ME 233 Spring 2016
Solution to Homework #4

1. (a) By the hint, we can prove the argument if we can show nullity(X) = nullity(Z), because the columns of a matrix are linearly independent if and only if the dimension of its null space is zero. In the following argument, we will show $\mathcal{N}(X) = \mathcal{N}(Z)$.

   - Let $v \in \mathcal{N}(X)$, then

   $$Xv = 0 \Rightarrow \begin{bmatrix} I \\ M \end{bmatrix} Xv = 0 \Rightarrow v \in \mathcal{N}\left(\begin{bmatrix} I \\ M \end{bmatrix} X\right) .$$

     Thus, we know $\mathcal{N}(X) \subseteq \mathcal{N}\left(\begin{bmatrix} I \\ M \end{bmatrix} X\right)$.

   - Let $v \in \mathcal{N}\left(\begin{bmatrix} I \\ M \end{bmatrix} X\right)$, then

   $$\begin{bmatrix} I \\ M \end{bmatrix} Xv = \begin{bmatrix} Xv \\ MXv \end{bmatrix} = 0 \Rightarrow Xv = 0 \Rightarrow v \in \mathcal{N}(X) .$$

     Thus, we know $\mathcal{N}\left(\begin{bmatrix} I \\ M \end{bmatrix} X\right) \subseteq \mathcal{N}(X)$.

   Therefore, $\mathcal{N}(X) = \mathcal{N}(Z)$ which implies the columns of X are linearly independent if and only if the columns of Z are linearly independent.

   (b) - We first assume that we have $\begin{bmatrix} \hat{A} - \lambda I \\ \hat{C} \end{bmatrix} X = 0$ and the column of X are linearly independent. Then we obtain:

   $$\begin{bmatrix} \hat{A} - \lambda I \\ \hat{C} \end{bmatrix} X = 0 \Rightarrow \begin{cases} (A - \lambda I)X - B\left(D^T D\right)^{-1} D^T CX = 0 \\ CX - D\left(D^T D\right)^{-1} D^T CX = 0 \end{cases}$$

     Define $Y = -\left(D^T D\right)^{-1} D^T CX$, we have

   $$\begin{cases} (A - \lambda I)X + BY = 0 \\ CX + DY = 0 \end{cases} \Rightarrow \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = 0 .$$

     And thanks to the result of part (a), we know that the columns of $\begin{bmatrix} X \\ Y \end{bmatrix}$ are linearly independent. So the first condition implies the second one.

   - Now we assume that $\exists Y$ such that $\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = 0$ and the columns of $\begin{bmatrix} X \\ Y \end{bmatrix}$ are linearly independent. Then we obtain:

   $$\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = 0 \Rightarrow \begin{cases} (A - \lambda I)X + BY = 0 \\ CX + DY = 0 \end{cases}$$

   $$\Rightarrow D^T DY = -D^T CX \Rightarrow Y = -\left(D^T D\right)^{-1} D^T CX \Rightarrow \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} I \\ -\left(D^T D\right)^{-1} D^T C \end{bmatrix} X$$

With the result from the previous part, we know the columns of X are linearly independent since the columns of $\begin{bmatrix} X \\ Y \end{bmatrix}$ are linearly independent. Thus,

$$\begin{cases} (A-\lambda I)X - B\left(D^T D\right)^{-1} D^T C X = 0 \\ CX - D\left(D^T D\right)^{-1} D^T C X = 0 \end{cases} \Rightarrow \begin{cases} (\hat{A}-\lambda I)X = 0 \\ \hat{C}X = 0 \end{cases}$$
$$\Rightarrow \begin{bmatrix} \hat{A}-\lambda I \\ \hat{C} \end{bmatrix} X = 0$$

Then the second condition implies the first one. We can deduce that both conditions are independent.

(c) • We first show nullity $\begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix} \leq$ nullity $\begin{bmatrix} \hat{A}-\lambda I \\ \hat{C} \end{bmatrix}$. Since this is trivial if the left-hand side is 0, we assume that it is nonzero. Letting the columns of $\begin{bmatrix} X \\ Y \end{bmatrix}$ be a basis for $\mathcal{N}\begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix}$, we obtain from the previous part that $\begin{bmatrix} \hat{A}-\lambda I \\ \hat{C} \end{bmatrix} X = 0$.

Also, the columns of X are linearly independent vectors in $\mathcal{N}\begin{bmatrix} \hat{A}-\lambda I \\ \hat{C} \end{bmatrix}$, which implies that nullity $\begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix} \leq$ nullity $\begin{bmatrix} \hat{A}-\lambda I \\ \hat{C} \end{bmatrix}$.

• We now show that nullity $\begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix} \geq$ nullity $\begin{bmatrix} \hat{A}-\lambda I \\ \hat{C} \end{bmatrix}$. Again since this is trivial if the right-hand side is 0, we assume that it is nonzero. Letting the columns of X be a basis for $\mathcal{N}\begin{bmatrix} \hat{A}-\lambda I \\ \hat{C} \end{bmatrix}$, we obtain from the previous part that $\exists Y$ such that $\begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = 0$.

Also, the number of linearly independent columns in $\begin{bmatrix} X \\ Y \end{bmatrix}$ is the same as that of linear independent columns in X, which implies nullity $\begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix} \geq$ nullity $\begin{bmatrix} \hat{A}-\lambda I \\ \hat{C} \end{bmatrix}$.

Since the preceding arguments hold for any $\lambda \in \mathcal{C}$, we conclude that nullity $\begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix} =$ nullity $\begin{bmatrix} \hat{A}-\lambda I \\ \hat{C} \end{bmatrix}$, for $\forall \lambda \in \mathcal{C}$.

(d) Using the rank-nullity theorem, we have that

$$\text{rank}\begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix} = n_x + n_u - \text{nullity}\begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix}$$

With the result from Part (b), we get

$$\begin{aligned} \text{rank}\begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix} &= n_x + n_u - \text{nullity}\begin{bmatrix} \hat{A}-\lambda I \\ \hat{C} \end{bmatrix} = n_x + n_u - \left(n_x - \text{rank}\begin{bmatrix} \hat{A}-\lambda I \\ \hat{C} \end{bmatrix}\right) \\ &= n_u + \text{rank}\begin{bmatrix} \hat{A}-\lambda I \\ \hat{C} \end{bmatrix} \end{aligned}$$

We thus see that

$$\text{rank}\begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix} < \text{normalrank}\begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix} \quad \Leftrightarrow \quad \text{rank}\begin{bmatrix} \hat{A}-\lambda I \\ \hat{C} \end{bmatrix} < \text{normalrank}\begin{bmatrix} \hat{A}-\lambda I \\ \hat{C} \end{bmatrix}.$$

Since rank $\begin{bmatrix} \hat{A}-\lambda I \\ \hat{C} \end{bmatrix} = n_x$ whenever λ is not an eigenvalue of $\hat{A}$, we see that

$$\text{rank}\begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix} < \text{normalrank}\begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix} \quad \Leftrightarrow \quad \text{rank}\begin{bmatrix} \hat{A}-\lambda I \\ \hat{C} \end{bmatrix} < n_x$$

which implies that λ is a transimission zero of the state-space realization $G(z) = C(zI - A)^{-1}B + D$ if and only if λ corresponds to an unobservable mode of $(\hat{C}, \hat{A})$.

2. From the previous problem, we have seen that λ is a transimission zero of the state-space realization $G(z) = C(zI - A)^{-1}B + D$ if and only if λ corresponds to an unobservable mode of $(\hat{C}, \hat{A})$, where $\hat{A} = A - B(D^T D)^{-1}D^T C$ and $\hat{C} = C - D(D^T D)^{-1}D^T C$.

   In this problem, we have $C^T D = 0$, then we have $\hat{A} = A$ and $\hat{C} = C$. So we obtain that the transmission zeros of the state space realization $G(z) = C(zI - A)^{-1}B + D$ are the unobservable modes of (C, A).

3. (a) One set of conditions to guarantee the existence of the stationary Kalman filter for Sensor Configuration A is:

   - $(A, B_w W^{1/2})$ is stabilizable;
   - (C, A) is detectable.

   From the lecture notes, we know (C, A) is detectable if and only if

$$\text{rank}\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n, \text{ whenever } |\lambda| \geq 1$$

$$\Rightarrow \quad \text{rank}\begin{bmatrix} A - \lambda I \\ C \\ C \end{bmatrix} = \text{rank}\begin{bmatrix} A - \lambda I \\ \begin{bmatrix} C \\ C \end{bmatrix} \end{bmatrix} = n, \text{ whenever } |\lambda| \geq 1$$

   Then, $\left(\begin{bmatrix} C \\ C \end{bmatrix}, A\right)$ is also detectable if (C, A) is detectable. Thus, the set of conditions specified for the existence of the stationary Kalman filter for Sensor Configuration A also guarantees that the stationary Kalman filter exists for Sensor Configuration B.

   (b) We are given these equations:

$$\begin{aligned} (I + MC^T V^{-1} C)^{-1} &= I - MC^T(CMC^T + V)^{-1}C \\ A - LC &= A[I - MC^T(CMC^T + V)^{-1}C] \\ M &= (A - LC)MA^T + B_w W B_w^T \end{aligned}$$

   Let $V_1 = V_A$, $V_2 = \begin{bmatrix} V_B & 0 \\ 0 & V_B \end{bmatrix}$, $C_1 = C$, $C_2 = \begin{bmatrix} C \\ C \end{bmatrix}$, $M_1 = M_A$, and $M_2 = M_B$. Then, we have DAREs for the Kalman filters of the two sensor configurations:

$$\begin{aligned} M_i &= AM_iA^T + B_w W B_w^T - AM_iC_i^T \left(C_i M_i C_i^T + V_i\right)^{-1} C_i M_i A^T, \ i = 1, 2 \\ L_i &= AM_iC_i^T(C_iM_iC_i^T + V_i)^{-1} \end{aligned}$$

   We notice, and using the results from the hint, that:

$$\begin{aligned} M_i &= AM_iA^T + B_w W B_w^T - AM_iC_i^T \left(C_i M_i C_i^T + V_i\right)^{-1} C_i M_i A^T, \ i = 1, 2 \\ M_i &= A(I - M_iC_i^T(C_iM_iC_i^T + V_i)^{-1}C_i)M_iA^T + B_w W B_w^T \\ M_i &= A(I + M_iC_i^T V_i^{-1} C_i)^{-1} M_i A^T + B_w W B_w^T \end{aligned}$$

   We can observe that all quantities in both DAREs defining M_1 and M_2 are the same except the quantities $C_i^T V_i^{-1} C_i$. Then let assume that $C_1^T V_1^{-1} C_1 = C_2^T V_2^{-1} C_2$. We obtain:

$$\begin{aligned} C_1^T V_1^{-1} C_1 &= C_2^T V_2^{-1} C_2 \\ C^T V_A^{-1} C &= \begin{bmatrix} C^T & C^T \end{bmatrix} \begin{bmatrix} V_B & 0 \\ 0 & V_B \end{bmatrix}^{-1} \begin{bmatrix} C \\ C \end{bmatrix} \\ C^T V_A^{-1} C &= 2C^T V_B^{-1} C \end{aligned}$$

   At this point, we choose $V_A = \frac{1}{2}V_B$ which guarantees that M solves the DARE for Sensor Configuration A if and only if M solves the DARE for Sensor Configuration B. For the conditions listed in part (a), this is enough to guarantee that $M_A = M_B$ because the DARE has a unique positive semi-definite solution in each case.

However, if the first condition is relaxed to the condition that $(A, B_w W^{1/2})$ has no uncontrollable modes on the unit circle, then we have one additional condition to check: that $A - L_1 C_1$ is Schur for $M_1 = M$ if and only if $A - L_2 C_2$ is Schur for $M_2 = M$. To show this, we notice from the hint that

$$A - L_i C_i = A(I + M_i C_i^T V_i^{-1} C_i)^{-1}$$

Since $C_1^T V_1^{-1} C_1 = C_2^T V_2^{-1} C_2$ by the restriction $V_A = \frac{1}{2} V_B$, we see that when $M_1 = M_2$, we have $A - L_1 C_1 = A - L_2 C_2$. This establishes the result that M is the stabilizing solution of the DARE for Sensor Configuration A if and only if M is the stabilizing solution of the DARE for Sensor Configuration B. This implies that $M_A = M_B$ for these relaxed existence conditions.

4. The proof is devided into two parts. First we are going to check that (A_e, B_e) is stabilizable, and then we are going to check the transmission zero condition.

   (a) We first consider:

$$\begin{bmatrix} A^T - \lambda I & B_1^T & 0 \\ 0 & A_1^T - \lambda I & 0 \\ 0 & 0 & A_2^T - \lambda I \\ B^T & 0 & B_2^T \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = 0$$

   where $|\lambda| \geq 1$. We then get:

$$(A_1^T - \lambda I)Y = 0$$
$$(A_2^T - \lambda I)Z = 0$$

   Then by stability of A_1 and A_2, we know that λ is not an eigenvalue of A_1 or A_2, which implies that $Y = 0$ and $Z = 0$. After, we have:

$$\begin{bmatrix} A^T - \lambda I \\ B^T \end{bmatrix} X = 0$$

   Since (A, B) is stabilizable, we obtain $X = 0$. So we deduce that:

$$\text{nullity} \begin{bmatrix} A_e^T - \lambda I \\ B_e^T \end{bmatrix} = \text{nullity} \begin{bmatrix} A^T - \lambda I & B_1^T & 0 \\ 0 & A_1^T - \lambda I & 0 \\ 0 & 0 & A_2^T - \lambda I \\ B^T & 0 & B_2^T \end{bmatrix} = 0$$

   Since this equality holds $\forall \lambda$ such that $|\lambda| \geq 1$, we have that (A_e, B_e) is stabilizable.

   (b) Now let consider:

$$\begin{bmatrix} A - \lambda I & 0 & 0 & B \\ B_1 & A_1 - \lambda I & 0 & 0 \\ 0 & 0 & A_2 - \lambda I & B_2 \\ D_1 & C_1 & 0 & 0 \\ 0 & 0 & C_2 & D_2 \end{bmatrix} \begin{bmatrix} W \\ X \\ Y \\ Z \end{bmatrix} = 0, \qquad |\lambda| = 1$$

   We notice that:

$$\begin{bmatrix} A_2 - \lambda I & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} Y \\ Z \end{bmatrix} = 0$$

   And because nullity $\begin{bmatrix} A_2 - \lambda I & B_2 \\ C_2 & D_2 \end{bmatrix} = 0$ whenever $|\lambda| = 1$, then we obtain $\begin{bmatrix} Y \\ Z \end{bmatrix} = 0$. Thanks to this result, we have:

$$\begin{bmatrix} A - \lambda I & 0 \\ B_1 & A_1 - \lambda I \\ D_1 & C_1 \end{bmatrix} \begin{bmatrix} W \\ X \end{bmatrix} = 0$$

   We can consider two different cases.

i. First we consider that λ is an eigenvalue of A. Then we get:

$$\begin{bmatrix} A_1 - \lambda I & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} X \\ W \end{bmatrix} = 0$$

And because we have nullity $\begin{bmatrix} A_1 - \lambda I & B_1 \\ C_1 & D_1 \end{bmatrix} = 0$ whenever λ is an eigenvalue of A satisfying $|\lambda| = 1$, then we obtain $\begin{bmatrix} X \\ W \end{bmatrix} = 0$.

ii. Now let consider that λ is not an eigenvalue of A. So we have:

$$\begin{aligned} (A - \lambda I)W &= 0 \\ \Rightarrow W &= 0 \end{aligned}$$

This implies that:

$$(A_1 - \lambda I)X = 0$$

Then by the stability condition, we obtain $X = 0$.

Combining all the results we obtained above, we deduce:

$$\text{nullity} \begin{bmatrix} A^T - \lambda I & 0 & 0 & B \\ B_1 & A_1^T - \lambda I & 0 & 0 \\ 0 & 0 & A_2^T - \lambda I & B_2 \\ D_1 & C_1 & 0 & 0 \\ 0 & 0 & C_2 & D_2 \end{bmatrix} = 0, \forall \lambda \text{ such that } |\lambda| = 1$$

Then we conclude that $C_e(zI - A_e)^{-1}B_e + D_e$ has no transmission zeros on the unit circle.