ME 233 Spring 2012
Solution to Homework #8

1. (a) We want $y(k)$ to be driven to 0 in finite time. Thus, the closed loop transfer function should have all of its poles at the origin. To achieve this, we choose the closed loop characteristic polynomial to be $A_c = 1$. Therefore, the Bezout (Diophantine) equation for this system is given by

$$\begin{aligned} A_c &= A_d A R' + q^{-1} B S \\ 1 &= \left(1 - q^{-8}\right)\left(1 - 1.5q^{-1} + 0.56q^{-2}\right)(1) + q^{-1}(0.1) S \end{aligned}$$

Thus

$$\begin{aligned} R' &= 1 \\ S &= 10q\left[1 - \left(1 - q^{-8}\right)\left(1 - 1.5q^{-1} + 0.56q^{-2}\right)\right] \\ &= 15 - 5.6q^{-1} + 10q^{-7} - 15q^{-8} + 5.6q^{-9} \end{aligned}$$

(b) See Figure 1 for simulation results.

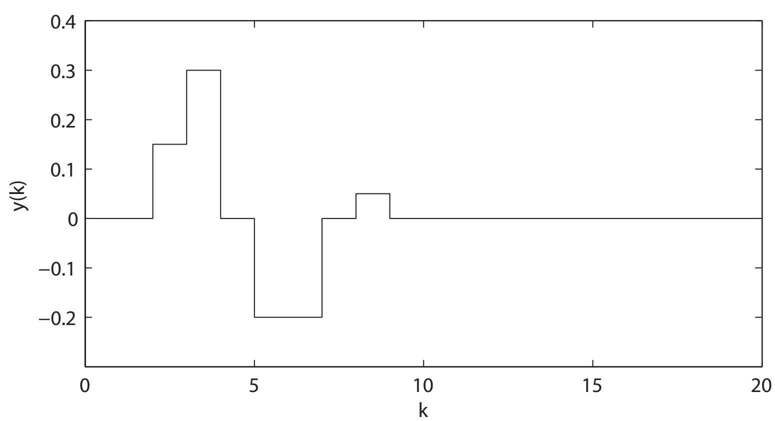

Figure 1: Closed loop simulation results

(c) See Figure 2 for simulation results.

(d) With this controller, the closed loop system has a pole at z if and only if

$$\begin{aligned} 0 &= 1 + \left[\frac{k_r z^{-(N-1)} \bar{A}(z^{-1})}{0.1 A_d(z^{-1})}\right]\left[\frac{0.1z^{-2}}{\bar{A}(z^{-1})}\frac{0.8}{(1 - 0.2q^{-1})}\right] \\ &= 1 + \frac{0.8 k_r z^{-N-1}}{(1 - z^{-N})(1 - 0.2z^{-1})} \\ &= 1 + \frac{0.8 k_r}{(z - 0.2)(z^8 - 1)} = 0 \end{aligned}$$

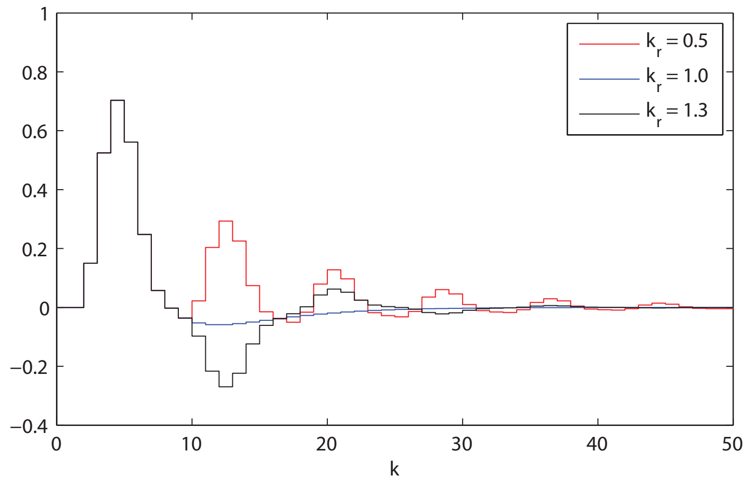

Figure 2: Closed loop simulation results for $k_r = 0.5$, 1, and 1.3

Figure 3 shows the root locus plot of this system. Clearly, the closed loop system is unstable for all $k_r > 0$.

(e) i. For this system, z is a closed loop pole if and only if

$$\begin{aligned} 0 &= 1 + \frac{0.8k_r}{(z-0.2)\left(z^8 - 0.25\left[z + 2 + z^{-1}\right]\right)} \\ &= 1 + \frac{0.8k_r z}{(z-0.2)\left(z^9 - 0.25\left[z^2 + 2z + 1\right]\right)} \end{aligned}$$

Figure 4 shows the root locus plot of the closed loop poles. For $k_r = 0.5$, the system is asymptotically stable.

ii. See Figure 5 for simulation results.

2. First, we need to design Kalman filter. Solve the following Riccati equation

$$M = \bar{A}M\bar{A}^T + \bar{B}_w W \bar{B}_w^T - \bar{A}M\bar{C}^T\left[\bar{C}M\bar{C}^T + V\right]^{-1}\bar{C}M\bar{A}^T$$

where

$$\bar{A} = \begin{bmatrix} 0.8 & 1 \\ 0 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \bar{B}_w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad W = 0.225, \quad V = 0.625\ .$$

Let

$$M = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} \succ 0\ .$$

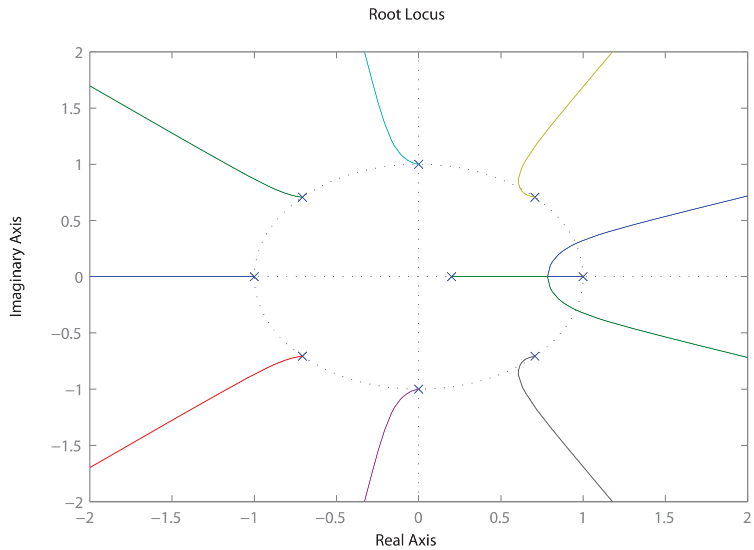

Figure 3: Root locus of repetitive control system with unmodeled plant dynamics

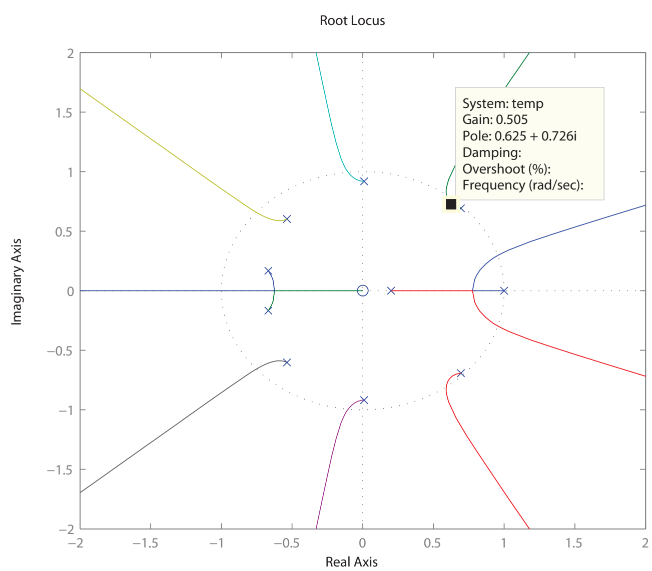

Figure 4: Root locus of Q-filter repetitive control system with unmodelled plant dynamics

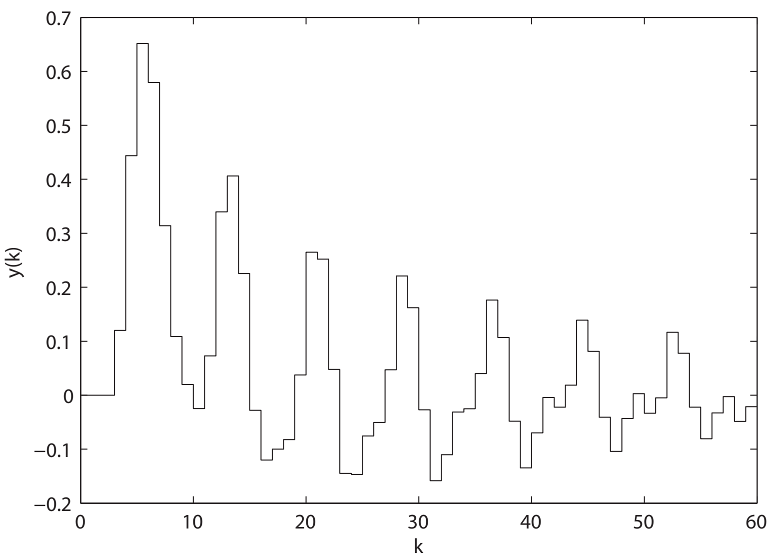

Figure 5: Closed loop simulation results for controller with Q-filter

Then,

$$
\begin{aligned}
\begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} &= \begin{bmatrix} 0.8 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} \begin{bmatrix} 0.8 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0.225 & 0 \\ 0 & 0 \end{bmatrix} - \frac{\begin{bmatrix} 0.8 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \begin{bmatrix} m_1 & m_2 \end{bmatrix} \begin{bmatrix} 0.8 & 0 \\ 1 & 0 \end{bmatrix}}{m_1 + 0.625} \\
&= \begin{bmatrix} 0.8^2 m_1 + 1.6 m_2 + m_3 + 0.225 & 0 \\ 0 & 0 \end{bmatrix} - \frac{1}{m_1 + 0.625} \begin{bmatrix} 0.8^2 m_1^2 + 1.6 m_1 m_2 + m_2^2 & 0 \\ 0 & 0 \end{bmatrix} \\
&\Rightarrow \left\{ \begin{array}{l} m_2 = 0 \\ m_3 = 0 \\ m_1 = 0.8^2 m_1 + 0.225 - \frac{0.8^2 m_1^2}{m_1 + 0.625} = 0.225 + \frac{0.4 m_1}{m_1 + 4} \end{array} \right. \\
&\Rightarrow \quad m_1 = 0.375 \\
&\Rightarrow \quad L = \bar{A} M \bar{C}^T \left[\bar{C} M \bar{C}^T + V \right]^{-1} = \begin{bmatrix} 0.3 \\ 0 \end{bmatrix} .
\end{aligned}
$$

Then, the output is

$$
Y(z) \quad = \quad \bar{C} \left(zI - \bar{A} \right)^{-1} \bar{B} U(z) + \left[1 + \bar{C} \left(zI - \bar{A} \right)^{-1} L \right] E(z) .
$$

Thus, we can get the transfer function

$$
\begin{aligned}
y(k) &= \bar{C} \left(qI - \bar{A} \right)^{-1} \bar{B} u(k) + \left[1 + \bar{C} \left(qI - \bar{A} \right)^{-1} L \right] \epsilon(k) \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q - 0.8 & -1 \\ 0 & q \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -2 \end{bmatrix} u(k) + \left\{ 1 + \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q - 0.8 & -1 \\ 0 & q \end{bmatrix}^{-1} \begin{bmatrix} 0.3 \\ 0 \end{bmatrix} \right\} \epsilon(k) \\
&= \frac{q - 2}{q(q - 0.8)} u(k) + \frac{q - 0.5}{q - 0.8} \epsilon(k) \\
&= \frac{q^{-1}(1 - 2q^{-1})}{1 - 0.8q^{-1}} u(k) + \frac{1 - 0.5q^{-1}}{1 - 0.8q^{-1}} \epsilon(k) .
\end{aligned}
$$

Thus, we know

$$
\begin{array}{lll}
A(q^{-1}) = 1 - 0.8q^{-1}, & B^u(q^{-1}) = q^{-1} - 0.5, & B^s(q^{-1}) = -2, \\
C(q^{-1}) = 1 - 0.5q^{-1}, & \bar{B}^u(q^{-1}) = 1 - 0.5q^{-1}, & d = 1 .
\end{array}
$$

Solve the Diophantine equation

$$C(q^{-1})\bar{B}^u(q^{-1}) = A(q^{-1})R(q^{-1}) + q^{-1}B^u(q^{-1})S(q^{-1})$$

where

$$R(q^{-1}) = 1 + r_1 q^{-1}, \qquad S(q^{-1}) = s_0 \ .$$

Then,

$$\begin{aligned}
(1-0.5q^{-1})(1-0.5q^{-1}) &= (1-0.8q^{-1})(1+r_1q^{-1}) + q^{-1}(q^{-1}-0.5)s_0 \\
&\Rightarrow \begin{cases} -1 = r_1 - 0.8 - 0.5s_0 \\ 0.25 = -0.8r_1 + s_0 \end{cases} \\
&\Rightarrow \begin{cases} s_0 = 0.15 \\ r_1 = -0.125 \end{cases} \\
&\Rightarrow R(q^{-1}) = 1 - 0.125q^{-1},\ S(q^{-1}) = 0.15 \ .
\end{aligned}$$

Finally, we can get the minimum variance regulator feedback law:

$$u(k) = \frac{-S(q^{-1})}{B^s(q^{-1})R(q^{-1})}y(k) = \frac{0.075}{1-0.125q^{-1}}y(k) \ .$$

3. **Proof**

Applying the signal $y(k)$ to both sides of the Diophantine equation $C(q^{-1})A_m(q^{-1}) = A(q^{-1})R(q^{-1}) + q^{-d}S(q^{-1})$, we see that

$$C(q^{-1})A_m(q^{-1})y(k) = R(q^{-1})A(q^{-1})y(k) + q^{-d}S(q^{-1})y(k) \ .$$

Since $A(q^{-1})y(k) = q^{-d}B(q^{-1})u(k) + C(q^{-1})\epsilon(k)$ by the plant dynamics, we see that

$$\begin{aligned}
C(q^{-1})A_m(q^{-1})y(k) &= R(q^{-1})\Big[q^{-d}B(q^{-1})u(k) + C(q^{-1})\epsilon(k)\Big] + q^{-d}S(q^{-1})y(k) \\
&= q^{-d}\Big[R(q^{-1})B(q^{-1})u(k) + S(q^{-1})y(k)\Big] + R(q^{-1})C(q^{-1})\epsilon(k) \ . \qquad (1)
\end{aligned}$$

Applying the $C(q^{-1})$ polynomial to both sides of the reference model equation, we see that

$$C(q^{-1})A_m(q^{-1})y_m(k) = q^{-d}C(q^{-1})B_m(q^{-1})r_m(k) \ .$$

Subtracting this equation from (1), we see that

$$\begin{aligned}
C(q^{-1})A_m(q^{-1})\Big[y(k) - y_m(k)\Big] &= q^{-d}\Big[R(q^{-1})B(q^{-1})u(k) + S(q^{-1})y(k) - C(q^{-1})B_m(q^{-1})r_m(k)\Big] \\
&\quad + R(q^{-1})C(q^{-1})\epsilon(k) \ .
\end{aligned}$$

Defining the signals $w(k)$, $z(k)$, and $\epsilon_f(k)$ respectively by

$$\begin{aligned}
w(k) &= A_m(q^{-1})\Big[y(k) - y_m(k)\Big] \\
C(q^{-1})z(k) &= R(q^{-1})B(q^{-1})u(k) + S(q^{-1})y(k) - C(q^{-1})B_m(q^{-1})r_m(k) \\
\epsilon_f(k) &= R(q^{-1})\epsilon(k)
\end{aligned}$$

we write

$$C(q^{-1})w(k) = q^{-d}C(q^{-1})z(k) + C(q^{-1})\epsilon_f(k) \ .$$

Since the polynomial $C(q^{-1})$ is anti-Schur, we therefore write

$$w(k) = z(k-d) + \epsilon_f(k) \ .$$

By Least Squares property 1, we know that $E\{y(k-j)\epsilon(k)\} = 0,\ \forall j > 0$. Since $u(k)$ is a function of $y(k), y(k-1), y(k-2), \ldots$, and $r_m(k), r_m(k-1), r_m(k-2), \ldots$, we also see that $E\{u(k-j)\epsilon(k)\} = 0,\ \forall j > 0$. By definition, $z(k)$ is a causal function of $u(k)$, $y(k)$, and $r_m(k)$. Therefore, we see that $E\{z(k-j)\epsilon(k)\} = 0,\ \forall j > 0$. This implies that

$$\begin{aligned} E\{z(k-d)\epsilon_f(k)\} = E\{z(k-d)\epsilon(k)\} + r_1 E\{z(k-d)\epsilon(k-1)\} \\ + \cdots + r_{d-1} E\{z(k-d)\epsilon(k-d+1)\} = 0 \ . \end{aligned}$$

Therefore,

$$\begin{aligned} J = E\{w^2(k)\} &= E\{z^2(k-d)\} + E\{\epsilon_f^2(k)\} + 2E\{z(k-d)\epsilon_f(k)\} \\ &= E\{z^2(k-d)\} + E\{\epsilon_f^2(k)\} \ . \end{aligned}$$

Up until this point, we have not used any information involving $u(k)$; the preceding equality holds regardless of how $u(k)$ is chosen, so long as the closed-loop system is stable and $u(k)$ is WSS (i.e. it is a time-invariant function of $y(k)$ and $r_m(k)$). Since the choice of $u(k)$ does not affect the correlation of $\epsilon_f(k)$, we see that minimizing the correlation of $z(k-d)$ also minimizes J. Therefore, choosing

$$u(k) = \frac{1}{R(q^{-1})B(q^{-1})}\Big[C(q^{-1})B_m(q^{-1})r_m(k) - S(q^{-1})y(k)\Big]$$

makes $C(q^{-1})z(k)$ zero, which in turn makes $z(k)$ zero (because $C(q^{-1})$ is anti-Schur), which obviously minimizes $E\{z(k)\}$. Therefore, this control law minimizes J.