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ME 233 – Homework 4

1.a)

$$p_{XY}(x,y)=\begin{cases}1 & 0\le y\le 2\text{x} \quad 0\le x\le 1\\ 0 & \text{elsewhere}\end{cases}$$

alternately, $p_{XY}(x,y)=\begin{cases}1 & 0\le y\le 2 \quad y/2\le x\le 1\\ 0 & \text{elsewhere}\end{cases}$

$$p_Y(y)=\int_{-\infty}^{\infty} p_{XY}(x,y)\,dx=\begin{cases}\int_{y/2}^{1} dx & 0\le y\le 2\\ 0 & \text{elsewhere}\end{cases}$$

$$p_Y(y)=\begin{cases}1-y/2 & 0\le y\le 2\\ 0 & \text{elsewhere}\end{cases}$$

$$p_X(x)=\int_{-\infty}^{\infty} p_{XY}(x,y)\,dy=\begin{cases}\int_{0}^{2\text{x}} dy & 0\le x\le 1\\ 0 & \text{elsewhere}\end{cases}$$

$$p_X(x)=\begin{cases}2\text{x} & 0\le x\le 1\\ 0 & \text{elsewhere}\end{cases}$$

1.b)

$$m_X=E\{X\}=\int_{-\infty}^{\infty} x\,p_X(x)\,dx=\int_0^1 2x^2\,dx=\frac{2}{3}$$

1.c)

$$\Lambda_{XX}=\int_{-\infty}^{\infty}(x-m_X)^2\,p_X(x)\,dx=\int_0^1 (x-2/3)^2\cdot 2x\,dx=\int_0^1 (2x^3-8x^2/3+8x/9)\,dx=\frac{1}{2}-\frac{8}{9}+\frac{4}{9}=\frac{1}{18}$$

1.d)

$$p_{X|Y}(x|y)=\frac{p_{XY}(x,y)}{p_Y(y)}=\begin{cases}2/(2-y) & 0\le y\le 2x\le 2\\ 0 & \text{elsewhere}\end{cases}$$

1.e)

$$E\{X|Y=y\}=\int_{-\infty}^{\infty} x\,p_{X|Y}(x|y)\,dx=\begin{cases}\int_{y/2}^{1}(2x/(2-y))\,dx & 0\le y\le 2\\ 0 & \text{elsewhere}\end{cases}$$

$$\int_{y/2}^{1}(2x/(2-y))\,dx=\frac{1-y^2/4}{2-y}=\frac{4-y^2}{4(2-y)}=\frac{(2+y)(2-y)}{4(2-y)}=\frac{2+y}{4}$$

$$E\{X|Y=y\}=\begin{cases}(2+y)/4 & 0\le y\le 2\\ 0 & \text{elsewhere}\end{cases}$$

1.f)

$$E\{X|Y=0.5\}=2.5/4=0.625$$

1.g)

$$m_{X|Y}(y)=E\{X|Y=y\}=\int_{-\infty}^{\infty} x\,p_{X|Y}(x|y)\,dx$$

$$E\{m_{X|Y}(y)\}=\int_{-\infty}^{\infty} m_{X|Y}(y)\,p_Y(y)\,dy=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x\,p_{X|Y}(x|y)\,dx\right)p_Y(y)\,dy$$

$$E\{m_{X|Y}(y)\}=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x\,\frac{p_{XY}(x,y)}{p_Y(y)}\,dx\right)p_Y(y)\,dy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\,p_{XY}(x,y)\,dx\,dy$$

$$E\{m_{X|Y}(y)\}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\,p_{XY}(x,y)\,dx\,dy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\,p_{XY}(x,y)\,dy\,dx=\int_{-\infty}^{\infty} x\left(\int_{-\infty}^{\infty} p_{XY}(x,y)\,dy\right)dx$$

$$E\{m_{X|Y}(y)\}=\int_{-\infty}^{\infty} x\left(\int_{-\infty}^{\infty} p_{XY}(x,y)\,dy\right)dx=\int_{-\infty}^{\infty} x\,p_X(x)\,dx$$

For this example, $m_{X|Y}(y)=E\{X|Y=y\}=\begin{cases}(2+y)/4 & 0\le y\le 2\\ 0 & \text{elsewhere}\end{cases}$

$$E\{m_{X|Y}(y)\}=\int_{-\infty}^{\infty} m_{X|Y}(y)\,p_Y(y)\,dy=\int_0^2 \frac{2+y}{4}\left(1-\frac{y}{2}\right)dy=\int_0^2 \frac{(2+y)(2-y)}{8}dy=\int_0^2 \frac{4-y^2}{8}dy$$

$$E\{m_{X|Y}(y)\}=\int_0^2\left(\frac{1}{2}-\frac{y^2}{8}\right)dy=1-\frac{2^3}{3\cdot 8}=1-\frac{8}{24}=\frac{2}{3}=m_X$$

1.h)

$$\Lambda_{m_{X|Y}m_{X|Y}}=\int_{-\infty}^{\infty}(m_{X|Y}(y)-m_X)^2\,p_Y(y)\,dy=\int_0^2\left(\frac{2+y}{4}-\frac{2}{3}\right)^2\left(1-\frac{y}{2}\right)dy=\int_0^2\left(\frac{3y-2}{12}\right)^2\frac{(2-y)}{2}dy$$

$$\Lambda_{m_{X|Y}m_{X|Y}}=\int_0^2\left(\frac{9y^2-12y+4}{144}\right)\frac{(2-y)}{2}dy=\int_0^2\frac{8-28y+30y^2-9y^3}{288}dy=\frac{8\cdot 2-14\cdot 2^2+10\cdot 2^3-9\cdot 2^4/4}{288}$$

$$\Lambda_{m_{X|Y}m_{X|Y}}=\frac{16-56+80-36}{288}=\frac{1}{72}$$

1.i)

$$\Lambda_{X|Y\,X|Y}(y)=E\{(X-m_{X|Y}(y))^2\,|\,Y=y\}=\int_{-\infty}^{\infty}(x-m_{X|Y}(y))^2\,p_{X|Y}(x|y)\,dx$$

$$\Lambda_{X|Y\,X|Y}(y)=\begin{cases}\displaystyle\int_{y/2}^{1}\left(x-\frac{2+y}{4}\right)^2\frac{2}{2-y}\,dx & 0\le y\le 2\\ 0 & \text{elsewhere}\end{cases}$$

$$\int_{y/2}^{1}\left(x-\frac{2+y}{4}\right)^2\frac{2}{2-y}\,dx=\frac{2}{3(2-y)}\left(\left(1-\frac{2+y}{4}\right)^3-\left(\frac{y}{2}-\frac{2+y}{4}\right)^3\right)$$

$$\int_{y/2}^{1}\left(x-\frac{2+y}{4}\right)^2\frac{2}{2-y}\,dx=\frac{2}{3(2-y)}\left(\left(\frac{2-y}{4}\right)^3-\left(\frac{-2+y}{4}\right)^3\right)=\frac{2}{3(2-y)}\left(\left(\frac{2-y}{4}\right)^3+\left(\frac{2-y}{4}\right)^3\right)$$

$$\int_{y/2}^{1}\left(x-\frac{2+y}{4}\right)^2\frac{2}{2-y}\,dx=\frac{4}{3(2-y)}\left(\frac{2-y}{4}\right)^3=\frac{(2-y)^2}{3\cdot 4^2}=\frac{(2-y)^2}{48}$$

$$\Lambda_{X|Y\,X|Y}(y)=\begin{cases}(2-y)^2/48 & 0\le y\le 2\\ 0 & \text{elsewhere}\end{cases}$$

1.j)

$$E\{\Lambda_{X|Y\,X|Y}(y)\}=\int_{-\infty}^{\infty}\Lambda_{X|Y\,X|Y}(y)\,p_Y(y)\,dy=\int_0^2\frac{(2-y)^2}{48}(1-y/2)\,dy=\int_0^2\frac{(2-y)^3}{96}dy$$

$$E\{\Lambda_{X|Y\,X|Y}(y)\}=\frac{2^4}{4\cdot 96}=\frac{1}{24}$$

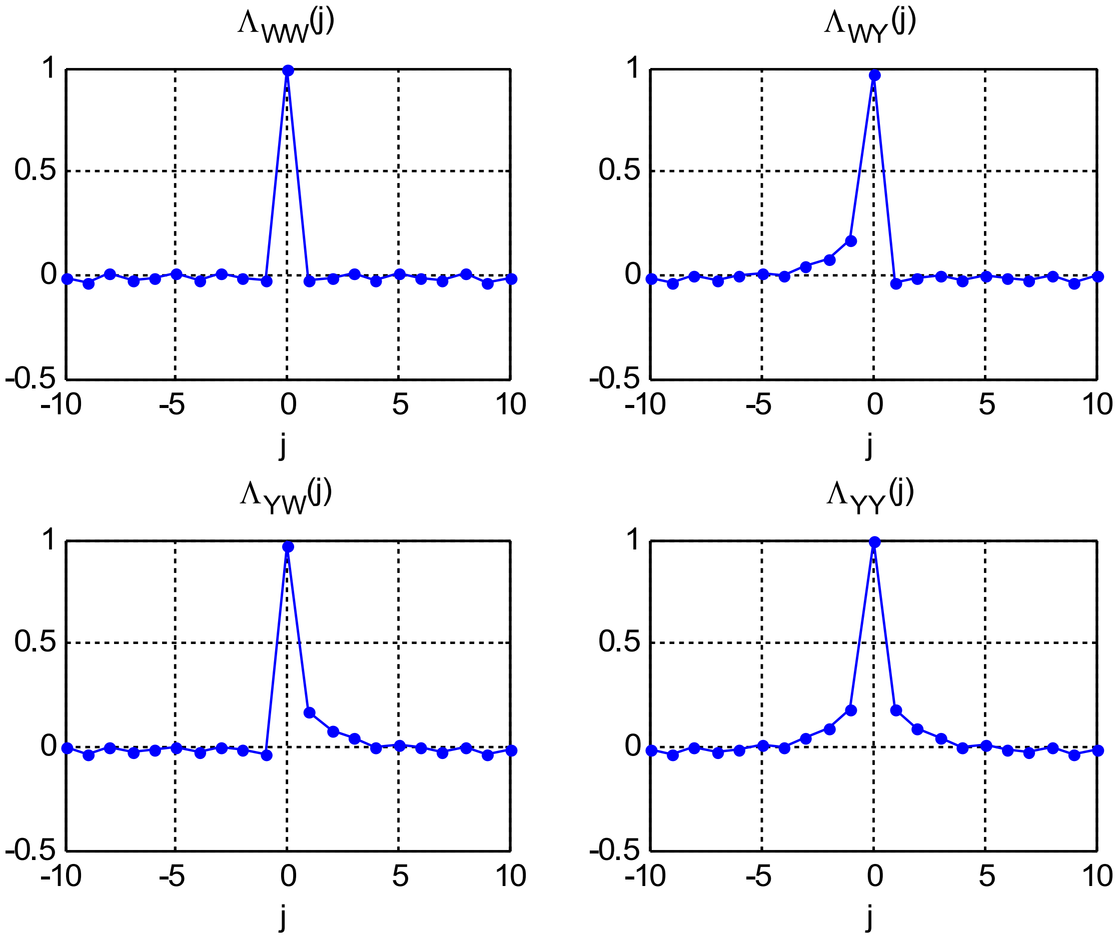
$$\Lambda_{m_{X|Y}m_{X|Y}}+E\{\Lambda_{X|Y\,X|Y}(y)\}=\frac{1}{72}+\frac{1}{24}=\frac{4}{72}=\frac{1}{18}=\Lambda_{XX}$$

2.a)

$$Y(k)-0.5\,Y(k-1)=W(k)-0.3\,W(k-1)$$

$$m_W=0,\ \Lambda_{WW}(l)=E\{W(k+l)W(k)\}=\delta(l)$$

Z-transform gives $Y(z)-0.5\,z^{-1}Y(z)=W(z)-0.3\,z^{-1}W(z)$



2.b)

$$\frac{Y(z)}{W(z)}=G(z)=\frac{z-0.3}{z-0.5}=\frac{0.4z}{z-0.5}+\frac{0.6z-0.3}{z-0.5}=\frac{0.4z}{z-0.5}+0.6$$

Inverse Z-transform, $G(k)=\begin{cases}0.4\cdot 0.5^{k}+0.6\,\delta(k) & k\geq 0\\ 0 & k<0\end{cases}$

$$\Lambda_{YW}(l)=E\{Y(k+l)W(k)\}=\sum_{i=-\infty}^{\infty}G(i)\Lambda_{WW}(l-i)=\sum_{i=-\infty}^{\infty}G(i)\delta(l-i)=G(l)=\begin{cases}0.4\cdot 0.5^{l}+0.6\,\delta(l) & l\geq 0\\ 0 & l<0\end{cases}$$

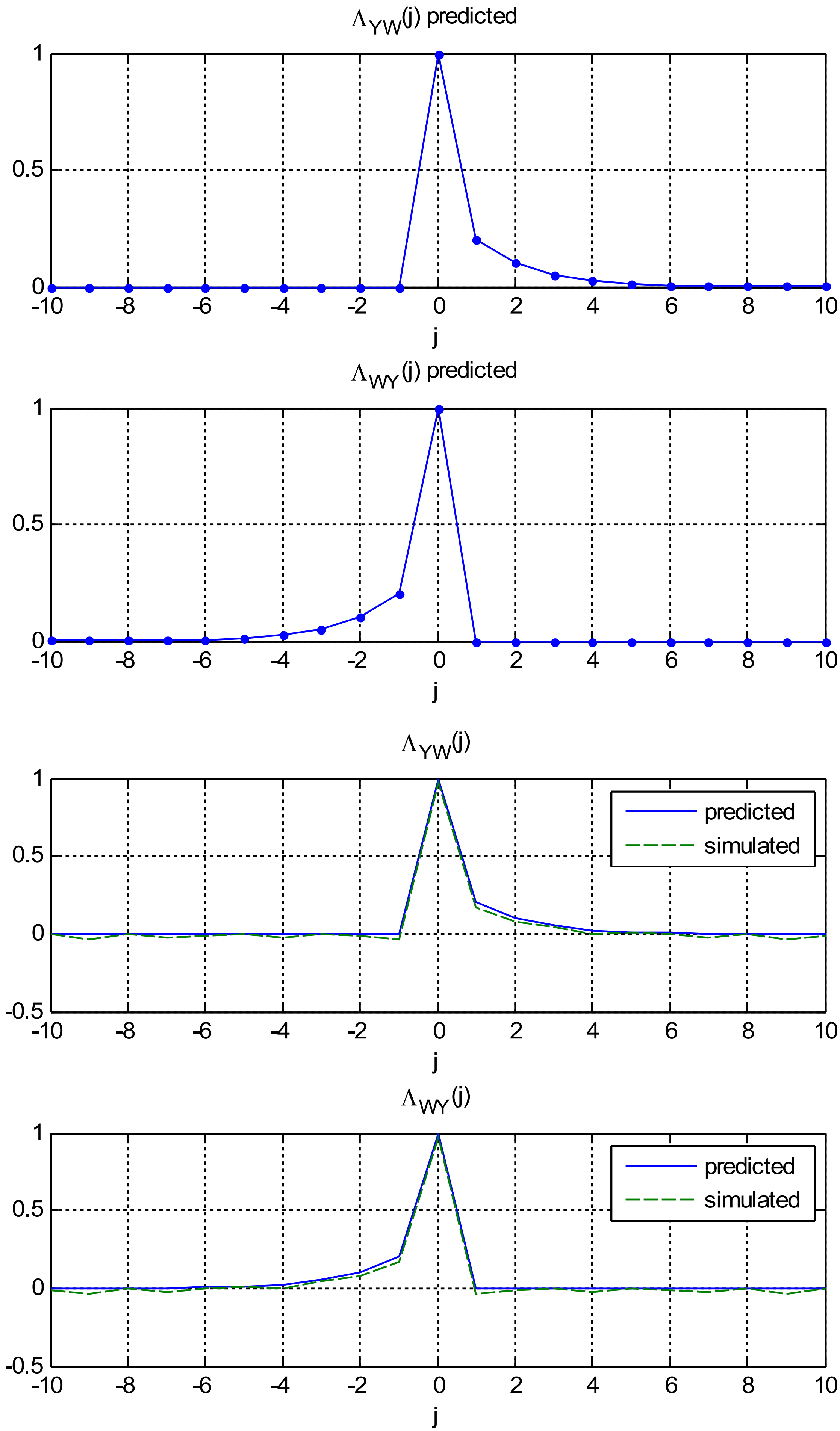
$$\Lambda_{YW}(z)=\sum_{l=-\infty}^{\infty}z^{-l}\Lambda_{YW}(l)=\frac{z-0.3}{z-0.5}$$

plots on next page

2.c)

$$\Lambda_{WY}(l)=E\{W(k+l)Y(k)\}=\Lambda_{YW}^{T}(-l)=G(-l)=\begin{cases}0.4\cdot 0.5^{-l}+0.6\,\delta(-l) & l\leq 0\\ 0 & l>0\end{cases}$$

$$\Lambda_{WY}(z)=\sum_{l=-\infty}^{\infty}z^{-l}\Lambda_{WY}(l)=\Lambda_{YW}(z^{-1})=\frac{z^{-1}-0.3}{z^{-1}-0.5}=\frac{0.3z-1}{0.5z-1}$$



$\Lambda_{YW}(j)$ predicted
$\Lambda_{WY}(j)$ predicted
$\Lambda_{YW}(j)$
$\Lambda_{WY}(j)$
predicted
simulated
j
1
0.5
0
-0.5
-10
-8
-6
-4
-2
0
2
4
6
8
10