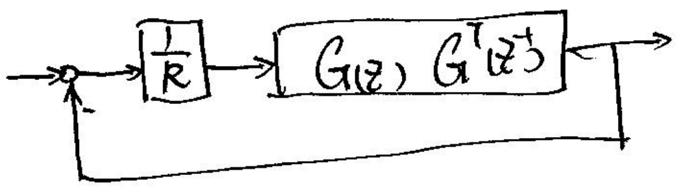
1 Transfer fcn for this system $\dfrac{z-1}{z^2-0.5z-0.6} = G(z)$

symmetric root locus

$$1 + \frac{1}{R} G(z) G^T(z^{-1}) = \text{return difference}$$

plot the



$$G(z)\,G(z^{-1}) = \frac{z-1}{z^2-0.5z-0.6} \cdot \frac{-z^2+z}{1-0.5z-0.6z^2}$$

Check matlab publish for root locus

2. the $R \to 0$, close loop poles $=$ stable openloop poles

$R \to \infty$ close loop poles $=$ stable open loop zeros

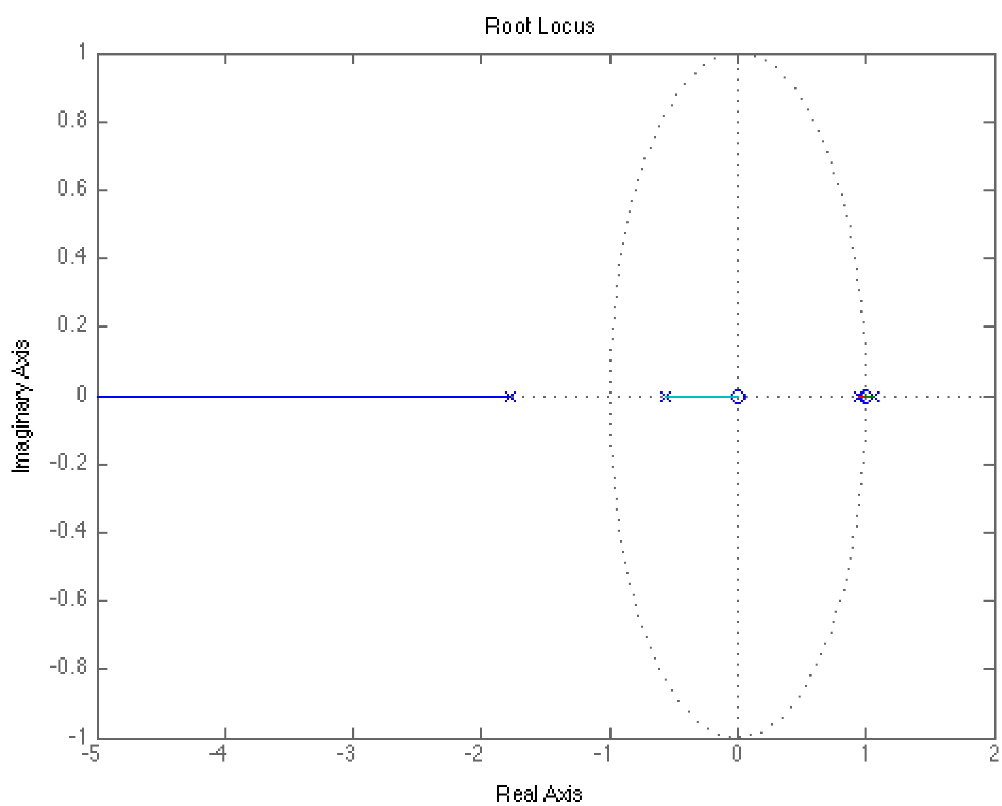
problem1

part 1 Since the problem is in controllable canonical form. The transfer function is

```
num1 = [1 -1];
den1 = [1 -0.5 -0.6];
sys1= tf(num1,den1,-1);

num2 = [-1 1 0];
den2 = [-0.6 -0.5 1];
sys2 = tf(num2,den2,-1);

rlocus(sys1*sys2)
```



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problem2

Suppose a causal output feedback controller achieves $E\{u^2(k)\} \leq \alpha$. Since the control law is suboptimal in terms of the cost function $J(\rho)$, we have

$$\begin{aligned} J(\rho) &\leq E\{x^T(k)Qx(k) + \rho u^T(k)u(k)\} \\ &= E\{x^T(k)Qx(k)\} + \rho E\{u^T(k)u(k)\} \\ &\leq E\{x^T(k)Qx(k)\} + \alpha\rho \end{aligned}$$

Rearranging terms, we have

$$E\{x^T(k)Qx(k)\} \geq J(\rho) - \alpha\rho$$

Problem3

(20 points) Show the given PAA is asymptotically stable
Stability analysis based on Hyperstability generally has three steps:
step 1: Transform the MRAS into the form of an equivalent feedback system composed of two blocks, LTI block in the feedforward path and Nonlinear block in the feedback path.
step 2: Find solutions for the part of the adaptation laws which appears in the feedback path of the equivalent system such that the Popov integral inequality is satisfied.
step 3: Check if the remaining part of the adaptation law which appears in the feedforward path is PR or SPR for hyperstability or asymptotic hyperstability.
We will apply this procedure in this problem.

i. Step 1: separate the adaptation law into feedforward and feedback components.

$$e(k+1) = y(k+1) - \hat{y}(k+1) = -\begin{bmatrix}\hat{a}(k+1) - a & \hat{b}(k+1) - b\end{bmatrix}\begin{bmatrix}y(k)\\u(k)\end{bmatrix} = -\tilde{\theta}^T(k+1)\phi(k) = -w(k+1)$$

with $\tilde{\theta}(k+1) = \begin{bmatrix}\tilde{a}(k+1)\\\tilde{b}(k+1)\end{bmatrix} = \begin{bmatrix}\hat{a}(k+1) - a\\\hat{b}(k+1) - b\end{bmatrix}$, $\phi(k) = \begin{bmatrix}y(k)\\u(k)\end{bmatrix}$, and $w(k+1) = \tilde{\theta}^T(k+1)\phi(k)$ is the output of the feedback components. Therefore the feedforward component is just 1, which is obvious SPR (step 3 can be skipped).
Find the feedback components:

$$\begin{aligned} \tilde{a}(k+1) &= \hat{a}(k+1) - a = \hat{a}_I(k) - a + (k_{aI} + k_{aP})y(k)e(k+1) \\ &= \tilde{a}(k) - \hat{a}_P(k) + (k_{aI} + k_{aP})y(k)e(k+1) \\ &= \tilde{a}(k) - \hat{a}_P(k) + \hat{a}_P(k+1) + k_{aI}y(k)e(k+1) \\ \Longrightarrow \underbrace{\begin{bmatrix}\tilde{a}(k+1)\\\tilde{b}(k+1)\end{bmatrix}}_{\tilde{\theta}(k+1)} &= \underbrace{\begin{bmatrix}\tilde{a}(k)\\\tilde{b}(k)\end{bmatrix}}_{\tilde{\theta}(k)} + \begin{bmatrix}\hat{a}_P(k+1)\\\hat{b}_P(k+1)\end{bmatrix} - \begin{bmatrix}\hat{a}_P(k)\\\hat{b}_P(k)\end{bmatrix} + \begin{bmatrix}k_{aI} & 0\\0 & k_{bI}\end{bmatrix}\phi(k)e(k+1) \end{aligned}$$

$$
\vdots
$$

$$
= \tilde{\theta}(0) + \begin{bmatrix} \hat{a}_P(k+1) \\ \hat{b}_P(k+1) \end{bmatrix} - \begin{bmatrix} \hat{a}_P(0) \\ \hat{b}_P(0) \end{bmatrix} + F\sum_{i=0}^{k} \phi(i)e(i+1), \quad F = \begin{bmatrix} k_{aI} & 0 \\ 0 & k_{bI} \end{bmatrix}
$$

ii. Step 2: Check if the feedback component satisfies the Popov inequality.

$$
\begin{aligned}
\eta(0,k_1) &= \sum_{k=0}^{k_1} e(k+1)w(k+1) = \sum_{k=0}^{k_1} e(k+1)\phi^T(k)\tilde{\theta}(k+1) \\
&= \sum_{k=0}^{k_1} e(k+1)\phi^T(k)\left(\tilde{\theta}(0) + \begin{bmatrix} \hat{a}_P(k+1) \\ \hat{b}_P(k+1) \end{bmatrix} - \begin{bmatrix} \hat{a}_P(0) \\ \hat{b}_P(0) \end{bmatrix} + F\sum_{i=0}^{k} \phi(i)e(i+1)\right) \\
&= \sum_{k=0}^{k_1} e(k+1)\phi^T(k)\left(F\sum_{i=0}^{k} \phi(i)e(i+1) + \underbrace{\tilde{\theta}(0) - \begin{bmatrix} \hat{a}_P(0) \\ \hat{b}_P(0) \end{bmatrix}}_{\triangleq c} + \begin{bmatrix} k_{aP} & 0 \\ 0 & k_{bP} \end{bmatrix}\phi(k)e(k+1) \right) \\
&= \sum_{k=0}^{k_1} e(k+1)\phi^T(k)\left(F\sum_{i=0}^{k} \phi(i)e(i+1) + c\right) + \sum_{k=0}^{k_1} \underbrace{e(k+1)\phi^T(k)\begin{bmatrix} k_{aP} & 0 \\ 0 & k_{bP} \end{bmatrix}\phi(k)e(k+1)}_{\geq 0} \\
&\geq \sum_{k=0}^{k_1} e(k+1)\phi^T(k)\left(F\sum_{i=0}^{k} \phi(i)e(i+1) + c\right) \\
&\geq -\frac{1}{2}c^T F^{-1} c
\end{aligned}
$$

Here, see equation PIAC-49 in the Reader for the last step in the derivation. The feedback component thus satisfies the Popov inequality.

An alternative way to show the Popov inequality is as follows: we have

$$
\begin{aligned}
\hat{\theta}_I(k+1) &= \hat{\theta}_I(k) + F_I\phi(k)e(k+1) \\
\hat{\theta}_p(k+1) &= F_p\phi(k)e(k+1)
\end{aligned}
$$

Let

$$
\theta = \theta_I + \theta_p,\ \theta_p = 0
$$

then

$$
\begin{aligned}
\tilde{\theta}_I(k+1) = \hat{\theta}_I(k+1) - \theta_I &= \tilde{\theta}_I(k) + F_I\phi(k)e(k+1) \\
\tilde{\theta}_p(k+1) &= F_p\phi(k)e(k+1)
\end{aligned}
$$

These two sub systems both satisfy the Popov inequality (the first one is in standard form as an example in the reader; for the second one, note $\sum_{k=0}^{k_1}\tilde{\theta}_p^T(k+1)\phi(k)e(k+1) = \sum_{k=0}^{k_1}\tilde{\theta}_p^T(k+1)F_p^{-1}\tilde{\theta}_p(k+1) \geq 0$). Parallel connections of two passive systems are still passive. Hence the nonlinear block satisfies Popov inequality.

Since the feedback part satisfied the Popov inequality, and the feedforward part is SPR, the PAA is asymptotically hyperstable.

(b) (5 points) express $e(k+1)$ in terms of $e^o(k+1)$.

$$\begin{aligned}
e(k+1) &= y(k+1) - \hat{y}(k+1) \\
&= (a - \hat{a}(k+1))y(k) + (b - \hat{b}(k+1))u(k) \\
&= (a - \hat{a}_I(k))y(k) - (k_{aI} + k_{aP})y^2(k)e(k+1) + (b - \hat{b}_I(k))u(k) - (k_{bI} + k_{bP})u^2(k)e(k+1) \\
&= e^o(k+1) - ((k_{aI} + k_{aP})y^2(k) + (k_{bI} + k_{bP})u^2(k))e(k+1) \\
e(k+1) &= \frac{e^o(k+1)}{1 + (k_{aI} + k_{aP})y^2(k) + (k_{bI} + k_{bP})u^2(k)}
\end{aligned}$$

Problem 4

(20 points) A discrete time system is described by

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) + b_0 u(k-2) + w_1(k-2) + w_2(k-2) \tag{1}$$

(a) (10 points) All parameters are known.
This case is just a standard pole placement design.

$y(k)$ can be written as:

$$y(k) = \frac{z^{-d}B(z^{-1})}{A(z^{-1})}\left(u(k) + \frac{1}{b_0}w_1(k) + \frac{1}{b_0}w_2(k)\right) \tag{2}$$

where $d = 2$, $B(z^{-1}) = b_0$, $A(z^{-1}) = 1 + a_1 z^{-1} + a_2 z^{-2}$, and z^{-1} is understood as the delay operator. Choose u to cancel out w_1, w_2 and achieve the required closed loop characteristic equation.

$$u(k) = -\frac{1}{b_0}w_1(k) - \frac{1}{b_0}w_2(k) + \frac{1}{S(z^{-1})}\Big(- R(z^{-1})y(k) + D'(z^{-1})y_d(k+2)\Big) \tag{3}$$

where R and S have the following structures:

$$R(z^{-1}) = r_0 + r_1 z^{-1}, \quad S(z^{-1}) = s_0 + s_1 z^{-1} = b_0(\underbrace{1 + s_1' z^{-1}}_{S'(z^{-1})}) \tag{4}$$

R and S are solutions of the Diophantine equation:

$$A(z^{-1})S'(z^{-1}) + z^{-2}R(z^{-1}) = D'(z^{-1}) \tag{5}$$

where $D'(z^{-1}) = 1 + d_1 z^{-1} + d_2 z^{-2}$. Solving for r_0, r_1 and s_1', we get:

$$\begin{aligned}
s_1' &= d_1 - a_1 \\
r_0 &= d_2 - a_2 - a_1(d_1 - a_1) \\
r_1 &= -a_2(d_1 - a_1)
\end{aligned} \tag{6}$$

(b) (10 points) All parameters are unknown.
In this case, we pose the problem as an extended LS problem. From (5), consider:

$$
\begin{aligned}
D'(z^{-1})y(k) &= \Big(A(z^{-1})S'(z^{-1}) + z^{-2}R(z^{-1})\Big)y(k) \\
&= S(z^{-1})u(k-2) + R(z^{-1})y(k-2) + S'(z^{-1})\big(\gamma + \alpha\cos\omega(k-2) + \beta\sin\omega(k-2)\big) \\
&= s_0u(k-2) + s_1u(k-3) + r_0y(k-2) + r_1y(k-3) + \gamma(1+s_1') + \alpha\cos\omega(k-2) \\
&\quad + \alpha s_1'\cos\omega(k-3) + \beta\sin\omega(k-2) + \beta s_1'\sin\omega(k-3) \\
&= s_0u(k-2) + s_1u(k-3) + r_0y(k-2) + r_1y(k-3) + \gamma(1+s_1') \\
&\quad + (\alpha + \alpha s_1'\cos\omega - \beta s_1'\sin\omega)\cos\omega(k-2) + (\beta + \alpha s_1'\sin\omega + \beta s_1'\cos\omega)\sin\omega(k-2)
\end{aligned}
$$

Define $\gamma' := \gamma(1+s_1')$, $\alpha' := \alpha + \alpha s_1'\cos\omega - \beta s_1'\sin\omega$, and $\beta' := \beta + \alpha s_1'\sin\omega + \beta s_1'\cos\omega$. Then the adaptive control law becomes

$$
D'(z^{-1})y_d(k+2) = \hat{\theta}_e(k)\phi_e(k) \tag{7}
$$

where

$$
\theta_e := \begin{bmatrix} s_0 \\ s_1 \\ r_0 \\ r_1 \\ \gamma' \\ \alpha' \\ \beta' \end{bmatrix}; \quad \phi_e(k) := \begin{bmatrix} u(k) \\ u(k-1) \\ y(k) \\ y(k-1) \\ 1 \\ \cos\omega k \\ \sin\omega k \end{bmatrix}
$$

and $\hat{\theta}$ is updated as:

$$
\hat{\theta}_e(k+1) = \hat{\theta}_e(k) + \frac{F(k)\phi_e(k-1)}{1+\phi_e^T(k-1)F(k)\phi_e(k-1)}\Big(D'(z^{-1}y(k+1) - \hat{\theta}_e^T(k)\phi_e(k-1)\Big) \tag{8}
$$

For implementation, the explicit form of the adaptive control becomes:

$$
u(k) = \frac{1}{\hat{s_0}}\Big[D'(z^{-1})y_d(k+2) - \hat{s_1}u(k-1) - \hat{r_0}y(k) - \hat{r_1}y(k-1) - \hat{\gamma'} - \hat{\alpha'}\cos\omega k - \hat{\beta'}\sin\omega k\Big] \tag{9}
$$

problem 5

Cancel this problem

Additional problem 1

Consider the following stationary stochastic system

$$
\begin{aligned}
\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} &= \begin{bmatrix} 0.8 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}u(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}w(k) \\
y(k) &= x_1(k) + v(k)
\end{aligned} \tag{2}
$$

where $u(k)$ is a deterministic (known) input, $y(k)$ is the measured output, $w(k)$ and $v(k)$ are zero-mean, jointly Gaussian WSS random sequences with

$$
E\left\{\begin{bmatrix} w(k+j) \\ v(k+j) \end{bmatrix}\begin{bmatrix} w(k) & v(k) \end{bmatrix}\right\} = \begin{bmatrix} 0.225 & 0 \\ 0 & 0.625 \end{bmatrix}\delta(j)
$$

Design a minimum variance regulator for this system.

Solution to additional problem 1

First, we need to design Kalman filter. Solve the following Riccati equation

$$M = \bar{A}M\bar{A}^T + \bar{B}_w W \bar{B}_w^T - \bar{A}M\bar{C}^T \left[\bar{C}M\bar{C}^T + V\right]^{-1} \bar{C}M\bar{A}^T$$

where

$$\bar{A} = \begin{bmatrix} 0.8 & 1 \\ 0 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \bar{B}_w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad W = 0.225, \quad V = 0.625 \, .$$

Let

$$M = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} \succ 0 \, .$$

Then,

$$\begin{aligned}
\begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} &= \begin{bmatrix} 0.8 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} \begin{bmatrix} 0.8 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0.225 & 0 \\ 0 & 0 \end{bmatrix} - \frac{\begin{bmatrix} 0.8 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \begin{bmatrix} m_1 & m_2 \end{bmatrix} \begin{bmatrix} 0.8 & 0 \\ 1 & 0 \end{bmatrix}}{m_1 + 0.625} \\
&= \begin{bmatrix} 0.8^2 m_1 + 1.6 m_2 + m_3 + 0.225 & 0 \\ 0 & 0 \end{bmatrix} - \frac{1}{m_1 + 0.625} \begin{bmatrix} 0.8^2 m_1^2 + 1.6 m_1 m_2 + m_2^2 & 0 \\ 0 & 0 \end{bmatrix} \\
&\Rightarrow \begin{cases} m_2 = 0 \\ m_3 = 0 \\ m_1 = 0.8^2 m_1 + 0.225 - \frac{0.8^2 m_1^2}{m_1 + 0.625} = 0.225 + \frac{0.4 m_1}{m_1 + 4} \end{cases} \\
&\Rightarrow m_1 = 0.375 \\
&\Rightarrow L = \bar{A}M\bar{C}^T \left[\bar{C}M\bar{C}^T + V\right]^{-1} = \begin{bmatrix} 0.3 \\ 0 \end{bmatrix} .
\end{aligned}$$

Then, the output is

$$Y(z) = \bar{C}\left(zI - \bar{A}\right)^{-1} \bar{B}U(z) + \left[1 + \bar{C}\left(zI - \bar{A}\right)^{-1} L\right] E(z) \, .$$

Thus, we can get the transfer function

$$\begin{aligned}
y(k) &= \bar{C}\left(qI - \bar{A}\right)^{-1} \bar{B}u(k) + \left[1 + \bar{C}\left(qI - \bar{A}\right)^{-1} L\right] \epsilon(k) \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q - 0.8 & -1 \\ 0 & q \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -2 \end{bmatrix} u(k) + \left\{ 1 + \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q - 0.8 & -1 \\ 0 & q \end{bmatrix}^{-1} \begin{bmatrix} 0.3 \\ 0 \end{bmatrix} \right\} \epsilon(k) \\
&= \frac{q - 2}{q(q - 0.8)} u(k) + \frac{q - 0.5}{q - 0.8} \epsilon(k) \\
&= \frac{q^{-1}(1 - 2q^{-1})}{1 - 0.8q^{-1}} u(k) + \frac{1 - 0.5q^{-1}}{1 - 0.8q^{-1}} \epsilon(k) \, .
\end{aligned}$$

Thus, we know

$$A(q^{-1}) = 1 - 0.8q^{-1}, \qquad B^u(q^{-1}) = q^{-1} - 0.5, \qquad B^s(q^{-1}) = -2,$$
$$C(q^{-1}) = 1 - 0.5q^{-1}, \qquad \bar{B}^u(q^{-1}) = 1 - 0.5q^{-1}, \qquad d = 1\ .$$

Solve the Diophantine equation

$$C(q^{-1})\bar{B}^u(q^{-1}) = A(q^{-1})R(q^{-1}) + q^{-1}B^u(q^{-1})S(q^{-1})$$

where

$$R(q^{-1}) = 1 + r_1q^{-1}, \qquad S(q^{-1}) = s_0\ .$$

Then,

$$\begin{aligned}(1-0.5q^{-1})(1-0.5q^{-1}) &= (1-0.8q^{-1})(1+r_1q^{-1}) + q^{-1}(q^{-1}-0.5)s_0 \\ &\Rightarrow \begin{cases} -1 = r_1 - 0.8 - 0.5s_0 \\ 0.25 = -0.8r_1 + s_0 \end{cases} \\ &\Rightarrow \begin{cases} s_0 = 0.15 \\ r_1 = -0.125 \end{cases} \\ &\Rightarrow R(q^{-1}) = 1 - 0.125q^{-1},\ S(q^{-1}) = 0.15\ .\end{aligned}$$

Finally, we can get the minimum variance regulator feedback law:

$$u(k) = \frac{-S(q^{-1})}{B^s(q^{-1})R(q^{-1})}y(k) = \frac{0.075}{1-0.125q^{-1}}y(k)\ .$$