ME 233 Spring 2010
Solution to Midterm #2

Problem 1

1. (a)

$$\begin{aligned} G_w(z) &= \frac{2(z+3.5)}{(z-1)(z+2)} \\ G_w(z^{-1})G_w(z) &= -7\frac{z(z+3.5)(z+\frac{1}{3.5})}{(z-1)^2(z+2)(z+0.5)} \end{aligned}$$

Because $G_w(z^{-1})G_w(z)$ has a negative gain, we use positive feedback rules for the root locus plot, as shown in Figure 1.

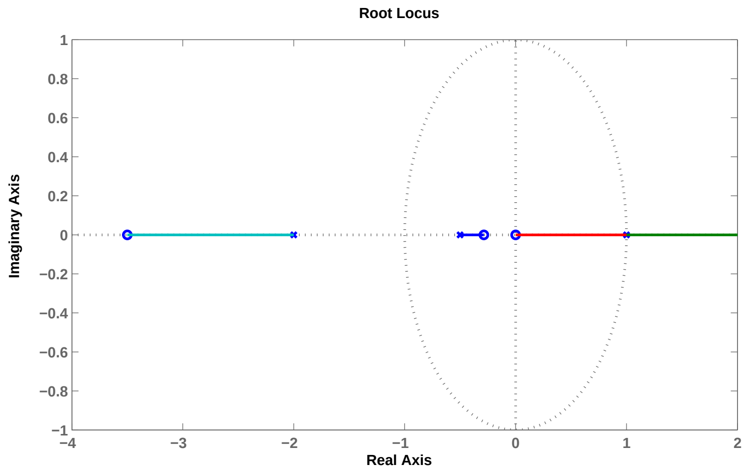

Figure 1: Root locus of closed loop Kalman filter poles and their reciprocals as W is varied

(b) Let $L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$, then the closed loop system has

$$\begin{aligned} \det(zI - A + LC) &= \det \begin{bmatrix} z-1+l_1 & -1 \\ l_2 & z+2 \end{bmatrix} \\ &= z^2 + (l_1+1)z + 2(l_1+1) + l_2 \end{aligned}$$

Compare the the coefficients with the ones of $C(q^{-1})$, we get

$$\left\{ \begin{array}{l} l_1 + 1 = 0.25 \\ l_2 + 2l_1 - 2 = -0.035 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} l_1 = -0.75 \\ l_2 = 3.465 \end{array} \right. \Rightarrow L = \begin{bmatrix} -0.75 \\ 3.465 \end{bmatrix}$$

Thus,

$$F = A^{-1}L = \begin{bmatrix} 1 & 0.5 \\ 0 & -0.5 \end{bmatrix} \begin{bmatrix} -0.75 \\ 3.465 \end{bmatrix} = \begin{bmatrix} 0.9825 \\ -1.7325 \end{bmatrix}$$

(c)

$$\varepsilon = CMC^T + V = m_{11} + 1 = 56 \Rightarrow m_{11} = 55$$

(d)

$$\begin{aligned}\hat{x}_1(k) &= \hat{x}_1^o(k) + f_1\epsilon(k) = \hat{x}_1^o(k) + f_1\left(\tilde{x}_1^o(k) + v(k)\right) \\ \Rightarrow \tilde{x}_1(k) &= (1-f_1)\tilde{x}_1^o(k) - f_1 v(k) \\ \Rightarrow z_{11} &= (1-f_1)^2 m_{11} + f_1^2 = 0.9821\end{aligned}$$

2. (a)

$$\begin{aligned}G(z) &= \frac{z}{(z-1)(z+2)} \\ G(z^{-1})G(z) &= -0.5\frac{z^2}{(z-1)^2(z+2)(z+0.5)}\end{aligned}$$

Because $G(z^{-1})G(z)$ has a negative gain, we use positive feedback rules for the root locus plot, as shown in Figure 2.

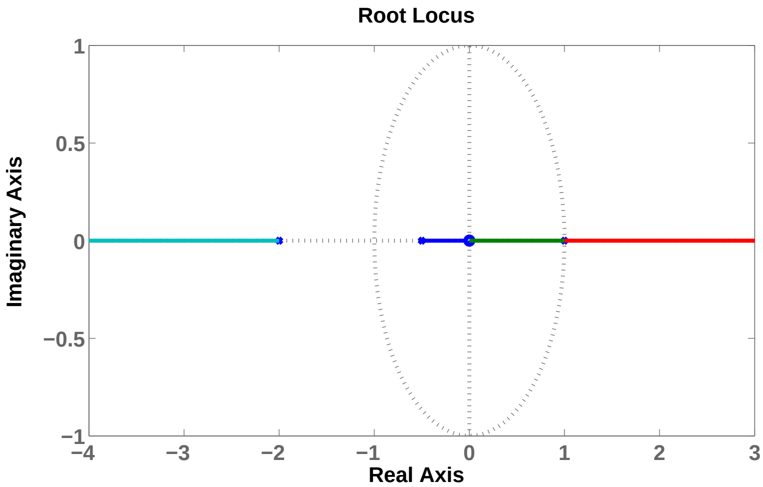

Figure 2: Root locus of closed loop LQ control poles and their reciprocals as ρ is varied

(b) From the previous part, we know the closed-loop poles go to $\{0, 0\}$ as $\rho \to 0$.
Let $K_o = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$, then the closed loop system has

$$\det(zI - A + BK_o) = \det\begin{bmatrix} z-1+k_1 & -1+k_2 \\ -2k_1 & z+2-2k_2 \end{bmatrix}$$

In order to make the two closed-loop poles be the origin, it is clear that we must have

$$\begin{cases} k_1 = 1 \\ k_2 = 1 \end{cases} \Rightarrow K_o = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

3. (a) As $\rho \to 0$, the LQR Riccati equation becomes:

$$A^T P_o A - P_o + C^T C - \alpha K_o^T K_o = 0 \text{ with } \alpha = B^T P_o B$$

From the previous part, we know $K_o = \begin{bmatrix} 1 & 1 \end{bmatrix}$. In addition, Let $P_o = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$.

Thus,

$$\begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \alpha \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 0$$
$$\Rightarrow \begin{bmatrix} 0 & P_{11} - 3P_{12} \\ P_{11} - 3P_{12} & P_{11} - 4P_{12} + 3P_{22} \end{bmatrix} + \begin{bmatrix} 1-\alpha & -\alpha \\ -\alpha & -\alpha \end{bmatrix} = 0$$

Obviously, $\alpha = 1$. As a result, we have:

$$\begin{cases} P_{11} - 3P_{12} = \alpha = 1 \\ P_{11} - 4P_{12} + 3P_{22} = \alpha = 1 \\ P_{11} - 4P_{12} + 4P_{22} = \alpha = 1 \end{cases} \Rightarrow P_o = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

(b)

$$\begin{aligned} J_o^o &= z_{11} + V + F^T P_o F \varepsilon = z_{11} + 1 + f_1^2 \varepsilon \\ &= 56.0393 \end{aligned}$$

Problem 2

In this problem, we define

$$A(q^{-1}) := 1 + 0.9q^{-1}, \qquad B(q^{-1}) := 0.5(1 + q^{-1}), \qquad g := 1$$

so that the plant dynamics are described by

$$A(q^{-1})y(k) = q^{-g} \left[B(q^{-1})u(k) + d \right].$$

Here, we have denoted the relative degree of the dynamics from $u(k)$ to $y(k)$ as g to avoid confusion with the constant disturbance, d. Note that this system has zeros at $q = -1$. Obviously, we can not have zero-pole cancelation. Thus, we define

$$B^s(q^{-1}) := 1, \qquad B^u(q^{-1}) := 0.5(1 + q^{-1}).$$

Now we need to specify $A_c'(q^{-1})$. We are given that the closed loop poles of the feedback system (in terms of q) should only include one pole at $q = 0.5$. This means that

$$A_c'(q^{-1}) = 1 - 0.5q^{-1}$$

because any other choice would create poles that are not at the origin (in terms of q). Finally, we note that the constant disturbance, d, is annihilated by the polynomial

$$A_d(q^{-1}) = 1 + q^{-2}.$$

The first step in designing the controller is designing the feedback loop to achieve pole placement and disturbance rejection. This is done by choosing the feedback control law

$$R'(q^{-1}) A_d(q^{-1}) B^s(q^{-1}) u(k) = r(k) - S(q^{-1}) y(k) \tag{1}$$

where $R'(q^{-1})$ and $S(q^{-1})$ solve the Diophantine equation

$$A_c'(q^{-1}) = A(q^{-1}) A_d(q^{-1}) R'(q^{-1}) + q^{-g} B^u(q^{-1}) S(q^{-1}). \tag{2}$$

Since the order of $R'(q^{-1})$ is $n_u + g - 1$ where n_u is the order of $B^u(q^{-1})$, we see that $R'(q^{-1})$ should have the form

$$R'(q^{-1}) = 1 + r_1 q^{-1}.$$

Since the order of $A_c'(q^{-1})$ is 1 and the order of $A(q^{-1})A_d(q^{-1})R'(q^{-1})$ is 4, the order of $q^{-g}B^u(q^{-1})S(q^{-1})$ is larger of these two which is 4 in this case. This implies that the order of $S(q^{-1})$ is 2, which in turn implies that $S(q^{-1})$ has the form

$$S(q^{-1}) = s_0 + s_1 q^{-1} + s_2 q^{-2}.$$

Thus, Eq. (2) can be written

$$\begin{aligned}
1 - 0.5q^{-1} &= (1+q^{-2})(1+0.9q^{-1})(1+r_1q^{-1}) + 0.5(q^{-1}+q^{-2})(s_0+s_1q^{-1}+s_2q^{-2}) \\
&= 1 + q^{-1}(0.9+r_1+0.5s_0) + q^{-2}(1+0.9r_1+0.5s_0+0.5s_1) + q^{-3}(0.9+r_1+0.5s_1+0.5s_2) \\
&\quad + q^{-4}(0.9r_1+0.5s_2).
\end{aligned}$$

Equating coefficients gives

$$\begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0.9 & 0.5 & 0.5 & 0 \\ 1 & 0 & 0.5 & 0.5 \\ 0.9 & 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} r_1 \\ s_0 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} -1.4 \\ -1 \\ -0.9 \\ -0.9 \end{bmatrix}.$$

In this problem, it was not necessary to actually solve this system of equations; it was only necessary to find this system of linear equations. This system of equations determines $R'(q^{-1})$ and $S(q^{-1})$ which in turn determines the feedback control law in Eq. (1).

Now, we need to find the feedforward control law which guarantees zero-phase tracking of $y_d(k)$. To do this, we choose the feedforward control law

$$\begin{aligned}
r(k) &= q^g A_c'(q^{-1}) \frac{B^u(q)}{[B^u(1)]^2} y_d(k) \\
&= 0.5q^2(1+0.5q^{-1}+0.5q^{-2})y_d(k).
\end{aligned}$$

As desired, we only need to know $y_d(k)$ two steps in advance.