ME 233 Advance Control II

Lecture 9
Kalman Filters Stationary Properties and LQR-KF Duality

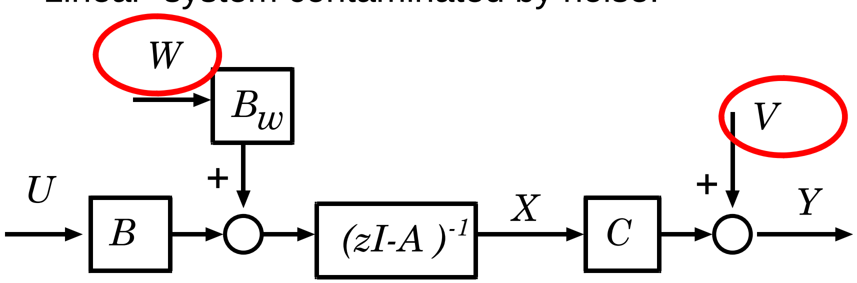
(ME233 Class Notes pp.KF1-KF6)

Summary

- Kalman filter algorithm:
  - using a-priori state estimate only
  - using a-priori and a-posteriori state estimates
- Stationary Kalman filters (KF):
  - KF algebraic Riccati equation
  - KF as an innovations filter
  - KF return difference equality

Stochastic State Estimation

Linear system contaminated by noise:



Two random disturbances:

- Input noise $w(k)$ - contaminates the state $x(k)$
- Measurement noise $v(k)$ - contaminates the output $y(k)$

Stochastic state model

State estimation of LT system:

$$x(k+1) = A\,x(k) + B\,u(k) + B_w\,w(k)$$

$$y(k) = C\,x(k) + v(k)$$

Where:

- $u(k)$ **known control input**
- $w(k)$ Gaussian, white noise, zero mean, input noise
- $v(k)$ Gaussian, white noise, zero mean, meas. noise
- $x(0)$ Gaussian

Assumptions

- Initial conditions:

$$E\{x(0)\} = x_o \qquad E\{\tilde{x}^o(0)\tilde{x}^{oT}(0)\} = X_o$$

- Noise properties:

$$\left.\begin{aligned} &E\{w(k+l)w^T(k)\} = W(k)\,\delta(l) \\ &E\{v(k+l)v^T(k)\} = V(k)\,\delta(l) \\ &E\{w(k+l)v^T(k)\} = 0 \end{aligned}\right\}$$

Zero-mean uncorrelated Gaussian white noises

$$E\{\tilde{x}^o(0)w^T(k)\} = 0 \qquad E\{\tilde{x}^o(0)v^T(k)\} = 0$$

Kalman Filter Solution

1) Compute a-priori output estimation error:

$$\tilde{y}^o(k) \;=\; y(k) - C\,\hat{x}^o(k) \qquad \hat{x}^o(0) = x_o$$

2) Compute a-posteriori state estimate:

$$\hat{x}(k) \;=\; \hat{x}^o(k) + F(k)\,\tilde{y}^o(k)$$

$$F(k) = M(k)C^T\left[C\,M(k)C^T + V(k)\right]^{-1}$$

$$M(0) \;=\; X_o$$

Kalman Filter Solution V-1

3) Compute a-posteriori state estimation error covariance:

$$Z(k) \;=\; M(k) - M(k)C^T\left[CM(k)C^T + V(k)\right]^{-1}CM(k)$$

$$M(0) \;=\; X_o$$

Kalman Filter Solution V-1

4) Update a-priori state estimate

$$\hat{x}^o(k+1) \;=\; A\,\hat{x}(k) + B\,u(k)$$

5) Update a-priori state estimation error covariance:

$$M(k+1) \;=\; A\,Z(k)A^T + B_wW(k)B_w^T$$

Kalman Filter Solution V-2

- Kalman filter algorithm can be written in a different manner, which only involves the a-priori state estimate and the a-priori estimation error covariance.

$$\hat{x}(k) \;=\; \hat{x}^o(k) + F(k)\,\tilde{y}^o(k)$$

$$F(k) = M(k)C^T\left[C\,M(k)C^T + V(k)\right]^{-1}$$

$$M(0) \;=\; X_o$$

Kalman Filter Solution V-2

Plug

$$\hat{x}(k) \;=\; \underbrace{\hat{x}^o(k) + F(k)\,\tilde{y}^o(k)}$$

Into

$$\hat{x}^o(k+1) \;=\; A\,\hat{x}(k) + B\,u(k)$$

$$\hat{x}^o(k+1) \;=\; A\,[\hat{x}^o(k) + F(k)\tilde{y}^o(k)] + B\,u(k)$$

Results in

$$\hat{x}^o(k+1) \;=\; A\,\hat{x}^o(k) + B\,u(k) + \overbrace{L(k)}^{A\,F(k)}\,\tilde{y}^o(k)$$

Kalman Filter Solution V-2

$$\hat{x}^o(k+1) \;=\; A\,\hat{x}^o(k) + B\,u(k) + L(k)\,\tilde{y}^o(k)$$

where

$$L(k) \;=\; A\,F(k)$$

$$L(k) \;=\; A\,\underbrace{M(k)C^T\left[C\,M(k)C^T + V(k)\right]^{-1}}_{F(k)}$$

Kalman Filter Solution V-2

Plugging

$$Z(k) \;=\; \underbrace{M(k) - M(k)C^T\left[CM(k)C^T + V(k)\right]^{-1}CM(k)}$$

Into

$$M(k+1) \;=\; A\,Z(k)A^T + B_wW(k)B_w^T$$

- Results in the following Riccati equation:

$$\begin{aligned} M(k+1) \;&=\; AM(k)A^T + B_wW(k)B_w^T \\ &\quad - AM(k)C^T\left[CM(k)C^T + V(k)\right]^{-1}CM(k)A^T \end{aligned}$$

Kalman Filter Solution V-2

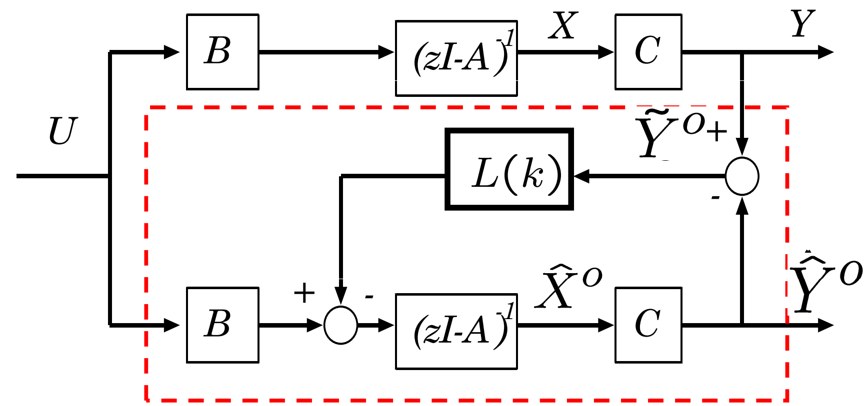
A-priori state observer structure:

$$\begin{aligned}\hat{x}^o(k+1) &= A\,\hat{x}^o(k) + B\,u(k) + L(k)\,\tilde{y}^o(k)\\ \tilde{y}^o(k) &= y(k) - C\,\hat{x}^o(k)\end{aligned}$$

$$\begin{aligned}L(k) &= A\,M(k)C^T\left[C\,M(k)C^T + V(k)\right]^{-1}\\ M(k+1) &= AM(k)A^T + B_wW(k)B_w^T\\ &\quad - AM(k)C^T\left[CM(k)C^T+V(k)\right]^{-1}CM(k)A^T\\ M(0) &= X_o\end{aligned}$$

Kalman Filter Solution V-2

- Same structure as deterministic a-priori observer



Kalman Filter Solution V-1

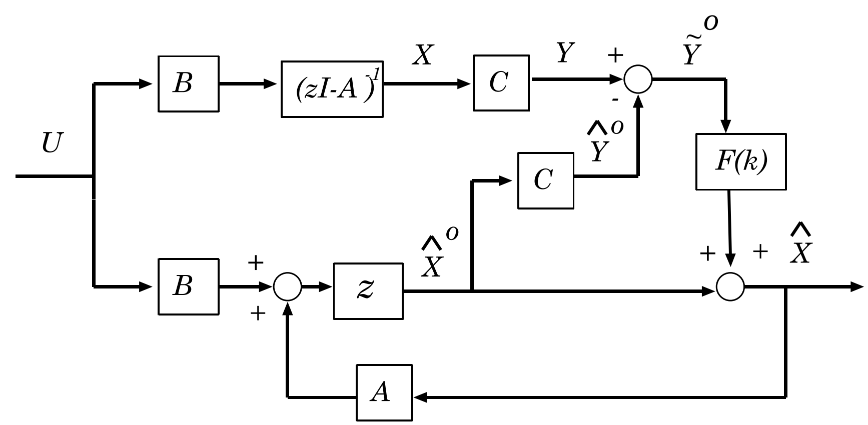
A-posteriori state observer structure:

$$\begin{aligned}\hat{x}(k) &= \hat{x}^o(k) + F(k)\,\tilde{y}^o(k)\\ \hat{x}^o(k+1) &= A\,\hat{x}(k) + B\,u(k)\\ \tilde{y}^o(k) &= y(k) - C\,\hat{x}^o(k)\end{aligned}$$

$$\begin{aligned}F(k) &= M(k)C^T\left[C\,M(k)C^T + V(k)\right]^{-1}\\ M(k+1) &= AM(k)A^T + B_wW(k)B_w^T\\ &\quad - AM(k)C^T\left[CM(k)C^T+V(k)\right]^{-1}CM(k)A^T\end{aligned}$$

Kalman Filter Solution V-1

- A-posteriori estimator as output



Kalman Filter (KF) Properties

- The KF is a linear time varying estimator.
- The KF is the **optimal state estimator** when the input and measurement noises are Gaussian.
- The KF is still the **optimal *linear* state estimator** even when the input and measurement noises are **not** Gaussian.
- The KF covariance Riccati equation is iterated in a forward manner, rather than in a backwards manner as in the LQR.

$$M(0) \rightarrow M(k)$$

Kalman Filter (KF) Properties

The KF a-priori output error (*a-priori output residual*)

$$\tilde{y}^o(k) \;=\; y(k) - C\,\hat{x}^o(k)$$

is often called the ***innovation***

it extracts from ***y(k)*** all the "information" that is new from $Y_{k-1} = \{y(0),\, y(1),\, \cdots,\, y(k-1)\}$

Moreover,

$$E\{\tilde{y}^o(k)\tilde{y}^{oT}(j)\} = 0 \qquad j < k$$

Kalman Filter (KF) Properties

Since

$$\hat{x}^o(k) = E\left\{x(k)|Y_{k-1}\right\}$$

$$Y_{k-1} = \{y(0),\, y(1),\, \cdots,\, y(k-1)\}$$

by the LS orthogonality property (1)

$$\tilde{x}^o(k) = x(k) - \hat{x}^o(k)$$

$$E\{\tilde{x}^o(k)\tilde{y}^{oT}(j)\} = 0 \qquad j < k$$

$$E\{C\,\tilde{x}^o(k)\tilde{y}^{oT}(j)\} = 0 \qquad j < k$$

$$\Rightarrow \quad E\{\tilde{y}^o(k)\tilde{y}^{oT}(j)\} = 0 \qquad j < k$$

Steady State Kalman Filter

- Assume now that we want to estimate the state under zero-mean, stationary input and output Gaussian white noise, I.e.

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) + B_w\,w(k) \\ y(k) &= C\,x(k) + v(k) \end{aligned}$$

$$\left.\begin{aligned} E\{w(k+l)w^T(k)\} &= W\,\delta(l) \\ E\{v(k+l)v^T(k)\} &= V\,\delta(l) \\ E\{w(k+l)v^T(k)\} &= 0 \end{aligned}\right\} \;\textit{\textbf{WSS Gaussian Noise}}$$

Steady State Kalman Filter

Theorem 1:

If the pair ***[A,C]*** is observable (or detectable):

the solution of

$$\begin{aligned} M(k+1) &= AM(k)A^T + B_w W B_w^T \\ &\quad - AM(k)C^T\left[CM(k)C^T + V\right]^{-1} CM(k)A^T \end{aligned}$$

with $M(0) = 0$

Converges to a stationary solution, ***M,*** which satisfies

$$M = AMA^T + B_w W B_w^T - AMC^T\left[CMC^T + V\right]^{-1} CMA^T$$

Steady State Kalman Filter

Weighted noise input vector:

$$B_w' = B_w W^{1/2}$$

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) + B_w w(k) \\ y(k) &= C\,x(k) + v(k) \end{aligned}$$

input noise intensity

$$E\{w(k+l)w^T(k)\} = W\delta(l)$$

Steady State Kalman Filter

Theorem 2:

If the pair ***[A, B'***$_w$***]*** is controllable (*or stabilizable*), where

$$B_w' = B_w W^{1/2}$$

*Then **[A,C]** is observable (or detectable) if and only if:*

1) The solution of

$$\begin{aligned} M(k+1) &= AM(k)A^T + B_w W B_w^T \\ &\quad - AM(k)C^T\left[CM(k)C^T + V\right]^{-1} CM(k)A^T \end{aligned} \qquad M(0) \succeq 0$$

*Converges to a **unique** stationary solution **M**,, which satisfies*

$$M = AMA^T + B_w W B_w^T - AMC^T\left[CMC^T + V\right]^{-1} CMA^T$$

Steady State Kalman Filter

Theorem 2: (continuation)

2) M is positive definite (*semi-definite*)

3) The close loop matrix $A_c = A - LC$ is **Schur**

$$L = A\,MC^T\left[C\,MC^T + V\right]^{-1}$$

Steady State Kalman Filter

Theorem 3:

Under stationary noise and the conditions in theorems 1) and 2),

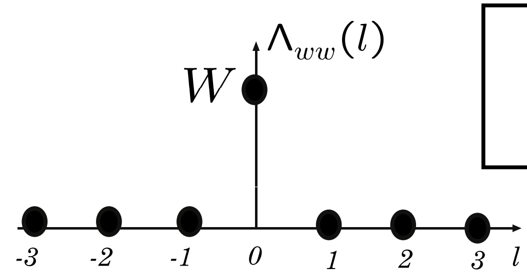
The observer a-priori residual (innovations)

$$\tilde{y}^o(k) \;=\; y(k) - C\,\hat{x}^o(k)$$

is white

$$\boxed{E\left\{\tilde{y}^o(k+l)\tilde{y}^{oT}(k)\right\} = \left[C\,MC^T + V\right]\delta(l)}$$

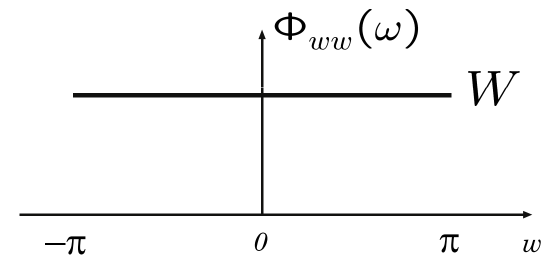
White noise



$$\boxed{\Lambda_{ww}(l) = W\,\delta(l)}$$

$$\Phi_{ww}(\omega) = W$$

constant power spectral density



Steady State Kalman Filter

Subtract observer state Eq. from plant state Eq.:

$$x(k+1) \;=\; A\,x(k) + B\,u(k) + B_w\,w(k)$$

$$\begin{aligned}\hat{x}^o(k+1) \;&=\; (A - LC)\,\hat{x}^o(k) + B\,u(k) \\ &\quad + L[Cx(k) + v(k)]\end{aligned}$$

$$\boxed{\begin{aligned}\tilde{x}^o(k+1) \;&=\; (A - LC)\,\tilde{x}^o(k) + B_w\,w(k) \\ &\quad - L\,v(k)\end{aligned}}$$

Kalman Filter & LQR Duality

Proof: Recall Steady state LQR:

$$x(k+1) \;=\; A\,x(k) + B\,u(k)$$

$$u(k) = -K\,x(k) + r(k)$$

$$J = \frac{1}{2}\sum_{k=0}^{\infty}\left\{x^T(k)\,C_Q^T\,C_Q\,x(k) + u^T(k)\,R\,u(k)\right\}$$

$$Q = C_Q^T\,C_Q \geq 0 \qquad\qquad R = R^T > 0$$

Note:

We need to distinguish between:

- **LQR:** state cost weight $Q = C_Q^T C_Q \geq 0$

$$J = \frac{1}{2} \sum_{k=0}^{\infty} \left\{ x^T(k)\, C_Q^T\, C_Q\, x(k) + u^T(k)\, R\, u(k) \right\}$$

- **KF:** output matrix C

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) + B_w\,w(k) \\ y(k) &= C\,x(k) + v(k) \end{aligned}$$

Kalman Filter & LQR Duality

Steady State LQR Close loop dynamics:

$$x(k+1) = (A - B\,K)\,x(k) + B\,r(k)$$

$$K = \left[R + B^T P B\right]^{-1} B^T P A$$

$$\begin{aligned} A^T P A - P &= -C_Q^T C_Q \\ &\quad + A^T P B \left[B^T P B + R\right]^{-1} B^T P A \end{aligned}$$

Kalman Filter & LQR Duality

Steady State KF Noise propagation dynamics

$$\tilde{x}^o(k+1) = (A - LC)\,\tilde{x}^o(k) + B_w\,w(k) - Lv(k)$$

$$L = A\,M C^T \left[C\,M C^T + V\right]^{-1}$$

$$\begin{aligned} A M A^T - M &= -B_w W B_w^T \\ &\quad + A M C^T \left[C M C^T + V\right]^{-1} C M A^T \end{aligned}$$

Kalman Filter & LQR Duality

Lets compare the AREs:

$$\begin{aligned} A^T P A - P &= -C_Q^T C_Q \\ &\quad + A^T P B \left[B^T P B + R\right]^{-1} B^T P A \end{aligned} \qquad \boxed{LQR}$$

$$\begin{aligned} A M A^T - M &= -B_w W B_w^T \\ &\quad + A M C^T \left[C M C^T + V\right]^{-1} C M A^T \end{aligned} \qquad \boxed{KF}$$

$$\boxed{P \Rightarrow M}$$

Kalman Filter & LQR Duality

Lets compare the AREs:

$$A^T P A - P = -C_Q^T C_Q + A^T P B \left[B^T P B + R\right]^{-1} B^T P A \qquad \boxed{LQR}$$

$$A M A^T - M = -B_w W B_w^T + A M C^T \left[C M C^T + V\right]^{-1} C M A^T \qquad \boxed{KF}$$

$$\boxed{C_Q^T \Rightarrow B_w W^{1/2} = B_w'}$$

Kalman Filter & LQR Duality

Lets compare the AREs:

$$A^T P A - P = -C_Q^T C_Q + A^T P B \left[B^T P B + R\right]^{-1} B^T P A \qquad \boxed{LQR}$$

$$A M A^T - M = -B_w W B_w^T + A M C^T \left[C M C^T + V\right]^{-1} C M A^T \qquad \boxed{KF}$$

$$\boxed{A \Rightarrow A^T}$$

Kalman Filter & LQR Duality

Lets compare the AREs:

$$A^T P A - P = -C_Q^T C_Q + A^T P B \left[B^T P B + R\right]^{-1} B^T P A \qquad \boxed{LQR}$$

$$A M A^T - M = -B_w W B_w^T + A M C^T \left[C M C^T + V\right]^{-1} C M A^T \qquad \boxed{KF}$$

$$\boxed{B \Rightarrow C^T}$$

Kalman Filter & LQR Duality

Lets compare the ARE's:

$$A^T P A - P = -C_Q^T C_Q + A^T P B \left[B^T P B + R\right]^{-1} B^T P A \qquad \boxed{LQR}$$

$$A M A^T - M = -B_w W B_w^T + A M C^T \left[C M C^T + V\right]^{-1} C M A^T \qquad \boxed{KF}$$

$$\boxed{R \Rightarrow V}$$

Kalman Filter & LQR Duality

Lets compare the Feedback gains:

$$K = \left[R + B^T P B\right]^{-1} B^T P A \qquad \boxed{LQR}$$

$$L^T = \left[V + C M C^T\right]^{-1} C M A^T \qquad \boxed{KF}$$

$$P \Rightarrow M \qquad B \Rightarrow C^T \quad A \Rightarrow A^T \qquad R \Rightarrow V$$

Kalman Filter & LQR Duality

Lets compare the Feedback gains:

$$K^T = A P B \left[R + B^T P B\right]^{-1} \qquad \boxed{LQR}$$

$$L = A M C^T \left[V + C M C^T\right]^{-1} \qquad \boxed{KF}$$

$$\boxed{K^T \Rightarrow L}$$

Kalman Filter & LQR Duality

Comparing ARE's and feedback gains, we obtain the following duality

duality $\longrightarrow$

LQR	KF
P	M
A	A^T
B	C^T
R	V
C_Q^T	$B'_w = B_w W^{1/2}$
K	L^T
$(A-BK)$	$(A-LC)^T$

Kalman Filter & LQR Duality

duality $\longrightarrow$

LQR	KF
P	M
A	A^T
B	C^T
R	V
C_Q^T	$B'_w = B_w W^{1/2}$
K	L^T
$(A-BK)$	$(A-LC)^T$

$$A^T P A - P + C_Q^T C_Q - A^T P B \left[B^T P B + R\right]^{-1} B^T P A = 0$$

$$A M A^T - M + B'_w B'^T_w - A M C^T \left[C M C^T + V\right]^{-1} C M A^T = 0$$

Kalman Filter & LQR Duality

duality $\longrightarrow$

LQR	KF
P	M
A	A^T
B	C^T
R	V
C_Q^T	$B'_w = B_w W^{1/2}$
K	L^T
$(A-BK)$	$(A-LC)^T$

$$K = \left[B^T P B + R\right]^{-1} B^T P A$$

$$L^T = \left[C M C^T + V\right]^{-1} C M A^T$$

Steady State LQR

Theorem 1):

If the pair ***[A, B]*** is controllable (or stabilizable), the solution of the DRE

$$\begin{aligned} -P(k) &= A^T P(k+1) A + C_Q^T C_Q \\ &\quad - A^T P(k+1) B \left[B^T P(k+1) B + R\right]^{-1} B^T P(k+1) A \end{aligned}$$

with $P(N) = 0$

converges, as $N \to \infty$, to a constant that satisfies

$$P = A^T P A + C_Q^T C_Q - A^T P B \left[B^T P B + R\right]^{-1} B^T P A$$

Steady State LQR

Theorem 2:

If the pair ***[A, C_q]*** is observable *(or detectable)*

*Then **[A,B]** is controllable (or stabilizable) if and only if:*

1) The solution of

$$\begin{aligned} -P(k) &= A^T P(k+1) A + C_Q^T C_Q \\ &\quad - A^T P(k+1) B \left[B^T P(k+1) B + R\right]^{-1} B^T P(k+1) A \end{aligned}$$

with $P(N) \succeq 0$

*Converges to a **unique** stationary solution **P**, which satisfies*

$$P = A^T P A + C_Q^T C_Q - A^T P B \left[B^T P B + R\right]^{-1} B^T P A$$

Steady State LQ

Theorem 2: (continuation)

2) P is positive definite (*semi-definite*)

3) The close loop matrix $A_c = A - BK$ is **Schur**

$$K = \left[B^T P B + R\right]^{-1} B^T P A$$

Kalman Filter & LQR Duality

LQR	KF
P	M
A	A^T
B	C^T
R	V
C_Q^T	$B'_w = B_w W^{1/2}$
K	L^T
$(A\text{-}BK)$	$(A\text{-}LC)^T$

LQR $[A, B]$ controllable $\xrightarrow{\textit{duality}}$ ***KF*** $[A^T, C^T]$ controllable

$\downarrow$

KF $[A, C]$ observable

Kalman Filter & LQR Duality

LQR	KF
P	M
A	A^T
B	C^T
R	V
C_Q^T	$B'_w = B_w W^{1/2}$
K	L^T
$(A\text{-}BK)$	$(A\text{-}LC)^T$

LQR $[A, C_Q]$ observable $\xrightarrow{\textit{duality}}$ ***KF*** $[A^T, B_w'^T]$ observable

$\downarrow$

KF $[A, B_w']$ controllable

Steady State Kalman Filter

Theorem 1:

If the pair ***[A,C]*** is observable (or detectable):

the solution of

$$\begin{aligned} M(k+1) &= AM(k)A^T + B_w W B_w^T \\ &\quad - AM(k)C^T\left[CM(k)C^T + V\right]^{-1} CM(k)A^T \end{aligned}$$

with $M(0) = 0$

Converges to a stationary solution, ***M,*** which satisfies

$$M = AMA^T + B_w W B_w^T - AMC^T\left[CMC^T + V\right]^{-1} CMA^T$$

Steady State Kalman Filter

Theorem 2:

If the pair ***[A, B'_w]*** is controllable (*or stabilizable*), where

$$B_w' = B_w W^{1/2}$$

Then ***[A,C]*** *is observable (or detectable) if and only if:*

1) The solution of

$$\begin{aligned} M(k+1) &= AM(k)A^T + B_w W B_w^T \\ &\quad - AM(k)C^T\left[CM(k)C^T + V\right]^{-1} CM(k)A^T \end{aligned} \qquad M(0) \succeq 0$$

Converges to a ***<u>unique</u>*** *stationary solution* ***M****, which satisfies*

$$M = AMA^T + B_w W B_w^T - AMC^T\left[CMC^T + V\right]^{-1} CMA^T$$

Steady State Kalman Filter

Theorem 2: (continuation)

2) M is positive definite (*semi-definite*)

3) The close loop matrix $A_c = A - LC$ is **Schur**

$$L \;=\; A\,MC^T\left[C\,MC^T + V\right]^{-1}$$

Steady State Kalman Filter

Theorem 3:

Under stationary noise and the conditions in theorems 1) and 2),

The observer a-priori residual (innovations)

$$\tilde{y}^o(k) \;=\; y(k) - C\,\hat{x}^o(k)$$

is white

$$\boxed{E\left\{\tilde{y}^o(k+l)\tilde{y}^{oT}(k)\right\} = \left[C\,MC^T + V\right]\delta(l)}$$

KF as an innovations filter

We will assume, without loss of generality that the control input is zero, I.e.

$$u(k) = 0 \qquad\qquad k = 0,\,1,\,\cdots$$

•**Plant:**

$$\begin{aligned} x(k+1) &= A\,x(k) + B_w\,w(k) \\ y(k) &= C\,x(k) + v(k) \end{aligned}$$

KF as an innovations (whitening) filter

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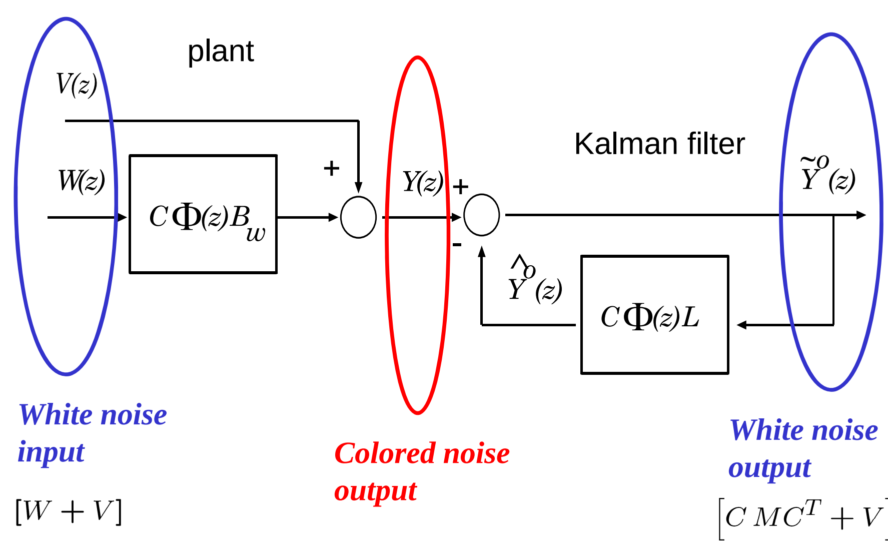
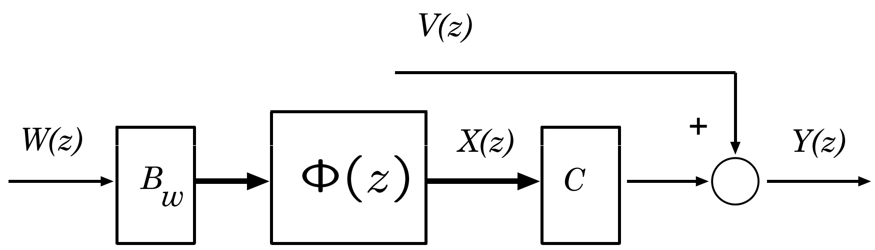


Output *Y(k)* is colored noise

- **Plant:**

$$x(k+1) = A\,x(k) + B_w\,w(k)$$

$$y(k) = C\,x(k) + v(k)$$



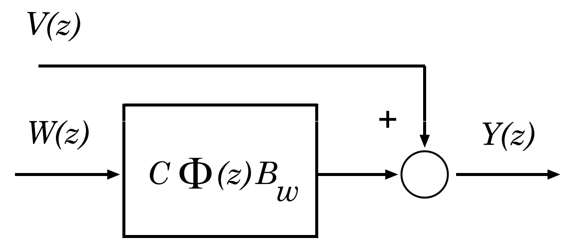
$$\Phi(z) = (zI - A)^{-1}$$

Output *Y(k)* is colored noise

- **Plant:**

$$Y(z) = [C\Phi(z)B_w]\;W(z) + V(z)$$

$$\Phi(z) = (zI - A)^{-1}$$

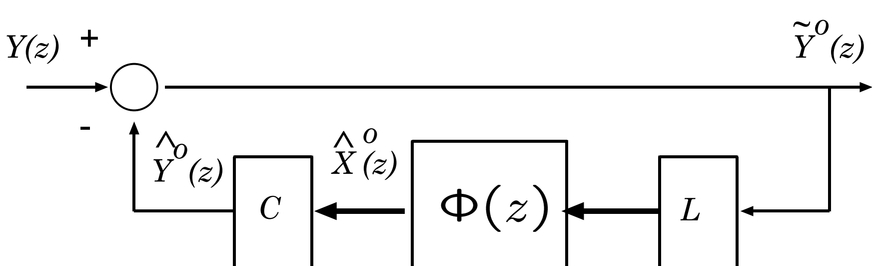


KF as an innovations filter

- **A-priori KF:**

$$\hat{x}^o(k+1) = A\,\hat{x}^o(k) + L\,\tilde{y}^o(k)$$

$$\tilde{y}^o(k) = y(k) - C\,\hat{x}^o(k)$$



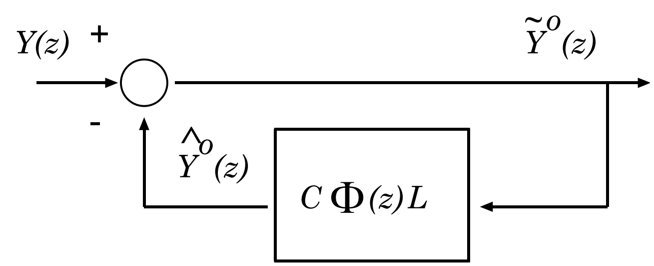
$\Phi(z) = (zI - A)^{-1}$

KF as an innovations filter

- **A-priori KF:**

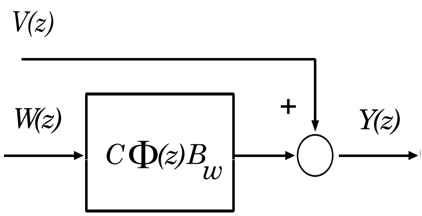
$$\tilde{Y}^o(z) = [I + C\Phi(z)L]^{-1}\,Y(z)$$

$$\Phi(z) = (zI - A)^{-1}$$



KF as an innovations filter

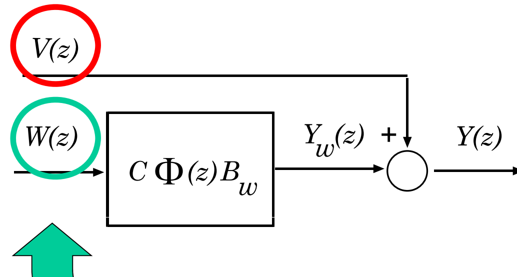
- **Plant** $$Y(z) = [C\Phi(z)B_w]\; W(z) + V(z)$$
- **A-priori KF:** $$\tilde{Y}^o(z) \;=\; [I + C\Phi(z)L]^{-1}\, Y(z)$$



Y(k) Power spectrum

Power spectrum of $y(k)$

$$Y(z) = \underbrace{[C\Phi(z)B_w]\; W(z)}_{Y_w(z)} + V(z)$$

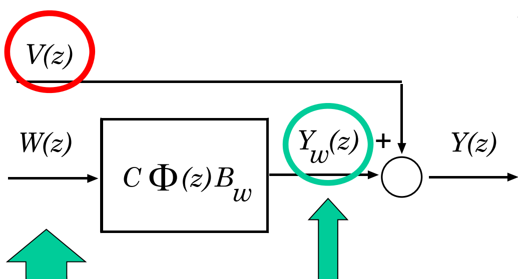


w(k) and ***v(k)*** are uncorrelated!

Y(k) Power spectrum

Power spectrum of $y(k)$

$$Y(z) = \underbrace{[C\Phi(z)B_w]\; W(z)}_{Y_w(z)} + V(z)$$

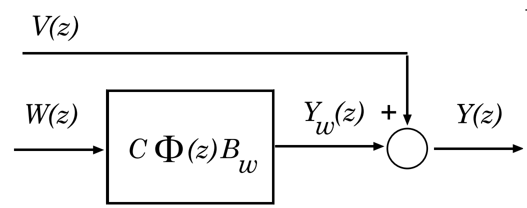


y_w(k) and ***v(k)*** are also uncorrelated!

Y(k) Power spectrum

Power spectrum of $y(k)$

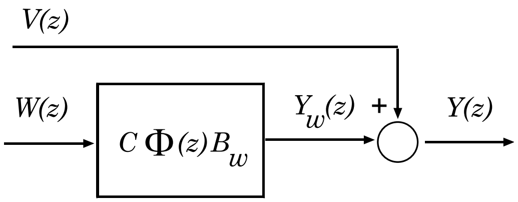
$$Y(z) = \underbrace{[C\Phi(z)B_w]\; W(z)}_{Y_w(z)} + V(z)$$



$$\Lambda_{yy}(z) = \Lambda_{y_w y_w}(z) + \Lambda_{vv}(z)$$

Y(k) Power spectrum

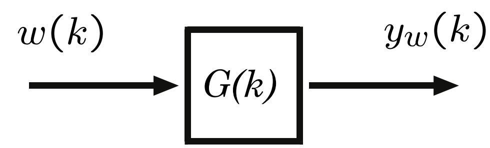
Power spectrum of $y(k)$



$$\Lambda_{yy}(z) = \Lambda_{y_w y_w}(z) + \underbrace{\Lambda_{vv}(z)}_{V}$$

v(k) is white noise

Y(k) Power spectrum

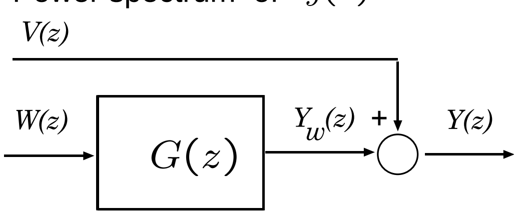


$$\Phi_{y_w y_w}(\omega) = G(\omega)\,\Phi_{ww}(\omega)\,G^T(-\omega)$$

$$\Lambda_{y_w y_w}(z) = G(z)\,\Lambda_{ww}(z)\,G^T(z^{-1})$$

Y(k) Power spectrum

Power spectrum of $y(k)$



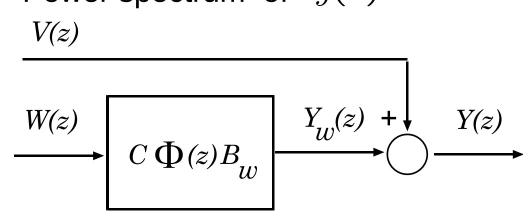
$$\Lambda_{yy}(z) = \underbrace{\Lambda_{y_w y_w}(z)} + V$$

$$\Lambda_{y_w y_w}(z) = [C\Phi(z)B_w]\,\underbrace{W}\left[C\Phi(z^{-1})B_w\right]^T$$

w(k) is white noise

Y(k) Power spectrum

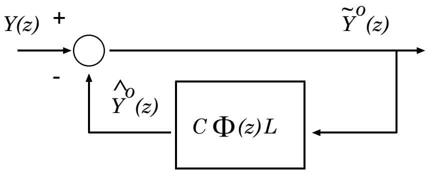
Power spectrum of $y(k)$



$$\boxed{\Lambda_{yy}(z) = V + [C\Phi(z)B_w]\;W\left[C\Phi(z^{-1})B_w\right]^T}$$

KF as an innovations filter

Power spectrum of $\tilde{y}^o(k)$



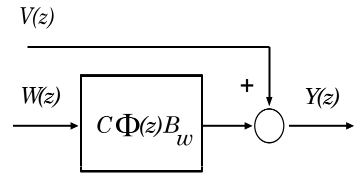
$$\tilde{Y}^o(z) \;=\; [I + C\Phi(z)L]^{-1}\, Y(z)$$

$$\Lambda_{\tilde{y}^o\tilde{y}^o}(z) = [I + C\Phi(z)L]^{-1}\; \Lambda_{yy}(z)\left[I + C\Phi(z^{-1})L\right]^{-T}$$

$$\Lambda_{yy}(z) = [I + C\Phi(z)L]\; \Lambda_{\tilde{y}^o\tilde{y}^o}(z)\left[I + C\Phi(z^{-1})L\right]^{T}$$

KF as an innovations filter

Combining two results:



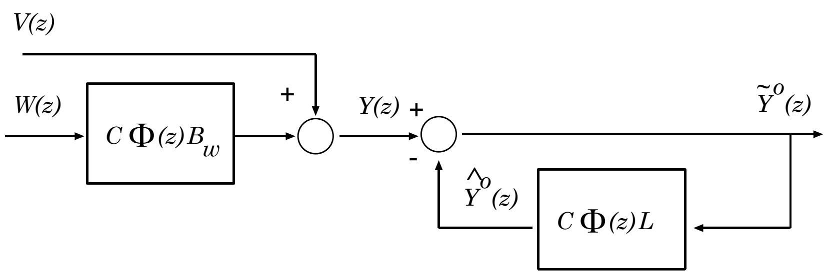
$$\Lambda_{yy}(z) = V + [C\Phi(z)B_w]\; W\left[C\Phi(z^{-1})B_w\right]^{T}$$

and

$$\Lambda_{yy}(z) = [I + C\Phi(z)L]\; \Lambda_{\tilde{y}^o\tilde{y}^o}(z)\left[I + C\Phi(z^{-1})L\right]^{T}$$

KF as an innovations filter

Combining two results:



$$\boxed{[I + C\Phi(z)L]\; \Lambda_{\tilde{y}^o\tilde{y}^o}(z)\left[I + C\Phi(z^{-1})L\right]^{T} = V + [C\Phi(z)B_w]\; W\left[C\Phi(z^{-1})B_w\right]^{T}}$$

KF as an innovations filter

Recall what Theorem part 3) says about the a-priori output error (the innovation sequence)

$$\Lambda_{\tilde{y}^o\tilde{y}^o}(l) \;=\; E\left\{\tilde{y}^o(k+l)\tilde{y}^{oT}(k)\right\}$$

$$=\; \left[C\,MC^T + V\right]\delta(l)$$

$\tilde{y}^o(k)$ is also white noise!!

KF as an innovations filter

Recall what Theorem part 3) says about the a-priori output error (the innovation sequence)

$$\Lambda_{\tilde{y}^o\tilde{y}^o}(l) \;=\; \left[C\,MC^T + V\right]\delta(l)$$

$$\boxed{\begin{aligned}\Lambda_{\tilde{y}^o\tilde{y}^o}(z) \;&=\; \left[C\,MC^T + V\right]\\ \Phi_{\tilde{y}^o\tilde{y}^o}(\omega) \;&=\; \left[C\,MC^T + V\right]\end{aligned}}$$

KF as a innovations (whitening) filter

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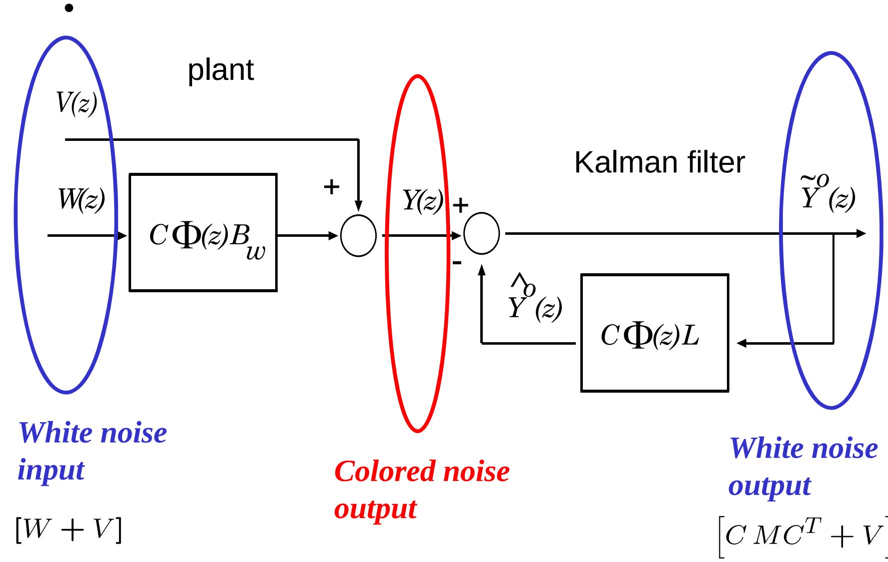

KF as a innovations (whitening) filter

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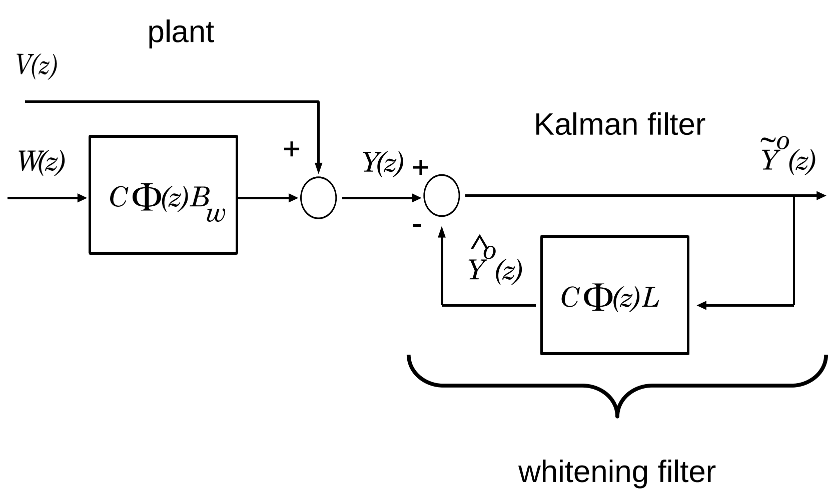
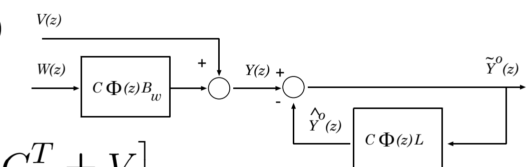


KF as an innovations filter

From Theorem, part 3)



$$\Lambda_{\tilde{y}^o\tilde{y}^o}(z) \;=\; \left[C\,MC^T + V\right]$$

therefore,

$$[I + C\Phi(z)L]\,\underbrace{\left[C\,MC^T + V\right]}_{=\Lambda_{\tilde{y}^o\tilde{y}^o}(z)}\left[I + C\Phi(z^{-1})L\right]^T =$$

$$\underbrace{V + [C\Phi(z)B_w]\;W\left[C\Phi(z^{-1})B_w\right]^T}_{=\Lambda_{yy}(z)}$$

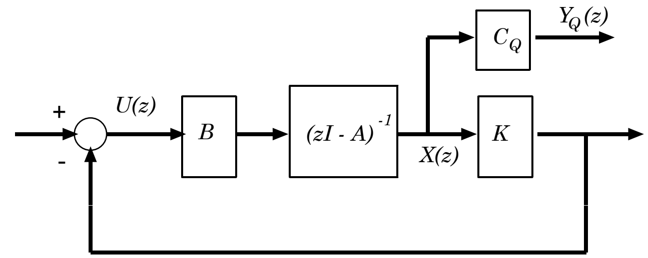
KF as an innovations filter

Thus, to prove Theorem, part 3), we need to prove that:

$$\boxed{\begin{aligned}&[I + C\Phi(z)L]\left[C\,MC^T + V\right]\left[I + C\Phi(z^{-1})L\right]^T = \\ &\qquad V + [C\Phi(z)B_w]\;W\left[C\Phi(z^{-1})B_w\right]^T\end{aligned}}$$

To do so, we will start from the LQR return difference equality and again use the LQR-KF duality.

Return difference equality for LQR



$$[I + G_o(z^{-1})]^T\,[R + B^TPB]\,[I + G_o(z)] = R + G_Q^T(z^{-1})\,G_Q(z)$$

Open loop transfer function:

$$G_o(z) = K\Phi(z)B$$

TF from $U(z)$ to $Y_Q(z)$:

$$G_Q(z) = C_Q\Phi(z)B$$

Return difference equality for LQR

Substituting, $G_o(z) = K\Phi(z)B$ $\quad G_Q(z) = C_Q\Phi(z)B$

into

$$[I + G_o(z^{-1})]^T\,[R + B^TPB]\,[I + G_o(z)] = R + G_Q^T(z^{-1})\,G_Q(z)$$

We obtain,

$$\left[I + K\Phi(z^{-1})B\right]^T\left[B^TPB + R\right][I + K\Phi(z)B] =$$

$$R + \left[C_Q\Phi(z^{-1})B\right]^T\left[C_Q\Phi(z)B\right]$$

Kalman Filter & LQR Duality

$$[I + K\Phi(z)B]^T\left[B^TPB + R\right]\left[I + K\Phi(z^{-1})B\right] =$$

$$R + \left[C_Q\Phi(z)B\right]^T\left[C_Q\Phi(z^{-1})B\right]$$

LQR	KF
P	M
A	A^T
B	C^T

LQR	KF
R	V
C_Q^T	$B'_w = B_w W^{1/2}$
K	L^T

$$\left[I + L^T\Phi^T(z)C^T\right]^T\left[CMC^T + V\right]\left[I + L^T\Phi^T(z^{-1})C^T\right] =$$

$$V + \left[B_w'^T\Phi^T(z)C^T\right]^T\left[B_w'^T\Phi^T(z^{-1})C^T\right]$$

Kalman Filter & LQR Duality

From,

$$\left[I + L^T\Phi^T(z)C^T\right]^T \left[CMC^T+V\right]\left[I+L^T\Phi^T(z^{-1})C^T\right] =$$

$$V + \left[B_w'^T\Phi^T(z)C^T\right]^T\left[B_w'^T\Phi^T(z^{-1})C^T\right]$$

- performing transpose operations and noticing that:

$$B_w' B_w'^T = B_w W B_w^T$$

we obtain the desired result

$$\boxed{\begin{aligned}&\left[I + C\Phi(z)L\right]\left[C\,MC^T+V\right]\left[I+C\Phi(z^{-1})L\right]^T = \\ &\qquad V + \left[C\Phi(z)B_w\right]\,W\left[C\Phi(z^{-1})B_w\right]^T\end{aligned}}$$

KF as a innovations (whitening) filter

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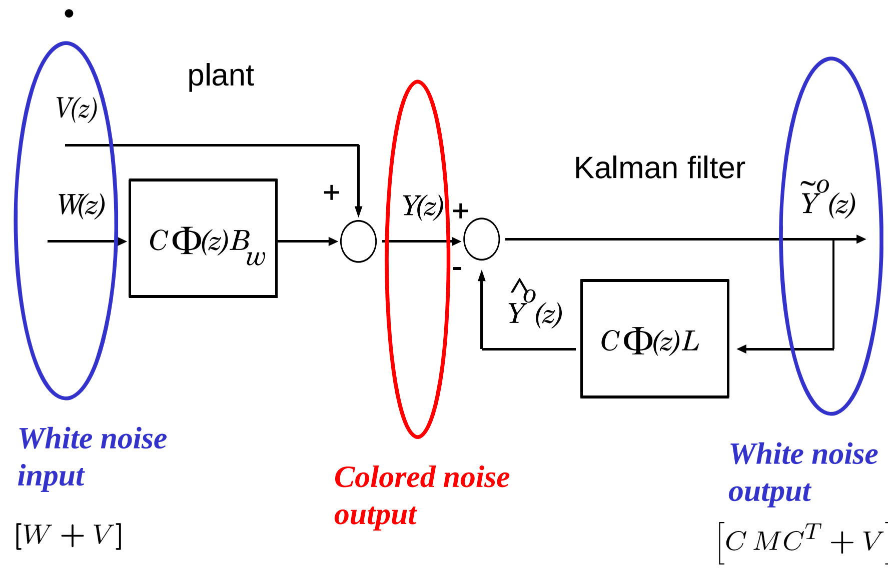
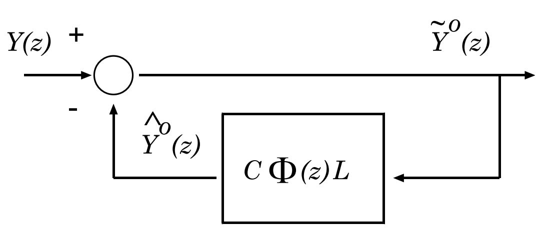


Kalman filter close loop eigenvalues

- A-priori KF

$$\hat{x}^o(k+1) \;=\; A\,\hat{x}^o(k) + L\,\tilde{y}^o(k)$$

$$\tilde{y}^o(k) \;=\; y(k) - C\,\hat{x}^o(k)$$



$$\tilde{Y}^o(z) \;=\; \left[I + C\Phi(z)L\right]^{-1} Y(z)$$

Kalman filter close loop eigenvalues

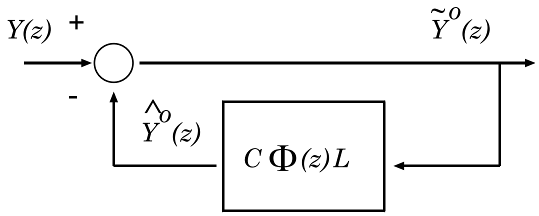
$$\hat{x}^o(k+1) \;=\; \underbrace{(A - LC)}_{A_c}\,\hat{x}^o(k) + L\,y(k)$$

•KF close loop eigenvalues

$$C(z) \;=\; \det\{(zI - A_c)\} = 0$$

$$\;=\; \det\{(zI - A + LC)\} = 0$$

Kalman filter return difference



$$\tilde{Y}^o(z) \;=\; [I + C\Phi(z)L]^{-1}\,Y(z)$$

Return difference: $[I + C\Phi(z)L]$

Kalman filter return difference

$$\det\{[I + C\Phi(z)L]\} = \frac{C(z)}{A(z)}$$

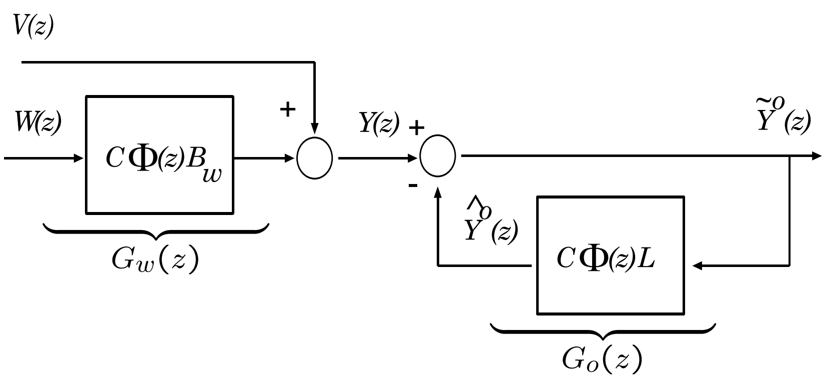
- KF close loop eigenvalues

$$C(z) \;=\; \det\{(zI - A + LC)\} = 0$$

- KF open loop eigenvalues

$$A(z) \;=\; \det\{(zI - A)\} = 0$$

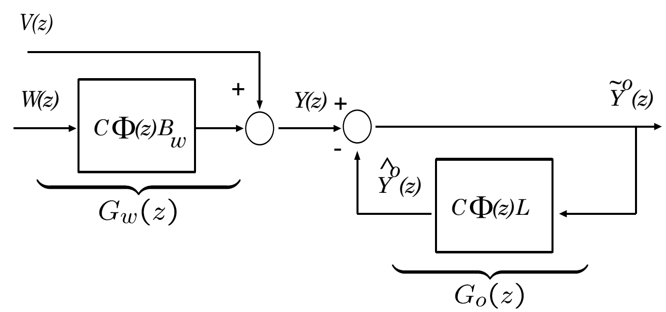
KF return difference equality



$$[I + G_o(z)]\;\underbrace{\left[C\,MC^T + V\right]}_{\Lambda_{\tilde{y}^o\tilde{y}^o}(0)}\left[I + G_o(z^{-1})\right]^T = V + G_w(z)\,W G_w^T(z^{-1})$$

KF return difference equality (SISO)

Assume that both, $w(k) \in \mathcal{R}$ and $y(k),\, v(k) \in \mathcal{R}$



$$\boxed{[1 + G_o(z)][1 + G_o(z^{-1})] = \gamma\,[1 + \frac{W}{V}G_w(z)G_w(z^{-1})]}$$

$$\gamma = \frac{V}{V + C\,MC^T}$$

Summary

- Kalman filter algorithm:
  - using a-priori state estimate only
  - using a-priori and a-posteriori state estimates

- Stationary Kalman filters (KF):
  - KF algebraic Riccati equation
  - KF as an innovations filter
  - KF return difference equality