UNIVERSITY OF CALIFORNIA AT BERKELEY
Department of Mechanical Engineering
ME233 Advanced Control Systems II, Spring 2010

Homework #7

Assigned: Th., March 18
Due: Th., April 1

1. (Former midterm problem) Consider the following stationary stochastic system

$$\begin{aligned} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} &= \begin{bmatrix} 0.8 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(k) \qquad (1) \\ y(k) &= x_1(k) + v(k) \\ x_o &= E\{x(0)\} \qquad X_o = E\{(x(0)-x_o)(x(0)-x_o)^T\} \end{aligned}$$

where $u(k)$ is a deterministic (known) input, $y(k)$ is the measured output, $w(k)$ and $v(k)$ are both white, zero mean, Gaussian and stationary random noises, and

$$E\left\{ \begin{bmatrix} w(k) & v(k) \end{bmatrix} \begin{bmatrix} w(k) \\ v(k) \end{bmatrix} \right\} = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix} \qquad E\left\{ \begin{bmatrix} w(k) & v(k) \end{bmatrix} (x(0)-x_o)^T \right\} = 0$$

(a) Determine if there exists a unique steady state solution to the Kalman filter a-priori state estimation error covariance, $M(k)$, Riccati equation.

(b) Draw the root locus of the stationary Kalman filter close loop poles and their reciprocals for $\frac{W}{V} \in (0 \in \infty)$.

(c) The system in Eqs. (1) can be described by the following ARMAX model

$$(1-0.8z^{-1})\,Y(z) = z^{-2}\,U(z) + (1-0.5z^{-1})\,\tilde{Y}^o(z)$$

where

- $Y(z) = \mathcal{Z}\{y(k)\}$, $U(z) = \mathcal{Z}\{y(k)\}$, $\tilde{Y}(z) = \mathcal{Z}\{\tilde{y}^o(k)\}$.
- $\tilde{y}^o(k) = y(k) - \hat{y}^o(k)$ is the steady state Kalman filter residual (i.e. a-priori output estimation error) and

$$E\{\tilde{(}y^o(k))^2\} = 1\,.$$

Determine the values of noise variances W and V.

2. Consider the design of a steady state Kalman filter for a positioning system described by

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -2 & -0.5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u(t) + w(t)) \qquad (2) \\ y(t) &= x_1(t) + v(t) \qquad (3) \end{aligned}$$

where x_1 is the position, x_2 is the velocity, and u is the input. $x(0)$, $w(t)$ and $v(t)$ are random, uncorrelated and satisfy all standard assumptions. Covariances are given as follows

$$X_o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad W = \rho\,, \qquad V = 0.5$$

To get some physical sense of this problem, imagine an a tracking antenna problem. $w(t)$ may represent the disturbance due to wind (and/or vibration if the antenna is mounted on the top of a moving vehicle), which can be best modeled to be random. The measurement noise may be due to electrical noise picked up in the transmission line.

(a) Find the steady state KF gain and estimation error covariance matrix when $\rho = 2$. (You are encouraged to use matlab).

(b) Utilize the return difference equality and draw the Kalman filter symmetric root locus plot for ρ varying from 0 to ∞.

(c) Let $\rho = 2$. The system in Eqs. (2)-(3) can be described by the following input/output ARMAX model

$$Y(s) = \frac{B(s)}{A(s)}\,U(s) + \frac{C(s)}{A(s)}\,\epsilon(s) \tag{4}$$

where $\epsilon(s)$ is the Laplace transform of the Kalman filter *innovations* signal $\epsilon(t) = \tilde{y}(t) = y(t) - C\,\hat{x}(t)$ (i.e. the output estimation error signal).

(i) Derive Eq. (4) using the following innovations driven model of the plant output

$$\begin{aligned} \frac{d}{dt}\hat{x}(t) &= A\hat{x}(t) + Bu(t) + L\epsilon(t) \\ y(t) &= C\hat{x}(t) + \epsilon(t) \end{aligned}$$

where L is the steady state Kalman filter gain.

(ii) Calculate the order and coefficients of the polynomials $A(s)$, $B(s)$.

(iii) Show that $C(s) = det[sI - (A - LC)]$ and calculate its coefficients.

3. Consider a continuous time process described by

$$\frac{d}{dt}x(t) = A\,x(t) + B\,u(t) + B_w\,w(t)$$

where $x \in \mathcal{R}^n$, $u \in \mathcal{R}^m$ and the stationary, Gaussian white noise random process, $w \in \mathcal{R}^s$ staisfies

$$E\{w(t)\} = 0, \qquad E\{w(t+\tau)w^T(t)\} = W\delta(\tau)$$

Assume also that the initial condition is also Gaussian and satisfies

$$E\{x(o)\} = x_o, \qquad E\{(x(0) - x_o)(x(0) - x_o)^T\} = X_o$$

By using a set of sensors (Sensor Configuration A: one set of sensors), the output equation is

$$y(t) = Cx(t) + v_{_A}(t)$$

where $y \in \mathcal{R}^r$ is output vector and $v_A \in \mathcal{R}^r$ is a stationary Gaussian measurement noise. The measurement noise is independent from the initial state and input noise, and the following quantities are given

$$E\{v_A(t)\} = 0, \qquad E\{v_A(t+\tau)v_A^T(t)\} = V_A \delta(\tau)$$

For the purpose of preparing for any sensor failures, it has been decided to use to identical sets of sensors and measure the output twice (Sensor Configuration B: two sets of sensors). With this sensor configuration the measurement vector is 2r dimensional, and it is given by

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} Cx(t) \\ Cx(t) \end{bmatrix} + \begin{bmatrix} v_{B1}(t) \\ v_{B2}(t) \end{bmatrix}$$

where the measurement noises are independent form the initial state and the input noise, and the following quantities are given, for $i = 1, 2$.

$$E\{v_{Bi}(t)\} = 0, \qquad E\{v_{Bi}(t+\tau)v_{Bi}^T(t)\} = V_B \delta(\tau), \qquad E\{v_{B1}(t+\tau)v_{B2}^T(t)\} = 0$$

While the duplication of the output measurement increases the hardware cost, it is also true that the specification for each sensor may be relaxed if the same output is measured twice and the two measurements are used in the Kalman filter.

(a) Assume that you design and use a Kalman filter for each case (Configuration A: single set of sensors, or Configuration B: two sets of sensors) and determine the relationship between the sensor noise covariances V_A and V_B that must exist so that the steady state state estimation error covariance using Sensor Configuration A, M_A, is identical to the steady state state estimation error covariance using Sensor Configuration B, M_B.

(b) Assume now that Sensor Configuration B is in operation and a fault is detected in the first set of sensors, making the output measurement $y_1(t)$ useless. Consider the following fault handling strategy: meausrement $y_1(t)$ is discarded and measurement $y_2(t)$ is also fed in the location where $y_1(t)$ was being fed (i.e. measurement $y_2(t)$ is now fed twice into the Kalman filter). Obtain an expression for the steady state state estimation error covariance for this case and comment if this strategy is a reasonable engineering decision when one set of measurements becomes useless.

4. A farmer produces at year k $x(k)$ units of a certain crop and stores $(1 - u(k))x(k)$ of his production, where $u(k) \in [0, 1]$. He invests the remaining units $u(k)x(k)$, thus increasing the next year's production to a level $x(k+1)$ given by

$$x(k+1) = x(k) + w(k)u(k)x(k)$$

where the random sequence $w(k)$ is assumed to be white and, in this case $E\{w(k)\} = \bar{w} > 0$. The farmer wishes to find an optimal investment policy that maximizes the total expected product stored over N years.

$$J = E_{\mathcal{W}_0}\left\{x(N) + \sum_{k=0}^{N-1}(1 - u(k))x(k)\right\}$$

where $\mathcal{W}_k = \{w(k), \cdots, w(N-1)\}$.

(a) Show that, if $\bar{w} > 1$, the optimal policy is $u^o(k) = 1$ for $k = 0, \cdots, N-1$.

(b) Show that, if $0 < \bar{w} < 1/N$, the optimal policy is $u^o(k) = 0$ for $k = 0, \cdots, N-1$.

(c) Show that, if $1/N \leq \bar{w} \leq 1$, the optimal policy is as follows

$$u^o(k) = \begin{cases} 1 & \text{for} \quad k = 0, \cdots, N - \bar{k} - 1 \\ 0 & \text{for} \quad k = N - \bar{k}, \cdots, N - 1 \end{cases}$$

where $\bar{k}$ is such that

$$\frac{1}{\bar{k}+1} < \bar{w} \leq \frac{1}{\bar{k}}.$$