ME 233 Advance Control II

Lecture 4
Random Sequences

(ME233 Class Notes pp. PR6-PR9)

Outline

- Random Sequences
- 2nd order statistics
  - Mean
  - Covariance and Cross-Covariance
- Stationarity and Periodicity
- White and Color Noise
- A white noise input example
- Power Spectral Density Function
- Stationary SISO LTI Systems

Random Sequences

A two-sided random sequence is a collection of random variables

$$X = \{\cdots \; X(-1), \;\; X(0), \;\; X(1),$$

$$X(2), \;\; X(3), \;\; \cdots \}$$

each $X(k) \in \mathcal{R}$ is itself a random variable

defined over the same probability space $(\Omega, \, \mathcal{S}, \, Pr)$

Random Sequences

We either will use

$$\{X(k)\}_{k=-\infty}^{\infty} \qquad\qquad X(k)$$

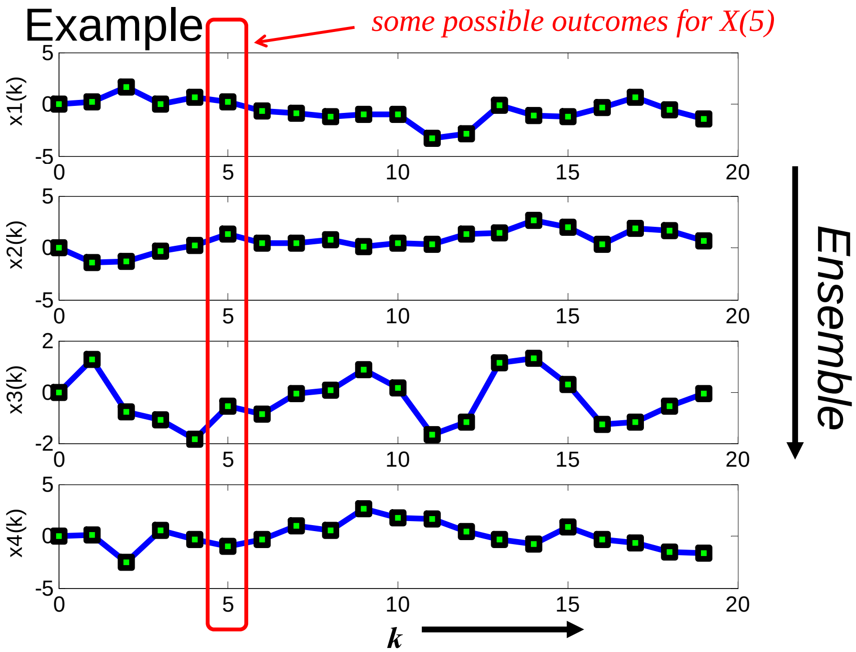
Shorthand (sloppy) notation

to denote the two-sided random sequence.

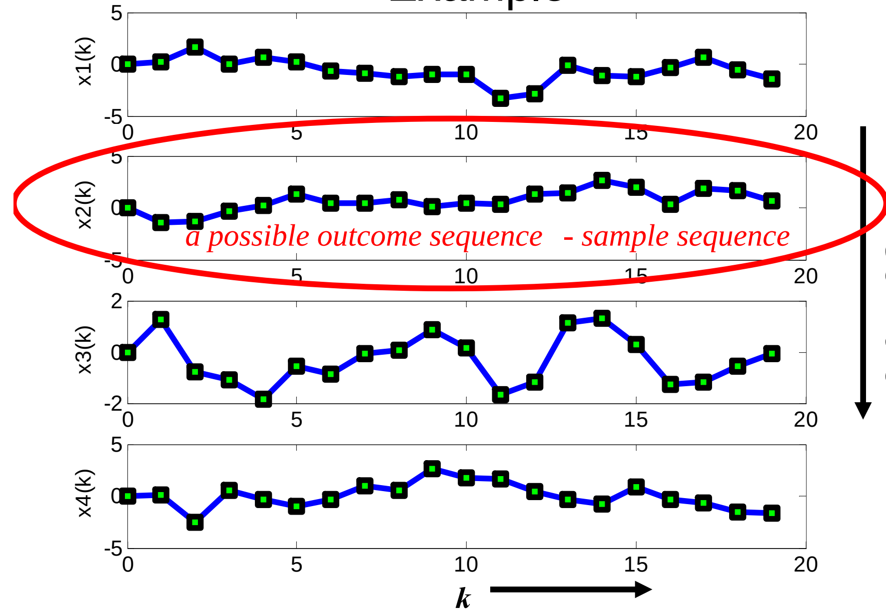
Each element $X(k)$ of the sequence is a random variable:

$$X(k) : \Omega \rightarrow \mathcal{R}$$

Example



Example



Random Sequences

Let $\{1,\, 2,\, 3,\, 4\}$ be some indices of the sequence

$$p_{X(1)X(2)X(3)X(4)}(x_1,\, x_2,\, x_3,\, x_4)$$

is the joint PDF of

$$\{X(1),\, X(2),\, X(3),\, X(4)\,\}$$

This is often a huge amount of redundant information

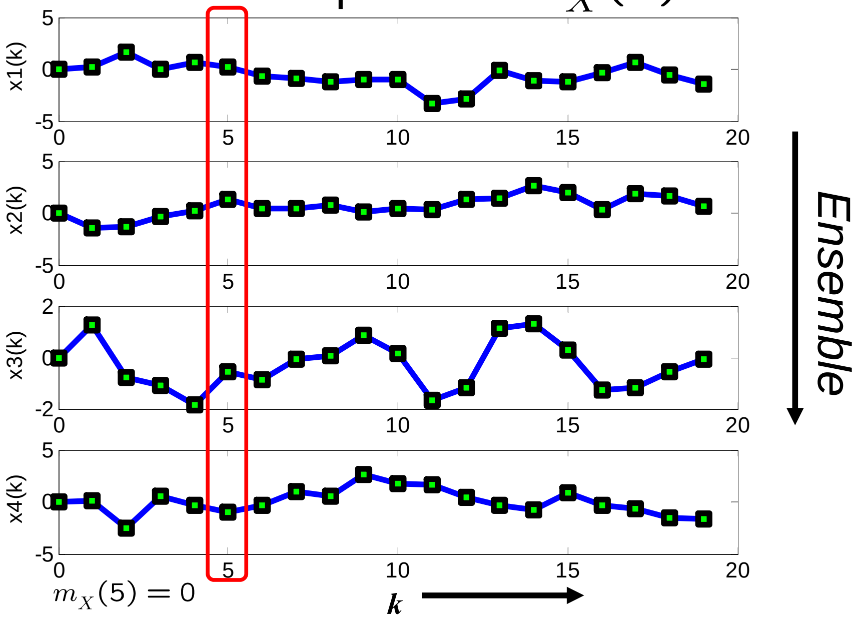
2nd order statistics

Fortunately, in many cases we only need to keep track of two things:

1) The mean:

$$m_X(k) = E\{X(k)\}$$

Example $m_X(k) = 0$



2nd order statistics

Fortunately, in many cases we only need to keep track of two things:

2) The covariance:

$$\Lambda_{XX}(k,j) = E\{\,\tilde{X}(k+j)\,\tilde{X}(k)\,\}$$

$$\tilde{X}(k) = X(k) - m_X(k)$$

2nd order statistics

Fortunately, in many cases we only need to keep track of two things:

2) The covariance:

$$\Lambda_{XX}(k,\underline{j}) = E\{\,\tilde{X}(k+\underline{j})\,\tilde{X}(k)\,\}$$

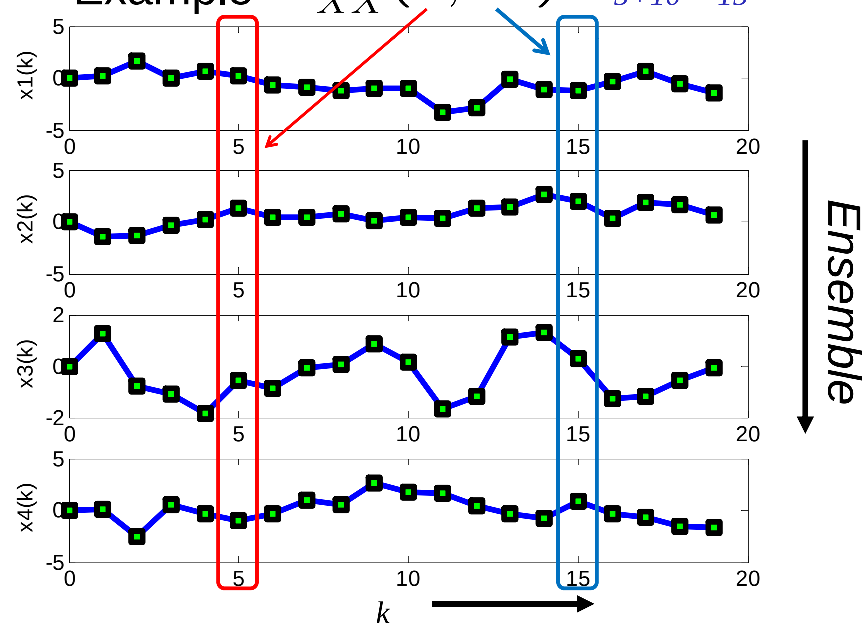
future

Correlation between $\tilde{X}(k)$ and the ***future*** $\tilde{X}(k+\underline{j})$

$\boldsymbol{j}$ is the correlation coefficient (negative for past)

Example $\Lambda_{XX}(5,10)$

5+10 = 15



Strict Sense Stationarity

A random sequence

is **Strict Sense Stationary (SSS)**

if its joint probability is time invariant, e.g.

$$P(X(j) \leq x_1,\, X(k) \leq x_2,\, X(m) \leq x_3,) =$$

$$P(X(j \underline{+ M}) \leq x_1,\, X(k \underline{+ M}) \leq x_2,\, X(m \underline{+ M}) \leq x_3)$$

for any M

Strict Sense Stationarity (SSS)

Let $X(k)$ be a SSS random sequence

Then

$$E\{X(k)\} = m_X$$

$$\Lambda_{XX}(k,j) = \Lambda_{XX}(k+M,j)$$

for any M

Ergodicity

A **Strict Sense Stationary** random sequence

is **ergodic**

if its ensemble average = time average (constant)

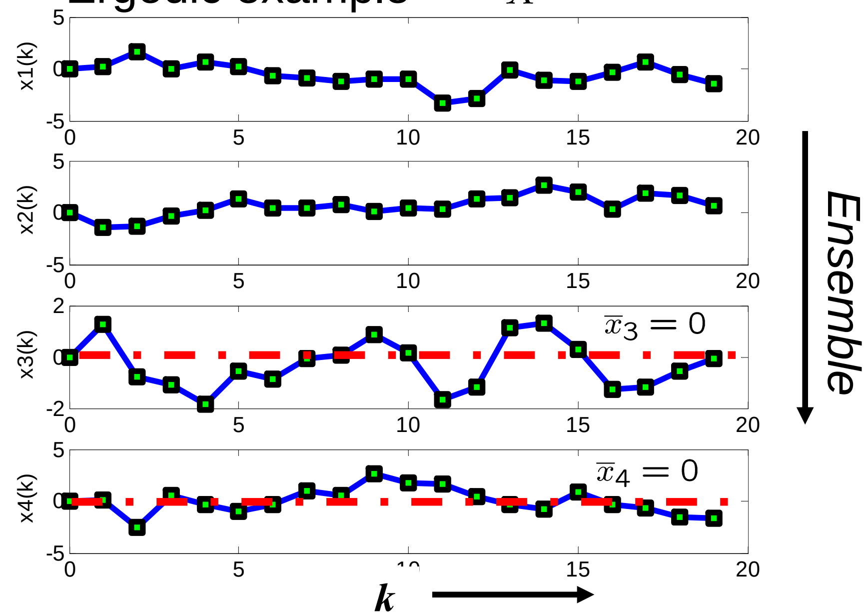
$$E\{X(k)\} = m_X$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} x(k)$$

with probability 1
(almost surely)

sample sequence

Ergodic example $m_X(k) = 0$



Ergodicity

For and **ergodic** random sequence

we can approximate the covariance as a "time average"

$$\Lambda_{XX}(k,j) = \quad E\{\, \tilde{X}(k+j)\, \tilde{X}(k)\, \}$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} \tilde{x}(k+j)\tilde{x}(k)$$

with probability 1
(almost surely)

$$\tilde{x}(k) = x(k) - m_X$$

sample sequence

Ergodicity

For and **ergodic** random sequence

$$\Lambda_{XX}(k,j) = \ \Lambda_{XX}(k+L,j)$$

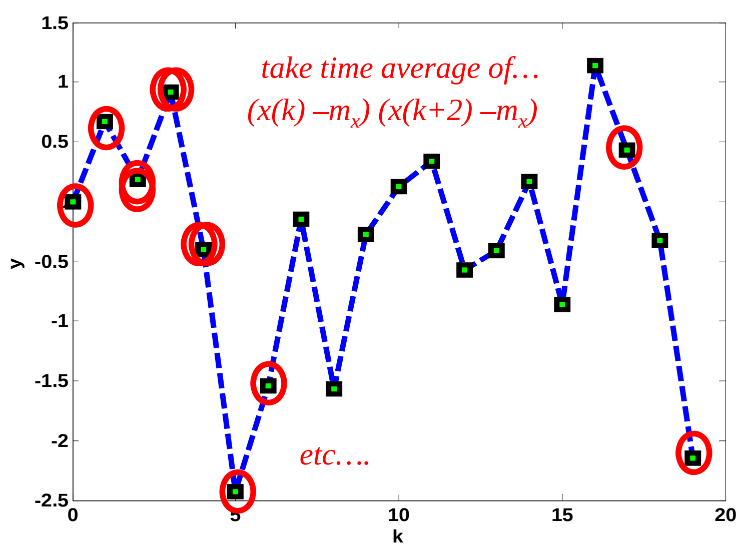
only a function of the correlation coefficient $\boldsymbol{j}$

$$\Lambda_{XX}(j) =$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} \tilde{x}(k+j)\tilde{x}(k)$$

with probability 1
(almost surely)

Example $\quad \Lambda_{YY}(2)$



White noise

A SSS random sequence $\quad W(k) \quad$ **is white** if:

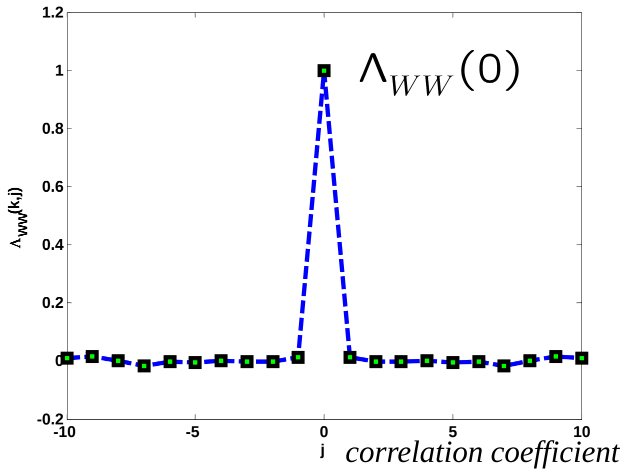
$$\boxed{\begin{aligned} &\Lambda_{WW}(0) = \sigma_W^2 \\ &\Lambda_{WW}(j) = 0 \qquad j \neq 0 \end{aligned}}$$

white noise is zero mean if $\quad E\{W(k)\} = 0$

Illustration

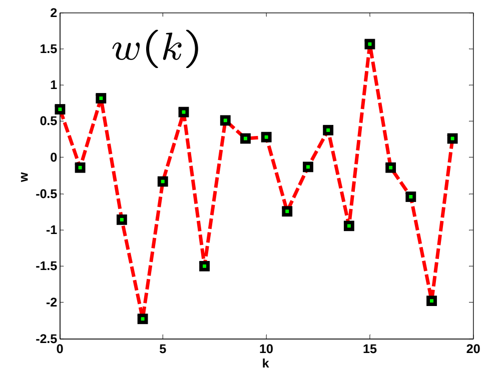
- matlab-generated

zero-mean white noise $W(k)$



Illustration

- zero-mean white noise $W(k)$



first 20 samples

Matlab commands:

N=5000;

w = randn(N,1);

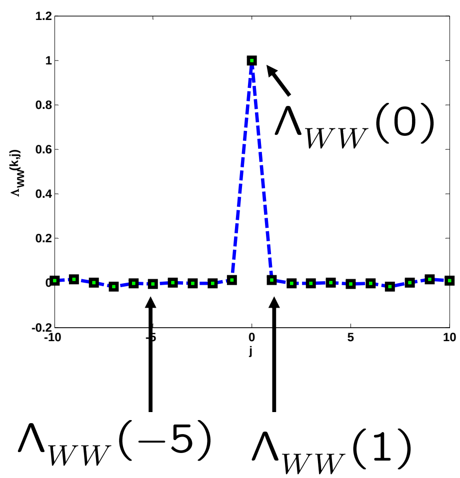
m_w = mean(w)

m_w = 0.0020

$$m_W \approx \frac{1}{N}\sum_{k=0}^{N} w(k)$$

Illustration

- zero-mean white noise $W(k)$ covariance



Matlab command:

normalizes

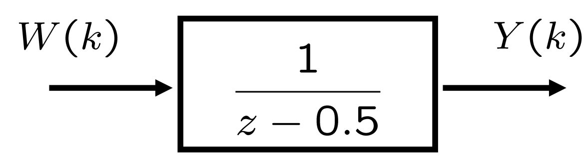
cov_ww =
xcov(w,10,'coeff');

compute only correlation coefficients: j ε [-10,10]

$$\Lambda_{WW}(j) \approx \frac{1}{N}\sum_{k=0}^{N} \tilde{w}(k+j)\tilde{w}(k)$$

Illustration – One sample sequence

- Feed zero-mean white noise to a first order system using Matlab



Matlab commands:

sys1=tf(1,[1 -.5],1)

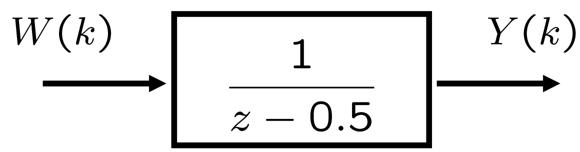
k = (0:1:N-1)';

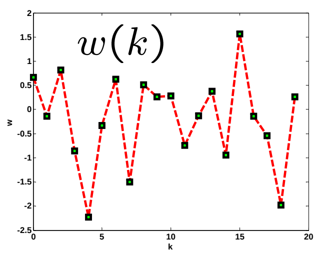
[y,k] = lsim(sys1,w,k);

```
k = (0:1:N-1)';
figure(1),plot(k(1:20),ww(1:20),'--rs','LineWidth',4,...
        'MarkerEdgeColor','k',...
        'MarkerFaceColor','g',...
        'MarkerSize',10)
xlabel('k')
ylabel('w')
```

Illustration – One sample sequence

- Feed zero-mean white noise to a first order system using Matlab





One sample sequence

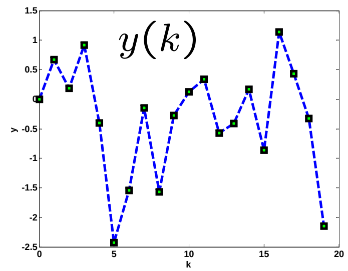
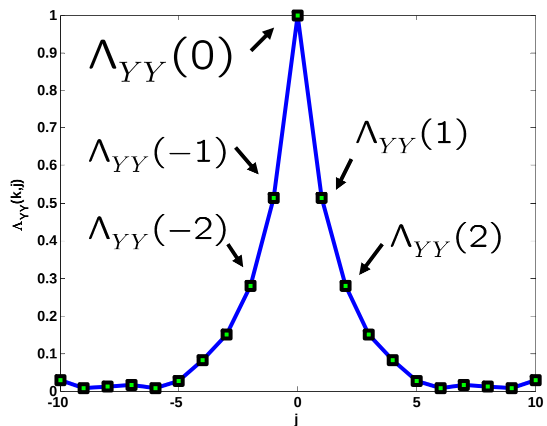


Illustration – compute $\Lambda_{YY}(j)$

- output $y(k)$



colored noise

Matlab commands:

m_y = mean(y)

m_y = 0.0147

cov_yy =....

xcov(y,10,'coeff');

$$\Lambda_{YY}(j) \approx \frac{1}{N}\sum_{k=0}^{N} \tilde{y}(k+j)\tilde{y}(k)$$

Wide Sense Stationarity

A random sequence

is **Wide Sense Stationary (WSS)** if:

1) Its mean is time invariant

$$E\{X(k)\} = m_X$$

$$\text{SSS} \Rightarrow \text{WSS}$$

Wide Sense Stationarity

A random sequence

is **Wide Sense Stationary (WSS)** if:

2) Its covariance only depends on the correlation index ***j*** e.g.

$$\Lambda_{XX}(k,j) = \Lambda_{XX}(k+M,j)$$

$$\text{SSS} \Rightarrow \text{WSS}$$

Wide Sense Stationarity

A random sequence

is **Wide Sense Stationary (WSS)** if:

2) Its covariance only depends on the correlation index j e.g.

$$E\left\{\tilde{X}(\underline{k+j})\tilde{X}(k)\right\} = E\left\{\tilde{X}(k)\tilde{X}(\underline{k-j})\right\}$$

$$\text{SSS} \Rightarrow \text{WSS}$$

For SSS and WSS random sequences

The auto-covariance function can be defined only as a function of the correlation index j

$$\Lambda_{XX}(\underline{j}) = E\left\{\tilde{X}(k+\underline{j})\tilde{X}(k)\right\}$$

for ***any*** time index k

Notice that:

$$\boxed{\begin{gathered}\Lambda_{XX}(j) = \Lambda_{XX}(-j) \\ \Lambda_{XX}(0) \geq |\Lambda_{XX}(j)|\end{gathered}}$$

$$\Lambda_{XX}(j) = \Lambda_{XX}(-j)$$

Proof:

$$\Lambda_{XX}(j) = E\left\{\tilde{X}(k+j)\tilde{X}(k)\right\}$$

(invariance under a TIME shift – j)

$$= E\left\{\tilde{X}(k+j-j)\tilde{X}(k-j)\right\}$$

$$= E\left\{\tilde{X}(k)\tilde{X}(k-j)\right\}$$

$$= \Lambda_{XX}(-j)$$

$$\Lambda_{XX}(0) \geq |\Lambda_{XX}(j)|$$

Proof: Define: the 2 dimensional random vector

$$Z(k) = \begin{bmatrix} X(k) \\ X(k+j) \end{bmatrix} \qquad m_Z = \begin{bmatrix} m_X \\ m_X \end{bmatrix}$$

$$\Lambda_{ZZ}(0) = E\left\{\tilde{Z}(k)\tilde{Z}^T(k)\right\} \qquad (\tilde{Z} = Z - m_Z)$$

$$= \begin{bmatrix} E\{\tilde{X}(k)\tilde{X}(k)\} & E\{\tilde{X}(k)\tilde{X}(k+j)\} \\ E\{\tilde{X}(k+j)\tilde{X}(k)\} & E\{\tilde{X}(k+j)\tilde{X}(k+j)\} \end{bmatrix}$$

$$\Lambda_{XX}(0) \geq |\Lambda_{XX}(j)|$$

- Since $\Lambda_{ZZ}(0) \succeq 0$

$$\Lambda_{ZZ}(0) = \begin{bmatrix} \Lambda_{XX}(0) & \Lambda_{XX}(j) \\ \Lambda_{XX}(j) & \Lambda_{XX}(0) \end{bmatrix} \succeq 0$$

$$\text{Det}\,[\Lambda_{ZZ}(0)] = \Lambda_{XX}(0)^2 - \Lambda_{XX}(j)^2 \geq 0$$

Power Spectral Density Function

Fourier transform of the auto-covariance function:

$$\Phi_{XX}(\omega) = \mathcal{F}\{\Lambda_{XX}(\cdot)\}$$

$$= \sum_{l=-\infty}^{\infty} \Lambda_{XX}(l) e^{-j\omega l}$$

l: correlation index

Note:

The power spectral density function is periodic, with period $T = 2\pi$

$$e^{-j\omega} = \cos(\omega) - j\sin(\omega)$$

Power Spectral Density Function

Using the inverse Fourier transform we obtain:

$$\Lambda_{XX}(l) = \mathcal{F}^{-1}\{\Phi_{XX}(\omega)\}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega l}\Phi_{XX}(\omega)\,d\omega$$

Power Spectral Density Function

Properties of the power spectral density function:

1. $\Phi_{XX}(\omega) \in \mathcal{R}_+ \qquad \omega \in [-\pi, \pi]$
2. $\Lambda_{XX}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{XX}(\omega)\,d\omega$
3. $\Phi_{XX}(\omega) \geq 0 \qquad \omega \in [-\pi, \pi]$
4. $\Phi_{XX}(\omega) = \Phi_{XX}(-\omega)$

WSS White noise

The power spectral density function for a WSS white noise is:

$$\boxed{\Phi_{WW}(w) = \sigma_W^2}$$

The power spectral density function is ***CONSTANT*** for all frequency

Hence the name ***"white noise"***

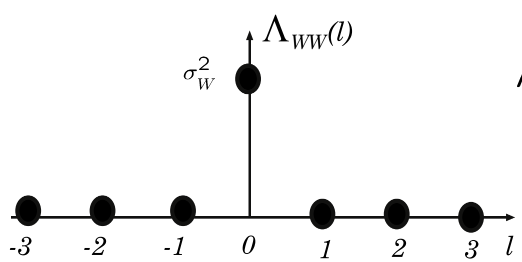
white light is the combination of all colors

For white noise: $\Phi_{WW}(w) = \sigma_W^2$

Proof:

$$\begin{aligned}
\Phi_{WW}(\omega) &= \sum_{l=-\infty}^{\infty} \Lambda_{WW}(l) e^{-j\omega l} \\
&= \sigma_W^2 \sum_{l=-\infty}^{\infty} \delta(l) e^{-j\omega l} \\
&= \sigma_W^2 \cancelto{1}{\delta(0) e^{-j\omega 0}} \\
&= \sigma_W^2
\end{aligned}$$

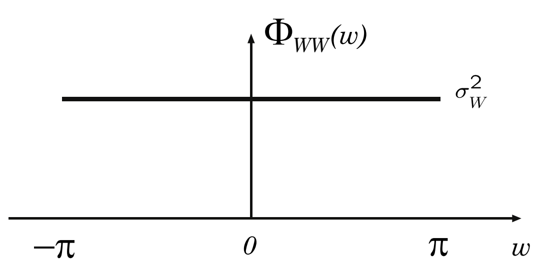
White noise



$$\Lambda_{WW}(l) = \sigma_W^2\, \delta(l)$$

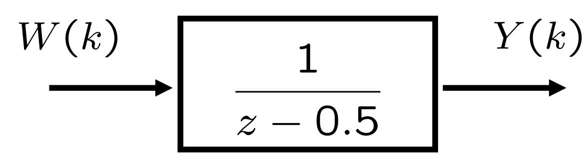
$$\Phi_{WW}(w) = \sigma_W^2$$

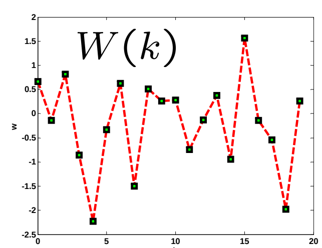
same magnitude over entire bandwidth

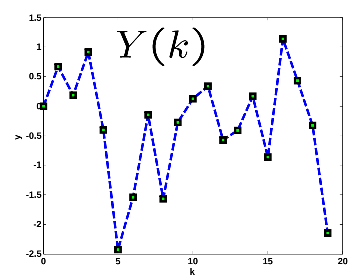


Color noise

- Feeding white noise through a linear filter generates color noise







Color noise

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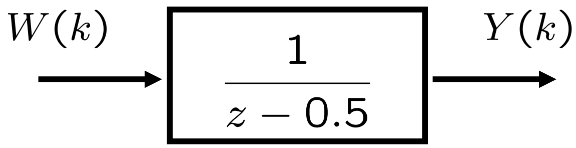
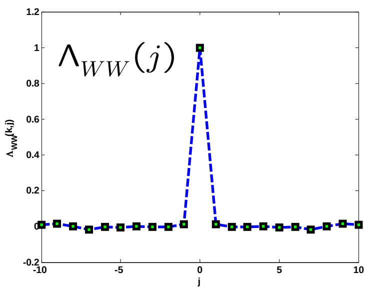
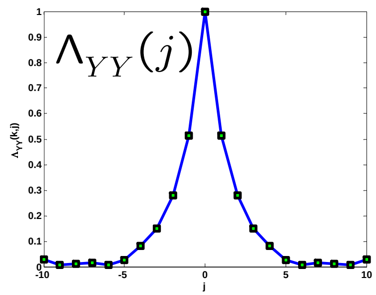






Cross-covariance function

Two scalar WSS random sequences: $X(k)$ $Y(k)$

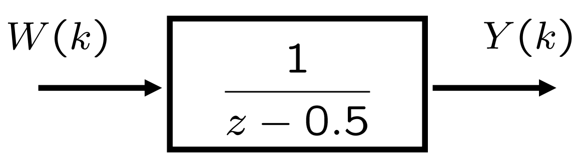
The cross-covariance function:

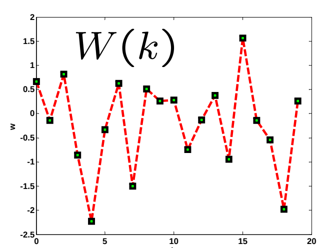
$$\Lambda_{XY}(j) = E\left\{\tilde{X}(k+j)\tilde{Y}(k)\right\}$$

Satisfies $\Lambda_{XY}(j) = \Lambda_{YX}(-j)$

Illustration

- Feed zero-mean white noise to a first order system using Matlab





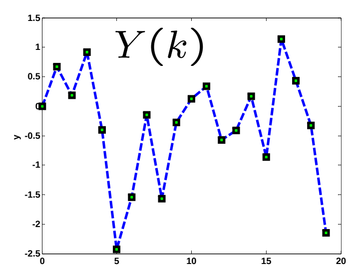
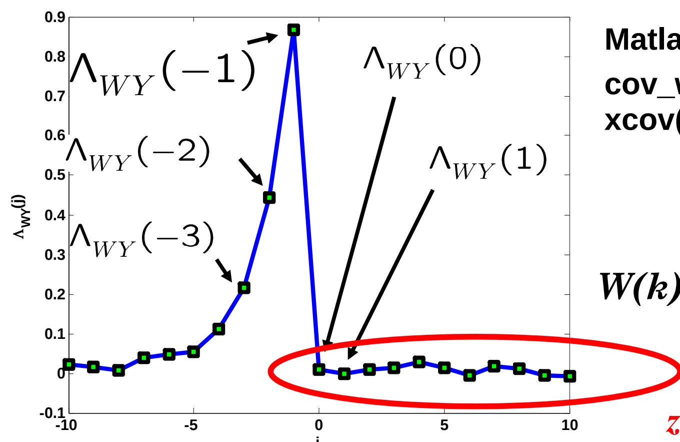


Illustration- Cross-covariance

- $\Lambda_{WY}(j) = E\{\tilde{W}(k+j)\tilde{Y}(k)\}$



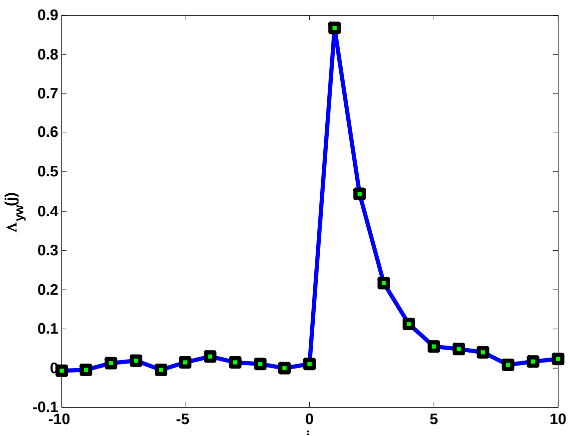
Matlab:
cov_wy = …
xcov(w,y,10,'coeff');

W(k) *is white noise*

zero for j ≥ 0

Illustration- Cross-covariance

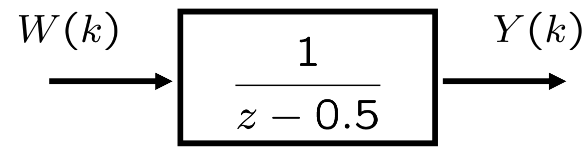
- $$\Lambda_{YW}(j) = \Lambda_{WY}(-j)$$



W(k) *is white noise*

Illustration

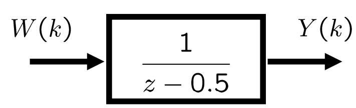
- Feed zero-mean white noise to a first order system using Matlab



$$Y(k+1) = 0.5\,Y(k) + W(k)$$

$$Y(0) = 0$$

Illustration



$$Y(k+1) = 0.5\,Y(k) + W(k)$$

- Taking expectations:

$$m_Y(k) = E\{Y(k)\}$$
$$m_W(k) = E\{W(k)\}$$

$$m_Y(k+1) = 0.5\,m_Y(k) + m_W(k)$$

Illustration

$$Y(k+1) = 0.5\,Y(k) + W(k)$$

$$m_Y(k+1) = 0.5\,m_Y(k) + m_W(k)$$

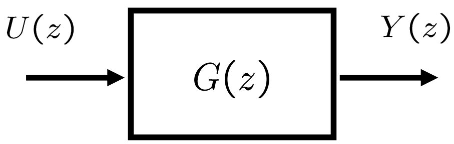
- subtracting the two equations,

$$\boxed{\tilde{Y}(k+1) = 0.5\,\tilde{Y}(k) + \tilde{W}(k)}$$

$$m_{\tilde{Y}}(k) = 0 \qquad m_{\tilde{W}}(k) = 0$$

SISO WSS Linear Time Invariant Systems

Two WSS random sequences: $U(k)$ $Y(k)$

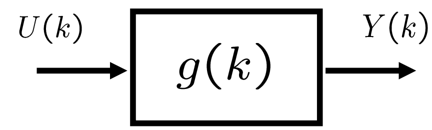


Transfer function

$$G(z) = \mathcal{Z}\{g(k)\} = \sum_{k=-\infty}^{\infty} g(k)\, z^{-k}$$

SISO WSS Linear Time Invariant Systems

Two WSS random sequences: $U(k)$ $Y(k)$



Input-output response is a convolution:

$$Y(k) = \sum_{i=-\infty}^{\infty} g(i)U(k-i)$$

SISO WSS Linear Time Invariant Systems

Note: we will assume without loss of generality that

The input WSS random sequence is zero mean, i.e.

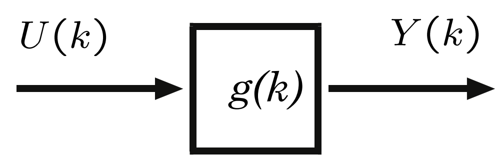
$$E\{U(k)\} = m_U = 0$$

Thus, the output random sequence is also zero mean

$$E\{Y(k)\} = m_Y = 0$$

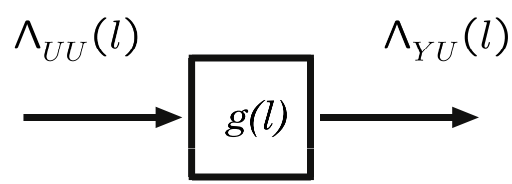
SISO WSS Linear Time Invariant Systems

If



"time domain"

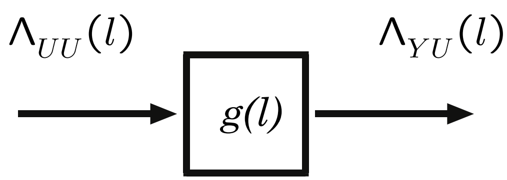
Then:



"correlation domain"

SISO WSS Linear Time Invariant Systems

What does this mean?



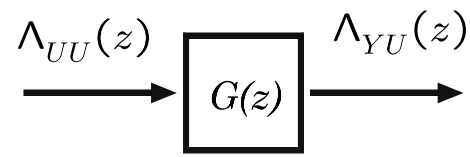
"correlation domain"

Convolution:

$$\Lambda_{YU}(l) = \sum_{i=-\infty}^{\infty} g(i)\,\Lambda_{UU}(l-i)$$

SISO WSS Linear Time Invariant Systems

Taking Z transform:



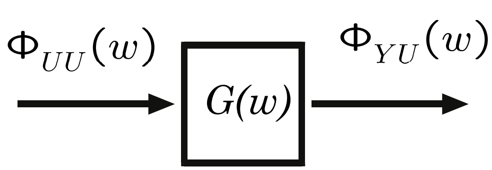
"correlation domain"

e.g.

$$\Lambda_{YU}(z) = \sum_{j=-\infty}^{j=\infty} z^{-j}\,\Lambda_{YU}(j)$$

SISO WSS Linear Time Invariant Systems

Taking Fourier transform:



"correlation domain"

e.g.

$$\Phi_{UU}(\omega) = \sum_{j=-\infty}^{j=\infty} e^{-j\omega}\,\Lambda_{UU}(j)$$

SISO WSS Linear Time Invariant Systems

If $$Y(k) = \sum_{i=-\infty}^{\infty} g(i) U(k-i)$$

Then:

$$\boxed{\Lambda_{YU}(l) = \sum_{i=-\infty}^{\infty} g(i)\,\Lambda_{UU}(l-i)}$$

$$\boxed{\Phi_{YU}(w) = G(w)\,\Phi_{UU}(w)}$$

$$\boxed{\Lambda_{YU}(l) = \sum_{i=-\infty}^{\infty} g(i)\,\Lambda_{UU}(l-i)}$$

Proof:

$$Y(k) = \sum_{i=-\infty}^{\infty} g(i)U(k-i) \qquad (m_U = 0)$$

Then:

$$\begin{aligned}\Lambda_{YU}(l) &= E\{Y(k+l)U(k)\}\\ &= E\left\{\left[\sum_{i=-\infty}^{\infty} g(i)U(k+l-i)\right]U(k)\right\}\end{aligned}$$

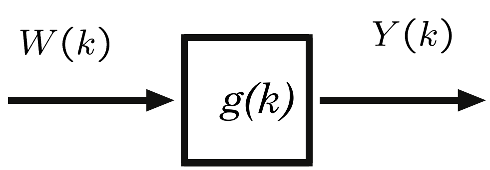
$$\boxed{\Lambda_{YU}(l) = \sum_{i=-\infty}^{\infty} g(i)\,\Lambda_{UU}(l-i)}$$

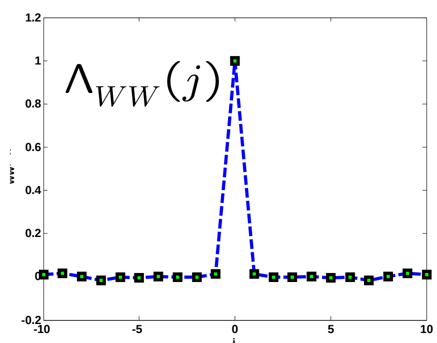
Proof:

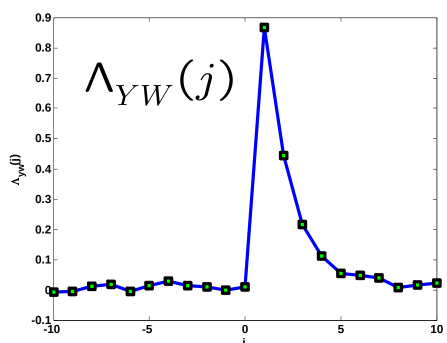
$$\begin{aligned}\Lambda_{YU}(l) &= E\left\{\left[\sum_{i=-\infty}^{\infty} g(i)U(k+l-i)\right]U(k)\right\}\\ &= \sum_{i=-\infty}^{\infty} g(i)\,\underbrace{E\{U(k+l-i)U(k)\}}\\ &= \sum_{i=-\infty}^{\infty} g(i)\Lambda_{UU}(l-i)\end{aligned}$$

Q.E.D

WSS white noise input example







WSS white noise input example

$$\Lambda_{YW}(l) = \sum_{i=-\infty}^{\infty} g(i)\,\Lambda_{WW}(l-i)$$

for white noise $\qquad \Lambda_{WW}(l) = \delta(l)$

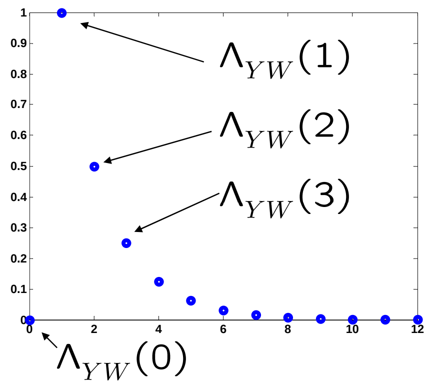
$$= \sum_{i=-\infty}^{\infty} g(i)\,\delta(l-i) = g(l)$$

$$= g(l)$$

WSS white noise input example

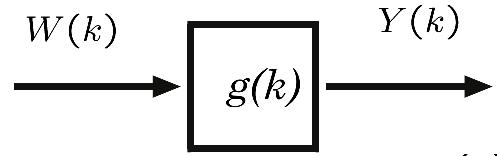
$$\boxed{\Lambda_{YW}(l) = g(l)}$$

$$G(z) = \frac{1}{z - 0.5}$$

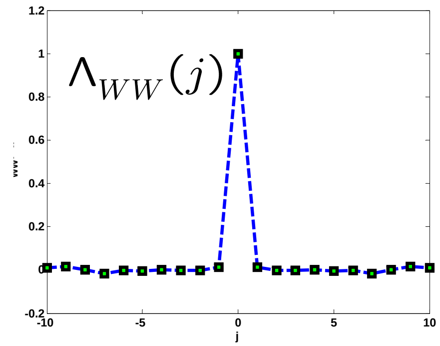


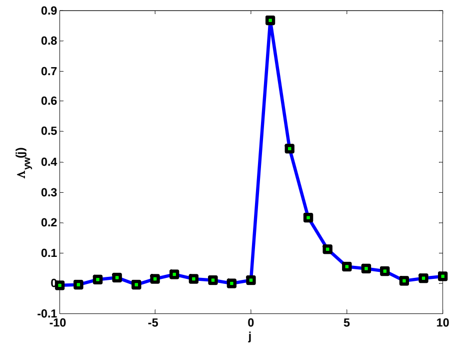
$$\begin{aligned} g(k) &= \mathcal{Z}^{-1}\{G(z)\} \\ &= \begin{cases} 0 & k < 1 \\ 0.5^{k-1} & k \geq 1 \end{cases} \end{aligned}$$

WSS white noise input example

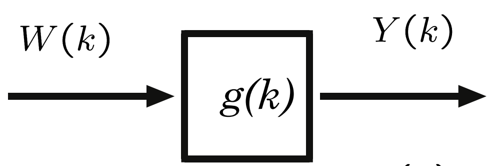


$$\Lambda_{YW}(j) = g(j)$$

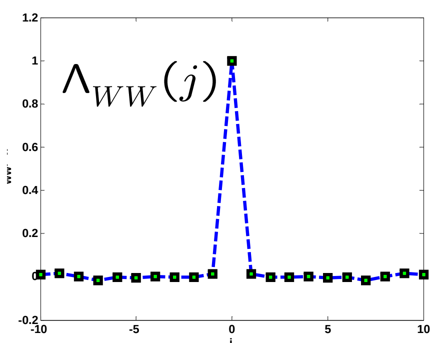


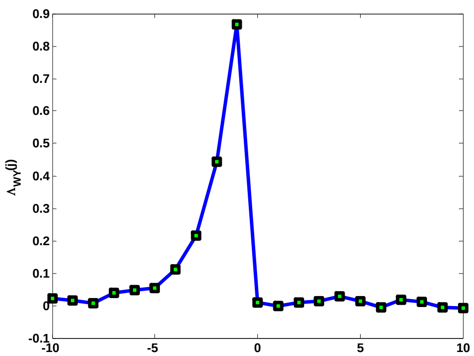


WSS white noise input example



$$\Lambda_{WY}(j) = \Lambda_{YW}(-j)$$





WSS white noise input example

$$\Lambda_{WW}(l) = \delta(l) \qquad Y(k) = 0.5Y(k-1) + W(k-1)$$

Remember that *W(k)* is white noise:

$$\Lambda_{WW}(l) = \delta(l) \qquad \begin{aligned} \Lambda_{YW}(-l) &= 0 \quad l > 0 \\ &= E\{Y(k-l)W(k)\} \end{aligned}$$

Thus,

$$\begin{aligned} \Lambda_{YW}(0) &= E\{Y(k)W(k)\} \\ &= E\{[0.5Y(k-1) + W(k-1)]W(k)\} \\ &= 0.5\,E\{Y(k-1)W(k)\} + E\{W(k-1)W(k)\} \\ &= 0.5\,\cancel{\Lambda_{YW}(-1)}^{0} + \cancel{\Lambda_{WW}(-1)}^{0} \\ &= 0 \end{aligned}$$

WSS white noise input example

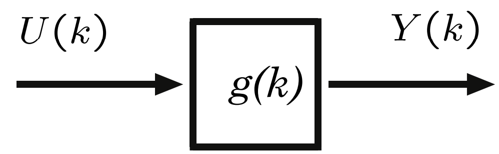
$$\Lambda_{WW}(l) = \delta(l) \qquad Y(k) = 0.5Y(k) + W(k-1)$$

$$\begin{aligned}
\Lambda_{YW}(1) &= E\{Y(k+1)W(k)\} \\
&= E\{[0.5Y(k) + W(k)]W(k)\} \\
&= 0.5\,E\{Y(k)W(k)\} + E\{W(k)W(k)\} \\
&= 0.5\,\cancelto{0}{\Lambda_{YW}}(0) + \cancelto{1}{\Lambda_{WW}}(0) \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\Lambda_{YW}(2) &= E\{Y(k+2)W(k)\} \\
&= E\{[0.5Y(k+1) + W(k+1)]W(k)\} \\
&= 0.5\,E\{Y(k+1)W(k)\} + E\{W(k+1)W(k)\} \\
&= 0.5\,\cancelto{1}{\Lambda_{YW}}(1) + \cancelto{0}{\Lambda_{WW}}(1) \\
&= 0.5
\end{aligned}$$

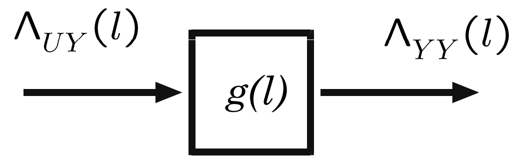
WSS SISO Linear Time Invariant Systems

If



"time domain"

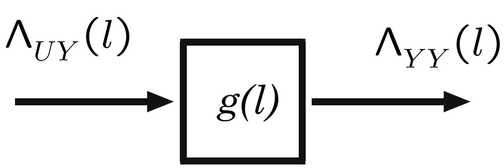
Then:



"correlation domain"

WSS SISO Linear Time Invariant Systems

What does this mean?



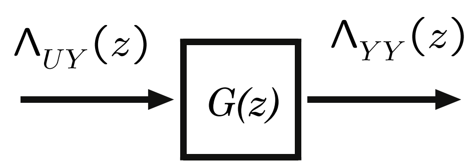
"correlation domain"

Convolution:

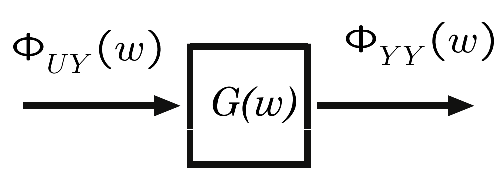
$$\Lambda_{YY}(l) = \sum_{i=-\infty}^{\infty} g(i)\,\Lambda_{UY}(l-i)$$

WSS SISO Linear Time Invariant Systems

Taking Z transform:



Taking Fourier transform:



WSS SISO Linear Time Invariant Systems

If
$$Y(k) = \sum_{i=-\infty}^{\infty} g(i) U(k-i)$$

Then:

$$\boxed{\Lambda_{YY}(l) = \sum_{i=-\infty}^{\infty} g(i)\, \Lambda_{UY}(l-i)}$$

$$\boxed{\Phi_{YY}(w) = G(w)\, \Phi_{UY}(w)}$$

$$\boxed{\Lambda_{YY}(l) = \sum_{i=-\infty}^{\infty} g(i)\, \Lambda_{UY}(l-i)}$$

Proof:

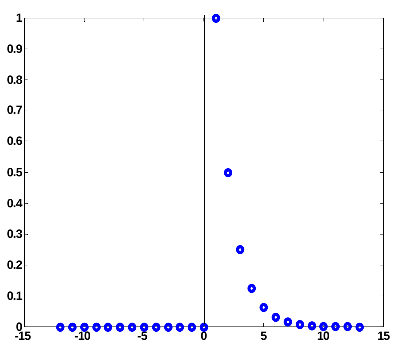
$$Y(k) = \sum_{i=-\infty}^{\infty} g(i) U(k-i) \qquad (m_U = 0)$$

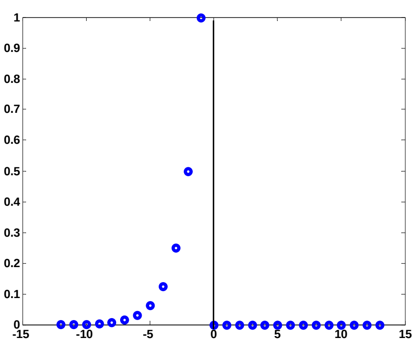
Then:

$$\begin{aligned}
\Lambda_{YY}(l) &= E\{Y(k+l)Y(k)\} \\
&= E\left\{ \left[\sum_{i=-\infty}^{\infty} g(i) U(k+l-i) \right] Y(k) \right\} \\
&= \sum_{i=-\infty}^{\infty} g(i)\, E\{U(k+l-i)Y(k)\} \\
&= \sum_{i=-\infty}^{\infty} g(i) \Lambda_{UY}(l-i)
\end{aligned}$$

White noise input example

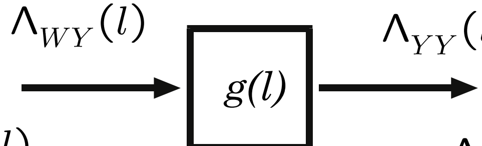
$$\boxed{\Lambda_{YW}(l) = g(l)} \qquad \boxed{\Lambda_{WY}(l) = g(-l)}$$



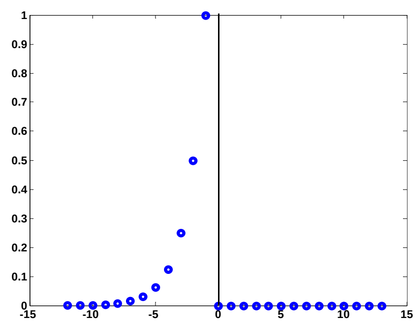


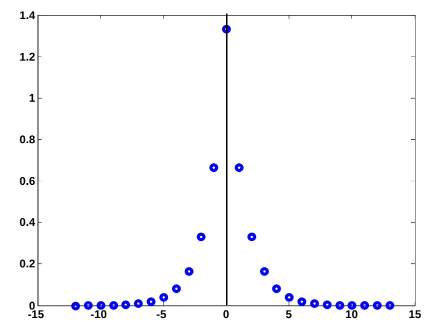
White noise input example

$\Lambda_{WY}(l)$ $\qquad$ $\Lambda_{YY}(l)$



$\Lambda_{WY}(l)$ $\qquad$ $\Lambda_{YY}(l)$

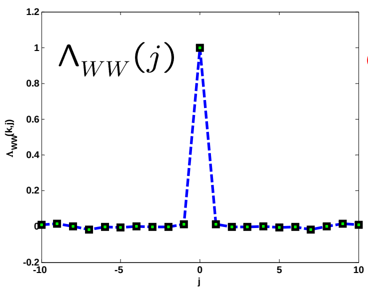




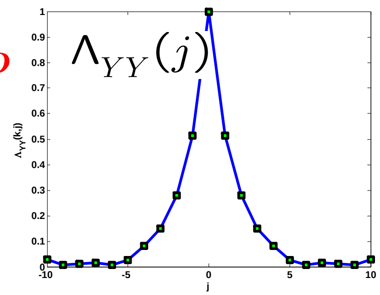
WSS white noise input Matlab example

- Feeding white noise through a linear filter generates color noise

$$W(k) \longrightarrow \boxed{\frac{1}{z-0.5}} \longrightarrow Y(k)$$



COMPUTED WITH MATLAB



Correlation Domain Overview

1. $\Lambda_{UU}(l) \longrightarrow \boxed{g(l)} \longrightarrow \Lambda_{YU}(l)$
2. $\Lambda_{YU}(l) = \Lambda_{UY}(-l)$
3. $\Lambda_{UY}(l) \longrightarrow \boxed{g(l)} \longrightarrow \Lambda_{YY}(l)$

Correlation Domain Overview

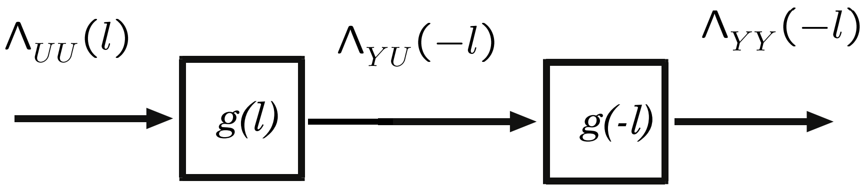
1. $\Lambda_{UU}(l) \longrightarrow \boxed{g(l)} \longrightarrow \Lambda_{UY}(-l)$
2. $\Lambda_{YU}(l) = \Lambda_{UY}(-l)$
3. $\Lambda_{UY}(l) \longrightarrow \boxed{g(l)} \longrightarrow \Lambda_{YY}(l)$

Correlation Domain Overview

1. $\Lambda_{UU}(l) \longrightarrow \boxed{g(l)} \longrightarrow \Lambda_{YU}(-l)$
2. $\Lambda_{YU}(l) = \Lambda_{UY}(-l)$
3. $\Lambda_{UY}(-l) \longrightarrow \boxed{g(-l)} \longrightarrow \Lambda_{YY}(-l)$

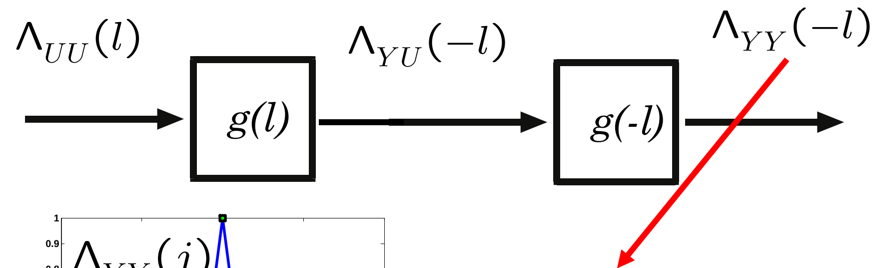
Correlation Domain Overview

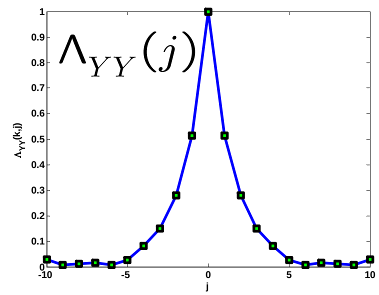
Thus



Correlation Domain Overview

Thus



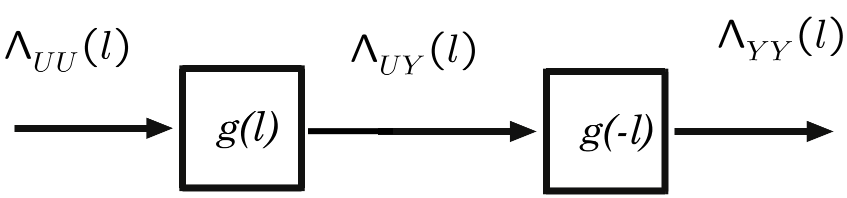


$$\Lambda_{YY}(-l) = \Lambda_{YY}(l)$$

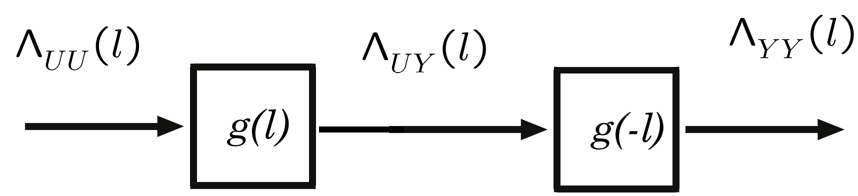
$$\Lambda_{YU}(l) = \Lambda_{UY}(-l)$$

Correlation Domain Overview

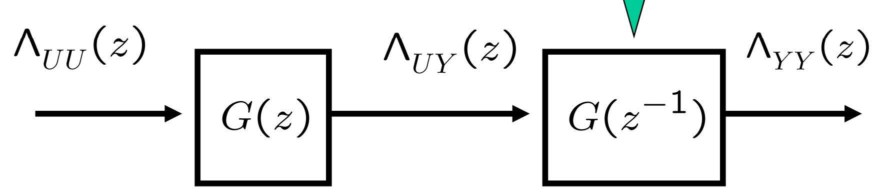
Thus



Z Domain Overview



Taking Z transforms:



Z-transform of *g(l)*

$$\mathcal{Z}\{g(l)\} = \sum_{j=-\infty}^{\infty} z^{-j} g(j)$$

(causality) $\quad (g(-j) = 0,\ j > 0)$

$$= \sum_{j=0}^{\infty} z^{-j} g(j)$$

$$= G(z)$$

Z-transform of *g(-l)*

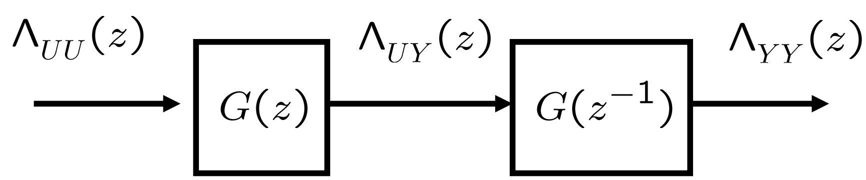
$$\mathcal{Z}\{g(-l)\} = \sum_{j=-\infty}^{\infty} z^{-j} g(-j)$$

(causality) $\quad (g(-j) = 0,\ j > 0)$

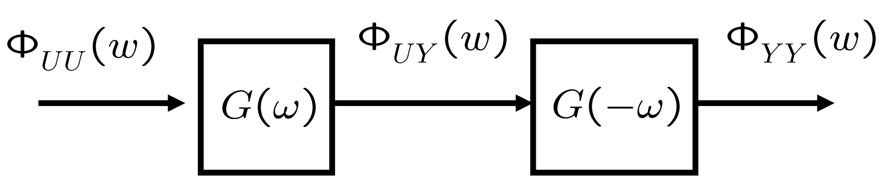
$$= \sum_{j=-\infty}^{0} z^{-j} g(-j)$$

$$= \sum_{j=0}^{\infty} z^{j} g(j) \quad = G(z^{-1})$$

Frequency Domain Overview



$$z = e^{j\omega} \qquad z^{-1} = e^{-j\omega}$$



WSS white noise input Matlab example

Let $U(k)$ be a scalar WSS random sequences

If $$Y(k) = \sum_{i=-\infty}^{\infty} g(i) U(k-i)$$

Then:

$$\boxed{\begin{aligned} \Phi_{YY}(\omega) &= G(\omega) G(-\omega)\, \Phi_{UU}(\omega) \\ \Phi_{YY}(\omega) &= |G(\omega)|^2\, \Phi_{UU}(\omega) \end{aligned}}$$

SISO Linear Time Invariant Systems

Notice that for

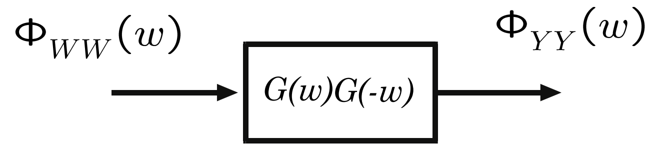
$$G(z) = \frac{B(z)}{A(z)} = \frac{b_m z^m + \cdots + b_o}{z^n + \cdots + a_o}$$

Then:

$$G(w) = \left[\frac{B(z)}{A(z)}\right]_{z=e^{j\omega}} \quad G(-w) = \left[\frac{B(z^{-1})}{A(z^{-1})}\right]_{z=e^{j\omega}}$$

$$G(\omega)G(-\omega) = \left[\frac{B(z)}{A(z)}\frac{B(z^{-1})}{A(z^{-1})}\right]_{z=e^{j\omega}} = |G(\omega)|^2$$

WSS White Noise Input Example

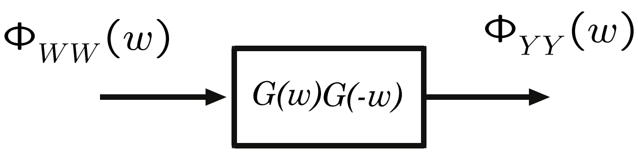


$$\Phi_{WW}(w) = \sigma_W^2 \qquad G(z) = \frac{1}{z - 0.5}$$

$$G(w) = \frac{1}{e^{j\omega} - 0.5} \qquad G(-w) = \frac{1}{e^{-j\omega} - 0.5}$$

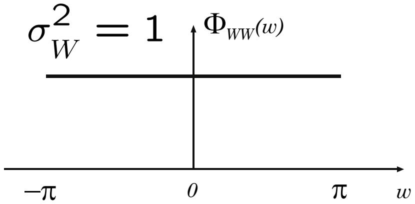
$$\boxed{G(\omega)G(-\omega) = \frac{1}{1 - \cos(w) + 0.25} > 0}$$

WSS White Noise Input Example

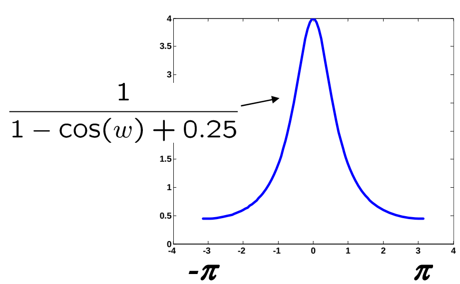


$\Phi_{WW}(w)$ $\Phi_{YY}(w)$

$\sigma_W^2 = 1$



white noise



colored noise