1. Sol:

(a) The plant is given by

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

The time domain cost function is

$$J = \int_0^\infty \left(y_f(t)^2 + R u_f^2(t) \right) dt$$

+1 points

No additional shaping in the control input is specified in the problem. Hence we can select $u_f(t) = u(t)$. For the rejection of sinusoidal inputs, the frequency shaping of the states can be selected as $Y_f(s) = Q_f(s)Y(s)$ where $Q_f(s) = 1/(s^2 + \omega_d^2)$.

+3 points

The frequency domain cost function is then

$$\begin{aligned} J &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[Y_f(-j\omega)^T Y_f(j\omega) + RU(-j\omega)^T U(j\omega) \right] d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[X(-j\omega)^T C^T Q_f(-j\omega)^T Q_f(j\omega) C X(j\omega) + RU(-j\omega)^T U(j\omega) \right] d\omega \end{aligned}$$

+2 points

Let the state $x(t) \in \mathbb{R}^n$. One minimal state-space realization for the filtered state is that

$$\begin{aligned} \dot{z}_1(t) &= \underbrace{\begin{bmatrix} 0 & 1 \\ -\omega_d^2 & 0 \end{bmatrix}}_{A_1} z_1(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B_1} C\,x(t) \\ y_f(t) &= \underbrace{[1, 0]}_{C_1} z_1(t) + \underbrace{0_{1\times n}}_{D_1} x(t) \end{aligned}$$

The enlarged system is then

$$\begin{aligned} \frac{d}{dt} \underbrace{\begin{bmatrix} x(t) \\ z_1(t) \end{bmatrix}}_{x_e} &= \underbrace{\begin{bmatrix} A & 0_{n\times 2} \\ B_1 & A_1 \end{bmatrix}}_{A_e} \underbrace{\begin{bmatrix} x(t) \\ z_1(t) \end{bmatrix}}_{x_e} + \underbrace{\begin{bmatrix} B \\ 0_{2\times 1} \end{bmatrix}}_{B_e} u(t) \\ y_f(t) &= \underbrace{\begin{bmatrix} D_1 & C_1 \end{bmatrix}}_{C_e} x_e \end{aligned}$$

+5 points

We now have a standard LQ problem with the cost function given by

$$\begin{aligned} J &= \int_0^\infty \left(x_e^T \begin{bmatrix} D_1 & C_1 \end{bmatrix}^T \begin{bmatrix} D_1 & C_1 \end{bmatrix} x_e + Ru^2(t) \right) dt \\ &= \int_0^\infty \left(x_e^T \underbrace{\begin{bmatrix} D_1^T D_1 & D_1^T C_1 \\ C_1^T D_1 & C_1^T C_1 \end{bmatrix}}_{Q_e} x_e + Ru^2(t) \right) dt \end{aligned}$$

+2 points

From the standard solution of LQ problems, the optimal control law and the Riccati equation are

$$u^o(t) = -R^{-1} B_e^T P_e x_e(t)$$
$$P_e A_e + A_e^T P_e + Q_e - P_e B_e R^{-1} B_e^T P_e = 0$$

+2 points

In a second-order example with $y = x_1$, we have $Q_f(s)y(s) = Q_f(s)[1,0]x(s)$ and we can select the realization of $Q_f(s)$ to be

$$A_1 = \begin{bmatrix} 0 & 1 \\ -\omega_d^2 & 0 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}[1,0] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ C_1 = [1,0]\,,\ D_1 = [0,0]$$

yielding

$$A_e = \begin{bmatrix} A_{2\times 2} & 0_{2\times 2} \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ -\omega_d^2 & 0 \end{bmatrix} \end{bmatrix},\ B_e = \begin{bmatrix} B_{2\times 1} \\ 0_{2\times 1} \end{bmatrix},\ C_e = \begin{bmatrix} 0, 0, & 1, 0 \end{bmatrix},\ D_e = 0$$

$$Q_e = \begin{bmatrix} D_1^T D_1 & D_1^T C_1 \\ C_1^T D_1 & C_1^T C_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

+5 points

(b) The controller is not unique, we can choose, for instance

$$G_c(s) = k_c \frac{s^2 + 1.414s + 1}{s^2 + 4}$$

and properly select k_c to make the closed loop stable (for this particular controller, a positive k_c will guarantee closed-loop stability. We can use e.g. root locus or the closed-loop characteristic equation to obtain this conclusion). Other answers are also acceptable as long as it is second-order and you provide proper stability conditions.

+7 points

The sensitivity function is

$$S(s) = \frac{1}{1 + G_c G_p} = \frac{1}{1 + k_c \frac{s+0.5}{s^2+4}} = \frac{s^2 + 4}{s^2 + 4 + k_c\,(s + 0.5)}$$

$\Rightarrow$

$$S\,(j\omega)\big|_{\omega=\omega_d} = \left.\frac{-\omega^2 + 4}{-\omega^2 + 4 + k_c\,(j\omega + 0.5)}\right|_{\omega=\omega_d} = \frac{-4 + 4}{-4 + 4 + k_c\,(j2 + 0.5)} = 0$$

+3 points

2. Sol:

(a) Substituting the control law into the system equation, we have

$$m\ddot{e} + k_d \dot{e} + k_p e = 0$$

+3 points

Applying Laplace transform, we get

$$ms^2 E(s) - mse(0) - m\dot{e}(0) + k_d s E(s) - k_d e(0) + k_p E(s) = 0$$

$$\Rightarrow E(s) = \frac{mse(0) + k_d e(0) + m\dot{e}(0)}{ms^2 + k_d s + k_p}$$

Notice that $m > 0$ by assumption. A second-order system is asymptotic stable if and only the coefficients of the denominator have the same sign (you can do e.g. a Routh test). The closed-loop asymptotic stability condition is that k_d and k_p should be positive real numbers.

+5 points

(b) The plant and the controller are given by

$$m\frac{d^2 y\,(t)}{dt^2} = u\,(t)\,;\ m > 0$$

$$u\,(t) = u_{ff}\,(t) - k_d \frac{dy\,(t)}{dt} - k_p y\,(t)$$

where we have used the the definition of feedforward control law $u_{ff}(t)$. Combine the two equations and remove $u(t)$. We have

$$m\ddot{y} + k_d\dot{y} + k_p y = u_{ff}$$

The transfer function from u_{ff} to y is thus

$$G_{u_{ff}\to y}(s) = \frac{1}{ms^2 + k_d s + k_p}$$

+3 points

Similarly, we can obtain the transfer function from y_d to u_{ff}

$$G_{y_d\to u_{ff}}(s) = ms^2 + k_d s + k_p$$

Clearly, $G_{y_d\to u_{fff}}(s)$ is an inverse-based feedforward controller as $G_{y_d\to u_{ff}}(s) = G_{u_{ff}\to y}(t)^{-1}$.

+4 points

3. Sol:

(a) The fastest way to obtain the result is to use block diagram transformation, or to compare the block diagram with a standard DOB and detect the difference between them. Detailed steps are shown during discussion session.

$$G_{d\to v}(z^{-1}) = \frac{\frac{z^{-1}B(z^{-1})}{A(z^{-1})}\left(1 - z^{-1}B_n(z^{-1})Q(z^{-1})\right)}{1 + z^{-1}Q(z^{-1})\left[\frac{B(z^{-1})}{A(z^{-1})}A_n(z^{-1}) - B_n(z^{-1})\right]}$$

$$G_{u^*\to v}(z^{-1}) = \frac{\frac{z^{-1}B(z^{-1})}{A(z^{-1})}}{1 + z^{-1}Q(z^{-1})\left[\frac{B(z^{-1})}{A(z^{-1})}A_n(z^{-1}) - B_n(z^{-1})\right]}$$

+4 points

(b) The characteristic polynomial comes from

$$1 + z^{-1}Q(z^{-1})\left[\frac{B(z^{-1})}{A(z^{-1})}A_n(z^{-1}) - B_n(z^{-1})\right] = 0$$

$$\Rightarrow A(z^{-1}) + z^{-1}Q(z^{-1})\left[B(z^{-1})A_n(z^{-1}) - A(z^{-1})B_n(z^{-1})\right] = 0 \quad (1)$$

If $Q(z^{-1}) = B_Q(z^{-1})/A_Q(z^{-1})$, we have

$$A(z^{-1}) + z^{-1}\frac{B_Q(z^{-1})}{A_Q(z^{-1})}\left[B(z^{-1})A_n(z^{-1}) - A(z^{-1})B_n(z^{-1})\right] = 0$$

$$\Rightarrow A_Q(z^{-1})\left\{A(z^{-1}) + z^{-1}B_Q(z^{-1})\left[B(z^{-1})A_n(z^{-1}) - A(z^{-1})B_n(z^{-1})\right]\right\} = 0 \quad (2)$$

For stability, the roots of (2) needs to be inside the unit circle. When $A(z^{-1}) = A_n(z^{-1})$ and $B(z^{-1}) = B_n(z^{-1})$, the characteristic equation simplifies to

$$A_Q(z^{-1})A(z^{-1}) = 0$$

Hence the stability condition is that the original plant $z^{-1}B(z^{-1})/A(z^{-1})$ and the filter $Q(z^{-1})$ need to be both stable (the second condition is easy to forget).

+4 points

Direct substitution of $A(z^{-1}) = A_n(z^{-1})$ and $B(z^{-1}) = B_n(z^{-1})$ gives

$$G_{d\to v}(z^{-1}) = \frac{z^{-1}B(z^{-1})}{A(z^{-1})}\left(1 - z^{-1}B_n(z^{-1})Q(z^{-1})\right)$$

$$= \frac{z^{-1}B(z^{-1})}{A(z^{-1})}\left(1 - z^{-1}B(z^{-1})Q(z^{-1})\right)$$

Note: from here you can also see that $Q(z^{-1})$ has to be stable.

+1 points

To achieve similar characteristic as a standard disturbance observers, we need $1 - z^{-1}B(z^{-1})Q(z^{-1})$ to behave like a high-pass filter. Since the plant is minimum-phase, all the roots of $B(z^{-1})$ and $A(z^{-1})$ are within the unit circle. Hence, we can select

$$Q(z^{-1}) = \frac{1}{B(z^{-1})} \times \text{a low pass filter}$$

which is stable.

+3 points

(c) To reject $d(k)$ from $v(k)$, we need $G_{d\to v}(q^{-1})d(k) = 0$, namely

$$\frac{q^{-1}B(q^{-1})}{A(q^{-1})}\left(1 - q^{-1}B(q^{-1})Q(q^{-1})\right)d(k) = 0 \tag{3}$$

From the hint we know that

$$(1 - 2\cos\omega_d q^{-1} + q^{-2})d(k) = 0 \tag{4}$$

Comparing (3) and (4), we can select

$$Q(q^{-1}) = \frac{1}{B(q^{-1})} \times \left(2\cos\omega_d - q^{-1}\right)$$

or in the z-domain transfer-function notation

$$Q(z^{-1}) = \frac{1}{B(z^{-1})} \times \left(2\cos\omega_d - z^{-1}\right)$$

Again, the minimum-phase assumption assures that the above transfer function is stable, and hence the stability of the closed loop.

+4 points

(d) Substituting $B(z^{-1})/A(z^{-1}) = 1$ into the transfer function in (c), we have

$$G_{d\to v}(z^{-1}) = z^{-1}\left(1 - z^{-1}Q(z^{-1})\right) = z^{-1}\left(1 - 2\cos\omega_d z^{-1} + z^{-2}\right)$$

The DC gain and the gain at Nyquist frequency are

$$|G_{d\to v}(1)| = 1$$
$$|G_{d\to v}(-1)| = |-3| = 3$$

+2 points

Disturbances at high frequencies are thus amplified! +2 points