Tony Kelman
ME 233 – Homework 11

1.a)

$A(q^{-1})y(k)=q^{-d}B(q^{-1})\{u(k)+d(k)\},\ A(q^{-1})=(1-0.8q^{-1})(1-0.7q^{-1}),\ n=2,\ d=1,\ B(q^{-1})=0.1$

$d(k+8)=d(k)$ so $A_d(q^{-1})d(k)=0$ where $A_d(q^{-1})=1-q^{-8},\ n_d=N=8$

$u(k)=\dfrac{-S(q^{-1})}{A_d(q^{-1})R'(q^{-1})}y(k)$ and we want $A_c(q^{-1})=1$, so Diophantine equation is

$A_c(q^{-1})=A_d(q^{-1})A(q^{-1})R'(q^{-1})+q^{-d}B(q^{-1})S(q^{-1})$

$B(q^{-1})=B^s(q^{-1})B^u(q^{-1})$, so $B^s(q^{-1})=1,\ B^u(q^{-1})=0.1,\ m_u=m_s=0$

$A_c(q^{-1})=B^s(q^{-1})A_c'(q^{-1})$, so $A_c'(q^{-1})=1,\ n_c'=0$

$n_r'=d+m_u-1=0,\ n_s=\max(n+n_d-1,\ n_c'-d-m_u)=9$

$R'(q^{-1})=1,\ S(q^{-1})=s_0+s_1q^{-1}+s_2q^{-2}+s_3q^{-3}+s_4q^{-4}+s_5q^{-5}+s_6q^{-6}+s_7q^{-7}+s_8q^{-8}+s_9q^{-9}$

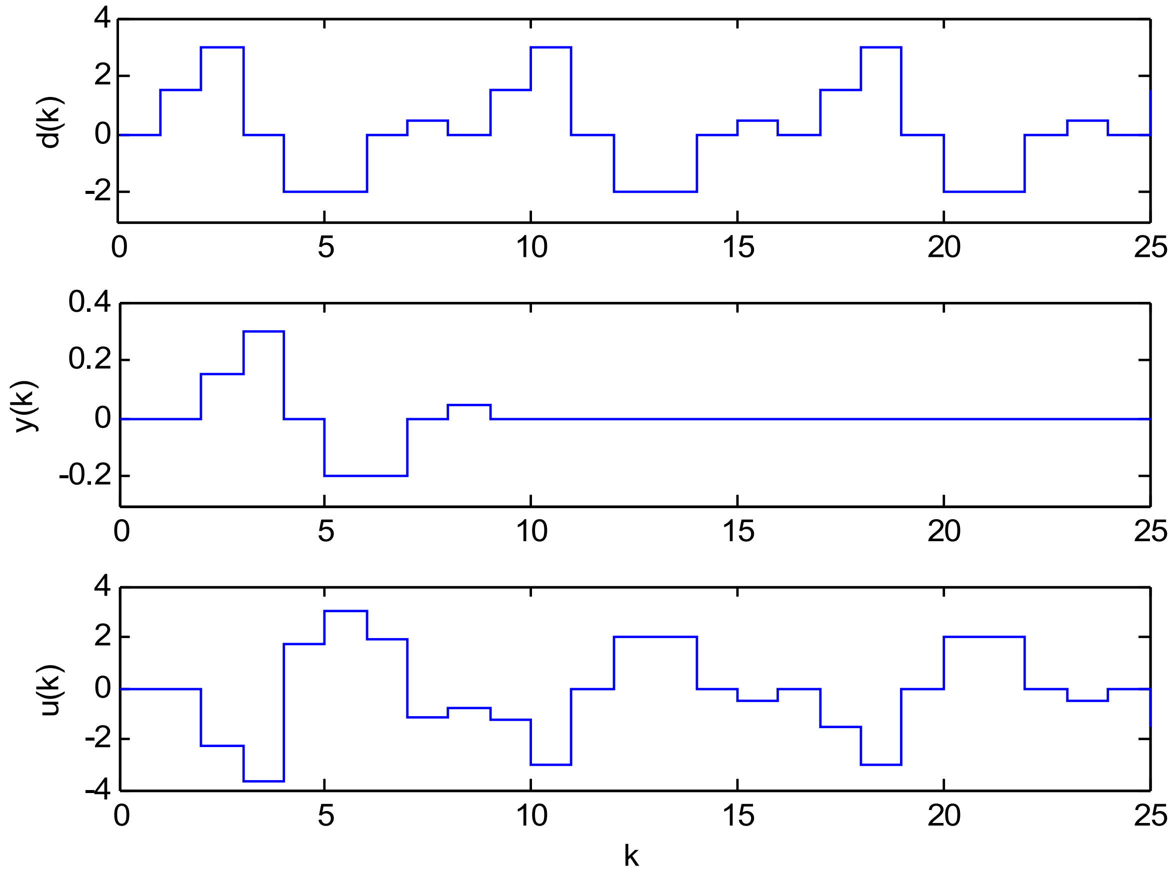
$1=(1-q^{-8})(1-0.8q^{-1})(1-0.7q^{-1})+0.1q^{-1}S(q^{-1})$

$1=(1-q^{-8})(1-1.5q^{-1}+0.56q^{-2})+0.1q^{-1}S(q^{-1})$

$1=1-1.5q^{-1}+0.56q^{-2}-q^{-8}+1.5q^{-9}-0.56q^{-10}+0.1q^{-1}S(q^{-1})$

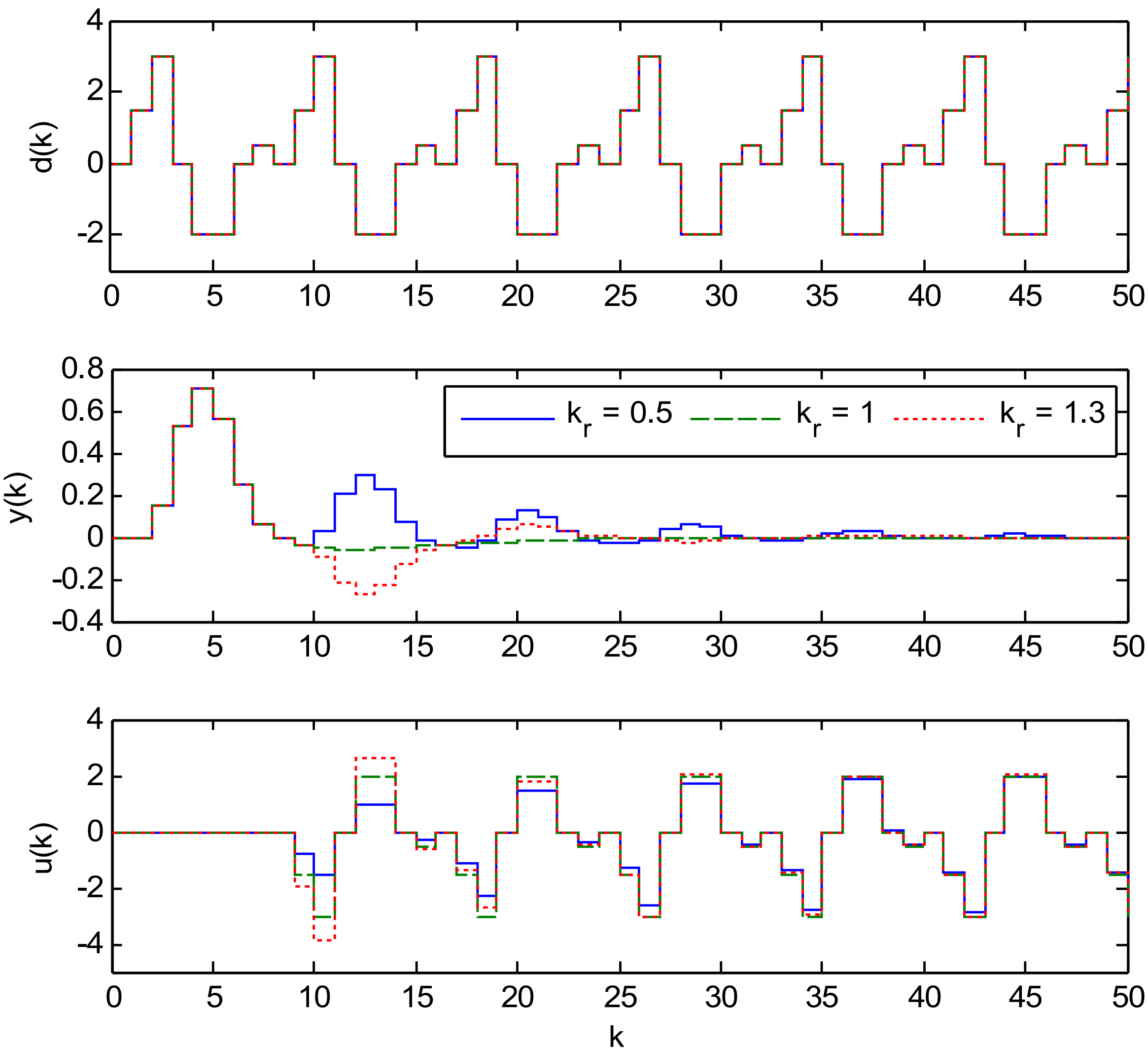
So $S(q^{-1})=15-5.6q^{-1}+10q^{-7}-15q^{-8}+5.6q^{-9}$

1.b)



1.c)

Results shown on next page for $u(k)=\dfrac{-k_r q^{-(N-d)}A(q^{-1})}{A_d(q^{-1})B(q^{-1})}y(k)$



1.d)

Now $A(q^{-1})=(1-0.2\,q^{-1})\bar{A}(q^{-1})$, $\bar{A}(q^{-1})=(1-0.8\,q^{-1})(1-0.7\,q^{-1})$, $B(q^{-1})=0.08\,q^{-1}$, $d=1$

$u(k)=\dfrac{-k_r q^{-(N-d)}\bar{A}(q^{-1})}{0.1\,A_d(q^{-1})}\,y(k)$, actual plant $G_A(s)=\dfrac{0.1\,q^{-1}}{(1-0.8\,q^{-1})(1-0.7\,q^{-1})}\dfrac{0.8\,q^{-1}}{(1-0.2\,q^{-1})}$

simplified plant $G(s)=\dfrac{0.1\,q^{-1}}{(1-0.8\,q^{-1})(1-0.7\,q^{-1})}$ used to design the control

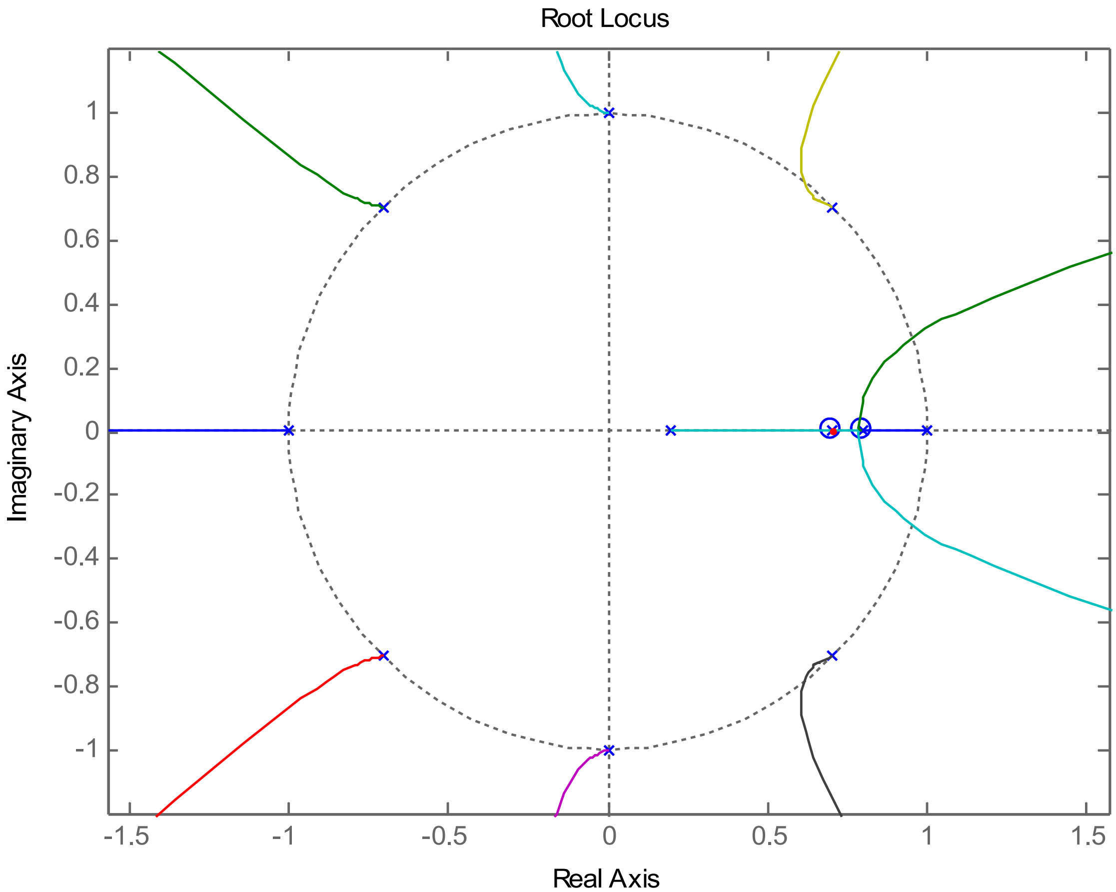
Root locus of $\dfrac{q^{-(N-d)}\bar{A}(q^{-1})}{0.1\,A_d(q^{-1})}G_A(s)=\dfrac{0.8\,q^{-9}(1-0.8\,q^{-1})(1-0.7\,q^{-1})}{(1-0.8\,q^{-1})(1-0.7\,q^{-1})(1-0.2\,q^{-1})}$ shown on next page

There are two pole-zero cancellations, at 0.7 and 0.8, that result in stable eigenvalues for all k_r

5 eigenvalues are on the unit circle for $k_r=0$ and unstable for all $k_r>0$

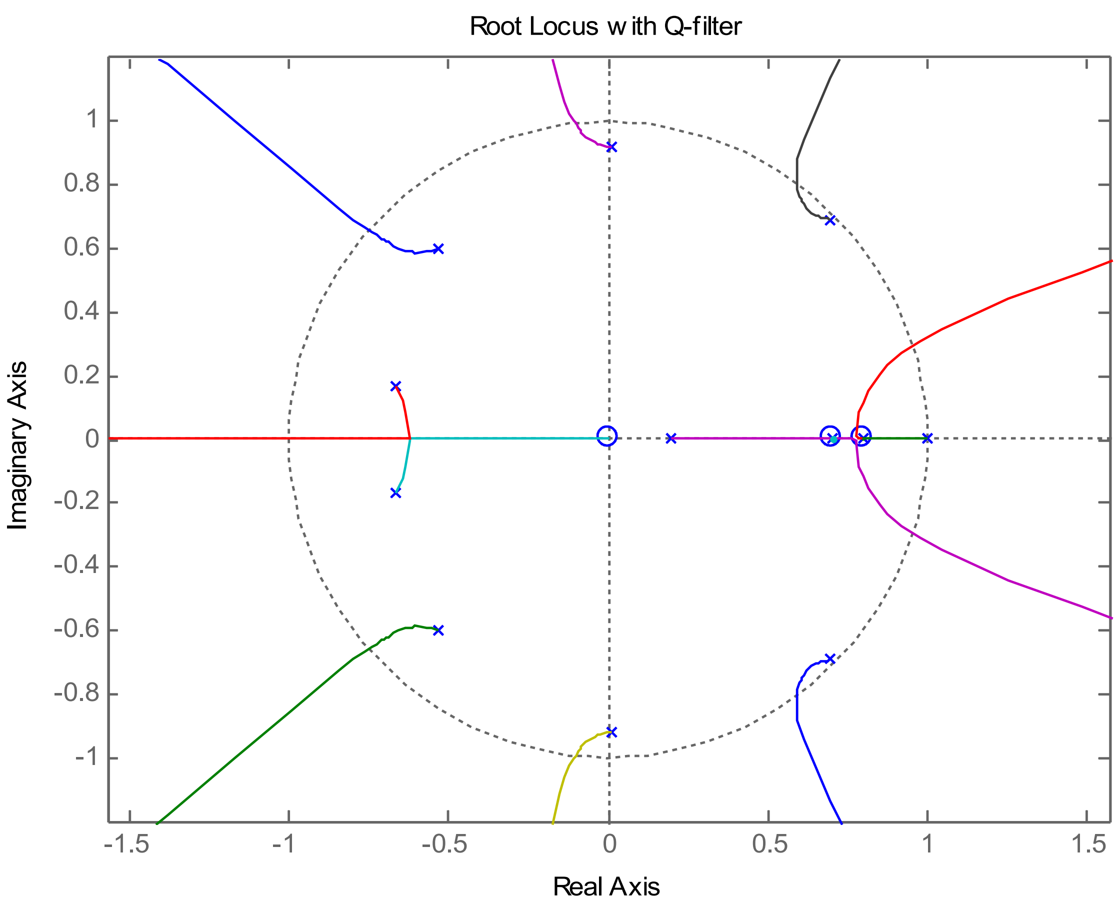
The 2 eigenvalues that start at 0.2 and 1 when $k_r=0$ are only stable for $k_r\leq 1.888$

The last 2 eigenvalues that start at $0.707\pm 0.707\,j$ when $k_r=0$ are only stable for $k_r\leq 1.04$



Root Locus
Imaginary Axis
Real Axis
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.5
-1
-0.5
0
0.5
1
1.5

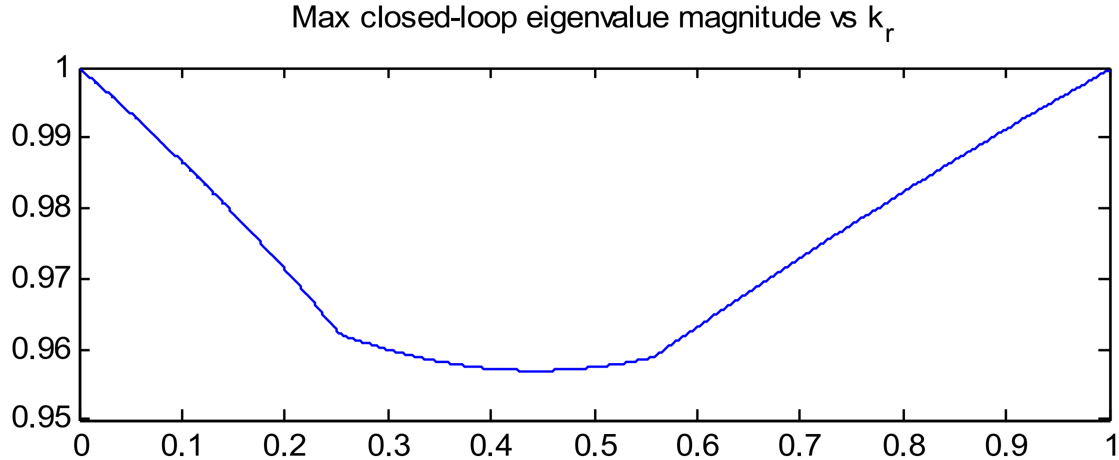
1.e)i)



Root Locus with Q-filter
Imaginary Axis
Real Axis
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.5
-1
-0.5
0
0.5
1
1.5

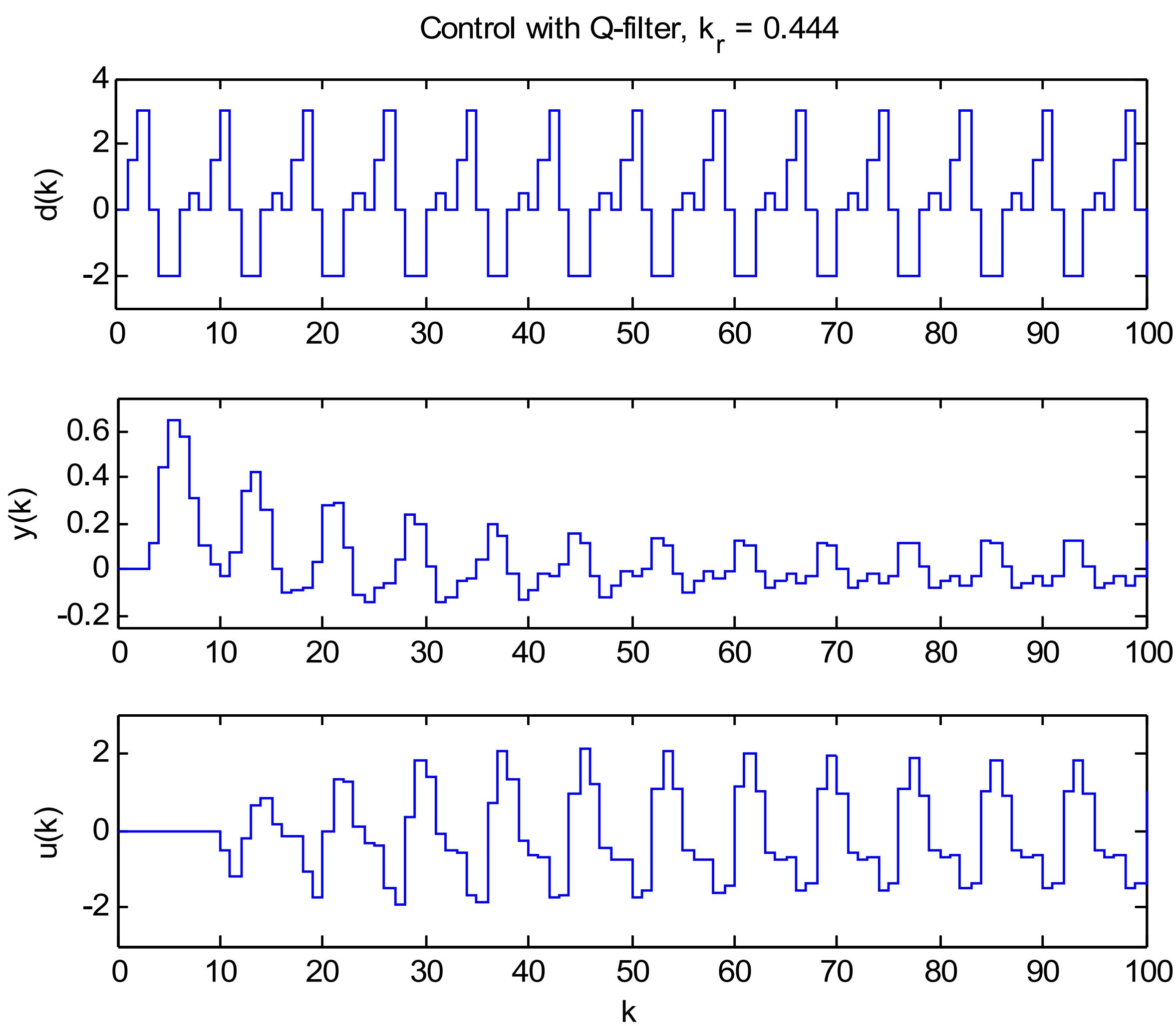
$u(k) = \dfrac{-k_r q^{-(N-d)} \bar{A}(q^{-1})}{0.1(1 - Q(q, q^{-1}) q^{-N})} y(k)$, with Q-filter $Q(q, q^{-1}) = \dfrac{q + 2 + q^{-1}}{4}$

Now the closed-loop eigenvalues are all inside the unit circle for $k_r \leq 1.004$



We can see that the closed-loop eigenvalues are as far inside the unit circle as possible for $k_r = 0.444$

1.e)ii)



2.

$y_1(t)=N_1(u_1(t)),\ y_2(t)=N_2(u_2(t)),\ \int_0^t y_i^T(\tau)u_i(\tau)d\tau \geq -\gamma_i^2$ for $i=1,2$

Lemma 1, $y(t)=N_1(u(t))+N_2(u(t)),\ \int_0^t y^T(\tau)u(\tau)d\tau=\int_0^t (N_1(u(\tau))+N_2(u(\tau)))^T u(\tau)d\tau$

$\int_0^t y^T(\tau)u(\tau)d\tau=\int_0^t (N_1^T(u(\tau))+N_2^T(u(\tau)))u(\tau)d\tau=\int_0^t (N_1^T(u(\tau))u(\tau)+N_2^T(u(\tau))u(\tau))d\tau$

$\int_0^t y^T(\tau)u(\tau)d\tau=\int_0^t (y_1^T(\tau)u(\tau)+y_2^T(\tau)u(\tau))d\tau=\int_0^t y_1^T(\tau)u(\tau)d\tau+\int_0^t y_2^T(\tau)u(\tau)d\tau$

$\int_0^t y^T(\tau)u(\tau)d\tau \geq -\gamma_1^2-\gamma_2^2$, so for $\gamma^2=\gamma_1^2+\gamma_2^2$ we have $\int_0^t y^T(\tau)u(\tau)d\tau \geq -\gamma^2$

Lemma 2, $y(t)=N_1(u(t)-y_1(t)),\ y_1(t)=N_2(y(t))$

$\int_0^t y^T(\tau)u(\tau)d\tau=\int_0^t y^T(\tau)(u(\tau)-y_1(\tau))d\tau+\int_0^t y^T(\tau)y_1(\tau)d\tau$

$\int_0^t y^T(\tau)u(\tau)d\tau=\int_0^t N_1^T(u(\tau)-y_1(\tau))(u(\tau)-y_1(\tau))d\tau+\int_0^t y^T(\tau)N_1(y(\tau))d\tau$

Let $z(t)=u(t)-y_1(t)$, so we have $\int_0^t y^T(\tau)u(\tau)d\tau=\int_0^t N_1^T(z(\tau))z(\tau)d\tau+\int_0^t y^T(\tau)N_1(y(\tau))d\tau$

Since y and N_1 are the same dimension, the 2nd integral results in a scalar and we can transpose it

$\int_0^t y^T(\tau)u(\tau)d\tau=\int_0^t N_1^T(z(\tau))z(\tau)d\tau+\int_0^t N_1^T(y(\tau))y(\tau)d\tau$

The Popov inequality for N_1 can be expressed as $\int_0^t N_1^T(u_1(\tau))u_1(\tau)d\tau \geq -\gamma_1^2$

This holds for any function $u_1(t)$, including $z(t)$ and $y(t)$, so $\int_0^t y^T(\tau)u(\tau)d\tau \geq -2\gamma_1^2$

3.a)

$Y(s)=\frac{b}{s-a}U(s),\ b>0,\ Y_r(s)=\frac{b_r}{s-a_r}R(s),\ b_r>0,\ a_r<0,\ r(t)$ bounded

$u(t)=\phi(t)^T\hat{\theta}(t),\ \phi(t)=[y(t)\quad r(t)]^T,\ \hat{\theta}(t)=[\hat{\alpha}(t)\quad \hat{\beta}(t)]^T$

$\frac{d}{dt}\hat{\theta}(t)=F\phi(t)e(t),\ F=F^T>0,\ e(t)=y_r(t)-y(t)$

$\dot{y}(t)=a\,y(t)+b\,u(t)=a\,y(t)+b(\hat{\alpha}(t)y(t)+\hat{\beta}(t)r(t)),\ \dot{y}_r(t)=a_r y_r(t)+b_r r(t)$

$\dot{e}(t)=\dot{y}_r(t)-\dot{y}(t)=a_r y_r(t)+b_r r(t)-a\,y(t)-b(\hat{\alpha}(t)y(t)+\hat{\beta}(t)r(t))$

$\dot{e}(t)=a_r y_r(t)-a_r y(t)+a_r y(t)+b_r r(t)-a\,y(t)-b\hat{\alpha}(t)y(t)-b\hat{\beta}(t)r(t)$

$\dot{e}(t)=a_r e(t)+(a_r-a-b\hat{\alpha}(t))y(t)+(b_r-b\hat{\beta}(t))r(t)$

$\frac{\dot{e}(t)-a_r e(t)}{b}=\left(\frac{a_r-a}{b}-\hat{\alpha}(t)\right)y(t)+\left(\frac{b_r}{b}-\hat{\beta}(t)\right)r(t)$

$E(s)=G(s)M(s)$ where $G(s)=\frac{b}{s-a_r}$, $m(t)=\tilde{\theta}^T(t)\phi(t),\ \tilde{\theta}(t)=\theta-\hat{\theta}(t),\ \theta=[\alpha\quad \beta]^T$

$m(t)=[\alpha-\hat{\alpha}(t)\quad \beta-\hat{\beta}(t)]\begin{bmatrix} y(t) \\ r(t) \end{bmatrix}=(\alpha-\hat{\alpha}(t))y(t)+(\beta-\hat{\beta}(t))r(t)$

So $\alpha=(a_r-a)/b,\ \beta=b_r/b$

3.b)

$V(\tilde{\theta}(t))=\frac{1}{2}\tilde{\theta}^T(t)F^{-1}\tilde{\theta}(t),\ \frac{d}{dt}V(\tilde{\theta}(t))=\tilde{\theta}^T(t)F^{-1}\dot{\tilde{\theta}}(t)=-\tilde{\theta}^T(t)F^{-1}\dot{\hat{\theta}}(t)=-\tilde{\theta}^T(t)F^{-1}(F\phi(t)e(t))$

$\frac{d}{dt}V(\tilde{\theta}(t))=-\tilde{\theta}^T(t)\phi(t)e(t)=-m(t)e(t)=w(t)e(t)$

$\int_0^t w(\tau)e(\tau)d\tau=\int_0^t \frac{d}{d\tau}V(\tilde{\theta}(\tau))d\tau=V(\tilde{\theta}(t))-V(\tilde{\theta}(0))=\frac{1}{2}\tilde{\theta}^T(t)F^{-1}\tilde{\theta}(t)-V(\tilde{\theta}(0))$

If F is strictly positive definite then F^{-1} is also strictly positive definite, so

$$\int_0^t w(\tau)e(\tau)d\tau = \frac{1}{2}\tilde{\theta}^T(t)F^{-1}\tilde{\theta}(t) - V(\tilde{\theta}(0)) \geq -V(\tilde{\theta}(0)) = \frac{-1}{2}\tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0) = -\gamma_0^2$$

3.c)

The transfer function from $r(t)$ to $y_r(t)$ is stable, so since $r(t)$ is bounded then $y_r(t)$ is bounded

From hyperstability we have that $|e(t)| < \infty$ assuming $|e(0)| < \infty$

$y_r(t)$ is bounded and the difference between $y_r(t)$ and $y(t)$ is bounded, so $y(t)$ must be bounded

$\phi(t) = [y(t) \quad r(t)]^T$ so since $y(t)$ and $r(t)$ are bounded, $\|\phi(t)\| < \infty$

3.d)

$\int_0^t m(\tau)e(\tau)d\tau \geq -\gamma_1^2$ since $G(s)$ is SPR

$m(t) = -w(t)$, so $\int_0^t w(\tau)e(\tau)d\tau \leq \gamma_1^2$

$$\int_0^t \frac{d}{d\tau}V(\tilde{\theta}(\tau))d\tau \leq \gamma_1^2$$

$$V(\tilde{\theta}(t)) - V(\tilde{\theta}(0)) \leq \gamma_1^2$$

$$\tilde{\theta}^T(t)F^{-1}\tilde{\theta}(t) - \tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0) \leq 2\gamma_1^2$$

$$\tilde{\theta}^T(t)F^{-1}\tilde{\theta}(t) \leq 2\gamma_1^2 + \tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0) \leq 2\gamma_1^2 + \lambda_{\max}(F^{-1})\|\tilde{\theta}(0)\|^2$$

$$\lambda_{\min}(F^{-1})\|\tilde{\theta}(t)\|^2 \leq \tilde{\theta}^T(t)F^{-1}\tilde{\theta}(t) \leq 2\gamma_1^2 + \lambda_{\max}(F^{-1})\|\tilde{\theta}(0)\|^2$$

$F > 0$ so it has no zero eigenvalues, therefore the max eigenvalue of F^{-1} is finite

So if $\|\tilde{\theta}(0)\| < \infty$, we have $\|\tilde{\theta}(t)\| \leq \sqrt{\dfrac{2\gamma_1^2 + \lambda_{\max}(F^{-1})\|\tilde{\theta}(0)\|^2}{\lambda_{\min}(F^{-1})}} < \infty$

3.e)

$w(t) = -m(t) = -\tilde{\theta}^T(t)\phi(t)$, from part c we have $\|\phi(t)\| < \infty$, and from part d we have $\|\tilde{\theta}(t)\| < \infty$

$w(t)$ is the product of 2 bounded vectors so it is bounded. The system is asymptotically hyperstable so $\lim\limits_{t\to\infty} e(t) = 0$