ME 233 Advance Control II

Lecture 17

Stability Analysis Using The Hyperstability Theorem

Adaptive Control

Basic Adaptive Control Principle

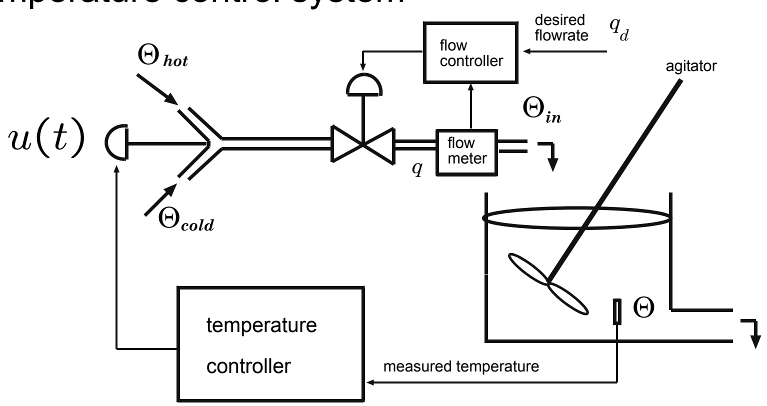
Controller parameters **are not constant,** rather, they are adjusted in an online fashion by a ***Parameter Adaptation Algorithm (PAA)***

When is adaptive control used?

- Plant parameters are unknown
- Plant parameters are time varying

Example of a system with varying parameters

- Temperature control system

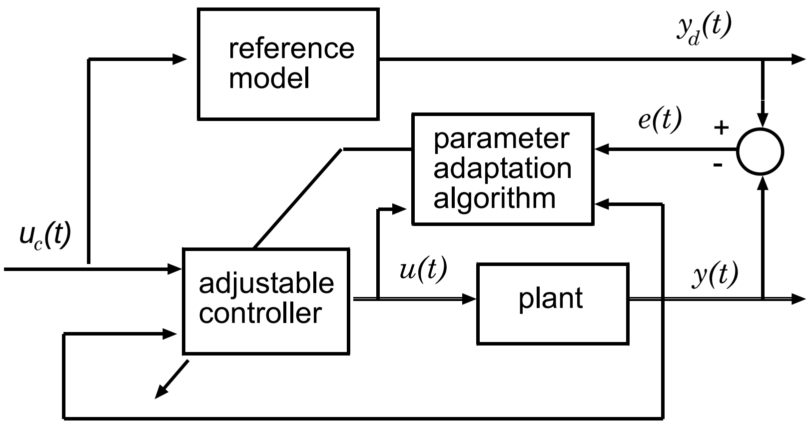


$$\frac{d}{dt}\theta(t) = -\underbrace{\frac{q}{V}}_{a(q)}\ \theta(t) + \underbrace{\frac{kq}{V}}_{b(q)}\ u(t-t_d)$$

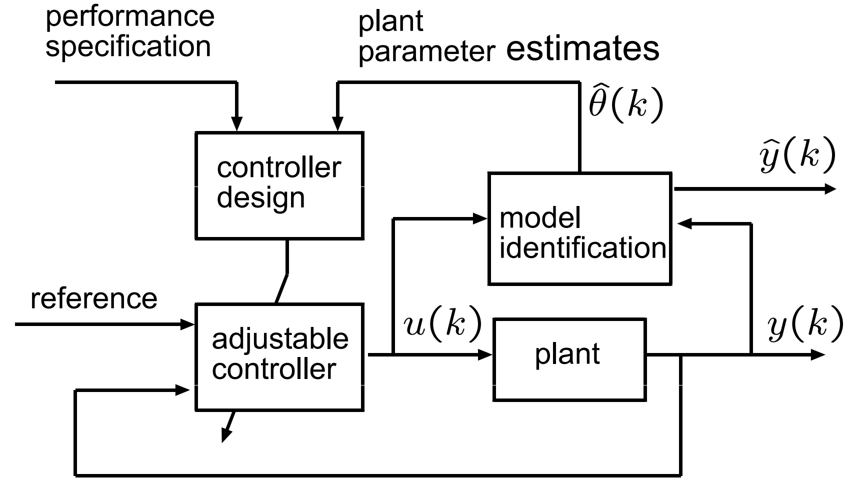
Adaptive Control Classification

- Continuous time VS **discrete time**
- Direct VS indirect
- MRAS VS **STR**

Model Reference Adaptive Systems (MRAS)

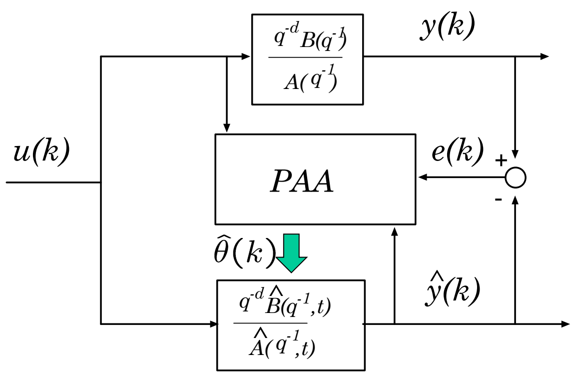


Self-Tuning Regulators (STR)



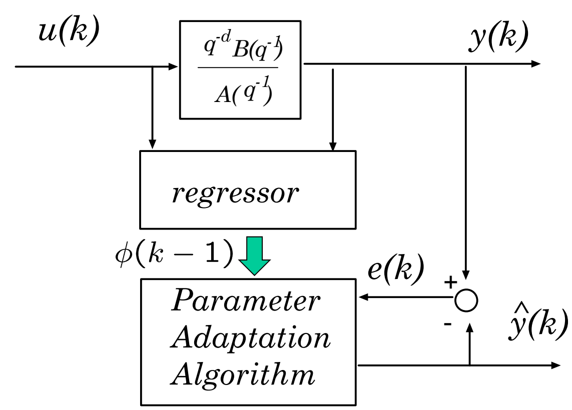
$$e(k) = y(k) - \widehat{y}(k)$$

Identification of a LTI system



Parallel model

Identification of a LTI system



Series-parallel model

Plant ARMA Model

Plant model

$$A(q^{-1})y(k) = q^{-d}B(q^{-1})u(k)$$

where

$$A(q^{-1}) = 1 + a_1 q^{-1} + \cdots + a_n q^{-n}$$
$$B(q^{-1}) = b_o + b_1 q^{-1} + \cdots + b_m q^{-m}$$

Unknown plant parameters

Assume ARMA model parameters are **unknown**

$$y(k) = -\underline{a_1}\, y(k-1) \cdots - \underline{a_n}\, y(k-n) + \underline{b_o} u(k-d) \cdots + \underline{b_m}\, u(k-d-m)$$

Define:

$$\theta = [\, \underline{a_1 \; \cdots \; a_n} \;\; \underline{b_o \; \cdots \; b_m} \,]^T$$

As the ***unknown*** parameter vector

Regressor vector

Collect all measurable signals in one vector

$$y(k) = -a_1 \underline{y(k-1)} \cdots - a_n \underline{y(k-n)} + b_o \underline{u(k-d)} \cdots + b_m \underline{u(k-d-m)}$$

We define

$$\phi(k-1) = [\underline{-\; y(k-1) \;\cdots\; -\; y(k-n)} \;\; \underline{u(k-d) \;\cdots\; u(k-d-m)}]^T$$

as the ***known*** regressor vector

Plant ARMA Model

Plant model

$$y(k) = \phi^T(k-1)\,\theta$$

where

$$\theta = [\, a_1 \; \cdots \; a_n \;\; b_o \; \cdots \; b_m \,]^T$$

$$\phi(k-1) = [-\; y(k-1) \;\cdots\; -\; y(k-n) \;\; u(k-d) \;\cdots\; u(k-d-m)]^T$$

Plant ARMA Model

Plant model

$$\hat{y}(k) = \phi^T(k-1)\,\hat{\theta}(k)$$

where

$$\hat{\theta}(k) = \left[\begin{array}{cccccc} \hat{a}_1(k) & \cdots & \hat{a}_n(k) & \hat{b}_o(k) & \cdots & \hat{b}_m(k) \end{array}\right]^T$$

$$\phi(k-1) = [-\ y(k-1)\ \cdots -\ y(k-n)$$
$$u(k-d)\ \cdots\ u(k-d-m)]^T$$

Plant output estimate

Plant a-posteriori estimate

$$\hat{y}(k) = \phi^T(k-1)\,\hat{\theta}(k)$$

Plant a-priori estimate

$$\hat{y}^o(k) = \phi^T(k-1)\,\hat{\theta}(k-1)$$

Plant a-posteriori error

$$y(k) = \phi^T(k-1)\,\theta$$

$$\hat{y}(k) = \phi^T(k-1)\,\hat{\theta}(k)$$

error: $$e(k) = y(k) - \hat{y}(k)$$

$$e(k) = \phi^T(k-1)\left[\theta - \hat{\theta}(k)\right]$$

$$= \phi^T(k-1)\tilde{\theta}(k)$$

$$\tilde{\theta}(k) = \theta - \hat{\theta}(k)$$

Parameter Adaptation Algorithm

PAA $$F = F^T \succ 0$$

$$\hat{\theta}(k) = \hat{\theta}(k-1) + F\,\phi(k-1)e(k)$$

Parameter error update law: $\tilde{\theta}(k) = \theta - \hat{\theta}(k)$

$$\tilde{\theta}(k) = \tilde{\theta}(k-1) - F\,\phi(k-1)e(k)$$

Adaptation Dynamics

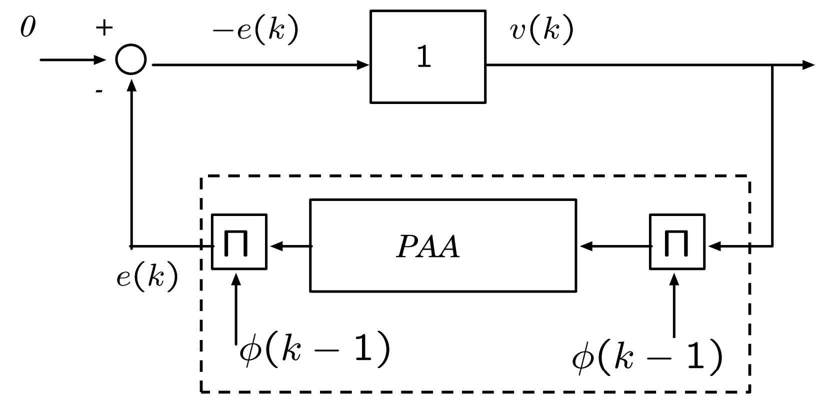
a-posteriori error: $e(k) = y(k) - \widehat{y}(k)$

$$e(k) = \phi^T(k-1)\widetilde{\theta}(k)$$

Parameter error update law: $\widetilde{\theta}(k) = \theta - \widehat{\theta}(k)$

$$\widetilde{\theta}(k) = \widetilde{\theta}(k-1) - F\,\phi(k-1)e(k)$$

Adaptation Dynamics



PAA: $\widetilde{\theta}(k) = \widetilde{\theta}(k-1) + F\,\phi(k-1)v(k)$

Convergence of Adaptive Systems

Adaptive systems are nonlinear

We need to prove that the algorithms converge:

- **Output error convergence** $\boxed{e(k) \to 0}$

$$e(k) = y(k) - \widehat{y}(k)$$

- **Parameter error convergence** $\boxed{\widetilde{\theta}(k) \to 0}$

$$\widetilde{\theta}(k) = \theta - \widehat{\theta}(k)$$

Output error Convergence

Our first goal will be to prove the asymptotic convergence of the output error:

$$e(k) \to 0$$

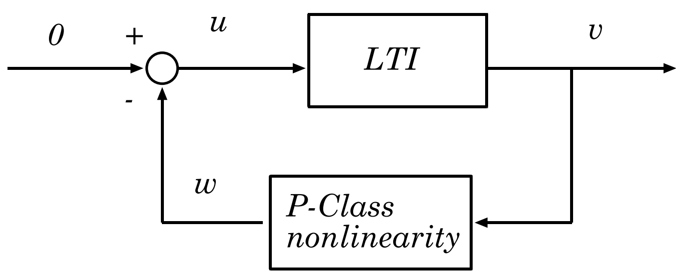
Two frequently used methods of stability analysis are**:**

- **Stability analysis using Lyapunov's direct method**
  - State space approach
- **Stability analysis using the Passivity or Hyperstability theorems**
  - Input/output approach

Hyperstability

Hyperstability Theory

- Developed by V.M. Popov to analyze the stability of a class of feedback systems (monograph published in 1973)

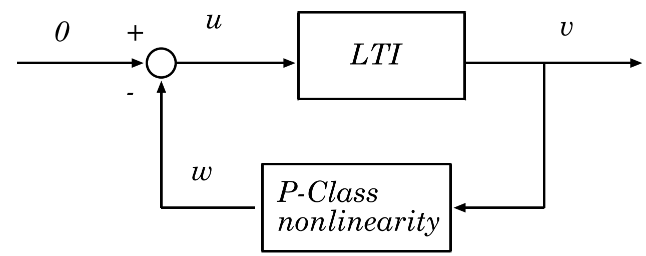


- Popularized by I.D. Landau for the analysis of adaptive systems (first book published in 1979)

Hyperstability Theory

Hyperstability Theory

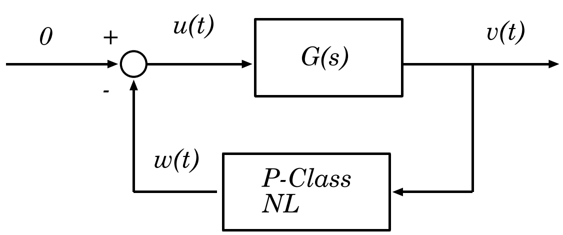
- Applies to both continuous time and discrete time systems



- **Abuse of notation:** We will denote the LTI block by its transfer function

CT Hyperstability Theory

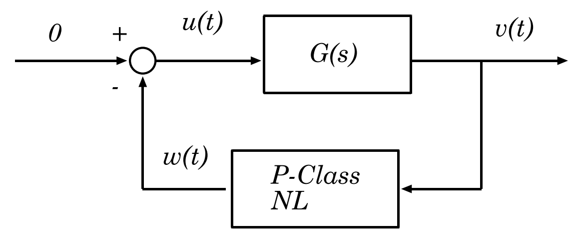
$$G(s) = C(sI - A)^{-1}B + D$$



- A state space description of the LTI Block:

$$\begin{aligned} \frac{d}{dt}x(t) &= A\,x(t) + B\,u(t) \\ v(t) &= C\,x(t) + D\,u(t) \end{aligned}$$

CT Hyperstability Theory

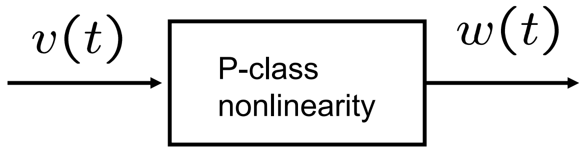


- P-class nonlinearity: (passive nonlinearities)

$$\boxed{\int_0^t w^T v \, d\tau \geq -\gamma_o^2 \qquad \forall\, t \geq 0}$$

Where γ_o is a constant which is a function of the initial conditions

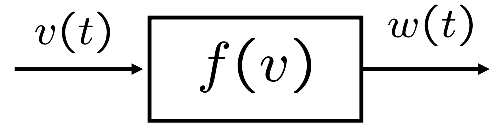
CT Hyperstability Theory



$$\boxed{\int_0^t w^T v \, d\tau \geq -\gamma_o^2 \qquad \forall\, t \geq 0}$$

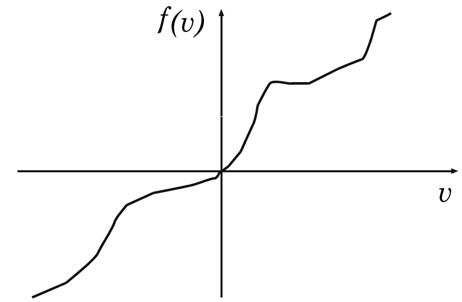
Where γ_o is a constant which is a function of the initial conditions

Example: Static P-class NL



$$w = f(v)$$

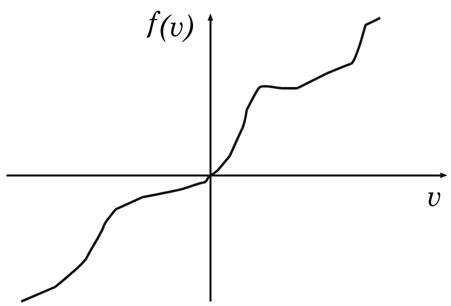
$$v\, f(v) \geq 0$$



Example: Static P-class NL

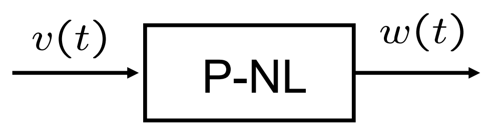
$$w = f(v)$$

$$v\, f(v) \geq 0$$



$$\int_0^t wv \, d\tau = \int_0^t \underbrace{f(v)v \, d\tau}_{\geq 0} \geq 0 > -\gamma_o^2$$

Example: Dynamic P-class block



$$\begin{cases} \dfrac{d}{dt}\tilde{\theta}(t) = F\,\phi(t)v(t) \\ w(t) = \phi^T(t)\tilde{\theta}(t) \end{cases}$$

$$\phi(t) \in \mathcal{R}^n$$

$$\tilde{\theta}(0) \in \mathcal{R}^n$$

$$|\tilde{\theta}(0)| < \infty$$

$$|\phi(t)| < \infty$$

$$F = F^T \succ 0$$

Example: Dynamic P-class block

$$w(t) = \phi^T(t)\tilde{\theta}(t) \qquad \dot{\tilde{\theta}}(t) = F\,\phi(t)v(t)$$

$$\int_0^t w(\tau)v(\tau)\,d\tau = \int_0^t \phi^T(\tau)\tilde{\theta}(\tau)v(\tau)d\tau$$

$$= \int_0^t \tilde{\theta}^T(\tau)\underbrace{[\phi(\tau)\tilde{v}(\tau)]}_{F^{-1}\dot{\tilde{\theta}}(\tau)}d\tau$$

$$= \frac{1}{2}\int_0^t \frac{d}{d\tau}\left\{\tilde{\theta}^T(\tau)F^{-1}\tilde{\theta}(\tau)\right\}d\tau$$

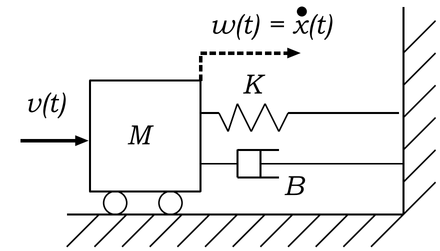
$$= \frac{1}{2}\tilde{\theta}^T(t)F^{-1}\tilde{\theta}(t) - \underbrace{\frac{1}{2}\tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0)}_{\gamma_o^2}$$

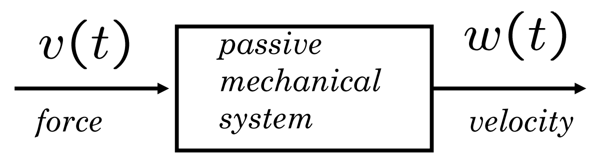
$$\geq -\gamma_o^2$$

Example: Passive mechanical system

Input is force and output is velocity

$$M\dot{w} + Bw + Kx = v$$

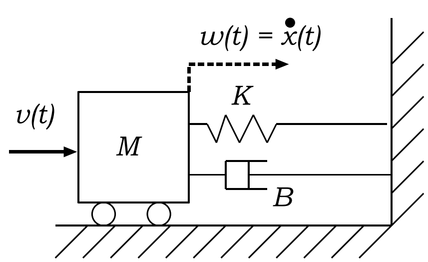




Example: Passive mechanical system

Input is force and output is velocity

$$M\dot{w} + Bw + Kx = v$$
$$\dot{x} = w$$



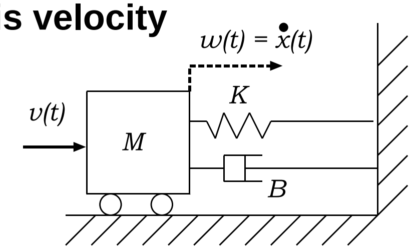
System Energy:

$$E(t) = \frac{1}{2}Mw^2(t) + \frac{1}{2}Kx^2(t) \geq 0$$

Example: Passive mechanical system

Input is force and output is velocity

$$E(t) = \frac{1}{2}Mw^2(t) + \frac{1}{2}Kx^2(t) \geq 0$$



Differentiating energy

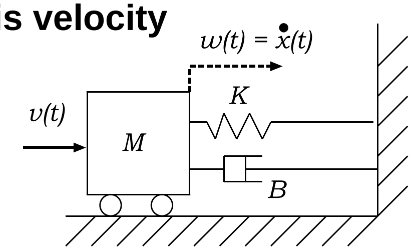
$$\dot{E} = M\dot{w}w + Kxw$$

$$= [-Kx - Bw + v]\,w + Kxw$$

$$= -\cancel{Kxw} - Bw^2 + wv + \cancel{Kxw}$$

Example: Passive mechanical system

Input is force and output is velocity



$$E(t) = \frac{1}{2}Mw^2(t) + \frac{1}{2}Kx^2(t) \geq 0$$

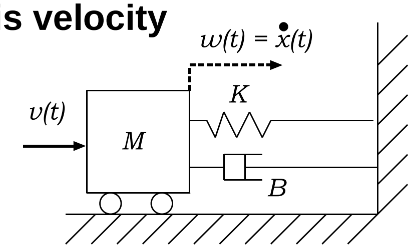
Differentiating energy

$$\dot{E} = -Bw^2 + w\,v$$

$$\underbrace{w\,v}_{power\ input} = \dot{E} + Bw^2$$

Example: Passive mechanical system

Input is force and output is velocity



$$E(t) = \frac{1}{2}Mw^2(t) + \frac{1}{2}Kx^2(t) \geq 0$$

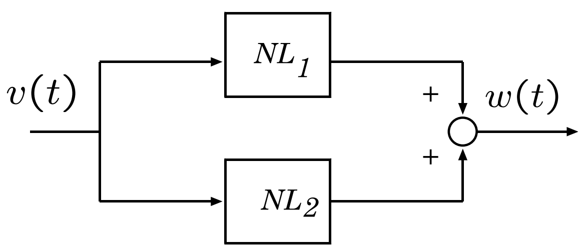
integrating power,

$$\boxed{\begin{aligned} \int_0^t wv\,d\tau \;&=\; E(t) - E(0) + \int_0^t Bw^2(\tau)\,d\tau \\ &\geq\; -\gamma_o^2 \\ &\qquad\qquad \gamma_o^2 = E(0) \geq 0 \end{aligned}}$$

Examples of P-class NL

Lemma:

- The parallel combination of two P-class nonlinearities is also a P-class nonlinearity.

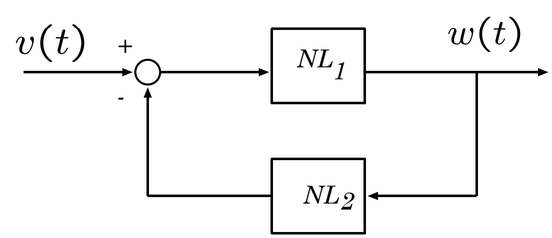


$$\int_0^t w^T v\,d\tau \geq -\gamma_o^2$$

Examples of P-class NL

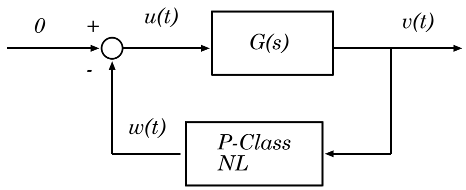
Lemma:

- The feedback combination of two P-class nonlinearities is also a P-class nonlinearity.



$$\int_0^t w^T v\,d\tau \geq -\gamma_o^2$$

CT Hyperstability

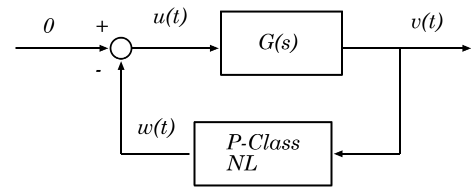


Hyperstability: The above feedback system is hyperstable if there exist positive bounded constants δ_1, δ_2 such that, for any state space realization of *G(s)*,

$$\boxed{|x(t)| < \delta_1 \left[|x(0)| + \delta_2\right] \qquad \forall\, t \geq 0}$$

<u>FOR ALL</u> P-class nonlinearities

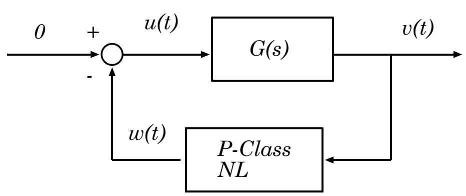
CT Asymptotic Hyperstability



Asymptotic Hyperstability: The above feedback system is asymptotically hyperstable if

1. It is hyperstable
2. For all signals $|w(t)| < \infty$ (I.e. bounded output of any P-class nonlinearity), and any state space realization of *G(s)*,
$$\lim_{t\to\infty} x(t) = 0$$

CT Hyperstability Theorems



Hyperstability Theorem: The above feedback system is hyperstable **iff** the transfer function *G(s)* of the LTI block is **Positive Real.**

Asymptotical Hyperstability Theorem: The above feedback system is asymptotically hyperstable **iff** the transfer function *G(s)* of the LTI block is **Strictly Positive Real.**

CT Positive Real TF

$$G(s) = C(sI - A)^{-1}B + D$$

Is **Positive Real** iff:

1. *G(s)* does not have any unstable poles (I.e. no Re{s} > 0).
2. Any pole of *G(s)* that is in the imaginary axis <u>does not repeat</u> and its associated residue (I.e. the coefficient appearing in the partial fraction expansion) is non-negative.
3. $$2\,\mathrm{Re}\{G(j\omega)\} = G(j\omega) + G^T(-j\omega) \geq 0$$

for all real ω's for which $s = j\,\omega$ is not a pole of *G(s)*

Strictly Positive Real (SPR) TF

$$G(s) = C(sI - A)^{-1}B + D$$

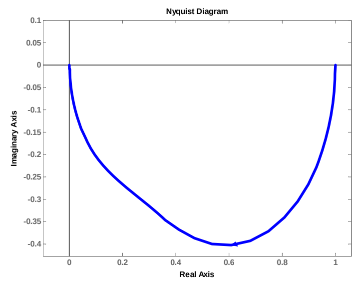
Is **Strictly Positive Real (SPR)** iff:

1. All poles of $G(s)$ are asymptotically stable.
2. $2\,\mathrm{Re}\{G(j\omega)\} = G(j\omega) + G^T(-j\omega) > 0$

   for all $\omega, \quad 0 \leq \omega < \infty$

Example:

$$G(s) = \frac{s+1}{s^2 + 3s + 1}$$



Strictly Positive Real (SPR) TF

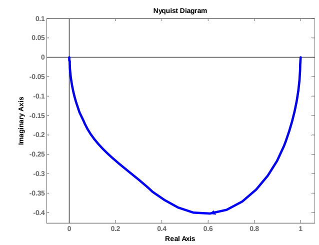
For scalar rational transfer functions

$$G(s) = \frac{B(s)}{A(s)}$$

1. All poles of $G(s)$ are asymptotically stable.
2. $\mathrm{Re}\{G(j\omega)\} > 0$ for all $\omega, \quad 0 \leq \omega < \infty$

Note:

A necessary (but not sufficient) condition for $G(s)$ to be SPR is that its relative degree must be less than or equal to 1.



Kalman Yakubovich Popov Lemma

$$G(s) = C(sI - A)^{-1}B + D$$

Is **Strictly Positive Real (SPR)** if and only if

- there exist a symmetric and positive definite matrix P,
- matrices L and K,
- and a constant $\varepsilon > 0$ such that

$$\begin{aligned} A^TP + PA &= -L^TL - \epsilon P \\ B^TP - C &= -K^TL \\ D + D^T &= K^TK \end{aligned}$$

Kalman Yakubovich Popov Lemma

$$G(s) = C(sI - A)^{-1}B$$

Is **Strictly Positive Real (SPR) iff** there exist symmetric and positive definite matrices P and Q, such that:

$$\begin{aligned} A^TP + PA &= -Q \\ B^TP &= C \end{aligned}$$

SPR TF implies Possitivity

Let $G(s) = C(sI - A)^{-1}B + D$ be SPR

Then there exist positive definite functions

$V(x) \succ 0 \quad \lambda_1(x) \succ 0$

and a positive semi-definite function $\lambda_2(x,u) \succeq 0$

Such that the input *u(t)* output *y(t)* pair satisfies

$$\boxed{\begin{aligned}\int_0^t y^T u\, d\tau &= V(x(t)) - V(x(0)) + \int_0^t (\lambda_1(x) + \lambda_2(x,u))\, d\tau \\ &\geq -\gamma_o^2 \\ &\qquad\qquad \gamma_o^2 = V(x(0))\end{aligned}}$$

SPR TF implies Passivity

Proof: We consider a strictly causal transfer function

$$G(s) = C(sI - A)^{-1}B$$

which is SPR, with state space realization

$$\begin{aligned}\frac{d}{dt}x &= A\,x + B\,u \\ v &= C\,x\end{aligned}$$

By the Kalman Yakubovich, Popov lemma, there exist symmetric and positive definite matrices P and Q, such that

$$\begin{aligned}A^T P + PA &= -Q \\ B^T P &= C\end{aligned}$$

SPR TF implies Passivity

Proof: Define the PD function $V(x) = \frac{1}{2}x^T P x$

and compute:

$$\begin{aligned}2\,\dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= (Ax + Bu)^T P x + x^T P(Ax + Bu) \\ &= x^T \left[\underbrace{A^T P + PA}_{-Q}\right] x + 2u^T \underbrace{\underbrace{B^T P}_{C} x}_{v}\end{aligned}$$

by the Kalman Yakubovich, Popov lemma.

$$\begin{aligned}A^T P + PA &= -Q \\ B^T P &= C\end{aligned}$$

SPR TF implies Passivity

Proof: Thus, since $v = C\,x$

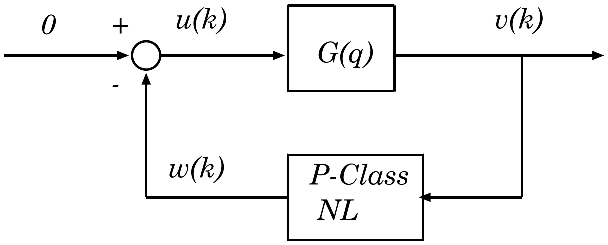
$$u^T v = \dot{V} + \frac{1}{2}x^T Q x$$

Define the PD function $\lambda_1(x) = \frac{1}{2}x^T Q x$ and integrate

$$\begin{aligned}\int_0^t u^T v\, d\tau &= \int_0^t \dot{V}\, d\tau + \int_0^t \lambda_1(x)\, d\tau \\ &= V(x(t)) - V(x(0)) + \int_0^t \lambda_1(x)\, d\tau\end{aligned}$$

DT Hyperstability Theory

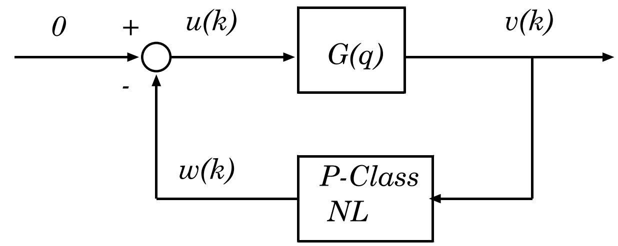
$$G(z) = C(zI - A)^{-1}B + D$$



- State space description of the LTI Block:

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ v(k) &= C\,x(k) + D\,u(k) \end{aligned}$$

DT Hyperstability Theory

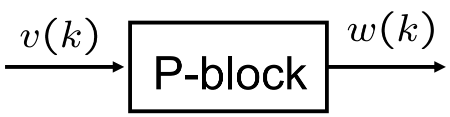


- P-class nonlinearity: (passive nonlinearities)

$$\boxed{\sum_{j=0}^{k} w^T(j)v(j) \geq -\gamma_o^2 \qquad \forall\, k \geq 0}$$

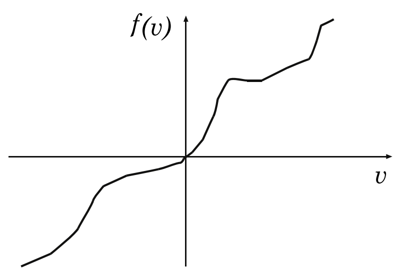
Where γ_o is a bounded constant.

Example: Static nonlinearity:



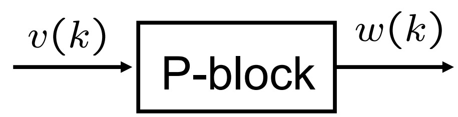
$$w = f(v)$$

$$v\,f(v) \geq 0$$



$$\sum_{j=0}^{k} w^T(j)v(j) = \sum_{j=0}^{k} \underbrace{f(v(j))v(j)}_{\geq 0} \geq 0 > -\gamma_o^2$$

Example: Dynamic P-class block



$$\begin{cases} \tilde{\theta}(k) = \tilde{\theta}(k-1) + F\,\phi(k)v(k) \\ w(k) = \phi^T(k)\tilde{\theta}(k) \end{cases}$$

$$F = F^T \succ 0$$

$$\phi(k) \in \mathcal{R}^n$$

$$\tilde{\theta}(-1) \in \mathcal{R}^n$$

$$|\tilde{\theta}(-1)| < \infty$$

$$|\phi(k)| < \infty$$

Example: Dynamic P-class block

$$w(k) = \phi^T(k)\tilde{\theta}(k) \qquad \tilde{\theta}(k) = \tilde{\theta}(k-1) + F\,\phi(k)v(k)$$

$$\sum_{j=0}^{k} w(j)v(j) = \sum_{j=0}^{k} \phi^T(j)\tilde{\theta}(j)v(j)$$

$$= \sum_{j=0}^{k} \tilde{\theta}^T(j)\,\underbrace{[\phi(j)v(j)]}_{F^{-1}[\tilde{\theta}(j)-\tilde{\theta}(j-1)]}$$

$$= \sum_{j=0}^{k} \tilde{\theta}^T(j)F^{-1}\left[\tilde{\theta}(j) - \tilde{\theta}(j-1)\right]$$

$$= \sum_{j=0}^{k} \left\{\tilde{\theta}^T(j)F^{-1}\tilde{\theta}(j) - \tilde{\theta}^T(j)F^{-1}\tilde{\theta}(j-1)\right\}$$

Example: Dynamic P-class block

$$\sum_{j=0}^{k} w(j)v(j) = \sum_{j=0}^{k} \left\{\tilde{\theta}^T(j)F^{-1}\tilde{\theta}(j) - \tilde{\theta}^T(j)F^{-1}\tilde{\theta}(j-1)\right\}$$

$$+ \frac{1}{2}\sum_{j=0}^{k} \tilde{\theta}^T(j-1)F^{-1}\tilde{\theta}(j-1) - \frac{1}{2}\sum_{j=0}^{k} \tilde{\theta}^T(j-1)F^{-1}\tilde{\theta}(j-1)$$

$$\sum_{j=0}^{k} w(j)v(j) = \frac{1}{2}\sum_{j=0}^{k} \left\{\tilde{\theta}^T(j)F^{-1}\tilde{\theta}(j) - \tilde{\theta}^T(j-1)F^{-1}\tilde{\theta}(j-1)\right\}$$

$$\underbrace{+ \frac{1}{2}\sum_{j=0}^{k} \left[\tilde{\theta}(j) - \tilde{\theta}(j-1)\right]^T F^{-1}\left[\tilde{\theta}(j) - \tilde{\theta}(j-1)\right]}_{\geq 0}$$

Example: Dynamic P-class block

$$\sum_{j=0}^{k} w(j)v(j) \geq \frac{1}{2}\sum_{j=0}^{k} \left\{\tilde{\theta}^T(j)F^{-1}\tilde{\theta}(j) - \tilde{\theta}^T(j-1)F^{-1}\tilde{\theta}(j-1)\right\}$$

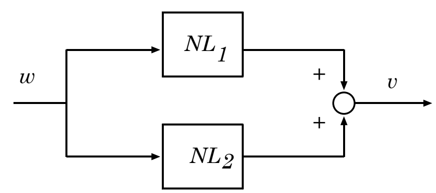
$$\geq \frac{1}{2}\tilde{\theta}^T(k)F^{-1}\tilde{\theta}(k) - \underbrace{\frac{1}{2}\tilde{\theta}^T(-1)F^{-1}\tilde{\theta}(-1)}_{\gamma_o^2}$$

$$\geq -\gamma_o^2$$

Examples of P-class NL

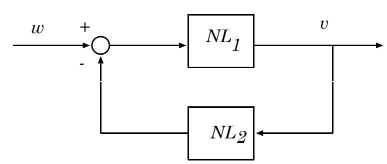
Lemma:

- The parallel combination of two P-class nonlinearities is also a P-class nonlinearity.



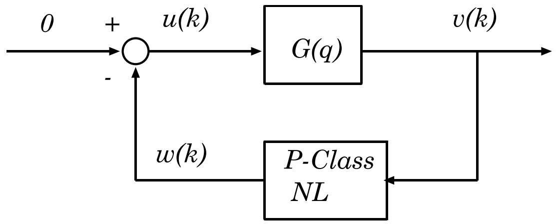
$$\sum_{j=0}^{k} w^T(j)v(j) \geq -\gamma_o^2$$

- The feedback combination of two P-class nonlinearities is also a P-class nonlinearity.



$$\sum_{j=0}^{k} w^T(j)v(j) \geq -\gamma_o^2$$

DT Hyperstability

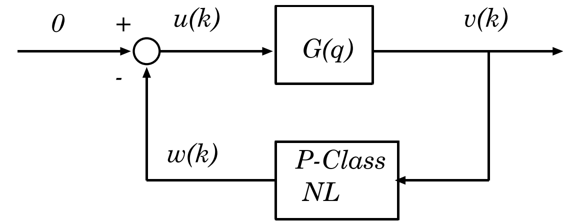


Hyperstability: The above feedback system is hyperstable if there exist positive bounded constants δ_1, δ_2 such that, for any state space realization of *G(q)*,

$$\boxed{|x(k)| < \delta_1 \left[|x(0)| + \delta_2 \right] \qquad \forall\, k \geq 0}$$

FOR ALL P-class nonlinearities

DT Asymptotic Hyperstability

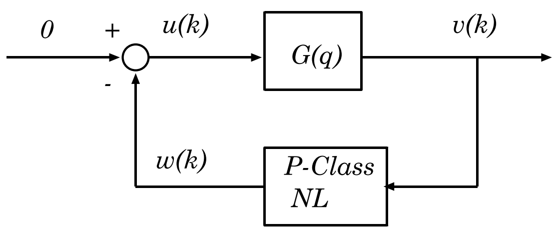


Asymptotic Hyperstability: The above feedback system is asymptotically hyperstable if

1. It is hyperstable
2. for any state space realization of *G(s)*,

$$\lim_{k \to \infty} x(k) = 0$$

DT Hyperstability Theorems



Hyperstability Theorem: The above feedback system is hyperstable **iff** the transfer function *G(z)* of the LTI block is **Positive Real.**

Asymptotical Hyperstability Theorem: The above feedback system is asymptotically hyperstable **iff** the transfer function *G(z)* of the LTI block is **Strictly Positive Real.**

Positive Real TF

$$G(z) = C(zI - A)^{-1}B + D$$

Is **Positive Real** iff:

1. *G(z)* does not have any unstable poles (I.e. no $|z| > 1$).
2. Any pole of *G(z)* that is in the unit circle does not repeat and its associated residue (I.e. the coefficient appearing in the partial fraction expansion) is non-negative.
3. $2\,\mathrm{Re}\{G(e^{j\omega})\} = G(e^{j\omega}) + G^T(e^{-j\omega}) \geq 0$

   for all $\omega, \quad 0 \leq \omega \leq \pi$ for which $z = e^{j\omega}$ is not a pole of *G(s)*

Strictly Positive Real (SPR) TF

$$G(z) = C(zI - A)^{-1}B + D$$

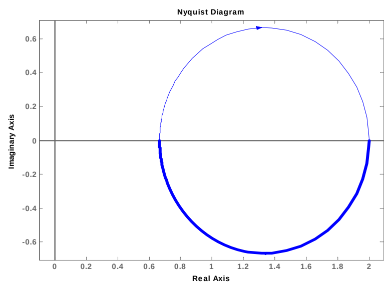
Is **Strictly Positive Real (SPR)** iff:

1. All poles of $G(z)$ are asymptotically stable.
2. $2\,\mathrm{Re}\{G(e^{j\omega})\} = G(e^{j\omega}) + G^T(e^{-j\omega}) > 0$

   for all $0 \le \omega \le \pi$

   Example:

$$G(z) = \frac{z}{z + 0.5}$$



Strictly Positive Real (SPR) TF

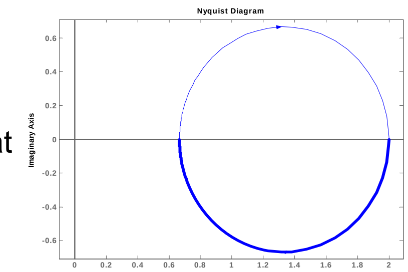
For scalar rational transfer functions

$$G(z) = \frac{B(z)}{A(z)}$$

1. All poles of $G(s)$ are asymptotically stable.
2. $\mathrm{Re}\{G(e^{j\omega})\} > 0$ for all $\omega, \quad 0 \le \omega \le \pi$

Note:

A necessary (but not sufficient) condition for $G(z)$ to be SPR is that its relative degree must be 0.



Kalman Szegö Popov Lemma

$$G(z) = C(zI - A)^{-1}B + D$$

Is **Strictly Positive Real (SPR)** if and only if

- there exist a symmetric and positive definite matrix P,
- matrices L and K,
- and a constant ε > 0 such that

$$\begin{aligned} A^T P A - P &= -L^T L - \epsilon P \\ B^T P A - C &= -K^T L \\ D + D^T - B^T P B &= K^T K \end{aligned}$$

SPR TF implies Possitivity

Let $G(z) = C(zI - A)^{-1}B + D$ be SPR

Then there exist positive definite functions

$$V(x) \succ 0 \qquad \lambda_1(x) \succ 0$$

and a positive semi-definite function $\lambda_2(x, u) \succeq 0$

Such that the input $u(k)$ output $y(k)$ pair satisfies

$$\boxed{\begin{aligned} \sum_{j=0}^{k} y^T(j) u(j) &= V(x(k+1)) - V(x(0)) + \sum_{j=0}^{k} \lambda_1(x(j)) \\ &\quad + \sum_{j=0}^{k} \lambda_2(x(j), u(j))) \\ &\ge -\gamma_o^2 \qquad\qquad \gamma_o^2 = V(x(0)) \end{aligned}}$$

Proof

Let $G(z) = C(zI - A)^{-1}B + D$ be SPR

Then by the Kalman Szegö Popov Lemma

$$\begin{aligned} A^T PA - P &= -L^T L - \epsilon P \\ B^T PA - C &= -K^T L \\ D + D^T - B^T PB &= K^T K \end{aligned}$$

$$P = P^T \succ 0$$

$$\epsilon P \succ 0$$

Define the Lyapunov function

$$V(x_k) = \frac{1}{2} x_k^T P x_k \succ 0$$

Proof

$$V(x_k) = \frac{1}{2} x_k^T P x_k \succ 0$$

$$V(x_{k+1}) - V(x_k) = \frac{1}{2}(Ax_k + Bu_k)^T P(Ax_k + Bu_k) - \frac{1}{2} x_k^T P x_k$$

$$= \frac{1}{2} x_k^T \underbrace{(A^T PA - P)}_{-\epsilon P - L^T L} x_k$$

$\boxed{A^T PA - P = -L^T L - \epsilon P}$

$$+ \frac{1}{2} x_k^T \underbrace{A^T PB}_{C^T - L^T K} u_k + \frac{1}{2} u_k^T \underbrace{B^T PA}_{C - K^T L} x_k$$

$\boxed{B^T PA - C = -K^T L}$

$$+ \frac{1}{2} u_k^T \underbrace{B^T PB}_{D + D^T - K^T K} u_k$$

$\boxed{D + D^T - B^T PB = K^T K}$

Proof

$$V(x_k) = \frac{1}{2} x_k^T P x_k \succ 0$$

$$V(x_{k+1}) - V(x_k) = -\underbrace{\epsilon \frac{1}{2} x_k^T P x_k}_{\lambda_1(x_k) \succ 0}$$

$$+ \frac{1}{2} \underbrace{(Cx_k + Du_k)^T}_{y_k^T} u_k + \frac{1}{2} u_k^T \underbrace{(Cx_k + u_k)}_{y_k}$$

$$\underbrace{- \frac{1}{2} \begin{bmatrix} x_k^T & u_k^T \end{bmatrix} \begin{bmatrix} L^T \\ K^T \end{bmatrix} \begin{bmatrix} L & K \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}}_{\lambda_2(x_k, u_k) \succeq 0}$$

Proof

$$V(x_k) = \frac{1}{2} x_k^T P x_k \succ 0$$

$$V(x_{k+1}) - V(x_k) = + y_k^T u_k - \underbrace{\epsilon \frac{1}{2} x_k^T P x_k}_{\lambda_1(x_k) \succ 0} - \underbrace{\frac{1}{2} \|Lx_k + Ku_k\|^2}_{\lambda_2(x_k, u_k) \succeq 0}$$

$$V(x_{k+1}) - V(x_k) = + y_k^T u_k - \lambda_1(x_k) - \lambda_2(x_k, u_k)$$

Proof

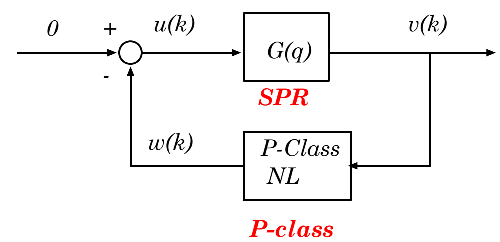
$$V(x_{k+1}) - V(x_k) = \quad + y_k^T u_k \quad - \lambda_1(x_k) - \lambda_2(x_k, u_k)$$

Taking summation:

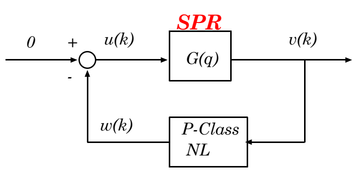
$$\sum_{j=0}^{k} y_j^T u_j \;=\; \sum_{j=0}^{k} \{V(x_{j+1}) - V(x_j))\}$$
$$+ \sum_{j=0}^{k} \lambda_1(x_j) + \sum_{j=0}^{k} \lambda_2(x_j, u_j)$$

$$\boxed{\sum_{j=0}^{k} y_j^T u_j \;=\; V(x_{k+1}) - V(x_0) + \sum_{j=0}^{k} \lambda_1(x_j) + \sum_{j=0}^{k} \lambda_2(x_j, u_j)}$$

Proof of the sufficiency part of the Asymptotic Hyperstability Theorem - Discrete Time



- Since the nonlinearity is P-class, $\sum_{j=0}^{k} w_j^T v_j \geq -\gamma_1^2$
- Since LTR block is SPR, we can use the Kalman Szegö Popov Lemma



Hyperstability

Using the Kalman Szegö Popov Lemma, for any minimal realization,

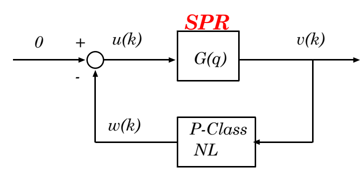
$$G(z) = C(zI - A)^{-1}B + D$$

we have

$$\sum_{j=0}^{k} v_j^T u_j \;=\; V(x_{k+1}) - V(x_0) + \sum_{j=0}^{k} \lambda_1(x_j) + \sum_{j=0}^{k} \lambda_2(x_j, u_j)$$

where

$$\lambda_1(x_k) = \epsilon V(x_k) \succ 0 \qquad V(x_k) = \frac{1}{2} x_k^T P x_k \succ 0 \qquad \lambda_2(x_k, u_k) \succeq 0$$



Hyperstability

$$G(z) = C(zI - A)^{-1}B + D$$

rearranging terms,

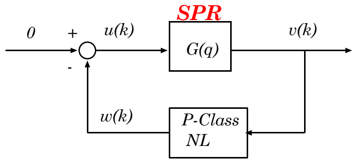
$$V(x_{k+1}) \;=\; V(x_0) + \sum_{j=0}^{k} v_j^T u_j \; - \sum_{j=0}^{k} \lambda_1(x_j) - \sum_{j=0}^{k} \lambda_2(x_j, u_j)$$

- P-class:

$$\sum_{j=0}^{k} w_j^T v_j \geq -\gamma_1^2 \quad \Rightarrow \quad \sum_{j=0}^{k} v_j^T u_j \leq \gamma_1^2$$

Therefore,

$$V(x_{k+1}) \;\leq\; V(x_0) + \gamma_1^2 \quad - \sum_{j=0}^{k} \lambda_1(x_j) - \sum_{j=0}^{k} \lambda_2(x_j, u_j)$$



Hyperstability

$$\lambda_1(x_k) = \epsilon V(x_k) \succ 0$$

$$\lambda_2(x_k, u_k) \succeq 0$$

$$V(x_{k+1}) \;\leq\; V(x_0) + \gamma_1^2 \underbrace{- \sum_{j=0}^{k} \lambda_1(x_j) - \sum_{j=0}^{k} \lambda_2(x_j, u_j)}_{\leq 0}$$

Therefore,

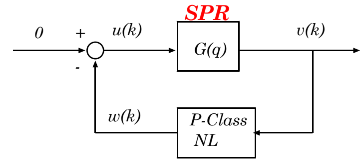
$$V(x_{k+1}) \;\leq\; V(x_0) + \gamma_1^2$$

Since $V(x_k) = \frac{1}{2} x_k^T P x_k \succ 0$

$$\Rightarrow \quad \lambda_{min}(P)|x_k|^2 \;\leq\; \lambda_{max}(P)|x_0|^2 + 2\gamma_1^2$$

$$\Rightarrow \quad |x_k|^2 \;\leq\; \underbrace{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}_{\delta_1} \left[|x_0|^2 + \underbrace{\frac{2}{\lambda_{max}(P)} \gamma_1^2}_{\delta_2} \right]$$

Thus, the feedback system is Hyperstable



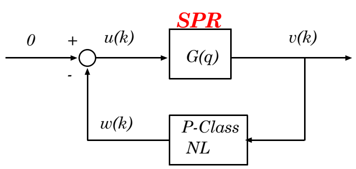
Asymptotic Hyperstability

$$G(z) = C(zI - A)^{-1} B + D$$

$$V(x_{k+1}) \;\leq\; V(x_0) + \gamma_1^2 \; - \sum_{j=0}^{k} \lambda_1(x_j) - \sum_{j=0}^{k} \lambda_2(x_j, u_j)$$

Taking the limit as k→∞

$$\lim_{k\to\infty} V(x_{k+1}) \;\leq\; V(x_0) + \gamma_1^2 \; - \sum_{j=0}^{\infty} \lambda_1(x_j) - \sum_{j=0}^{\infty} \lambda_2(x_j, u_j)$$



Asymptotic Hyperstability

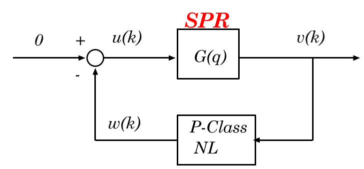
$$G(z) = C(zI - A)^{-1} B + D$$

$$\lim_{k\to\infty} V(x_{k+1}) \;\leq\; \underbrace{V(x_0) + \gamma_1^2}_{\gamma_2^2} \; - \sum_{j=0}^{\infty} \lambda_1(x_j) \underbrace{- \sum_{j=0}^{\infty} \lambda_2(x_j, u_j)}_{\leq 0}$$

$$\lim_{k\to\infty} V(x_{k+1}) \;\leq\; \gamma_2^2 - \sum_{j=0}^{\infty} \lambda_1(x_j)$$

where

$$V(x_k) = \frac{1}{2} x_k^T P x_k \succ 0 \qquad \lambda_1(x_k) = \epsilon V(x_k) \succ 0$$



Asymptotic Hyperstability

$$\lambda_1(x_k) = \epsilon V(x_k) \succ 0$$

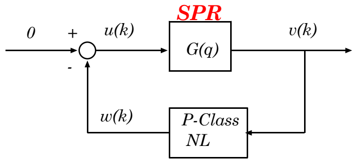
From our previous page, $\lim_{k\to\infty} V(x_{k+1}) \;\leq\; \gamma_2^2 - \sum_{j=0}^{\infty} \underbrace{\lambda_1(x_j)}_{\geq 0}$

Since $V(x_k) = \frac{1}{2} x_k^T P x_k \succ 0$ $\Rightarrow$ it cannot become negative

Moreover, the term $- \sum_{j=0}^{\infty} \lambda_1(x_j)$ can only become more negative or converge to a constant

Therefore,

$$0 \leq \lim_{k\to\infty} V(x_k) = V_\infty \leq \gamma_2^2 \quad \Rightarrow \quad \sum_{j=0}^{\infty} \lambda_1(x_j) = C_1 \leq \gamma_2^2$$



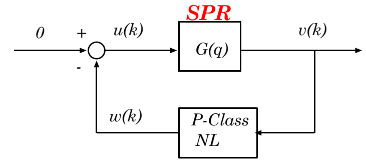
Asymptotic Hyperstability

Since

$$\sum_{j=0}^{\infty} \lambda_1(x_j) = \epsilon \sum_{j=0}^{\infty} \underbrace{V(x_k)}_{\geq 0} = C_1 \quad \Rightarrow \quad \lim_{k\to\infty} V(x_k) = 0$$

Moreover, since $V(x_k) = \underbrace{\frac{1}{2} x_k^T P x_k}_{\succ 0}$ $\Rightarrow$ $\lim_{k\to\infty} x_k = 0$

Q.E.D.



Asymptotic Hyperstability

Additional result

If in addition

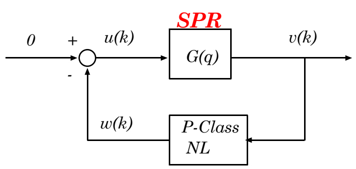
$$D + D^T \;=\; K^T K + B^T P B \succ 0$$

$$\boxed{\begin{aligned} &\lim_{k\to\infty} v(k) = 0 \\ &\lim_{k\to\infty} u(k) = 0 \end{aligned}}$$

We have already shown that $\lim_{k\to\infty} x_k = 0$

Since $v_k = C x_k + D u_k$

We need to prove that: $\lim_{k\to\infty} D\, u_k = 0$



Asymptotic Hyperstability

$$\lim_{k\to\infty} v(k) = 0$$

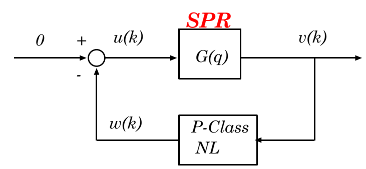
Using the Kalman Szegö Popov Lemma, we obtained

$$\lim_{k\to\infty} V(x_{k+1}) \;\leq\; V(x_0) + \gamma_1^2 \;-\sum_{j=0}^{\infty} \lambda_1(x_j) - \sum_{j=0}^{\infty} \lambda_2(x_j, u_j)$$

where: $\lambda_1(x_j) = \; \epsilon \frac{1}{2} x_k^T P x_k \succ 0$

$$\lambda_2(x_j, u_j) = \|L x_k + K u_k\|^2 \succeq 0$$

Thus: $\sum_{j=0}^{\infty} \lambda_1(x_j) < \infty$, $\sum_{j=0}^{\infty} \lambda_2(x_j, u_j) < \infty$ $\Rightarrow$ $\lim_{k\to\infty} \lambda_1(x_k) = 0$, $\lim_{k\to\infty} \lambda_2(x_k, u_k) = 0$



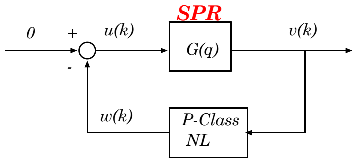
Asymptotic Hyperstability

$$\lim_{k\to\infty} v(k) = 0$$

$\lim_{k\to\infty} \lambda_1(x_k) = 0$, $\lambda_1(x_j) = \; \epsilon \frac{1}{2} x_k^T P x_k \succ 0$ $\Rightarrow$ $\lim_{k\to\infty} x_k = 0$

$\lim_{k\to\infty} \lambda_2(x_k, u_k) = 0$, $\lambda_2(x_j, u_j) = \|L x_k + K u_k\|^2 \succeq 0$ $\Rightarrow$ $\lim_{k\to\infty} \|L x_k + K u_k\|^2 = 0$

$\Rightarrow$ $\lim_{k\to\infty} K u_k = 0$



Asymptotic Hyperstability

$$\lim_{k\to\infty} v(k) = 0$$

So far we have: $\lim_{k\to\infty} x_k = 0 \qquad \lim_{k\to\infty} K u_k = 0$

The state equation $\quad x(k+1) = A\,x(k) + B\,u(k)$

$\Rightarrow \quad \lim_{k\to\infty} B u_k = 0$

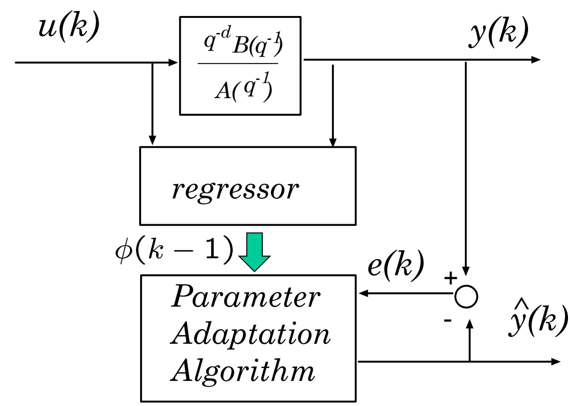
From the Kalman Szegö Popov Lemma: $\quad D + D^T = K^T K + B^T P B$

Thus, $\quad u_k^T (D + D^T) u_k = u_k^T (K^T K + B^T P B) u_k = 0$

$(D + D^T) \succ 0 \quad \Rightarrow \quad \lim_{k\to\infty} u_k = 0 \quad \Rightarrow \quad \lim_{k\to\infty} v_k = 0$

Q.E.D.

Stability analysis of Series-parallel ID



Series-Parallel ID Dynamics

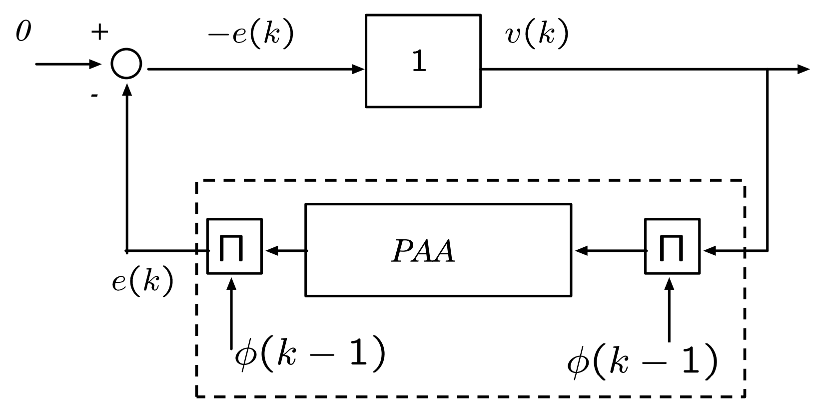
a-posteriori error: $\quad e(k) = y(k) - \hat{y}(k)$

$$e(k) = \phi^T(k-1)\tilde{\theta}(k)$$

Parameter error update law: $\quad \tilde{\theta}(k) = \theta - \hat{\theta}(k)$

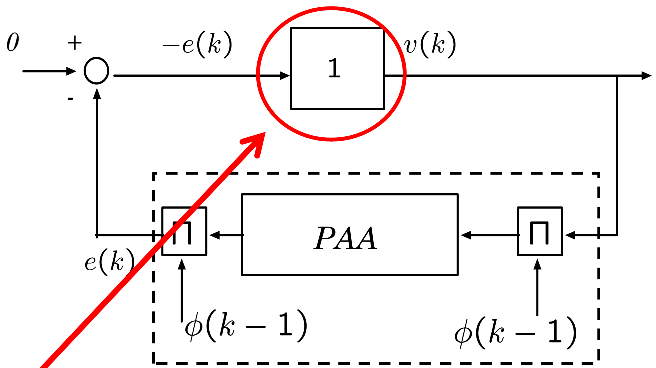
$$\tilde{\theta}(k) = \tilde{\theta}(k-1) - F\,\phi(k-1)e(k)$$

Series-Parallel ID Dynamics



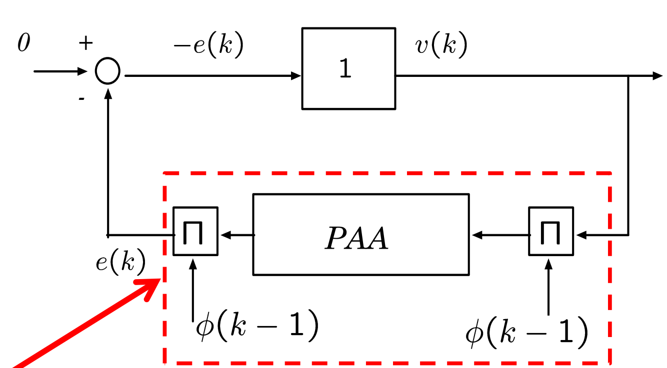
PAA: $\quad \tilde{\theta}(k) = \tilde{\theta}(k-1) + F\,\phi(k-1)v(k)$

Stability analysis of Series-parallel ID



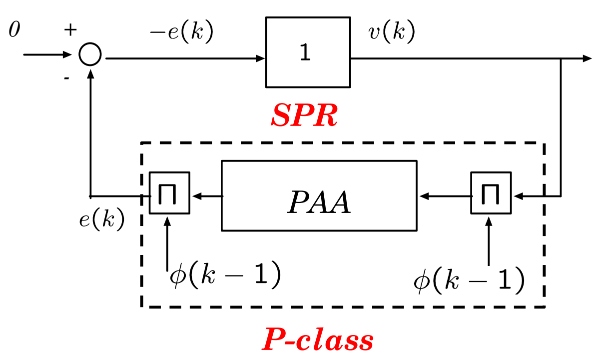
Strictly Positive Real

Stability analysis of Series-parallel ID



$$\left\{\begin{array}{l}\tilde{\theta}(k) = \tilde{\theta}(k-1) + F\,\phi(k-1)v(k) \\ e(k) = \phi^T(k-1)\tilde{\theta}(k)\end{array}\right. \Rightarrow \sum_{j=0}^{k} e^T(j)v(j) \geq -\gamma_o^2$$

Stability analysis of Series-parallel ID

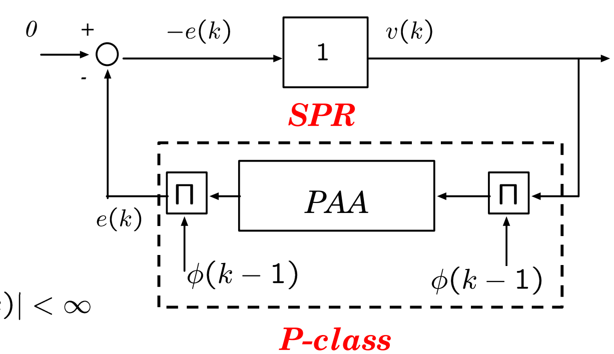


By the sufficiency portion of Hyperstability Theorem:

$|v(k)| < \infty$

$|e(k)| < \infty$

Stability analysis of Series-parallel ID



$|e(k)| < \infty$

By the sufficiency portion of Asymptotic Hyperstability Theorem:

$|v(k)| \rightarrow 0$

$|e(k)| \rightarrow 0$

Q.E.D.

How to we implement the PAA?

a-posteriori error & PAA:

$$\left.\begin{aligned} e(k) &= \phi^T(k-1)\tilde{\theta}(k) \\ \tilde{\theta}(k) &= \tilde{\theta}(k-1) - F\,\phi(k-1)e(k) \end{aligned}\right\} \textit{Static coupling}$$

Solution: Use the a-priori error

$$e^o(k) = \phi^T(k-1)\tilde{\theta}(k-1)$$

How to we implement the PAA?

$$\tilde{\theta}(k) = \tilde{\theta}(k-1) - F\,\phi(k-1)e(k)$$

Multiply by $\quad \phi^T(k-1) = \phi_{k-1}$

$$\underbrace{\phi_{k-1}^T\tilde{\theta}(k)}_{e(k)} = \underbrace{\phi_{k-1}^T\tilde{\theta}(k-1)}_{e^o(k)} - \phi_{k-1}^T F\,\phi_{k-1}e(k)$$

$$e(k) = e^o(k) - \phi_{k-1}^T F\,\phi_{k-1}e(k)$$

Therefore, $\quad e(k) = \dfrac{e^o(k)}{1+\phi^T(k-1)F\,\phi(k-1)}$

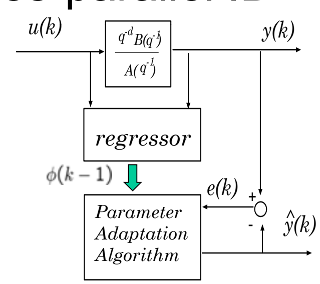
How to we implement the PAA?

$$e^o(k) = \phi^T(k-1)\tilde{\theta}(k-1)$$

$$e(k) = \frac{e^o(k)}{1+\phi^T(k-1)F\,\phi(k-1)}$$

$$\tilde{\theta}(k) = \tilde{\theta}(k-1) - F\,\phi(k-1)e(k)$$

Stability analysis of Series-parallel ID



We have shown that

$e(k) \to 0$

Now we will shown that

$e^o(k) \to 0$

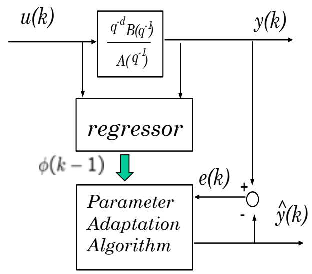
Under the following assumptions:

$u(k) < \infty \qquad A(q^{-1})$ is Schur

Since $\quad y(k) = \dfrac{q^{-d}B(q^{-1})}{A(q^{-1})}u(k) \;\Rightarrow\; y(k) < \infty$

Since $\quad \phi(k-1) = \begin{bmatrix} y(k-1) \\ \vdots \\ u(k-d) \\ \vdots \end{bmatrix} \;\Rightarrow\; |\phi(k-1)| < \infty$

Stability analysis of Series-parallel ID



Thus,

$e(k) \to 0$

$|\phi(k-1)| < \infty$

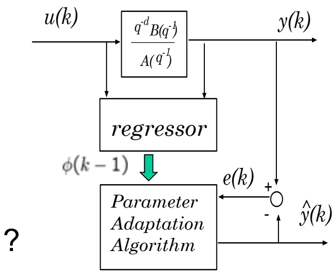
remember that,

$$e(k) = \frac{e^o(k)}{1 + \phi^T(k-1)F\,\phi(k-1)}$$

$$\Rightarrow \quad e^o(k) = \underbrace{e(k)}_{\to 0}\underbrace{\{1 + \phi^T(k-1)F\,\phi(k-1)\}}_{<\infty}$$

$$\Rightarrow \quad e^o(k) \to 0$$

Stability analysis of Series-parallel ID



We have shown that

$e(k) \to 0 \quad e^o(k) \to 0$

$|\phi(k-1)| < \infty$

What about the parameter error $\tilde{\theta}(k)$?

since

$$\underbrace{e^o(k)}_{\to 0} = \phi^T(k-1)\tilde{\theta}(k-1) \quad \Rightarrow \quad |\phi^T(k)\tilde{\theta}(k)| \to 0$$

However, this **does not imply** that the parameter error goes to zero

We need to impose another condition on *u(k)* (*persistence of excitation*) to guarantee that the parameter error goes to zero.