1 [12 points]

$$\hat{y}(k+3|k) = \frac{L(z^{-1})}{C(z^{-1})}y(k) + \frac{B(z^{-1})F(z^{-1})}{C(z^{-1})}u(k)$$

(+2 points for the equation, +2 points for specifying $A(z^{-1})$ and $C(z^{-1})$)

where $A(z^{-1}) = (1-z^{-1})^2$, $C(z^{-1}) = 1-0.5z^{-1}$ and $L(z^{-1}), F(z^{-1})$ are the solutions for the Diophantine equation:

$$A(z^{-1})F(z^{-1}) + z^{-3}L(z^{-1}) = C(z^{-1})$$

(+3 points for the Diophantine equation)

So $L(z^{-1}) = 2.5 - 2z^{-1}$, $F(z^{-1}) = 1 + 1.5z^{-1} + 2z^{-2}$.

(+2 points for $L(z^{-1})$, +2 points for $F(z^{-1})$)

Thus

$$\hat{y}(k+3|k) = \frac{2.5-2z^{-1}}{1-0.5z^{-1}}y(k) + \frac{(0.5+z^{-1})(1+1.5z^{-1}+2z^{-2})}{1-0.5z^{-1}}u(k)$$

(+1 point for the final answer)

2 [16 points]

(a) [8 points]

This is a second order system. Define the state $x(t) = \left[\theta(t), \dot{\theta}(t)\right]^T$.

(+1 point for the state definition)

The state equations can be written as:

$$\begin{aligned}
\dot{x}(t) &= \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B_w} w(t) \\
y(t) &= \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x(t) + v(t)
\end{aligned}$$

(+1 point)

So the steady state Kalman filter is:

$$\frac{d\hat{x}(t|t)}{dt} = A\hat{x}(t|t) + Bu(t) + MC^TV^{-1}\left[y(t) - C\hat{x}(t|t)\right], \ \hat{x}(0|0) = x_0$$

(+2 points)

M is the solution of the Riccati equation:

$$AM + MA^T + B_wWB_w^T - MC^TV^{-1}CM = 0$$

(+1 point)

To get steady state Kalman Filter poles:

Method 1:

When $W = 0.1$, $V = 0.4$,

$$M = \begin{bmatrix} 0.4 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}$$

(+1 point)

The closed loop matrix is

$$A - MC^TV^{-1}C = \begin{bmatrix} -1 & 1 \\ -0.5 & 0 \end{bmatrix}$$

And the steady state Kalman Filter poles are $\frac{-1 \pm i}{2}$.

(+2 points)

Method 2:

The closed loop Kalman Filter poles come from the RDE:

$$1 + \frac{W}{V} G(s) G(-s) = 0$$

(+1 point)

where $G(s) = C(sI - A)^{-1} B_w = \frac{1}{s^2}$. The solutions are $\frac{-1 \pm i}{2}, \frac{1 \pm i}{2}$. Since the two Kalman Filter poles should be stable, they are $\frac{-1 \pm i}{2}$.

(+2 points)

(b) [8 points]

Since the ratio of $\frac{W}{V}$ remains the same, the optimal Kalman Filter gain is the same as designed in (a).

(+1 point for pointing out that the Kalman Filter gain is the same; +3 points for the reasoning)

Thus the steady-state Kalman Filter poles remain the same.

(+2 points)

The estimation error covariance changed due to the change of the covariance of the noises. When $W = 1$, $V = 4$,

$$M = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$

(+2 points)

3 [22 points]

(a) [10 points]

The state matrices of the system are

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -c & -b & 1 \\ \hline 1 & 0 & 0 \end{array}\right]$$

(+1 point)

To reject the disturbance, we design an output filter $Q_f(s) = \frac{1}{s\left(s^2 + w_0^2\right)}$.

(+2 points)

So the frequency domain cost function is

$$\begin{aligned} J &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ Q_f(jw) Q_f(-jw) Y(jw) Y(-jw) + RU(-jw) U(jw) \right\} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \frac{1}{w^2 \left(w_0^2 - w^2\right)^2} Y(jw) Y(-jw) + RU(-jw) U(jw) \right\} \end{aligned}$$

(+2 points)

The state space representation of the filter is

$$\begin{aligned} \dot{z}_1(t) &= A_1 z_1(t) + B_1 x(t) \\ x_f(t) &= C_1 z_1(t) + D_1 x(t) \end{aligned}$$

where

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -w_0^2 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

(+2 points; If $y(t)$ is used as the input in the filter, then $B_1 = [0, 0, 1]^T$, $D_1 = 0$)

The extended state is $x_e(t) = \begin{bmatrix} x(t) \\ z_1(t) \end{bmatrix} \in \mathbb{R}^5$ and the time domain cost function is

$$J = \int_0^\infty \left\{ x_e^T(t) Q_e x_e(t) + u_e^T(t) R u_e(t) \right\} dt$$

where

$$Q_e = \begin{bmatrix} D_1^T D_1 & D_1^T C_1 \\ C_1^T D_1 & C_1^T C_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The dynamics of x_e follows from $\dot{x}_e(t) = A_e x_e(t) + B_e u(t)$, where

$$A_e = \begin{bmatrix} A & 0 \\ B_1 & A_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -c & -b & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -w_0^2 & 0 \end{bmatrix}$$

$$B_e = \begin{bmatrix} B \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(+1 point for A_e, B_e, Q_e)

The optimal law is

$$u(t) = -R^{-1} B_e^T P_e x_e(t)$$

(+1 point)

The Riccati equation is

$$P_e A_e + A_e^T P_e - P_e B_e R^{-1} B_e^T P_e + Q_e = 0$$

(+1 point)

(b) [7 points]

Step 1: Find the internal model of the disturbance $A_d(s) = s\left(s^2 + w_0^2\right)$ and let $C(s) = \frac{S(s)}{A_d(s)R(s)}$.

(+2 points for A_d; +1 point for the controller structure)

Step 2: Solve the following Diophantine equation for $R(s)$ and $S(s)$

$$A_d(s) R(s) A_p(s) + S(s) B_p(s) = D(s)$$

where $B_p(s) = 1$, $A_p(s) = s^2 + bs + c$.

(+2 points for the Diophantine equation)

To have a solution for the pole-placement design, $A_d A_p$ and B_p need to be coprime. Since $B_p = 1$, this condition is always satisfied. Thus the pole-placement is guaranteed to have a solution.

(+2 points for a correct answer; +1 point if mentioning coprime)

(c)

From part (a), we have

$$u(t) = -R^{-1} B_e^T P_e x_e(t) = -K_1 x(t) - K_2 z_1(t)$$

Taking the Laplace transform, we have

$$\begin{aligned} U(s) &= -K_1 X(s) - K_2 Z_1(s) \\ &= -K_1 X(s) - K_2 (sI - A_1)^{-1} B_1 X(s) \\ &= \frac{-K_1 \det(sI - A_1) - K_2 adj(sI - A_1) B_1}{\det(sI - A_1)} X(s) \end{aligned}$$

(+3 points)

Since $\det(sI - A_1) = s\left(s^2 + w_0^2\right)$, we can see that the internal model is included in the feedback controller.

(+2 points)

4 [30 points]

(a) [14 points]

$$y(k+1) - a_1 y(k) - a_2 y(k-1) = 2u(k) + b_1 u(k-1) + 2\begin{bmatrix} c_1 & c_2 \end{bmatrix} x_d(k) + b_1 \begin{bmatrix} c_1 & c_2 \end{bmatrix} x_d(k-1)$$

Notice

$$2\begin{bmatrix} c_1 & c_2 \end{bmatrix} x_d(k) + b_1 \begin{bmatrix} c_1 & c_2 \end{bmatrix} x_d(k-1) = \begin{bmatrix} 2c_2 + b_1 c_1 & 2c_1 + b_1 c_2 \end{bmatrix} x_d(k-1)$$

Let $\begin{bmatrix} d_1 & d_2 \end{bmatrix} = \begin{bmatrix} 2c_2 + b_1 c_1 & 2c_1 + b_1 c_2 \end{bmatrix}$.

(+5 points for handelling the noise correctly)

Since $\frac{1}{A_p(z^{-1})} = \frac{1}{1-a_1 z^{-1} - a_2 z^{-2}}$ is SPR, we can run parallel PAA with fixed updation gain. The parallel algorithm will guarantee an unbiased estimation in the presence of the disturbances. Using a fixed updation gain will guarantee that the system is hyperstable ($\frac{1}{A_p(z^{-1})} - \frac{\lambda}{2}$ is SPR as $\lambda = 0$ for fixed updation gain).

(+5 points for choosing the correct PAA and explanations)

Let $y_e(k+1) = y(k+1) - 2u(k)$. Define $\hat{\theta}^T(k) = \begin{bmatrix} \hat{a}_1(k) & \hat{a}_2(k) & \hat{b}_1(k) & \hat{d}_1(k) & \hat{d}_2(k) \end{bmatrix}$, $\phi^T(k) = [\ \hat{y}(k) \quad \hat{y}(k-1) \quad u($
Then

$$\begin{aligned} \hat{\theta}(k+1) &= \hat{\theta}(k) + \frac{F\phi(k)}{1+\phi^T(k) F \phi(k)} \epsilon^o(k+1) \\ \epsilon^o(k+1) &= y_e(k+1) - \hat{y}_e^o(k+1) = y_e(k+1) - \hat{\theta}^T(k)\phi(k) \end{aligned}$$

where $F \in \mathbb{R}^{5\times 5}$ is the constant updation gain.

(+4 points for the PAA equations)

Note: (1) To handel the noise, we can also let $y_f(k) = y(k)\left[1 - z^{-2}\right]$, $u_f(k) = u(k)\left[1 - z^{-2}\right]$. Then $y_f(k+1) = \frac{2+b_1 z^{-1}}{1-a_1 z^{-1} - a_2 z^{-2}} u_f(k)$. (2) Parallel PAA with adjustable compensator also serves as an unbiased hyperstable algorithm in this problem.

(b) [16 points]

$d = 1$, $B_p(z^{-1}) = 2 + b_1 z^{-1}$. Since the zero is stable, we can use a direct adaptive control algorithm. Let $D'(z^{-1}) = 1 + d_1 z^{-1} + d_2 z^{-2}$.

$$z^{-1} R(z^{-1}) + S'(z^{-1}) A_p(z^{-1}) = D'(z^{-1})$$

(+2 points for the Diophantine equation)

Since the order of A_p is 2, $R(z^{-1}) = r_0 + r_1 z^{-1}$, $S'(z^{-1}) = 1$. Then $S(z^{-1}) = S'(z^{-1}) B_p(z^{-1}) = 2 + b_1 z^{-1}$.

(+2 points for the order of $R(z^{-1})$; +2 points for the structure of $S(z^{-1})$)

By disturbance cancellation, the deterministic control is

$$u(k) = \frac{1}{S(z^{-1})}\left[-R(z^{-1}) y(k) + D'(z^{-1}) y_d(k+1)\right] - \begin{bmatrix} c_1 & c_2 \end{bmatrix} x_d(k)$$

(+2 points for the deterministic control law)

$$\begin{aligned} D'(z^{-1}) y_d(k+1) &= S(z^{-1}) u(k) + R(z^{-1}) y(k) + S(z^{-1}) \begin{bmatrix} c_1 & c_2 \end{bmatrix} x_d(k) \\ &= 2u(k) + b_1 u(k-1) + r_0 y(k) + r_1 y(k-1) + \begin{bmatrix} d_1 & d_2 \end{bmatrix} x_d(k-1) \\ &= 2u(k) + \theta_c^T \phi(k) \end{aligned}$$

where $\theta_c^T = \begin{bmatrix} b_1 & r_0 & r_1 & d_1 & d_2 \end{bmatrix}$, $\phi^T(k) = \begin{bmatrix} u(k-1) & y(k) & y(k-1) & x_d^T(k-1) \end{bmatrix}$.

(+2 points for the deterministic version of $D' y_d$ and the definition of θ_c, ϕ)

In the adaptive case,

$$D'(z^{-1}) y_d(k+1) - 2u(k) = \hat{\theta}_c^T(k+1)\phi(k)$$

(+2 points for the adaptive version of $D' y_d$ and the definition of $\hat{\theta}_c$)

where $\hat{\theta}_c^T(k+1) = \begin{bmatrix} \hat{b}_1(k+1) & \hat{r}_0(k+1) & \hat{r}_1(k+1) & \hat{d}_1(k+1) & \hat{d}_2(k+1) \end{bmatrix}$ is updated by RLS PAA as follows

$$\begin{aligned} \hat{\theta}_c(k+1) &= \hat{\theta}_c(k) + \frac{F(k)\phi(k)}{1+\phi^T(k) F(k) \phi(k)}\left[D'(z^{-1}) y(k+1) - 2u(k) - \hat{\theta}_c^T(k)\phi(k)\right] \\ F^{-1}(k) &= \lambda_1 F^{-1}(k-1) + \lambda_2 \phi(k-1)\phi^T(k-1) \end{aligned}$$

(+2 points for the parameter updation; +1 point for the gain updation)

The adaptive control law is

$$u(k) = \frac{1}{2}\left[D'\left(z^{-1}\right) y_d(k+1) - \hat{b}_1(k) u(k-1) - \hat{r}_0(k) y(k) - \hat{r}_1(k) y(k-1) - \begin{bmatrix} \hat{d}_1(k) & \hat{d}_2(k) \end{bmatrix} x_d(k-1)\right]$$

(+1 point)

Note: if we do not use disturbance cancellation, we can include the internal model in $S\left(z^{-1}\right)$, i.e. $S\left(z^{-1}\right) = S'\left(z^{-1}\right) B_p\left(z^{-1}\right)\left(1-z^{-2}\right)$. In this way, $R\left(z^{-1}\right) = r_0 + r_1 z^{-1} + r_2 z^{-2} + r_3 z^{-3}$, $S\left(z^{-1}\right) = 2 + b_1 z^{-1} - 2z^{-2} - b_1 z^{-3}$. Then we can use standard adaptive control method.

5 [20 points] (key steps only, the solution is not complete)

(a)

$A\left(z^{-1}\right) = 1 - 2\cos w_0 z^{-1} + z^{-2}$. Need to obtain $y(k+2) = L\left(z^{-1}\right) y(k)$ for some $L\left(z^{-1}\right)$. It is equivalent to solve the following Diophantine equation:

$$1 = z^{-2} L\left(z^{-1}\right) + F\left(z^{-1}\right) A\left(z^{-1}\right)$$

Then $L\left(z^{-1}\right) = 4\left(\cos w_0\right)^2 - 1 - 2\cos w_0 z^{-1}$. So $y(k+2) = \left[4\left(\cos w_0\right)^2 - 1\right] y(k) - 2\cos w_0 y(k-1)$.

(b)

Method 1:

Let $\theta^T = \begin{bmatrix} 4\left(\cos w_0\right)^2 - 1 & -2\cos w_0 \end{bmatrix} = \begin{bmatrix} a & b \end{bmatrix}$, $\phi^T(k) = \begin{bmatrix} y(k) & y(k-1) \end{bmatrix}$, $\hat{\theta}^T(k) = \begin{bmatrix} \hat{a}(k) & \hat{b}(k) \end{bmatrix}$. Then

$$y(k+1) = \theta^T \phi(k-1)$$

The direct RLS PAA is:

$$\begin{aligned}
\hat{\theta}(k+1) &= \hat{\theta}(k) + F(k+1)\phi(k-1)\epsilon^o(k+1) \\
\epsilon^o(k+1) &= y(k+1) - \hat{\theta}^T(k)\phi(k-1) \\
F(k+1) &= \frac{1}{\lambda}\left[F(k) - \frac{F(k)\phi(k-1)\phi^T(k-1)F(k)}{\lambda + \phi^T(k-1)F(k)\phi(k-1)}\right], \ \lambda = 0.999
\end{aligned}$$

Then

$$\hat{y}(k+2|k) = \hat{\theta}^T(k)\phi(k)$$

Method 2:

Notice

$$y(k) + y(k-2) = 2\cos w_0 y(k-1)$$

Let $y_e(k) = y(k) + y(k-2)$, $\theta = 2\cos w_0$, $\hat{\theta}(k) = 2\widehat{\cos w_0}(k)$, $\phi(k-1) = y(k-1)$. The indirect RLS PAA is:

$$\begin{aligned}
\hat{\theta}(k+1) &= \hat{\theta}(k) + F(k+1)\phi(k)\epsilon^o(k+1) \\
\epsilon^o(k+1) &= y_e(k+1) - \hat{\theta}^T(k)\phi(k) \\
F(k+1) &= \frac{1}{\lambda}\left[F(k) - \frac{F(k)\phi(k)\phi^T(k)F(k)}{\lambda + \phi^T(k)F(k)\phi(k)}\right], \ \lambda = 0.999
\end{aligned}$$

Then

$$\hat{y}(k+2|k) = \left[\hat{\theta}(k)^2 - 1\right] y(k) - \hat{\theta}(k) y(k-1)$$

(c)

Need to obtain $y(k+50) = L\left(z^{-1}\right) y(k)$ for some $L\left(z^{-1}\right)$. It is equivalent to solve the following Diophantine equation:

$$1 = z^{-50} L\left(z^{-1}\right) + F\left(z^{-1}\right) A\left(z^{-1}\right)$$