2016 Spring Practice Final.

Problem 1

Consider a system in controllable canonical form.

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0.6 & 0.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \qquad x(0) = x_0,$$

$$y(k) = C\,x(k)$$

the optimal linear quadratic regulator problem

$$J = \min_{U} \left\{ \sum_{k=0}^{\infty} \left[x^T(k) C^T C\, x(k) + R u^2(k) \right] \right\}$$

This system is SISO.

1. Plot root locus for $C = [-1,\ 1]$

2. closed loop pole for $R = 0$ and $R = \infty$?

Problem 3.

Consider discrete LTI system.

$$x(k+1) = Ax(k) + Bu(k) + B_w w(k)$$
$$y(k) = Cx(k) + v(k)$$

$w(k)$ and $v(k)$ are independent, zero-mean, white Gaussian.

System is SISO, (A, B) is stabilizable & (C, A) is detectable.

For a standard optimal linear quadratic control problem

$$J(\rho) = \min_{U} E\{x^T(k) Q x(k) + \rho u^2(k)\}$$

~~Problem~~

Assume for a particular value of $\rho > 0$, we have a determined value of $J(\rho)$.

Prove that if a controller achieves $E\{u^2(k)\} \leq \alpha$,
it must ~~be~~ hold that

$$E\{x^T(k) Q x(k)\} \geq J(\rho) - \alpha\rho.$$

Parameter Adaptation Algorithm with PI-Adaptation Law
Consider a system described by

$$y(k+1) = ay(k) + bu(k)$$

where a and b are constant. The predictor output is
a priori:

$$\hat{y}^0(k+1) = \hat{a}_I(k)y(k) + \hat{b}_I(k)u(k)$$

a posteriori:

$$\hat{y}(k+1) = \hat{a}(k+1)y(k) + \hat{b}(k+1)u(k)$$

The prediction error is
a priori:

$$e^0(k+1) = y(k+1) - \hat{y}^0(k+1)$$

a posteriori:

$$e(k+1) = y(k+1) - \hat{y}(k+1)$$

The parameters are updated by

$$\hat{a}(k+1) = \hat{a}_I(k+1) + \hat{a}_P(k+1); \qquad \hat{b}(k+1) = \hat{b}_I(k+1) + \hat{b}_P(k+1)$$

where the subscripts, I and P, denote the integral and proportional parts of the adaptation law given by

$$\begin{gathered}
\hat{a}_I(k+1) = \hat{a}_I(k) + k_{aI}y(k)e(k+1); \quad \hat{b}_I(k+1) = \hat{b}_I(k) + k_{bI}u(k)e(k+1) \\
\hat{a}_P(k+1) = k_{aP}y(k)e(k+1); \quad \hat{b}_P(k+1) = k_{bP}u(k)e(k+1) \\
k_{aI} > 0; \quad k_{bI} > 0; \quad k_{aP} \geq 0; \quad k_{bP} \geq 0
\end{gathered}$$

(a) Show that the PAA as defined above is asymptotically stable.

(b) Express $e(k+1)$ in terms of $e^0(k+1)$.

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(a) Show that the PAA as defined above is asymptotically stable.

(b) Express $e(k+1)$ in terms of $e^0(k+1)$.

A discrete time system is described by

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) + b_0 u(k-2) + w_1(k-2) + w_2(k-2)$$

where

- $y(k)$: system output,
- $u(k)$: system input,
- $w_1(k) = \gamma$, constant disturbance
- $w_2(k) = \alpha cos(\omega k) + \beta sin(\omega k)$ where ω is known

(a) Assuming that the plant parameters as well as α, β and γ are known, obtain the control law to achieve

$$(1 + d_1 z^{-1} + d_2 z^{-2})[y_d(k+2) - y(k+2)] = 0$$

where z^{-1} is the one-step delay operator.

(b) Now assume that the plant parameters as well as α, β and γ are not known. Obtain the adaptive control law to achieve the control objective asymptotically.

Consider the state-space system

$$x(k+1) = Ax(k) + Bu(k) + B_w w(k) \tag{1}$$
$$y(k) = Cx(k) + v(k) \tag{2}$$

where the sequences $w(k)$ and $v(k)$ are jointly Gaussian WSS random sequences that satisfy the usual assumptions:

$$E\{w(k)\} = 0 \qquad E\{v(k)\} = 0$$
$$E\{w(k+j)w(k)\} = W\delta(j) \qquad E\{v(k+j)v(k)\} = V\delta(j)$$
$$E\{w(k+j)v(k)\} = 0\ .$$

By the internal model principle, a reasonable way to reject constant disturbances is to impose the control structure

$$u(k+1) = u(k) + \bar{u}(k)\ . \tag{3}$$

where $\bar{u}(k)$ is the incremental control to be designed. We measure the performance of the closed-loop system using the cost function

$$J = E\{x^T(k)Qx(k) + u^T(k)Ru(k)\}\ .$$

1. Append the controller dynamics (3) to the system dynamics (1)–(2) and express the resulting system in the form

$$x_e(k+1) = A_e x_e(k) + B_e \bar{u}(k) + B_{we} w(k) \tag{4}$$
$$y(k) = C_e x_e(k) + v(k) \tag{5}$$

2. Show that, for the system (4)–(5), there does not exist a Kalman filter with asymptotically stable estimation error dynamics. Assume that the Kalman filter only has access to the measurements $y(k)$ and the incremental control $\bar{u}(k)$; do not treat $u(k)$ as measurable.

3. Suppose we now modify the control structure to instead be

$$u(k+1) = u(k) + \bar{u}(k) + \eta(k)$$

where $\eta(k)$ is a Gaussian WSS random sequence that is independent from $w(k)$ and $v(k)$ and satisfies

$$E\{\eta(k)\} = 0 \qquad E\{\eta^T(k+j)\eta(k)\} = \alpha I \delta(j)$$

where $\alpha \in \mathcal{R}$. It can be shown under some reasonable assumptions on Q, R, and the system (1)–(2) that we can construct LQG controllers that optimize J for the system (4)–(5) whenever $\alpha > 0$. (You do not need to find the corresponding conditions or prove existence of an optimal controller.) Thus, adding the noise $\eta(k)$ into the control law makes the optimal LQG control problem solvable. Is is possible, via choice of α, to design a high-performance controller (in terms of J) that rejects constant disturbances? Give a brief justification of your answer.