ME 233 Advance Control II

Lecture 11
Continuous Time Kalman Filters

(ME233 Class Notes pp.KF7-KF10)

Outline

- Continuous time Kalman Filter
- LQ-KF duality
- KF return difference equality
  – symmetric root locus
- ARMAX models

Stochastic state model

Consider the following nth order LTI system with stochastic input and measurement noise:

$$\frac{d}{dt}x(t) = A\,x(t) + B\,u(t) + B_w\,w(t)$$

$$y(t) = C\,x(t) + v(t)$$

Where:

- $u(t)$ **deterministic (known) input**
- $w(t)$ Gaussian, white noise, zero mean, input noise
- $v(t)$ Gaussian, white noise, zero mean, meas. noise
- $x(0)$ Gaussian

Assumptions

- Initial conditions:

$$E\{x(0)\} = x_o \qquad E\{\tilde{x}^o(0)\tilde{x}^{oT}(0)\} = X_o$$

- Noise properties (in addition to Gaussian),:

$$E\{w(t+\tau)w^T(t)\} = W(t)\,\delta(\tau)$$

$$E\{v(t+\tau)v^T(t)\} = V(t)\,\delta(\tau)$$

$$E\{w(t+\tau)v^T(t)\} = 0$$

$$E\{\tilde{x}^o(0)w^T(t)\} = 0 \qquad E\{\tilde{x}^o(0)v^T(t)\} = 0$$

Conditional estimation

- Conditional state estimate

$$Y_t \;=\; \{y(\tau)\} \qquad \tau \in [0,t]$$

$$\widehat{x}(t) = E\{x(t)|Y_t\}$$

- Conditional state estimation error covariance

$$M(t) = E\{\tilde{x}(t)\tilde{x}^T(t)\}$$

$$\tilde{x}(t) = x(t) - \widehat{x}(t)$$

CT Kalman Filter

Kalman filter:

$$\frac{d}{dt}\widehat{x}(t) \;=\; A\,\widehat{x}(t) + B\,u(t) + L(t)\,\tilde{y}(t)$$

$$\tilde{y}(t) \;=\; y(t) - C\,\widehat{x}(t) \qquad \widehat{x}^o(0) = x_o$$

Where:

$$L(t) = M\,C^T\,V^{-1}$$

$$\frac{d}{dt}M(t) \;=\; AM + MA^T + B_w W B_w^T - \; MC^T V^{-1} CM$$

$$M(0) \;=\; X_o$$

Steady State KF

Theorem:

1) If the pair ***[A, C]*** is observable (or detectable):

The solution of the Riccati differential equation

$$\frac{d}{dt}M(t) \;=\; AM + MA^T + B_w W B_w^T - \; MC^T V^{-1} CM$$

$$M(0) = 0$$

Converges to a stationary solution, which satisfies the Algebraic Riccati Equation (ARE):

$$AM + MA^T = -B_w W B_w^T + MC^T V^{-1} CM$$

Steady State KF

Theorem:

2) If in addition to 1) the pair ***[A,B'$_w$]*** is controllable (stabilizable), where

$$B_w'\,B_w'^T \;= B_w W B_w^T$$

The solution of the Algebraic Riccati Equation (ARE):

$$AM + MA^T = -B_w W B_w^T + MC^T V^{-1} CM$$

is unique, positive definite (semi-definite) , and the close loop observer matrix

$$A_c = A - LC$$

is **Hurwitz**.

$$L = M\,C^T\,V^{-1}$$

Steady State Kalman Filter

Theorem:

3) Under stationary noise and the conditions in 1) and 2),
The observer residual

$$\tilde{y}(t) = y(t) - C\,\hat{x}(t)$$

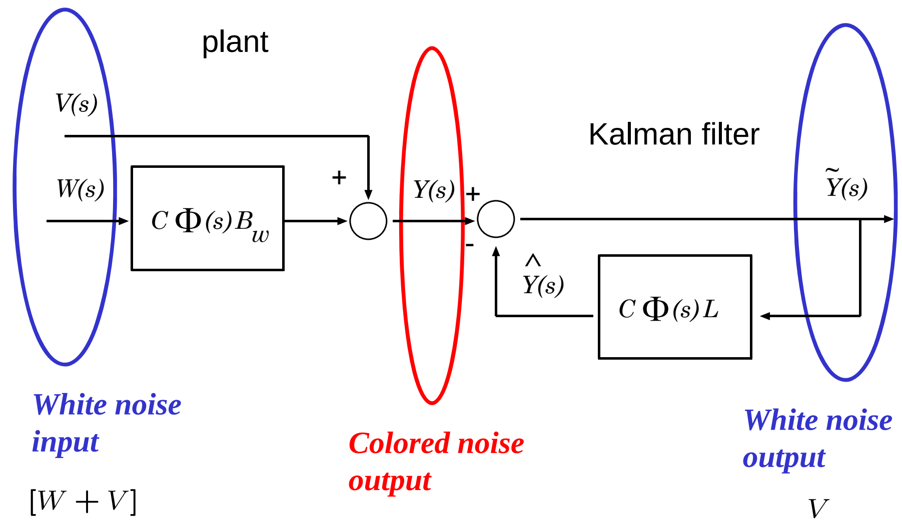
of the KF:

$$\frac{d}{dt}\hat{x}(t) = A\,\hat{x}(t) + B\,u(t) + L\,\tilde{y}(t)$$

becomes white

$$\boxed{E\left\{\tilde{y}(t+\tau)\tilde{y}^T(t)\right\} = V\,\delta(\tau)}$$

KF as a innovations (whitening) filter



LQR duality

Cost:

$$J = \frac{1}{2}x^T(t_f)\,S\,x(t_f) + \frac{1}{2}\int_0^{t_f}\left\{x^T\,C_Q^T C_Q\,x + u^T\,R\,u\right\}\,dt$$

$$u(t) = -K(t)\,x(t)$$

Where:

$$K(t) = R^{-1}\,B^T\,P$$

$$-\frac{d}{dt}P(t) = A^T\,P + P\,A + C_Q^T C_Q - P\,B\,R^{-1}\,B^T P$$

$$P(t_f) = S$$

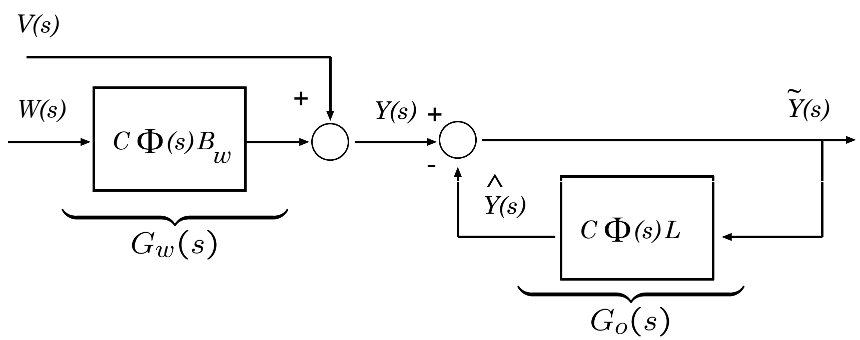
Kalman Filter & LQR Duality

Comparing ARE's and feedback gains, we obtain the following duality

duality $\longrightarrow$

LQR	KF
P	M
A	A^T
B	C^T
R	V
$C_Q{}^T$	$B'_w = B_w W^{1/2}$
K	L^T
$(A-BK)$	$(A-LC)^T$

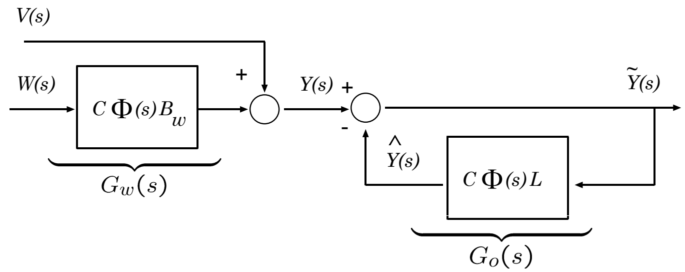
KF return difference equality



$$[I+G_o(s)]\;\underbrace{V}_{\Lambda_{\tilde y\tilde y}(s)}\;[I+G_o(-s)]^T = V + G_w(s)\,W G_w^T(-s)$$

$$\Lambda_{\tilde y\tilde y}(\tau) = E\left\{\tilde y(t+\tau)\tilde y^T(t)\right\}$$

KF root locus for SISO Systems



$$\begin{aligned}[1+G_o(s)] &= [1+C\Phi(s)L] \\ &= \frac{\det(sI-A+LC)}{\det(sI-A)} \\ &= \frac{C(s)}{A(s)}\end{aligned}$$

roots are Kalman filter poles

roots are plant poles

KF symetric root locus for SISO Systems

$$\frac{C(-s)C(s)}{A(-s)A(s)} = \left[1+\rho\,\frac{B_w(-s)B_w(s)}{A(-s)A(s)}\right]$$

$$\frac{C(s)}{A(s)} = \frac{\det(sI-A+LC)}{\det(sI-A)}$$

roots are Kalman filter poles

roots are plant poles

$$\frac{B_w(s)}{A(s)} = G_w(s) = C\Phi(s)B_w$$

$$\rho = \frac{W}{V} \geq 0$$

SISO ARMAX stochastic models

SISO ARMAX model:

$$A(s)\,Y(s) = B(s)\,U(s) + C(s)\,\tilde Y(s)$$

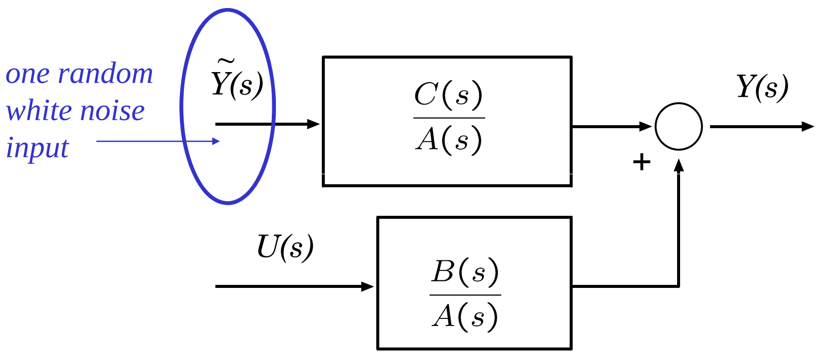
$$C(s) = \det\{(sI-A+LC)\} = 0$$ **(Hurwitz)**

$$A(s) = \det\{(sI-A)\} = 0$$

$\tilde y(t)$ Kalman filter innovations (residual)

SISO ARMAX stochastic models

$$Y(s) = \frac{B(s)}{A(s)} U(s) + \frac{C(s)}{A(s)} \tilde{Y}(s)$$



Outline

- – Continuous time Kalman Filter
- – LQ-KF duality
- – KF return difference equality
  - symmetric root locus
- – ARMAX models

Additional material:

- Derivation of continuous time Kalman Filter

Derivation of the CT Kalman Filter

1. Approximate the CT state estimation problem by a DT state estimation problem .
2. Obtain the DT Kalman filter for the DT state estimation problem.
3. Obtain the CT Kalman filter from the DT Kalman filter by taking the limit as the sampling time approaches to zero.

CT Kalman Filter

Consider the following nth order LTI system with stochastic input and measurement noise:

$$\begin{aligned} \frac{d}{dt}x(t) &= A\,x(t) + B\,u(t) + B_w\,w(t) \\ y(t) &= C\,x(t) + v(t) \end{aligned}$$

Where:

- $u(t)$ **deterministic input**
- $w(t)$ Gaussian, white noise, zero mean, input noise
- $v(t)$ Gaussian, white noise, zero mean, meas. noise
- $x(0)$ Gaussian

Derivation of the CT Kalman Filter

1. Approximate the CT state estimation problem by a DT state estimation problem .

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3. Obtain the CT Kalman filter from the DT Kalman filter by taking the limit as the sampling time approaches to zero.

Derivation of the CT Kalman Filter

1. Approximate the CT state estimation problem by a DT state estimation problem :

- **State and output equations:**

$$x(k+1) \approx \underbrace{[I + \Delta t\, A]}_{A_d} x(k) + \underbrace{B\,\Delta t}_{B_d}\, u(k) + \underbrace{B_w\,\Delta t}_{B_{dw}}\, w(k)$$

$$y(k) \approx Cx(k) + v(k)$$

$$w(k) \approx \frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t} w(t)dt \qquad v(k) \approx \frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t} v(t)dt$$

Derivation of the CT Kalman Filter

- **Covariances** (from pages 48-52 in random process lecture 8):

$$\Lambda_{ww}(k,l) = W_d(k)\,\delta(l)$$

$$W_d(k) = \frac{1}{\Delta t}\bar{W}(k)$$

$$\bar{W}(k) = \frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t} W(t)dt$$

Notice that:

$$\lim_{\Delta t \to 0} = \bar{W}(k) = W(t)$$

Derivation of the CT Kalman Filter

- **Covariances:**

$$\Lambda_{vv}(k,l) = V_d(k)\,\delta(l)$$

$$V_d(k) = \frac{1}{\Delta t}\bar{V}(k)$$

$$\bar{V}(k) = \frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t} V(t)dt$$

Notice that:

$$\lim_{\Delta t \to 0} = \bar{V}(k) = V(t)$$

Derivation of the CT Kalman Filter

2. Obtain the DT Kalman filter for the DT state estimation problem.

$$\hat{x}^o(k+1) = A_d\,\hat{x}^o(k) + B_d\,u(k) + L_d(k)\,\tilde{y}^o(k)$$

$$\tilde{y}^o(k) = y(k) - C\,\hat{x}^o(k)$$

$$L_d(k) = A_d\,M(k)C^T\left[C\,M(k)C^T + V_d(k)\right]^{-1}$$

$$M(k+1) = A_d M(k) A_d^T + B_{dw} W_d(k) B_{dw}^T - A M(k) C^T\left[C M(k) C^T + V_d(k)\right]^{-1} C M(k) A_d^T$$

Derivation of the CT Kalman Filter

3) Obtain the CT Kalman filter from the DT Kalman filter.

- **State Equation:**

$$\hat{x}^o(k+1) = A_d\,\hat{x}^o(k) + B_d\,u(k) + L_d(k)\,\tilde{y}^o(k)$$

$$\hat{x}^o(k+1) = \underbrace{[I + \Delta t\, A]}_{A_d}\hat{x}^o(k) + \underbrace{B\,\Delta t}_{B_d}\,u(k) + L_d(k)\,\tilde{y}^o(k)$$

$$\frac{\hat{x}^o(k+1) - \hat{x}^o(k)}{\Delta t} = A\,\hat{x}^o(k) + B\,u(k) + \frac{1}{\Delta t}L_d(k)\,\tilde{y}^o(k)$$

Derivation of the CT Kalman Filter

Taking limit as $\Delta t \to 0$

$$\frac{d}{dt}\hat{x}(t) = A\,\hat{x}(t) + B\,u(t) + \lim_{\Delta t \to 0}\{\frac{1}{\Delta t}L_d(k)\}\,\tilde{y}(t)$$

$$\tilde{y}(t) = y(t) - C\,\hat{x}(t)$$

$$\lim_{\Delta t \to 0}\{\frac{1}{\Delta t}L_d(k)\} = L(t) = M(t)C^T V(t)^{-1}$$

Derivation of the CT Kalman Filter

Kalman filter gain

$$L_d(k) = A_d\,M(k)C^T\left[C\,M(k)C^T + V_d(k)\right]^{-1}$$

$$L_d(k) = (1 + \Delta t A)\,M(k)C^T\left[C\,M(k)C^T + \frac{1}{\Delta t}\bar{V}(k)\right]^{-1}$$

$$L_d(k) = \Delta t(1 + \Delta t A)\,M(k)C^T\left[\Delta t C\,M(k)C^T + \bar{V}(k)\right]^{-1}$$

$$\lim_{\Delta t \to 0}\{\frac{1}{\Delta t}L_d(k)\} = L(t) = M(t)C^T V(t)^{-1}$$

Derivation of the CT Kalman Filter

Riccati equation

$$\begin{aligned} M(k+1) &= A_d M(k) A_d^T + B_{dw} W_d(k) B_{dw}^T \\ &\quad - AM(k)C^T \left[CM(k)C^T + V_d(k) \right]^{-1} CM(k)A_d^T \end{aligned}$$

Subtracting $M(k)$ from both sides and dividing by Δt

$$\begin{aligned} \frac{M(k+1) - M(k)}{\Delta t} &= AM(k) + M(k)A^T + \Delta t AM(k)A^T \\ &\quad + B_w \bar{W}(k) B_w^T - M(k)C^T \left[\Delta t CM(k)C^T + \bar{V}(k) \right]^{-1} CM(k) \\ &\quad - \Delta t\, AM(k)C^T \left[\Delta t CM(k)C^T + \bar{V}(k) \right]^{-1} CM(k)A_d^T \end{aligned}$$

Derivation of the CT Kalman Filter

Taking $\Delta t \rightarrow 0$

$$\begin{aligned} \frac{M(k+1) - M(k)}{\Delta t} &= AM(k) + M(k)A^T + \Delta t AM(k)A^T \\ &\quad + B_w \bar{W}(k) B_w^T - M(k)C^T \left[\Delta t CM(k)C^T + \bar{V}(k) \right]^{-1} CM(k) \\ &\quad - \Delta t\, AM(k)C^T \left[\Delta t CM(k)C^T + \bar{V}(k) \right]^{-1} CM(k)A_d^T \end{aligned}$$

we obtain

$$\begin{aligned} \frac{d}{dt} M(t) &= AM(t) + M(t)A^T + B_w W(t) B_w^T \\ &\quad - M(t)C^T V^{-1}(t) CM(t) \end{aligned}$$