ME 233 Spring 2010
Solution to Homework #9

1. (a) $\Delta(s)$ can be obtained by the MATLAB command

```
>> Delta = tf([wr*zr wr^2], [1 2*zr*wr wr^2]) ...
   * tf([(wt/wn)^2 2*zt*wt^2/wn wt^2], [1 2*zt*wt wt^2]) - 1;
```

In zero-pole-gain form, it takes the value

$$\Delta(s) = \frac{-s(s+14.6)(s^2+32.88s+1.205\times10^6)}{(s^2+30s+10^6)(s^2+36s+1.44\times10^6)}.$$

(b) Since $G_p(s)$ is given by

```
>> Gp = tf(wb^2,[1 2*zb*wb wb^2]);
```

$G_e(s)$ and $G_{EA}(s)$ are respectively given by

```
>> Ge = ss(Gp*tf(1,[1 0]));
>> Gea = ss(Gp*(1+Delta)*tf(1,[1 0]));
```

(c) Figure 1 shows the bode plot of $G_w(s) = C_e\Phi_e(s)B_w$. Note that since $B_w = B_e$, it has the same

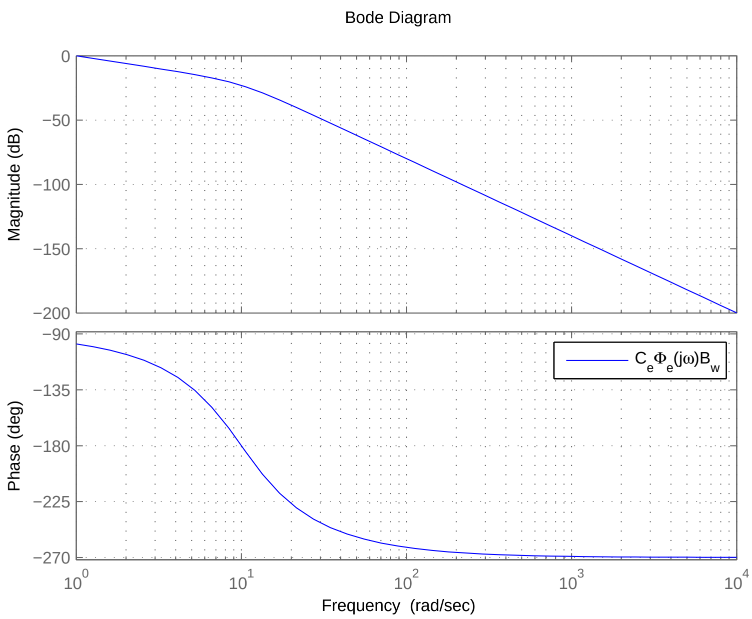

Figure 1: Bode plot of $G_w(s) = C_e\Phi_e(s)B_w$

realization as $G_p(s)$, which means that its Bode plot can be generated by the command

```
>> bode(Gp)
```

For any value of μ, the Kalman Filter gain can be computed by

```
>> M = care(Ae', Ce', Bw*Bw', mu^2);
>> Le = mu^-2 * M*Ce';
```

The Bode magnitude plot of $G_{okf}(s)$ can then be generated by

```
>> Gokf = ss(Ae,Le,Ce,0);
>> bodemag(Gokf)
```

Also note that the Bode magnitude plots of the robustness constraint, $1/|\Delta(j\omega)|$, and the slightly more conservative robustness constraint, $1/|\Delta(j\omega)| - 5db$, can be respectively generated by

```
>> bodemag(inv(Delta))
>> bodemag(inv(Delta)*10^(-5/20))
```

Figure 2 shows this plot for several values of μ along with $1/|\Delta(j\omega)|$ and $1/|\Delta(j\omega)| - 5db$. From

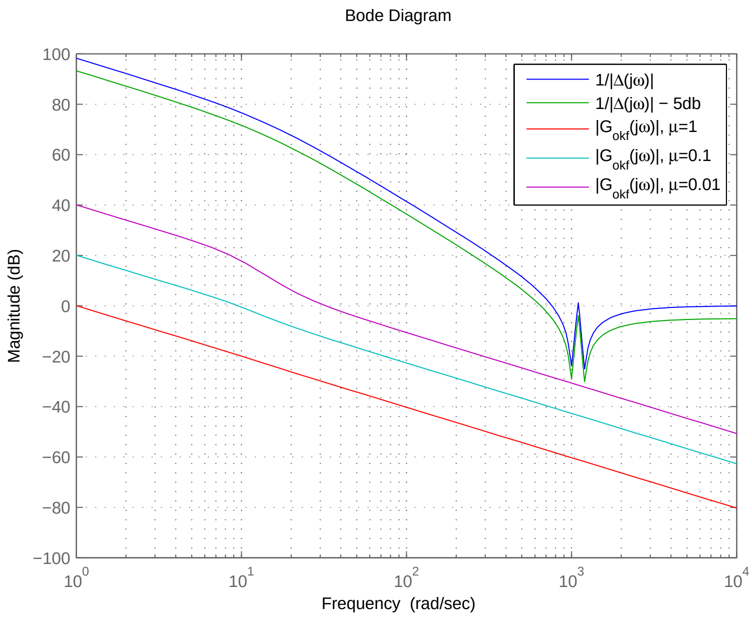

Figure 2: Bode magnitude plot of open loop Kalman Filter $G_{okf}(s)$ for several values of μ, robust stability constraint $1/|\Delta(j\omega)|$, and slightly conservative robustness constraint $1/|\Delta(j\omega)| - 5db$

this Bode plot, we choose $\mu_1 = 0.01$. Figure 3 shows a Bode magnitude plot of the complementary sensitivity function $T_{kf}(s)$, $G_{okf}(s)$, and $1/|\Delta(j\omega)| - 5db$ for μ_1. These were generated by the commands

```
>> Tkf = feedback(Gokf,1);
>> bodemag(Tkf, Gokf, inv(Delta)*10^(-5/20))
```

As expected, $|T_{kf}(j\omega)|$ is close to $|G_{okf}(j\omega)|$ at high frequencies and $|T_{kf}(j\omega)|$ satisfies the (conservative) robustness constraint. Figure 4 shows the step response of the closed loop Kalman Filter for μ_1, which was generated by

```
>> step(feedback(Gokf,1))
```

(d) For a given value of ρ, the LQR feedback gain K_e, the LQG controller $C_{LQG}(s)$, and the open loop LQG system $G_o(s)$ can be obtained using the commands

```
>> P = care(Ae, Be, Ce'*Ce, rho);
>> Ke = 1/rho * Be'*P;
>> CLQG = ss(Ae - Be*Ke - Le*Ce, Le, Ke, 0);
>> Go = Ge*CLQG;
```

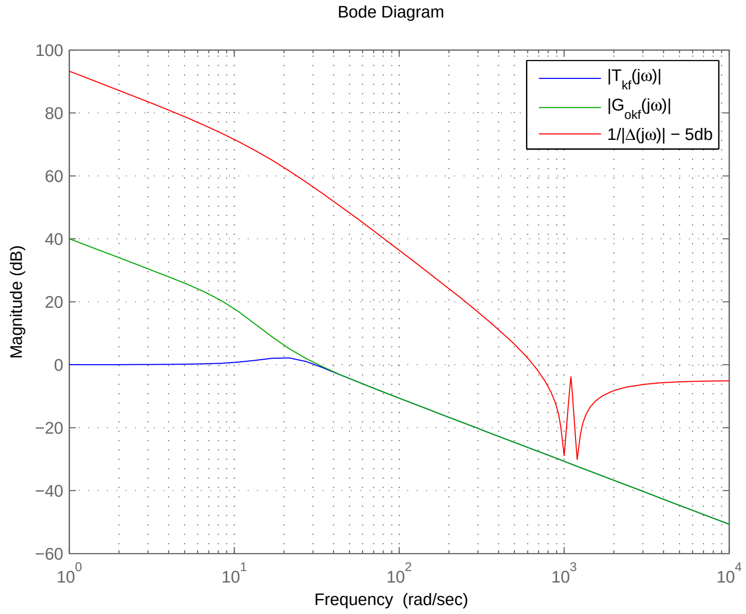

Figure 3: Bode magnitude plot of complementary sensitivity function $T_{kf}(s)$, open loop Kalman Filter $G_{okf}(s)$, and consertive robust stability constraint $1/|\Delta(j\omega)| - 5db$

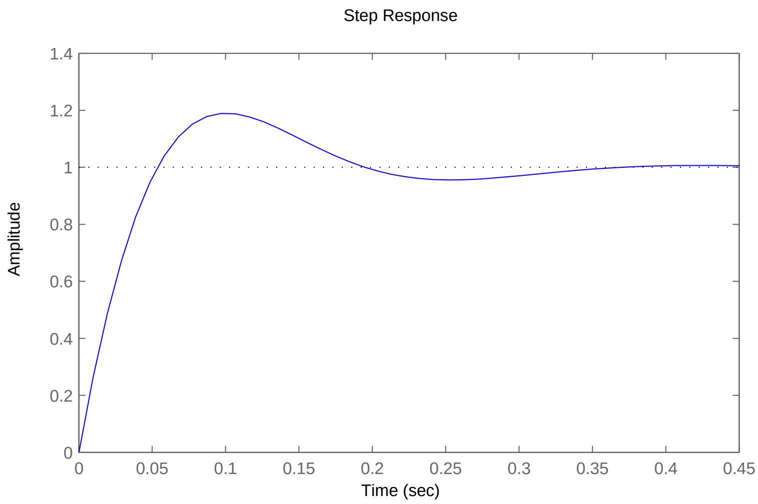

Figure 4: Step response for closed loop Kalman Filter, which is equivalent to $T_{kf}(s)$

Figure 5 shows Bode plots of $G_{okf}(s)$ and open loop LQG system $G_o(s)$ for several values of ρ. Note that the recovery process begins at low frequencies and recovers magnitude more quickly than phase. Choosing the value $\rho = 10^{-15}$, the nominal closed loop step response generated by

```
>> step(feedback(Ge*CLQG, 1))
```

is shown in Figure 8.

(e) Figure 6 shows the Bode magnitude plot of the nominal complementary sensitivity function $T_p(s)$ and $1/|\Delta(j\omega)|$ generated by

```
>> Tp = feedback(Go, 1);
>> bodemag(Tp, inv(Delta))
```

Note that the robustness constraint is satisfied.

(f) Figure 7 shows the Bode magnitude plots for nominal and actual open loop LQG systems ($G_e C_{LQG}(s)$ and $G_{EA} C_{LQG}(s)$ respectively) generated by

```
>> bodemag(Ge*CLQG, Gea*CLQG)
```

The phase margin for both systems is $60.5°$. The gain margin for the nominal system is $31.7db$ and the gain margin for the actual system is $16db$. Figure 8 shows the step responses for the nominal and actual closed loop LQG systems generated by

```
>> step(feedback(Ge*CLQG,1), feedback(Gea*CLQG,1))
```

(g) i. For $\mu = 0.001$, Figure 9 shows $|G_{okf}(j\omega)|$, $1/|\Delta(j\omega)|$, and $1/|\Delta(j\omega)| - 5db$. From looking at this plot, it is clear that for the value $\mu = 0.001$, the robustness constraint is slightly violated. Figure 10 shows the recovery process, i.e. the Bode plot of $G_{okf}(s)$ and $G_o(s)$ for several values of ρ. As before, the recovery process starts at low frequencies and recovers gain more quickly than phase. Choosing the value $\rho = 10^{-15}$ gives the complementary sensitivity function shown in Figure 11. Clearly, the LQG system also violates the robustness constraint. Despite this, the resulting actual closed loop system is stable for this case. This reflects the conservatism of the small gain theorem, which was used to generate the robust stability constraint. Figure 12 shows the Bode magnitude plots and stability margins for the nominal and actual open loop LQG systems. The nominal system has one gain crossover frequency and one phase crossover frequency. These respectively correspond to a phase margin of $54.9°$ and a gain margin of $23.3db$. The actual system has five gain crossover frequencies and one phase crossover frequency. The gain margin is $7.22db$ and the minimum phase margins in each direction are $54.9°$ and $-34.2°$. Figure 13 shows the closed loop step response for the nominal and actual LQG system. In this case, the closed loop actual step response noticeably excites the high frequency dynamics that were neglected in the controller design.

ii. Now we consider $\mu_2 = 10^{-4}$, which results in the Bode plot of $G_{okf}(s)$ shown in Figure 14. Clearly, this violates the robustness constraint. Figure 15 shows the LTR recovery process, which again starts at low frequency and recovers gain before phase. As we would expect, $T_p(s)$ violates the robustness constraint, as shown in Figure 16. However, unlike the previous design, which was still stable, this closed loop system is unstable. Thus, if we consider both of these designs, we can see that although the small gain theorem is a little bit conservative in this case, it is a reasonably accurate condition for stability.

2. First, we define

$$v := u + Lx_e.$$

With this, we can write the system dynamics as

$$\begin{aligned} \dot{x}_e &= A_e x_e + B_e\left(-Lx_e + v\right) \\ &= \bar{A}_e x_e + B_e v \end{aligned}$$

where

$$\bar{A}_e := A_e - B_e L.$$

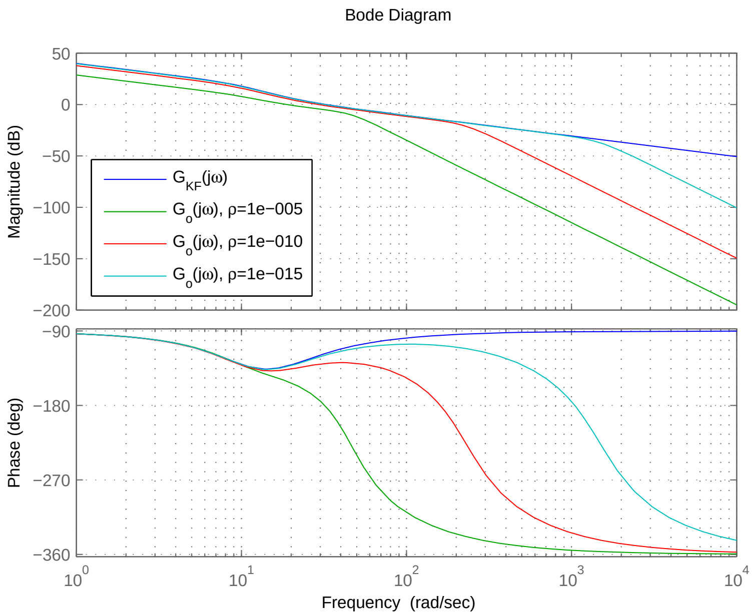

Figure 5: Bode plot of open loop Kalman filter $G_{okf}(s)$ and open loop LQG system $G_o(s)$ for several values of ρ

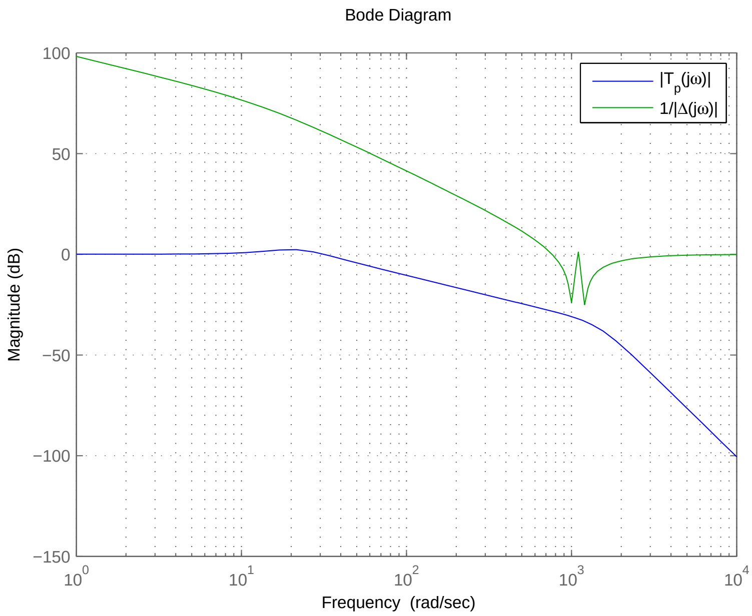

Figure 6: Bode magnitude plot of nominal complementary sensitivity function $T_p(s)$ and robustness constraint, $1/|\Delta(j\omega)|$

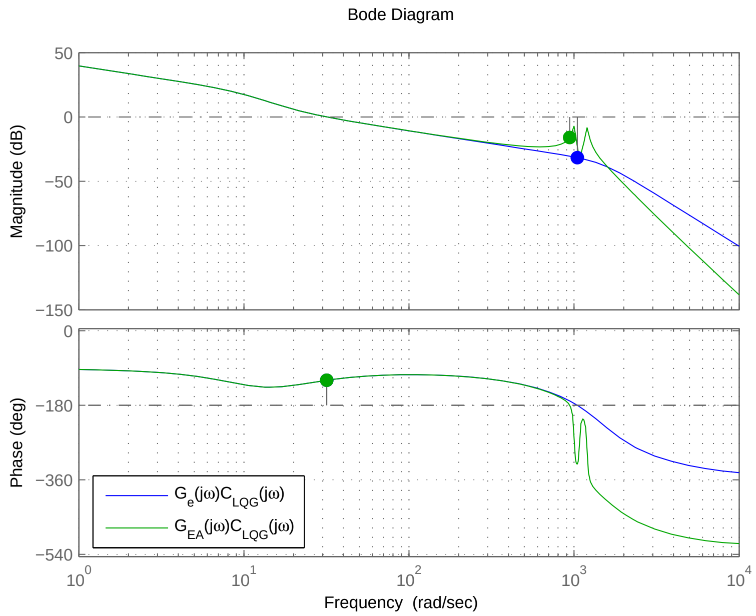

Figure 7: Bode magnitude plots for nominal and actual open loop LQG systems, $G_eC_{LQG}(s)$ and $G_{EA}C_{LQG}(s)$ respectively

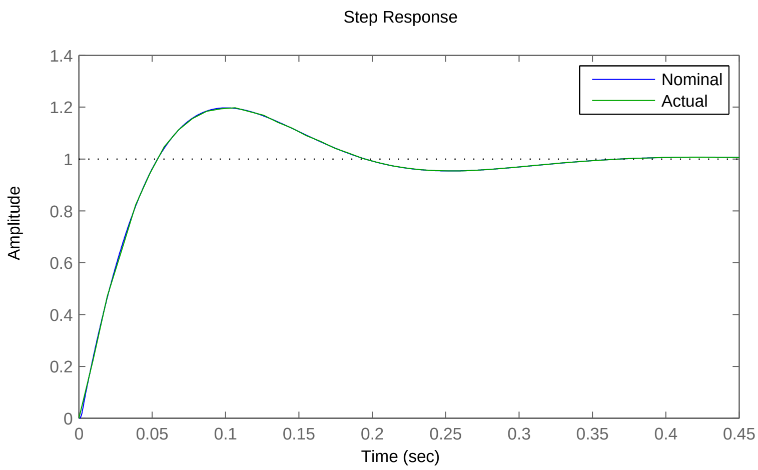

Figure 8: Step response for nominal and actual closed loop LQG systems

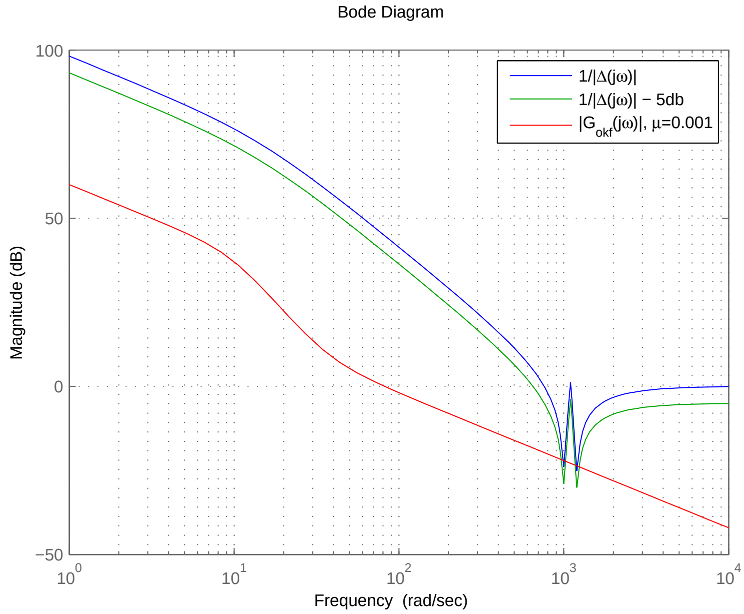

Figure 9: Bode magnitude plot of open loop Kalman Filter $G_{okf}(s)$ for several values of μ, robust stability constraint $1/|\Delta(j\omega)|$, and slightly conservative robustness constraint $1/|\Delta(j\omega)| - 5db$

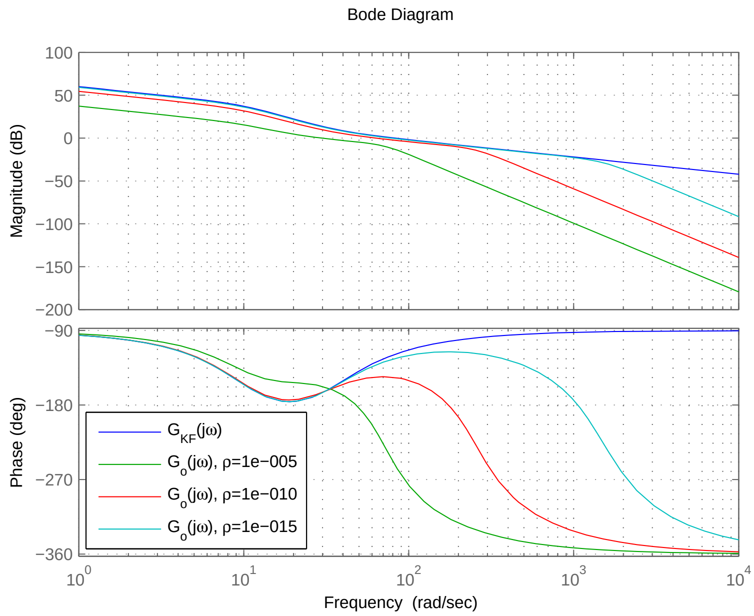

Figure 10: Bode plot of open loop Kalman filter $G_{okf}(s)$ and open loop LQG system $G_o(s)$ for several values of ρ

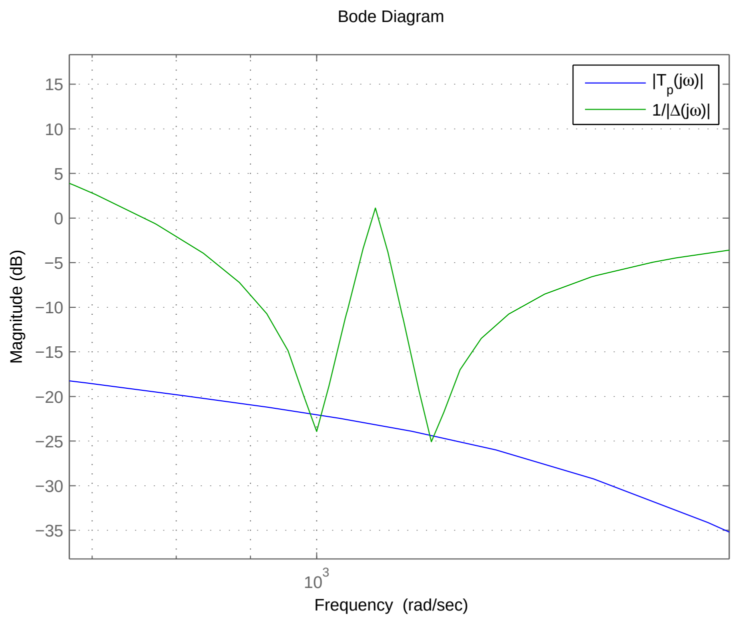

Figure 11: Bode magnitude plot of nominal complementary sensitivity function $T_p(s)$ and robustness constraint, $1/|\Delta(j\omega)|$

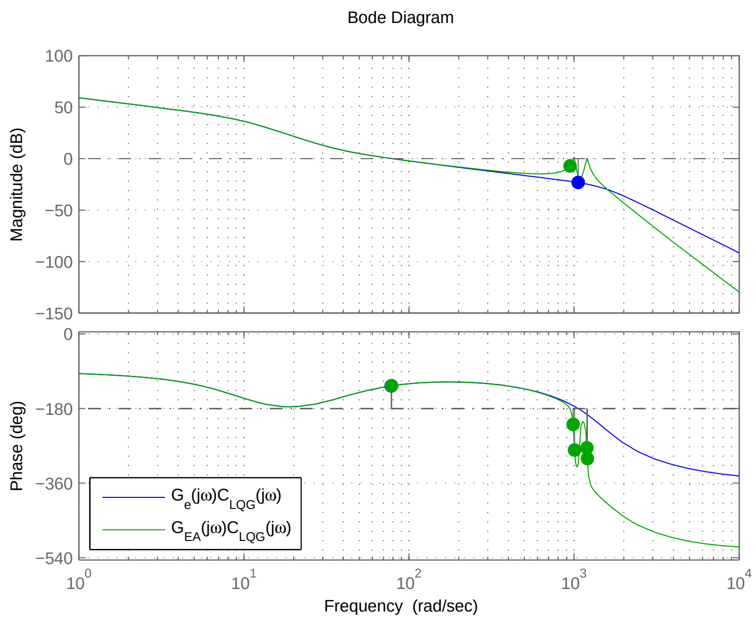

Figure 12: Bode magnitude plots for nominal and actual open loop LQG systems, $G_e C_{LQG}(s)$ and $G_{EA} C_{LQG}(s)$ respectively

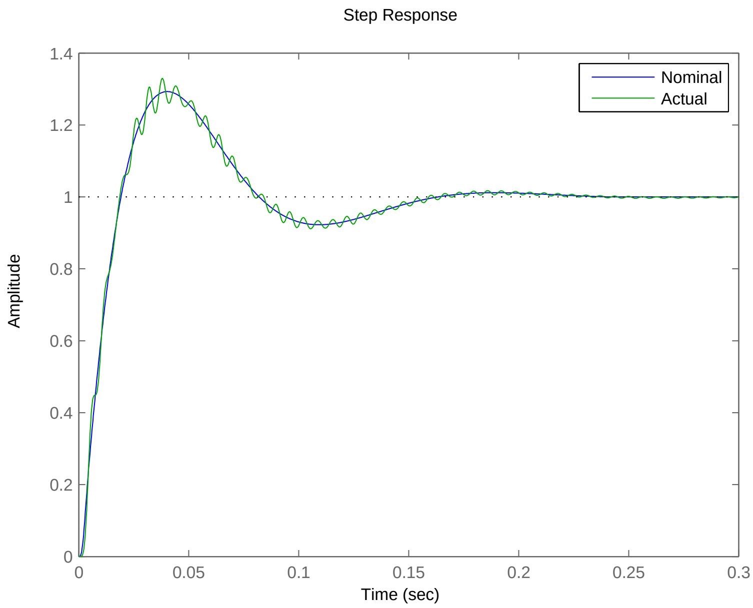

Figure 13: Step response for nominal and actual closed loop LQG systems

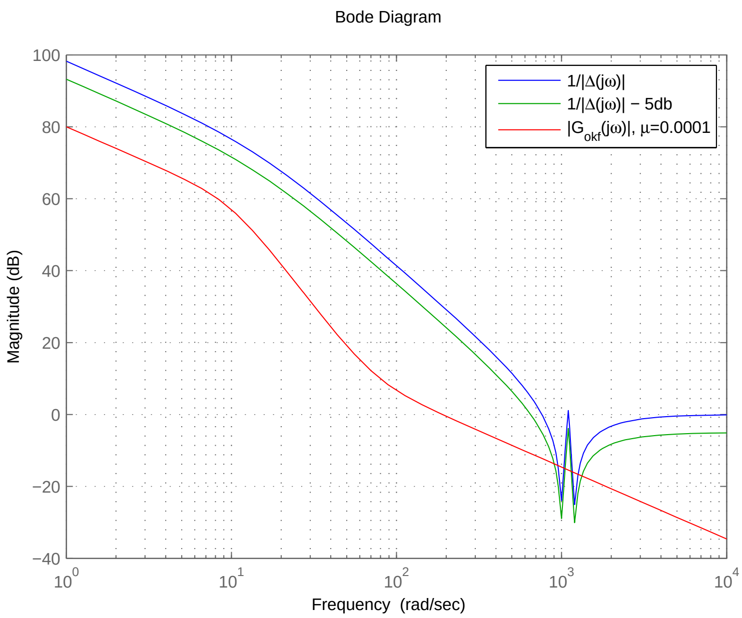

Figure 14: Bode magnitude plot of open loop Kalman Filter $G_{okf}(s)$ for $\mu = 10^{-4}$, robust stability constraint $1/|\Delta(j\omega)|$, and slightly conservative robustness constraint $1/|\Delta(j\omega)| - 5db$

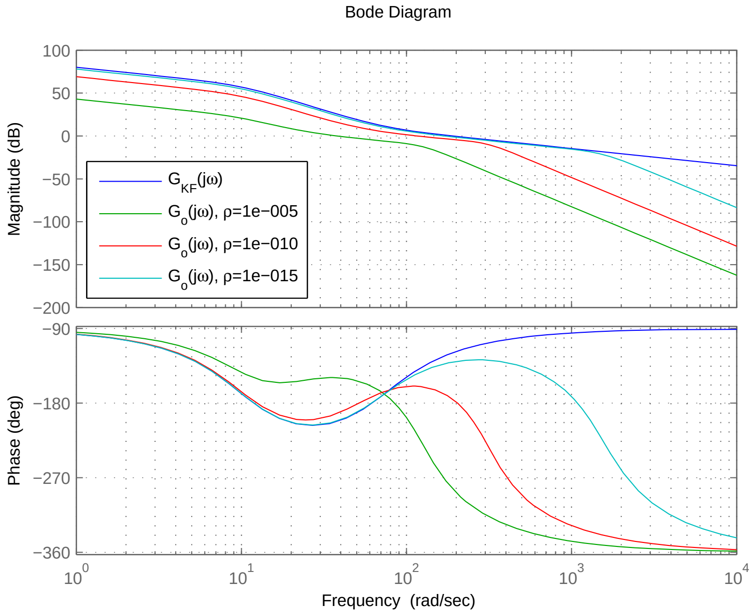

Figure 15: Bode plot of open loop Kalman filter $G_{okf}(s)$ and open loop LQG system $G_o(s)$ for several values of ρ

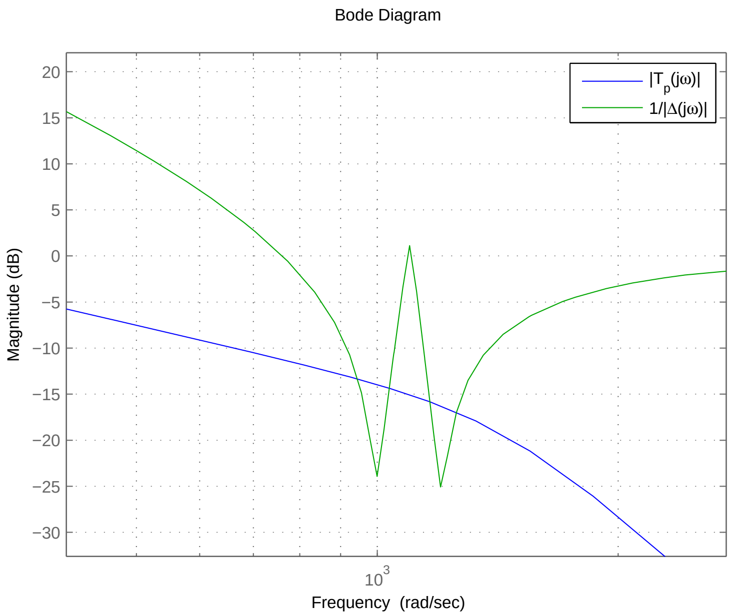

Figure 16: Bode magnitude plot of nominal complementary sensitivity function $T_p(s)$ and robustness constraint, $1/|\Delta(j\omega)|$

Also, if we define

$$R_e := \rho D_2^T D_2$$

we can write the cost function as

$$\begin{aligned} J &= \int_0^\infty \left\{ x_e^T C_e^T C_e x_e + 2x_e^T N_e \left(-Lx_e + v\right) + \left(-Lx_e + v\right)^T R_e \left(-Lx_e + v\right) \right\} dt \\ &= \int_0^\infty \left\{ x_e^T \left(C_e^T C_e - 2N_e L + L^T R_e L\right) x_e + 2x_e^T \left(N_e - L^T R_e\right) v + v^T R_e v \right\} dt. \end{aligned}$$

Thus, to cancel the cross term between x_e and v, we should choose

$$L = R_e^{-1} N_e^T.$$

With this choice of L, we see that

$$\begin{aligned} \bar{A}_e &= A_e - B_e R_e^{-1} N_e^T \\ J &= \int_0^\infty \left\{ x_e^T \left(C_e^T C_e - N_e R_e^{-1} N_e^T\right) x_e + v^T R_e v \right\} dt. \end{aligned}$$

Now we concentrate on simplifying the term in parentheses. Noting that

$$C_e^T C_e = \begin{bmatrix} C_q^T & \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sqrt{\rho} C_2^T \end{bmatrix} \end{bmatrix} \begin{bmatrix} C_q \\ \begin{bmatrix} 0 & 0 & 0 & \sqrt{\rho} C_2 \end{bmatrix} \end{bmatrix} = C_q^T C_q + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho C_2^T C_2 \end{bmatrix}$$

and

$$N_e R_e^{-1} N_e^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sqrt{\rho} C_2^T D_2 \end{bmatrix} D_2^{-1} D_2^{-T} \begin{bmatrix} 0 & 0 & 0 & \sqrt{\rho} D_2^T C_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho C_2^T C_2 \end{bmatrix}$$

we see that

$$C_e^T C_e - N_e R_e^{-1} N_e^T = C_q^T C_q.$$

Thus, we have reduced our system and cost function to

$$\begin{aligned} \dot{x}_e &= \left(A_e - B_e R_e^{-1} N_e^T\right) x_e + B_e v \\ J &= \int_0^\infty \left\{ x_e^T C_q^T C_q x_e + v^T R_e v \right\} dt. \end{aligned}$$

Note that this is a standard LQR problem. Thus under the assumptions that $[A_e - B_e R_e^{-1} N_e^T, B_e]$ is stabilizable and $[A_e - B_e R_e^{-1} N_e^T, C_q]$ is detectable, the optimal control law can be found using a Riccati equation solution. (We will return to the problem of finding the optimal control after restating the existence conditions.) Noting that

$$\left(A_e - B_e R_e^{-1} N_e^T\right) + B_e K \text{ is Schur} \quad \Leftrightarrow \quad A_e + B_e \left(K - B_e R_e^{-1}\right) \text{ is Schur}$$

we see that the stabilizability of $[A_e - B_e R_e^{-1} N_e^T, B_e]$ is equivalent to the stabilizability of $[Ae, Be]$. Thus, the conditions for the existence of a unique optimal control are:

- $[A_e, B_e]$ is stabilizable
- $[A_e - B_e R_e^{-1} N_e^T, C_q]$ is detectable

Now, we turn our attention to finding the optimal controller. The optimal control for our reformulated system is given by

$$\begin{aligned} v &= -\bar{K}_e x_e \\ \bar{K}_e &= R_e^{-1} B_e^T P_e \\ 0 &= \bar{A}_e^T P_e + P_e \bar{A}_e + C_q^T C_q - P_e B_e R_e^{-1} B_e^T P_e. \end{aligned}$$

Thus, the optimal control (in terms of u) is given by

$$\begin{aligned} u &= -\bar{K}_e x_e - L x_e \\ &= -\left(\bar{K}_e + R_e^{-1} N_e^T\right) x_e \\ &= -R_e^{-1}\left(B_e^T P_e + N_e^T\right) x_e \end{aligned}$$

where P_e is the solution of the Riccati equation

$$\begin{aligned} 0 &= \left(A_e - B_e R_e^{-1} N_e^T\right)^T P_e + P_e\left(A_e - B_e R_e^{-1} N_e^T\right) + \left(C_e^T C_e - N_e R_e^{-1} N_e^T\right) - P_e B_e R_e^{-1} B_e^T P_e \\ &= A_e^T P_e + P_e A_e + C_e^T C_e - \left[P_e B_e R_e^{-1} B_e^T P_e + N_e R_e^{-1} B_e^T P_e + P_e B_e R_e^{-1} N_e^T + N_e R_e^{-1} N_e^T\right] \\ &= A_e^T P_e + P_e A_e + C_e^T C_e - \left(P_e B_e + N_e\right) R_e^{-1}\left(B_e^T P_e + N_e^T\right). \end{aligned}$$

Thus, in summary, when

- $[A_e, B_e]$ is stabilizable
- $[A_e - B_e R_e^{-1} N_e^T, C_q]$ is detectable

the optimal control is given by

$$\begin{aligned} u &= -K_e x_e \\ K_e &= R_e^{-1}\left(P_e B_e + N_e\right)^T \\ 0 &= A_e^T P_e + P_e A_e + C_e^T C_e - \left(P_e B_e + N_e\right) R_e^{-1}\left(P_e B_e + N_e\right)^T. \end{aligned}$$