UNIVERSITY OF CALIFORNIA AT BERKELEY
Department of Mechanical Engineering
ME233 Advanced Control Systems II, Spring 2010

Homework #8

Assigned:	Th., April 1
Due:	Th., April 8

1. A LTI discrete time system is described by

$$x(k+1) = A\,x(k) + B\,u(k) + B_w\,w(k)$$

where $x(k) \in \mathcal{R}^{p_1}$ is the state, $u(k)\mathcal{R}^{p_2}$ is the control input and $w(k) \in \mathcal{R}^{p_3}$ is stationary colored noise given by

$$\begin{aligned} w(k) &= C_w\,x_w(k) \\ x_w(k+1) &= A_w\,x_w(k) + B_n\,\eta(k)\,. \end{aligned}$$

where $x_w(k) \in \mathcal{R}^{p_4}$ and $n(k) \in \mathcal{R}^{p_5}$ is stationary and white. Assume that $x(0)$, $x_w(0)$ and $\eta(k)$ are normally distributed with

$$\begin{aligned} &E\{x(0)\} = x_o, \quad E\{(x(0)-x_o)(x(0)-x_o)^T\} = X_o \\ &E\{x_w(0)\} = x_{wo}, \quad E\{(x_w(0)-x_{wo})(x_w(0)-x_{wo})^T\} = X_{wo} \\ &E\{\eta(k)\} = 0, \quad E\{\eta(k)\eta(k+l)^T\} = \Gamma\delta(l) \end{aligned}$$

The performance index to be minimized is

$$J = \frac{1}{2}\,E\left\{x^T(N)\,S\,x(N) + \sum_{k=0}^{N-1}\left[x^T(k)\,Q\,x(k) + u^T(k)\,R\,u(k)\right]\right\}$$

where the expectation must be taken over all underlying random quantities in each of the following situations that will be described below. For each situation obtain the equations that must be solved to find the optimal control and the resulting optimal cost:

(a) $x(k)$ and $x_w(k)$ are measurable for all k.

(b) $y(k) = C\,x(k) + v(k)$ is the only available measurement for all k; where $v(k)$ is a zero-mean, white and Gaussian noise with covariance $E\{v(k)v(k+l)\} = V\,\delta(l)$ and is independent from $x(0)$, $X_w(0)$ and $\eta(k)$.

(c) $x(k)$ is measurable for all k. Notice that in this case, since

$$x(k) - A\,x(k-1) - B\,u(k-1) = B_w C_w\,x_w(k-1)$$

$x_w(k-1)$ can be determined without being contaminated by measurement noise at the instance k.Therfore, if you set the measurement covariance to zero and obtain the equations for the Kalman filter, you will obtain $\hat{x}_w(k-1|k)$.

2. In a hard disk drive, the read-write head is located on a flexible arm, or suspension, which may vibrate due to airflow turbulence generated by the spinning disk. We want to install a MEMS actuator on the tip of the suspension that will move the head relative to the suspension, in order to cancel out this vibration. A schematic drawing of the system is shown in Fig. 1.

In this problem we will use a simple model of a disk drive suspension, by only considering a single vibration mode. The transfer function from the windage disturbance $w(t)$ to the tip of the suspension can be expressed as:

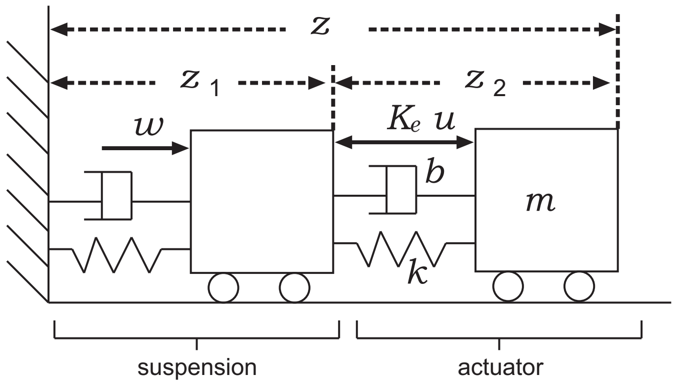

Figure 1: Disk Drive suspension/micro-actuator model

$$Z_1(s) = \frac{1.367e4}{s^2 + 134s + 4.52e9} W(s);$$

where W(s) is the windage disturbance. At the tip of the suspension we add an actuator, which can be modeled as a mass-spring-damper system with mass $m = (2e-6)\,kg$, spring constant $k = 400N/m$, and damping constant $b = (5e-3)\,N \cdot s/m$, and is driven by a force proportional to our control input voltage, u, times gain $K_e =\ (2.5e-5)\ N/V$.

Note that the plant is equivalent to a double mass-spring-damper system as shown in the figure, with the output being the absolute position of the actuator mass,

$$z(t) = z_1(t) + z_2(t).$$

Suppose the windage disturbance to the system, which is due to air turbulence, can be modeled as a white, zero-mean Gaussian signal with $W = E\{w(t+\tau)w(t)\} =\ (1.01e-7)\ N^2$ and the noise $v(t)$ from the sensor reading the output

$$y(t) = z(t) + v(t)$$

is white, zero-mean, Gaussian, with $V = E\{v(t+\tau)v(t)\} = 1.66\ (e-14)$. The control objective is to design a LQG regulator that will minimize

$$J = \frac{1}{2} E\{z^2(t) + Ru^2(t)\}$$

You are encouraged to use matlab to answer some of the questions below.

(a) Obtain a state space description for the system.

(b) Draw the symmetric root locus that includes the eigenvalues of $A - BK$ for the LQG regulator, when $R \in (0, \infty)$. Normally from this plot, you would select a value of R, which you would use in your LQG controller design. However, in this case, we will use the values $R = 1$ and $R = 1e-10$, so that everyone comes up with the same answers.

(c) Design an LQG regulator and determine the values of the optimal cost function J^o for $R = 1$ and $R = 1e-10$.

(d) The optimal cost function J^o can be computed as follows

$$J^o = \text{Tr}\left\{C_Q^T C_Q\, M + L^T PLV\right\}$$

where P is the solution of the algebraic Riccati equation associated with the LQR, M is the estimation error covariance, which is the algebraic Riccati equation associated with the Kalman filter, L is the Kalman filter gain. Show that $\text{Tr}\left\{C_Q^T C_Q\, M\right\} \geq 0$ and $\text{Tr}\left\{L^T PLV\right\} \geq 0$ and separately compute each term for $R = 1$ and $R = 1e - 10$. From these results, you should be able to obtain an estimate of J^o as $R \to 0$ (cheap control). We will discuss this result further in the LQG Loop Transfer Recovery lecture.

(e) Because the mass of the microactuator M is normally significantly smaller than the mass of the suspension and the voice coil motor, we may be able to assume that the microactuator control effort $u(t)$ and motion do not significantly affect the rest of the drive dynamics. Thus, the model or this system may be simplified as shown in Fig. 2.

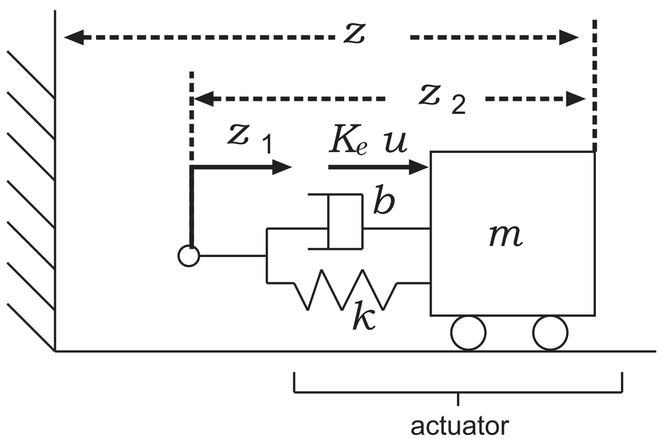

Figure 2: Disk Drive suspension/micro-actuator model

Repeat steps a) - c) with the new model and comment on your results.

3. Consider the following second order system

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ m \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t) \quad (1)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + v(t)\,, \quad (2)$$

where $w(t)$ and $v(t)$ are zero mean, white Gaussian uncorrelated signals with covariances given by $E[w(t)w(t-\tau)] = \delta(\tau)$ and $E[v(t)v(t-\tau)] = \sigma\,\delta(\tau)$.

Assume that the nominal value of m is $m_o = 1$ and perform your LQR, Kalman filter and LQG designs using this nominal value.

Consider also the quadratic cost criteria

$$J = E\left\{x^T(\tau)\, Q\, x(\tau) + u^2(\tau)\right\} = \lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T \left\{x^T(\tau)\, Q\, x(\tau) + u^2(\tau)\right\} d\tau\right\}$$

$$Q = \rho\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} = \rho\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

where $\sigma > 0$ is the measurement noise covariance and $\rho > 0$ is a scalar input weighting factor in the LQ cost function.

(a) Consider first the case when the state $x(t)$ is measurable. Obtain an analytical expression for the optimal control state feedback gain K_{LQ} in

$$u(t) = -K_{LQ}\, x(t)\,,$$

and show that it can be written in the form

$$K_{LQ} = \alpha \begin{bmatrix} 1 & 1 \end{bmatrix}\,,$$

where α is a function of ρ.

(b) Calculate the closed loop eigenvalues of

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 \\ m \end{bmatrix} K_{LQ} \right\} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

for $\rho = 5$ and $m = 1$. Verify that the system is stable for $\infty \geq m > .5$.

Hint: This result should be immediate from the known LQ robustness results which are derived from the return difference equality.

(c) Using duality, determine the analytical solution for the Kalman Filter gain matrix K_{KF} in terms of σ when $m = 1$. It should be in the form

$$K_{KF} = \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Hint: Write the state equation for the Kalman filter using the state $\hat{x} = [\hat{x}_2\ \hat{x}_1]^T$ (with the estimates of x_1 and x_2 swapped).

(d) The optimal LQG gains K_{LQ} and K_{KF} where obtained for the case when $m = 1$. Show that the closed loop eigenvalues of the actual LQG system (i.e. any value m) are the eigenvalues of the matrix

$$\bar{A}_c = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -m\alpha & -m\alpha \\ \beta & 0 & 1-\beta & 1 \\ \beta & 0 & -\beta-\alpha & 1-\alpha \end{bmatrix}$$

(e) Consider the case when $\rho = \sigma = 5$ and $m = 1$. Compute using matlab the closed loop eigenvalues of the LQG system and confirm the separation theorem.

(f) Consider the case when $\rho = \sigma = 5$ and $m = 1.1$. Compute using matlab the closed loop eigenvalues of the LQG system. The result may surprise you.