ME233 Advanced Control II
Lecture 11

Infinite-horizon LQR
PART II

(ME232 Class Notes pp. 135-137)

Outline

Previous lecture :

- Solution of infinite-horizon LQR

This Lecture: Review ME232 results on

- Infinite-horizon LQR properties
  - Stability margins
  - Reciprocal root locus

Infinite Horizon LQ regulator

LTI system:

$$x(k+1) \;=\; A\,x(k) + B\,u(k) \qquad x(0) = x_o$$

LQR that minimizes the cost:

$$J[x_o] = \sum_{k=0}^{\infty} \left\{ x^T(k) Q x(k) + u^T(k) R u(k) \right\}$$

$$Q = C^T C \succeq 0 \qquad\qquad R \succ 0$$

Infinite Horizon (IH) LQ regulator

Assume that ***(A,B)*** stabilizable and ***(C,A)*** detectable,

- Optimal, asymptotically stable, closed-loop system

$$x(k+1) \;=\; [A - B\,K]\;\; x(k) \qquad x(0) = x_o$$

$$K = \left[R + B^T P B \right]^{-1} B^T P A$$

Discrete Algebraic Riccati Equation (DARE)

$$P = Q + A^T P A - A^T P B \left[R + B^T P B \right]^{-1} B^T P A$$

Infinite Horizon LQ Regulator

Lets analyze the stability and robustness properties of the closed-loop system:

$$x(k+1) \;=\; A\,x(k) + B\,u(k)$$

$$u(k) \;=\; -K\,x(k) + v(k)$$

With fictitious reference input $v(k)$

$$v(k) \;=\; v_o \;= 0$$

Infinite Horizon LQ Regulator

Use the Z-transform:

$$X(z) \;=\; (zI - A)^{-1} B\,U(z)$$

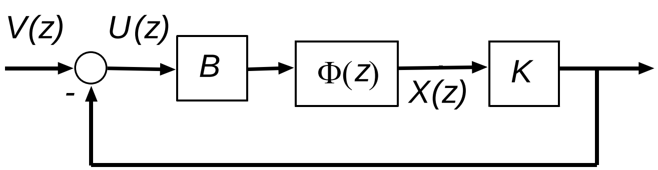
$$U(z) = -K\,X(z) + V(z)$$

Define

$$\Phi(z) = (zI - A)^{-1}$$

Infinite Horizon LQ Regulator

Closed-loop system block diagram:



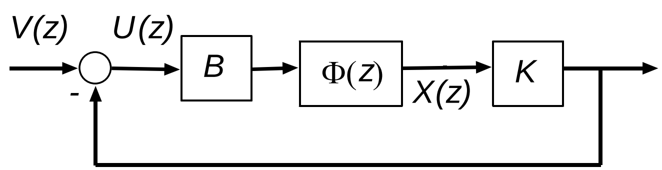
$$\Phi(z) = (zI - A)^{-1}$$

$$X(z) \;=\; \Phi(z) B\,U(z)$$

$$U(z) \;=\; V(z) - K\,X(z)$$

Infinite Horizon LQ Regulator

Closed-loop system block diagram:



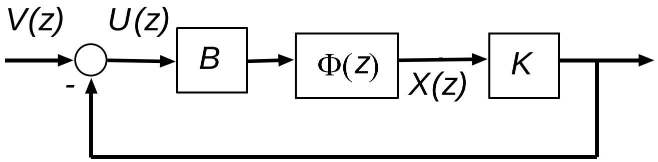
Open-loop transfer function:

$$G_o(z) = K\Phi(z)B$$

9

Infinite Horizon LQ Regulator

Closed-loop system block diagram:



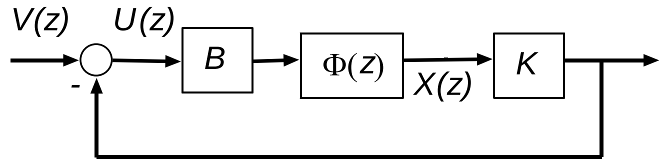
Closed-loop sensitivity transfer function
(from $V(z)$ to $U(z)$):

$$S(z) = [I + K\Phi(z)B]^{-1}$$

10

Infinite Horizon LQ Regulator

Closed-loop system block diagram:



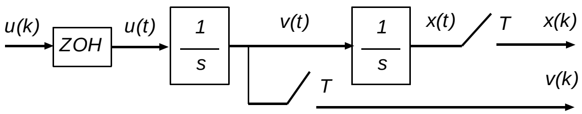
For Single Input Systems $u(k) \in \mathcal{R}$

$$S(z) = \frac{1}{1 + K\Phi(z)B}$$

11

Example – Double Integrator

Double integrator with ZOH and sampling time T=1:



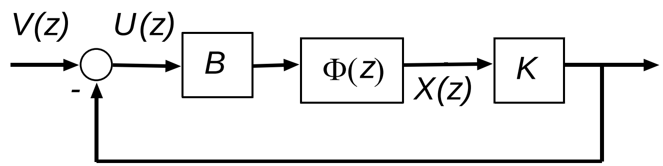
$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} u(k)$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u(k)$$

12

Example – Double Integrator

Closed-loop system block diagram:



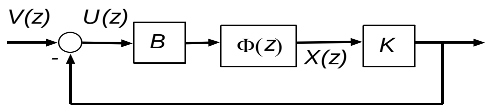
$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)$$

$$J[x_o] = \sum_{k=0}^{\infty} \left\{ y^2(k) + Ru^2(k) \right\}$$

Example – Double Integrator

Closed-loop system block diagram:



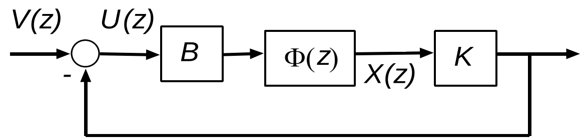
For $R = 10$ we obtained $K = \begin{bmatrix} 0.21 & 0.65 \end{bmatrix}$

Open-loop transfer function:

$$G_o(z) = K\Phi(z)B = \begin{bmatrix} 0.21 & 0.61 \end{bmatrix} \begin{bmatrix} (z-1) & -1 \\ 0 & (z-1) \end{bmatrix}^{-1} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

$$= \frac{0.76\,z - 0.55}{(z-1)^2}$$

Example – Double Integrator

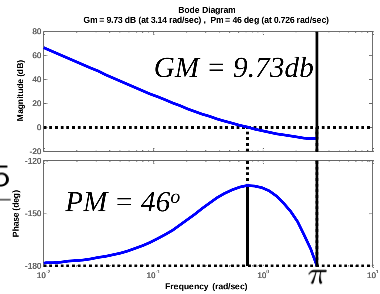


$R = 10$

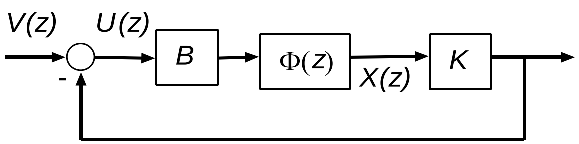
Open-loop transfer function:

$$G_o(j\omega) = K\Phi(j\omega)B$$

$$= \frac{0.76\,e^{j\omega} - 0.55}{(e^{j\omega} - 1)^2}$$



Example – Double Integrator

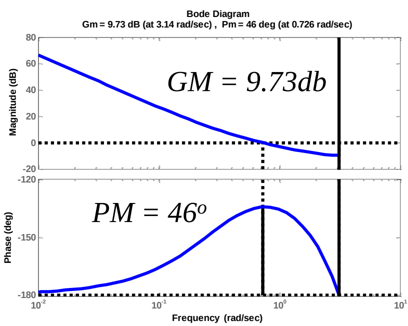


We already saw results for:

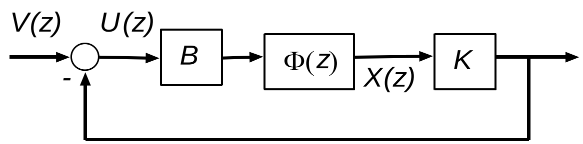
$R = 10$

Bode Plot

$$G_o(j\omega) = \frac{0.76\,e^{j\omega} - 0.55}{(e^{j\omega} - 1)^2}$$



Example – Double Integrator

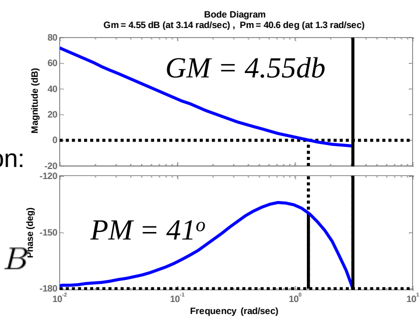


Now consider:

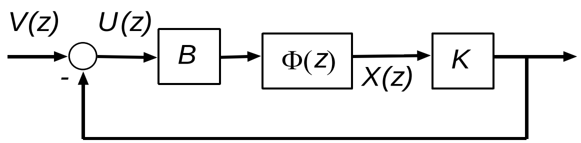
$R = 0.1$

Open-loop transfer function:

$$G_o(j\omega) = K\Phi(j\omega)B$$



Example – Double Integrator



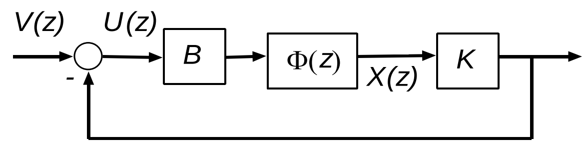
$$J[x_o] = \sum_{k=0}^{\infty} \left\{ y^2(k) + Ru^2(k) \right\}$$

$R = 10$

Sensitivity transfer function:

$$S(z) = \frac{1}{1 + K\Phi(z)B} = \frac{z^2 - 2z + 1}{z^2 - 1.24z + 0.45}$$

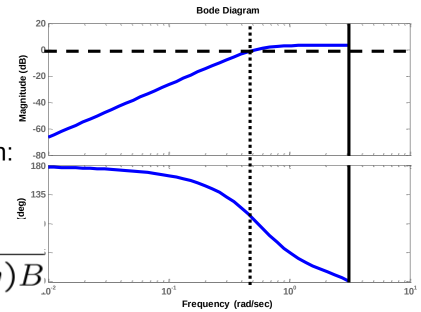
Example – Double Integrator



$$J[x_o] = \sum_{k=0}^{\infty} \left\{ y^2(k) + Ru^2(k) \right\}$$

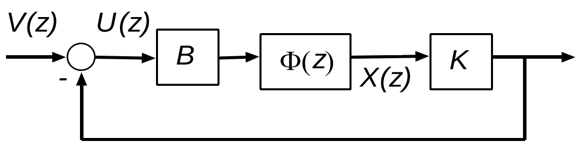
$R = 10$

Sensitivity transfer function:



$$S(j\omega) = \frac{1}{1 + K\Phi(j\omega)B}$$

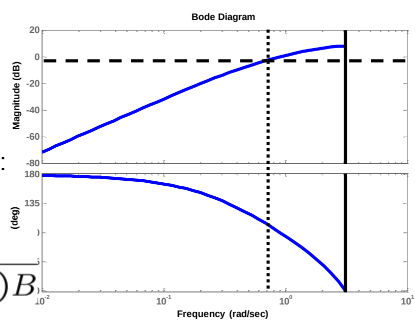
Example – Double Integrator



$$J[x_o] = \sum_{k=0}^{\infty} \left\{ y^2(k) + Ru^2(k) \right\}$$

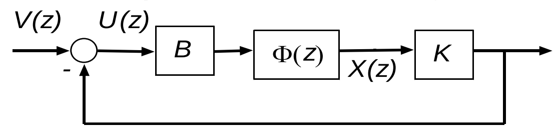
$R = 0.1$

Sensitivity transfer function:



$$S(j\omega) = \frac{1}{1 + K\Phi(j\omega)B}$$

Example – Double Integrator

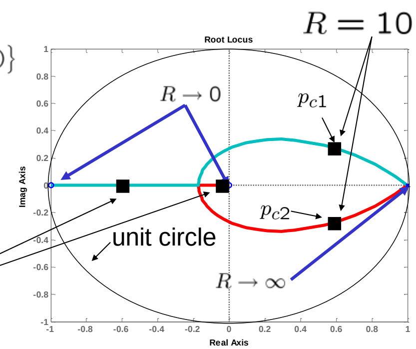


$$J[x_o] = \sum_{k=0}^{\infty} \left\{ y^2(k) + Ru^2(k) \right\}$$

Closed-loop poles for varying values of

$R \in (\infty, 0)$

$R = 0.1$



24

Stability and Robustness of LQR

We will see for Single input LQR systems,
$(u(k) \in \mathcal{R})$

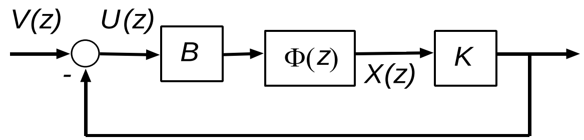
- Guaranteed open-loop frequency response gain and phase margins can be determined in closed form.

- Locus of the LQR closed-loop poles as a function of varying $R \in (\infty, 0)$ can be easily plotted

LQR Return difference equality

25

Return difference for LQR



Open-loop transfer function:

$$G_o(z) = K\Phi(z)B$$

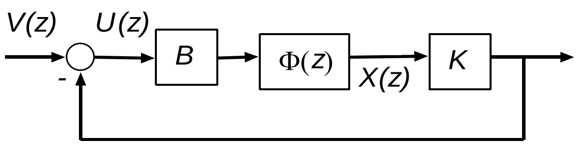
Closed-loop sensitivity transfer function ($V(z)$ to $U(z)$)

$$S(z) = [I + K\Phi(z)B]^{-1} = [I + G_o(z)]^{-1}$$

$$S(z) = [\text{return difference}]^{-1}$$

26

Return difference for LQR



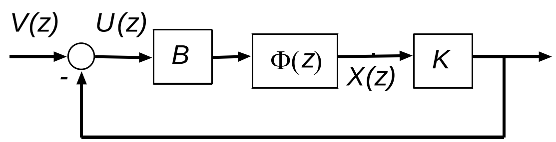
Open-loop transfer function:

$$G_o(z) = K\Phi(z)B$$

Return difference: $[I + K\Phi(z)B] = [I + G_o(z)]$

27

Output weighting in LQ cost



Open-loop transfer function

$$G_o(z) = K\Phi(z)B$$

LQ cost:

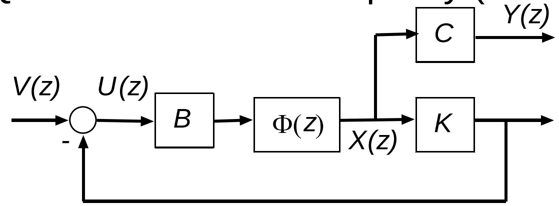
$$J[x_o] = \sum_{k=0}^{\infty} \left\{ \underline{y^2}(k) + Ru^2(k) \right\}$$

Open-loop transfer function from $U(z)$ to $Y(z)$:

$$\boxed{G(z) = C\Phi(z)B}$$

28

LQ Return Difference Equality (RDE)



Return difference equality (see ME232 class notes):

$$\left[I + G_o(z^{-1})\right]^T \left[R + B^T P B\right] \left[I + G_o(z)\right] = R + G^T(z^{-1})\, G(z)$$

Open-loop transfer function:

$$G_o(z) = K\Phi(z)B$$

TF from $U(z)$ to $Y(z)$:

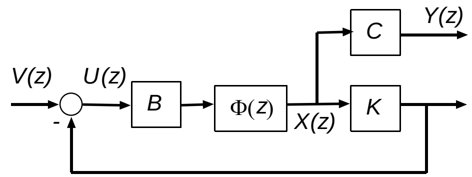
$$G(z) = C\Phi(z)B$$

29

RDE for Single Input Systems

When:

$$u(k) \in \mathcal{R}$$



$$(1 + G_o(z^{-1}))(1 + G_o(z)) = \frac{R}{R + B^T P B}\left[1 + \frac{1}{R} G(z^{-1})^T G(z)\right]$$

Open-loop transfer function:

$$G_o(z) = K\Phi(z)B$$

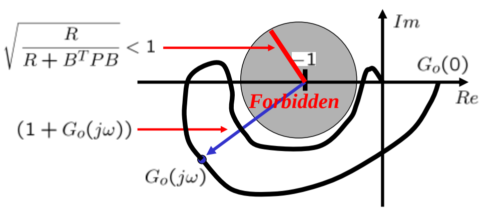
TF from $U(z)$ to $Y(z)$:

$$G(z) = C\Phi(z)B$$

30

Guaranteed stability margins

- From the return difference equality, it can be shown that $|(1 + G_o(e^{j\omega}))| \geq \sqrt{\dfrac{R}{R + B^T P B}}$



- From this picture, we can derive guaranteed gain and phase margins

31

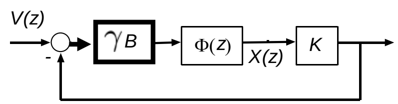
Guaranteed stability margins

Worst possible phase margin: $PM \geq 2\sin^{-1}\left\{0.5\sqrt{\dfrac{R}{R + B^T P B}}\right\}$

Worst possible gain margins:

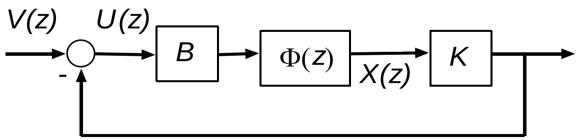
The loop below is asymptotically stable for

$$\frac{1}{1 + \sqrt{R/(R + B^T P B)}} < \gamma < \frac{1}{1 - \sqrt{R/(R + B^T P B)}}$$



44

Poles of an LQR



Closed-loop poles are the zeros of the return difference

Open-loop poles are the poles of the return difference

$$\boxed{\text{Det}[I+G_o(z)]=\frac{\text{Det}[zI-A+BK]}{\text{Det}[zI-A]}}$$

45

Poles of an LQR

$$\text{Det}[I+G_o(z)]=\frac{\text{Det}[zI-A+BK]}{\text{Det}[zI-A]}$$

Proof:

$$\begin{aligned}\text{Det}[I+G_o(z)] &= \text{Det}[I+K\Phi(z)B]\\ &= \text{Det}[I+BK\Phi(z)]\\ &= \text{Det}[\Phi^{-1}(z)+BK]\text{Det}\Phi(z)\\ &= \text{Det}[zI-A+BK]\text{Det}[zI-A]^{-1}\end{aligned}$$

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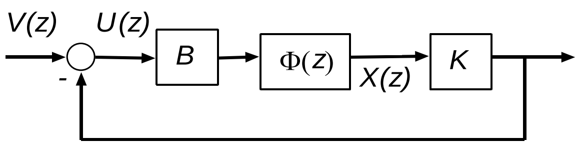
46

Poles of an LQR



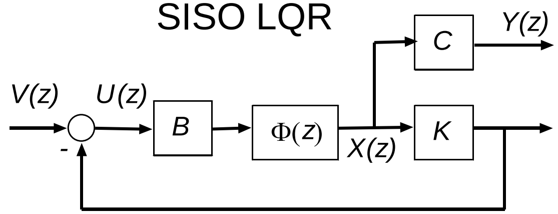
Open-loop polynomial: $\widehat{A}(z)=\det(zI-A)$

Closed-loop polynomial: $\widehat{A}_c(z)=\det(zI-A+BK)$

$$\text{Det}[I+G_o(z)]=\frac{\text{Det}[zI-A+BK]}{\text{Det}[zI-A]}$$

47

SISO LQR



Open-loop transfer function from $U(z)$ to $Y(z)$:

$$G(z)=C\Phi(z)B$$

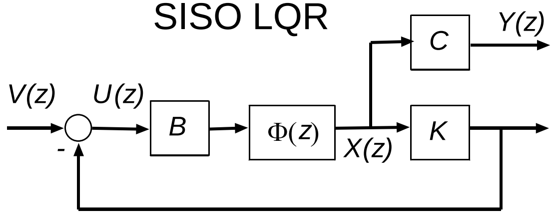
when

$$u(k)\in\mathcal{R}$$

$$y(k)\in\mathcal{R}$$

$$G(z)=\frac{\widehat{B}(z)}{\widehat{A}(z)}$$

SISO LQR



Open-loop poles: $\hat{A}(z) = 0$

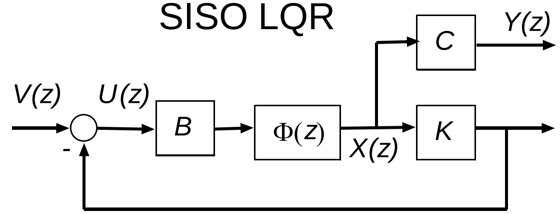
Closed-loop poles: $\hat{A}_c(z) = 0$

Open-loop zeros: $\hat{B}(z) = 0$

TF from U ➔ Y

$$G(z) = C\Phi(z)B = \frac{\hat{B}(z)}{\hat{A}(z)}$$

SISO LQR



Open-loop polynomial: $\hat{A}(z) = z^n + a_1 z^{n-1} + \cdots + a_0$

Closed-loop polynomial: $\hat{A}_c(z) = z^n + a_{c1} z^{n-1} + \cdots + a_{c0}$

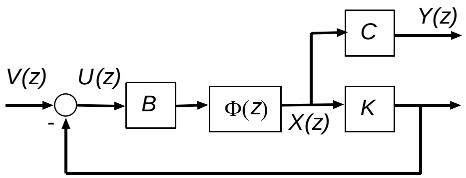
Open-loop plant zero polynomial: $\hat{B}(z) = \bar{b}_m (z^m + b_1 z^{m-1} + \cdots + b_0)$

TF from U ➔ Y

$$G(z) = C\Phi(z)B = \frac{\hat{B}(z)}{\hat{A}(z)}$$

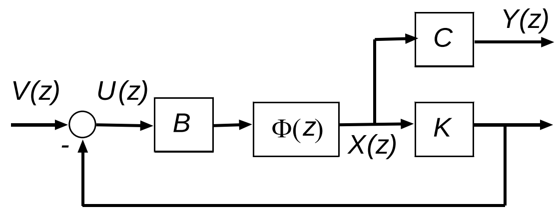
SISO Return Difference Equality (RDE)

$$u(k) \in \mathcal{R}$$
$$y(k) \in \mathcal{R}$$



$$\underbrace{(1+G_o(z^{-1}))}_{\frac{\hat{A}_c(z^{-1})}{\hat{A}(z^{-1})}}\underbrace{(1+G_o(z))}_{\frac{\hat{A}_c(z)}{\hat{A}(z)}} = \underbrace{\frac{R}{R+B^TPB}}_{\gamma > 0 \quad \text{for} \quad R \in (0,\infty)}\left[1+\frac{1}{R}\underbrace{G(z^{-1})}_{\frac{\hat{B}(z^{-1})}{\hat{A}(z^{-1})}}\underbrace{G(z)}_{\frac{\hat{B}(z)}{\hat{A}(z)}}\right]$$

SISO Return Difference Equality (RDE)



$$\frac{\hat{A}_c(z^{-1})\hat{A}_c(z)}{\hat{A}(z^{-1})\hat{A}(z)} = \gamma\left[1+\frac{1}{R}\frac{\hat{B}(z^{-1})\hat{B}(z)}{\hat{A}(z^{-1})\hat{A}(z)}\right]$$

Basis for the Reciprocal root locus technique

52

RDE Left hand side:

$2n$ zeros of the transfer function:

$$\frac{\widehat{A}_c(z^{-1})\widehat{A}_c(z)}{\widehat{A}(z^{-1})\widehat{A}(z)} = 0$$

n closed-loop poles: $\widehat{A}_c(z) = (z - p_{c1}) \cdots (z - p_{cn})$

$$\underbrace{n \text{ zeros of: } \widehat{A}_c(z^{-1})}_{} = \left(z - \frac{1}{p_{c1}}\right) \cdots \left(z - \frac{1}{p_{cn}}\right) \frac{a_{co}}{z^n}$$

***n** reciprocals of closed-loop poles*

$$a_{co} = (-1)^n p_{c1} p_{c2} \cdots p_{cn}$$

53

RDE Left hand side:

$2n$ zeros of the transfer function:

$$\frac{\widehat{A}_c(z^{-1})\widehat{A}_c(z)}{\widehat{A}(z^{-1})\widehat{A}(z)} = 0$$

n closed-loop poles: $p_{c1}, p_{c2}, \cdots p_{cn}$

n reciprocals of closed-loop poles: $\frac{1}{p_{c1}}, \frac{1}{p_{c2}}, \cdots \frac{1}{p_{cn}}$

$$\boxed{|p_{ci}| < 1 \qquad \left|\frac{1}{p_{ci}}\right| > 1 \qquad R \in (0, \infty)}$$

54

LQ Reciprocal Root Locus

$$\frac{\widehat{A}_c(z^{-1})\widehat{A}_c(z)}{\widehat{A}(z^{-1})\widehat{A}(z)} = \gamma \left[1 + \frac{1}{R} \frac{\widehat{B}(z^{-1})\widehat{B}(z)}{\widehat{A}(z^{-1})\widehat{A}(z)}\right]$$

$$\Downarrow$$

$$\boxed{\frac{\prod_{i=1}^n (z - p_{ci}) \prod_{i=1}^n (z - \frac{1}{p_{ci}})}{\prod_{i=1}^n (z - p_{oi}) \prod_{i=1}^n (z - \frac{1}{p_{oi}})} = \beta \left[1 + \frac{b_o}{a_o} \frac{\overline{b}_m^2}{R} \frac{z^{n-m} \prod_{i=1}^m (z - z_{oi}) \prod_{i=1}^m (z - \frac{1}{z_{oi}})}{\prod_{i=1}^n (z - p_{oi}) \prod_{i=1}^n (z - \frac{1}{p_{oi}})}\right]}$$

$$\beta = \left(\frac{a_o}{a_{co}}\right) \frac{R}{R + B^T P B}$$

Is a constant, which does not affect the Reciprocal root locus

55

LQ Reciprocal Root Locus

$$\frac{\prod_{i=1}^n (z - p_{ci}) \prod_{i=1}^n (z - \frac{1}{p_{ci}})}{\prod_{i=1}^n (z - p_{oi}) \prod_{i=1}^n (z - \frac{1}{p_{oi}})} = \beta \left[1 + \frac{b_o}{a_o} \frac{\overline{b}_m^2}{R} \frac{z^{n-m} \prod_{i=1}^m (z - z_{oi}) \prod_{i=1}^m (z - \frac{1}{z_{oi}})}{\prod_{i=1}^n (z - p_{oi}) \prod_{i=1}^n (z - \frac{1}{p_{oi}})}\right]$$

$\left|\frac{1}{p_{ci}}\right| < 1$ inverses of close loop eigenvalues (**always unstable**)

$|p_{ci}| < 1$ closed loop eigenvalues **(always asymptotically stable)**

LQ Reciprocal Root Locus

$$\frac{\prod_{i=1}^n (z-p_{ci}) \prod_{i=1}^n \left(z-\frac{1}{p_{ci}}\right)}{\prod_{i=1}^n (z-p_{oi}) \prod_{i=1}^n \left(z-\frac{1}{p_{oi}}\right)} = \beta\left[1+\frac{b_o}{a_o}\frac{\bar b_m^2}{R}\frac{z^{n-m}\prod_{i=1}^m (z-z_{oi})\prod_{i=1}^m \left(z-\frac{1}{z_{oi}}\right)}{\prod_{i=1}^n (z-p_{oi})\prod_{i=1}^n \left(z-\frac{1}{p_{oi}}\right)}\right]$$

open loop eigenvalues

Inverses of open loop eigenvalues

LQ Reciprocal Root Locus

$$\frac{\prod_{i=1}^n (z-p_{ci}) \prod_{i=1}^n \left(z-\frac{1}{p_{ci}}\right)}{\prod_{i=1}^n (z-p_{oi}) \prod_{i=1}^n \left(z-\frac{1}{p_{oi}}\right)} = \beta\left[1+\frac{b_o}{a_o}\frac{\bar b_m^2}{R}\frac{z^{n-m}\prod_{i=1}^m (z-z_{oi})\prod_{i=1}^m \left(z-\frac{1}{z_{oi}}\right)}{\prod_{i=1}^n (z-p_{oi})\prod_{i=1}^n \left(z-\frac{1}{p_{oi}}\right)}\right]$$

n-m zeros at the origin

zeros of $\hat{B}(z)$

Inverses of zeros of $\hat{B}(z)$

LQ Reciprocal Root Locus

$$\frac{\prod_{i=1}^n (z-p_{ci}) \prod_{i=1}^n \left(z-\frac{1}{p_{ci}}\right)}{\prod_{i=1}^n (z-p_{oi}) \prod_{i=1}^n \left(z-\frac{1}{p_{oi}}\right)} = \beta\left[1+\frac{b_o}{a_o}\frac{\bar b_m^2}{R}\frac{z^{n-m}\prod_{i=1}^m (z-z_{oi})\prod_{i=1}^m \left(z-\frac{1}{z_{oi}}\right)}{\prod_{i=1}^n (z-p_{oi})\prod_{i=1}^n \left(z-\frac{1}{p_{oi}}\right)}\right]$$

$$\hat{B}(z) = \bar{b}_m\left(z^m + \cdots + b_o\right) \qquad \hat{A}(z) = z^n + \cdots + a_o,$$

$\frac{b_o}{a_o} > 0 \quad \Rightarrow$ negative feedback

$\frac{b_o}{a_o} < 0 \quad \Rightarrow$ positive feedback

Sketching the Reciprocal RL Plot

1. Find poles for RL plot
   – Open-loop poles and their inverses
2. Find zeros for RL plot
   – Zeros of $C(zI-A)^{-1}B$, their inverses, and an extra *n-m* zeros at z=0
3. Determine sign of feedback rules
   – If $b_0/a_0 > 0$, use (-) feedback rules
   – If $b_0/a_0 < 0$, use (+) feedback rules
4. Draw RL plot

This procedure requires $C(zI-A)^{-1}B$ to have no poles or zeros at the origin

60

Sketching the Reciprocal RL Plot

When $C(zI-A)^{-1}B$ has poles or zeros at the origin, the rules generalize as follows:

- For any poles or zeros of $C(zI-A)^{-1}B$ at the origin, do not include their inverse in the RL poles and zeros
- To determine the sign of the feedback rules
  - In place of b_0, use the coefficient of the smallest power of z in $\hat{B}(z)$
  - In place of a_0, use the coefficient of the smallest power of z in $\hat{A}(z)$

61

Reciprocal RL Plots in MATLAB

- Let `sys` be a `tf` object representing the transfer function $G(z)$
- Two useful MATLAB commands:
  - `>> sys.'` ⟵ `tf` object representing $G^T(z)$
  - `>> sys'` ⟵ `tf` object representing $G^T(z^{-1})$
- Make sure to specify the sampling time of `sys` otherwise the command `sys'` will interpret $G(z)$ as the continuous-time transfer function $G(s)$ and then return $G^T(-s)$

62

Reciprocal RL Plots in MATLAB

Code for plotting a reciprocal root locus plot:

```
>> sys = ss(A,B,C,0, -1);
```

discrete-time model with unspecified sampling time

```
>> sys = tf(sys);
```

converts the model to a transfer function

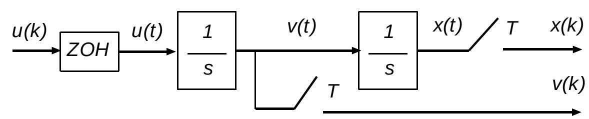
```
>> rlocus(sys' * sys);
```

plots reciprocal root locus plot

63

Example – Double Integrator

Double integrator with ZOH and sampling time T=1:

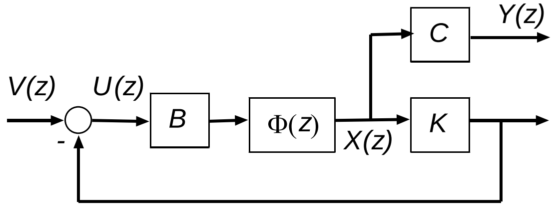


$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)$$

$$J[x_o] = \sum_{k=0}^{\infty} \left\{ y^2(k) + R u^2(k) \right\} \qquad R > 0$$

Example – Double Integrator



$$G(z) = C\Phi(z)B$$

$$G(z) = \frac{\hat{B}(z)}{\hat{A}(z)} = \frac{\frac{1}{2}(z+1)}{(z-1)^2} = \frac{\frac{1}{2}(z+1)}{(z^2-2z+1)}$$

$$\frac{b_o}{a_o} = \frac{1}{1}$$

Example – Double Integrator

$$\frac{\hat{B}(z)}{\hat{A}(z)} = \frac{\frac{1}{2}(z+1)}{(z^2-2z+1)} \quad \left\{ \begin{aligned} \bar{b}_m &= \tfrac{1}{2} \\ a_o &= 1 \qquad b_o = 1 \\ n &= 2 \qquad m = 1 \\ n-m &= 1 \\ \frac{a_o}{b_o} &= 1 \end{aligned} \right.$$

$$1 + \frac{1}{R}\frac{\hat{B}(z^{-1})\hat{B}(z)}{\hat{A}(z^{-1})\hat{A}(z)} = 0$$

$$1 + \frac{(1/2)^2}{R}\left(\frac{1}{1}\right)\frac{z(z+1)(z+1)}{(z-1)(z-1)(z-1)(z-1)} = 0$$

Example – Double Integrator

$$1 + \frac{1}{R}\frac{\hat{B}(z^{-1})\hat{B}(z)}{\hat{A}(z^{-1})\hat{A}(z)} = 0$$

1 open loop zero @ 0 ***2 open loop zeros @ -1***

$$1 + \frac{0.25}{R}\frac{z\,(z+1)^2}{(z-1)^4} = 0$$

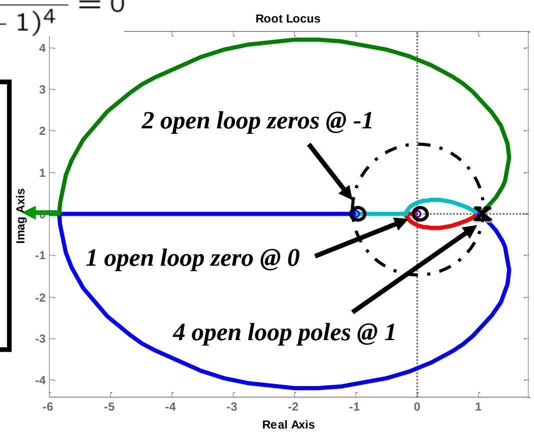
4 open loop poles @ 1

Example – Double Integrator

$$1 + \frac{0.25}{R}\frac{z\,(z+1)^2}{(z-1)^4} = 0$$

LQR poles always inside unit circle

$R \in (0, \infty)$

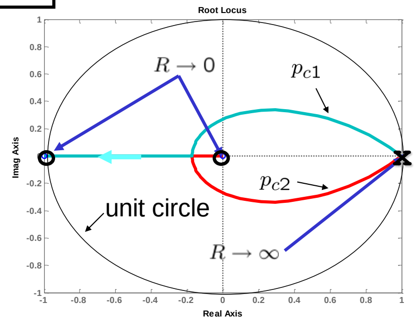


70

LQR Closed-Loop poles

$$\boxed{1+\frac{0.25}{R}\frac{z\,(z+1)^2}{(z-1)^4}=0}\qquad R\to\infty\;\Rightarrow\;\begin{cases}p_{c1}\to 1\\ p_{c2}\to 1\end{cases}$$

$$R\to 0\;\Rightarrow\;\begin{cases}p_{c1}\to -1\\ p_{c2}\to 0\end{cases}$$



71

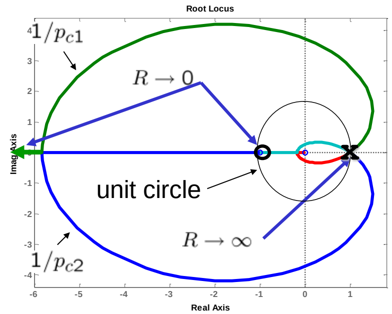
LQR Closed-Loop poles and reciprocals

$$\boxed{1+\frac{0.25}{R}\frac{z\,(z+1)^2}{(z-1)^4}=0}\qquad R\to\infty\;\Rightarrow\;\begin{cases}1/p_{c1}\to 1\\ 1/p_{c2}\to 1\end{cases}$$

$$R\to 0\;\Rightarrow$$

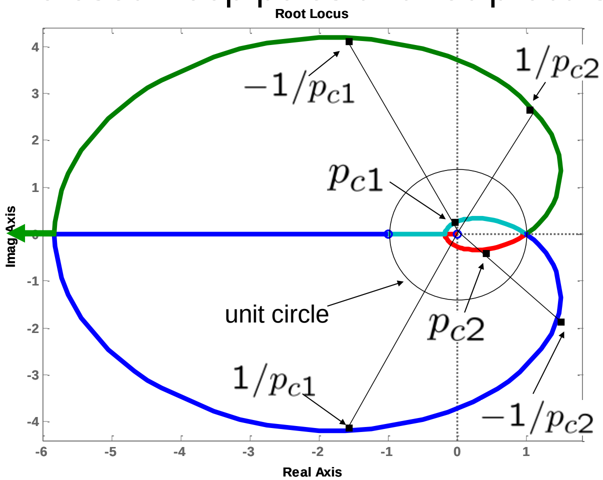
$$\begin{cases}1/p_{c1}\to -1\\ 1/p_{c2}\to -\infty\end{cases}$$

always unstable



72

LQR Closed-Loop poles and reciprocals



73

Summary

- Return difference equality
  - Guaranteed gain and phase margins of LQR
  - Reciprocal root locus (LQR closed-loop poles)