Tony Kelman
ME 233 – Homework 7

1.a)

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.8 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(k)$$

$$y(k) = x_1(k) + v(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + v(k),\ x_0 = E\{x(0)\},\ X_0 = E\{(x(0)-x_0)(x(0)-x_0)^T\}$$

$w(k)$ and $v(k)$ both white, zero mean, Gaussian and stationary

$$E\left\{ \begin{bmatrix} w(k) & v(k) \end{bmatrix} \begin{bmatrix} w(k) \\ v(k) \end{bmatrix} \right\} = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix},\ E\{[w(k) \quad v(k)](x(0)-x_0)^T\} = 0$$

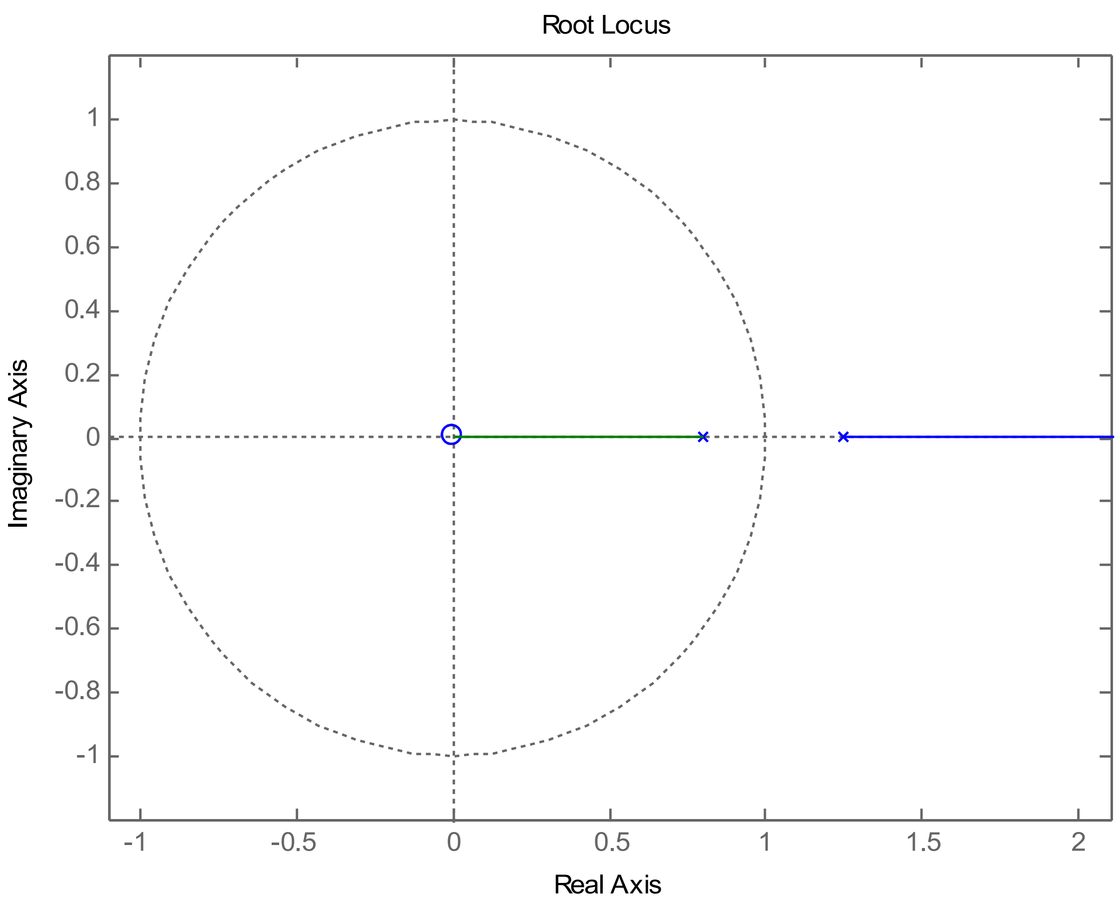
A has eigenvalues 0.8 and 0 so all modes are stable, therefore $[A, C]$ is detectable and $[A, B_w W^{1/2}]$ is stabilizable, so yes the Kalman filter Riccati equation converges to a unique steady state solution.

1.b)

$$G_w(z) = C(zI-A)^{-1} B_w = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z-0.8 & -1 \\ 0 & z \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{z(z-0.8)} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z & 1 \\ 0 & z-0.8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$G_w(z) = \frac{1}{z(z-0.8)} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z \\ 0 \end{bmatrix} = \frac{z}{z(z-0.8)}$$

1 open-loop zero and 1 open-loop pole at the origin, so reciprocal root locus is of $\dfrac{z/(-0.8)}{(z-0.8)(z-1/0.8)}$ and there is 1 closed-loop pole at the origin regardless of W/V



1.c)

ARMAX model $(1-0.8z^{-1})Y(z)=z^{-2}U(z)+(1-0.5z^{-1})\tilde{Y}^o(z)$

$$Y(z)=\frac{z^{-2}}{1-0.8z^{-1}}U(z)+\frac{1-0.5z^{-1}}{1-0.8z^{-1}}\tilde{Y}^o(z)=\frac{1}{z(z-0.8)}U(z)+\frac{z(z-0.5)}{z(z-0.8)}\tilde{Y}^o(z)$$

$Y(z)=\frac{B(z)}{A(z)}U(z)+\frac{C(z)}{A(z)}\tilde{Y}^o(z)$, where $A(z)=\det(zI-A)$, $C(z)=\det(zI-A+LC)$

$A(z)=\det(zI-A)=z(z-0.8)$, so $C(z)=\det(zI-A+LC)=z(z-0.5)$

$$z(z-0.5)=\det\left(\begin{bmatrix} z & 0 \\ 0 & z\end{bmatrix}-\begin{bmatrix} 0.8 & 1 \\ 0 & 0\end{bmatrix}+\begin{bmatrix} l_1 \\ l_2\end{bmatrix}[1 \quad 0]\right)=\det\begin{bmatrix} z-0.8+l_1 & -1 \\ l_2 & z\end{bmatrix}=z(z-0.8+l_1)+l_2$$

clearly $l_1=0.3,\ l_2=0,\ L=\begin{bmatrix} 0.3 \\ 0\end{bmatrix}=AMC^T(CMC^T+V)^{-1}$

$E\{\tilde{y}^o(k+l)\tilde{y}^o(k)^T\}=(CMC^T+V)\delta(l)$, and in this problem with scalar $\tilde{y}^o(k)$ we are given $E\{(\tilde{y}^o(k))^2\}=1$, so $CMC^T+V=1$

$$L=\begin{bmatrix} 0.3 \\ 0\end{bmatrix}=AMC^T(CMC^T+V)^{-1}=\begin{bmatrix} 0.8 & 1 \\ 0 & 0\end{bmatrix}\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22}\end{bmatrix}\begin{bmatrix} 1 \\ 0\end{bmatrix}\cdot 1^{-1}=\begin{bmatrix} 0.8 & 1 \\ 0 & 0\end{bmatrix}\begin{bmatrix} m_{11} \\ m_{12}\end{bmatrix}=\begin{bmatrix} 0.8m_{11}+m_{12} \\ 0\end{bmatrix}$$

Steady-state Riccati equation $M=AMA^T+B_wWB_w^T-AMC^T(CMC^T+V)^{-1}CMA^T$

$$AMA^T=\begin{bmatrix} 0.8 & 1 \\ 0 & 0\end{bmatrix}\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22}\end{bmatrix}\begin{bmatrix} 0.8 & 0 \\ 1 & 0\end{bmatrix}=\begin{bmatrix} 0.8 & 1 \\ 0 & 0\end{bmatrix}\begin{bmatrix} 0.8m_{11}+m_{12} & 0 \\ 0.8m_{12}+m_{22} & 0\end{bmatrix}=\begin{bmatrix} 0.64m_{11}+1.6m_{12}+m_{22} & 0 \\ 0 & 0\end{bmatrix}$$

$$B_wWB_w^T=\begin{bmatrix} 1 \\ 0\end{bmatrix}W[1 \quad 0]=\begin{bmatrix} W & 0 \\ 0 & 0\end{bmatrix}$$

$$AMC^T(CMC^T+V)^{-1}CMA^T=\begin{bmatrix} 0.8 & 1 \\ 0 & 0\end{bmatrix}\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22}\end{bmatrix}\begin{bmatrix} 1 \\ 0\end{bmatrix}\cdot 1^{-1}\cdot[1 \quad 0]\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22}\end{bmatrix}\begin{bmatrix} 0.8 & 0 \\ 1 & 0\end{bmatrix}$$

$$AMC^T(CMC^T+V)^{-1}CMA^T=\begin{bmatrix} 0.8 & 1 \\ 0 & 0\end{bmatrix}\begin{bmatrix} m_{11} \\ m_{12}\end{bmatrix}[m_{11} \quad m_{12}]\begin{bmatrix} 0.8 & 0 \\ 1 & 0\end{bmatrix}=\begin{bmatrix} 0.8m_{11}+m_{12} \\ 0\end{bmatrix}[0.8m_{11}+m_{12} \quad 0]$$

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22}\end{bmatrix}=\begin{bmatrix} 0.64m_{11}+1.6m_{12}+m_{22}+W-(0.8m_{11}+m_{12})^2 & 0 \\ 0 & 0\end{bmatrix}$$

So $m_{12}=0,\ m_{22}=0,$ and from above $0.3=0.8m_{11}+m_{12}=0.8m_{11}+0,$ so $m_{11}=0.3/0.8=0.375$

$m_{11}=0.64m_{11}+1.6m_{12}+m_{22}+W-(0.8m_{11}+m_{12})^2=0.64\cdot 0.375+W-0.3^2$

$W=0.375(1-0.64)+0.09=0.225$

$$CMC^T+V=[1 \quad 0]\begin{bmatrix} 0.375 & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix} 1 \\ 0\end{bmatrix}+V=0.375+V=E\{(\tilde{y}^o(k))^2\}=1,\ \text{so}\ V=1-0.375=0.625$$

2.a)

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t)\end{bmatrix}=\begin{bmatrix} 0 & 1 \\ -2 & -0.5\end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t)\end{bmatrix}+\begin{bmatrix} 0 \\ 1\end{bmatrix}(u(t)+w(t)),\ y(t)=x_1(t)+v(t)=[1 \quad 0]\begin{bmatrix} x_1(t) \\ x_2(t)\end{bmatrix}+v(t)$$

$X_0=\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix},\ W=\rho,\ V=0.5$

Riccati equation for steady-state continuous Kalman filter: $AM+MA^T=-B_wWB_w^T+MC^TV^{-1}CM$

In Matlab, $M=\text{care}(A^T,\ C^T,\ B_wWB_w^T,\ V^{-1})=\begin{bmatrix} 0.4404 & 0.194 \\ 0.194 & 1.1488\end{bmatrix}$ when $\rho=2$

$L=MC^TV^{-1}=[0.8809 \quad 0.388]^T$

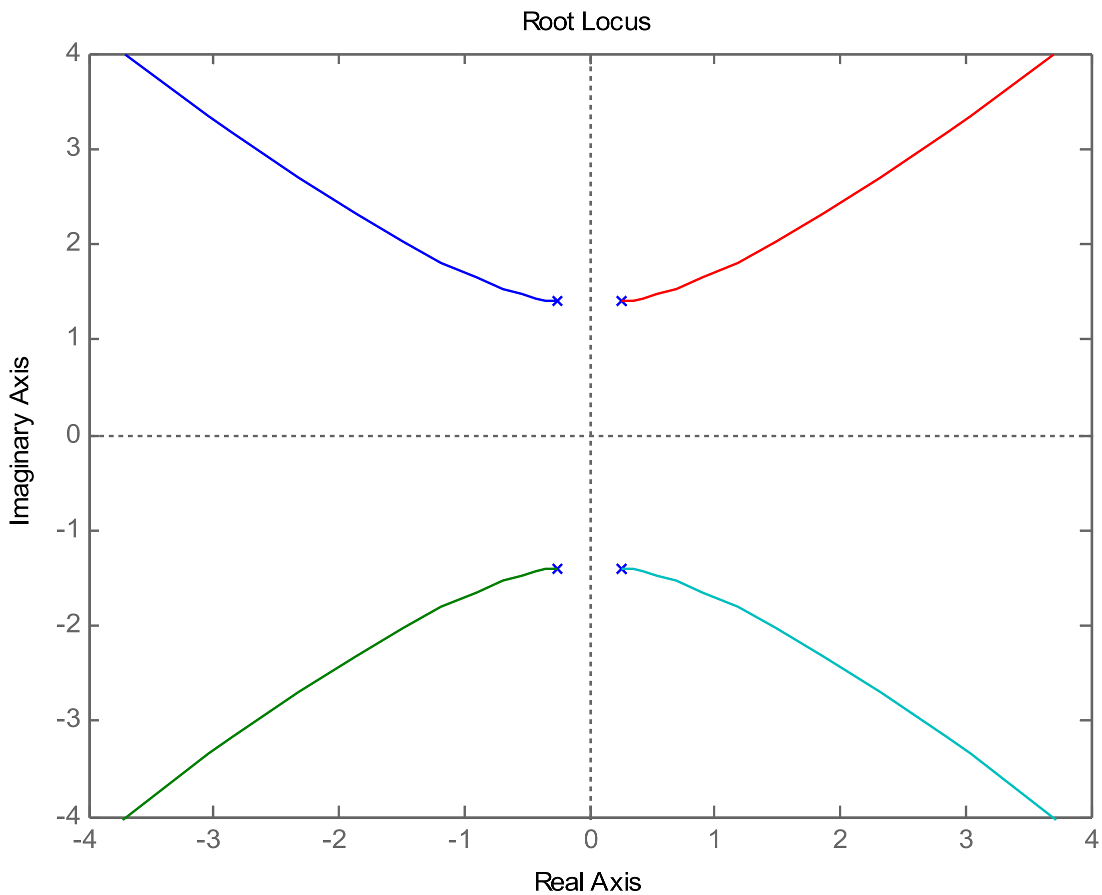
2.b)

$$G_w(s)=C(sI-A)^{-1}B_w=[1\quad 0]\begin{bmatrix} s & -1 \\ 2 & s+0.5\end{bmatrix}^{-1}\begin{bmatrix}0\\1\end{bmatrix}=\frac{1}{s^2+0.5s+2}[1\quad 0]\begin{bmatrix} s+0.5 & 1 \\ -2 & s\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$$

$$G_w(s)=\frac{1}{s^2+0.5s+2}[1\quad 0]\begin{bmatrix}1\\s\end{bmatrix}=\frac{1}{s^2+0.5s+2}$$

No open-loop zeroes, open-loop poles at $s=-0.25\pm j\sqrt{7.75/4}$

Symmetric root locus for $G_w(s)G_w(-s)$:



2.c)i)

$$\frac{d}{dt}\hat{x}(t)=A\hat{x}(t)+Bu(t)+L\epsilon(t),\ y(t)=C\hat{x}(t)+\epsilon(t)$$

Taking Laplace transforms, $s\hat{X}(s)=A\hat{X}(s)+BU(s)+L\epsilon(s),\ Y(s)=C\hat{X}(s)+\epsilon(s)$

$(sI-A)\hat{X}(s)=BU(s)+L\epsilon(s),\ Y(s)=C\hat{X}(s)+\epsilon(s)=C(sI-A)^{-1}(BU(s)+L\epsilon(s))+\epsilon(s)$

$Y(s)=C(sI-A)^{-1}BU(s)+(C(sI-A)^{-1}L+1)\epsilon(s)$

Then $Y(s)=\frac{B(s)}{A(s)}U(s)+\frac{C(s)}{A(s)}\epsilon(s)$ where $\frac{B(s)}{A(s)}$ is the transfer function $C(sI-A)^{-1}B$ which has denominator $A(s)=\det(sI-A)$, and we observe that $(C(sI-A)^{-1}L+1)$ can be expressed with the same denominator since it will also blow up at the poles where $(sI-A)$ is singular.

2.c)ii)

Since $B=B_w$ in this problem, $C(sI-A)^{-1}B=C(sI-A)^{-1}B_w=G_w(s)=\frac{1}{s^2+0.5s+2}=\frac{B(s)}{A(s)}$

$B(s)=1,\ A(s)=\det(sI-A)=s^2+0.5s+2$

2.c)iii)

By the matrix determinant lemma, $(C(sI-A)^{-1}L+1)\det(sI-A)=\det(sI-A+LC)$

so $C(sI-A)^{-1}L+1=\dfrac{\det(sI-A+LC)}{\det(sI-A)}=\dfrac{C(s)}{A(s)}$, and we have $C(s)=\det(sI-A+LC)$

With $\rho=2,\ L=[0.8809 \quad 0.388]^T$ from part a.

$$C(s)=\det(sI-A+LC)=\det\left(\begin{bmatrix} s & 0 \\ 0 & s\end{bmatrix}-\begin{bmatrix} 0 & 1 \\ -2 & -0.5\end{bmatrix}+\begin{bmatrix} 0.8809 \\ 0.388\end{bmatrix}[1 \quad 0]\right)$$

$$C(s)=\det\begin{bmatrix} s+0.8809 & -1 \\ 2.388 & s+0.5\end{bmatrix}=s^2+0.5s+0.8809s+0.4404+2.388=s^2+1.3809s+2.8284$$

3.a)

$$\frac{d}{dt}x(t)=Ax(t)+Bu(t)+B_w w(t),\ E\{w(t)\}=0,\ E\{w(t+\tau)w^T(t)\}=W\delta(\tau)$$

$$E\{x(0)\}=x_0,\ E\{(x(0)-x_0)(x(0)-x_0)^T\}=X_0$$

$$y(t)=Cx(t)+v_A(t),\ E\{v_A(t)\}=0,\ E\{v_A(t+\tau)v_A^T(t)\}=V_A\delta(\tau)$$

$$y_B(t)=\begin{bmatrix} y_1(t) \\ y_2(t)\end{bmatrix}=\begin{bmatrix} C \\ C\end{bmatrix}x(t)+\begin{bmatrix} v_{B1}(t) \\ v_{B2}(t)\end{bmatrix}$$

$$E\{v_{Bi}(t)\}=0,\ E\{v_{Bi}(t+\tau)v_{Bi}^T(t)\}=V_B\delta(\tau),\ E\{v_{B1}(t+\tau)v_{B2}^T(t)\}=0$$

continuous Riccati equation for configuration A: $AM_A+M_AA^T=-B_wWB_w^T+M_AC^TV_A^{-1}CM_A$

for configuration B: $AM_B+M_BA^T=-B_wWB_w^T+M_B[C^T \quad C^T]\begin{bmatrix} V_B & 0 \\ 0 & V_B\end{bmatrix}^{-1}\begin{bmatrix} C \\ C\end{bmatrix}M_B$

in order to have $M_A=M_B$, $C^TV_A^{-1}C=[C^T \quad C^T]\begin{bmatrix} V_B & 0 \\ 0 & V_B\end{bmatrix}^{-1}\begin{bmatrix} C \\ C\end{bmatrix}=[C^T \quad C^T]\begin{bmatrix} V_B^{-1}C \\ V_B^{-1}C\end{bmatrix}=2C^TV_B^{-1}C$

so $M_A=M_B$ if $V_A^{-1}=2V_B^{-1}$, or $V_B=2V_A$

3.b)

Kalman filter of configuration B is operating with gain L_B but instead of feeding in $y_B(t)$

we have $y(t)=\begin{bmatrix} y_2(t) \\ y_2(t)\end{bmatrix}=\begin{bmatrix} C \\ C\end{bmatrix}x(t)+\begin{bmatrix} v_{B2}(t) \\ v_{B2}(t)\end{bmatrix}$

$$\frac{d}{dt}\hat{x}(t)=A\hat{x}(t)+Bu(t)+L_B\tilde{y}(t)=A\hat{x}(t)+Bu(t)+L_B\left(y(t)-\begin{bmatrix} C \\ C\end{bmatrix}\hat{x}(t)\right)$$

$$\tilde{x}(t)=x(t)-\hat{x}(t),\ \frac{d}{dt}\tilde{x}(t)=A(x(t)-\hat{x}(t))+B_w w(t)-L_B\left(y(t)-\begin{bmatrix} C \\ C\end{bmatrix}\hat{x}(t)\right)$$

$$\frac{d}{dt}\tilde{x}(t)=A(x(t)-\hat{x}(t))+B_w w(t)-L_B\left(\begin{bmatrix} C \\ C\end{bmatrix}x(t)+\begin{bmatrix} v_{B2}(t) \\ v_{B2}(t)\end{bmatrix}-\begin{bmatrix} C \\ C\end{bmatrix}\hat{x}(t)\right)$$

$$\frac{d}{dt}\tilde{x}(t)=\left(A-L_B\begin{bmatrix} C \\ C\end{bmatrix}\right)\tilde{x}(t)+B_w w(t)-L_B\begin{bmatrix} v_{B2}(t) \\ v_{B2}(t)\end{bmatrix}$$

$$L_B=M_B[C^T \quad C^T]\begin{bmatrix} V_B & 0 \\ 0 & V_B\end{bmatrix}^{-1}=M_B[C^TV_B^{-1} \quad C^TV_B^{-1}]$$

$$\frac{d}{dt}\tilde{x}(t)=(A-2M_BC^TV_B^{-1}C)\tilde{x}(t)+B_w w(t)-2M_BC^TV_B^{-1}v_{B2}(t)$$

$$\frac{d}{dt}\tilde{x}(t)=(A-2M_BC^TV_B^{-1}C)\tilde{x}(t)+[B_w \quad -2M_BC^TV_B^{-1}]\begin{bmatrix} w(t) \\ v_{B2}(t) \end{bmatrix}$$

Measurement noises are independent from input noise, so covariance matrix of the augmented noise vector $\begin{bmatrix} w(t) \\ v_{B2}(t) \end{bmatrix}$ is $E\left\{\begin{bmatrix} w(t+\tau) \\ v_{B2}(t+\tau) \end{bmatrix}[w^T(t) \quad v_{B2}^T(t)]\right\}=\begin{bmatrix} W & 0 \\ 0 & V_B \end{bmatrix}\delta(\tau)$

State estimation error covariance for failure condition is $M_C(t)=E\{\tilde{x}(t)\tilde{x}^T(t)\}$

Covariance propagation equation gives the following, where $A_c=A-2M_BC^TV_B^{-1}C$

$$\frac{d}{dt}M_C(t)=A_cM_C(t)+M_C(t)A_c^T+[B_w \quad -2M_BC^TV_B^{-1}]\begin{bmatrix} W & 0 \\ 0 & V_B \end{bmatrix}[B_w \quad -2M_BC^TV_B^{-1}]^T$$

$$\frac{d}{dt}M_C(t)=A_cM_C(t)+M_C(t)A_c^T+[B_w \quad -2M_BC^TV_B^{-1}]\begin{bmatrix} W B_w^T \\ -2CM_B \end{bmatrix}$$ (since V_B, M_B are symmetric)

at steady state $0=A_cM_C+M_CA_c^T+B_wWB_w^T+4M_BC^TV_B^{-1}CM_B$

$0=(A-2M_BC^TV_B^{-1}C)M_C+M_C(A^T-2C^TV_B^{-1}CM_B)+B_wWB_w^T+4M_BC^TV_B^{-1}CM_B$

I couldn't get any simpler result for M_C by any combination of expanding, factoring, rearranging or relating to the Riccati equations from part a, so I have no idea here. The terms are similar but the coefficients just aren't working out. I may have made a sign error or messed up a factor of 2 somewhere, I looked everything over but I'm lost. I give up on this one.

4.a)

$x(k+1)=x(k)+w(k)u(k)x(k)=(1+w(k)u(k))x(k)$

$J_k^o[x(k)]=\max_{U_k}E_{W_k}\left\{x(N)+\sum_{k=0}^{N-1}(1-u(k))x(k)\right\}$

$J_N^o[x(N)]=x(N)$

Dynamic programming, stochastic Bellman equation

$J_k^o[x(k)]=\max_{u(k)}\left[(1-u(k))x(k)+E_{w(k)}\{J_{k+1}^o[x(k+1)]\}\right]$

$J_{N-1}^o[x(N-1)]=\max_{u(N-1)}\left[(1-u(N-1))x(N-1)+E_{w(N-1)}\{J_N^o[x(N)]\}\right]$

$J_{N-1}^o[x(N-1)]=\max_{u(N-1)}\left[(1-u(N-1))x(N-1)+E_{w(N-1)}\{x(N)\}\right]$

$J_{N-1}^o[x(N-1)]=\max_{u(N-1)}\left[(1-u(N-1))x(N-1)+E_{w(N-1)}\{(1+w(N-1)u(N-1))x(N-1)\}\right]$

$J_{N-1}^o[x(N-1)]=\max_{u(N-1)}\left[(1-u(N-1))x(N-1)+(1+\bar{w}u(N-1))x(N-1)\right]$

$J_{N-1}^o[x(N-1)]=\max_{u(N-1)}\left[(2+(\bar{w}-1)u(N-1))x(N-1)\right]$

Since $0\le u(N-1)\le 1$, if $\bar{w}>1$ then the maximum occurs at $u^o(N-1)=1$

$J_{N-1}^o[x(N-1)]=(2+(\bar{w}-1))x(N-1)=(1+\bar{w})x(N-1)$ for $\bar{w}>1$

$J_{N-2}^o[x(N-2)]=\max_{u(N-2)}\left[(1-u(N-2))x(N-2)+E_{w(N-2)}\{J_{N-1}^o[x(N-1)]\}\right]$

$J_{N-2}^o[x(N-2)]=\max_{u(N-2)}\left[(1-u(N-2))x(N-2)+E_{w(N-2)}\{(1+\bar{w})x(N-1)\}\right]$

$J_{N-2}^o[x(N-2)]=\max_{u(N-2)}\left[(1-u(N-2))x(N-2)+E_{w(N-2)}\{(1+\bar{w})(1+w(N-2)u(N-2))x(N-2)\}\right]$

$J_{N-2}^o[x(N-2)]=\max_{u(N-2)}\left[(1-u(N-2))x(N-2)+(1+\bar{w})(1+\bar{w}u(N-2))x(N-2)\right]$

$J_{N-2}^o[x(N-2)]=\max_{u(N-2)}\left[(2+\bar{w}+((1+\bar{w})\bar{w}-1)u(N-2))x(N-2)\right]$

If $\bar{w}>1$ then $((1+\bar{w})\bar{w}-1)>0$ and the maximum occurs at $u^o(N-2)=1$

$J_{N-2}^o[x(N-2)]=(2+\bar{w}+(1+\bar{w})\bar{w}-1)x(N-2)=(1+\bar{w})^2x(N-2)$

Pattern looks like $J_k^o[x(k)]=(1+\bar{w})^{N-k}x(k)$, assume that holds and prove for next k by induction

$J^o_{k-1}[x(k-1)]=\max_{u(k-1)}\left[(1-u(k-1))x(k-1)+E_{w(k-1)}\{J^o_k[x(k)]\}\right]$
$J^o_{k-1}[x(k-1)]=\max_{u(k-1)}\left[(1-u(k-1))x(k-1)+E_{w(k-1)}\{(1+\bar{w})^{N-k}x(k)\}\right]$
$J^o_{k-1}[x(k-1)]=\max_{u(k-1)}\left[(1-u(k-1))x(k-1)+E_{w(k-1)}\{(1+\bar{w})^{N-k}(1+w(k-1)u(k-1))x(k-1)\}\right]$
$J^o_{k-1}[x(k-1)]=\max_{u(k-1)}\left[(1-u(k-1))x(k-1)+(1+\bar{w})^{N-k}(1+\bar{w}u(k-1))x(k-1)\right]$
$J^o_{k-1}[x(k-1)]=\max_{u(k-1)}\left[(1+(1+\bar{w})^{N-k}+((1+\bar{w})^{N-k}\bar{w}-1)u(k-1))x(k-1)\right]$
If $\bar{w}>1$ then $((1+\bar{w})^{N-k}\bar{w}-1)>0$ and the maximum occurs at $u^o(k-1)=1$
$J^o_{k-1}[x(k-1)]=(1+(1+\bar{w})^{N-k}+(1+\bar{w})^{N-k}\bar{w}-1)x(k-1)=(1+\bar{w})^{N-k+1}x(k-1)$
This fits the assumed form and we demonstrated base cases above, so by induction, when $\bar{w}>1$
we have $u^o(k)=1$ and $J^o_k[x(k)]=(1+\bar{w})^{N-k}x(k)$ for all k from 0 to $N-1$

4.b)

backtrack to: $J^o_{N-1}[x(N-1)]=\max_{u(N-1)}\left[(2+(\bar{w}-1)u(N-1))x(N-1)\right]$
Since $0\le u(N-1)\le 1$, if $0<\bar{w}<1/N\le 1$ then the maximum occurs at $u^o(N-1)=0$
$J^o_{N-1}[x(N-1)]=2x(N-1)$ for $\bar{w}<1/N$
$J^o_{N-2}[x(N-2)]=\max_{u(N-2)}\left[(1-u(N-2))x(N-2)+E_{w(N-2)}\{J^o_{N-1}[x(N-1)]\}\right]$
$J^o_{N-2}[x(N-2)]=\max_{u(N-2)}\left[(1-u(N-2))x(N-2)+E_{w(N-2)}\{2x(N-1)\}\right]$
$J^o_{N-2}[x(N-2)]=\max_{u(N-2)}\left[(1-u(N-2))x(N-2)+E_{w(N-2)}\{2(1+w(N-2)u(N-2))x(N-2)\}\right]$
$J^o_{N-2}[x(N-2)]=\max_{u(N-2)}\left[(1-u(N-2))x(N-2)+2(1+\bar{w}u(N-2))x(N-2)\right]$
$J^o_{N-2}[x(N-2)]=\max_{u(N-2)}\left[(3+(2\bar{w}-1)u(N-2))x(N-2)\right]$
If $\bar{w}<1/N\le 1/2$ then the maximum occurs at $u^o(N-2)=0$, $J^o_{N-2}[x(N-2)]=3x(N-2)$
Pattern looks like $J^o_k[x(k)]=(N-k+1)x(k)$, assume that holds and prove for next k by induction
$J^o_{k-1}[x(k-1)]=\max_{u(k-1)}\left[(1-u(k-1))x(k-1)+E_{w(k-1)}\{J^o_k[x(k)]\}\right]$
$J^o_{k-1}[x(k-1)]=\max_{u(k-1)}\left[(1-u(k-1))x(k-1)+E_{w(k-1)}\{(1+\bar{w})^{N-k}x(k)\}\right]$
$J^o_{k-1}[x(k-1)]=\max_{u(k-1)}\left[(1-u(k-1))x(k-1)+E_{w(k-1)}\{(N-k+1)(1+w(k-1)u(k-1))x(k-1)\}\right]$
$J^o_{k-1}[x(k-1)]=\max_{u(k-1)}\left[(1-u(k-1))x(k-1)+(N-k+1)(1+\bar{w}u(k-1))x(k-1)\right]$
$J^o_{k-1}[x(k-1)]=\max_{u(k-1)}\left[(N-k+2+((N-k+1)\bar{w}-1)u(k-1))x(k-1)\right]$
If $\bar{w}<1/N\le 1/(N-k+1)$ then the max occurs at $u^o(k-1)=0$, $J^o_{k-1}[x(k-1)]=(N-k+2)x(k-1)$
This fits the assumed form and we demonstrated base cases above, so by induction, when $\bar{w}<1/N$
we have $u^o(k)=0$ and $J^o_k[x(k)]=(N-k+1)x(k)$ for all k from 0 to $N-1$

4.c)

If $1/N\le\bar{w}\le 1$ then everything begins from $k=N$ as in part b and follows the same pattern with
$u^o(k)=0$ until $(N-k+1)\bar{w}-1>0$, let the largest k where that is true equal $N-\bar{k}$
So $(\bar{k}+1)\bar{w}-1>0$ but for $k=N-\bar{k}+1$ we have $\bar{k}\bar{w}-1<0$, or equivalently $\dfrac{1}{\bar{k}+1}<\bar{w}<\dfrac{1}{\bar{k}}$
(exact equality at any point is ambiguous - expected value is insensitive to the choice at that step)
At $k=N-\bar{k}$: $J^o_{N-\bar{k}-1}[x(N-\bar{k}-1)]=\max_{u(N-\bar{k}-1)}\left[(\bar{k}+2+((\bar{k}+1)\bar{w}-1)u(N-\bar{k}-1))x(N-\bar{k}-1)\right]$
$\bar{w}>1/(\bar{k}+1)$ so the maximum occurs at $u^o(N-\bar{k}-1)=1$
$J^o_{N-\bar{k}-1}[x(N-\bar{k}-1)]=(\bar{k}+2+(\bar{k}+1)\bar{w}-1)x(N-\bar{k}-1)=(\bar{k}+1)(1+\bar{w})x(N-\bar{k}-1)$
$J^o_{N-\bar{k}-2}[x(N-\bar{k}-2)]=\max_{u(N-\bar{k}-2)}\left[(1-u(N-\bar{k}-2))x(N-\bar{k}-2)+E_{w(N-\bar{k}-2)}\{J^o_{N-\bar{k}-1}[x(N-\bar{k}-1)]\}\right]$
$\quad=\max_{u(N-\bar{k}-2)}\left[(1-u(N-\bar{k}-2))x(N-\bar{k}-2)+E_{w(N-\bar{k}-2)}\{(\bar{k}+1)(1+\bar{w})x(N-\bar{k}-1)\}\right]$
$J^o_{N-\bar{k}-2}[x(N-\bar{k}-2)]=\max_{u(N-\bar{k}-2)}[(1-u(N-\bar{k}-2))x(N-\bar{k}-2)$
$\quad+E_{w(N-\bar{k}-2)}\{(\bar{k}+1)(1+\bar{w})(1+w(N-\bar{k}-2)u(N-\bar{k}-2))x(N-\bar{k}-2)\}]$

$$J^o_{N-\bar{k}-2}[x(N-\bar{k}-2)]=\max_{u(N-\bar{k}-2)}[(1-u(N-\bar{k}-2))x(N-\bar{k}-2)$$
$$+(\bar{k}+1)(1+\bar{w})(1+\bar{w}u(N-\bar{k}-2))x(N-\bar{k}-2)]$$
$$J^o_{N-\bar{k}-2}[x(N-\bar{k}-2)]=\max_{u(N-\bar{k}-2)}\left[(1+(\bar{k}+1)(1+\bar{w})+((\bar{k}+1)(1+\bar{w})\bar{w}-1)u(N-\bar{k}-2))x(N-\bar{k}-2)\right]$$
$\bar{w}>1/(\bar{k}+1)$ so the maximum occurs at $u^o(N-\bar{k}-2)=1$
$$J^o_{N-\bar{k}-2}[x(N-\bar{k}-2)]=(1+(\bar{k}+1)(1+\bar{w})+(\bar{k}+1)(1+\bar{w})\bar{w}-1)x(N-\bar{k}-2)$$
$$J^o_{N-\bar{k}-2}[x(N-\bar{k}-2)]=(\bar{k}+1)(1+\bar{w})^2x(N-\bar{k}-2)$$
And we can see once again the pattern that when $u^o(k)=1$, $\dfrac{J^o_{k-1}[x(k-1)]}{x(k-1)}=(1+\bar{w})\dfrac{J^o_k[x(k)]}{x(k)}$
which leads to $u^o(k-1)=1$ as well, so if $\bar{w}>1/N$ then $u^o(k)=1$ for all $k\leq N-\bar{k}-1$