1. How do you split one number, say x_f, to N pieces so that the product of the N pieces is maximized? Think in the following way. We have an integrator described by

$$x(k+1) = x(k) + u(k), \quad x(0) = 0, \qquad x(N) = x_f$$

where $u(k) \geq 0$. Our goal is then to find $\{u(0), u(1), \ldots\ u(N-1)\}$ so that

$$J = \prod_{i=0}^{N-1} u(i) = u(0)u(1)\cdots u(N-1)$$

is maximized. You can solve this problem by applying the dynamic programming. For this purpose, define

$$J_k(x(k)) = \prod_{i=k}^{N-1} u(i)$$

Show that $J^0_{N-1}(x(N-1)) = x_f - x(N-1)$ and proceed.

2. Optimal Tracking Problem
The controlled plant is described by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \quad x(0) = x_0 \\ y(k) &= Cx(k) \end{aligned}$$

where the dimensions of x, u and y are as usual.
The optimal control is sought to minimize the quadratic performance index

$$J = \frac{1}{2}[y(N) - y_d(N)]^T S\,[y(N) - y_d(N)] + \frac{1}{2}\sum_{k=0}^{N-1}\left\{[y(k) - y_d(k)]^T Q_y\,[y(k) - y_d(k)] + u(k)^T R u(k)\right\}$$

where $S > 0$, $Q_y > 0$, $R > 0$ and $\{y_d(k)|0 \leq k \leq N\}$ is a prescribed desired output sequence.
Find the optimal control law by applying dynamic programming.
Hint:

$$J^0_k(x(k)) = \frac{1}{2}x^T(k)P(k)x(k) + x^T(k)b(k) + c(k)$$

3. Consider three independent random variables, x_1, x_2 and x_3. Each of them is uniformly distributed between 0 and 1. Obtain the probability density function for $x_1 + x_2$ and $x_1 + x_2 + x_3$. HINT: Let $z = x_1 + x_2$ and x_1 and x_2 are independent. Then, z and x_1 are dependent, and $p_z(z|x_1) = p_{x_2}(z - x_1)$. Furthermore,

$$p_z(z) = \int_{-\infty}^{\infty} p(z, x_1)dx_1 = \int_{-\infty}^{\infty} p_z(z|x_1)p_{x_1}(x_1)dx_1 = \int_{-\infty}^{\infty} p_{x_2}(z - x_1)p_{x1}(x_1)dx_1$$

Note: The sum of N independent random variables with uniform distributions approaches a Gaussian random variable as $N \to \infty$. For N sufficiently large, $n = n_1 + n_2 + ... + n_N$, where n_i's are independent and each n_i is uniformly distributed over [0, 1], the distribution of n is close to Gaussian (normal) with mean $N/2$ and variance $N/12$.

4. For a random vector x, the covariance matrix X is defined by

$$X = E[(x - m_x)(x - m_x)^T]$$

where $m_x = E[x]$. Notice that the matrix, $(x - m_x)(x - m_x)^T$, cannot be positive definite. Show X is at least positive semidefinite and that X is positive definite if the variance of $\alpha^T(x - m_x)$ is positive for any deterministic vector α.

5. Determine the autocovariance (autocorrelation) function $X_{yy}(k)$ and the spectral density $\Phi_{yy}(\omega)$, of the process $y(k)$ when

$$y(k) - 0.8y(k-1) = e(k) + 0.5e(k-1)$$

where $e(k)$ is a zero mean white random signal with unit variance.

6. A continuous time system is given by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t)$$
$$y(t) = \begin{bmatrix} 1 & 0.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where $x(0) = 0$ and $w(t)$ is a white zero mean random process with $E[w^2(t)] = 1$. Obtain the variance, auto-correlation and spectral density of $y(t)$ at the steady state.

7. MATLAB Exercise (If you do not have any experience on random process and would like to get some feel for it, work on the following problem. You don't need to turn in your solution for this problem.)
Consider a random sequence generated by

$$x(k+1) = ax(k) + \sqrt{1-a^2}w(k)$$

($w(k)$ is a zero mean, Gaussian white random sequence). This random process and its auto covariance function are described on page PR-10 in the course reader. You can plot the auto covariance function and the spectral density by MATLAB (try $a = 0.95$, 0.5 and 0.1 among others). Take a look of sample sequences for $a = 0.95$, 0.5 and 0.1. Obtain the variance of x by time average and check whether it comes close to the ensemble average (i.e. confirm the random sequence is ergodic).