UNIVERSITY OF CALIFORNIA AT BERKELEY
Department of Mechanical Engineering
ME233 Advanced Control Systems II

Adaptive Identification using a First Order Parallel Model

1 Continuous Time

We consider the model reference adaptive identification of the following first order system:

$$\dot{y}(t) = a\,y(t) + b\,u(t) \tag{1}$$

where

- $u(t)$ is a known *bounded* input, i.e. $|u(t)| < \infty$.
- $y(t)$ is the measured output.
- a and b are unknown *constant* parameters, with $a < 0$ and $b \neq 0$.

We now define the parallel estimation model

$$\dot{\hat{y}}(t) = \hat{a}(t)\,\hat{y}(t) + \hat{b}(t)\,u(t) \tag{2}$$

where $\hat{y}(t)$ is the estimate of $y(t)$ and $\hat{a}(t)$ and $\hat{b}(t)$ are the estimates of the unknown parameters a and b.

Let us introduce some notation:

Unknown parameter vector : $\theta = \begin{bmatrix} a & b \end{bmatrix}^T \in \mathcal{R}^2$.

Parameter vector estimate : $\hat{\theta}(t) = \begin{bmatrix} \hat{a}(t) & \hat{b}(t) \end{bmatrix}^T \in \mathcal{R}^2$.

Regressor vector : $\phi(t) = \begin{bmatrix} \hat{y}(t) & u(t) \end{bmatrix}^T \in \mathcal{R}^2$.

Parameter estimation error : $\tilde{\theta}(t) = \theta - \hat{\theta}(t)$.

Output estimation error : $e(t) = y(t) - \hat{y}(t)$.

Notice that, since $\dot{\theta} = 0$,

$$\dot{\tilde{\theta}} = -\dot{\hat{\theta}}$$

Thus, although θ and $\tilde{\theta}$ are unknown, we can adjust $\dot{\tilde{\theta}}$ by adjusting $\dot{\hat{\theta}}$.

Error dynamics

Let us now derive the estimation error dynamics. We start by adding and subtracting $a\,\hat{y}(t)$ to Eq. (2)

$$\dot{\hat{y}}(t) = a\,\hat{y}(t) - [a - \hat{a}(t)]\,\hat{y}(t) + \hat{b}(t)\,u(t)\,. \tag{3}$$

Subtracting Eq. (6) from Eq. (1) we obtain

$$\begin{aligned} \dot{e}(t) &= a\,e(t) + \begin{bmatrix} \tilde{a}(t) & \tilde{b}(t) \end{bmatrix}^T \begin{bmatrix} \hat{y}(t) \\ u(t) \end{bmatrix} \\ &= a\,e(t) + \tilde{\theta}(t)^T \phi(t) \end{aligned} \tag{4}$$

Parameter Adaptation Algorithm (PAA)

We will use the following PAA to update $\hat{\theta}(t)$:

$$\dot{\hat{\theta}}(t) = F\,\phi(t)\,e(t) \tag{5}$$

where $F = F^T$ and $F > 0$ is a positive definite and symmetric constant matrix.

1.1 Stability Analysis

In this section, we will prove that, under the assumptions stated in the previous section, the output estimation error, $e(t)$, converges to zero. To this end, we will use two different, but very similar approaches: the hyperstability approach and the Lyapunov function approach.

1.1.1 Hyperstability Approach

To use the Hyperstability theory in pages HS-1 to Hs-3 in the ME233 class notes, we define the signals

$$m(t) = -w(t) = \tilde{\theta}(t)^T \phi(t) \tag{6}$$

and notice that the combined error dynamics and PAA, can be described by the equivalent feedback loop in Fig. 1.
Fig. 1 has a linear time invariant system in its forward path ($G(s)$):

$$\dot{e} = a\,e(t) + m(t) \tag{7}$$

and a nonlinear system in its feedback path (NL):

$$\begin{aligned} \dot{\tilde{\theta}} &= -F\,\phi(t)\,e(t) \\ w(t) &= -\phi(t)^T \tilde{\theta}(t) \end{aligned} \tag{8}$$

The stability analysis is conducted as follows:

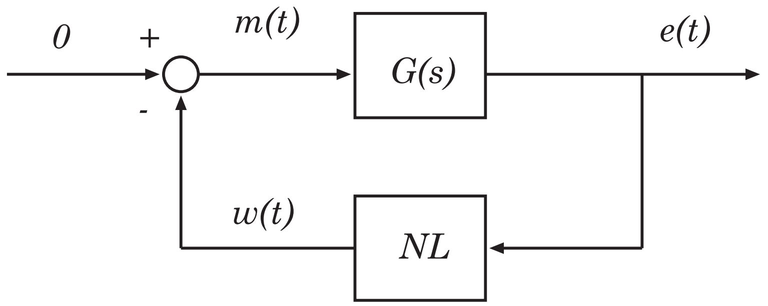

Figure 1: Equivalent Feedback Block Diagram

1. Verify that $G(s)$ in the forward path is a strictly positive real (SPR) transfer function.
2. Verify that the nonlinear block in the feedback path is a P-class (Passive) nonlinearity.
3. Use the sufficiency portion of the hyperstability theorem to show that $e(t)$ is bounded.
4. Show that $w(t)$ is bounded.
5. Convergence of $e(t)$ to zero follows from the sufficiency portion of the asymptotic hyperstability theorem in page HS-2 of the ME233 class notes.

Lets now perform each step in detail:

1) **Linear Forward Path:** From Eq. (7), $G(s)$ is given by

$$G(s) = \frac{1}{s-a} \, ,$$

which is SPR since $a < 0$.

2) **Nonlinear block in the feedback path:** It is necessary to verify that the input-output relation given by Eqs. (8) satisfies the Popov (passivity) innequality, i.e.

$$\int_0^t w(\tau)e(\tau)d\tau \geq -\gamma_o^2 \, .$$

To this end, notice that by Eq. (8)

$$\begin{aligned} \int_0^t w(\tau)e(\tau)dt &= -\int_0^t \tilde{\theta}^T(\tau)\,\phi(\tau)e(\tau)\,d\tau \\ &= \int_0^t \tilde{\theta}^T(\tau)F^{-1}\dot{\tilde{\theta}}(\tau)\,d\tau = \frac{1}{2}\int_0^t \frac{d}{d\tau}\left[\tilde{\theta}^T(\tau)F^{-1}\tilde{\theta}(\tau)\right]\,d\tau \\ &= \frac{1}{2}\,\tilde{\theta}^T(t)F^{-1}\tilde{\theta}(t) - \frac{1}{2}\,\tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0) \\ &\geq -\gamma_o^2 \end{aligned} \qquad (9)$$

where $\gamma_o^2 = \frac{1}{2}\tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0)$.

3) By the sufficiency portion of the hyperstability theorem, we now conclude from Eq. (H-4)in pages HS-1 of the ME233 class notes that

$$|e(t)| \leq \delta \left\{ |e(0)| + \gamma_2^2 \right\} ,$$

where both $\delta > 0$ and $\gamma_2^2 > 0$ are bounded. Thus, $e(t)$ is bounded, i.e. $|e(t)| < \infty$.

4) **Boundness of $w(t)$:** Notice that $|e(t)| < \infty$ does not imply that $|w(t)| < \infty$, since the impulse response of a strictly causal asymptotically stable system is bounded. From Eq. (15), to show the Boundness of $w(t)$, it is necessary to show the Boundness of the regressor vector, $\phi(t)$, and the Boundness of the parameter error vector, $\tilde{\theta}(t)$. We proceed as follows:

   (i) Show that $\phi(t)$ is bounded: Notice that, by assumption, $u(t)$ is bounded. Since, $G(s)$ is asymptotically stable, it is also bounded-input-bounded-output stable. Thus, $y(t)$ is also bounded. Now,

   $$e(t) = y(t) - \hat{y}(t)$$

   Therefore, by the triangle inequality,

   $$|\hat{y}(t)| \leq |y(t)| + |e(t)| < \infty$$

   Thus,

   $$||\phi(t)|| = \left(|\hat{y}(t)|^2 + |u(t)|^2 \right)^{\frac{1}{2}} < \infty$$

   (ii) Boundness of $\tilde{\theta}(t)$: Because $G(s)$ is an SPR transfer function, from the results in page HS-2 of the ME233 class notes, it follows that

   $$\int_0^t m(\tau) e(\tau) dt \geq -\gamma_3^2 \tag{10}$$

   For a first order system, this can be verified easily as follows:

   Multiply Eq. (7) by $e(t)$ and integrate with respect to time, to obtain

   $$\int_0^t e(\tau)\dot{e}(\tau)\, d\tau \;=\; a \int_0^t e^2(\tau) d\tau + \int_0^t m(\tau) e(\tau) dt$$

   Since, $a < 0$,

   $$\begin{aligned} \int_0^t m(\tau) e(\tau) dt &= |a| \int_0^t e^2(\tau) d\tau + \frac{1}{2}\, e^2(t) - \frac{1}{2}\, e^2(0) \\ &\geq -\frac{1}{2}\, e^2(0) \end{aligned}$$

   Setting $\gamma_3^2 = \frac{1}{2}\, e^2(0)$, we obtain (10).

From (14) and inequality (10) we obtain

$$\frac{1}{2}\,\tilde{\theta}^T(t)F^{-1}\tilde{\theta}(t) \;=\; \frac{1}{2}\,\tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0) + \int_0^t w(\tau)e(\tau)dt \tag{11}$$

$$\leq\; \frac{1}{2}\,\tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0) + \gamma_3^2 \tag{12}$$

Thus,

$$||\tilde{\theta}(t)|| \;\leq\; \lambda_{\max}\{F\}\left\{\frac{1}{\lambda_{\min}\{F\}}\,||\tilde{\theta}(0)|| + 2\,\gamma_3^2\right\} \tag{13}$$

where $\lambda_{\max}\{F\} = 1/\lambda_{\min}\{F^{-1}\}$ is the largest eigenvalue (and singular value) of F. Inequality (13) shows that $\tilde{\theta}(t)$ is bounded.

(iii) To prove that $w(t)$ is bounded, we utilize Eq. (15) and the Schwartz inequality,

$$||w(t)|| \leq ||\tilde{\theta}(t||\,||\phi(t)|| < \infty\,. \tag{14}$$

5) **Convergence of $e(t)$ to zero:** Since $G(s)$ is SPR, the convergence of $e(t)$ to zero follows from the sufficiency portion of the Asymptotic Hyperstability Theorem in page HS-2 of the ME233 class notes, and the fact that $w(t)$ is bounded.

1.1.2 Lyapunov Function Approach

We will now prove the stability of the adaptive identification system, using the Lyapunov function approach. To this end, let us re-write the error dynamics and PAA estimation equations:

$$\dot{e}(t) \;=\; a\,e(t) + \tilde{\theta}(t)^T\phi(t) \tag{15}$$

$$\dot{\tilde{\theta}} \;=\; -F\,\phi(t)\,e(t) \tag{16}$$

Defining the extended error by

$$e_e = \begin{bmatrix} e & \tilde{\theta} \end{bmatrix}^T \in \mathcal{R}^3$$

Eq. (15) can be written as

$$\dot{e}_e(t) = f(e_e(t), t)$$

Notice that this system is non-autonomous since $f(\cdot,\cdot)$ is a function of both e_e and time t (remember that $u(t)$ is exciting the system). We will first show, using Lyapunov's direct method that the origin $e_e = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$ is stable in the sense of Lyapunov. Subsequently we will show that $e(t) \to 0$.

1) **Stability in the Sense of Lyapunov:**

To show that the equilibrium state $e_e = 0$ is stable in the sense of Lyapunov, we use the following Lyapunov function candidate:

$$V(e_e) = \frac{1}{2}\, {e_e}^T \begin{bmatrix} 1 & 0 \\ 0 & F^{-1} \end{bmatrix} e_e \;=\; \frac{1}{2}\, e^2 + \frac{1}{2}\, \tilde{\theta}^T F^{-1} \tilde{\theta} \tag{17}$$

Notice that $V(\cdot)$ is a positive definite function (PDF) of the state e_e. We now compute $\dot{V}$ utilizing Eq. (15):

$$\begin{aligned} \dot{V}(e_e) &= e\,\dot{e} + \tilde{\theta}^T F^{-1} \dot{\tilde{\theta}} \\ &= e\left\{a\,e + \tilde{\theta}^T \phi\right\} + \tilde{\theta}^T\, F^{-1}\left\{-F\,\phi\, e\right\} \\ &= -|a|e^2 = {e_e}^T \begin{bmatrix} -|a| & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} e_e\,. \end{aligned} \tag{18}$$

Thus, $-\dot{V}$ is a positive semi-definite function (PSF) and by Lyapunov's theorem, the equilibrium state $e_e = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$ is stable in the sense of Lyapunov. Assuming that $||e_e(0)|| < \infty$, this in turn implies that

$$||e_e(t)|| < \infty \Rightarrow |e(t)| < \infty \qquad \text{and} \qquad ||\tilde{\theta}(t)|| < \infty\,.$$

2) **Convergence of $e(t)$ to zero.**

Because the system is not-autonomous, we cannot use L'Salle's theorem to show that the equilibrium state is asymptotically stable. Indeed it is not. We will now show that $e(t) \rightarrow 0$ (not e_e) by using a very useful result, which is known as Barbalat's lemma.

Barbalat's Lemma

Let $g(t) \in \mathcal{R}$ be **uniformly continuous** *for $t \geq 0$ and define*

$$G(t) = \int_0^t g(\tau)d\tau\,.$$

If

$$\lim_{t\to\infty} G(t) = G_o \qquad \text{and} \qquad |G_o| < \infty$$

then,

$$\lim_{t\to\infty} g(t) = 0.$$

Notice that, if $|\dot{g}(t)| < \infty$, then $g(t)$ is uniformly continuous.

Lets now integrate Eq. (18) with respect to time and denote $V(t) := V(e(t))$

$$\int_0^t \dot{V}(\tau)d\tau \;\; = \;\; -|a| \int_0^t e^2(\tau)d\tau\,.$$

Taking limits as $t \to \infty$, we obtain

$$\lim_{t\to\infty} V(t) = V(0) - |a| \int_0^\infty e^2(\tau)d\tau\,.$$

Because $V(t) \geq 0$,

$$\int_0^\infty e^2(t)dt = E_o \leq \frac{V(0)}{|a|} < \infty$$

Thus, by Barbalat's Lemma, to prove that $e(t)$ converges to zero, all we need to do is to prove that $e^2(t)$ is uniformly continuous. This can be accomplished by showing that $\dot{e}(t)$ is bounded.

From Eq. (15) and applying once again the triangle and the Schwartz inequalities, we obtain

$$|\dot{e}| \;\; \leq \;\; |a|\,|e| + ||\tilde{\theta}||\,||\phi||\,.$$

We have already shown, using Lyapunov's direct method, that $|e| < \infty$ and $||\tilde{\theta}|| < \infty$. The proof that $||\phi|| < \infty$ is the same as in the Hyperstability analysis.

Thus, since $|\dot{e}(t)| < \infty$, $e^2(t)$ is uniformly continuous and, by Barbalat's lemma,

$$\lim_{t\to\infty} e(t) = 0\,.$$

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Adaptive Identification - First Order Parallel Model

2 Discrete Time

We consider the model reference adaptive identification of the following first order system:

$$y(k+1) = a\, y(k) + b\, u(k) \tag{1}$$

where

- $u(k)$ is a known *bounded* input, i.e. $|u(k)| < \infty$.
- $y(k)$ is the measured out put.
- a and b are unknown *constant* parameter, with $|a| < 1$ and $b \neq 0$.

We now define the a-posteriori parallel estimation model

$$\hat{y}(k+1) = \hat{a}(k+1)\, \hat{y}(k) + \hat{b}(k+1)\, u(k) \tag{2}$$

where $\hat{y}(k)$ is the *a-posteriori* estimate of $y(k)$ and $\hat{a}(k+1)$ and $\hat{b}(k+1)$ are the estimates of the unknown parameters a and b, and the a-priori estimation model

$$\hat{y}^o(k+1) = \hat{a}(k)\, \hat{y}(k) + \hat{b}(k)\, u(k) \tag{3}$$

where $\hat{y}^o(k)$ is the *a-priori* estimate of $y(k)$, which utilizes the parameter estimates $\hat{a}(k)$ and $\hat{b}(k)$ instead of $\hat{a}(k+1)$ and $\hat{b}(k+1)$.

Let us introduce some notation:

Unknown parameter vector : $\theta = \begin{bmatrix} a & b \end{bmatrix}^T \in \mathcal{R}^2$.

Parameter vector estimate : $\hat{\theta}(k) = \begin{bmatrix} \hat{a}(k) & \hat{b}(k) \end{bmatrix}^T \in \mathcal{R}^2$.

Regressor vector : $\phi(k) = \begin{bmatrix} \hat{y}(k) & u(k) \end{bmatrix}^T \in \mathcal{R}^2$.

Parameter estimation error : $\tilde{\theta}(k) = \theta - \hat{\theta}(k)$.

A-posteriori output estimation error : $e(k) = y(k) - \hat{y}(k)$.

A-priori output estimation error : $e^o(k) = y(k) - \hat{y}^o(k)$.

Notice that, since $\theta(k+1) - \theta(k) = 0$,

$$\Delta\tilde{\theta}(k+1) = \tilde{\theta}(k+1) - \tilde{\theta}(k) = -\Delta\hat{\theta}(k+1) = -\left[\hat{\theta}(k+1) - \hat{\theta}(k)\right] = -\hat{\theta}(k+1) + \hat{\theta}(k)$$

Thus, although θ and $\tilde{\theta}$ are unknown, we can adjust $\Delta\tilde{\theta}(k+1)$ by adjusting $\Delta\hat{\theta}(k+1)$.

Error dynamics

Let us now derive the *a-priori* estimation error dynamics. We start by adding and subtracting $a\,\hat{y}(k)$ to Eq. (2)

$$\hat{y}^o(k+1) = a\,\hat{y}(k) - [a - \hat{a}(k)]\,\hat{y}(k) + \hat{b}(k)\,u(k)\,. \tag{4}$$

Substracting Eq. (4) from Eq. (1) we obtain

$$\begin{aligned} e^o(k+1) &= a\,e(k) + \begin{bmatrix} \tilde{a}(k) & \tilde{b}(k) \end{bmatrix}^T \begin{bmatrix} \hat{y}(k) \\ u(k) \end{bmatrix} \\ &= a\,e(k) + \tilde{\theta}(k)^T \phi(k)\,. \end{aligned} \tag{5}$$

Doing similar calculation for the *a-posteriori* error dynamics, we obain

$$\hat{y}(k+1) = a\,\hat{y}(k) - [a - \hat{a}(k+1)]\,\hat{y}(k) + \hat{b}(k+1)\,u(k)\,. \tag{6}$$

Substracting Eq. (4) from Eq. (1) we obtain

$$\begin{aligned} e(k+1) &= a\,e(k) + \begin{bmatrix} \tilde{a}(k+1) & \tilde{b}(k+1) \end{bmatrix}^T \begin{bmatrix} \hat{y}(k) \\ u(k) \end{bmatrix} \\ &= a\,e(k) + \tilde{\theta}(k+1)^T \phi(k)\,. \end{aligned} \tag{7}$$

Parameter Adaptation Algorithm (PAA)

We will use the following PAA to update $\hat{\theta}(k)$:

$$\hat{\theta}(k+1) = \hat{\theta}(k) + F\,\phi(k)\,e(k+1) \tag{8}$$

where $F = F^T$ and $F \succ 0$ is a positive definite and symmetric constant matrix.
Eq. (8) is not realizable, but useful for performing the stability analysis. However, from Eqs. (6), (7) and (8), we can obtain the following relation between the a-posteriori and a-priori estimation errors,

$$e(k+1) = \frac{e^0(k+1)}{1 + \phi(k)^T F \phi(k)} \tag{9}$$

and express the PAA as a function of the a-priori estimation error instead

$$\hat{\theta}(k+1) = \hat{\theta}(k) + F\,\phi(k)\,\frac{e^0(k+1)}{1 + \phi(k)^T F \phi(k)} \tag{10}$$

2.1 Stability Analysis

In this section, we will prove that, under the assumptions stated in the previous section, both the a-posteriori output estimation error, $e(k)$, and the a-priori output estimation error, $e(k)$ and converge to zero. To this end, we will use two different, but very similar approaches: the hyperstability approach and the Lyapunov function approach.

2.1.1 Hyperstability Approach

To use the Hyperstability theory in pages HS-1 to Hs-3 in the ME233 class notes, we define the signals

$$m(k) = -w(k) = \tilde{\theta}(k)^T \phi(k-1) \tag{11}$$

and notice that the combined error dynamics and PAA, can be described by the equivalent feedback loop in Fig. 4.

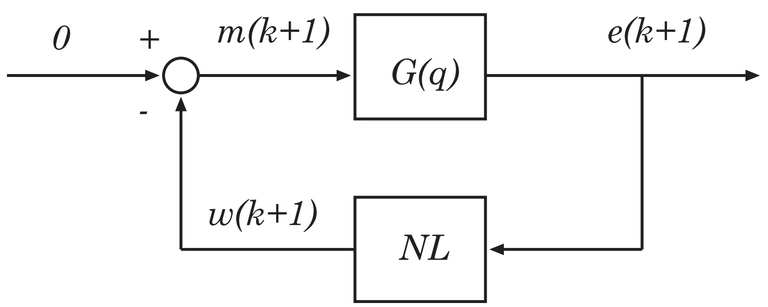

Figure 2: Equivalent Feedback Block Diagram

Fig. 1 has a linear time invariant system in its forward path ($G(q) = \frac{q}{q-a}$):

$$e(k+1) = a\, e(k) + m(k+1) \quad \Rightarrow \quad e(k) = \frac{q}{q-a}\, m(k)\,, \tag{12}$$

and a nonlinear system in its feedback path (NL):

$$\begin{aligned} \tilde{\theta}(k+1) &= \tilde{\theta}(k) - F\,\phi(k)\, e(k+1) \\ w(k) &= -\phi(k-1)^T \tilde{\theta}(k) \end{aligned} \tag{13}$$

The stability analysis is conducted as follows:

1. Verify that $G(q)$ in the forward path is a strictly positive real (SPR) transfer function.
2. Verify that the nonlinear block in the feedback path is a P-class (Passive) nonlinearity.
3. Use the sufficiency portion of the hyperstability and asymptotic hyperstability theorems to show that the a-posteriori error $e(k)$ is bounded and $\lim_{k\to\infty} e(k) = 0$.
4. Show that the a-priori error $e^o(k)$ is a bounded and converges to zero, by showing that that $\phi(k)$ is bounded.

There are three major differences between the discrete time stability approach and the continuous time counterpart:

(a) As in the case of the Kalman filter, it is necessary to first calculate the a-priori output estimation error $e^o(k)$ using the current parameter estimates, and subsequently update the a-posteriori output estimation error $e(k)$.

(b) In contrast to the continuous time case, it is not necessary to proof the boundness of the sequence $w(k)$ in Fig. 4, in order to use the sufficiency portion of the hyperstability theorem. Moreover, because the transfer function $G(z)$ must be causal but not strictly causal in order to be SPR, the boundness of $w(k)$ will follow from the boundness of $e(k)$.

(c) The hyperstability and asymptotic hyperstability theorems only grantee the boundness and convergence of the a-posteriori output error $e(k)$. The boundness and convergence of the a-priori output error $e^o(k)$ must be shown by further analysis.

Lets now perform each step in detail:

1) **Linear Forward Path:** From Eq. (12), $G(z)$ is given by

$$G(z) = \frac{z}{z-a}\,,$$

which is SPR since $|a| < 1$.

Notice that, in contrast to the continuous time case, $G(z)$ is causal, but not strictly causal, i.e. $G(z)$ has zero relative degree. This is a necessary condition for a transfer function to be SPR in discrete time systems.

2) **Nonlinear block in the feedback path:** It is necessary to verify that the input-output relation given by Eqs. (14) satisfies the Popov (passivity) innequality, i.e.

$$\sum_{j=0}^{k} w(j)e(j) \geq -\gamma_o^2\,.$$

To this end, notice that by Eq. (14) we have

$$\phi(k-1)\, e(k) = -F^{-1}\left[\tilde{\theta}(k) - \tilde{\theta}(k-1)\right]\,.$$

Therefore,

$$\begin{aligned}\sum_{j=0}^{k} w(j)e(j) &= -\sum_{j=0}^{k} \tilde{\theta}^T(j)\,\phi(j-1)e(j) \\ &= \sum_{j=0}^{k} \tilde{\theta}^T(j)F^{-1}\left\{\tilde{\theta}(j) - \tilde{\theta}(j-1)\right\}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{k} \left\{ \frac{1}{2}\tilde{\theta}^T(j)F^{-1}\tilde{\theta}(j) - \frac{1}{2}\tilde{\theta}^T(j-1)F^{-1}\tilde{\theta}(j-1) \right. \\
&\quad \left. + \left[\frac{1}{2}\tilde{\theta}^T(j)F^{-1}\tilde{\theta}(j) - \tilde{\theta}^T(j)F^{-1}\tilde{\theta}(j-1) + \frac{1}{2}\tilde{\theta}^T(j-1)F^{-1}\tilde{\theta}(j-1)\right]\right\} \\
&= \frac{1}{2}\,\tilde{\theta}^T(k)F^{-1}\tilde{\theta}(k) - \frac{1}{2}\,\tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0) + \sum_{j=0}^{k}\frac{1}{2}\,||F^{-1/2}[\tilde{\theta}(j) - \tilde{\theta}^T(j-1)]||^2 \\
&\geq -\gamma_o^2 \qquad (14)
\end{aligned}
$$

where $\gamma_o^2 = \frac{1}{2}\tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0)$.

3) By the sufficiency portion of the hyperstability theorem, we now conclude from Eq. (H-4)in pages HS-1 of the ME233 class notes that

$$
|e(k)| \leq \delta \left\{|e(0)| + \gamma_2^2\right\} ,
$$

where both $\delta > 0$ and $\gamma_2^2 > 0$ are bounded. Thus, $e(k)$ is bounded, i.e. $|e(k)| < \infty$.

Since $G(z)$ is SPR, the convergence of $e(k)$ to zero follows from the sufficiency portion of the Asymptotic Hyperstability Theorem in page HS-2 of the ME233 class notes.

4) **Convergence of $e^o(k)$ to zero:**

Notice that

$$
e(k+1) = \frac{e^0(k+1)}{1 + \phi(k)^T F \phi(k)} .
$$

Therefore, to show the convergence of the a-posteriori error $e^0(k)$ to zero, it is necessary to show that the repressor $\phi(k)$ remains bounded. This is done in an analogous manner to the continuous time case. Notice that, by assumption, $u(k)$ is bounded. Since, $G(z)$ is asymptotically stable, it is also bounded-input-bounded-output stable. Thus, $y(k)$ is also bounded. Now,

$$
e(k) = y(k) - \hat{y}(k)
$$

Therefore, by the triangle inequality,

$$
|\hat{y}(k)| \leq |y(k)| + |e(k)| < \infty
$$

Thus,

$$
||\phi(k)|| = \left(|\hat{y}(k)|^2 + |u(k)|^2\right)^{\frac{1}{2}} < \infty
$$

2.1.2 Lyapunov Function Approach

We will now prove the stability of the adaptive identification system, using the Lyapunov function approach. To this end, let us re-write the a-posteriori error dynamics and PAA estimation equations:

$$
\begin{aligned}
e(k+1) &= a\, e(k) + \phi^T(k)\, \tilde{\theta}(k+1) &\quad (15)\\
\tilde{\theta}(k+1) &= \tilde{\theta}(k) - F\, \phi(k)\, e(k+1) &\quad (16)
\end{aligned}
$$

2.1.3 A-posteriori error dynamics

Using the same notation as in Fig. 4, the a-posteriori error dynamics, Eq. (15) can be expressed by the following LTI state and output equations

$$
\begin{aligned}
x(k+1) &= A\, x(k) + B\, m(k) &\quad (17)\\
e(k) &= C\; x(k) + D\, m(k)
\end{aligned}
$$

where

$$
m(k) = \phi^T(k-1)\, \tilde{\theta}(k)\,, \quad (18)
$$

$x(k)$ is the state variable, and the constants are $A = a$, $|a| < 1$, $B = a$, $C = 1$ and $D = 1$.

Since the transfer function

$$
G(q^{-1}) = \frac{C\, B\; q^{-1}}{1 - A\, q^{-1}} + D = \frac{a\, q^{-1}}{1 - a\, q^{-1}} + 1 = \frac{q^{-1}}{1 - a\, q^{-1}}
$$

is SPR, then by the SPR4 lemma in section 2.3, there exist a Lyapunov positive constants $P > 0$ and $Q > 0$ and gains L and K such that the positive definite function

$$
V_1(x(k)) = \frac{1}{2}\, P x^2(k) \quad (19)
$$

satisfies

$$
V_1(x(k+1)) - V_1(x(k)) \;=\; e(k)m(k) - \frac{1}{2} Q x^2(k) - \frac{1}{2}\, |L\, x(k) + K\, m(k)|^2\,. \quad (20)
$$

Moreover, from Lemma SPR3 in Page 8 of the Hyperstability notes, the gain K satisfies

$$
K^2 \;=\; 2D - PB^2\,.
$$

2.1.4 Parameter adaptation algorithm (PAA)

Consider now the PAA in (16)

$$
\tilde{\theta}(k) \;=\; \tilde{\theta}(k-1) - F\, \phi(k-1)\, e(k)
$$

and notice that

$$\frac{1}{2}\tilde{\theta}^T(k-1)F^{-1}\tilde{\theta}(k-1) \;=\; \frac{1}{2}\tilde{\theta}^T(k)F^{-1}\tilde{\theta}(k) + \frac{1}{2}\phi^T(k-1)F\phi(k-1)\,e^2(k) + \tilde{\theta}^T(k)\phi(k-1)\,e(k)\,. \tag{21}$$

Define the positive definite function

$$V_2(\tilde{\theta}(k)) = \frac{1}{2}\tilde{\theta}^T(k)F^{-1}\tilde{\theta}(k) \tag{22}$$

From (21), and (18) we obtain

$$V_2(\tilde{\theta}(k)) - V_2(\tilde{\theta}(k-1)) \;=\; -m(k)e(k) - \frac{1}{2}\phi^T(k-1)F\phi(k-1)\,e^2(k)\,. \tag{23}$$

2.1.5 Stability in the sense of Lyapunov

Define the extended state by

$$x_e(k) = \begin{bmatrix} x(k) & \tilde{\theta}^T(k-1) \end{bmatrix}^T \in \mathcal{R}^3 \tag{24}$$

and define the Lyapunov function

$$V(x_e(k)) = V_1(x(k)) + V_2(\tilde{\theta}(k-1))\,. \tag{25}$$

From Eqs. (20) and (23) we obtain

$$\begin{aligned} V(x_e(k+1)) - V(x_e(k)) \;&=\; e(k)m(k) - \frac{1}{2}Qx^2(k) - \frac{1}{2}\,|L\,x(k) + K\,m(k)|^2 \\ &\quad -m(k)e(k) - \frac{1}{2}\phi^T(k-1)F\phi(k-1)\,e^2(k) \end{aligned} \tag{26}$$

$$\leq\; -\frac{1}{2}Qx^2(k) - \frac{1}{2}\,|L\,x(k) + K\,m(k)|^2\,. \tag{27}$$

Thus, by the Lyapunov direct method, since $V(x_e(k)) \succ 0$ is a positive definite function of the extended state $x_e(k)$ and $\Delta V(x_e(k)) = V(x_e(k+1)) - V(x_e(k)) \preceq 0$ is a negative **semi definite** function of the the extended state $x_e(k)$, we can conclude that the origin of the extended state space an equilibrium state that is stable in the sense of Lyapunov.

Moreover,

$$||x_e(0)|| \leq x_o < \infty \quad \Rightarrow \quad ||x_e(k)|| < \infty\,,$$

which implies that $|x(k)| < \infty$ and $\|\tilde{\theta}(k)\| < \infty$ remain bounded.

2.1.6 Convergence of $e(k)$ to zero

We first prove that $\lim_{k\to\infty} x(k) = 0$ and $\lim_{k\to\infty} |Km(k)| = 0$. From Eq. (26) we have

$$V(x_e(k+1)) - V(x_e(k)) \;\; \leq \;\; -\frac{1}{2}Qx^2(k) - \frac{1}{2}\left|L\,x(k) + K\,m(k)\right|^2 .$$

Since $Q > 0$ and summing both side of the equations from $k = 0$ to $k \to \infty$ we obtain

$$\lim_{k\to\infty} V(x_e(k)) = V(x_e(0)) - \frac{1}{2}Q\sum_{k=0}^{\infty} x^2(k) - \frac{1}{2}\sum_{k=0}^{\infty}\left|L\,x(k) + K\,m(k)\right|^2$$

Since $V(x_e(k)) \geq 0$, and $0 \leq V(x_e(0)) < \infty$, this implies that

$$\sum_{k=0}^{\infty} x^2(k) < \infty \quad \Rightarrow \quad \lim_{k\to\infty} x(k) = 0\,.$$

Moreover,

$$\sum_{k=0}^{\infty}\left|L\,x(k) + K\,m(k)\right|^2 < \infty \quad \Rightarrow \quad \lim_{k\to\infty}\left|L\,x(k) + K\,m(k)\right| = 0$$

and $\lim_{k\to\infty} |Km(k)| = 0$.

Since $\lim_{k\to\infty} x(k) = 0$, by the state equation (17) we also obtain that $\lim_{k\to\infty} |Bm(k)| = 0$. Finally from the last equation in (29)

$$K^2 \;\; = \;\; 2D - PB^2\,,$$

we also conclude that

$$\lim_{k\to\infty} Dm^2(k) = \frac{1}{2}\lim_{k\to\infty}\left[|Km(k)|^2 + P|Bm(k)|^2\right] = 0\,.$$

Therefore by the output equation (17), we conclude that

$$\lim_{k\to\infty} e(k) = \lim_{k\to\infty} Cx(k) + \lim_{k\to\infty} Dm(k) = 0\,.$$

Q.E.D.

2.1.7 Convergence of $e^o(k)$ to zero

See part 4 of section 2.1.1

2.2 SPR - 3 Lemma

Kalman Szegö, Popov Lemma (Page 8 in the Hyperstability notes).

Let the discrete-time LTI square MIMO transfer function $G(z) \in C^{p\times p}$ be strictly positive real (SPR). Then, for any minimal state space realization

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\;x(k) + D\,u(k)\,, \end{aligned} \tag{28}$$

there exists

(a) positive definite matrices $P \in R^{n\times n} \succ 0$, $Q \in R^{n\times n} \succ 0$,

(b) and matrices $K \in R^{m\times p}$, and $L \in R^{m\times n}$

such that

$$\begin{aligned} A^T P\,A - P &= -L^T L - Q \\ B^T P\,A + K^T L &= C \\ K^T K &= D + D^T - B^T P\,B\,. \end{aligned} \tag{29}$$

2.3 SPR - 4 Lemma

Let the discrete-time LTI square MIMO transfer function $G(z) \in C^{p\times p}$ be strictly positive real (SPR). Then, for any minimal state space realization there exists

(a) positive definite matrices $P \in R^{n\times n} \succ 0$, $Q \in R^{n\times n} \succ 0$,

(b) and matrices $K \in R^{m\times p}$, and $L \in R^{m\times n}$

such that the positive definite function

$$V(x(k)) = \frac{1}{2}\,x^T(k) P x(k) \tag{30}$$

satisfies

$$\begin{aligned} V(x(k+1)) - V(x(k)) &= y(k)^T u(k) - \frac{1}{2} x^T(k) Q x(k) \\ &\quad -\frac{1}{2}\left[L\,x(k) + K\,u(k)\right]^T \left[L\,x(k) + K\,u(k)\right]. \end{aligned} \tag{31}$$

.

2.3.1 Proof:

We utilize the Kalman Szegö, Popov Lemma SPR3 in S2.2.

Consider now the PDF $V(x(k))$ in Eq. (30) and calculate the difference $\Delta V(x(k+1))$ along the state trajectories

$$\begin{aligned} 2\,V(x(k+1)) - 2\,V(x(k)) &= [A\,x(k) + B\,u(k)]^T\,P\,[A\,x(k) + B\,u(k)] - 2\,V(x(k)) \\ &= x^T(k)[A^TPA - P]x(k) + x^T(k)A^TPBu(k) + u^T(k)B^TPAx(k) \\ &\quad +u^T(k)B^TPBu(k)\,. \end{aligned}$$

Utilizing the first and second equations in (29), we now obtain

$$\begin{aligned} 2\,V(x(k+1)) - 2\,V(x(k)) &= -x^T(k)\,[Q + L^T\,L]\,x(k) + x^T(k)[C^T - L^TK]u(k) \\ &\quad +u^T(k)[C - K^TL]x(k) + u^T(k)B^TPBu(k)\,. \end{aligned} \tag{32}$$

Adding and subtracting $u^T(k)[D^T + D]u(k)$ to the right hand side of (32) we obtain

$$\begin{aligned} 2\,V(x(k+1)) - 2\,V(x(k)) &= [Cx(k) + Du(k)]^Tu(k) + u^T(k)[Cx(k) + Du(k)] - x^T(k)Qx(k) \\ &\quad -x^T(k)L^TLx(k) - x^T(k)L^TKu(k) - u^T(k)K^TLx(k) \\ &\quad +u^T(k)[-D - D^T + B^TPB]u(k)\,. \end{aligned}$$

Finally, utilizing the output equation $y(k) = Cx(k) + Du(k)$ and the last equation in (29), we obtain

$$\begin{aligned} V(x(k+1)) - V(x(k)) &= y(k)^Tu(k) - \frac{1}{2}x^T(k)Qx(k) \\ &\quad -\frac{1}{2}\,[L\,x(k) + K\,u(k)]^T\,[L\,x(k) + K\,u(k)]\,. \end{aligned} \tag{33}$$

2.4 Sufficiency portion of the Asymptotic Hyperstability Theorem

Consider the feedback system in Fig. 3 where

(i) The discrete-time LTI square MIMO transfer function $G(z) \in C^{p\times p}$ be strictly positive real (SPR).

(ii) The discrete time square MIMO nonlinear block is P-class and satisfies, for any $k \geq 0$ and some constant γ_o, which is a function of the initial conditions,

$$\sum_{j=0}^{k} w^T(j)y(j) \geq -\gamma_o^2\,. \tag{34}$$

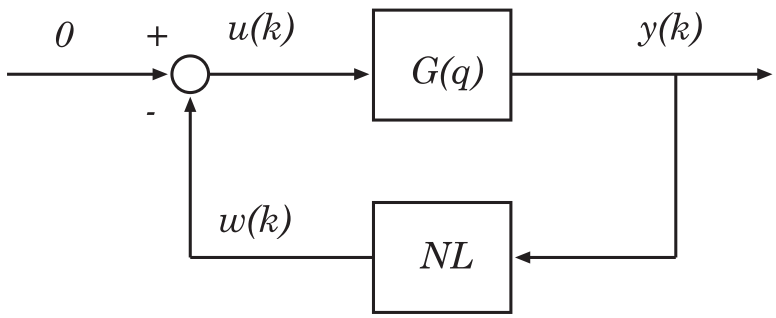

Figure 3: Asymptotically Hyperstable Feedback Block Diagram

Then, for any minimal state space realization of $G(z)$

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\;x(k) + D\,u(k)\,, \end{aligned} \tag{35}$$

the following are true:

1. The exist bounded constants $\delta_1 > 0$ and $\delta_2 \geq 0$, such that, for all $k > 0$,

$$\|x(k)\|^2 \leq \delta_1 \left[\|x(0)\|^2 + \delta_2\right]$$

2. $\lim_{k\to\infty} x(k) = 0$,
3. If $(D + D^T) \succ 0$, $\lim_{k\to\infty} u(k) = 0$ and $\lim_{k\to\infty} y(k) = 0$.

2.4.1 Proof:

From lemma SPR - 4, there exists

(a) positive definite matrices $P \in R^{n\times n} \succ 0$, $Q \in R^{n\times n} \succ 0$,

(b) and matrices $K \in R^{m\times p}$, and $L \in R^{m\times n}$

such that the positive definite function

$$V(x(k)) = \frac{1}{2}\,x^T(k)Px(k)$$

satisfies

$$V(x(k+1)) - V(x(k)) \;=\; y(k)^T u(k) - \frac{1}{2}x^T(k)Qx(k) - \frac{1}{2}\,\|L\,x(k) + K\,u(k)\|^2\,. \tag{36}$$

Rearranging (36), and summing from $k \in [0, m)$,

$$\begin{aligned} \sum_{k=0}^{m} [V(x(k+1)) - V(x(k))] &= -\sum_{k=0}^{m} y(k)^T w(k) - \frac{1}{2}\sum_{k=0}^{m} x^T(k)Qx(k) \\ &\quad -\frac{1}{2}\sum_{k=0}^{m} \|L\,x(k) + K\,u(k)\|^2\,. \end{aligned}$$

1. Utilizing (34),

$$\begin{aligned} V(x(m+1)) &= V(0) - \sum_{k=0}^{m} y(k)^T w(k) - \frac{1}{2}\sum_{k=0}^{m} x^T(k)Qx(k) - \frac{1}{2}\sum_{k=0}^{m} \|L\,x(k) + K\,u(k)\|^2 \\ &\leq V(0) + \gamma_o^2 - \frac{1}{2}\sum_{k=0}^{m} x^T(k)Qx(k) - \frac{1}{2}\sum_{k=0}^{m} \|L\,x(k) + K\,u(k)\|^2 \qquad (37) \end{aligned}$$

Therefore,

$$V(x(k)) \leq V(0) + \gamma_o^2$$

for all $k > 0$ and, defining the maximum and minimum eigenvalues of $P \succ 0$ respectively as $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$,

$$\lambda_{\min}(P)\|x(t)\|^2 \leq \lambda_{\max}(P)\|x(0)\|^2 + 2\gamma_o^2\,.$$

Thus,

$$\|x_k\|^2 \;\leq\; \frac{\lambda_{\max}(P)}{\lambda_{min}(P)}\left[\|x_0\|^2 + \frac{2}{\lambda_{max}(P)}\gamma_0^2\right]\,.$$

2. Taking the limit as $m \to \infty$ in (37), we obtain

$$\lim_{m\to\infty} V(x(m)) \;=\; V(0) + \gamma_o^2 - \frac{1}{2}\sum_{k=0}^{\infty} x^T(k)Qx(k) - \frac{1}{2}\sum_{k=0}^{\infty} \|L\,x(k) + K\,u(k)\|^2 \qquad (38)$$

Since $P \succ 0$ and $Q \succ 0$, which in turn implies that $V(x(m)) \succ 0$,

$$\frac{1}{2}\sum_{k=0}^{\infty} x^T(k)Qx(k) < 0 \quad\Rightarrow\quad \lambda_{\min}(Q)\frac{1}{2}\sum_{k=0}^{\infty}\|x(k)\|^2 < 0 \quad\Rightarrow\quad \lim_{k\to\infty} x(k) = 0\,.$$

3. From (38), we also obtain and

$$\frac{1}{2}\sum_{k=0}^{\infty}\|L\,x(k) + K\,u(k)\|^2 < 0 \quad\Rightarrow\quad \lim_{k\to\infty}\|L\,x(k) + K\,u(k)\| = 0$$

Since, $\lim_{k\to\infty} x(k) = 0$ the above equation implies that

$$\lim_{k\to\infty} Ku(k) = 0\,.$$

Moreover, from the state equation, we also obtain

$$\lim_{k\to\infty} Bu(k) = 0\,.$$

Now, from the the Kalman, Szegö, Popov Lemma SPR3 we have

$$D + D^T = K^T\, K + B^T\, P\, B$$

Therefore,

$$\lim_{k\to\infty} u^T(k) \left[D + D^T\right] u(k) = \lim_{k\to\infty} u^T(k) \left[K^T\, K + B^T\, P\, B\right] u(k) = 0\,.$$

and $[D + D^T] \succ 0$ implies that $\lim_{k\to\infty} u(k) = 0$.
Since $y(k) = Cx(k) + Du(k)$, then $\lim_{k\to\infty} y(k) = 0$.

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Recursive Least Squares Estimation

This handout contains notes from Chapter 2 of (Astrom and Wittenmark, 1995). However, the sampling index convention follows the ME233 class notes.

3 Derivation of the least squares estimation algorithm

Consider the identification of a set of unknown but constant parameters θ_1, $\theta_2 \cdots \theta_n$, which form part of a system which has the following structure

$$y(k) = \phi_1(k-1)\,\theta_1 + \phi_2(k-1)\,\theta_2 + \cdots + \phi_n(k-1)\,\theta_n\,, \tag{1}$$

where $y(k)$ is the observed system output and $\phi_i(k)$, $i = 1, \cdots, n$ are known functions and k is the sampling index.

Eq. (1) can be rewritten in vector form

$$y(k) = \phi^T(k-1)\,\theta \tag{2}$$

where $\phi(k-1) = \begin{bmatrix} \phi_1(k-1) & \cdots & \phi_n(k-1) \end{bmatrix}^T \in \Re_n$ is the regressor vector and $\theta = \begin{bmatrix} \theta_1 & \cdots & \theta_n \end{bmatrix}^T \in \Re_n$ is the unknow paramter vector.

Let us now consider the estimation of the unknown parameter vector $\hat{\theta}$ after collecting k measurements of y and the regressor vector ϕ. This problem can be cast as the determination of the parameter estimate vector $\hat{\theta}(k) = \begin{bmatrix} \hat{\theta}_1(k) & \cdots & \hat{\theta}_n(k) \end{bmatrix}^T \in \Re_n$ which minimizes the following least squares cost function

$$V(\hat{\theta}(k)) = \frac{1}{2} \sum_{j=1}^{k} \left[y(j) - \phi^T(j-1)\,\hat{\theta}(k) \right]^2 \tag{3}$$

Notice that the measured variable $y(k)$ is a linear function of the unknown parameter vector $\hat{\theta}$ and the least squares cost function $V(\hat{\theta}(k))$ is a quadratic function of $\hat{\theta}(k)$. Thus, it has an analytical solution. Let us rewrite Eq. (3) in matrix form

$$V(\hat{\theta}(k)) = \frac{1}{2} \left[Y(k) - \Phi^T(k-1)\,\hat{\theta}(k) \right]^T \left[Y(k) - \Phi^T(k-1)\,\hat{\theta}(k) \right] \tag{4}$$

where

$$Y(k) \;=\; \begin{bmatrix} y(1) & \cdots & y(k) \end{bmatrix}^T \in \Re^k \tag{5}$$

$$\Phi(k-1) \;=\; = \begin{bmatrix} \phi(0) \cdots \phi(k-1) \end{bmatrix} = \begin{bmatrix} \phi_1(0) & \cdots & \phi_1(k-1) \\ \phi_2(0) & \cdots & \phi_2(k-1) \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \phi_n(0) & \cdots & \phi_n(k-1) \end{bmatrix} \in \Re^{k \times n}\,. \tag{6}$$

Eq. (4) can be minimized be solving $\frac{\partial V(\hat{\theta}(k))}{\partial \hat{\theta}(k)} = 0$, which yields

$$\Phi(k-1)\Phi(k-1)^T \hat{\theta}(k) = \Phi(k-1)\,Y(k) \tag{7}$$

which can be solved as follows

$$\hat{\theta}(k) = \left[\Phi(k-1)\Phi(k-1)^T\right]^\sharp \, \Phi(k-1)\,Y(k)$$

where $A^\sharp$ denotes the pseudoinverse of A and satisfies the following properties:

- $A^\sharp$ has the same dimensions as A^T
- $A\,A^\sharp\,A = A$ and $A^\sharp\,A\,A^\sharp = A^\sharp$
- $A\,A^\sharp = (A^\sharp)^T A^T$.

Let us now define the symmetric and positive semi-definite matrix $F(k)$ by its pseudo-inverse:

$$F^\sharp(k) = \Phi(k-1)\Phi^T(k-1) = \phi(0)\phi(0)^T + \cdots \phi(k-1)\phi(k-1)^T \in \Re^{n\times n}\,. \tag{8}$$

Assume that $k \geq n$ and that the sequence of regressor vectors $\phi(i)$ $i = \{1, \cdots, k\}$ is persistently exciting [1] so that the matrix $\Phi(k-1)$ in (6) has rank n and $F(k)$ in Eq. (8) is nonsingular. In this case we have

$$F^{-1}(k) = \Phi(k-1)\Phi^T(k-1) = \phi(0)\phi(0)^T + \cdots \phi(k-1)\phi(k-1)^T \in \Re^{n\times n} \tag{9}$$

and

$$\hat{\theta}(k) = F(k)\,\Phi(k-1)\,Y(k) = F(k)\sum_{j=1}^{k}\phi(j-1)\,y(j)\,. \tag{10}$$

3.1 Recursive Least Squares Parameter Adaptation Algorithm

Assume that we have calculated Eq. (10) for $k-1$, i.e.

$$\hat{\theta}(k-1) = F(k-1)\,\Phi(k-2)\,Y(k-1)\,,$$

and we subsequently collect a new data set $y(k)$ and $\phi(k)$. We wish to calculate the parameter estimate $\theta(k)$ which minimizes Eq. (3) utilizing $\theta(k-1)$, $F(k-1)$, $y(k)$ and $\phi(k)$ and without having to recalculate Eq. (10). This can be done utilizing the following recursive least squares (RLS) parameter adaptation algorithm (PAA):

$$\begin{aligned} e^o(k) &= y(k) - \phi(k-1)^T\,\hat{\theta}(k-1) && (11)\\ \hat{\theta}(k) &= \hat{\theta}(k-1) + F(k)\,\phi(k-1)\,e^o(k) && (12)\\ F(k) &= F(k-1) - \frac{F(k-1)\,\phi(k-1)\,\phi(k-1)^T\,F(k-1)}{1+\phi(k-1)^T\,F(k-1)\,\phi(k-1)} && (13)\end{aligned}$$

[1] We will discuss persistence of excitation in the next sections.

Derivation of Eq. (24)

Assume that $F(k-1)$ is nonsingular. Thus, from Eq. (9),

$$\begin{aligned} F(k)^{-1} &= \sum_{i=0}^{k-1} \phi(i)\,\phi(i)^T \\ &= F(k-1)^{-1} + \phi(k-1)\,\phi(k-1)^T\,. \end{aligned}$$

Also, from Eq. (10) we have

$$\begin{aligned} \hat{\theta}(k) &= F(k) \sum_{i=1}^{k} \phi(i-1)\,y(i) = F(k) \left[\sum_{i=1}^{k-1} \phi(i-1)\,y(i) + \phi(k-1)\,y(k)\right] \\ &= F(k)\,\left[F(k-1)^{-1}\,\hat{\theta}(k-1) + \phi(k-1)\,y(k)\right] \\ &= F(k)\,\left[\left(F(k)^{-1} - \phi(k-1)\,\phi(k-1)^T\right)\,\hat{\theta}(k-1) + \phi(k-1)\,y(k)\right] \\ &= \hat{\theta}(k-1) + F(k)\,\phi(k-1)\,\left[y(k) - \phi(k-1)^T\,\hat{\theta}(k-1)\right] \end{aligned}$$

Eq. (24) follows from the definition of $e^o(k)$ given by Eq. (11).

Derivation of Eq. (13)

Assume that $F(k-1)$ is nonsingular. Then, from Eq. (9),

$$F(k)^{-1} = F(k-1)^{-1} + \phi(k-1)\,\phi(k-1)^T\,. \tag{14}$$

We will now derive a simplified form of the well-known matrix inversion lemma. Multiplying (14) by $F(k)$ on the left and $F(k-1)$ on the right we obtain

$$F(k-1) = F(k) + F(k)\,\phi(k-1)\,\phi(k-1)^T\,F(k-1)\,. \tag{15}$$

Multiplying (15) by $\phi(k)$ to the right and doing a bit of algebra we obtain

$$F(k)\,\phi(k) = \frac{F(k-1)\,\phi(k-1)}{1 + \phi(k-1)^T\,F(k-1)\,\phi(k-1)} \tag{16}$$

Substituting (16) into (15) we obtain Eq. (13).

4 Statistical Interpretation and Persistence of Excitation

Assume that a process is described by the following equation

$$y(k) = \phi^T(k-1)\,\theta + \epsilon(k) \tag{17}$$

where $\phi(k) = \begin{bmatrix} \phi_1(k) & \cdots & \phi_n(k) \end{bmatrix}^T \in \Re_n$ is the regressor vector, $\theta = \begin{bmatrix} \theta_1 & \cdots & \theta_n \end{bmatrix}^T \in \Re_n$ is the unknow paramter vector and $\epsilon(k)$ is a random sequence which is zero mean. Moreover assume that ϵ is *independent* of the regressor vector ϕ for all k.

Define, as in Eqs. (5) and (6),

$$
\begin{aligned}
Y(k) &= \begin{bmatrix} y(1) & \cdots & y(k) \end{bmatrix}^T \\
\mathcal{E}(k) &= \begin{bmatrix} \epsilon(1) & \cdots & \epsilon(k) \end{bmatrix}^T \\
\Phi(k-1) &= \begin{bmatrix} \phi(0)\cdots\phi(k-1) \end{bmatrix}.
\end{aligned}
$$

Then Eq. (17) can be rewriten as

$$
Y(k) = \Phi^T(k-1)\,\theta + \mathcal{E}(k)\,. \tag{18}
$$

The least squares parameter estimate is given by Eq. (7), which is rewritten below

$$
\Phi(k-1)\Phi^T(k-1)\,\hat{\theta}(k) = \Phi(k-1)\,Y(k)\,.
$$

Define the parameter estimate error

$$
\tilde{\theta}(k) = \theta - \hat{\theta}(k)\,. \tag{19}
$$

from Eqs. (18) and (7) we obtain

$$
\Phi(k-1)\Phi^T(k-1)\,\hat{\theta}(k) = \Phi(k-1)\Phi^T(k-1)\,\theta + \Phi(k-1)\mathcal{E}(k)\,.
$$

Thus,

$$
\Phi(k-1)\Phi^T(k-1)\,\tilde{\theta}(k) = -\Phi(k-1)\mathcal{E}(k)\,. \tag{20}
$$

Assume now that $\hat{\theta}(k)$ converges and define

$$
\bar{\theta} = \lim_{k\to\infty} \tilde{\theta}(k)\,. \tag{21}
$$

Multiplying (20) by $1/k$ and taking limits as $k \to \infty$ we obtain

$$
\begin{aligned}
\lim_{k\to\infty}\left\{\frac{1}{k}\Phi(k-1)\Phi^T(k-1)\,\tilde{\theta}(k)\right\} &= -\lim_{k\to\infty}\left\{\frac{1}{k}\,\Phi(k-1)\mathcal{E}(k)\right\} \\
\lim_{k\to\infty}\left\{\frac{1}{k}\Phi(k-1)\Phi^T(k-1)\right\}\,\bar{\theta} &= -\lim_{k\to\infty}\left\{\frac{1}{k}\,\Phi(k-1)\mathcal{E}(k)\right\}
\end{aligned}
$$

Let us now assume *ergodicity* and recall that $E\{\mathcal{E}\} = 0$ and $E\{\phi\epsilon\} = E\{\phi\}\,E\{\epsilon\} = 0$ (independence). Thus,

$$
E\{\phi\phi^T\}\,\bar{\theta} = -E\{\phi\epsilon\} = 0
$$

or

$$
\lim_{k\to\infty}\left\{\frac{1}{k}\Phi(k-1)\Phi^T(k-1)\right\}\,\bar{\theta} = 0\,. \tag{22}
$$

Eq. (22) provides necessary and sufficient conditions for the parameter estimates to converge to the true parameter θ. We now define the *excitation matrix* $C_n \in \Re^{n\times n}$ by

$$
\begin{aligned}
C_n &= \lim_{k\to\infty}\left\{\frac{1}{k}\Phi(k-1)\Phi^T(k-1)\right\} = \lim_{k\to\infty}\left\{\frac{1}{k}\sum_{j=0}^{k-1}\phi(j)\phi^T(j)\right\} \\
&= E\{\phi\phi^T\}
\end{aligned} \tag{23}
$$

Notice that C_n is symmetric and at least positive semi definite.

Theorem

Assume that data is generated by Eq. (17), where $\epsilon(k)$ is a random sequence which is zero mean and *independent* of the regressor vector ϕ for all k. Than the necessary and sufficient condition for

$$\bar{\theta} = \lim_{k\to\infty} \tilde{\theta}(k) = 0$$

is that the matrix C_n defined in (23) be positive definite.

The following are equivalent definitions

Definition Persistently Exciting Regressor $\phi(k)$

1) The regressor vector $\phi(k) \in \Re^n$ is persistently exciting iff the matrix C_n is positive definite.

2) The regressor vector $\phi(k) \in \Re^n$ is persistently exciting iff there exists an integer $m > 0$ and constants $0 < \rho_1 < \rho_2 < \infty$ such that

$$\rho_2 \, I_n \geq \sum_{j=k}^{k+m} \phi(j)\phi(j)^T \geq \rho_1 \, I_n \tag{24}$$

for all $k \geq 0$, where I_n is the $n \times n$ identity matrix.

Eq. (24) means that the matrix $\sum_{j=k}^{k+m} \phi(j)\phi(j)^T$ is positive definite and bounded. [2]

4.1 Conditions for Persistence of Excitation (PE)

We now examine what conditions must the input to a system satisfy in order to satisfy the PE condition. To do so, we need to make our input/output model representation more concrete. We will thus, consider two dynamic input/output models: 1) finite impulse response (FIR) models and 2) auto regressive moving average (ARMA) models.

4.1.1 PE for Finite Impulse Response (FIR) Models

Consider the following FIR model

$$y(k+1) = b_o \, u(k) + b_1 \, u(k-1) + \cdots + b_{n-1} \, u(k-n+1) \tag{25}$$

Eq. (25) can be rewritten as (2) with

$$\begin{aligned} \phi(k) &= \begin{bmatrix} u(k) & u(k-1)\cdots & u(k-n+1) \end{bmatrix}^T \in \Re^n && (26) \\ \theta &= \begin{bmatrix} b_o & b_1 \cdots & b_{n-1} \end{bmatrix}^T \in \Re^n && (27) \end{aligned}$$

[2] Consider a square matrix P. Then, $P > \rho I_n \Rightarrow \sigma_{\min}(P) > \rho$, where $\sigma_{\min}(P)^2 = \lambda_{\min}(P^T P)$ is the minimum singular value of the matrix P. Similarly, $P < \rho I_n \Rightarrow \sigma_{\max}(P) < \rho$, where $\sigma_{\max}(P)^2 = \lambda_{\max}(P^T P)$ is the maximum singular value of the matrix P.

Definition Persistently exciting FIR input sequence $u(k)$

The input sequence $u(k)$ is persistently exciting of order n iff the regressor vector $\phi(k) \in \Re^n$ in (26) is persistently exciting.

For the FIR model, the excitation matrix C_n defined in (23) is a Toeplitz matrix

$$C_n = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ c_{1n} & \cdots & c_{nn} \end{bmatrix}$$

where its elements

$$\begin{aligned} c_{ij} &= c_{ji} = \lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} u(k)\, u(k+i-j) \\ &= E\{u(k)\, u(k+i-j)\} = R_{uu}(i-j) \end{aligned}$$

are the correlation of the input signal $u(k)$.

The following theorem provides a method for determining the persistence of excitation order of a signal $u(k)$.

Theorem

The signal $u(k)$ is PE of order n iff

$$U = E\{(A(q^{-1})u(k))^2\} = \lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} \Big((A(q^{-1})u(j) \Big)^2 > 0 \tag{28}$$

for all non-zero polynomials $A(q^{-1})$ of order $n-1$ or less.

Proof:

Let

$$A(q^{-1}) = a_o + a_1 q^{-1} + \cdots + a_{n-1}\, q^{n-1}$$

define $a = \begin{bmatrix} a_o \cdots a_{n-1} \end{bmatrix}^T \in \Re^n$ and notice that

$$A(q^{-1})u(k) = a^T \phi(k)\,.$$

From (28) and (23)

$$U = E\{(A(q^{-1})u(k))^2\} = E\{a^T\, \phi(k)\phi(k)^T\, a\} = a^T\, E\{\phi(k)\phi(k)^T\}\, a = a^T\, C_n\, a\,.$$

Thus, for all non-zero vectors $a \in \Re^n$, $U \neq 0$ iff C_n is positive definite, which implies that $\phi(k)$ is PE.

Example Pulse

Consider the pulse input

$$u(k) = \delta(k) = \begin{cases} 1 & \text{for} \quad k = 0 \\ 0 & \text{for} \quad k > 0 \end{cases} .$$

Notice that

$$\lim_{k\to\infty} \frac{1}{k}\delta(k)^2 = 0 .$$

Thus, the pulse input is not PE for any n.

Example Step input

Consider the unit step input

$$u(k) = 1(k) = \begin{cases} 1 & \text{for} \quad k \geq 0 \\ 0 & \text{for} \quad k < 0 \end{cases} .$$

Since

$$(1 - q^{-1})u(k) = 0$$

for $k > 0$, the step input is at most PE of order n = 1. Since

$$C_1 = \lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} 1 = 1 ,$$

it follows that the step input is PE or order 1.

Example Sinusoid input

Consider the pure sinusoid input

$$u(k) = \sin(\omega\, k) .$$

Since

$$(1 - 2\cos(\omega)q^{-1} + q^{-2})u(k) = 0 \tag{29}$$

for $k > 2$, the pure sinusoid input is at most PE of order n = 2. [3]

[3]Notice that

$$\mathcal{Z}\{\sin(\omega\, k)\} = \sum_{k=0}^{\infty} z^{-k} \sin(\omega\, k) = \frac{\sin(\omega)\, z}{z^2 - 2\cos(\omega)\, z + 1} .$$

Therefore, $z^2 - 2\cos(\omega)\, z + 1$ is the characteristic polynomial of a dynamic system which has as a response mode $\sin(\omega\, k)$. Thus, Eq. (29) must be satisfied since every response mode satisfies its own characteristic

Let $\phi(k) = \begin{bmatrix} u(k) & u(k-1) \end{bmatrix}^T$. Since

$$
\begin{aligned}
C_2 &= \lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} \phi(j)\phi(j)^T = \lim_{k\to\infty} \frac{1}{k} \begin{bmatrix} \sum_{j=1}^{k} u(j)^2 & \sum_{j=1}^{k} u(j)u(j-1) \\ \sum_{j=1}^{k} u(j)u(j-1) & \sum_{j=1}^{k} u(j-1)^2 \end{bmatrix} \\
&= \lim_{k\to\infty} \frac{1}{k} \begin{bmatrix} \sum_{j=1}^{k} \sin^2(\omega\, j) & \sum_{j=1}^{k} \sin(\omega\, j)\sin(\omega\,(j-1)) \\ \sum_{j=1}^{k} \sin(\omega\, j)\sin(\omega\,(j-1)) & \sum_{j=1}^{k} \sin^2(\omega\,(j-1)) \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 1 & \cos(\omega) \\ \cos(\omega) & 1 \end{bmatrix}
\end{aligned}
$$

is positive definite, it follows that the pure sinusoid input is PE of order n = 2.

Notice that

$$
c_{11} = \lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} \sin^2(\omega\, j) = \lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} \frac{1}{2}\left(1 - \cos(2\omega k)\right) = \frac{1}{2}
$$

and

$$
\begin{aligned}
c_{12} &= \lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} \sin(\omega\, j)\sin(\omega\,(j-1)) = \lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} \Big(\sin^2(\omega\, j)\, \cos(\omega) - \sin(\omega\, j)\, \cos(\omega\, j)\sin(\omega)\Big) \\
&= \frac{1}{2}\cos(\omega)\,.
\end{aligned}
$$

PE of filtered signals

Let $u(k)$ be PE of order n. Assume that $A(q^{-1})$ is a polynomial of degree $m < n$. Then, the signal

$$
v(k) = A(q^{-1})u(k)
$$

is PE of order r where $n - m \le r \le n$. Assume that $A^*(q) = q^m A(q^{-1})$ has all roots inside the unit circle (i.e. $A(q^{-1})$ is Schur), then the signal

$$
w(k) = \frac{1}{A(q^{-1})}\, u(k)
$$

is also PE of order n.

---

polynomial. Indeed, direct substitution reveals that

$$
\begin{aligned}
&(1 - 2\cos(\omega)q^{-1} + q^{-2})\sin(\omega\, k) = \sin(\omega\, k) - 2\cos(\omega)\sin(\omega\,(k-1)) + \sin(\omega\,(k-2)) \\
&= \sin(\omega\, k) - 2\cos(\omega)\left(\sin(\omega k)\cos(\omega) - \cos(\omega k)\sin(\omega)\right) + \left(\sin(\omega k)\cos(2\omega) - \cos(\omega k)\sin(2\omega)\right) \\
&= \sin(\omega\, k)(1 - 2\cos(\omega)^2) + 2\cos(\omega k)\sin(\omega)\cos(\omega) + \sin(\omega k)\cos(2\omega) - \cos(\omega k)\sin(2\omega) \\
&= 0
\end{aligned}
$$

by the fact that $(1 - 2\cos(\omega)^2) = -\cos(2\omega)$ and $\sin(2\omega) = 2\sin(\omega)\cos(\omega)$.

4.1.2 PE for auto regressive moving average (ARMA)models

Consider the following ARMA model

$$A(q^{-1})y(k) = q^{-d}\,B(q^{-1})u(k) \tag{30}$$

where

$$\begin{aligned} A(q^{-1}) &= 1 + a_1 q^{-1} + \cdots + a_n q^{-n} && (31)\\ B(q^{-1}) &= b_o + b_1 q^{-1} + \cdots + b_m q^{-m} && (32) \end{aligned}$$

and assume that $A^*(q) = q^n A(q^{-1})$ has all roots inside the unit circle (i.e. $A(q^{-1})$ is Schur). Eq. (1) can be rewritten as (2) with

$$\begin{aligned} \phi(k-1) &= \left[\begin{array}{cccccc} -y(k-1) & \cdots & -y(k-n) & u(k-d)\cdots & u(k-d-m) \end{array}\right]^T \in \Re^{n+m+1} && (33)\\ \theta &= \left[\begin{array}{ccccc} a_1 & \cdots a_n & b_o & \cdots & b_m \end{array}\right]^T \in \Re^{n+m+1} && (34) \end{aligned}$$

Theorem

Consider the parameter estimation of the ARMA system (1), using the LS estimation algorithm (10) or the recursive PAA given by Eqs. (11)-(13) with

$$\hat{\theta}(k) = \left[\begin{array}{ccc} \hat{a}_1(k)\cdots\hat{a}_n(k)\,\hat{b}_o(k) & \cdots & \hat{b}_m(k) \end{array}\right]^T \in \Re^{n+m+1}$$

and $\phi(k)$ given by (33). If $A(q^{-1})$ is Schur, $A(q^{-1})$ and $B(q^{-1})$ are co-prime (i.e. no pole-zero cancellation) and $u(k)$ is PE of order $n+m+1$ then $\phi(k)$ is PE of order $n+m+1$ and

$$\lim_{k\to\infty} \tilde{\theta}(k) = \lim_{k\to\infty} (\theta - \hat{\theta}(k)) = 0$$

Proof:

Assume that

$$\lim_{k\to\infty} \tilde{\theta}(k) = \lim_{k\to\infty} (\theta - \hat{\theta}(k)) = \theta - \hat{\theta} = \bar{\theta}$$

and define the signal

$$v(k) = \phi(k-1)^T \bar{\theta}$$

Notice that

$$\lim_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k} v^2(k) = \bar{\theta}^T \left\{ \lim_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k} \phi(j-1)\phi(j-1)^T \right\} \bar{\theta} = \bar{\theta}^T\, C_n\, \bar{\theta}\,.$$

Thus, we need to show that

$$\lim_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k} v^2(k) = 0 \Longleftrightarrow \bar{\theta} = 0$$

From Eqs. (33)-(35) We have

$$v(k) = q^{-d}\,\bar{B}(q^{-1})\,u(k) - \bar{A}(q^{-1})y(k)\,,$$

where $\bar{A}(q^{-1}) = A(q^{-1}) - \hat{A}(q^{-1})$, $\bar{B}(q^{-1}) = B(q^{-1}) - \hat{B}(q^{-1})$ and

$$\begin{aligned}\hat{A}(q^{-1}) &= 1 + \hat{a}_1 q^{-1} + \cdots + \hat{a}_n q^{-n}\\ \hat{B}(q^{-1}) &= \hat{b}_o + \hat{b}_1 q^{-1} + \cdots + \hat{b}_m q^{-m}\end{aligned}$$

are the polynomials formed by the parameter estimates. Thus,

$$\begin{aligned}v(k) &= q^{-d}\,\bar{B}(q^{-1})\,u(k) - \bar{A}(q^{-1})\frac{q^{-d}\,B(q^{-1})}{A(q^{-1})}u(k)\\ &= q^{-d}\left[\bar{B}(q^{-1})\,A(q^{-1}) - \bar{A}(q^{-1})\,B(q^{-1})\right]\frac{1}{A(q^{-1})}\,u(k)\,.\end{aligned}$$

Notice that the polynomial inside the braquets, $\bar{B}(q^{-1})\,A(q^{-1}) - \bar{A}(q^{-1})\,B(q^{-1})$, is order $n+m$. Thus, since the signal $u(k)$ is PE of order $n+m+1$ and $A(q^{-1})$ is Schur by the assumptions, the signal

$$w(k) = \frac{1}{A(q^{-1})}\,u(k)$$

is also PE of order $n+m+1$. Therefore, the signal $v(k)$ will be of PE of order r, where $1 \le r \le n+m+1$, unless

$$\bar{B}(q^{-1})\,A(q^{-1}) - \bar{A}(q^{-1})\,B(q^{-1}) = 0\,. \tag{35}$$

It remains to be shown that Eq. (35) is true iff $\bar{\theta} = 0$.

For simplicity assume that

$$A(q^{-1}) = 1 + a_1 q^{-1} + a_2 q^{-2}$$

and

$$B(q^{-1}) = b_o + q^{-1} b_1$$

Then, expanding (35), we obtain

$$(\bar{b}_o + q^{-1}\,\bar{b}_1)(1 + a_1 q^{-1} + a_2 q^{-2}) - (\bar{a}_1 q^{-1} + \bar{a}_2 q^{-2})(b_o + q^{-1} b_1) = 0$$

which can be written in vector form as follow

$$\begin{bmatrix} 0 & 0 & 1 & 0\\ -b_o & 0 & a_1 & 1\\ -b_1 & -b_0 & a_2 & a_1\\ 0 & -b_1 & 0 & a_2 \end{bmatrix}\begin{bmatrix}\bar{a}_1\\ \bar{a}_2\\ \bar{b}_0\\ \bar{b}1\end{bmatrix} = 0 \tag{36}$$

Eq. (36) is of the form

$$M\,\bar{\theta} = 0\,,$$

where the matrix M is as the Sylvester matrix of the polynomials $A(q^{-1})$ and $B(q^{-1})$. It is well known that M is nonsingular iff these polynomials are co-prime (i.e. they don't have common roots). As a consequence, the signal $v(k)$ is at least PE of order 1, unless $\bar{\theta} = 0$. Thus,

$$\lim_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k} v^2(k) = 0 \Longleftrightarrow \bar{\theta} = 0$$

Q.E.D.

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Adaptive Identification -Series Parallel Model

5 Discrete Time

We consider the recursive parameter identification of the following LTI, SISO, ARMA system:

$$A(q^{-1})y(k+1) = B(q^{-1})u(k) \tag{1}$$

where

- $u(k) \in \mathcal{R}$ is a known *bounded* input, i.e. $|u(k)| < \infty$.
- $y(k) \in \mathcal{R}$ is the measured output.
- $A(q^{-1}) = 1 + a_1 q^{-1} + \cdots + a_n q^{-n}$ is *Schur* and monic polynomial.
- $B(q^{-1}) = b_o + b_1 q^{-1} + \cdots + b_m q^{-m}$.
- a_i's and b_i's are unknown but *constant* parameters.

Eq. (1) can be rewritten as follows

$$\begin{aligned} y(k+1) &= -\sum_{i=1}^{n} a_i \, y(k-i+1) + \sum_{i=0}^{m} b_i \, u(k-i) \\ &= \theta^T \phi(k) \end{aligned} \tag{2}$$

where

Unknown parameter vector : $\theta = \begin{bmatrix} a_1 & \cdots & a_n & b_o \cdots & b_m \end{bmatrix}^T \in \mathcal{R}^{n+m+1}$.

Regressor vector : $\phi(k) = \begin{bmatrix} -y(k) & \cdots & -y(k-n) & u(k) & \cdots & u(k-m) \end{bmatrix}^T \in \mathcal{R}^{n+m+1}$.

We now define the a-posteriori series-parallel estimation model

$$\hat{y}(k+1) = -\sum_{i=1}^{n} \hat{a}_i(k+1) \, y(k-i+1) + \sum_{i=0}^{m} \hat{b}_i(k+1) \, u(k-i) \tag{3}$$

where $\hat{y}(k)$ is the *a-posteriori* estimate of $y(k)$ and $\hat{a}_i(k+1$'s and $\hat{b}_i(k+1$'s are the estimates of the unknown parameters a_i's and b_i's. Eq. (3) can be rewritten as follows

$$\hat{y}(k+1) = \hat{\theta}^T(k+1) \, \phi(k) \tag{4}$$

where

A-priori output estimate : $\hat{y}(k)$.

Parameter estimate vector : $\hat{\theta}(k) = \left[\begin{array}{cccc} \hat{a}_1(k) & \cdots & \hat{a}_n(k) & \hat{b}_o(k) \cdots & \hat{b}_m(k) \end{array} \right]^T \in \mathcal{R}^{n+m+1}$.

We also define the a-priori series-parallel estimation model

$$\hat{y}^o(k+1) = \hat{\theta}^T(k)\,\phi(k)\,, \tag{5}$$

where $\hat{y}^o(k)$ is the *a-priori* estimate of $y(k)$, which utilizes the parameter estimate vector $\hat{\theta}(k)$ instead of $\hat{\theta}(k+1)$.

Let us now introduce the following estimation errors:

Parameter error estimate : $\tilde{\theta}(k) = \theta - \hat{\theta}(k)$.

A-posteriori output estimation error : $e(k) = y(k) - \hat{y}(k)$.

A-priori output estimation error : $e^o(k) = y(k) - \hat{y}^o(k)$.

Notice that, since $\theta(k+1) - \theta(k) = 0$,

$$\Delta\tilde{\theta}(k+1) = \tilde{\theta}(k+1) - \tilde{\theta}(k) = \Delta\hat{\theta}(k+1) = \hat{\theta}(k+1) - \hat{\theta}(k)$$

Thus, although θ and $\tilde{\theta}$ are unknown, we can adjust $\Delta\tilde{\theta}(k+1)$ by adjusting $\Delta\hat{\theta}(k+1)$.

Series-parallel model error dynamics

Substracting Eq. (5) from Eq. (2) we obtain

$$e^o(k+1) = \tilde{\theta}(k)^T\phi(k)\,. \tag{6}$$

Doing similar calculation for the *a-posteriori* error dynamics, we obain

$$e(k+1) \;\; = \;\; \tilde{\theta}(k+1)^T\phi(k)\,. \tag{7}$$

Parameter Adaptation Algorithm (PAA) with time varying adaptation gain

We will use the following PAA to update $\hat{\theta}(k)$:

$$\hat{\theta}(k+1) = \hat{\theta}(k) + F(k)\,\phi(k)\,e(k+1) \tag{8}$$

where $F(k) = F(k)^T$ and $F(0) > 0$ is updated by the following algorithm:

$$F(k+1) = \frac{1}{\lambda_1(k)}\left[F(k) - \frac{F(k)\phi(k)\phi^T(k)F(k)}{\frac{\lambda_1(k)}{\lambda_2(k)} + \phi^T(k)F(k)\phi(k)}\right] \tag{9}$$

and $\lambda_1(k)$ and $\lambda_2(k)$ must satisfy the following conditions

$$\begin{aligned} 0 &< \lambda_1(k) \leq 1 \\ 0 &\leq \lambda_2(k) < 2 \end{aligned} \tag{10}$$

and are adjusted so that the maximum singular value of $F(k)$ is uniformly bounded, i.e.

$$0 \leq \lambda_{\max}\{F(k)\} < K_{max} < \infty\,. \tag{11}$$

By the matrix inversion lemma, the inverse of $F(k)$ in Eq. (9) is updated by

$$F^{-1}(k+1) = \lambda_1(k)\, F^{-1}(k) + \lambda_2(k)\, \phi(k)\phi^T(k) \tag{12}$$

Eq. (8) is not realizable, but useful for performing the stability analysis. However, from Eqs. (6), (7) and (8), we can obtain the following relation between the a-posteriori and a-priori estimation errors,

$$e(k+1) = \frac{e^0(k+1)}{1 + \phi(k)^T F(k)\phi(k)} \tag{13}$$

and express the PAA as a function of the a-priori estimation error instead

$$\hat{\theta}(k+1) = \hat{\theta}(k) + \frac{F(k)\phi(k)\, e^0(k+1)}{1 + \phi(k)^T F(k)\phi(k)} \tag{14}$$

5.1 Stability Analysis

In this section, we will prove utilizing the Hyperstability approach, that under the assumptions stated in the previous section, both the a-posteriori output estimation error, $e(k)$, and the a-priori output estimation error, $e^o(k)$ and converge to zero.

To use the Hyperstability theory in pages HS-1 to HS-3 in the ME233 class notes, we define the signals

$$m(k) = -w(k) = \tilde{\theta}(k)^T \phi(k-1) \tag{15}$$

and notice that the combined error dynamics and PAA, can be described by the equivalent feedback loop in Fig. 4.
Fig. 4 has a linear time invariant system in its forward path ($G(q) = 1$):

$$e(k+1) = m(k+1) \tag{16}$$

and a nonlinear system in its feedback path (NL):

$$\begin{aligned} \tilde{\theta}(k+1) &= \tilde{\theta}(k) - F(k)\,\phi(k)\, e(k+1) \\ F(k+1) &= \frac{1}{\lambda_1(k)}\left[F(k) - \frac{F(k)\phi(k)\phi^T(k)F(k)}{\frac{\lambda_1(k)}{\lambda_2(k)} + \phi^T(k)F(k)\phi(k)}\right] \\ w(k+1) &= -\phi(k)^T\tilde{\theta}(k+1) \end{aligned} \tag{17}$$

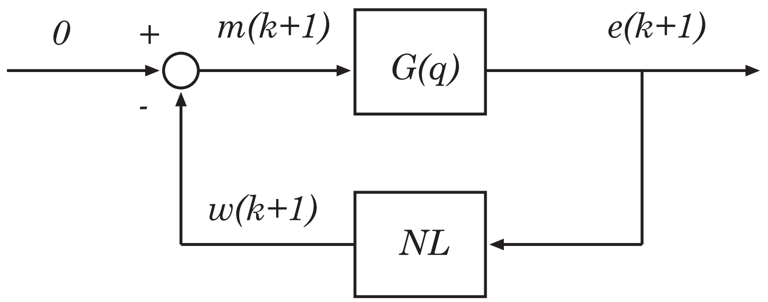

Figure 4: Equivalent Feedback Block Diagram

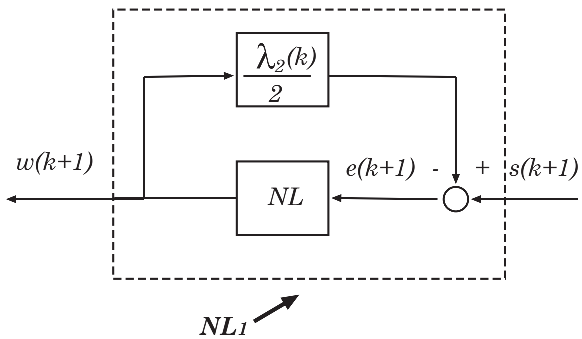

Figure 5: Modified nonlinear block NL_1

Unfortunately, it is not possible to show the nonlinear block NL, which is given by Eqs. (17) is passive. However, it is possible to show the passivity of the modified nonlinear block NL_1 in Fig. 5.
Notice that the block NL_1 in Fig. 5 has the following dynamics:

$$\begin{aligned}
e(k) &= s(k) - \frac{\lambda_2(k-1)}{2}\, w(k) \\
\tilde{\theta}(k) &= \tilde{\theta}(k-1) - F(k-1)\,\phi(k-1)\, e(k) \\
F(k+1) &= \frac{1}{\lambda_1(k)}\left[F(k) - \frac{F(k)\phi(k)\phi^T(k)F(k)}{\frac{\lambda_1(k)}{\lambda_2(k)} + \phi^T(k)F(k)\phi(k)}\right] \\
w(k) &= -\phi(k-1)^T\tilde{\theta}(k)
\end{aligned} \tag{18}$$

From Eqs. (18) and (12) we obtain

$$\begin{aligned}
\sum_{j=0}^{k} w(j)s(j) &= \sum_{j=0}^{k} w(j)\left\{e(j) + \lambda_2(j-1)w(j)\right\} \\
&= \sum_{j=0}^{k} w(j)\, e(j) + \frac{1}{2}\sum_{j=0}^{k} \lambda_2(j-1)w^2(j) \\
&= -\sum_{j=0}^{k} \tilde{\theta}^T(j)\,\phi(j-1)e(j) + \frac{1}{2}\sum_{j=0}^{k} \lambda_2(j-1)\tilde{\theta}^T(j)\phi(j-1)\phi^T(j-1)\tilde{\theta}(j)
\end{aligned} \tag{19}$$

$$= \sum_{j=0}^{k} \tilde{\theta}^T(j)\, F^{-1}(j-1) \left[\tilde{\theta}(j) - \tilde{\theta}(j-1)\right] + \frac{1}{2} \sum_{j=0}^{k} \tilde{\theta}^T(j) \left[F^{-1}(j) - \lambda_1(j-1)F^{-1}(j-1)\right] \tilde{\theta}(j)$$

Completing the sum of squares in the first term in Eq. (44), we obtain

$$\begin{aligned}
\sum_{j=0}^{k} w(j)s(j) &= \frac{1}{2}\sum_{j=0}^{k} \Big[\tilde{\theta}^T(j)\, F^{-1}(j-1)\tilde{\theta}(j) + ||F^{-\frac{1}{2}}(j-1)(\tilde{\theta}^T(j) - \tilde{\theta}^T(j-1))||^2 - \tilde{\theta}^T(j-1)\, F^{-1}(j-1)\tilde{\theta}(j-1)\Big] \\
&\quad + \frac{1}{2}\sum_{j=0}^{k} \Big[\tilde{\theta}^T(j)F^{-1}(j)\tilde{\theta}(j) - \lambda_1(j-1)\tilde{\theta}^T(j)F^{-1}(j-1)\tilde{\theta}(j)\Big] \\
&= \frac{1}{2}\sum_{j=0}^{k} \Big[\tilde{\theta}^T(j)F^{-1}(j)\tilde{\theta}(j) - \tilde{\theta}^T(j-1)\, F^{-1}(j-1)\tilde{\theta}(j-1)\Big] \\
&\quad + \frac{1}{2}\sum_{j=0}^{k} \Big[(1-\lambda_1(j-1))\,\tilde{\theta}^T(j)F^{-1}(j-1)\tilde{\theta}(j) + ||F^{-\frac{1}{2}}(j-1)(\tilde{\theta}^T(j) - \tilde{\theta}^T(j-1))||^2\Big] \\
&\geq \frac{1}{2}\,\tilde{\theta}^T(k)F^{-1}(k)\tilde{\theta}(k) - \frac{1}{2}\,\tilde{\theta}^T(0)F^{-1}(0)\tilde{\theta}(0) \\
&\geq -\gamma_o^2
\end{aligned} \tag{20}$$

where $\gamma_o^2 = \frac{1}{2}\,\tilde{\theta}^T(0)F^{-1}(0)\tilde{\theta}(0)$ and $(1-\lambda_1(k)) \geq 0$ by Eq. (41).

We will now prove the stability of the adaptive system. This is accomplished by performing a series of block diagram operations to the feedback system in Fig. 4, so that we end up with a feedback system that has the nonlinearity NL_1. We will do this in step 1:

1. By adding and subtracting static time varying block $\frac{\lambda_2(k)}{2}$ and time invariant block $\frac{\lambda}{2}$, where

   $$\lambda = \max_k \lambda_2(k)\,, \tag{21}$$

   to the system in Fig. 4 we obtain the equivalent feedback system in Fig. 6-(a). Doing a bit of block diagram algebra, we obtain the equivalent feedback system in Fig. 6-(b). where $L_1(q) = G(q) - \frac{\lambda}{2}$, and the nonlinear block NL_2 is itself a feedback loop of two blocks, as shown in Fig. 6.

2. Let us analyze the properties of the linear block L_1 in the forward in Fig. 6:

   $$\begin{aligned} L_1(q) &= G(q) - \frac{\lambda}{2} \\ &= 1 - \frac{\lambda}{2} > 0, \end{aligned}$$

   since $\lambda < 2$ by Eqs. (41) and (42).

   Thus, the forward linear block L_1 is Strictly Positive Real (SPR).

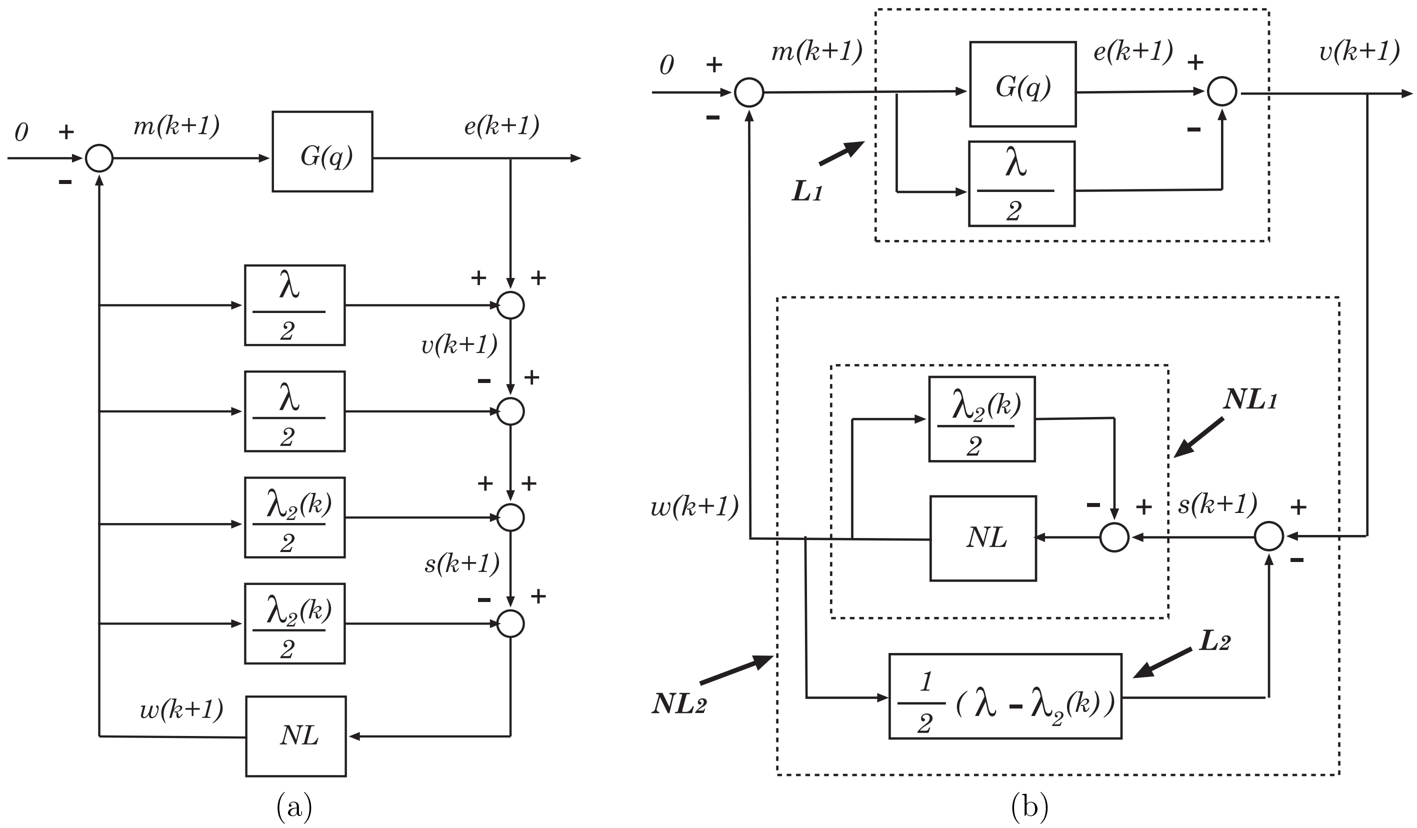

Figure 6: Equivalent Feedback Block Diagrams

3. Now we analyze the properties of the nonlinear block NL_2 in the feedback path in Fig. 6 and represented by the feedback loop shown in in Fig. 6:

   (i) We have already shown that the nonlinear block NL_1 is passive.

   (ii) The block L_2 satisfies

   $$L_2(k) = \frac{1}{2}\left[\lambda - \lambda_2(k)\right] \geq 0$$

   and therefore it is also passive.

   (iii) NL_2 is a feedback combination of two passive blocks.

   Therefore, NL_2 is also a passive block.

4. Since the block L_1 in Fig. 6 is SPR, and the block NL_2 in Fig. 6 is passive, by the sufficiency portion of the hyperstability theorem, we now conclude from Eq. (H-4) in page HS-1 of the ME233 class notes that

   $$|v(k)| \leq \delta\left\{|v(0)| + \gamma_2^2\right\},$$

   where both $\delta > 0$ and $\gamma_2^2 > 0$ are bounded.

5. Because the block L_1 is an SPR block which has a direct feed-through term and is invertible, the signal $w(k)$ is also bounded. Thus, $e(k)$ is also bounded, i.e. $|e(k)| < \infty$. and

$$\lim_{k \to \infty} e(k) = 0$$

follows from the sufficiency portion of the Asymptotic Hyperstability Theorem in page HS-2 of the ME233 class notes, and the fact that $w(k)$ is bounded.

6. Notice that

$$e(k) = \frac{e^0(k)}{1 + \phi(k-1)^T F(k-1)\phi(k-1)} \,.$$

Therefore, to show the convergence of the a-posteriori error $e^0(k)$ to zero, it is necessary to show that both the repressor $\phi(k)$ and the gain matrix $F(k)$ remain bounded. Notice that, by assumption, $u(k)$ is bounded. Since, the polynomial $A(q^{-1})$ in Eq. (1) is Schur, the transfer function

$$G(z) = \frac{z^{-1}B(z^{-1})}{A(z^{-1})}$$

is also bounded-input-bounded-output stable, which implies that both $|y(k)| < \infty$ and $||\phi(k)|| < \infty$.

Thus, under assumption (18), the a-priori error $e^o(k)$ also converges to zero.

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ME233 Advanced Control Systems II

Adaptive Pole-Placement, Tracking Control and Deterministic Disturbance Rejection for ARMA Models

6 Introduction

The control synthesis procedure that was presented in the *Pole-Placement, Tracking Control and Deterministic Disturbance Rejection for ARMA Models* handout will serve as the basis for deriving adaptive control algorithms for SISO ARMA models.

After reviewing the control algorithms presented in the Pole Placement, Tracking Control and Deterministic Disturbance Rejection handout, we will introduce indirect adaptive control schemes, which are based on the so-called certainty equivalence principle. In certainty equivalence indirect adaptive control schemes, the control system is first designed assuming that the plant parameters are known. The plant parameters are then estimated using on-line identification schemes and the controller parameters are determined by using the estimated plant parameters as if they were the true plant parameters.

Subsequently, we will present a direct adaptive control scheme, for plants that do not have unstable zeros, where the controller parameters are updated directly by an on-line identification algorithm.

7 Review of Pole-Placement, Tracking Control and Deterministic Disturbance Rejection Design

7.1 Plant

Consider the ARMA model with deterministic disturbance

$$A(q^{-1})\, y(k) = q^{-d} B(q^{-1})\ (\ u(k) + d(k)\ )\ , \tag{1}$$

where

- $u(k)$ is the control input,
- $y(k)$ is the system's output,
- $d(k)$ is an unknown but deterministic disturbance, which will be further characterized subsequently.

$$\begin{aligned} A(q^{-1}) &= 1 + a_1 q^{-1} + \cdots + a_n q^{-n} \\ B(q^{-1}) &= b_o + b_1 q^{-1} + \cdots + b_m q^{-m} \end{aligned}$$

are coprime.

- q^{-1} is the one-step delay operator, e.g. $q^{-1}\,y(k) = y(k-1)$
- d is the number of pure delays in the system.
- As before, we use the convention $A^*(q) = q^n A(q^{-1})$ and $B^*(q) = q^m B(q^{-1})$.
- There exists a monic *known* polynomial $A_d(q^{-1})$, of order n_d such that

  $$A_d(q^{-1})d(k) = 0\,, \tag{2}$$

  moreover, without loss of generality, the polynomials $A_d(q^{-1})$ and $B(q^{-1})$ are co-prime.

- Without loss of generality, we assume that the zero polynomial $B(q^{-1})$ will be factorized into two factors:

  $$B(q^{-1}) = B^s(q^{-1})\,B^u(q^{-1}) \tag{3}$$

  where

  - $B^u(q^{-1})$ is a monic polynomial[4] of order m_u,
  - $B^s(q^{-1})$ is a polynomial of order m_s and $B^{s*}(q^{-1}) = q^{m_s}\,B^s(q^{-1})$ is Schur.
  - $m = m_u + m_s$

  The polynomial $B^{s*}(q) = q^{m_s}\,B^s(q^{-1})$ includes all sufficiently damped plant zeros, *that will be canceled by the controller.*

  The polynomial $B^{u*}(q) = q^{m_u}\,B^u(q^{-1})$ includes all unstable or lightly damped plant zeros, which should not be canceled by the controller. Notice that, if $B^*(q) = q^m B(q^{-1})$ is Schur, and we want to cancell all plant zeros, then $B^{u*}(q) = 1$.

7.2 Control Objectives

In these notes, we will consider three different control objectives:

1) Pole Placement: The poles of the closed loop system should be placed at specific locations in the complex plane. Specifically, we define the *closed loop pole polynomial* $A_c(q^{-1})$

$$A_c(q^{-1}) = B^s(q^{-1})\,A_c^{'}(q^{-1}) \tag{4}$$

where

[4] i.e. The leading (i.e. q^0) coefficient of a monic polynomial is 1, e.g. $A(q^{-1})$ is monic.

- $B^{s*}(q)$ contains all cancelable plant zeros.
- $A_c^{'}(q^{-1})$ is a monic Schur polynomial chosen by the designer

$$A_c^{'}(q^{-1}) = 1 + a_{c1}^{'} q^{-1} + \cdots + a_{cn_c^{'}}^{'} q^{-n_c^{'}} \tag{5}$$

2) Tracking: The output sequence $y(k)$ must follow a *reference sequence* $y_d(k)$, which is known for all $k \geq 0$.

In some instances, we will assume that the reference sequence $y_d(k)$ may be generated by the following *reference model*

$$A_m(q^{-1}) y_d(k) = q^{-d} B_m(q^{-1}) u_d(k) \tag{6}$$

where d is the plant pure delay. $A_m(q^{-1})$, the reference model Schur pole polynomial, and $B_m(q^{-1})$, the reference model zero polynomial, are selected by the designer. $u_d(k)$ is a deterministic input for the reference model which is also selected by the designer.

Finally, it is assume that the reference model output $y)d(k)$ is available to the control system in advanced, i.e. $y_d(k+L)$ is available to the control system at the instance k for a sufficiently large $L \geq 0$, which will be defined subsequently. When the zero-phase error tracking compensator described in section 7.5.2 is used, $L = d + m_u$, the number of pure delays in the plant dynamics plus the number of plant zeros that are not being canceled by the feedback action.

3) Deterministic disturbance rejection: The closed loop system must reject deterministic disturbances $d(k)$, which satisfy $A_d(q^{-1})d(k) = 0$, where the annihilating polynomial $A_d(q^{-1})$ is *known*.

7.3 Control Law

The control system's block diagram is shown in Fig. 7 where $u(k)$ and $d(k)$ are respectively

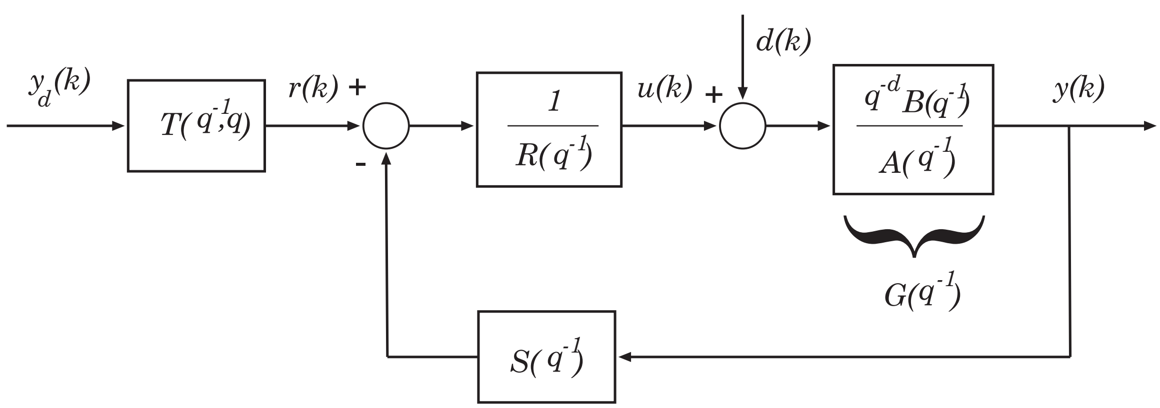

Figure 7: Feedback System

the control and disturbance plant inputs, $u_d(k)$ is the reference model's input, which was defined in (4), $r(k)$ is the reference input to the feedback block.

7.4 Feedback Control Synthesis using the Diophantine Equation

The $R(q^{-1})$ and $S(q^{-1})$ polynomials in the feedback control law are determined using the following Diophantine equation

$$A_c^{'}(q^{-1}) = A_d(q^{-1})\,A(q^{-1})\,R^{'}(q^{-1}) + q^{-d}B^u(q^{-1})\,S(q^{-1}) \tag{7}$$

where

- $A_c^{'}(q^{-1})$ is the desired closed loop polynomial defined in Eq. (3),
- $A(q^{-1})$ is the plant's pole polynomial,
- $A_d(q^{-1})$ is the disturbance annihilator polynomial defined in Eq. (2),
- $B^u(q^{-1})$ is the factor of the plant's zero polynomial, $B(q^{-1})$, that will not be cancelled by the controller and was defined in Eq. (3) and
- the polynomials $R^{'}(q^{-1})$ and $S(q^{-1})$ are determined so that Eq. (7) is satisfied.

$$\begin{aligned} R^{'}(q^{-1}) &= 1 + r_1^{'}q^{-1} + \cdots + r_{n_r^{'}}\,q^{-n_r^{'}} && (8) \\ S(q^{-1}) &= s_o + s_1q^{-1} + \cdots + s_{n_s}q^{-n_s} && (9) \end{aligned}$$

  Under the assumptions that $A(q^{-1})\,A_d(q^{-1})$ and $B^u(q^{-1})$ are co-prime, a unique minimum-degree solution of the Diophantine equation (7) is obtained when

$$\begin{aligned} n_r^{'} &= d + m_u - 1 \\ n_s &= \max[\,n + n_d - 1\,,\, n_c^{'} - d - m_u\,] \end{aligned}$$

- The polynomial $R(q^{-1})$ in Fig. 7 is given by

$$R(q^{-1}) = R^{'}(q^{-1})\,A_d(q^{-1})\,B^s(q^{-1})\,, \tag{10}$$

  where $B^s(q^{-1})$ is the factor in the plant's zero polynomial $B(q^{-1})$ that will be cancelled by the controller and $A_d(q^{-1})$, defined in Eq. (2), is the *known* polynomial that annihilates the disturbance $d(k)$.

Remark Notice that, if the degree of the disturbance annihilator polynomial $A_d(q^{-1})$, n_d, is large (e.g. N is a large number), then the degree of $S(q^{-1})$, n_s, will also be large and the solution of the Diophantine equation may be ill conditioned. There are several controller design alternatives for this case, which are discussed in pages IMP-5 to IMP-8 of the ME233 class notes.

The resulting control law is given by

$$
\begin{aligned}
u(k) &= \frac{1}{R(q^{-1})}\left\{r(k) - S(q^{-1})y(k)\right\} \\
&= \frac{1}{R'(q^{-1})\,A_d(q^{-1})\,B^s(q^{-1})}\left\{r(k) - S(q^{-1})y(k)\right\}
\end{aligned}
\tag{11}
$$

which results in the following closed-loop dynamics

$$
A_c'(q^{-1})\,y(k) = q^{-d}B^u(q^{-1})\,r(k) \tag{12}
$$

7.5 Feedforward Control Synthesis

The design of the feedforward compensator $T(q^{-1}, q)$ in Fig. 7 depends on whether the plant has unstable zeros and on the assumptions made on the reference model given by Eq. (4).

7.5.1 Perfect tracking when all plant zeros are cancelable, i.e. $B^u(q^{-1}) = 1$

When $B^u(q^{-1}) = 1$, the closed loop dynamics given by Eq. (12) reduces to

$$
A_c'(q^{-1})\,y(k) = q^{-d}\,r(k) \tag{13}
$$

In this case, we set the feedforward compensator $T(q^{-1})$ in the block diagram of Fig. 7 to

$$
T(q^{-1}, q) = q^d A_c'(q^{-1})\,.
$$

where $A_c'(q^{-1})$ is the closed-loop pole polynomial in Eq. (3), which is selected by the designer.

The control law is

$$
\begin{aligned}
u(k) &= \frac{1}{R(q^{-1})}\left\{T(q, q^{-1})\,y_d(k) - S(q^{-1})y(k)\right\} \\
&= \frac{1}{R'(q^{-1})\,A_d(q^{-1})\,B^s(q^{-1})}\left\{A_c'(q^{-1})\,y_d(k+d) - S(q^{-1})y(k)\right\}
\end{aligned}
\tag{14}
$$

This results in

$$A_c'(q^{-1})\,\{\,y_d(k) - y(k)\,\} = 0\,. \tag{15}$$

7.5.2 Zero-phase Error Tracking when $B^u(q^{-1}) \neq 1$

When some of the plant zeros should not be canceled and $B^u(q^{-1}) \neq 1$, perfect tracking of the *reference sequence* $y_d(k)$ cannot be achieved. Here we review the zero-phase error tracking controller, which was originally proposed by Professor Tomizuka.

To achieve zero phase error tracking, we set the feedforward compensator $T(q^{-1}, q)$ in the block diagram of Fig. 7 to

$$T(q^{-1}, q) = \frac{B^u(q)\, q^d A_c'(q^{-1})}{[B^u(1)]^2}$$

where $B_m(q^{-1})$ is the zero polynomial of the reference model in Eq. (4).
Remark Notice that $T(q^{-1}, q)$ in this case is a-causal, i.e. future values of $y_d(k)$ up to $u_d(k + d + m_u)$ are required to generate $r(k)$.

The control law is given by

$$\begin{aligned} u(k) &= \frac{1}{R(q^{-1})}\left\{T(q^{-1}, q)\, y_d(k) - S(q^{-1})y(k)\right\} \qquad (16) \\ &= \frac{1}{R'(q^{-1})\, A_d(q^{-1})\, B^s(q^{-1})}\left\{\frac{B^u(q)\, A_c'(q^{-1})}{[B^u(1)]^2}\, y_d(k + d) - S(q^{-1})y(k)\right\} \end{aligned}$$

and the zero-phase tracking objective is achieved

$$A_c'(q^{-1})\,\{G_{zp}(q)y_d(k) - y(k)\} = 0\,. \tag{17}$$

where the transfer fucntion

$$G_{zp}(z) = \frac{B^u(z^{-1})\, B^u(z)}{[B^u(1)]^2}$$

satisfies

1. $\mathrm{Im}\,\{G_{zp}(e^{j\omega})\} = 0$, for $0 \leq \omega \leq \pi/T$, where T is the sampling time.
2. $\lim_{\omega\to 0} G_{zp}(e^{j\omega}) = 1$.

8 Indirect Adaptive Control

The control synthesis procedure outlined in the previous section can be used as the basis of an indirect adaptive control scheme, by using the so-called certainty equivalence principle. For this purpose we first estimate the unknown plant polynomials $A(q^{-1})$ and $B(q^{-1})$ in (1) and then we will compute the feedback controller parameters $R^{'}(q^{-1})$ and $S(q^{-1})$ by solving at each sampling instance the Diophantine equation (7), utilizing the plant polynomial estimates $\hat{A}(q^{-1}, k)$ and $\hat{B}(q^{-1}, k)$ instead of $A(q^{-1})$ and $B(q^{-1})$.

We assume that

$$
\begin{aligned}
|B^u(1)| &\geq B^u_{\min} > 0 \\
b_o &\geq b_{mino} > 0
\end{aligned}
$$

where the polynomial $B^u(q^{-1})$ contains all plant unstable zeros and was defined in Eq. (3). We also assume that $B^u_{\min}$ and b_{mino} are known.

8.1 Parameter estimation algorithm

We will use a series-parallel parameter adaptation algorithm to estimate the plant polynomials $A(q^{-1})$ and $B(q^{-1})$ in (1), we assume that the number of pure plant delays d is *known*.

8.1.1 Signal pre-filtering

In order to prevent biasing of the parameter estimates due to the adverse effect of the deterministic disturbance, the plant input $u(k)$ and output $y(k)$ sequences will be pre-filtered using the *known* disturbance annihilating polynomial $A_d(k)$. Thus, we defined the filtered plant input and output sequences $y_f(k)$ and $u_f(k)$ by

$$
\begin{aligned}
y_f(k) &= A_d(q^{-1})\, y(k) \\
u_f(k) &= A_d(q^{-1})\, u(k)
\end{aligned}
\tag{18}
$$

By Eqs. (2) and (1), we obtain

$$
A(q^{-1})\, y_f(k) = q^{-d} B(q^{-1})\, u_f(k)\,. \tag{19}
$$

We now define the regressor model of the plant based on the filtered ARMA model in Eq. (19)

$$
y_f(k) = \phi(k-1)^T\, \theta \tag{20}
$$

where

$$
\theta = \begin{bmatrix} a_1 & \cdots & a_n & b_o & \cdots & b_m \end{bmatrix}^T \in \Re^{n+m+1} \tag{21}
$$

$$
\phi(k-1) = \begin{bmatrix} -y_f(k-1) & \cdots & -y_f(k-n) & u_f(k-d) & \cdots & u_f(k-d-m) \end{bmatrix}^T \in \Re^{n+m+1} \tag{22}
$$

and $y_f(k)$ and $u_f(k)$ are defined in Eq. (18).

We now define the estimate of the unknown parameter θ by

$$\hat{\theta}(k) = \begin{bmatrix} \hat{a}_1(k) & \cdots & \hat{a}_n(k) & \hat{b}_o(k) & \cdots & \hat{b}_m(k) \end{bmatrix}^T \tag{23}$$

$\hat{\theta}(k)$ can be estimated with a series-parallel parameter adaptation algorithm (PAA) based on Eq. (20)

$$\begin{aligned}
e^o(k+1) &= y_f(k+1) - \phi^T(k)\hat{\theta}(k) \\
e(k+1) &= \frac{e^o(k+1)}{1+\phi^T(k)F(k)\phi(k)} \\
\hat{\theta}^o(k+1) &= \hat{\theta}(k) + F(k)\phi(k)\, e(k+1) \\
F(k+1) &= \frac{1}{\lambda_1(k)}\left[F(k) - \frac{F(k)\phi(k)\phi^T(k)F(k)}{\frac{\lambda_1(k)}{\lambda_2(k)} + \phi^T(k)F(k)\phi(k)}\right] \\
0 &< \lambda_1(k) \le 1 \\
0 &\le \lambda_2(k) < 2
\end{aligned} \tag{24}$$

where $y_f(k)$ and $\phi(k)$ are respectively defined in Eqs. (18) and (22), and

$$\hat{\theta}(k) = \begin{cases} \hat{\theta}^o(k) & \text{if } \hat{b}_o^o(k) \ge b_{mino} \\ \begin{bmatrix} \hat{a}_1(k)\cdots\hat{a}_n(k)\, b_{mino} \cdots \hat{b}_m(k) \end{bmatrix}^T & \text{if } \hat{b}_o^o(k) < b_{mino} \end{cases} \tag{25}$$

The projection algorithm in the above equation is used to prevent the control signal from becoming too large, if $\hat{b}_o^o(k)$ becomes too small, when the adaptive control algorithms in sections 8.2 and 8.3 are used.

8.2 Indirect adaptive control law for Zero-Phase Error Tracking (ZPET)

Instead of using the polynomials $R'(q^{-1})$, $S(q^{-1})$ and the filter $T(q^{-1},q)$ in the control law (16), we will use their estimates $\hat{R}'(q^{-1},k)$, $\hat{S}(q^{-1},k)$ and $\hat{T}(q^{-1},q,k)$. These estimates are calculated as follows:

1) The polynomials $\hat{A}(q^{-1},k)$ and $\hat{B}(q^{-1},k)$ are defined as follows

$$\begin{aligned}
\hat{A}(q^{-1},k) &= 1 + \hat{a}_1(k)\, q^{-1} + \cdots + \hat{a}_n(k)\, q^{-n} \\
\hat{B}(q^{-1},k) &= \hat{b}_o(k) + b_1(k)\, q^{-1} + \cdots + \hat{b}_m(k)\, q^{-m}
\end{aligned}$$

2) The polynomial $\hat{B}(q^{-1}, k)$ is factorized

$$\hat{B}(q^{-1}, k) = \hat{B}^s(q^{-1}, k)\hat{B}^u(q^{-1}, k)$$

where $\hat{B}^u(q^{-1}, k)$ is monic.

3) The polynomials $\hat{R}'(q^{-1}, k)$ and $\hat{S}(q^{-1}, k)$ are calculated by solving the Diophantine equation

$$A_c'(q^{-1}) = A_d(q^{-1})\,\hat{A}(q^{-1}, k)\,\hat{R}(q^{-1}, k) + q^{-d}\,\hat{B}^u(q^{-1}, k)\hat{S}(q^{-1}, k) \tag{26}$$

4) The feedforward filter $\hat{T}(q^{-1}, q)$ is calculated as follows

$$\hat{T}(q^{-1}, q, k) = \frac{\hat{B}^u(q, k)\, q^d A_c'(q^{-1})}{[\bar{B}^u(k)]^2}$$

where

$$\bar{B}^u(k) = \begin{cases} \hat{B}^u(1, k) & \text{if} \quad |\hat{B}^u(1, k)| \geq B^u_{\min} \\ B^u_{\min} & \text{if} \quad |\hat{B}^u(1, k)| < B^u_{\min} \end{cases} .$$

5) The polynomial $\hat{R}(q^{-1}, k)$ is calculated as follows

$$\hat{R}(q^{-1}, k) = A_d(q^{-1})\,\hat{B}^s(q^{-1}, k)\,\hat{R}'(q^{-1}, k) \tag{27}$$

6) Finally, the adaptive control law is given by

$$\hat{R}(q^{-1}, k)\,u(k) = \hat{T}(q^{-1}, q, k)\,y_d(k) - \hat{S}(q^{-1}, k)y(k)\,. \tag{28}$$

Notice that the polynomials monic $A_c'(q^{-1})$ and $B_m(q^{-1})$ in Eq. (4) are set by the designer and $A_c'(q^{-1})$ must be Schur.

8.3 Indirect adaptive control of plants with stable zeros

In this section we assume that all plant zeros can be cancelled by the control law, i.e. $B(q^{-1}) = B^s(q^{-1})$ (i.e. $B^u(q^{-1}) = 1$). In this case the Diophantine Eq. (7) simplifies to

$$A_c^{'}(q^{-1}) = A_d(q^{-1})A(q^{-1})\, R^{'}(q^{-1}) + q^{-d}\, S(q^{-1}) \tag{29}$$

and the feedforward filter $T(q^{-1})$ is now strictly causal and given by

$$T(q^{-1}, q) = q^d A_c^{'}(q^{-1})\,, \tag{30}$$

where the polynomial $B_m^{'}(q^{-1})$ was defined in Eq. (4). Therefore, $T(q^{-1}, q)$ is now completely independent of the unknown plant.

Using the certainty equivalence principle, the indirect adaptive controller is implemented as follows:

1) The polynomials $\hat{R}^{'}(q^{-1}, k)$ and $\hat{S}(q^{-1}, k)$ are calculated by solving the Diophantine equation

$$A_c^{'}(q^{-1}) = A_d(q^{-1})\hat{A}(q^{-1}, k)\, \hat{R}^{'}(q^{-1}, k) + q^{-d}\, \hat{S}(q^{-1}, k) \tag{31}$$

2) The polynomial $\hat{R}(q^{-1}, k)$ is calculated as follows

$$\hat{R}(q^{-1}, k) \;=\; A_d(q^{-1})\, \hat{B}(q^{-1}, k)\, \hat{R}^{'}(q^{-1}, k) \tag{32}$$

Notice that, since both $A_d(q^{-1})$ and $\hat{R}^{'}(q^{-1})$ are monic polynomials,

$$\begin{aligned} \hat{R}(q^{-1}, k) &= \hat{r}_o(k) + \cdots + \hat{r}_{n_r}(k) q^{-n_r} \\ \hat{S}(q^{-1}, k) &= \hat{s}_o(k) + \cdots + \hat{s}_{n_s}(k) q^{-n_s} \end{aligned}$$

and

$$\begin{aligned} n_r &= d + m - 1 \\ n_s &= \max(n + n_d - 1, n_c^{'} - d - m) \\ \hat{r}_o(k) &= \hat{b}_o(k) \end{aligned}$$

where $\hat{b}_o(k)$ is defined in Eq. (25).

3) Finally, the adaptive control law is given by

$$\hat{R}(q^{-1}, k)\, u(k) = A_c^{'}(q^{-1})\, y_d(k + d) - \hat{S}(q^{-1}, k) y(k)\,. \tag{33}$$

9 Direct Adaptive Control of Plants With Stable Zeros

If $B^u(q^{-1}) = 1$, the compensator $T(q^{-1}, q)$ is given by Eq. (30) and the polynomials $R(q^{-1})$ and $S(q^{-1})$ in the control law (14) can be updated directly.

Multiplying Eq. (9) by $y(k)$ and utilizing the plant dynamics Eq. (1) and Eq. (2) we obtain

$$A_c^{'}(q^{-1})\, y(k) = q^{-d} \left(R(q^{-1})\, u(k) + S(q^{-1}) y(k) \right) . \tag{34}$$

where

$$R(q^{-1}) = A_d(q^{-1})\, B(q^{-1})\, R^{'}(q^{-1})$$

and

$$\begin{aligned} R^{'}(q^{-1}) &= 1 + r_1^{'} q^{-1} + \cdots + r_{n_r^{'}}^{'}\, q^{-n_r^{'}} \\ S(q^{-1}) &= s_o + s_1 q^{-1} + \cdots + s_{n_s} q^{-n_s} \\ \\ n_r^{'} &= d - 1 \\ n_s &= \max(n + n_d - 1, n_c^{'} - d - m) \\ \\ R(q^{-1}) &= r_o + r_1 q^{-1} + \cdots + r_{n_r}\, q^{-n_r} \\ n_r &= d + m + n_d - 1 \end{aligned}$$

where $r_o = b_o$.

We will now consider two cases:

(a) $d(k) = 0$ and $A_d(q^{-1}) = 1$

(b) $d(k) \neq 0$ with $A_d(q^{-1})$ *known.*

9.1 Direct adaptive control of plants with stable zeros and no disturbances

When $A_d(q^{-1}) = 1$, The polynomials $R(q^{-1})$ and $S(q^{-1})$ in Eq. (10) are

$$\begin{aligned} R(q^{-1}) &= B(q^{-1})\, R^{'}(q^{-1}) \\ &= b_o + r_1 q^{-1} + \cdots + r_{n_r}\, q^{-n_r} \\ S(q^{-1}) &= s_o + s_1 q^{-1} + \cdots + s_{n_s} q^{-n_s} \\ \\ n_r &= d + m - 1 \\ n_s &= \max(n + n_d - 1, n_c^{'} - d - m) \end{aligned}$$

Eq. (10) can be written as follows

$$y(k) \;=\; \phi_f^T(k-d)\theta_c \tag{35}$$

where the parameter vector θ_c is

$$\theta_c = \left[\begin{array}{ccc} s_o \cdots s_{n_s} \; r_o & \cdots & r_{n_r} \end{array}\right]^T \in \Re^{N_c} \; ,$$

where

- $s_o \cdots s_{n_s}$ are the coefficients of $S(q^{-1})$.
- $r_o \, , \cdots \, , r_{n_r}$ are the coefficients of $R(q^{-1})$.
- $N_c = n_s + n_r + 2$.
- The filtered regressor vector $\phi_f(k)$ is given by

$$\begin{aligned} \phi_f(k) &= \frac{1}{A_c'(q^{-1})}\, \phi(k) \qquad (36) \\ \phi(k) &= \left[\begin{array}{cccccc} y(k) & \cdots & y(k-n_s) & u(k) & \cdots & u(k-n_r) \end{array}\right]^T \in \Re^{N_c} \end{aligned}$$

Notice that the pole-placement control law Eq. (7) can also be rewritten as

$$\theta_c^T \, \phi(k) = A_c'(q^{-1}) \; y_d(k+d) \; .$$

The following direct adaptive controller can be implemented.

1) The controller parameter vector estimate $\hat{\theta}_c(k)$ is updated by the series-parallel PAA, based on Eq. (13)

$$\begin{aligned} e^o(k+1) &= y(k+1) - \phi_f^T(k-d+1)\hat{\theta}_c(k) \qquad (37) \\ e(k+1) &= \frac{e^o(k+1)}{1+\phi_f^T(k-d+1)F(k)\phi_f(k-d+1)} \\ \hat{\theta}_c^o(k+1) &= \hat{\theta}_c(k) + F(k)\phi_f(k-d+1)\; e(k+1) \\ F(k+1) &= \frac{1}{\lambda_1(k)} \left[F(k) - \frac{F(k)\phi_f(k-d+1)\phi_f^T(k-d+1)F(k)}{\frac{\lambda_1(k)}{\lambda_2(k)} + \phi_f^T(k-d+1)F(k)\phi_f(k-d+1)} \right] \end{aligned}$$

where $\phi_f(k)$ is defined in Eq. (14) and

$$\begin{aligned} 0 &< \lambda_1(k) \leq 1 \\ 0 &\leq \lambda_2(k) < 2 \end{aligned}$$

and are adjusted so that the maximum singular value of $F(k)$ is uniformly bounded, and

$$0 < K_{\min} \leq \lambda_{\min}\{F(k)\} \leq \lambda_{\max}\{F(k)\} < K_{max} < \infty \,. \tag{38}$$

Also,

$$\hat{\theta}_c(k) \;=\; \begin{cases} \hat{\theta}_c^o(k) & \text{if} \;\; \hat{r}_o^o(k) \geq b_{mino} \\ \left[\; \hat{s}_o(k) \cdots \hat{s}_{n_s}(k)\, b_{mino} \;\cdots\; \hat{r}_{n_r}(k) \;\right] & \text{if} \;\; \hat{r}_o^o(k) < b_{mino} \end{cases} \tag{39}$$

The projection algorithm in the above equation is used to prevent the control signal from becoming too large if $\hat{r}_o^o(k)$ becomes too small.

2) The adaptive control law is implemented by

$$\hat{\theta}_c^{\,T}(k)\,\phi(k) = A_c^{'}(q^{-1})\; y_d(k+d)\;,$$

which can be rewritten as

$$\hat{R}(q^{-1},k)\,u(k) = A_c^{'}(q^{-1})\; y_d(k+d) - \hat{S}(q^{-1},k)y(k)\,. \tag{40}$$

9.2 Direct adaptive control with integral action for plants with stable zeros

Let us now consider the case when $d(k) \neq 0$ and $A_d(q^{-1})d(k) = 0$, with $A_d(q^{-1})$ *known*. Although the algorithm presented in the previous section can be used, only that in this case $n_r = d + m + n_d - 1$, since $R(q^{-1}) = A_d(q^{-1})\,B(q^{-1})\,R^{'}(q^{-1})$, it may be advantageous to reduce the number of parameters being estimated, by utilizing the fact that the disturbance annihilating polynomial $A_d(q^{-1})$ is known.

We consider now the specific case when the disturbance $d(k)$ is constant:

- $d(k) = d(k-1)$
- $A_d(q^{-1}) = 1 - q^{-1}\,.$

A similar approach can be followed for repetitive disturbances, such that $A_d(q^{-1})\,d(k) = 0$ and $A_d(1) = 0$.

Let us first again write the Diophantine Eq. (9),

$$A_c^{'}(q^{-1}) = A_d(q^{-1})A(q^{-1})\,R^{'}(q^{-1}) + q^{-d}\,S(q^{-1}) \tag{41}$$

Notice that, because $A_d(1) = 0$, the polynomial $S(q^{-1})$ must satisfy

$$A_c^{'}(1) = S(1)\,.$$

Thus, $S(q^{-1})$ can be expressed as

$$S(q^{-1}) = A_c^{'}(1) + A_d(q^{-1})\,\mathcal{S}(q^{-1})\,. \tag{42}$$

Substituting Eq. (42) into the Diophantine Eq. (41), we obtain

$$\left[A_c^{'}(q^{-1}) - q^{-d}A_c^{'}(1)\right] = A_d(q^{-1})A(q^{-1})\,R^{'}(q^{-1}) + q^{-d}\,A_d(q^{-1})\,\mathcal{S}(q^{-1})\,. \tag{43}$$

Notice that $A_d(q^{-1})$ is also a factor of $A_c^{'}(q^{-1}) - q^{-d}A_c^{'}(1)$

$$\left[A_c^{'}(q^{-1}) - q^{-d}A_c^{'}(1)\right] = A_d(q^{-1})\,\bar{A}_c^{'}(q^{-1})\,,$$

Thus, the common term $A_d(q^{-1}) = 1 - q^{-1}$ can be factored from Eq. (43) to obtain

$$\bar{A}_c^{'}(q^{-1}) = A(q^{-1})\,R^{'}(q^{-1}) + q^{-d}\,\mathcal{S}(q^{-1})\,, \tag{44}$$

where

$$\begin{aligned}
R^{'}(q^{-1}) &= 1 + r_1^{'}q^{-1} + \cdots + r_{n_r^{'}}\,q^{-n_r^{'}} \\
\mathcal{S}(q^{-1}) &= \bar{s}_o + \bar{s}_1 q^{-1} + \cdots + \bar{s}_{n_{\bar{s}}}q^{-n_{\bar{s}}} \\
n_r^{'} &= d - 1 \\
n_{\bar{s}} &= n_s - n_d = \max[\,n-1\,,\, n_c^{'} - d - n_d\,] \\
&= \max[\,n-1\,,\, n_c^{'} - d - 1\,] \qquad (n_d = 1)
\end{aligned}$$

Let us now define the $\mathcal{R}(q^{-1})$ polynomial by

$$\begin{aligned}
\mathcal{R}(q^{-1}) &= B(q^{-1})\,R^{'}(q^{-1}) \\
&= \bar{r}_o + \bar{r}_1 q^{-1} + \cdots + \bar{r}_{n_{\bar{r}}}\,q^{-n_{\bar{r}}} \\
n_{\bar{r}} &= d + m - 1,.
\end{aligned} \tag{45}$$

We now

- multiply both sides of Eq. (43) by $y(k)$,
- substitute the plant dynamics, Eq. (1),
- substitute Eq. (45),
- utilize the fact that $A_d(q^{-1})d(k) = 0$,

to obtain

$$\left[A_c^{'}(q^{-1}) - q^{-d}A_c^{'}(1)\right]y(k) = \mathcal{R}(q^{-1})\,A_d(q^{-1})\,u(k-d) + \mathcal{S}(q^{-1})\,A_d(q^{-1})\,y(k-d)\,. \tag{46}$$

Finally, multiplying Eq. (46) by $1/A_c^{'}(q^{-1})$ we obtain

$$\eta(k) \;=\; \mathcal{R}(q^{-1})\, u_f(k-d) + \mathcal{S}(q^{-1})\, y_f(k-d)\,, \tag{47}$$

where the sequences $\eta(k)$, $u_f(k)$ and $y_f(k)$ are given by

$$\begin{aligned} \eta(k) &= y(k) - \frac{A_c^{'}(1)}{A_c^{'}(q^{-1})}\, y(k-d) \qquad (48) \\ u_f(k) &= \frac{A_d(q^{-1})}{A_c^{'}(q^{-1})}\, u(k) \\ y_f(k) &= \frac{A_d(q^{-1})}{A_c^{'}(q^{-1})}\, y(k)\,. \end{aligned}$$

Eq. (47) is the basis for the on-line identification of the coefficients for the polynomials $\mathcal{R}(q^{-1})$ and $\mathcal{S}(q^{-1})$, since it can be written as follows:

$$\eta(k) \;=\; \phi_f^T(k-d)\theta_c\,, \tag{49}$$

where the parameter vector θ_c is

$$\theta_c = \left[\; \bar{s}_o \cdots \bar{s}_{n_{\bar{s}}}\, \bar{r}_o \;\cdots\; \bar{r}_{n_{\bar{r}}} \;\right]^T \in \Re^{N_c}\,,$$

and

- $\eta(k)$ is defined in Eq. (48)
- $\bar{s}_o \cdots \bar{s}_{n_{\bar{s}}}$ are the coefficients of $\mathcal{S}(q^{-1})$.
- $\bar{r}_o\,, \cdots\,, \bar{r}_{n_{\bar{r}}}$ are the coefficients of $\mathcal{R}(q^{-1})$.
- $N_c = n_{\bar{s}} + n_{\bar{r}} + 2$.
- The filtered regressor vector $\phi_f(k)$ is given by

$$\phi_f(k) \;=\; \frac{A_d(q^{-1})}{A_c^{'}(q^{-1})}\, \phi(k) \tag{50}$$

$$\phi(k) \;=\; \left[\; y(k) \;\;\cdots\;\; y(k-n_s) \;\;\; u(k) \;\;\cdots\;\; u(k-n_r) \;\right]^T \in \Re^{N_c}$$

Notice that, from Eq. (43), the pole-placement control law Eq. (7) can also be rewritten as

$$\begin{aligned} \mathcal{R}(q^{-1})\, A_d(q^{-1})\, u(k) + \mathcal{S}(q^{-1})\, A_d(q^{-1})\, y(k) &= A_c^{'}(q^{-1})\, y_d(k+d) - A_c^{'}(1) y(k) \qquad (51) \\ \theta_c^T \left\{ A_d(q^{-1}) \phi(k) \right\} &= A_c^{'}(q^{-1})\, y_d(k+d) - A_c^{'}(1) y(k)\,. \end{aligned}$$

The following direct adaptive controller can be implemented:

1. The controller parameter vector estimate $\hat{\theta}_c(k)$ is updated by the series-parallel PAA, based on Eq. (49)

$$\eta(k+1) = y(k+1) - \frac{A_c^{'}(1)}{A_c^{'}(q^{-1})}\, y(k-d+1) \tag{52}$$

$$\begin{aligned}
e^o(k+1) &= \eta(k+1) - \phi_f^T(k-d+1)\hat{\theta}_c(k) \\
e(k+1) &= \frac{e^o(k+1)}{1+\phi_f^T(k-d+1)F(k)\phi_f(k-d+1)} \\
\hat{\theta}_c^o(k+1) &= \hat{\theta}_c(k) + F(k)\phi_f(k-d+1)\, e(k+1) \\
F(k+1) &= \frac{1}{\lambda_1(k)}\left[F(k) - \frac{F(k)\phi_f(k-d+1)\phi_f^T(k-d+1)F(k)}{\frac{\lambda_1(k)}{\lambda_2(k)} + \phi_f^T(k-d+1)F(k)\phi_f(k-d+1)}\right]
\end{aligned}$$

where $\phi_f(k)$ is given by Eq. (50) and

$$\begin{aligned}
0 &< \lambda_1(k) \le 1 \\
0 &\le \lambda_2(k) < 2
\end{aligned}$$

and are adjusted so that the maximum singular value of $F(k)$ is uniformly bounded, and

$$0 < K_{\min} \le \lambda_{\min}\{F(k)\} \le \lambda_{\max}\{F(k)\} < K_{max} < \infty\,. \tag{53}$$

Also,

$$\hat{\theta}_c(k) = \begin{cases} \hat{\theta}_c^o(k) & \text{if } \hat{r}_o^o(k) \ge b_{mino} \\ \left[\ \hat{s}_o(k) \cdots \hat{s}_{n_s}(k)\, b_{mino}\ \cdots\ \hat{r}_{n_r}(k)\ \right] & \text{if } \hat{r}_o^o(k) < b_{mino} \end{cases} \tag{54}$$

The projection algorithm in the above equation is used to prevent the control signal from becoming too large if $\hat{r}_o^o(k)$ becomes too small.

2. The adaptive control law is implemented by

$$\hat{\theta}_c^{\,T}(k)\,\left\{A_d(q^{-1})\,\phi(k)\right\} = A_c^{'}(q^{-1})\, y_d(k+d) - A_c^{'}(1)y(k)\,,$$

which can be rewritten as

$$\hat{\mathcal{R}}(q^{-1},k)\,\left\{A_d(q^{-1})\,u(k)\right\} = A_c^{'}(q^{-1})\, y_d(k+d) - A_c^{'}(1)y(k) - \hat{\mathcal{S}}(q^{-1},k)\left\{A_d(q^{-1})\,y(k)\right\}\,, \tag{55}$$

or

$$\hat{R}(q^{-1},k)\,u(k) = A_c^{'}(q^{-1})\;y_d(k+d) - \hat{S}(q^{-1},k)y(k)\,,$$

where

$$\begin{aligned}\hat{R}(q^{-1},k) &= \hat{\mathcal{R}}(q^{-1},k)\,A_d(q^{-1})\\ \hat{S}(q^{-1},k) &= \hat{\mathcal{S}}(q^{-1},k)\,A_d(q^{-1}) + A_c^{'}(1)\,.\end{aligned}$$

References

Astrom, K. and Wittenmark, B. (1995). *Adaptive Control*. Addison Wesley.

Goodwin, G. C. and Sin, K. S. (1984). *Adaptive Filtering Prediction and Control*. Prentice-Hall.

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Stability Analysis of a Direct Adaptive Controller

10 Introduction

In this handout we will analyze the stability of a direct adaptive control algorithm that is very similar to the one presented in Section 4.1 of the *Adaptive Pole-Placement, Tracking Control and Deterministic Disturbance Rejection for ARMA Models* handout. The proof closely follows that presented in (Goodwin and Sin, 1984).

We first briefly review the plant assumptions and the adaptive algorithms.

11 Review

We consider the control of a SISO ARMA model

$$A(q^{-1})\, y(k) = q^{-d} B(q^{-1})\, u(k) \tag{1}$$

where

$$\begin{aligned} A(q^{-1}) &= 1 + a_1 q^{-1} + \cdots + a_n q^{-n} \\ B(q^{-1}) &= b_o + b_1 q^{-1} + \cdots + b_m q^{-m}\,, \end{aligned}$$

and q^{-1} is the one-step delay operator, e.g. $q^{-1}\, y(k) = y(k-1)$ d is the number of pure delays in the system.

We assume that all plant zeros can be canceled by the control law, i.e. $B(q^{-1})$ is Schur and all zeros are sufficiently damped. Moreover, we also assume that the sign and a lower bound of the leading coefficient in $B(q^{-1})$ is known, i.e. without loss of generality,

$$b_o > b_{omin} > 0$$

where b_{omin} is known.

11.1 Control Objectives

We consider two control objectives:

1) Pole Placement: The poles of the closed loop system should be placed at specific locations in the complex plane. Specifically, we define the *closed loop pole polynomial* $A_c(q^{-1})$

$$A_c(q^{-1}) = B(q^{-1})\, A_c'(q^{-1}) \tag{2}$$

where

- $B^*(q)$ contains all plant zeros.
- $A_c^{'}(q^{-1})$ is a monic Schur polynomial chosen by the designer

$$A_c^{'}(q^{-1}) = 1 + a_{c1}^{'} q^{-1} + \cdots + a_{cn_c^{'}}^{'} q^{-n_c^{'}} \tag{3}$$

2) Tracking: The output sequence $y(k)$ must follow a *reference sequence* $y_d(k)$, which is known for all $k \geq 0$ and is generated by the following *reference model*

$$A_m(q^{-1}) y_d(k) = q^{-d}\, B_m(q^{-1})\, u_d(k) \tag{4}$$

where d is the plant pure delay, the Schur polynomial $A_m(q^{-1})$ the zero polynomial $B_m(q^{-1})$ and $u_d(k)$ is a deterministic input for the reference model are all selected by the designer. It is assume that the reference output $y_d(k+d)$ is available to the control system at the instance k.

Filter output and tracking error sequences

In order to simplify the stability analysis of the adaptive control scheme, we will define the following three filter sequences:

1. Filtered output sequence:

$$\eta(k) = A_c^{'}(q^{-1})\, y(k) \tag{5}$$

2. Desired filtered output sequence:

$$\eta_d(k) = A_c^{'}(q^{-1})\, y_d(k) \tag{6}$$

3. Filtered tracking error sequence:

$$\epsilon(k) = A_c^{'}(q^{-1})\ [y(k) - y_d(k)] = \eta(k) - \eta_d(k) \tag{7}$$

Thus, the overall control objective is the asymptotic converge of the filtered tracking error sequence

$$\lim_{k \to \infty} \epsilon(k) = 0 \tag{8}$$

Notice that, since $A_c^{'}(q^{-1})$ is Schur, Eq. (8) implies that

$$\lim_{k \to \infty} y(k) = y_d(k)\,.$$

11.2 Non-adaptive controller

Consider the following Diophantine equation

$$A_c^{'}(q^{-1}) = A(q^{-1})\, R^{'}(q^{-1}) + q^{-d}\, S(q^{-1}) \tag{9}$$

where

$$\begin{aligned} R^{'}(q^{-1}) &= 1 + r_1^{'} q^{-1} + \cdots + r_{n_r^{'}}\, q^{-n_r^{'}} \\ S(q^{-1}) &= s_o + s_1 q^{-1} + \cdots + s_{n_s} q^{-n_s} \end{aligned}$$

The minimum degree solution of (9) is obtained when

$$\begin{aligned} n_r^{'} &= d-1 \\ n_s &= \max(n-1, n_c^{'} - d) \end{aligned}$$

Multiplying Eq. (9) by $y(k)$ and utilizing the plant dynamics Eq. (1) we obtain

$$\eta(k) = A_c^{'}(q^{-1})\, y(k) = q^{-d} \left\{ R(q^{-1})\, u(k) + S(q^{-1}) y(k) \right\} . \tag{10}$$

where

$$\begin{aligned} R(q^{-1}) &= B(q^{-1})\, R^{'}(q^{-1}) \\ &= r_o + r_1 q^{-1} + \cdots + r_{n_r}\, q^{-n_r} \end{aligned}$$

and has order $n_r = m + d - 1$.

Setting the control law to

$$\begin{aligned} R(q^{-1})\, u(k) + S(q^{-1}) y(k) &= B_m(q^{-1})\, u_d(k)\,, \\ &= \eta_d(k+d) \end{aligned} \tag{11}$$

where $\eta_d(k) = A_c^{'}(q^{-1})\, y_d(k)$, achieves the tracking and regulation objective:

$$\eta(k) = \eta_d(k) \Longrightarrow \epsilon(k) = 0\,. \tag{12}$$

Eq. (10) can be written as follows

$$\eta(k) = \phi^T(k-d)\theta_c \tag{13}$$

where

- The parameter vector θ_c is

$$\theta_c = \left[\begin{array}{c} s_o \cdots s_{n_s} \, r_o \ \cdots \ r_{n_r} \end{array} \right]^T \in \Re^{N_c} \, ,$$

- $s_o \cdots s_{n_s}$ are the coefficients of $S(q^{-1})$.
- $r_o \, , \cdots \, , r_{n_r}$ are the coefficients of $R(q^{-1})$.
- $N_c = n_s + n_r + 2$.
- The regressor vector is given by

$$\phi(k) \;=\; \left[\begin{array}{cccccc} y(k) & \cdots & y(k-n_s) & u(k) & \cdots & u(k-n_r) \end{array} \right]^T \in \Re^{N_c} \qquad (14)$$

Notice that the control law in Eq. (11) can also be written as

$$\begin{aligned} \phi(k)^T\theta_c &= A_c'(q^{-1})y_d(k+d) \\ &= \eta_d(k+d)\,. \end{aligned} \qquad (15)$$

11.3 Adaptive controller

If the parameter vector θ_c is known, the fixed control law given by Eqs. (11) or (15) achieves (12). However, we now assume that θ_c, is unknown and we replace it by the parameter estimate at $\theta_c(k)$. Thus, the control law becomes

$$\begin{aligned} \hat{\theta}_c^{\,T}(k)\,\phi(k) &= A_c'(q^{-1})y_d(k+d)\,, \\ &= \eta_d(k+d) \end{aligned} \qquad (16)$$

The controller parameter vector estimate $\hat{\theta}_c(k)$ is updated by the series-parallel PAA, which based on Eq. (13)

$$\begin{aligned} e^o(k+1) &= \eta(k+1) - \phi^T(k-d+1)\hat{\theta}_c(k) \\ e(k+1) &= \frac{e^o(k+1)}{1+\phi^T(k-d+1)F(k)\phi(k-d+1)} \\ \hat{\theta}_c^o(k+1) &= \hat{\theta}_c(k) + F(k)\phi(k-d+1)\; e(k+1) \\ F(k+1) &= \frac{1}{\lambda_1(k)}\left[F(k) - \frac{F(k)\phi(k-d+1)\phi^T(k-d+1)F(k)}{\frac{\lambda_1(k)}{\lambda_2(k)} + \phi^T(k-d+1)F(k)\phi(k-d+1)}\right] \end{aligned} \qquad (17)$$

where $\eta(k)$ is defined in (5), $\phi(k)$ is defined in Eq. (14) and the adaptation gain matrix parameters $\lambda_1(k)$ and $\lambda_2(k)$ satisfy

$$\begin{aligned} 0 &< \lambda_1(k) \leq 1 \\ 0 &\leq \lambda_2(k) < 2 \end{aligned}$$

and are adjusted so that the maximum singular value of $F(k)$ is uniformly bounded, and

$$0 < K_{\min} \leq \lambda_{\min}\{F(k)\} \leq \lambda_{\max}\{F(k)\} < K_{max} < \infty \tag{18}$$

Also,

$$\hat{\theta}_c(k) \;=\; \begin{cases} \hat{\theta}^o_c(k) & \text{if} \quad \hat{r}^o_o(k) \geq b_{mino} \\ \left[\; \hat{s}_o(k) \cdots \hat{s}_{n_s}(k)\, b_{mino} \;\cdots\; \hat{r}_{n_r}(k) \;\right] & \text{if} \quad \hat{r}^o_o(k) < b_{mino} \end{cases} \tag{19}$$

The projection algorithm in the above equation is used to prevent the control signal from becoming too large if $\hat{r}^o_o(k)$ becomes too small.

Remarks:

The PAA in Eqs. (17) is slightly different from the one presented in Section 4.1 of the adaptive pole-placement and tracking control handout, where $y(k) = \theta_c^T \phi_f(k-d)$ and $\phi_f(k) = \frac{1}{A_c'}\,\phi(k)$ was used as the basis of the PAA. In general, $\frac{1}{A_c'(q^{-1})}$ will be a low-pass filter while $A_c'(q^{-1})$ will be a high-pass filter. Thus, the PAA presented in Section 4.1 of the adaptive pole-placement and tracking control handout will be less sensitive to high-frequency noise than the PAA presented in this handout. However, the stability analysis is simplified slightly by using the PAA presented in this handout.

12 Stability Analysis

Before beginning the stability analysis, let us define:

- Parameter estimate error:

$$\tilde{\theta}_c(k) = \theta_c - \hat{\theta}_c(k) \tag{20}$$

- A-posteriori output estimation error sequence:

$$e(k) \;=\; \eta(k) - \hat{\theta}_c^T(k)\phi(k-d) \tag{21}$$

where $\eta(k) = A_c'(q^{-1})\, y(k)$.

Notice that, from Eqs. (13) and (21), the a-posteriori error sequence can be expressed as follows

$$e(k) \;=\; \tilde{\theta}_c^T(k)\phi(k-d)\,. \tag{22}$$

On the other hand, from Eq. (17) the a-priori error sequence can be expressed as follows

$$e^o(k) \quad = \quad \tilde{\theta}_c^T(k-1)\phi(k-d)\,, \tag{23}$$
$$\tag{24}$$

while the filtered tracking error sequence can be expressed as follows

$$\epsilon(k) = \tilde{\theta}_c^T(k-d)\,\phi(k-d)\,. \tag{25}$$

Eq. (25) is obtained by re-writing Eq. (16) as

$$\eta_d(k) = A_c^{'}(q^{-1})\,y_d(k) = \hat{\theta}_c^{\,T}(k-d)\,\phi(k-d)\,, \tag{26}$$

and subtracting Eq. (26) from Eq. (13).

The control objective is to drive the tracking error to zero, which is equivalent to guaranteeing that

$$\lim_{k\to\infty} \epsilon(k) = 0 \tag{27}$$

since $A_c^{'}(q^{-1})$ is Schur.

Eq. (27) will be proven in four steps:

1. Prove that

$$\lim_{k\to\infty} e(k) = 0 \tag{28}$$

   using the Asymptotic Hyperstability theorem, where $e(k)$ is the a-posteriori output error signal and is related to the a-priori output error signal $e^o(k)$ by

$$e(k) = \frac{e^o(k)}{1+\phi(k-d)^T F(k-1)\phi(k-d)} \tag{29}$$

   This part of the proof is almost identical to the one in that was presented in the series-parallel identification notes.

2. Prove that

$$\lim_{k\to\infty} ||\tilde{\theta}_c(k) - \tilde{\theta}_c(k-1)|| = 0 \tag{30}$$

and

$$\lim_{k\to\infty} \frac{(e^o(k))^2}{1+\phi(k-d)^T F(k-1)\phi(k-d)} = 0 \tag{31}$$

and

$$\lim_{k\to\infty} \frac{\epsilon^2(k)}{1+\phi(k-d)^T F(k-1)\phi(k-d)} = 0\,, \tag{32}$$

where $\epsilon(k)$ is the filtered tracking error. Notice that Eq. (31) is somewhat stricter convergence condition than Eq. (28).

3. Prove that the vector norm of the filtered regressor vector is an affine function of the truncated infinity norm of the filtered tracking error, i.e. there exists finite positive constants C_1 and C_2 such that

$$||\phi(k-d)|| \le C_1 + C_2 \max_{j\in[0,k]} |\epsilon(j)|\,. \tag{33}$$

Eq. (33) is often called the *linear boundedness condition*.

4. Finally, utilize Goodwin's Lemma (Goodwin and Sin, 1984) to prove that

$$\lim_{k\to\infty} \epsilon(k) = 0\,. \tag{34}$$

Eq. (34) will be proven without proving that the parameter error vector $\tilde{\theta}_c(k)$ goes to zero and hence without requiring persistence of excitation.

Goodwin's technical Lemma:

Given sequences $\epsilon(k) \in \mathcal{R}$, $\phi(k) \in \mathcal{R}^{N_c}$, $b_1(k) \in \mathcal{R}$, and $b_2(k) \in \mathcal{R}$ which satisfy:

1.

$$\lim_{k\to\infty} \frac{\epsilon^2(k)}{b_1(k) + b_2(k)\,\phi(k-d)^T\phi(k-d)} = 0\,, \tag{35}$$

2. Uniform boundedness condition:

$$\begin{aligned} 0 &< b_1(k) < K < \infty \\ 0 &\le b_2(k) < K < \infty \end{aligned} \tag{36}$$

3. Linear boundedness condition:

$$||\phi(k-d)|| \leq C_1 + C_2 \max_{j\in[0,k]} |\epsilon(j)| \,. \tag{37}$$

where

$$||\phi(k)|| = \sqrt{\phi^T(k)\phi(k)}\,, \tag{38}$$

and $0 < C_1 < \infty$ and $0 < C_2 < \infty$.

Then, it follows that

(i) $\lim_{k\to\infty} \epsilon(k) = 0$.

(ii) $||\phi(k)|| < \infty$.

Proof of Goodwin's technical lemma:

The result is trivial if $|\epsilon(k)|$ and $||\phi(k)||$ are bounded, so assume that $\lim_{k\to\infty} |\epsilon(k)| = \infty$. In this case there exists a subsequence $\{k_n\}$ of the sampling index sequence $\{k\}$ such that

$$\lim_{k_n\to\infty} |\epsilon(k_n)| = \infty$$

and $|\epsilon(k)| \leq |\epsilon(k_n)|$ for $k \leq k_n$.

Now, along the subsequence $\{k_n\}$ we have

$$\lim_{k_n\to\infty} \frac{(\epsilon(k_n))^2}{b_1(k_n) + b_2(k_n)\phi^T(k_n-d)\phi(k_n-d)} \;\geq\; \lim_{k_n\to\infty} \frac{(\epsilon(k_n))^2}{K + K(C_1 + C_2|\epsilon(k_n)|)^2} \geq \frac{1}{K\,C_2^2} > 0\,,$$

where we have utilized Eqs. (36) and (37). This contradicts the right hand side of Eq. (35). Thus, $\epsilon(k)$ cannot be unbounded and the lemma is proven. Q.E.D.

12.1 Proof of Eq. (28)

To simplify the stability analysis, we will ignore the projection algorithm and re-write the error dynamics and PAA in Eqs. (13) in terms of the a-posteriori output error $e(k)$, the parameter error $\tilde{\theta}_c(k)$, and the inverse of the gain matrix $F(k)$:

Linear Block 1**:**

$$\begin{aligned} e(k) &= \tilde{\theta}_c^T(k)\,\phi(k-d) \\ &= m(k) \end{aligned} \tag{39}$$

Nonlinear Block NL**:**

$$\begin{aligned} \tilde{\theta}_c(k) &= \tilde{\theta}_c(k-1) - F(k-1)\phi(k-d)\,e(k) \\ F^{-1}(k) &= \lambda_1(k-1)F^{-1}(k-1) + \lambda_2(k-1)\,\phi(k-d)\phi^T(k-d) \\ w(k) &= -m(k) = -\tilde{\theta}_c(k)^T\,\phi(k-d) \end{aligned} \tag{40}$$

where $\phi(k)$ is defined in Eq. (14) and the adaptation gain matrix parameters $\lambda_1(k)$ and $\lambda_2(k)$ satisfy

$$\begin{aligned} 0 &< \lambda_1(k) \leq 1 \\ 0 &\leq \lambda_2(k) < 2 \end{aligned} \tag{41}$$

and are adjusted so that Eq. (18) is satisfied.

Defining

$$\lambda = \max_k \lambda_2(k) < 2\,, \tag{42}$$

and following the identical series of block diagram operations that were performed in the stability analysis presented in the handout *Adaptive Identification -Series Parallel Model*, Eqs. (39) and (40) can be described by the equivalent feedback block in Fig. 8 where,

$$s(k) = e(k) + \frac{\lambda_2(k-1)}{2} w(k) \tag{43}$$

12.1.1 Proof that block NL_1 in Fig. 8 is passive

In order to use the Asymptotic Hyperstability Theorem with the equivalent feedback system in Fig. 8, it is necesssary to show that the nonlinear block NL_1 in the figure is passive. Using Eq. (43) and Eqs. (40), we have:

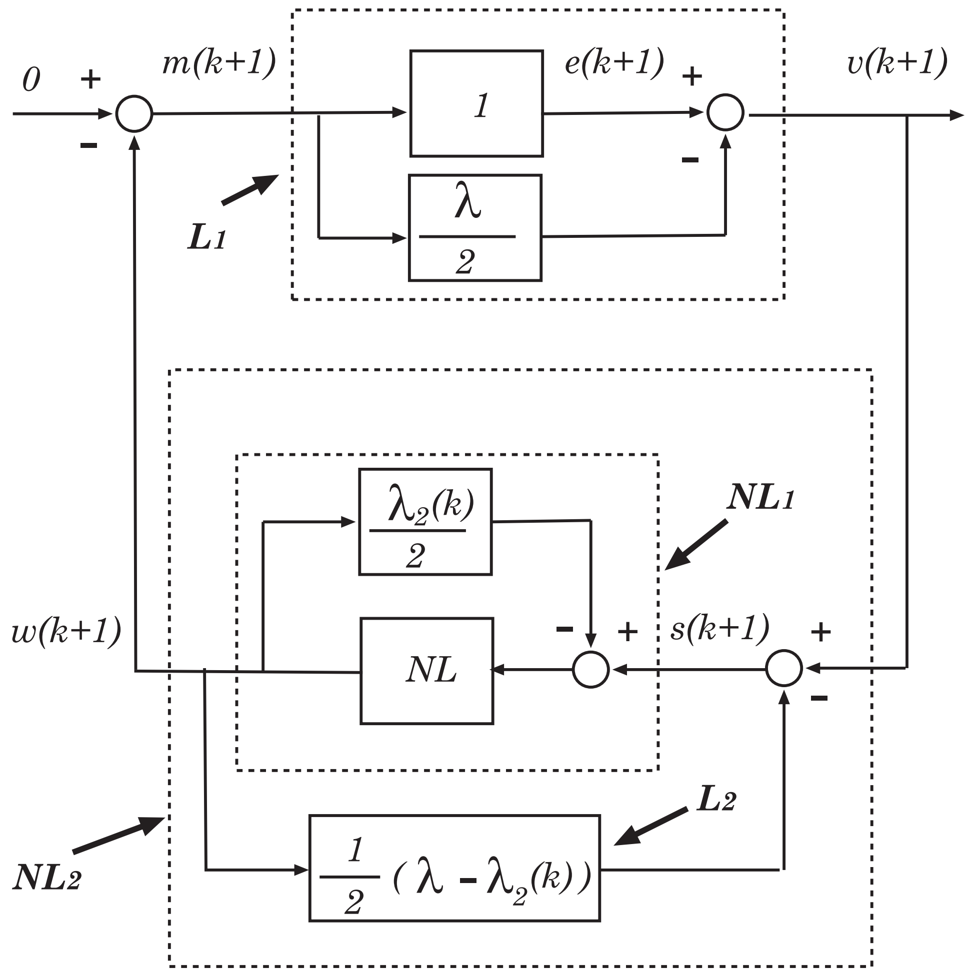

Figure 8: EquivalentFeedback System

$$
\begin{aligned}
\sum_{j=0}^{k} w(j)s(j) &= \sum_{j=0}^{k} w(j)\left\{e(j)+\lambda_2(j-1)w(j)\right\} \qquad (44)\\
&= \sum_{j=0}^{k} w(j)\,e(j)+\frac{1}{2}\sum_{j=0}^{k}\lambda_2(j-1)w^2(j)\\
&= -\sum_{j=0}^{k}\tilde{\theta}_c^T(j)\,\phi(j-d)e(j)+\frac{1}{2}\sum_{j=0}^{k}\lambda_2(j-1)\tilde{\theta}_c^T(j)\phi(j-d)\phi^T(j-d)\tilde{\theta}_c(j)\\
&= \sum_{j=0}^{k}\tilde{\theta}_c^T(j)\,F^{-1}(j-1)\left[\tilde{\theta}_c(j)-\tilde{\theta}_c(j-1)\right]+\frac{1}{2}\sum_{j=0}^{k}\tilde{\theta}_c^T(j)\left[F^{-1}(j)-\lambda_1(j-1)F^{-1}(j-1)\right]\tilde{\theta}_c(j)
\end{aligned}
$$

Completing the sum of squares in the first term in Eq. (44), we obtain

$$
\sum_{j=0}^{k} w(j)s(j) = \frac{1}{2}\sum_{j=0}^{k}\left[\tilde{\theta}_c^T(j)\,F^{-1}(j-1)\tilde{\theta}_c(j)\right. \qquad (45)
$$

$$
\begin{aligned}
& +||F^{-\frac{1}{2}}(j-1)(\tilde{\theta}_c^T(j) - \tilde{\theta}_c^T(j-1))||^2 - \tilde{\theta}_c^T(j-1)\,F^{-1}(j-1)\tilde{\theta}_c(j-1)\Big] \\
& + \frac{1}{2}\sum_{j=0}^{k}\Big[\tilde{\theta}_c^T(j)F^{-1}(j)\tilde{\theta}_c(j) - \lambda_1(j-1)\tilde{\theta}_c^T(j)F^{-1}(j-1)\tilde{\theta}_c(j)\Big] \\
= \;& \frac{1}{2}\sum_{j=0}^{k}\Big[\tilde{\theta}_c^T(j)F^{-1}(j)\tilde{\theta}_c(j) - \tilde{\theta}_c^T(j-1)\,F^{-1}(j-1)\tilde{\theta}_c(j-1)\Big] \\
& + \frac{1}{2}\sum_{j=0}^{k}\Big[(1-\lambda_1(j-1))\,\tilde{\theta}_c^T(j)F^{-1}(j-1)\tilde{\theta}_c(j) + ||F^{-\frac{1}{2}}(j-1)(\tilde{\theta}_c(j) - \tilde{\theta}_c(j-1))||^2\Big] \\
\geq \;& \frac{1}{2}\,\tilde{\theta}_c^T(k)F^{-1}(k)\tilde{\theta}_c(k) - \frac{1}{2}\,\tilde{\theta}_c^T(0)F^{-1}(0)\tilde{\theta}_c(0) \\
\geq \;& -\gamma_o^2 \qquad (46)
\end{aligned}
$$

where $\gamma_o^2 = \frac{1}{2}\,\tilde{\theta}_c^T(0)F^{-1}(0)\tilde{\theta}_c(0)$ and $(1-\lambda_1(k)) \geq 0$ by Eq. (41).

12.1.2 Proof that block NL_2 in Fig. 8 is passive

From Eqs. (42) and (41), the linear time varying but static block L_2 in Fig. 8 satisfies

$$\frac{1}{2}\,(\lambda - \lambda_2(k)) \geq 0$$

which makes it a passive block. Therefore, since block NL_2 is a feedback combination of two passive blocks: NL_1 and L_2, it is also a passive block.

12.1.3 Proof that block L_1 in Fig. 8 is SPR

From Eq. (42) the static and time invariant block L_1 satisfies

$$1 - \frac{\lambda}{2} > 0 \qquad (47)$$

Therefore, it is SPR.

12.1.4 Proof that $e(k)$ and $w(k)$ are bounded

From the sufficiency portion of the Hyperstability theorem and Eq. (47), it follows that

$$|v(k)| = \left|\left\{1 - \frac{\lambda}{2}\right\} w(k)\right| < \infty \Longrightarrow |w(k)| < \infty \Longrightarrow |e(k)| < \infty$$

where $v(k)$ is the output of the LTI block L_1 in Fig. 8.

12.1.5 Proof that $e(k)$ and $w(k)$ converge to zero

From the sufficiency portion of the Asymptotic Hyperstability theorem and Eq. (47), it follows that

$$\lim_{k\to\infty} v(k) = 0 \Longrightarrow \lim_{k\to\infty} w(k) = 0 \quad \text{and} \quad \lim_{k\to\infty} e(k) = 0$$

Q.E.D.

12.2 Proof of Eqs. (30) and (31)

Let us define

$$\Delta\tilde{\theta}_c(k) = \tilde{\theta}_c(k) - \tilde{\theta}_c(k-1) \tag{48}$$

and return to Eq. (45), focusing on the term

$$\frac{1}{2}\sum_{j=0}^{k} ||F^{-\frac{1}{2}}(j-1)(\tilde{\theta}_c(j) - \tilde{\theta}_c(j-1))||^2 = \frac{1}{2}\sum_{j=0}^{k} \Delta\tilde{\theta}_c^T(j)F^{-1}(j-1)\Delta\tilde{\theta}_c(j)$$

in that equation.

From Eq. (45), we have

$$\frac{1}{2}\sum_{j=0}^{k} \Delta\tilde{\theta}_c^T(j)F^{-1}(j-1)\Delta\tilde{\theta}_c(j) \le \sum_{j=0}^{k} w(j)s(j) + \frac{1}{2}\tilde{\theta}_c^T(0)F^{-1}(0)\tilde{\theta}_c(0)\,. \tag{49}$$

From Fig. 8 we notice that

$$s(k) = v(k) - \frac{1}{2}\left[\lambda - \lambda_2(k-1)\right] w(k)\,.$$

Therefore, since $[\lambda - \lambda_2(k)] \ge 0$,

$$\begin{aligned}\sum_{j=0}^{k} w(j)v(j) &= \sum_{j=0}^{k} w(j)\left\{s(j) + \frac{1}{2}\left[\lambda - \lambda_2(j-1)\right] w(j)\right\} \\ &\ge \sum_{j=0}^{k} w(j)\, s(j)\end{aligned}$$

and, from Eq. (49), we obtain

$$\frac{1}{2}\sum_{j=0}^{k} \Delta\tilde{\theta}_c^T(j)F^{-1}(j-1)\Delta\tilde{\theta}_c(j) \le \sum_{j=0}^{k} w(j)v(j) + \frac{1}{2}\tilde{\theta}_c^T(0)F^{-1}(0)\tilde{\theta}_c(0)\,. \tag{50}$$

On the other hand, because the LTI block L_1 in Fig. 8 is SPR, we also have

$$\sum_{j=0}^{k} v(j)m(j) \geq -\gamma_1^2 \tag{51}$$

where $\gamma_1^2 < \infty$ is a constant.

Noticing that $m(k) = -w(k)$ and combining Eqs. (51) and (50) we obtain

$$\frac{1}{2}\sum_{j=0}^{k} \Delta\tilde{\theta}_c^T(j)F^{-1}(j-1)\Delta\tilde{\theta}_c(j) \leq \gamma_1^2 + \frac{1}{2}\tilde{\theta}_c^T(0)F^{-1}(0)\tilde{\theta}_c(0) < \infty\,. \tag{52}$$

Thus, since $\Delta\tilde{\theta}_c^T(j)F^{-1}(j-1)\Delta\tilde{\theta}_c(j) \geq 0$,

$$\lim_{k\to\infty} \Delta\tilde{\theta}_c^T(k)F^{-1}(k-1)\Delta\tilde{\theta}_c(k) = 0\,. \tag{53}$$

From assumption (18), we have

$$\lambda_{\min}\left\{F^{-1}(k)\right\} = \frac{1}{\lambda_{\max}\left\{F(k)\right\}} > \frac{1}{K_{\max}} > 0\,. \tag{54}$$

Thus, From Eq. (53) we obtain

$$\frac{1}{K_{\max}}\lim_{k\to\infty} ||\Delta\tilde{\theta}_c(k)||^2 \leq \lim_{k\to\infty} \Delta\tilde{\theta}_c^T(k)F^{-1}(k-1)\Delta\tilde{\theta}_c(k) = 0$$

from where Eq. (30) is obtained. It should be emphasized that Eq. (30)implies that the parameter estimate error converges to a constant value, but not necessarily zero, i.e.

$$\lim_{k\to\infty} \tilde{\theta}_c(k) = \bar{\theta}\,.$$

From the PAA we have

$$\Delta\tilde{\theta}_c(k) = F(k-1)\phi(k-d)\,e(k)\,.$$

Inserting this equation into Eq. (53) we obtain

$$\lim_{k\to\infty} \phi^T(k-d)F(k-1)\phi(k-d)e^2(k) = 0\,. \tag{55}$$

We have already established Eq. (28). Therefore,

$$\lim_{k\to\infty} \left[1 + \phi^T(k-d)F(k-1)\phi(k-d)\right] e^2(k) = 0$$

and by Eq. (29), we obtain Eq. (32).

Q.E.D.

12.3 Proof of Eq. (32)

From Eqs. (25) and (23) we obtain

$$\epsilon(k) = e^o(k) - \phi(k-d)^T \left(\hat{\theta}_c(k-1) - \hat{\theta}_c(k-d)\right)$$

Thus,

$$\begin{aligned}\frac{\epsilon(k)}{(1+\phi^T(k-d)F(k-1)\phi(k-d))^{\frac{1}{2}}} &= \frac{e^o(k)}{(1+\phi^T(k-d)F(k-1)\phi(k-d))^{\frac{1}{2}}} \\ &- \frac{\phi(k-d)^T \left(\hat{\theta}_c(k-1) - \hat{\theta}_c(k-d)\right)}{(1+\phi^T(k-d)F(k-1)\phi(k-d))^{\frac{1}{2}}}\end{aligned}$$

Notice that the convergence of the parameter estimates, Eq. (30), implies that

$$\lim_{k\to\infty} ||\hat{\theta}_c(k) - \hat{\theta}_c(k-L)|| = 0$$

for any finite L. Thus,

$$\lim_{k\to\infty} \frac{\epsilon(k)}{(1+\phi^T(k-d)F(k-1)\phi(k-d))^{\frac{1}{2}}} = 0 \tag{56}$$

since by assumption (18),

$$\begin{aligned}\lim_{k\to\infty} \left|\frac{\phi(k-d)^T \left(\hat{\theta}_c(k-1) - \hat{\theta}_c(k-d)\right)}{(1+\phi^T(k-d)F(k-1)\phi(k-d))^{\frac{1}{2}}}\right| &\le \lim_{k\to\infty} \frac{||\phi(k-d)||}{(1+\phi^T(k-d)F(k-1)\phi(k-d))^{\frac{1}{2}}} ||\hat{\theta}_c(k) - \hat{\theta}_c(k-d)|| \\ &\le \frac{1}{\sqrt{K_{min}}} \lim_{k\to\infty} ||\hat{\theta}_c(k) - \hat{\theta}_c(k-d)|| = 0\,,\end{aligned}$$

where $0 < K_{\min} \le \lambda_{\min}\{F(k)\}$.

12.4 Proof of Eq. (33)

Under the following conditions:

(a) $|u_d(k)| \leq \bar{u}_d < \infty$ for all $k \geq 0$

(b) $A_c'(q^{-1})$ is Schur

(c) $B(q^{-1})$ is Schur,

the norm of the regressor vector is linearly bounded by the tracking error, i.e. there exists constants $0 \leq C_1 < \infty$ and $0 \leq C_2 < \infty$, such that

$$||\phi(k-d)|| \leq C_1 + C_2 \max_{j \in [0,k]} |\epsilon(j)| \tag{57}$$

The first two conditions are necessary to guarantee that $y_d(k)$ remains bounded. The third condition is necessary to guarantee that the plant has a stable inverse.

Proof: Given in the appendix.

12.5 Proof of Eq. (34) using Goodwin's technical lemma

The only step that remains to be proven, in order to use Goodwin's technical lemma, is to prove Eq. (35). This can easily be proven from Eq. (32) and assumption (18):

$$\frac{(e^o(k))^2}{1+\phi(k-d)^T F(k-1)\phi(k-d)} \geq \frac{(e^o(k))^2}{1+\lambda_{\max}\{F(k)\}\ \phi(k-d)^T\phi(k-d)}$$

Thus, Eq. (35) is obtained from Eq. (32) by setting

$$b_1(k) = 1 \qquad \text{and} \qquad b_2(k) = \lambda_{\max}\{F(k)\}\ .$$

Thus, all conditions in Goodwin's technical lemma apply and Eq. (34) follows.

Q.E.D.

It is important to note that the convergence of the tracking error to zero was proven without requiring persistence of excitation nor requiring that the parameter estimate vector $\hat{\theta}(k)$ converge to the true parameter vector θ_c.

12.6 Appendix

Proof of the Boundness Theorem in section 12.4

The proof is based on the following basic result from linear system's theory:

12.6.1 Lemma

Consider the discrete time SISO LTI system

$$y(k) \quad = \quad \frac{B(q^{-1})}{A(q^{-1})}\, u(k) \tag{58}$$

where $A^*(q) = q^n A(q^{-1})$ is Schur. Then

(a) There exist constants $0 \leq K_1 < \infty$ and $0 \leq K_2 < \infty$, which are independent of N such that

$$\sum_{k=1}^{N} |y(k)|^2 \leq K_1 + K_2 \sum_{k=1}^{N} |u(k)|^2 \,. \tag{59}$$

(b) There exist constants $0 \leq m_1 < \infty$ and $0 \leq m_2 < \infty$, which are independent of k such that

$$|y(k)| \leq m_1 + m_2 \max_{1 \leq j \leq N} |u(j)| \qquad \text{for all } 1 \leq k \leq N\,. \tag{60}$$

Proof:

The proof of this lemma can be found in most linear system theory textbooks. Here we prove only part b).

Define a state space realization of the SISO LTI system (58)

$$\begin{aligned} x(k+1) &= A\, x(k) + B\, u(k) \\ y(k) &= C\, x(k) + D\, u(k)\,, \end{aligned}$$

where all eigenvalues of A are strictly inside the unit circle. The solution to this system is

$$y(k) = C\, A^k x(0) + C\, Du(k) + \sum_{j=1}^{k} C\, A^{j-1}\, Bu(k-j)\,.$$

Taking absolute value and utilizing some basic matrix algebra we obtain

$$|y(k)| \leq ||C||\, ||A^k||\, ||x(0)|| + ||C||\, |D|\, \bar{u}_m + \left\{ \sum_{j=1}^{k} ||C||\, ||A^{j-1}||\, ||B|| \right\} \bar{u}_m$$

where $\bar{u}_m = \max_{0 \leq j \leq k} |u(j)|$. Since all eigenvalues of A are strictly inside the unit circle we have

$$||A^j|| \leq K\, \lambda^j$$

where $0 \leq K < \infty$ and $0 \leq \lambda < 1$. Thus,

$$\begin{aligned} |y(k)| &\leq K_1 \lambda^k + K_2 \bar{u}_m + \left\{ \sum_{j=1}^{k} \lambda^{j-1} \right\} K_3 \bar{u}_m \\ &\leq K_1 \lambda^k + \left(K_2 + \frac{K_3}{1-\lambda} \right) \bar{u}_m \\ &\leq m_1 + m_2 \bar{u}_m \end{aligned}$$

where $0 \leq m_1 < \infty$ and $0 \leq m_2 < \infty$. Q.E.D.

We now proceed with our proof.

(i) Boundness of $y_d(k)$:

Since $A_c'(q^{-1})$ is Schur and $|u_d(k)| \leq \bar{u}_d < \infty$, we obtain from Lemma 12.6.1 and Eq. (4)

$$|y_d(k)| \leq m_{m1} + m_{m2} \max_{1 \leq j \leq N} |u_d(j)| \leq m_{m1} + m_{m2} \bar{u}_d \leq \bar{y}_m < \infty \,. \tag{61}$$

(ii) $y(k)$ is linearly bounded by $\epsilon(k)$:

Recall the definition of $\epsilon(k)$:

$$\epsilon(k) \;=\; A_c'(q^{-1}) \left(y(k) - y_d(k) \right) \,.$$

Then, since $A_c'(q^{-1})$ is Schur we obtain

$$|y(k)| - |y_d(k)| \leq \left| \frac{1}{A_c'(q^{-1})} \epsilon(k) \right|$$

which implies that

$$|y(k)| \leq \bar{y}_m + \left(m_1 + m_2 \max_{1 \leq j \leq k} |\epsilon(j)| \right) = m_{y1} + m_{y2} \max_{1 \leq j \leq k} |\epsilon(j)| \,, \tag{62}$$

where we have used Eq. (61).

(iii) $u(k-d)$ is linearly bounded by $y(k)$:

Here we use the fact that $B^*(q) = q^m B(q^{-1})$ is Schur. Thus, the plant has a stable inverse:

$$u(k-d) = \frac{A(q^{-1})}{B(q^{-1})} y(k)$$

Thus,

$$|u(k-d)| \leq m_{u1} + m_{u2} \max_{1 \leq j \leq k} |y(j)| \tag{63}$$

(iv) $||\phi(k-d)||$ IS LINEARLY BOUNDED BY $y(k)$:

From the definition of $\phi(k)$ given in Eq. (14),

$$||\phi(k-d)||^2 = \sum_{j=0}^{n_s} |y(k-d-j)|^2 + \sum_{j=0}^{n_r} |u(k-d-j|^2$$

Therefore,

$$\begin{aligned} ||\phi(k-d)|| &\leq (1+n_s) \max_{1\leq j\leq k-d} |y(j)| + (1+n_r) \max_{1\leq j\leq k-d} |u(j)| \\ &\leq (1+n_s) \max_{1\leq j\leq k} |y(j)| + (1+n_r) \left[m_{u1} + m_{u2} \max_{1\leq j\leq k} |y(j)| \right] \\ &\leq m_{p1} + m_{p2} \max_{1\leq j\leq k} |y(j)| \end{aligned} \tag{64}$$

where we have used Eq. (63). Combining (38) and (62) we obtain our desired result Eq. (57).

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