1. Solution:

1) This is the standard Kalman filter

$$\begin{aligned}
\hat{x}(k+1|k+1) &= \hat{x}(k+1|k) + f_1(k+1)\left[y_1(k+1) - c_1\hat{x}(k+1|k)\right] \\
&= a\hat{x}(k|k) + bu(k) + f_1(k+1)\left[y_1(k+1) - c_1\hat{x}(k+1|k)\right] \\
f_1(k+1) &= m_1(k+1)\,c_1\left[c_1^2 m_1(k+1) + \sigma_1^2\right]^{-1} \\
m_1(k+1) &= a^2 z_1(k) + b_w^2\sigma_w^2 \\
z_1(k+1) &= m_1(k+1) - m_1(k+1)\,c_1\left[c_1^2 m_1(k+1) + \sigma_1^2\right]^{-1} c_1 m_1(k+1)
\end{aligned}$$

where

$$\hat{x}(0|-1) = x_0,\ m_1(0) = X_0$$

2) In the presence of the bias term, we have

$$\begin{aligned}
\hat{y}(k+1|k) &= E\left[Cx(k+1) + v(k+1)\,|y(0), y(1), \dots, y(k)\right] = C\hat{x}(k+1|k) + E\left[v(k+1|k)\right] \\
&= C\hat{x}(k+1|k) + \begin{bmatrix} 0 \\ l_0 \end{bmatrix} \\
&= \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\hat{x}(k+1|k) + \begin{bmatrix} 0 \\ l_0 \end{bmatrix}
\end{aligned}$$

The Kalman filter is

$$\begin{aligned}
\hat{x}(k+1|k+1) &= \hat{x}(k+1|k) + f_2(k+1)\left[y(k+1) - C\hat{x}(k+1|k) - \begin{bmatrix} 0 \\ l_0 \end{bmatrix}\right] \\
&= a\hat{x}(k|k) + bu(k) + f_2(k+1)\left[\begin{bmatrix} y_1(k+1) \\ y_2(k+1) \end{bmatrix} - \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\hat{x}(k+1|k) - \begin{bmatrix} 0 \\ l_0 \end{bmatrix}\right] \\
f_2(k+1) &= m_2(k+1)\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}^T\left[\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} m_2(k+1)\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}^T + \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}\right]^{-1} \\
m_2(k+1) &= a^2 z_2(k) + b_w^2\sigma_w^2 \\
z_2(k+1) &= m_2(k+1) - m_2(k+1)\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}^T\left[\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} m_2(k+1)\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}^T + \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}\right]^{-1}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} m_2(k+1)
\end{aligned} \tag{1}$$

where

$$\hat{x}(0|-1) = x_0,\ m_2(0) = X_0$$

and in the noise covariance matrix we have used the assumption that $v_1(k)$ and $v_2(k)$ are independent.

3) For part 1), the conditions are (a, b_w) should be controllable (disturbable) or stabilizable; and (a, c_1) should be observable or detectable. For part 2), we need (a, b_w) should be controllable (disturbable) or stabilizable; and (a, C) should be observable or detectable. Under the assumption that $a,\ b,\ c_1,\ c_2,\ b_w$ are nonzero, the controllability and observability conditions are all satisfied, since the rank of the controllability matrix $[a, ab_w]$ is one; and the rank of the observability matrices $\begin{bmatrix} a \\ c_1 a \end{bmatrix}$ and $\begin{bmatrix} a \\ Ca \end{bmatrix}$ are both one.

2. Solution: we can write

$$u(k-1) = \begin{bmatrix} y(k) \\ y(k-1) \\ y(k-2) \\ -u(k-2) \\ -u(k-3) \end{bmatrix}^T \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{bmatrix} \triangleq \phi(k)^T\theta$$

The reconstructed $u(k-1)$ is then

$$\hat{u}(k-1) = \begin{bmatrix} y(k) \\ y(k-1) \\ y(k-2) \\ -\hat{u}(k-2) \\ -\hat{u}(k-3) \end{bmatrix}^T \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \psi(k)^T\hat{\theta}(k)$$

We have the following given PAA:

$$\hat{\theta}(k) = \hat{\theta}(k-1) + F\psi(k)\epsilon(k)$$

where

$$F = diag\{k_{\alpha_0} k_{\alpha_1}, k_{\alpha_2}, k_{\beta_1}, k_{\beta_2}\}$$

a) Construction of the feedback block diagram: letting $\tilde{\theta}(k) = \hat{\theta}(k) - \theta$ we have

$$\begin{aligned}
\epsilon(k) &= u(k-1) - \hat{u}(k-1) \\
&= \phi(k)^T \theta \pm \psi(k)^T \theta - \psi(k)^T \hat{\theta}(k) \\
&= -\psi(k)^T \tilde{\theta}(k) + (\phi(k) - \psi(k))^T \theta \\
&= -\psi(k)^T \tilde{\theta}(k) - \beta_1 (u(k-2) - \hat{u}(k-2)) - \beta_2 (u(k-3) - \hat{u}(k-3)) \\
&= -\psi(k)^T \tilde{\theta}(k) - \beta_1 \epsilon(k-1) - \beta_2 \epsilon(k-2)
\end{aligned}$$

yielding

$$\epsilon(k) = \frac{1}{1 + \beta_1 q^{-1} + \beta_2 q^{-2}} \left[-\phi(k)^T \tilde{\theta}(k)\right] \tag{2}$$

In addition

$$\tilde{\theta}(k) = \tilde{\theta}(k-1) + F\psi(k)\epsilon(k) \tag{3}$$

Equation (2) and (3) form a standard feedback block diagram for hyperstability analysis, with the LTI block being

$$G\left(z^{-1}\right) = \frac{1}{1 + \beta_1 z^{-1} + \beta_2 z^{-2}}$$

and the nonlinear block specified by (3).
From the problem statements we know that $\beta_1 = b_1/b_0$ and $\beta_2 = b_2/b_0$. From prolem 1 in homework 6, $G\left(z^{-1}\right)$ is SPR. The proof of the satisfaction of Popov Inequality is a standard one in the reader (students however need to cite the result clearly). Hence hyperstability holds. After analyzing the boundedness of the signals (students need to provide the detailed analysis here), we can confirm convergence of $\epsilon(k)$ to zero.
b) to obtain $\epsilon(k)$, we need the parameter estimate $\hat{\theta}(k)$, which is unavailable before computing the PAA equation at time k. To obtain the implementable version using the *a priori* adaptation error $\epsilon^o(k)$, we notice that

$$\epsilon^o(k) = u(k-1) - \hat{u}^o(k-1) = -\psi(k)^T \tilde{\theta}(k-1) - \beta_1 \epsilon(k-1) - \beta_2 \epsilon(k-2)$$

Equation (3) gives

$$\begin{aligned}
\psi(k)^T \tilde{\theta}(k) &= \psi(k)^T \tilde{\theta}(k-1) + \psi(k)^T F\psi(k)\epsilon(k) \\
\Leftrightarrow -\epsilon(k) - \beta_1 \epsilon(k-1) - \beta_2 \epsilon(k-2) &= -\epsilon^o(k) - \beta_1 \epsilon(k-1) - \beta_2 \epsilon(k-2) + \psi(k)^T F\psi(k)\epsilon(k) \\
\Rightarrow \epsilon^o(k) &= \left[1 + \psi(k)^T F\psi(k)\right] \epsilon(k) \\
\Rightarrow \epsilon(k) &= \frac{1}{1 + \psi(k)^T F\psi(k)} \epsilon^o(k)
\end{aligned}$$

Hence

$$\hat{\theta}(k) = \hat{\theta}(k-1) + \frac{F\psi(k)\epsilon^o(k)}{1 + \psi(k)^T F\psi(k)}$$

This is the implementable PAA.

3. Solution: the final-value condition gives

$$\begin{aligned}
2 &= -2a_1 - 2a_2 + b_0 + b_1 \\
\Rightarrow b_1 &= 2 - b_0 + 2a_1 + 2a_2
\end{aligned}$$

yielding

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) + b_0 u(k-1) + (2 - b_0 + 2a_1 + 2a_2) u(k-2)$$
$$\Rightarrow y(k) - 2u(k-2) = [a_1, a_2, b_0] \begin{bmatrix} -y(k-1) + 2u(k-2) \\ -y(k-2) + 2u(k-2) \\ u(k-1) - u(k-2) \end{bmatrix}$$

Hence the system can be written as

$$\begin{aligned} w(k) = y(k) - 2u(k-2) &= [a_1, a_2, b_0] \begin{bmatrix} -y(k-1) + 2u(k-2) \\ -y(k-2) + 2u(k-2) \\ u(k-1) - u(k-2) \end{bmatrix} \\ &\triangleq \phi(k-1)^T \theta \end{aligned}$$

Standard RLS can now be applied. We have

$$\begin{aligned} \hat{w}^o(k) &= \phi(k-1)^T \hat{\theta}(k-1) \\ \hat{w}(k) &= \phi(k-1)^T \hat{\theta}(k) \\ e^o(k) &= w(k) - \hat{w}^o(k) \\ \hat{\theta}(k) &= \hat{\theta}(k-1) + \frac{F(k)\phi(k-1)}{1 + \phi(k-1)^T F(k)\phi(k-1)} e^o(k) \\ F(k) &= F(k-1) - \frac{F(k)\phi(k-1)\phi(k-1)^T F(k)}{1 + \phi(k-1)^T F(k)\phi(k-1)} \end{aligned}$$

4. Solution:
a) high gain at low frequencies and low gain at high frequencies.
b) let $Q(z^{-1}) = R(z^{-1})/M(z^{-1})$, with $M(z^{-1}) = (1 - z^{-1})M^{'}(z^{-1})$, then the Diophantine equation is

$$b_0 z^{-1}(1 + z^{-1})R(z^{-1}) + (1 - z^{-1})^2(1 - z^{-1})M^{'}(z^{-1}) = D(z^{-1}) = (1 - 0.8z^{-1})^2$$

Let $R(z^{-1}) = r_0 + r_1 z^{-1} + r_2 z^{-2}$ and $M^{'}(z^{-1}) = 1 + mz^{-1}$. We get

$$b_0 z^{-1}(1 + z^{-1})\left(r_0 + r_1 z^{-1} + r_2 z^{-2}\right) + (1 - 2z^{-1} + z^{-2})(1 - z^{-1})\left(1 + mz^{-1}\right) = D(z^{-1}) = (1 - 0.8z^{-1})^2$$

Matching the coefficients of z^{-1}, z^{-2}, z^{-3}, and z^{-4} gives

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ -3 & 1 & 1 & 0 \\ 3 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m \\ \frac{r_0}{b_0} \\ \frac{r_1}{b_0} \\ \frac{r_2}{b_0} \end{bmatrix} + \begin{bmatrix} -3 \\ 3 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1.6 \\ 0.64 \\ 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} m \\ \frac{r_0}{b_0} \\ \frac{r_1}{b_0} \\ \frac{r_2}{b_0} \end{bmatrix} = \begin{bmatrix} 0.68 \\ 0.72 \\ -1.04 \\ 0.68 \end{bmatrix}$$

Note: to solve the equation set, we can first simplify it to

$$\begin{bmatrix} 1 & 1 & 0 \\ -3 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} m \\ \frac{r_0}{b_0} \\ \frac{r_1}{b_0} \end{bmatrix} + \begin{bmatrix} -3 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -1.6 \\ 0.64 \\ 0 \end{bmatrix}$$
$$\frac{r_2}{b_0} = m$$

c-1) This controller structure is called internal model control (not to be confused with internal model principle). We can derive the sensitivity and complementary sensitivity functions:

$$\begin{aligned} T(z^{-1}) &= P(z^{-1})Q(z^{-1}) \\ S(z^{-1}) &= 1 - T(z^{-1}) = 1 - P(z^{-1})Q(z^{-1}) \end{aligned}$$

Hence to make $S(z^{-1})$ small at low frequencies, we need $Q(z^{-1})$ to approximate $P(z^{-1})^{-1}$. Since $S\left(z^{-1}\right)$ and $T\left(z^{-1}\right)$ are affine in $Q(z^{-1})$, $Q(z^{-1})$ must be stable for closed-loop stability.
c-2) since

$$P(z^{-1}) = \frac{b_0 z^{-1}(1+z^{-1})}{(1-0.9z^{-1})^2}$$

we have

$$P_{ZPET}^{-1}(z^{-1}) = \frac{\left(1-0.9z^{-1}\right)^2(1+z)z^{-1}}{b_0 2^2}$$

where the one-step delay term was to make the filter implementable.
c-3) with Q being the ZPET inverse we have

$$T(z^{-1}) = P(z^{-1})Q(z^{-1}) = \frac{z^{-2}\left(1+z\right)\left(1+z^{-1}\right)}{2^2}$$

Note this is a zero phase filter cascaded with two steps of delays. Hence we need to select $r\left(k\right) = y_d\left(k+2\right)$.
d) the nominal complementary sensitivity function is

$$T(z^{-1}) = P(z^{-1})Q(z^{-1})$$

Using the result from the reader, we require the closed loop to have nominal stability and

$$\begin{aligned} &\left|T(e^{-j\omega})\Delta(e^{-j\omega})\right| < 1 \\ \Leftrightarrow &\left|P(e^{-j\omega})Q(e^{-j\omega})\Delta(e^{-j\omega})\right| < 1 \end{aligned}$$

5. Solution:
a) the proof is analogous to the continuous-time case provided in the class
b) the disturbance signal contains two components

$$d\left(k\right) = c\sin\left(\omega k+\phi\right)+l$$

We have

$$\underbrace{\left(1-2\cos\omega z^{-1}+z^{-2}\right)\left(1-z^{-1}\right)}_{A_d(z^{-1})}d(k) = 0$$

after three steps of transient. Here z^{-1} is the one-step delay operator.
If we choose an FIR Q filter, we need

$$\begin{aligned} 1-z^{-1}Q(z^{-1}) &= A_d(z^{-1}) \\ &= 1-\left(2\cos\omega+1\right)z^{-1}+\left(2\cos\omega+1\right)z^{-2}-z^{-3} \end{aligned}$$

which gives

$$Q(z^{-1}) = \left(2\cos\omega+1\right)-\left(2\cos\omega+1\right)z^{-1}+z^{-2}$$

c)[extra credit problem] In this case

$$\begin{aligned} y\left(k\right) &= z^{-1}P\left(z^{-1}\right)S^n(z^{-1})(1-z^{-1}Q(z^{-1}))d(k) \\ &= (1-z^{-1}Q(z^{-1}))P\left(z^{-1}\right)S^n(z^{-1})d(k-1) \end{aligned}$$

From the construction of DOB, we have $w\left(k\right) = d\left(k-1\right)$. Define $w_1(k) = P\left(z^{-1}\right)S^n(z^{-1})d(k-1)$. We have

$$\begin{aligned} y\left(k\right) &= (1-z^{-1}Q(z^{-1}))w_1(k) \\ &= \left[1-\left(2\cos\omega+1\right)z^{-1}+\left(2\cos\omega+1\right)z^{-2}-z^{-3}\right]w_1\left(k\right) \\ &= \left(w_1\left(k\right)-w_1\left(k-3\right)\right)+\left(2\cos\omega+1\right)\left[-w_1\left(k-1\right)+w_1\left(k-2\right)\right] \end{aligned}$$

Define $\theta = 2\cos\omega+1$ and $\phi\left(k-1\right) = -w_1\left(k-1\right)+w_1\left(k-2\right)$. We adapt θ to find

$$\hat{y}^o\left(k\right) = \phi\left(k-1\right)^T\hat{\theta}\left(k-1\right)+\left(w_1\left(k\right)-w_1\left(k-3\right)\right)$$

The ideal output is $y_d(k) = 0$. Hence

$$\phi(k-1)^T \theta + (w_1(k) - w_1(k-3)) = 0$$

We can define the error signal

$$\begin{aligned} e^o(k) &= 0 - \hat{y}^o(k) \\ &= \phi(k-1)^T \theta + (w_1(k) - w_1(k-3)) - \phi(k-1)^T \hat{\theta}(k-1) + (w_1(k) - w_1(k-3)) \\ &= -\phi(k-1)^T \tilde{\theta}(k-1) \end{aligned}$$

and apply RLS for parameter adaptation:

$$\begin{aligned} \hat{\theta}(k+1) &= \hat{\theta}(k) + \frac{F(k)\phi(k)}{1 + \phi(k)^T F(k)\phi(k)} e^o(k+1) \\ F(k+1) &= F(k) - \frac{F(k)\phi(k)\phi^T(k)F(k)}{1 + \phi(k)^T F(k)\phi(k)} \end{aligned}$$

Remark: in

$$\underbrace{\left(1 - 2\cos\omega z^{-1} + z^{-2}\right)\left(1 - z^{-1}\right)}_{A_d(z^{-1})} d(k) = 0$$

the term $1 - z^{-1}$ will amplify high-frequency noises,[1] if students can observe this point and use, e.g., filtered signals in the PAA, or if students provide an IIR construction, bonus points are proposed to be granted.

[1] $1 - 2\cos\omega z^{-1} + z^{-2}$ actually also have high gains at high frequencies in the common situations.