ME 233 Advance Control II

Continuous time results 1

Random processes

(ME233 Class Notes pp. PR6-PR13)

Random Process

A random processes is a **continuous** function of time
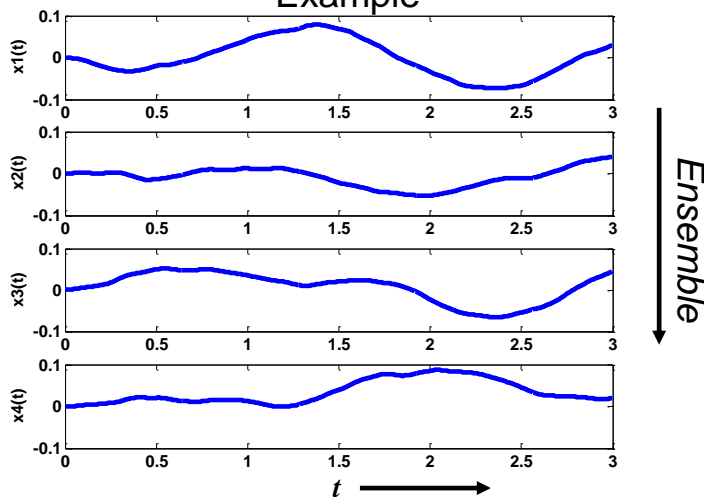
$$X(\cdot) : \mathcal{R} \to \mathcal{R}$$
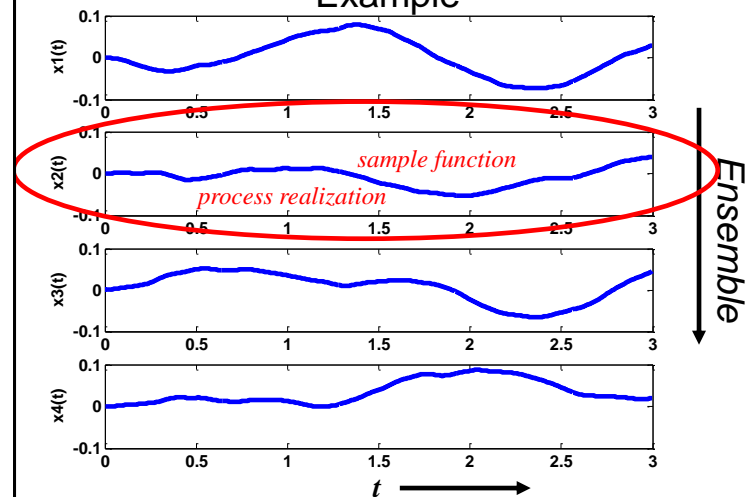
Such that for any time t_o ,

$$X(t_o)$$

Is a random variable defined over the same probability space

$$(\Omega, \mathcal{S}, Pr)$$

Example

Ensemble

Example

sample function

process realization

Ensemble

1

2nd order statistics

Let $X(t)$ be a random vector process

Expected value or mean of $X(t)$,

$$E\{X(t)\} = m_X(t)$$

Auto-covariance function:

$$\Lambda_{XX}(t, \underline{\tau}) =$$
$$E\left\{\left[X(t + \underline{\tau}) - m_X(t + \underline{\tau})\right]\left[X(t) - m_X(t)\right]^T\right\}$$

Auto-covariance function

Define: $\tilde{X}(t) = X(t) - m_X(t)$

$$\Lambda_{XX}(t, \tau) = E\left\{\tilde{X}(t + \tau)\tilde{X}^T(t)\right\}$$

$$\Lambda_{XX}(t + \tau) = E\left\{\begin{bmatrix} \tilde{X}_1(t + \tau) \\ \vdots \\ \tilde{X}_n(t + \tau) \end{bmatrix} \begin{bmatrix} \tilde{X}_1(t) & \cdots & \tilde{X}_n(t) \end{bmatrix}\right\}$$

Wide Sense Stationarity

A random sequence

is **Wide Sense Stationary (WSS)** if:

1) Its mean is time invariant

$$E\{X(t)\} = m_X$$

Wide Sense Stationarity

A random sequence

is **Wide Sense Stationary (WSS)** if:

2) Its covariance only depends on the correlation shift τ

$$\Lambda_{XX}(t, \tau) = \Lambda_{XX}(t + T, \tau)$$

Wide Sense Stationarity

The auto-covariance function can be defined only as
a function of the correlation time shift τ

$$\Lambda_{XX}(\underline{\tau}) = E\left\{\tilde{X}(t+\underline{\tau})\tilde{X}^T(t)\right\}$$

Notice that:

$$\Lambda_{XX}(\tau) = \Lambda_{XX}^T(-\tau)$$
$$\text{trace}\{\Lambda_{XX}(0)\} \geq |\text{trace}\{\Lambda_{XX}(\tau)\}|$$

Cross-covariance function

Let $X(t) \in \mathcal{R}^n$ and $Y(t) \in \mathcal{R}^m$
be two **WSS** random vector processes

The cross-covariance function:

$$\Lambda_{XY}(\tau) = E\left\{\tilde{X}(t+\tau)\tilde{Y}^T(t)\right\}$$

for *any* time t

Cross-covariance function

$$\Lambda_{XY}(\tau) = E\left\{\tilde{X}(t+\tau)\tilde{Y}^T(t)\right\}$$

$$\Lambda_{XY}(\tau) = \Lambda_{YX}^T(-\tau)$$

Power Spectral Density Function

For WSS random process, the power spectral density
function is the Fourier transform of the auto-
covariance function:

$$\Phi_{XX}(\omega) = \mathcal{F}\{\Lambda_{XX}(\tau)\}$$

$$= \int_{-\infty}^{\infty} \Lambda_{XX}(\tau)e^{-j\omega\tau}d\tau$$

Power Spectral Density Function

Since,

$$\Lambda_{XX}(\tau) = \mathcal{F}^{-1}\{\Phi_{XX}(\omega)\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega\tau}\Phi_{XX}(\omega)\, d\omega$$

$$\Lambda_{XX}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{XX}(\omega)\, d\omega$$

White noise

A WSS random process $W(t) \in \mathcal{R}$ **is white** if:

$$\boxed{\Lambda_{WW}(t) = \sigma_W^2\, \delta(t)}$$

Where $\delta(t)$ is the **Dirac delta impulse**

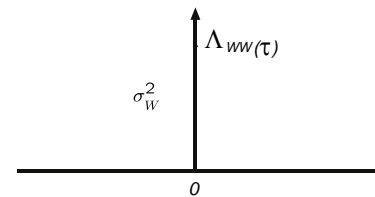white noise is zero mean if $E\{W(t)\} = 0$

White noise

The power spectral density function for white noise is:
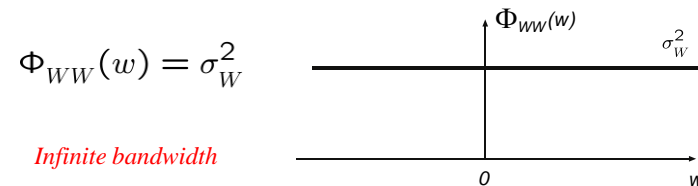
$$\boxed{\Phi_{WW}(w) = \sigma_W^2}$$

Proof:

$$\Phi_{WW}(\omega) = \int_{-\infty}^{\infty}\Lambda_{WW}(\tau)e^{-j\omega\tau}d\tau$$

$$= \sigma_W^2 \int_{-\infty}^{\infty} e^{-j\omega\tau}\,\delta(\tau)\, d\tau$$

$$= \sigma_W^2$$

White noise

$$\Lambda_{WW}(\tau) = \sigma_W^2\, \delta(\tau)$$

$$\Phi_{WW}(w) = \sigma_W^2$$

Infinite bandwidth

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White noise vector process

A **WSS** random vector sequence $\quad W(t) \in \mathcal{R}^n \quad$ **is white** if:

$$\Lambda_{WW}(\tau) = \Sigma_{WW}\, \delta(\tau)$$

where

$$\Sigma_{WW} = \Sigma_{WW}^T \succeq 0$$

and $\quad \delta(t) \quad$ is the Dirac delta impulse

MIMO Linear Time Invariant Systems

Let $\quad G(t) \in \mathcal{R}^{p \times m}$

be the impulse response of an LTI SISO system with transfer function

$$G(s) = \mathcal{L}\{G(t)\} = \int_{-\infty}^{\infty} e^{-st}\, G(t)\, dt$$

MIMO Linear Time Invariant Systems

Let $\quad U(t) \in \mathcal{R}^m \quad$ be WSS

Then the forced response (zero initial state)

$$Y(t) = \int_{-\infty}^{\infty} G(\tau)U(t - \tau)d\tau$$

$Y(t) \in \mathcal{R}^p \quad$ is also WSS

MIMO Linear Time Invariant Systems

We will assume that

- The WSS random process $\quad U(t)$ is zero mean, I.e.

$$E\{U(t)\} = m_U = 0$$
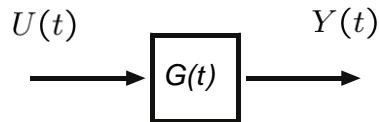
Thus, the output random process is also zero mean
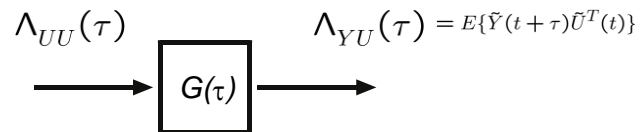
$$E\{Y(t)\} = m_Y = 0$$

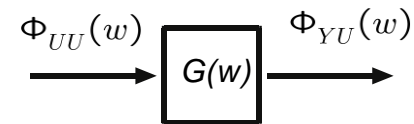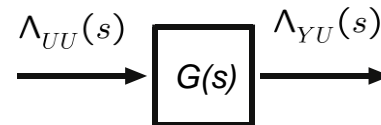# MIMO Linear Time Invariant Systems

Let $U(t)$ be WSS

If

$U(t)$ → $G(t)$ → $Y(t)$

Then:

$\Lambda_{UU}(\tau)$ → $G(\tau)$ → $\Lambda_{YU}(\tau) = E\{\tilde{Y}(t+\tau)\tilde{U}^T(t)\}$

MIMO Linear Time Invariant Systems

Let $U(t)$ be a WSS random process

$\Lambda_{UU}(s)$ → $G(s)$ → $\Lambda_{YU}(s)$

$\Phi_{UU}(w)$ → $G(w)$ → $\Phi_{YU}(w)$

$\Phi_{UU}(w) = \Lambda_{UU}(s)|_{s=j\omega}$ $\Phi_{YU}(w) = \Lambda_{YU}(s)|_{s=j\omega}$

MIMO Linear Time Invariant Systems

Let $U(t)$ be a WSS random process

$\Lambda_{UU}(s)$ → $G(s)$ → $\Lambda_{YU}(s)$

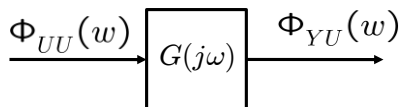$\Phi_{UU}(w)$ → $G(j\omega)$ → $\Phi_{YU}(w)$

$\Phi_{UU}(w) = \Lambda_{UU}(s)|_{s=j\omega}$ $\Phi_{YU}(w) = \Lambda_{YU}(s)|_{s=j\omega}$

MIMO Linear Time Invariant Systems

Let $U(t)$ be WSS

If

$U(t)$ → $G(t)$ → $Y(t)$

Then:

$E\{\tilde{U}(t+\tau)\tilde{Y}^T(t)\} = \Lambda_{UY}(\tau)$ → $G(\tau)$ → $\Lambda_{YY}(\tau)$

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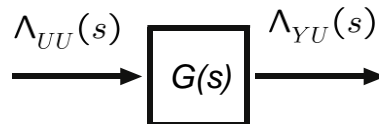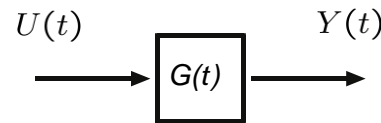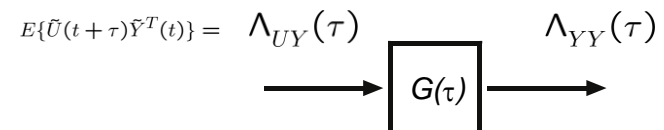# MIMO Linear Time Invariant Systems

Let $U(t)$ be a WSS random process
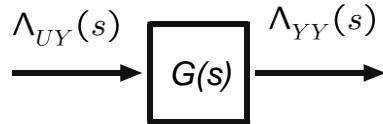
$$\Lambda_{UY}(s) \quad\quad \Lambda_{YY}(s)$$

$$\xrightarrow{\hspace{1.5cm}} \boxed{G(s)} \xrightarrow{\hspace{1.5cm}}$$

$$\xrightarrow{\Phi_{UY}(w)} \boxed{G(j\omega)} \xrightarrow{\Phi_{YY}(w)}$$

$$\Phi_{UY}(w) = \Lambda_{UY}(s)\big|_{s=j\omega} \quad\quad\quad \Phi_{YY}(w) = \Lambda_{YY}(s)\big|_{s=j\omega}$$

MIMO Linear Time Invariant Systems

Let $U(t)$ be WSS

If
$$Y(t) = \int_{-\infty}^{\infty} G(\tau)U(t-\tau)d\tau$$

Then:

$$\boxed{\Phi_{YY}(\omega) = G(j\omega)\,\Phi_{UU}(\omega)\,G^{T}(-j\omega)}$$

$$G^{*}(j\omega)$$

White noise driven state space systems

Consider a LTI system driven by white noise:

$$\frac{d}{dt}X(t) = A\,X(t) + B\,W(t)$$

$$Y(t) = C\,X(t)$$

$$X(t) \in \mathcal{R}^{n} \quad\quad\quad W(t) \in \mathcal{R}^{p}$$

$$Y(t) \in \mathcal{R}^{m}$$

White noise driven state space systems

$$\frac{d}{dt}X(t) = A\,X(t) + B\,W(t)$$
$$Y(t) = C\,X(t)$$

Assume that $W(t)$ is white, but not stationary

$$m_W(t) = E\{W(t)\}$$

$$\Lambda_{WW}(t,\tau) = \Sigma_{WW}(t)\,\delta(\tau)$$

White noise driven state space systems

$$\frac{d}{dt}X(t) = A\,X(t) + B\,W(t)$$

$$Y(t) = C\,X(t)$$

Assume state Initial Conditions (IC):

$$m_X(0) = E\{X(0)\}$$

$$\Lambda_{XX}(0,0) = E\left\{\tilde{X}(0)\tilde{X}^T(0)\right\}$$

$$E\left\{\tilde{X}(0)\tilde{W}^T(t)\right\} = 0$$

White noise driven state space systems

$$\frac{d}{dt}X(t) = A\,X(t) + B\,W(t)$$

$$Y(t) = C\,X(t)$$

Taking expectations on the equations above, we obtain:

$$\frac{d}{dt}m_X(t) = A\,m_X(t) + B\,m_W(t)$$

$$m_Y(t) = C\,m_X(t)$$

White noise driven state space systems

Subtracting the means,

$$\frac{d}{dt}\tilde{X}(t) = A\,\tilde{X}(t) + B\,\tilde{W}(t)$$

$$\tilde{Y}(t) = C\,\tilde{X}(t)$$

$$m_{\tilde{W}}(t) = 0 \quad m_{\tilde{X}}(t) = 0 \quad m_{\tilde{Y}}(t) = 0$$

White noise driven covariance propagation

$$\frac{d}{dt}\Lambda_{XX}(t,0) = A\,\Lambda_{XX}(t,0) + \Lambda_{XX}(t,0)\,A^T$$

$$+ B\,\Sigma_{WW}(t)\,B^T$$

with

$$\Lambda_{XX}(t,0) = E\left\{\tilde{X}(t)\tilde{X}^T(t)\right\}$$

$$\Lambda_{WW}(t,0) = E\left\{\tilde{W}(t)\tilde{W}^T(t)\right\} = \Sigma_{WW}(t)$$

White noise driven **covariance propagation**

Also,

$$\Lambda_{XX}(t,\tau) \;=\; e^{A\tau}\,\Lambda_{XX}(t,0) \qquad \tau \geq 0$$

where:

$$\Lambda_{XX}(t,\tau) \;=\; E\left\{\tilde{X}(t+\tau)\tilde{X}^T(t)\right\}$$

White noise driven **covariance propagation**

Also,

$$\Lambda_{XX}(t,-\tau) = \Lambda_{XX}(t-\tau,0)\,e^{A^T\tau} \quad \tau \geq 0$$

where:

$$\Lambda_{XX}(t,\tau) \;=\; E\left\{\tilde{X}(t+\tau)\tilde{X}^T(t)\right\}$$

Stationary covariance equation

For $W(t)$ WSS,

$$m_W(t) \;=\; m_W$$

$$\Lambda_{WW}(t,\tau) = \Sigma_{WW}\delta(\tau)$$

and A Hurwitz,

$$\bar{\Lambda}_{XX}(\tau) = \lim_{t\to\infty} E\{\tilde{X}(t+\tau)\tilde{X}^T(t)\}$$
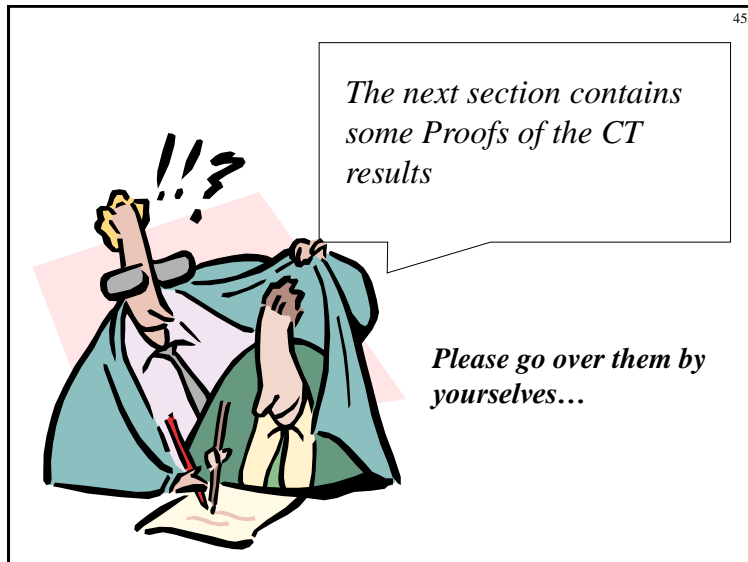
Stationary covariance equation

For $W(t)$ WSS, and A Hurwitz,

$$\bar{\Lambda}_{XX}(\tau) = \lim_{t\to\infty} E\{\tilde{X}(t+\tau)\tilde{X}^T(t)\}$$

Satisfies:

$$A\,\bar{\Lambda}_{XX}(0) + \bar{\Lambda}_{XX}(0)\,A^T = -B\,\Sigma_{WW}\,B^T$$

$$\bar{\Lambda}_{XX}(\tau) = e^{A\tau}\,\bar{\Lambda}_{XX}(0) \qquad \tau \geq 0$$

The next section contains some Proofs of the CT results

Please go over them by yourselves…

Proof of continuous time results – Method 1

We first prove that:

$$\frac{d}{dt}\Lambda_{XX}(t,0) = A\,\Lambda_{XX}(t,0) + \Lambda_{XX}(t,0)\,A^T$$

$$+ \; B\,\Sigma_{WW}(t)\,B^T$$

By starting from the Discrete Time (DT) results

Proof of continuous time results – Method 1

Approximate the state equation ODE

$$\frac{d}{dt}X(t) = A\,X(t) + B\,W(t)$$

using the Euler numerical integration method.

$$\frac{d}{dt}X(t) \approx \frac{1}{\Delta t}\{X((k+1)\Delta t) - X(k\Delta t)\}$$

- We have to be careful in dealing with white noise $W(t)$

Approximate $W(t)$

1. Define $W(k)$ as the ***time average*** of $W(t)$

$$W(k) \approx \frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t} W(t)dt$$

Similarly, taking expectations

$$m_W(k) \approx \frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t} m_W(t)dt$$

Approximate $\Lambda_{WW}(k,0)$ for $W(t)$ white

$$\Lambda_{WW}(k,0) = E\{\tilde{W}(k)\tilde{W}^T(k)\}$$

$$\approx E\left\{\left(\frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t}\tilde{W}(t)dt\right)\left(\frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t}\tilde{W}^T(\tau)d\tau\right)\right\}$$

$$\underbrace{}_{\approx \tilde{W}(k)} \qquad \underbrace{}_{\approx \tilde{W}^T(k)}$$

Approximate $\Lambda_{WW}(k,0)$ for $W(t)$ white

$$\Lambda_{WW}(k,0) = E\{\tilde{W}(k)\tilde{W}^T(k)\}$$

$$\approx \frac{1}{(\Delta t)^2}\int_{k\Delta t}^{(k+1)\Delta t}\int_{k\Delta t}^{(k+1)\Delta t}\underbrace{E\left\{\tilde{W}(t)\tilde{W}^T(\tau)\right\}}d\tau dt$$

$$\Sigma_{WW}(\tau)\underbrace{\delta(t-\tau)}_{\textit{Dirac impulse}}$$

since for $W(t)$ white

$$E\left\{\tilde{W}(t)\tilde{W}^T(\tau)\right\} = E\left\{\tilde{W}(\tau+t-\tau)\tilde{W}^T(\tau)\right\} = \Sigma_{WW}(\tau)\delta(t-\tau)$$

Approximate $\Lambda_{WW}(k,0)$ for $W(t)$ white

$$\Lambda_{WW}(k,0) = E\{\tilde{W}(k)\tilde{W}^T(k)\}$$

$$\approx \frac{1}{(\Delta t)^2}\int_{k\Delta t}^{(k+1)\Delta t}\left[\underbrace{\int_{k\Delta t}^{(k+1)\Delta t}\Sigma_{WW}(\tau)\delta(t-\tau)d\tau}_{\Sigma_{WW}(t)}\right]dt$$

$$\approx \frac{1}{(\Delta t)^2}\int_{k\Delta t}^{(k+1)\Delta t}\Sigma_{WW}(t)dt$$

Approximate $\Lambda_{WW}(k,0)$ for $W(t)$ white

$$\Lambda_{WW}(k,0) = E\{\tilde{W}(k)\tilde{W}^T(k)\}$$

$$\approx \frac{1}{(\Delta t)^2}\int_{k\Delta t}^{(k+1)\Delta t}\Sigma_{WW}(t)dt$$

$$\approx \frac{1}{(\Delta t)}\left[\underbrace{\frac{1}{(\Delta t)}\int_{k\Delta t}^{(k+1)\Delta t}\Sigma_{WW}(t)dt}_{\Sigma_{WW}(k)}\right]$$

Approximate $\Lambda_{WW}(k,0)$ for $W(t)$ white

$$\Lambda_{WW}(k,0) \approx \frac{1}{\Delta t}\Sigma_{WW}(k)$$

Where $\Sigma_{WW}(k)$ is the **time average** of $\Sigma_{WW}(t)$

$$\Sigma_{WW}(k) = \frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t}\Sigma_{WW}(\tau)d\tau$$

Numerical Integration

The state equation

$$\frac{d}{dt}X(t) = AX(t) + BW(t)$$

By the discrete time state equation

$$X(k+1) \approx \underbrace{[I + \Delta t\,A]}_{A_d}X(k) + \underbrace{B\,\Delta t}_{B_d}W(k)$$

where

$$W(k) \approx \frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t}W(t)dt$$

Proof of continuous time results – Method 1

1. Obtain DT state equations by approximating the CT state equation solution:

$$\frac{d}{dt}X(t) = AX(t) + BW(t)$$

$$\frac{d}{dt}X(t) \approx \frac{1}{\Delta t}\{X((k+1)\Delta t) - X(k\Delta t)\}$$

Thus,

$$X(k+1) \approx \underbrace{[I + \Delta t\,A]}_{A_d}X(k) + \underbrace{B\,\Delta t}_{B_d}W(k)$$

where $\qquad W(k) \approx \frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t}W(t)dt$

Proof of continuous time results – M1

2. Obtain the CT covariance propagation equation from from the DT covariance propagation, using the approximated DT state equation:

$$\Lambda_{XX}(k+1,0) \approx A_d\Lambda_{XX}(k,0)A_d^T + B_d\frac{1}{\Delta t}\Sigma_{WW}(k)B_d^T$$

$$\approx (I + \Delta t\,A)\Lambda_{XX}(k,0)(I + \Delta t\,A)^T + \Delta t\,B\,\Sigma_{WW}(k)B^T$$

$$\approx \Lambda_{XX}(k,0) + \Delta t\,A\Lambda_{XX}(k,0) + \Delta t\,\Lambda_{XX}(k,0)A^T$$

$$+ (\Delta t)^2\,A\Lambda_{XX}(k,0)A^T + \Delta t\,B\,\Sigma_{WW}(k)B^T$$

Proof of continuous time results – M1

3. Take the limit as $\Delta t \to 0$ of

$$\frac{\Lambda_{XX}((k+1)\Delta t, 0) - \Lambda_{XX}(k\Delta t, 0)}{\Delta t} \approx$$

$$A\Lambda_{XX}(k\Delta t, 0) + \Lambda_{XX}(k\Delta t, 0)\, A^T + B\, \Sigma_{WW}(k)\, B^T$$

$$+ \Delta t\, A\Lambda_{XX}(k\Delta t, 0)\, A^T$$

and noticing that

$$\lim_{\Delta t \to 0} \Sigma_{WW}(k) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{k\Delta t}^{(k+1)\Delta t} \Sigma_{WW}(t)\, dt$$

$$= \Sigma_{WW}(t)$$

Proof of continuous time results – M1

3. Take the limit as $\Delta t \to 0$ of

$$\frac{\Lambda_{XX}((k+1)\Delta t, 0) - \Lambda_{XX}(k\Delta t, 0)}{\Delta t} \xrightarrow{\;\;\frac{d}{dt}\Lambda_{WW}(t,0)\;\;} \approx$$

$$A\Lambda_{XX}(k\Delta t, 0) \xrightarrow{t} + \Lambda_{XX}(k\Delta t, 0)\, A^T \xrightarrow{t} + B\, \Sigma_{WW}(k)\, B^T \xrightarrow{\Sigma_{WW}(t)}$$

$$+ \Delta t\, A\Lambda_{XX}(k\Delta t, 0)\, A^T \xrightarrow{0}$$

Thus,

$$\frac{d}{dt}\Lambda_{XX}(t,0) = A\,\Lambda_{XX}(t,0) + \Lambda_{XX}(t,0)\, A^T$$

$$+ B\, \Sigma_{WW}(t)\, B^T$$

Proof of continuous time results – Method 2

We now proof that:

$$\frac{d}{dt}\Lambda_{XX}(t,0) = A\,\Lambda_{XX}(t,0) + \Lambda_{XX}(t,0)\, A^T$$

$$+ B\, \Sigma_{WW}(t)\, B^T$$

Directly from continuous time (CT) results

Proof of continuous time results – M2

1) Lets calculate $\dfrac{d}{dt}\Lambda_{XX}(t,0)$ using

$$\dot{\tilde{X}}(t) = A\,\tilde{X}(t) + B\,\tilde{W}(t)$$

$$\frac{d}{dt}\Lambda_{XX}(t,0) = \frac{d}{dt}E\{\tilde{X}(t)\tilde{X}^T(t)\}$$

$$= E\{\underbrace{\dot{\tilde{X}}(t)}_{A\tilde{X}(t)+B\tilde{W}(t)}\tilde{X}^T(t)\} + E\{\tilde{X}(t)\underbrace{\dot{\tilde{X}}^T(t)}_{\tilde{X}^T(t)A^T+\tilde{W}^T(t)B^T}\}$$

$$= A\Lambda_{XX}(t,0) + \Lambda_{XX}(t,0)A^T$$

$$+ B\, E\{\tilde{W}(t)\tilde{X}^T(t)\} + E\{\tilde{X}(t)\tilde{W}^T(t)\}B^T$$

Proof of continuous time results – M2

2) We now need to calculate

$$B\,E\{\tilde{W}(t)\tilde{X}^T(t)\} + E\{\tilde{X}(t)\tilde{W}^T(t)\}B^T$$

using

$$\tilde{X}(t) \;=\; e^{At}\,\tilde{X}(0) + \int_0^t e^{A(t-\tau)}B\,\tilde{W}(\tau)d\tau$$

$$B\,E\{\tilde{W}(t)\tilde{X}^T(t)\} \;=\; B\,E\{\tilde{W}(t)\tilde{X}(0)\}e^{A^Tt}$$

$$\;=\; +B\int_0^t E\{\tilde{W}(t)\tilde{W}(\tau)\}B^T e^{A^T(t-\tau)}d\tau$$

Proof of continuous time results – M2

2) We now need to calculate $\quad B\,E\{\tilde{W}(t)\tilde{X}^T(t)\}$

using

$$\tilde{X}(t) \;=\; e^{At}\,\tilde{X}(0) + \int_0^t e^{A(t-\tau)}B\,\tilde{W}(\tau)d\tau$$

$$B\,E\{\tilde{W}(t)\tilde{X}^T(t)\} \;=\; B\,\underbrace{E\{\tilde{W}(t)\tilde{X}(0)\}}_{=o}e^{A^Tt}$$

$$\;=\; +B\int_0^t \underbrace{E\{\tilde{W}(t)\tilde{W}^T(\tau)\}}_{\Sigma_{WW}(\tau)\delta(t-\tau)}B^T e^{A^T(t-\tau)}d\tau$$

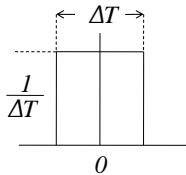$$\;=\; B\int_0^t \Sigma_{WW}(\tau)\delta(t-\tau)B^T e^{A^T(t-\tau)}d\tau$$

(notice that the Dirac impulse occurs at the edge t)

Proof of continuous time results – M2

2) Continuing,

$$B\,E\{\tilde{W}(t)\tilde{X}^T(t)\} \;=\; B\int_0^t \Sigma_{WW}(\tau)\delta(t-\tau)B^T e^{A^T(t-\tau)}d\tau$$

$$\;=\; B\int_0^t \Sigma_{WW}(t-\eta)\delta(\eta)B^T e^{A^T\eta}d\eta$$

(make integral symmetrical w/r 0)

$$\;=\; \frac{1}{2}B\int_{-t}^t \Sigma_{WW}(t-\eta)\delta(\eta)B^T e^{A^T\eta}d\eta$$

$$\;=\; \frac{1}{2}B\,\Sigma_{WW}(t)B^T$$

Proof of continuous time results – M2

2) A similar calculation for $\quad E\{\tilde{X}(t)\tilde{W}^T(t)\}\,B^T$
yields

$$E\{\tilde{X}(t)\tilde{W}^T(t)\}B^T \;=\; e^{At}\,\underbrace{E\{\tilde{X}(0)\tilde{W}^T(t)\}}_{=o}B^T$$

$$\;=\; +\int_0^t e^{A(t-\tau)}B\,\underbrace{E\{\tilde{W}(\tau)\tilde{W}^T(t)\}}_{\Sigma_{WW}(t)\delta(\tau-t)}\,d\tau\,B^T$$

$$\;=\; \int_0^t e^{A(t-\tau)}B\Sigma_{WW}(t)\delta(\tau-t)d\tau\,B^T$$

(notice that the Dirac impulse occurs at the edge t)

Proof of continuous time results – M2

2) Continuing,

$$E\{\tilde{X}(t)\tilde{W}^T(t)\}B^T = \int_0^t e^{A(t-\tau)}B\Sigma_{WW}(t)\delta(\tau - t)d\tau\, B^T$$

$$= \int_{-t}^0 e^{-A\eta}B\Sigma_{WW}(t)\delta(\eta)d\eta\, B^T$$

(make integral symmetrical w/r O)

$$= \frac{1}{2}\int_{-t}^t e^{-A\eta}B\Sigma_{WW}(t)\delta(\eta)d\eta\, B^T$$

$$= \frac{1}{2}B\,\Sigma_{WW}(t)\, B^T$$

Proof of continuous time results – M2

2) Thus

$$\boxed{B\,E\{\tilde{W}(t)\tilde{X}^T(t)\} + E\{\tilde{X}(t)\tilde{W}^T(t)\}B^T = B\,\Sigma_{WW}(t)\,B^T}$$

and

$$\frac{d}{dt}\Lambda_{XX}(t,0) = A\,\Lambda_{XX}(t,0) + \Lambda_{XX}(t,0)\,A^T$$

$$+ B\,\Sigma_{WW}(t)\,B^T$$

Proof of continuous time results – M2

Now we proof that:

$$\Lambda_{XX}(t,\tau) = e^{A\tau}\,\Lambda_{XX}(t,0) \qquad \tau \geq 0$$

Notice that:

$$\tilde{X}(t+\tau) = e^{A\tau}\,\tilde{X}(t) + \int_t^{t+\tau} e^{A(t+\tau-\eta)}\,B\,\tilde{W}(\eta)d\eta$$

where,

$$\tilde{X}(t) = X(t) - m_X(t)$$

$$\tilde{W}(t) = W(t) - m_W(t)$$

Proof of continuous time results – M2

Therefore,

$$\Lambda_{XX}(t,\tau) = E\{\tilde{X}(t+\tau)\tilde{X}^T(t)\}$$

$$= e^{A\tau}\underbrace{E\{\tilde{X}(t)\tilde{X}^T(t)\}}_{\Lambda_{WW}(t,0)}$$

$$+ \int_t^{t+\tau} e^{A(t+\tau-\eta)}B\,E\{\tilde{W}(\eta)\tilde{X}^T(t)\}\,d\eta$$

Notice that $\tilde{W}(\eta)$ and $\tilde{X}(t)$ are uncorrelated for $\eta > t$

$$E\{\tilde{W}(\eta)\tilde{X}^T(t)\} = \begin{cases} \frac{1}{2}\Sigma_{WW}(t)\,B^T & \eta = t \\ \\ 0 & \eta > t \end{cases}$$

Proof of continuous time results – M2

Thus,

$$\Lambda_{XX}(t,\tau) \;=\; e^{A\tau}\,\Lambda_{XX}(t,0) \qquad \tau \geq 0$$