \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\}.}$$

Example: $\mathcal{C}(q^{-1}) = \mathcal{A}(q^{-1})\,\mathcal{R}(q^{-1}) + q^{-1}\,\mathcal{B}(q^{-1})\mathcal{S}(q^{-1})$

Let

$$\mathcal{C}(q^{-1}) = (1-0.5q^{-1})(1-0.8q^{-1}) \qquad \text{order } n_c = 2$$
$$= (1-1.3q^{-1}0.4q^{-2})$$

$$\mathcal{A}(q^{-1}) = (1-q^{-1})(1-1.2q^{-1}) \qquad \text{order } n = 2$$
$$= (1-2.2q^{-1}+1.2q^{-2})$$

$$\mathcal{B}(q^{-1}) = (2q^{-1}+2.4q^{-2}) \qquad \text{order } m = 2$$

$$\text{Solve for } \begin{cases} \mathcal{R}(q^{-1}) = 1 + r_1q^{-1} + r_2q^{-2} & \text{order } m = 2 \\ \mathcal{S}(q^{-1}) & \text{order } n_s \end{cases}$$

Example: $\mathcal{C}(q^{-1}) = \mathcal{A}(q^{-1})\,\mathcal{R}(q^{-1}) + q^{-1}\,\mathcal{B}(q^{-1})\mathcal{S}(q^{-1})$

$$n_s = \max\{n-1, n_c-m-1\} = \max\{2-1, 2-2-1\} = 1$$

$$\underbrace{(1-1.3q^{-1}0.4q^{-2})}_{\mathcal{C}(q^{-1})} = \underbrace{(1-2.2q^{-1}+1.2q^{-2})}_{\mathcal{A}(q^{-1})}\underbrace{(1+r_1q^{-1}+r_2q^{-2})}_{\mathcal{R}(q^{-1})}$$

$$+q^{-1}\underbrace{(2q^{-1}+2.4q^{-2})}_{\mathcal{B}(q^{-1})}\underbrace{(s_o+s_1q^{-1})}_{\mathcal{S}(q^{-1})}$$

4 equations and 4 unknowns

Example: $\mathcal{C}(q^{-1}) = \mathcal{A}(q^{-1})\,\mathcal{R}(q^{-1}) + q^{-1}\,\mathcal{B}(q^{-1})\mathcal{S}(q^{-1})$

Equating coefficients of powers of q^{-1}

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2.2 & 1 & 2 & 0 \\ 1.2 & -2.2 & 2.4 & 2 \\ 0 & 1.2 & 0 & 2.4 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ s_0 \\ s_1 \end{bmatrix} = \begin{bmatrix} -1.3 \\ 0.4 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -2.2 \\ 1.2 \\ 0 \\ 0 \end{bmatrix}$$

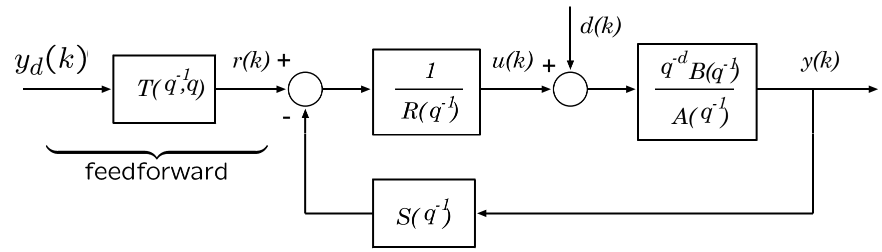
Solution:

$$\mathcal{R}(q^{-1}) = 1 + 0.9q^{-1} + 0.57q^{-2}$$

$$\mathcal{S}(q^{-1}) = 0.31 - 0.28q^{-1}$$

Control Law

- Feedback and feedforward actions:



$$u(k) = \frac{1}{R(q^{-1})}\left[r(k) - S(q^{-1})y(k)\right]$$

$$r(k) = T(q^{-1}, q)\,y_d(k)$$ *Feedforward (a-causal)*

Feedback Controller

Diophantine equation: Obtain polynomials $R(q^{-1}),\ S(q^{-1})$ which satisfy:

$$A_c(q^{-1}) = A(q^{-1})\,\underline{R(q^{-1})} + q^{-\mathrm{d}}B(q^{-1})\,\underline{S(q^{-1})}$$

Close loop poles *Plant poles* *plant zeros*

$$\boxed{\begin{aligned} A_c(q^{-1}) &= \underline{B^s(q^{-1})}\,A_c'(q^{-1}) \\ R(q^{-1}) &= R'(q^{-1})\,\underline{A_d(q^{-1})}\,\underline{B^s(q^{-1})} \end{aligned}}$$

We will factor out the $B^s(q^{-1})$ *polynomial next*

Controller Diophantine equation

Factor out $B^s(q^{-1})$ polynomial

$A_c(q^{-1}) = B^s(q^{-1})\,A_c'(q^{-1})$
$R(q^{-1}) = R'(q^{-1})\,A_d(q^{-1})\,B^s(q^{-1})$

$$A_c(q^{-1}) = A(q^{-1})\,R(q^{-1}) + q^{-\mathrm{d}}B(q^{-1})\,S(q^{-1})$$

$$\cancel{B^s(q^{-1})}\,A_c'(q^{-1}) = \cancel{B_s(q^{-1})}A(q^{-1})A_d(q^{-1})R'(q^{-1}) + q^{-\mathrm{d}}\cancel{B^s(q^{-1})}B^u(q^{-1})\,S(q^{-1})$$

$$A_c'(q^{-1}) = A_d(q^{-1})\,A(q^{-1})\,R'(q^{-1}) + q^{-\mathrm{d}}B^u(q^{-1})\,S(q^{-1})$$

Feedback Controller

Diophantine equation: Obtain polynomials $R^{'}(q^{-1}),\ S(q^{-1})$ which satisfy:

$$A_c^{'}(q^{-1}) = A_d(q^{-1})\,A(q^{-1})\,R^{'}(q^{-1}) + q^{-\mathrm{d}}B^u(q^{-1})\,S(q^{-1})$$

Close loop poles minus cancelled zeros

Plant poles

Unstable plant zeros

Disturbance annihilating polynomial

$$R(q^{-1}) = R^{'}(q^{-1})\,A_d(q^{-1})\,B^s(q^{-1})$$
$$A_c(q^{-1}) = B^s(q^{-1})\,A_c^{'}(q^{-1})$$

Use previous solution of the Diophantine equation

$$A_c^{'}(q^{-1}) = A_d(q^{-1})\,A(q^{-1})\,R^{'}(q^{-1}) + q^{-\mathrm{d}}B^u(q^{-1})\,S(q^{-1})$$

$$\underbrace{A_c^{'}(q^{-1})}_{\mathcal{C}(q^{-1})} = \underbrace{\left(A_d(q^{-1})\,A(q^{-1})\right)}_{\mathcal{A}(q^{-1})}\,\underbrace{R^{'}(q^{-1})}_{\mathcal{R}(q^{-1})}$$
$$+q^{-1}\underbrace{\left(q^{d-1}B^u(q^{-1})\right)}_{\mathcal{B}(q^{-1})}\,\underbrace{S(q^{-1})}_{\mathcal{S}(q^{-1})}$$

Diophantine equation

$$A_c^{'}(q^{-1}) = A_d(q^{-1})\,A(q^{-1})\,R^{'}(q^{-1}) + q^{-\mathrm{d}}B^u(q^{-1})\,S(q^{-1})$$

Solution:

$$R^{'}(q^{-1}) \;=\; 1 + r_1^{'}q^{-1} + \cdots + r_{n_r^{'}}\,q^{-n_r^{'}}$$
$$S(q^{-1}) \;=\; s_o + s_1 q^{-1} + \cdots + s_{n_s}q^{-n_s}$$
$$R(q^{-1}) = R^{'}(q^{-1})\,A_d(q^{-1})\,B^s(q^{-1})$$

$$n_r^{'} \;=\; d + m_u - 1$$
$$n_s \;=\; \max\{\, n + n_d - 1\,,\ n_c^{'} - d - m_u \,\}$$
$$n_r = n_r^{'} + n_d + m_s$$

Feedback Controller

$$u(k) = \frac{1}{R(q^{-1})}\left[r(k) - S(q^{-1})y(k)\right]$$

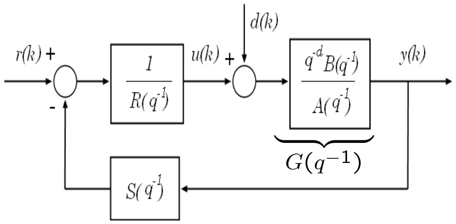
where

$$n_r^{'} \;=\; d + m_u - 1$$
$$n_s \;=\; \max\{\, n + n_d - 1\,,\ n_c^{'} - d - m_u \,\}$$
$$n_r = n_r^{'} + n_d + m_s$$

If the degree of the disturbance annihilator polynomial, n_d is large (e.g. N is large), then n_r and n_s are also large

Then, the solution of the Diophantine equation may be ill conditioned.

Example



$$G(q^{-1}) = \frac{q^{-d}B(q^{-1})}{A(q^{-1}))}$$

Plant:

$$G(q^{-1}) = \frac{q^{-2}(2+2.4q^{-1})}{(1-1.2q^{-1})}$$

Zeros: $B(q^{-1}) = (2+2.4q^{-1})$ ➡ $\begin{cases} B^s(q^{-1}) = 1 \\ B^u(q^{-1}) = (2+2.4q^{-1}) \end{cases}$

Disturbance: $d(k) = d(k-1)$ ➡ $A_d(q^{-1}) = 1 - q^{-1}$

Select close loop poles:

$$A_c'(q^{-1}) = (1-0.5q^{-1})(1-0.8q^{-1})$$
$$= (1-1.3q^{-1}0.4q^{-2})$$

Diophantine equation

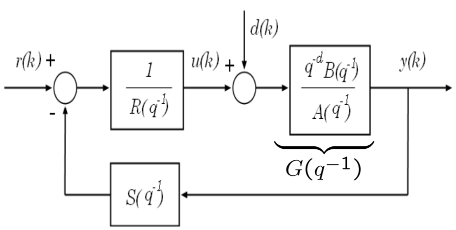
$$A_c'(q^{-1}) = A_d(q^{-1})\,A(q^{-1})\,R'(q^{-1}) + q^{-\mathrm{d}}B^u(q^{-1})\,S(q^{-1})$$

$$(1-1.3q^{-1}0.4q^{-2}) = \underbrace{(1-2.2q^{-1}+1.2q^{-2})}_{A(q^{-1})A_d(q^{-1})}\underbrace{(1+r_1'q^{-1}+r_2'q^{-2})}_{R'(q^{-1})}$$
$$+q^{-2}\underbrace{(2+2.4q^{-1})}_{B^u(q^{-1})}\underbrace{(s_o+s_1q^{-1})}_{S(q^{-1})}$$

Solution: $\begin{cases} R'(q^{-1}) = 1+0.9q^{-1}+0.57q^{-2} \\ S(q^{-1}) = 0.31-0.28q^{-1} \end{cases}$

$$R(q^{-1}) = A_d(q^{-1})R'(q^{-1}) = (1-q^{-1})(1+0.9q^{-1}+0.57q^{-2})$$

Example



$$G(q^{-1}) = \frac{q^{-2}(2+2.4q^{-1})}{(1-1.2q^{-1})}$$

$$d(k) = d(k-1)$$

Control: $u(k) = \dfrac{1}{R(q^{-1})}\left[r(k) - S(q^{-1})y(k)\right]$

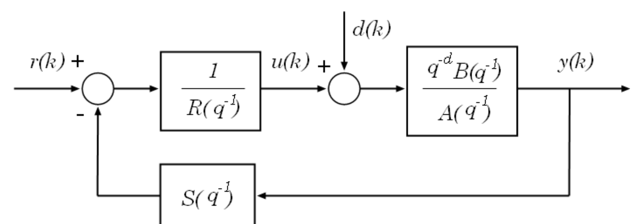
$$R(q^{-1}) = 1-0.1q^{-1}-0.33q^{-2}-0.57q^{-3}$$

$$S(q^{-1}) = 0.31-0.28q^{-1}$$

$$r(k) = r(k-1) = 1$$

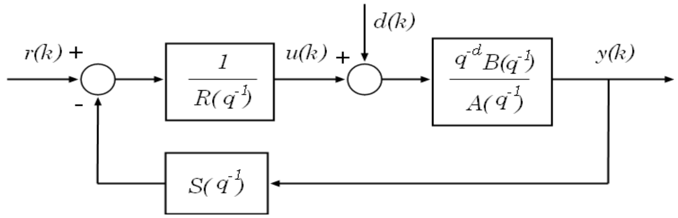
Feedback Control Law

Feedback control action:



$$y(k) = \frac{q^{-\mathrm{d}}B^u(q^{-1})}{A_c'(q^{-1})}\,r(k)$$

Proof – block diagram algebra



The close loop dynamics is from *r(k)* and *d(k)* to *y(k)*

$$y(k) = \frac{q^{-\mathrm{d}}B(q^{-1})}{A(q^{-1})R(q^{-1}) + q^{-\mathrm{d}}B(q^{-1})S(q^{-1})}\,r(k)$$

$$+ \frac{q^{-\mathrm{d}}B(q^{-1})\,R(q^{-1})}{A(q^{-1})R(q^{-1}) + q^{-\mathrm{d}}B(q^{-1})S(q^{-1})}\,d(k)$$

Proof – block diagram algebra

The close loop dynamics is from *d(k)* to *y(k)* (*r(k) = 0*)

$$y(k) = \frac{q^{-\mathrm{d}}B(q^{-1})\,R(q^{-1})}{A(q^{-1})R(q^{-1}) + q^{-\mathrm{d}}B(q^{-1})S(q^{-1})}\,d(k)$$

Substitute:

$$B(q^{-1}) = B^s(q^{-1})\,B^u(q^{-1})$$
$$R(q^{-1}) = R'(q^{-1})\,A_d(q^{-1})\,B^s(q^{-1})$$

$$y(k) = \underbrace{\frac{B^s(q^{-1})}{B^s(q^{-1})}}_{1}\,\frac{q^{-\mathrm{d}}B(q^{-1})R'(q^{-1})A_d(q^{-1})}{\underbrace{\left[A(q^{-1})A_d(q^{-1})R'(q^{-1}) + q^{-\mathrm{d}}B^u(q^{-1})S(q^{-1})\right]}_{A_c'(q^{-1})}}\,d(k)$$

pole-zero cancellation

Diophantine equation

Proof – block diagram algebra

The close loop dynamics is from *d(k)* to *y(k)* (*r(k) = 0*)

$$y(k) = \left[\frac{q^{-\mathrm{d}}B(q^{-1})R'(q^{-1})}{A_c'(q^{-1})}\right]\underbrace{A_d(q^{-1})d(k)}_{0}$$

$A_c'(q^{-1})$ is Schur ⟺ $r(k) \to 0 \Rightarrow y(k) \to 0$

$$y(k) = \underbrace{\frac{B^s(q^{-1})}{B^s(q^{-1})}}_{1}\,\frac{q^{-\mathrm{d}}B(q^{-1})R'(q^{-1})A_d(q^{-1})}{\underbrace{\left[A(q^{-1})A_d(q^{-1})R'(q^{-1}) + q^{-\mathrm{d}}B^u(q^{-1})S(q^{-1})\right]}_{A_c'(q^{-1})}}\,d(k)$$

pole-zero cancellation

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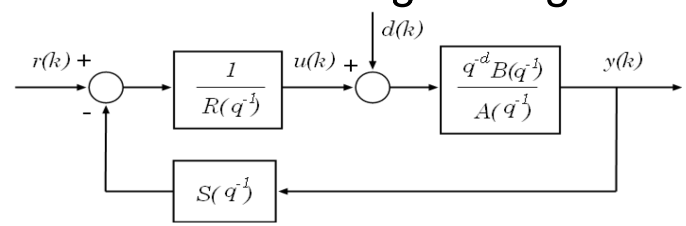
Proof – block diagram algebra



The close loop dynamics is from *r(k)* and *d(k)* to *y(k)*

$$y(k) = \frac{q^{-\mathrm{d}}B(q^{-1})}{A(q^{-1})R(q^{-1}) + q^{-\mathrm{d}}B(q^{-1})S(q^{-1})}\,r(k)$$

$$+ \underbrace{\frac{q^{-\mathrm{d}}B(q^{-1})\,R(q^{-1})}{A(q^{-1})R(q^{-1}) + q^{-\mathrm{d}}B(q^{-1})S(q^{-1})}\,d(k)}_{\to 0}$$

Proof – block diagram algebra

$$y(k) = \frac{q^{-\mathrm{d}}B(q^{-1})}{A(q^{-1})R(q^{-1}) + q^{-\mathrm{d}}B(q^{-1})S(q^{-1})}\, r(k)$$

Substitute:

$$B(q^{-1}) = B^s(q^{-1})\, B^u(q^{-1})$$

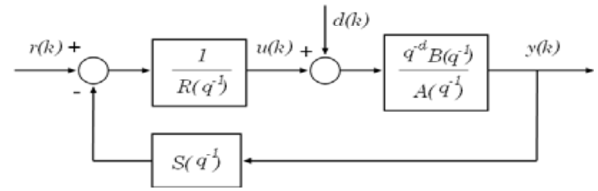
$$R(q^{-1}) = R'(q^{-1})\, A_d(q^{-1})\, B^s(q^{-1})$$

$$y(k) = \cancelto{1}{\frac{B^s(q^{-1})}{B^s(q^{-1})}}\, \frac{q^{-\mathrm{d}}B^u(q^{-1})}{\underbrace{\left[A(q^{-1})A_d(q^{-1})R'(q^{-1}) + q^{-\mathrm{d}}B^u(q^{-1})S(q^{-1})\right]}_{A_c'(q^{-1})}}\, r(k)$$

pole-zero cancellation

Diophontine equation

Proof – block diagram algebra



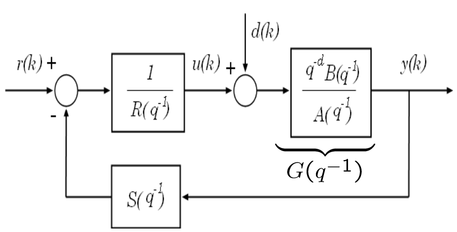
$$\boxed{y(k) = \frac{q^{-\mathrm{d}}B^u(q^{-1})}{A_c'(q^{-1})}\, r(k)}$$

$$y(k) = \cancelto{1}{\frac{B^s(q^{-1})}{B^s(q^{-1})}}\, \frac{q^{-\mathrm{d}}B^u(q^{-1})}{\underbrace{\left[A(q^{-1})A_d(q^{-1})R'(q^{-1}) + q^{-\mathrm{d}}B^u(q^{-1})S(q^{-1})\right]}_{A_c'(q^{-1})}}\, r(k)$$

pole-zero cancellation

Diophantine equation

Example



$$G(q^{-1}) = \frac{q^{-2}(2 + 2.4q^{-1})}{(1 - 1.2q^{-1})}$$

$$d(k) = d(k-1)$$

Control: $$u(k) = \frac{1}{R(q^{-1})}\left[r(k) - S(q^{-1})y(k)\right]$$

$$R(q^{-1}) = 1 - 0.1q^{-1} - 0.33q^{-2} - 0.57q^{-3}$$

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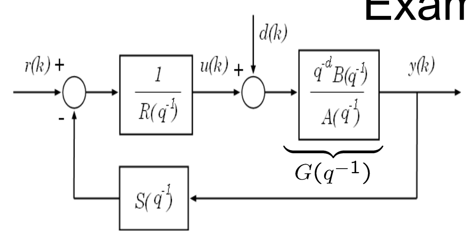
Example



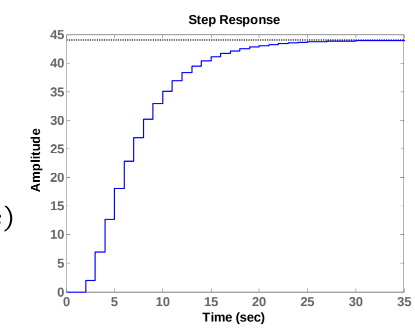
$$G(q^{-1}) = \frac{q^{-2}(2 + 2.4q^{-1})}{(1 - 1.2q^{-1})}$$

Close loop dynamics:

$$y(k) = \frac{q^{-\mathrm{d}}B^u(q^{-1})}{A_c'(q^{-1})}\, r(k)$$

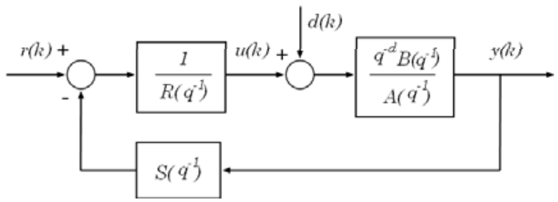
$$y(k) = \frac{q-2(2 + 2.4q^{-1})}{(1 - 0.8q^{-1})(1 - 0.5q^{-1})}\, r(k)$$

Unit step response



Feedback Control Law

The feedback control action:



Results in the following close loop input/output dynamics:

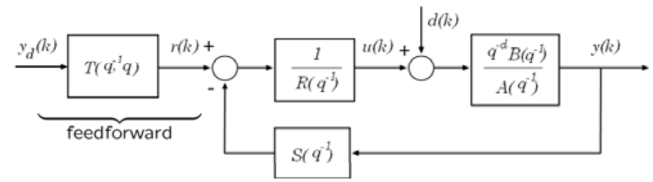
$$u(k) = \frac{A(q^{-1})}{\underbrace{B^s(q^{-1})}_{}A_c'(q^{-1})} r(k)$$

well-damped zeros

$$+ \frac{q^{-\mathrm{d}} B^u(q^{-1}) S(q^{-1})}{A_c'(q^{-1})} d(k)$$

Feedforward Control

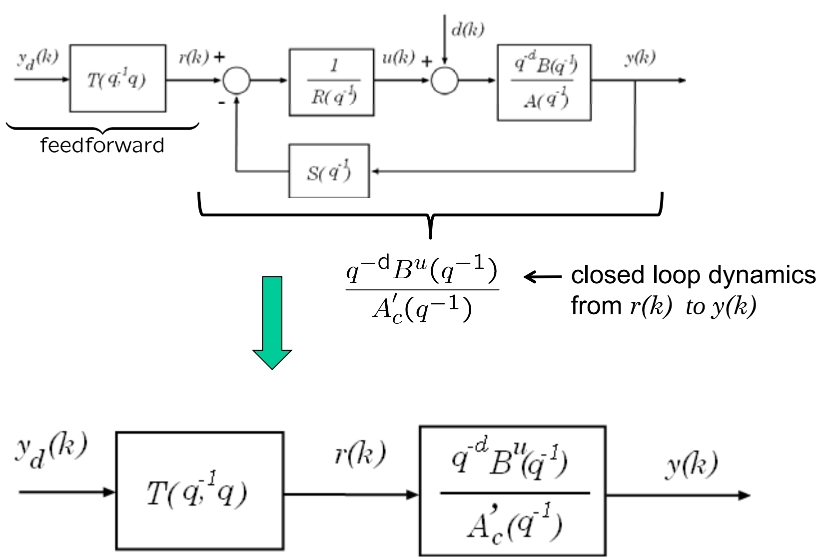
Feedforward control objective is to make $y(k)$ follow $y_d(k)$ as closely as possible.



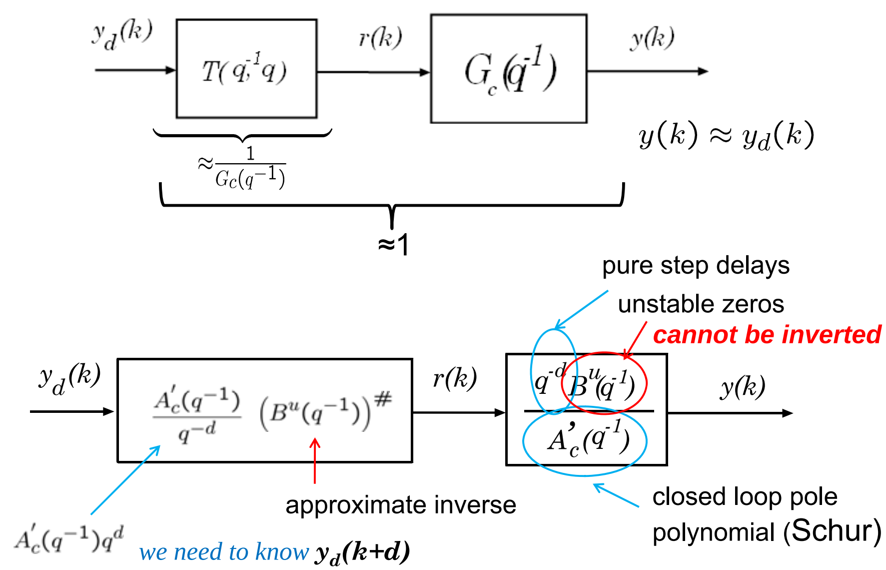
Goal: $y(k) = y_d(k)$ or $y(k) \approx y_d(k)$

how well the objective met depends on whether the plant has unstable zeros or not

Feedforward Control Synthesis



Feedforward control principle: **plant inversion**

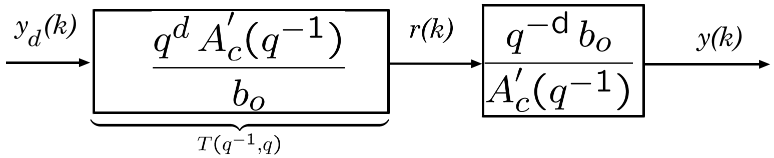


Perfect Tracking Feedforward Control

Perfect tracking can be achieved if all plant zeros are cancelable, i.e.

$$B^u(q^{-1}) = b_o$$

in this case



$$r(k) = \frac{1}{b_o} A_c'(q^{-1})\, y_d(k+d)$$

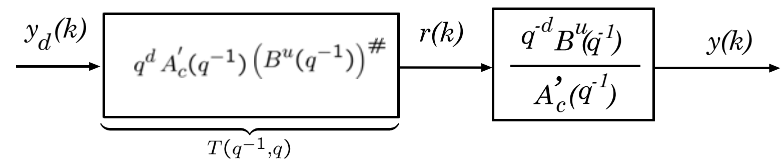
Tracking with unstable zeros

- When the plant has unstable zeros, i.e.

$$B^u(q^{-1}) \neq b_o$$

- We need to find an approximate inverse $\left(B^u(q^{-1})\right)^{\#}$

$$B^u(q^{-1}) \left(B^u(q^{-1})\right)^{\#} \approx 1$$



A-causal Bounded-Input Bounded-Output (BIBO) realization of a purely unstable operator

Let $B^u(p^{-1}) = 0 \Longleftrightarrow |p| > 1$

i.e. all roots of $B^{u*}(q) = q^{m_u} B^u(q^{-1})$ are <u>outside the unite circle</u>

Then we can interpret $\dfrac{1}{B^u(q^{-1})}$ in two ways:

- $\dfrac{1}{B^u(q^{-1})}$ is <u>**causal**</u> but unstable
- $\dfrac{1}{B^u(q^{-1})}$ is <u>**a-causal**</u> but BIBO

A-causal Bounded-Input Bounded-Output (BIBO) realization of a purely unstable operator

Example: $B^u(q^{-1}) = (2 + 2.4q^{-1}) = 2.4(0.8\bar{3} + q^{-1})$

$$\frac{1}{B^u(q^{-1})} = \frac{0.41\bar{6}}{0.8\bar{3} + q^{-1}} \longleftarrow \textit{unstable causal operator}$$

Using an infinite series expansion,

$$\frac{0.41\bar{6}}{(0.8\bar{3} + q^{-1})} = \frac{0.41\bar{6}q}{0.8\bar{3}q + 1}$$

$$= \underbrace{0.41\bar{6}q \left[1 - 0.8\bar{3}q + (0.8\bar{3}q)^2 - (0.8\bar{3}8q)^3 \cdots + \cdots (-1)^n (0.8\bar{3}q)^n\right]}_{\textit{infinite dimensional a-causal operator}}$$

A-causal BIBO realization of a purely unstable operator

Thus, $$y(k) = \frac{1}{2 + 2.4q^{-1}}\, u(k)$$

Can be realized either as:

$$y(k) = -1.2\, y(k-1) + 0.5\, u(k) \qquad \text{(unstable)}$$

or

$$\begin{aligned} y(k) \;=\; & 0.41\bar{6}\Big[u(k+1) - 0.8\bar{3}\, u(k+2) + (0.8\bar{3})^2\, u(k+3) \\ & -(0.8\bar{3})^3\, u(k+4) + \cdots (-0.8\bar{3})^n\, u(k+n+1) + \cdots\Big] \end{aligned}$$

(a-causal BIBO)

A-causal BIBO **approximation** of a purely unstable operator

We will now describe two methods of approximating a purely unstable operator:

1) Truncated a-casual series expansion:

$$\left(B^u(q^{-1})\right)^{\#} = \beta_1 q + \beta_2 q^2 + \cdots + \beta_3 q^M$$

2) Zero-phase error feedforward operator:
(develop by Prof. Tomizuka)

$$\left(B^u(q^{-1})\right)^{\#} = \frac{1}{[B^u(1)]^2} B^u(q)$$

Example: realizing $\left(B^u(q^{-1})\right)^{\#}$

Let, $B^u(q^{-1}) = (2 + 2.4q^{-1})$

1) Truncated a-casual series expansion:

$$\left(B^u(q^{-1})\right)^{\#} = 0.41\bar{6}\left[q - 0.8\bar{3}q^2 + (0.8\bar{3}q)^3 - (0.8\bar{3}8q)^4\right]$$

2) Zero-phase error feedforward operator:

$$\left(B^u(q^{-1})\right)^{\#} = \frac{1}{[4.4]^2}(2 + 2.4q)$$

Zero-phase error tracking

One of the most popular feedforward techniques for systems with unstable zeros.

$$\left(B^u(q^{-1})\right)^{\#} = \frac{1}{|B^u(0)|^2} B^u(q)$$

Define the zero-phase operator

$$\begin{aligned} G_{zp}(q^{-1}, q) &= B^u(q^{-1})\ \left(B^u(q^{-1})\right)^{\#} \\ &= \frac{B^u(q^{-1})\, B^u(q)}{[B^u(1)]^2} \end{aligned}$$

Zero-phase error transfer function

A-causal zero-phase transfer function:

$$G_{zp}(z^{-1}, z) = \frac{B^u(z^{-1})\, B^u(z)}{[B^u(1)]^2}$$

Properties:

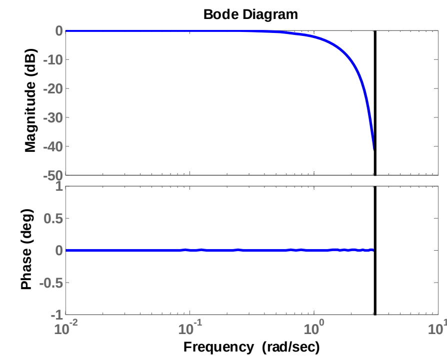
- It has zero-phase, i.e. $\text{Im}\left\{G_{zp}(e^{-j\omega}, e^{j\omega})\right\} = 0$
- It has unity dc gain, i.e. $G_{zp}(e^{-0}, e^{0}) = 1$

Example: realizing $\left(B^u(q^{-1})\right)^{\#}$

Let, $B^u(q^{-1}) = (2 + 2.4q^{-1})$

- Zero-phase feedforward: $\left(B^u(q^{-1})\right)^{\#} = \frac{1}{[4.4]^2}(2 + 2.4q)$
- Zero-phase transfer function:

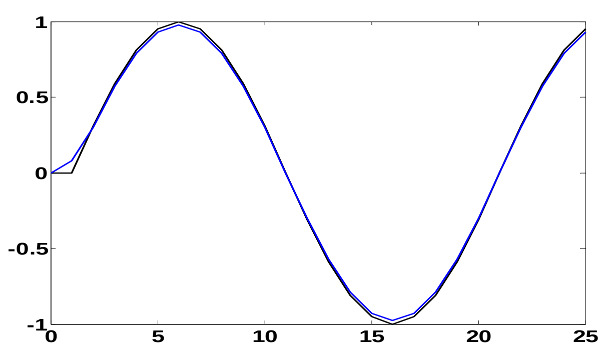
$$G_{zp}(z^{-1}, z) = \frac{(2 + 2.4z^{-1})\,(2z + 2.4)}{[4.4]^2}$$



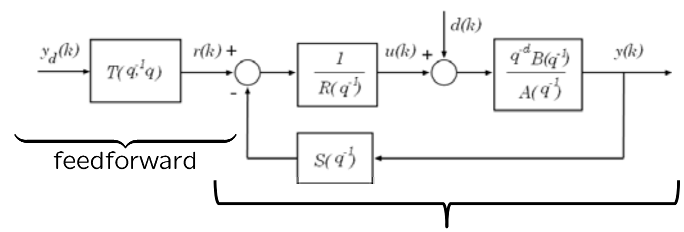
Sinusoidal zero-phase error tracking

If $y_d(k)$ is a sinusoidal, there will be no phase shift between $y_d(k)$ and $y(k)$

$$y(k) = \frac{B^u(q^{-1})\, B^u(q)}{[B^u(1)]^2}\, y_d(k)$$



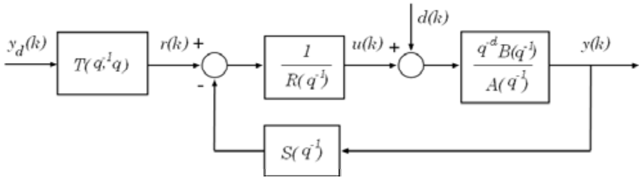
Zero-phase error feedforward



$$\frac{q^{-d} B^u(q^{-1})}{A_c'(q^{-1})}$$

$$\boxed{T(q^{-1}, q) = A_c'(q^{-1})\, q^d \frac{B^u(q)}{[B^u(1)]^2}}$$

Zero-phase error feedforward



$$y(k) = \frac{B^u(q^{-1})\,B^u(q)}{[B^u(1)]^2}\,y_d(k)$$