UNIVERSITY OF CALIFORNIA AT BERKELEY
Department of Mechanical Engineering
ME233 Advanced Control Systems II
Spring 2010

Homework #1

Assigned: Th., Jan. 28
Due: Th., Feb. 4

1. **Finite Horizon Optimal Tracking Problem:**

   Consider the discrete time system

$$
\begin{aligned}
x(k+1) &= A\,x(k) + B\,u(k)\ , \\
y(k) &= C\,x(k)
\end{aligned}
$$

   where $x \in \mathcal{R}^n$ and $y\ u \in \mathcal{R}^m$. Assume the existence of a *known* reference output sequence

$$
y_{d[0,N]} = \{y_d(0),\, y_d(1),\, \cdots,\, y_d(N)\}
$$

   The optimal control is sought to minimize the finite horizon quadratic performance index:

$$
\begin{aligned}
J = \quad & \frac{1}{2}\,[y_d(N) - y(N)]^T\, S\, [y_d(N) - y(N)] \\
& + \frac{1}{2}\sum_{k=0}^{N-1} \left\{ [y_d(k) - y(k)]^T\, T\, [y_d(k) - y(k)] + u^T(k)\, R u(k) \right\}
\end{aligned}
$$

   where S, T and R are symmetric and positive definite matrices of the appropriate dimensions. Find the optimal control law by applying dynamic programming and utilizing the Belmann equation

$$
J^o[x(k), k] = \min_{u(k)} \{L[x(k), u(k), k] + J^o[x(k+1), k+1]\}
$$

   where

$$
\begin{aligned}
L[x(k), u(k), k] &= \frac{1}{2}\,[y_d(k) - y(k)]^T\, T\, [y_d(k) - y(k)] + u^T(k)\, R u(k) \\
J^o[x(N), N] &= \frac{1}{2}\,[y_d(N) - y(N)]^T\, S\, [y_d(N) - y(N)].
\end{aligned}
$$

   HINT: Show that the optimal cost from state $x(k)$ to the final state can be expressed as

$$
J^o[x(k), k] = \frac{1}{2} x^T(k) P(k) x(k) + x^T(k) b(k) + c(k)\ .
$$

   Obtain recursive expressions for $P(k)$, $b(k)$ and $c(k)$ (from N to 0).

2. We wish to determine how to split a positive number L into N pieces, so that the product of the N pieces is maximized. The problem can be solved using dynamic programming by formulating it as follows. Consider a first-order "pure integrator"

$$
x(k+1) = x(k) + u(k) \qquad\qquad x(0) = 0,
$$

   we wish to determine the optimal control sequence

$$
U_o^o = \{u^o(0),\, u^o(1),\, \cdots,\, u^o(N-1)\}
$$

   such that:

(i) $u^o(k) \geq 0$.

(ii) $x(N) = L$.

(iii) The cost function

$$J = \prod_{k=0}^{N-1} u(k) = u(0)\, u(1) \,\cdots\, u(N-1)$$

is maximized.

To use dynamic programming, it is convenient to define the following optimal value function

$$J^o[x(m)] = \max_{U_m} \prod_{k=m}^{N-1} u(k)$$

where $U_m = \{u(m),\, u(m+1),\, \cdots,\, u(N-1)\}$ is the set of all feasible control sequences from the instance m.

Hint: Notice that, because of the terminal condition $x(N) = L$, and the state equation, the optimal value function at $x(N-1)$ is given by

$$J^o[x(N-1)] = u^o(N-1) = L - x(N-1)$$

Use the Belman equation starting from this boundary condition.

3. Consider the discrete time system

$$x(k+1) \;=\; A\, x(k) + B\, u(k)\;, \qquad\qquad x(0) \neq 0$$

with $x(k) \in \mathcal{R}^n$ and $u(k) \in \mathcal{R}^m$.

We wish to find the optimal state feedback control law that minimizes the cost functional

$$J[x_m, m, S, N] = \frac{1}{2}\, x^T(N) S x(N) + \frac{1}{2} \sum_{k=m}^{N-1} \left\{ x^T(k) Q x(k) + + u^T(k) R u(k) \right\}\;,$$

with $x(m) = x_m$, $m \in [0, N-1]$, $Q = Q^T \succeq 0$, $R = R^T \succ 0$ and $S \in \mathcal{R}^{n\times n}$. Define,

$$J^o[x_m, m, S, N] = \min_{U_m} J[x_m, m, S, N]$$

where $U_m = \{u(m), \cdots u(N-1)\}$ is the set of all possible control actions from $k = m$.

Use the principle of optimality to proof that, when $S = 0$, $J^o[x_m, m, S, N]$ is a monotonically nondecreasing function of N:

$$J^o[x_m, m, 0, N+1] \geq J^o[x_m, m, 0, N]$$