ME 233 Spring 2010
Solution to Homework #1

1. **Finite Horizon Optimal Tracking Problem**
The LQ tracking problem is formulated as follows:

$$\begin{aligned}
& \min_{U_0} \{J\} \\
J &= \frac{1}{2}\left[y_d(N) - y(N)\right]^T S \left[y_d(N) - y(N)\right] \\
& + \frac{1}{2}\sum_{k=0}^{N-1}\left\{[y_d(k) - y(k)]^T T [y_d(k) - y(k)] + u^T(k) R u(k)\right\} \\
U_k &= \{u(k), u(k+1), \cdots, u(N-1)\}
\end{aligned}$$

subject to

$$\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) \\
y(k) &= Cx(k) \\
x(0) &= x_0
\end{aligned}$$

where $y_d(k)$ is specified for all k.

This is analogous to the LQ regulator problem which has been discussed in detail in class. Define:

$$\begin{aligned}
J_k^o[x(k)] &= \min_{U_k}\left\{\frac{1}{2}\left[y_d(N) - y(N)\right]^T S \left[y_d(N) - y(N)\right]\right. \\
& \left. + \frac{1}{2}\sum_{i=0}^{N-1}\left\{[y_d(i) - y(i)]^T T [y_d(i) - y(i)] + u^T(i) R u(i)\right\}\right\}
\end{aligned}$$

First note that

$$\begin{aligned}
J_N^o[x(N)] &= \frac{1}{2}[(y_d(N) - y(N)]^T S [y_d(N) - y(N)] \\
&= \frac{1}{2}x^T(N) C^T S C x(N) - x^T(N) C^T S y_d(N) + \frac{1}{2} y_d^T(N) S y_d(N)
\end{aligned}$$

Defining

$$\begin{aligned}
P(N) &= C^T S C && (1) \\
b(N) &= -C^T S y_d(N) && (2) \\
c(N) &= \frac{1}{2} y_d^T(N) S y_d(N) && (3)
\end{aligned}$$

gives

$$J_N^o[x(N)] = \frac{1}{2}x^T(N) P(N) x(N) + x^T(N) b(N) + c(N)$$

which is in the form shown in the hint.

Now, we will prove using induction that $J_k^o[x(k)]$ has the form shown in the hint. Using Bellman's principle of optimality we can obtain a recursive relation between $J_{k-1}^o[x(k-1)]$, which is the optimal cost to go from $x(k-1)$ to $x(N)$, and $J_k^o[x(k)]$:

$$\begin{aligned} J_{k-1}^o[x(k-1)] &= \min_{u(k)} \left\{ \frac{1}{2} \left[y_d(k-1) - y(k-1)\right]^T T \left[y_d(k-1) - y(k-1)\right] \right. \\ &\quad \left. + \frac{1}{2} u^T(k-1) R u(k-1) + J_k^o(x(k)) \right\} \end{aligned}$$

Assuming that $J_k^o[x(k)]$ has the form shown in the hint gives

$$\begin{aligned} J_{k-1}^o[x(k-1)] &= \min_{u(k)} \left\{ \frac{1}{2} x^T(k-1) \left[C^T T C + A^T P(k) A\right] x(k-1) \right. \\ &\quad + x^T(k-1) \left[A^T b(k) - C^T T y_d(k-1)\right] + \frac{1}{2} u^T(k-1) \left[R + B^T P(k) B\right] u(k-1) \\ &\quad \left. + u^T(k-1) B^T \left[P(k) A x(k-1) + b(k)\right] + \frac{1}{2} y_d^T(k-1) T y_d(k-1) + c(k) \right\} \end{aligned}$$

Taking the partial derivative of the term in the curly braces with respect to $u(k-1)$ and setting it equal to 0 gives

$$\begin{align} u^o(k-1) &= -\left[R + B^T P(k) B\right]^{-1} B^T \left[P(k) A x(k-1) + b(k)\right] \tag{4} \\ \Rightarrow J_{k-1}^o[x(k-1)] &= \frac{1}{2} x^T(k-1) \left\{ C^T T C + A^T P(k) A - A^T P(k) B \left[R + B^T P(k) B\right]^{-1} B^T P(k) A \right\} x(k-1) \notag \\ &\quad + x^T(k-1) \left\{ A^T b(k) - C^T T y_d(k-1) - A^T P(k) B \left[R + B^T P(k) B\right]^{-1} B^T b(k) \right\} \notag \\ &\quad + \left\{ \frac{1}{2} y_d^T(k-1) T y_d(k-1) + c(k) - \frac{1}{2} b^T(k) B \left[R + B^T P(k) B\right]^{-1} B^T b(k) \right\} \notag \end{align}$$

Defining

$$\begin{align} P(k-1) &= C^T T C + A^T P(k) A - A^T P(k) B \left[R + B^T P(k) B\right]^{-1} B^T P(k) A \tag{5} \\ b(k-1) &= A^T b(k) - C^T T y_d(k-1) - A^T P(k) B \left[R + B^T P(k) B\right]^{-1} B^T b(k) \tag{6} \\ c(k-1) &= \frac{1}{2} y_d^T(k-1) T y_d(k-1) + c(k) - \frac{1}{2} b^T(k) B \left[R + B^T P(k) B\right]^{-1} B^T b(k) \tag{7} \end{align}$$

gives

$$J_{k-1}^o[x(k-1)] = \frac{1}{2} x^T(k-1) P(k-1) x(k-1) + x^T(k-1) b(k-1) + c(k-1)$$

which concludes our proof by induction. Thus our optimal control law is given by equations (1)–(7)

Notice that the control law can be written as

$$\begin{aligned} u^o(k) &= F(k) b(k+1) - K(k) x(k) \\ K(k) &= \left[R + B^T P(k+1) B\right]^{-1} B^T P(k+1) A \\ F(k) &= -\left[R + B^T P(k+1) B\right]^{-1} B^T \end{aligned}$$

where $K(k)$ is the feedback gain and $F(k)$ is the feedforward gain. Figure 1 shows the block diagram of the system.

As N goes to ∞, the performance index should be modified to

$$J = \frac{1}{2} \sum_{k=0}^{\infty} \left\{ [y_d(k) - y(k)]^T T [y_d(k) - y(k)] + u^T(k) R u(k) \right\}$$

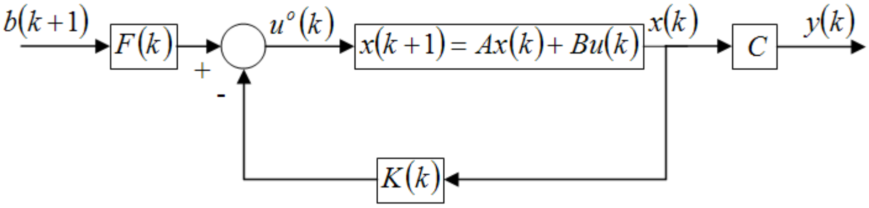

Figure 1: Block Diagram Showing Optimal Control

In this case, the solution becomes stationary, that is, $P(k)$ is a constant, therefore the feedback and feedforward gain become constant. However, since $b(k)$ and $c(k)$ also depend on $y_d(k)$, $b(k)$ and $c(k)$ may not be constant.

Intuitively, the control input should depend more on the immediate desired trajectory. The desired trajectory affects the control input through $b(k)$. For simplicity, we assume N goes to ∞, so that the feedforward gain is a constant. We have

$$\begin{aligned} b(k) &= \left\{A - B\left[R + B^TPB\right]^{-1}B^TPA\right\}^T b(k+1) - C^TTy_d(k) \\ &= \{A - BK\}^T b(k+1) - C^TTy_d(k) \end{aligned}$$

notice that $A_c = \{A - BK\}$ is exactly the 'A' matrix for the closed-loop system. Since the closed-loop is stable, A_c has all eigenvalues inside unit circle. Hence,

$$b(k) = -\left\{C^TTy_d(k) + A_c^TC^TTy_d(k+1) + (A_c^T)^2C^TTy_d(k+2) + \cdots\right\}$$

has larger coefficients on more imediate y_d.

2. **Application of Dynamic Programming**

Our goal is to solve the following problem:

$$\begin{aligned} &\max_{U_0}\{J\} \\ J &= \prod_{i=0}^{N-1} u(i), \qquad u(i) \geq 0 \\ U_k &= \{u(k), u(k+1), \ldots, u(N-1)\} \\ x(k+1) &= x(k) + u(k) \\ x(0) &= 0 \\ x(N) &= L \end{aligned}$$

Define

$$\begin{aligned} J_k[x(k)] &= \prod_{i=k}^{N-1} u(i) \\ J_k^o[x(k)] &= \max_{U_k}\left\{\prod_{i=k}^{N-1} u(i)\right\} \\ \Rightarrow J_{N-1}^o[x(N-1)] &= u^o(N-1) \\ &= L - x(N-1) \end{aligned}$$

The central idea in dynamic programming is to express the optimal cost at time step k as a function of of the optimal cost at time step $k+1$ so that a backward recursive scheme may be used. We will do that now.

$$\begin{aligned}
J_k^o[x(k)] &= \max_{U_k}\left\{\prod_{i=k}^{N-1} u(i)\right\} \\
&= \max_{u(k),U_{k+1}}\left\{u(k)\prod_{i=k+1}^{N-1} u(i)\right\} \\
&= \max_{u(k)}\left\{u(k)\max_{U_{k+1}}\left(\prod_{i=k+1}^{N-1} u(i)\right)\right\} \\
&= \max_{u(k)}\left\{u(k)J_{k+1}^o[x(k+1)]\right\}
\end{aligned}$$

You may need to convince yourself of some of the intermediate steps in the above set of equations. Consider the equation:

$$\begin{aligned}
J_{N-2}^o[x(N-2)] &= \max_{u(N-2)}\big(u(N-2)J_{N-1}^o[x(N-1)]\big) \\
\Rightarrow u^o(N-2) &= \arg\left(\max_{u(N-2)}\left\{u(N-2)J_{N-1}^o[x(N-1)]\right\}\right) \\
&= \arg\left(\max_{u(N-2)}\left\{u(N-2)\big[L-x(N-1)\big]\right\}\right) \\
&= \arg\left(\max_{u(N-2)}\left\{u(N-2)\big[L-x(N-2)-u(N-2)\big]\right\}\right) \\
&= \frac{L-x(N-2)}{2}
\end{aligned}$$

Similarly,

$$\begin{aligned}
u^o(N-3) &= \arg\left(\max_{u(N-3)}\left\{u(N-3)J_{N-2}^o[x(N-2)]\right\}\right) = \frac{L-x(N-3)}{3} \\
\vdots &= \qquad\qquad \vdots \\
u^o(0) &= \arg\left(\max_{u(0)}\left\{u(0)J_1^o[x(1)]\right\}\right) = \frac{L-x(0)}{N}
\end{aligned}$$

Given $u^o(0) = L/N$, the above set of equations yield $u(i) = L/N$ for all i.

3. Consider the discrete time system

$$x(k+1) = A\,x(k) + B\,u(k)\,, \qquad\qquad x(0) \neq 0$$

with $x(k) \in \mathcal{R}^{\backslash}$ and $u(k) \in \mathcal{R}^{\Updownarrow}$.
and define the cost function

$$J[x_m, m, S, N] = \frac{1}{2}\,x^T(N)Sx(N) + \frac{1}{2}\sum_{k=m}^{N-1}\left\{x^T(k)Qx(k) + u^T(k)Ru(k)\right\}\,,$$

with $x(m) = x_m$, $m \in [0, N-1]$, $Q = Q^T \succeq 0$, $R = R^T \succ 0$ and $S \in \mathcal{R}^{\backslash\times\backslash}$. Define,

$$J^o[x_m, m, S, N] = \min_{U_{[m,N]}} J[x_m, m, S, N]$$

where $U_{[m,N]} = \{u(m), \cdots u(N)\}$ is the set of all possible control actions from $k = m$ to $k = N$.

Use the principle of optimality to proof that, when $S = 0$, $J^o[x_m, m, S, N]$ is a monotonically nondecreasing function of N:

$$J^o[x_m, m, 0, N+1] \geq J^o[x_m, m, 0, N]$$

Solution:

Using the principle of optimality, we obtain

$$\begin{aligned}
2\,J^o[x_m, m, 0, N+1] &= \min_{U_{[m,N]}} \left\{ \sum_{k=m}^{N} \left\{ x^T(k)Qx(k) + u^T(k)Ru(k) \right\} \right\} \\
&= \min_{u(N)} \min_{U_{[m,N-1]}} \left\{ \sum_{k=m}^{N} \left\{ x^T(k)Qx(k) + u^T(k)Ru(k) \right\} \right\} \\
&= \min_{U_{[m,N-1]}} \left\{ \min_{u(N)} \left\{ x^T(N)Qx(N) + u^T(N)Ru(N) \right\} + \sum_{k=m}^{N-1} \left\{ x^T(k)Qx(k) + u^T(k)Ru(k) \right\} \right\} \\
&\geq \min_{U_{[m,N-1]}} \sum_{k=m}^{N-1} \left\{ x^T(k)Qx(k) + u^T(k)Ru(k) \right\} = 2\,J^o[x_m, m, 0, N]
\end{aligned}$$

Notice that this result does not generally apply when $S = P(N) \neq 0$ since,

$$\begin{aligned}
2\,J^o[x_m, m, S, N+1] &= \min_{U_{[m,N]}} x^T(N+1)Sx(N+1) + \left\{ \sum_{k=m}^{N} \left\{ x^T(k)Qx(k) + u^T(k)Ru(k) \right\} \right\} \\
&= \min_{u(N)} \min_{U_{[m,N-1]}} \left\{ x^T(N+1)Sx(N+1) + \sum_{k=m}^{N} \left\{ x^T(k)Qx(k) + u^T(k)Ru(k) \right\} \right\} \\
&= \min_{U_{[m,N-1]}} \left\{ \min_{u(N)} \left\{ x^T(N+1)Sx(N+1) + x^T(N)Qx(N) + u^T(N)Ru(N) \right\} \right. \\
&\qquad \left. + \sum_{k=m}^{N-1} \left\{ x^T(k)Qx(k) + u^T(k)Ru(k) \right\} \right\} \\
&\neq \min_{U_{[m,N-1]}} \left\{ \min_{u(N)} \left\{ x^T(N)Qx(N) + u^T(N)Ru(N) \right\} \right. \\
&\qquad \left. + x^T(N)Sx(N) + \sum_{k=m}^{N-1} \left\{ x^T(k)Qx(k) + u^T(k)Ru(k) \right\} \right\}
\end{aligned}$$