ME 233 Advance Control II

Lecture 20

Least Squares Estimation Parameter Convergence and Persistence of Excitation

Estimation of ARMA model

$$A(q^{-1})y(k) = q^{-1}\,B(q^{-1})u(k)$$

Where

- $u(k)$ known ***bounded*** input
- $y(k)$ measured output

$$A(q^{-1}) = 1 + a_1 q^{-1} + \cdots + a_n q^{-n}$$ **(Schur)**

$$B(q^{-1}) = b_o + b_1 q^{-1} + \cdots + b_m q^{-m}$$

ARMA Model

$$y(k) = \phi^T(k-1)\,\theta$$

Unknown parameter vector:

$$\theta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \\ b_o \\ \vdots \\ b_m \end{bmatrix} \quad \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} n \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} m+1 \end{matrix}$$

Known regressor vector:

$$\phi(k) = \left.\begin{bmatrix} -y(k) \\ \vdots \\ -y(k-n+1) \\ u(k-d) \\ \vdots \\ u(k-m-d) \end{bmatrix}\right\} n+m+1$$

ARMA series-parallel estimation

- A-priori output

$$\hat{y}^o(k) \;=\; \phi^T(k-1)\,\hat{\theta}(k-1)$$

$$\hat{\theta}(k) = \left[\; \hat{a}_1(k) \;\; \cdots \;\; \hat{a}_n(k) \;\; \hat{b}_o(k) \cdots \;\; \hat{b}_m(k) \;\right]^T$$

- A-priori error

$$e^o(k) \;=\; y(k) - \hat{y}^o(k)$$

ARMA series-parallel estimation

- A-priori error

$$e^o(k) \;=\; y(k) - \hat{y}^o(k)$$

$$e^o(k) \;=\; \phi^T(k-1)\,\tilde{\theta}(k-1)$$

- Parameter error

$$\tilde{\theta}(k) = \theta - \hat{\theta}(k)$$

RLS Estimation Algorithm

$$e^o(k) = y(k) - \phi^T(k-1)\hat{\theta}(k-1)$$

$$e(k) \;=\; \frac{e^o(k)}{1 + \phi^T(k-1)F(k-1)\phi(k-1)}$$

$$\hat{\theta}(k) \;=\; \hat{\theta}(k-1) + F(k-1)\phi(k-1)\,e(k)$$

$$F(k+1) = \frac{1}{\lambda_1(k)}\left[F(k) - \frac{F(k)\phi(k)\phi^T(k)F(k)}{\frac{\lambda_1(k)}{\lambda_2(k)} + \phi^T(k)F(k)\phi(k)}\right]$$

Overview

- In lecture 19 we learn how to analyze the stability of adaptive systems and proved:
  - Convergence of the a-priori output error

$$e^o(k) \rightarrow 0$$

- Today we will provide conditions on the input sequence $u(k)$ so that the

$$\tilde{\theta}(k) = \theta - \hat{\theta}(k)$$

also converges to zero.

Parameter error convergence

- Remember that $e^o(k) \rightarrow 0$

It can be shown that the n+m+*1* parameter error also converges:

$$\lim_{k\to\infty} \tilde{\theta}(k) = \bar{\theta}$$

$$\lim_{k\to\infty} \tilde{\theta}(k) = \bar{\theta} \;=\; \begin{bmatrix} \left.\begin{matrix} \bar{a}_1 \\ \vdots \\ \bar{a}_n \end{matrix}\right\} n \\ \left.\begin{matrix} \bar{b}_o \\ \vdots \\ \bar{b}_m \end{matrix}\right\} m+1 \end{bmatrix}$$

Parameter error convergence

The steady-state parameter error satisfies

$$\boxed{E\left\{\phi(k)\phi^T(k)\right\}\bar{\theta} \;=\; 0}$$

Regressor

$$\phi(k) = \begin{bmatrix} -y(k) \\ \vdots \\ -y(k-n+1) \\ u(k-d) \\ \vdots \\ u(k-m-d) \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\b\\c\end{matrix}}\right\} n \\ \left.\vphantom{\begin{matrix}a\\b\\c\end{matrix}}\right\} m+1 \end{matrix}$$

Parameter error convergence

The steady-state parameter error satisfies

$$\boxed{E\left\{\phi(k)\phi^T(k)\right\}\bar{\theta} \;=\; 0}$$

Where the regressor correlation $E\left\{\phi(k)\phi^T(k)\right\}$ is:

$$E\{\phi(k)\phi(k)^T\} = \lim_{N\to\infty}\left\{\frac{1}{2N+1}\sum_{j=-N}^{j=N}\phi(k+j)\phi^T(k+j)\right\}$$

Parameter error convergence

Since the steady-state parameter error satisfies

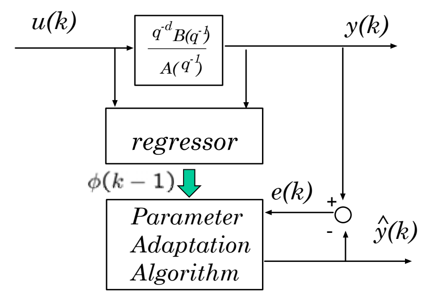
$$E\left\{\phi(k)\phi^T(k)\right\}\bar{\theta} \;=\; 0$$

$$E\left\{\phi(k)\phi^T(k)\right\} \succ 0 \quad \Rightarrow \quad \lim_{k\to\infty}\tilde{\theta}(k) = \bar{\theta} = 0$$

The regressor vector $\phi(k)$ is persistently exciting if

$$E\left\{\phi(k)\phi^T(k)\right\} \succ 0$$

Persistence of Excitation



$$\phi(k) = \begin{bmatrix} -y(k) \\ \vdots \\ -y(k-n) \\ u(k-d) \\ \vdots \\ u(k-m-d) \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\b\\c\end{matrix}}\right\} n \\ \left.\vphantom{\begin{matrix}a\\b\\c\end{matrix}}\right\} m+1 \end{matrix}$$

We need to find the conditions that the input sequence *u(k)* must satisfy to guarantee that $\phi(k)$ is persistently exciting.

$$E\left\{\phi(k)\phi^T(k)\right\} \succ 0$$

Excitation matrix

Given and input sequence $u(k) \in \mathcal{R}$

Define the u-regressor of order n:

$$\phi_{u_n}(k) = \begin{bmatrix} u(k) \\ u(k-1) \\ \vdots \\ u(k-n+1) \end{bmatrix} \in \mathcal{R}^n$$

only present and past values of u(k) are used

Excitation matrix

Given and input sequence $u(k) \in \mathcal{R}$

Define the n x n excitation matrix:

$$C_n = E\{\phi_{u_n}(k)\phi_{u_n}^T(k)\}$$

$$= \lim_{N\to\infty} \left\{ \frac{1}{2N+1} \sum_{j=-N}^{j=N} \phi_{u_n}(k+j)\phi_{u_n}^T(k+j) \right\}$$

Persistence of Excitation (PE)

The input sequence $u(k)$

is **<u>persistently exciting</u>** of order n **iff**
the n x n excitation matrix is **<u>positive definite</u>**

$$C_n = E\{\phi_{u_n}(k)\phi_{u_n}^T(k)\} \succ 0$$

$$\phi_{u_n}(k) = \begin{bmatrix} u(k) \\ u(k-1) \\ \vdots \\ u(k-n+1) \end{bmatrix} \in \mathcal{R}^n$$

PE inputs

To determine the PE order of a sequence $u(k)$

1. Find an annihilating polynomial $A_n(q^{-1})$ of order n such

$$A_n(q^{-1})u(k) = 0$$

this means that $u(k)$ <u>is at most PE of order n</u>

2. Compute the excitation matrix

$$C_n = E\{\phi_{u_n}(k)\phi_{u_n}^T(k)\} \succ 0$$

and verify that it is positive definite.

Conditions for PE

- Example 1: The step input

$$u(k) = 1(k) = \begin{cases} 1 & \text{for} \quad k \geq 0 \\ 0 & \text{for} \quad k < 0 \end{cases} .$$

1(k) is PE of order 1

Conditions for PE

Example 1: The step input ***1(k)***

1) $(1 - q^{-1})\, 1 = 0$ ⇒ ***1(k)*** is **at most** PE of order 1

2) $$C_1 = \lim_{k \to \infty} \frac{1}{k} \sum_{j=1}^{k} 1 = 1 > 0$$

⇒ ***1(k)*** is PE of order 1

Conditions for PE

Examples: Sum of Sinusoids

Consider an input that is a sum of m sinusoids, with m distinct frequencies

$$u(k) = \sum_{i=1}^{m} \sin(\omega_i\, k) \,.$$

$$0 < \omega_i < \pi$$
$$\omega_i \neq w_j$$

$u(k)$ is PE of order $n = 2m$.

Conditions for PE

Examples: Random sequence

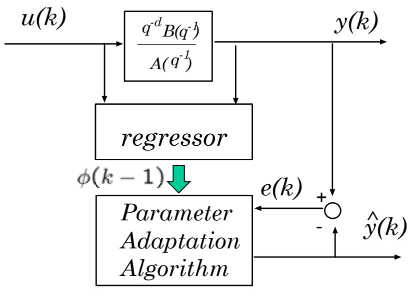
Consider a colored random sequence

$$u(k) = G(q)\, w(k)$$

where $w(k)$ is white noise.

$u(k)$ is PE of any order.

Persistence of excitation for ARMA model identification



$$\phi(k) = \begin{bmatrix} -y(k) \\ \vdots \\ -y(k-n+1) \\ u(k-d) \\ \vdots \\ u(k-m-d) \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a\\ b\\ c\end{matrix}}\right\} n \\ \left.\vphantom{\begin{matrix} a\\ b\\ c\end{matrix}}\right\} m+1 \end{matrix}$$

We need to find what conditions must the input sequence *u(k)* satisfy so that $\phi(k)$ is persistently exciting.

$$E\left\{\phi(k)\phi^T(k)\right\} \succ 0$$

PE in ARMA models

Given:

$$y(k) = \frac{q^{-d}B(q^{-1})}{A(q^{-1})}u(k) \qquad \phi(k) = \begin{bmatrix} -y(k) \\ \vdots \\ -y(k-n+1) \\ u(k-d) \\ \vdots \\ u(k-m-d) \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a\\ b\\ c\end{matrix}}\right\} n \\ \left.\vphantom{\begin{matrix} a\\ b\\ c\end{matrix}}\right\} m+1 \end{matrix}$$

- *u(k)* is bounded
- $A(q^{-1})$ is Schur
- $A(q^{-1})$ and $B(q^{-1})$ are co-prime

$$\boxed{u(k) \text{ is PE of order } n+m+1 \iff E\left\{\phi(k)\phi^T(k)\right\} \succ 0}$$

PE in ARMA models

Theorem:

Consider the parameter estimation of the ARMA system using the LS estimation algorithm. If

- $A(q^{-1})$ is Schur
- $A(q^{-1})$ and $B(q^{-1})$ are co-prime
- $\boxed{u(k) \text{ is PE of order } n+m+1}$

Parameter estimates convergence to the true values

Example

- Plant:

$$y(k) = \frac{q^{-1}\,0.1(1+0.5q^{-1})}{(1+0.9q^{-1})(1+0.8q^{-1})}u(k)$$

$$y(k+1) = \theta^T\phi(k)$$

$$\theta = \begin{bmatrix} 1.7 \\ 0.72 \\ 0.1 \\ 0.05 \end{bmatrix} \in \mathcal{R}^4 \qquad \phi(k) = \begin{bmatrix} -y(k) \\ -y(k-1) \\ u(k) \\ u(k-1) \end{bmatrix} \in \mathcal{R}^4$$

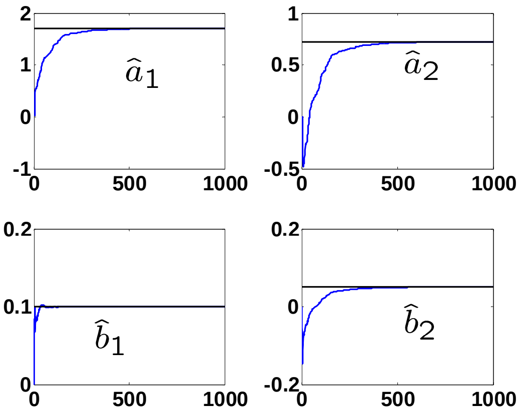
- We need *u(k)* to be a PE sequence of order 4 to guarantee parameter convergence

Example: Input Random Noise

u(k) **: zero mean uniform white noise between [-1,1]**

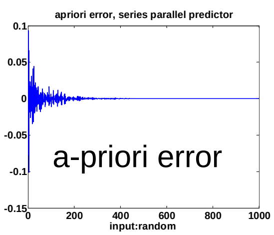
$\boxed{u(k) \text{ is PE of any order.}}$

parameter convergence



$F(0) = 100 * I_4$

$\lambda_1 = 0.99 \qquad \lambda_2 = 1$

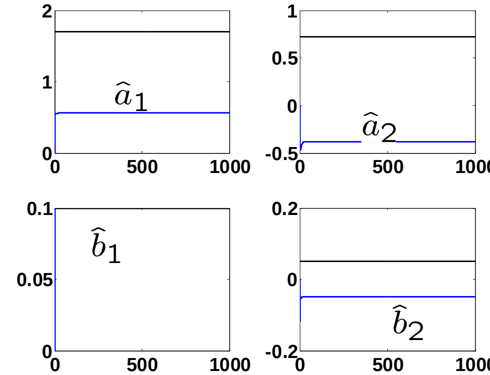


Example: Step Input

***u(k)* = 2*1(t)**

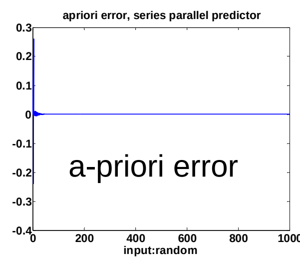
$\boxed{u(k) \text{ is PE of order 1.}}$

NO parameter convergence



$F(0) = 100 * I_4$

$\lambda_1 = 0.99 \qquad \lambda_2 = 1$

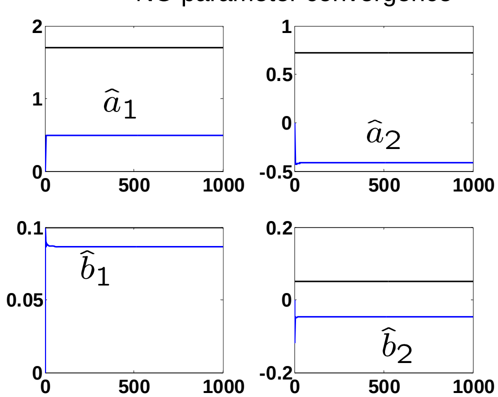


Example: Sinusoidal input – 1 frequency

***u(k)* = 2*sin(t)**

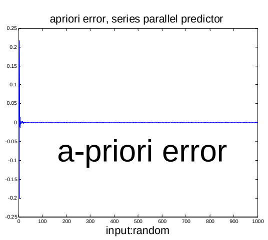
$\boxed{u(k) \text{ is PE of order 2.}}$

NO parameter convergence



$F(0) = 100 * I_4$

$\lambda_1 = 0.99 \qquad \lambda_2 = 1$

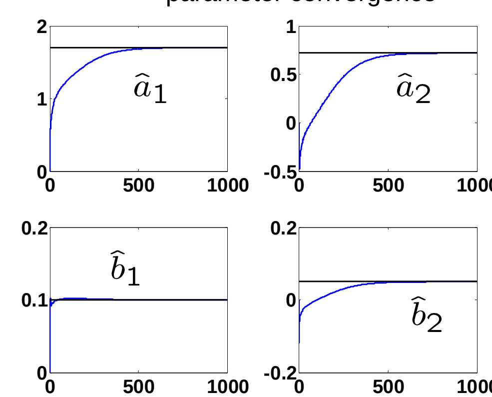


Example: Sinusoidal input – 2 frequencies

***u(k)* = 2*sin(t)+2cos(2*t)**

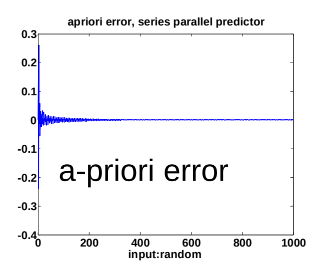
$\boxed{u(k) \text{ is PE of order 4.}}$

parameter convergence



$F(0) = 100 * I_4$

$\lambda_1 = 0.99 \qquad \lambda_2 = 1$



Derivation of Results

1. Determine conditions on the input sequence

$$u(k) \in \mathcal{R}$$

- For the parameter convergence of a Moving Average (MA) model

$$y(k) \;=\; q^{-\mathrm{d}}\, B(q^{-1})\, u(k)$$

- For the parameter convergence of an ARMA model

$$A(q^{-1})\, y(k) \;=\; q^{-\mathrm{d}}\, B(q^{-1})\, u(k)$$

Statistical Interpretation of LS Estimation

Stochastic Model

$$\boxed{y(k) = \phi^T(k-1)\,\theta + \epsilon(k)}$$

Where

- $y(k)$ observed output
- $\epsilon(k)$ **zero-mean noise**
- $\phi(k) = \begin{bmatrix} \phi_1(k) & \cdots & \phi_n(k) \end{bmatrix}^T$ regressor
- $\theta = \begin{bmatrix} \theta_1 & \cdots & \theta_n \end{bmatrix}^T$ unknown parameter vector

Statistical Interpretation of LS Estimation

Assumptions:

- $E\{\epsilon(k)\} = 0$ zero-mean
- Independence or orthogonality:

$$E\{\phi(k)\epsilon(k)\} = E\{\phi(k)\}E\{\epsilon(k)\} = 0$$

- Ergodicity

$$E\{\phi(k)\phi(k)^T\} = \lim_{N\to\infty}\left\{\frac{1}{2N+1}\sum_{j=-N}^{j=N}\phi(k+j)\phi^T(k+j)\right\}$$

Statistical Interpretation of LS Estimation

Collect data for k observations:

$$y(k) = \phi^T(k-1)\,\theta + \epsilon(k)$$

$$\underbrace{\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(k) \end{bmatrix}}_{Y(k)} = \underbrace{\begin{bmatrix} \phi_1(0) & \cdots & \phi_n(0) \\ \phi_1(1) & \cdots & \phi_n(1) \\ \cdot & \cdots & \cdot \\ \phi_1(k-1) & \cdots & \phi_n(k-1) \end{bmatrix}}_{\Phi^T(k-1)} \underbrace{\begin{bmatrix} \theta_1 \\ \vdots \\ \theta_n \end{bmatrix}}_{\theta} + \underbrace{\begin{bmatrix} \epsilon(1) \\ \epsilon(2) \\ \vdots \\ \epsilon(k) \end{bmatrix}}_{\mathcal{E}(k)}$$

LS Statistical Interpretation

Collect data for k observations:

$$Y(k) = \Phi^T(k-1)\,\theta + \mathcal{E}(k)$$

Where

- $$Y(k) \;=\; \left[\; y(1) \;\; \cdots \;\; y(k) \;\right]^T \in \mathcal{R}^k$$
- $$\Phi(k-1) \;=\; \left[\; \phi(0)\cdots\phi(k-1) \;\right] \in \mathcal{R}^{n\times k}$$
- $$\mathcal{E}(k) \;=\; \left[\; \epsilon(1) \;\; \cdots \;\; \epsilon(k) \;\right]^T \in \mathcal{R}^k$$
- $$\theta = \left[\; \theta_1 \;\; \cdots \;\; \theta_n \;\right]^T \in \mathcal{R}^n$$

LS Statistical Interpretation

$$\Phi(k-1) \;=\; \left[\; \phi(0)\cdots\phi(k-1) \;\right] \in \mathcal{R}^{n\times k}$$

$$= \begin{bmatrix} \phi_1(0) & \cdots & \phi_1(k-1) \\ \phi_2(0) & \cdots & \phi_2(k-1) \\ . & \cdots & . \\ . & \cdots & . \\ \phi_n(0) & \cdots & \phi_n(k-1) \end{bmatrix}$$

Deterministic Least Squares Estimation

Parameter estimate after k observations: $\widehat{\theta}(k)$

$$y(1),\,\cdots,\,y(k)$$
$$\phi(0),\,\cdots,\,\phi(k-1)$$

Which minimizes the following cost functional:

$$V(\widehat{\theta}(k)) = \frac{1}{2}\sum_{j=1}^{k}\left[y(j) - \phi^T(j-1)\,\widehat{\theta}(k)\right]^2$$

Notice that $\widehat{\theta}(k)$ is kept constant in the summation

Deterministic Least Squares Estimation

$\widehat{\theta}(k)$: Parameter estimate which minimizes

$$V(\widehat{\theta}(k))$$

Is given by the **Normal Equation:**

$$\boxed{\Phi(k-1)\Phi(k-1)^T\,\widehat{\theta}(k) = \Phi(k-1)\,Y(k)}$$

LS Statistical Interpretation

Normal equation:

$$\boxed{\Phi(k-1)\Phi(k-1)^T\,\hat{\theta}(k) = \Phi(k-1)\,Y(k)}$$

Stochastic model:

$$\boxed{Y(k) = \Phi^T(k-1)\,\theta + \mathcal{E}(k)}$$

Parameter error vector:

$$\boxed{\tilde{\theta}(k) = \theta - \hat{\theta}(k)}$$

LS Statistical Interpretation

Substitute the stochastic model

$$Y(k) = \Phi^T(k-1)\,\theta + \mathcal{E}(k)$$

Into the normal equation:

$$\Phi(k-1)\Phi(k-1)^T\,\hat{\theta}(k) = \Phi(k-1)\,Y(k)$$

To obtain:

$$\boxed{\Phi(k-1)\Phi^T(k-1)\,\tilde{\theta}(k) = -\Phi(k-1)\mathcal{E}(k)\,.}$$

LS Statistical Interpretation

$$\Phi(k-1)\Phi^T(k-1)\,\tilde{\theta}(k) = -\Phi(k-1)\mathcal{E}(k)\,.$$

Notice that

$$\begin{aligned}\Phi(k-1) &= \left[\ \phi(0)\cdots\phi(k-1)\ \right]\\ \mathcal{E}(k) &= \left[\ \epsilon(1)\ \ \cdots\ \ \epsilon(k)\ \right]^T\end{aligned}$$

Therefore,

$$\boxed{\left\{\sum_{j=0}^{k-1}\phi(j)\phi^T(j)\right\}\tilde{\theta}(k) = -\sum_{j=1}^{k}\phi(j-1)\epsilon(j)}$$

LS Statistical Interpretation

Assume now that the parameter error converges:

$$\boxed{\bar{\theta} = \lim_{k\to\infty}\tilde{\theta}(k)}$$

Multiply by $1/k$ and take limits as $k \to \infty$

$$\lim_{k\to\infty}\left\{\frac{1}{k}\sum_{j=0}^{k-1}\phi(j)\phi^T(j)\right\}\tilde{\theta}(k) = -\lim_{k\to\infty}\left\{\frac{1}{k}\sum_{j=1}^{k}\phi(j-1)\epsilon(j)\right\}$$

$$\boxed{\lim_{k\to\infty}\left\{\frac{1}{k}\sum_{j=0}^{k-1}\phi(j)\phi^T(j)\right\}\bar{\theta} = -\lim_{k\to\infty}\left\{\frac{1}{k}\sum_{j=1}^{k}\phi(j-1)\epsilon(j)\right\}}$$

LS Statistical Interpretation

$$\lim_{k\to\infty}\left\{\frac{1}{k}\sum_{j=0}^{k-1}\phi(j)\phi^T(j)\right\}\bar{\theta} = -\lim_{k\to\infty}\left\{\frac{1}{k}\sum_{j=1}^{k}\phi(j-1)\epsilon(j)\right\}$$

By Ergodicity,

$$\boxed{E\left\{\phi(k)\phi^T(k)\right\}\bar{\theta} \;=\; -E\left\{\phi(k)\epsilon(k+1)\right\}}$$

LS Statistical Interpretation

$$E\left\{\phi(k)\phi^T(k)\right\}\bar{\theta} \;=\; -E\left\{\phi(k)\epsilon(k+1)\right\}$$

If $\phi(k)$ and $\epsilon(k)$ are independent or orthogonal,

$$\begin{aligned} E\left\{\phi(k)\epsilon(k+1)\right\} &= -E\left\{\phi(k)\right\} E\left\{\epsilon(k+1)\right\} \\ &= 0 \end{aligned}$$

Since, $E\left\{\epsilon(k)\right\} = 0$

LS Statistical Interpretation

The parameter error vector satisfies:

$$\boxed{E\left\{\phi(k)\phi^T(k)\right\}\bar{\theta} \;=\; 0}$$

Thus, a sufficient condition for $\bar{\theta} = 0$ is that

$$E\left\{\phi(k)\phi^T(k)\right\} > 0$$ (positive definite)

LS Statistical Interpretation

We now define the Excitation matrix $C_n \in \mathcal{R}^{n\times n}$

$$\begin{aligned} C_n &= E\left\{\phi(k)\phi^T(k)\right\} \qquad \phi(k)\in\mathcal{R}^n \\ &= \lim_{k\to\infty}\left\{\frac{1}{k}\sum_{j=0}^{k-1}\phi(j)\phi^T(j)\right\} \end{aligned}$$

$$C_n = C_n^T \qquad\qquad C_n \geq 0$$

LS Statistical Interpretation

Theorem:

$$y(k) = \phi^T(k-1)\,\theta + \epsilon(k)$$

Under the conditions:

- $E\{\epsilon(k)\} \;=\; 0$
- $E\{\phi(k-1)\epsilon(k)\} \;=\; E\{\,\phi(k-1)\}\;E\{\epsilon(k)\} = 0 = 0$

If the excitation matrix C_n is positive definite,

the parameter error vector of the least square algorithm converges to zero.

$$\bar{\theta} = \lim_{k\rightarrow\infty} \tilde{\theta}(k) = 0$$

Persistence of Excitation (PE)

Persistently exciting regressor: $\phi(k) \in \mathcal{R}^n$

There exist finite constants:

- $0 < m$
- $0 < \rho_1 < \rho_2 < \infty$

For all k

$$\boxed{\rho_2\, I_n \geq \sum_{j=k}^{k+m} \phi(j)\phi(j)^T \geq \rho_1\, I_n}$$

Persistence of Excitation (PE)

Persistently exciting regressor: $\phi(k) \in \mathcal{R}^n$

$$\boxed{\rho_2\, I_n \geq \sum_{j=k}^{k+m} \phi(j)\phi(j)^T \geq \rho_1\, I_n}$$

$$0 < \rho_1 < \lambda_{min}\left\{\sum_{j=k}^{k+m} \phi(j)\phi(j)^T\right\}$$

$$\infty > \rho_2 > \lambda_{max}\left\{\sum_{j=k}^{k+m} \phi(j)\phi(j)^T\right\}$$

for all k
and a fixed m

PE in Moving Average (MA) models

Finite Impulse Response (FIR) model:

$$\begin{aligned} y(k+1) &= B(q^{-1})\,u(k) \\ &= b_o\,u(k) + \cdots + b_{n-1}\,u(k-n+1) \\ &= \theta^T\phi(k) \end{aligned}$$

where

$$\theta \;=\; \left[\; b_o \;\; b_1 \cdots \;\; b_{n-1} \;\right]^T \in \mathcal{R}^n$$

$$\phi(k) \;=\; \left[\; u(k) \;\; u(k-1)\cdots \;\; u(k-n+1) \;\right]^T \in \mathcal{R}^n$$

Conditions for PE in FIR Models

Persistently exciting input sequence:

$u(k)$ Is persistently exciting (PE) of order n

if the regressor vector

$$\phi(k) = \left[\, u(k) \;\; u(k-1) \cdots \;\; u(k-n+1) \,\right]^T \in \mathcal{R}^n$$

is persistently exciting

Conditions for PE in FIR Models

For a persistently exciting input sequence $u(k)$ with regressor

$$\phi(k) = \left[\, u(k) \;\; u(k-1) \cdots \;\; u(k-n+1) \,\right]^T \in \mathcal{R}^n$$

The excitation matrix C_n is a Positive Definite Toeplitz matrix

$$C_n = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ c_{1n} & \cdots & c_{nn} \end{bmatrix} \qquad \begin{aligned} c_{ij} &= c_{ji} \\ &= E\{u(k)\,u(k+i-j)\} \\ &= R_{uu}(i-j) \end{aligned}$$

PE inputs in FIR models

Theorem:

$u(k)$ Is persistently exciting (PE) of order $\boldsymbol{n}$ iff

$$\boxed{U = E\{[A(q^{-1})u(k)]^2\} > 0}$$

<u>for all</u> polynomials $A(q^{-1})$ of order n-1

PE inputs in FIR models

Proof: Let

$$A(q^{-1}) = a_o + a_1 q^{-1} + \cdots + a_{n-1}\, q^{n-1}$$

$$a = \left[\, a_o \cdots a_{n-1} \,\right]^T \in \mathcal{R}^n$$

Then

$$A(q^{-1})\, u(k) = \left[\, a_o \;\; a_1 \;\; \cdots \;\; a_{n-1} \,\right] \begin{bmatrix} u(k) \\ u(k-1) \\ \vdots \\ u(k-n-1) \end{bmatrix}$$

$$\boxed{A(q^{-1})\, u(k) = a^T \phi(k) = \phi^T(k) a}$$

PE inputs in FIR models

Proof:

$$\begin{aligned} U &= E\{[A(q^{-1})u(k)]^2\} \\ &= E\{[a^T\phi(k)]^2\} = E\{a^T\,\phi(k)\phi(k)^T\,a\} \\ &= a^T\,E\{\phi(k)\phi(k)^T\}\,a \\ &= a^T\,C_n\,a \end{aligned}$$

$$\boxed{U > 0 \quad \text{for all} \quad a \in \mathcal{R}^n \quad \Leftrightarrow \quad C_n > 0}$$

PE inputs in FIR models

To determine the PE order of a sequence $u(k)$

1. Find an annihilating polynomial $A(q^{-1})$ of order n such

$$A(q^{-1})u(k) = 0$$

this means that $u(k)$ is at most PE of order n

2. Compute the excitation matrix C_n and verify that it is positive definite.

Conditions for PE in FIR Models

Examples: Step

Consider the unit step input

$$u(k) = 1(k) = \begin{cases} 1 & \text{for} \quad k \geq 0 \\ 0 & \text{for} \quad k < 0 \end{cases} .$$

$$(1 - q^{-1})u(k) = 0$$

Thus, the step input is at most PE of order n = 1.

Conditions for PE in FIR Models

Examples: Step

Since

$$C_1 = \lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} 1 = 1\,,$$

$$\boxed{\text{The step input is PE or order 1.}}$$

Conditions for PE in FIR Models

Examples: Step

Since

$$C_1 = \lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} 1 = 1\,,$$

The step input is PE or order 1.

Conditions for PE in FIR Models

Examples: Sinusoid input

Consider the pure sinusoid input

$$u(k) = \sin(\omega\, k)\,. \qquad 0 < \omega < \pi$$

Since

$$(1 - 2\cos(\omega) q^{-1} + q^{-2}) u(k) = 0$$

the pure sinusoid input is at most PE of order n = 2.

Conditions for PE in FIR Models

Examples: Sinusoid input

Let $\phi(k) = \begin{bmatrix} u(k) & u(k-1) \end{bmatrix}^T$. $\qquad u(k) = \sin(\omega\, k)\,.$

$$\begin{aligned} C_2 &= \lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} \phi(j)\phi(j)^T \\ &= \lim_{k\to\infty} \frac{1}{k} \begin{bmatrix} \sum_{j=1}^{k} u(j)^2 & \sum_{j=1}^{k} u(j)u(j-1) \\ \sum_{j=1}^{k} u(j)u(j-1) & \sum_{j=1}^{k} u(j-1)^2 \end{bmatrix} \\ &= \lim_{k\to\infty} \frac{1}{k} \begin{bmatrix} \sum_{j=1}^{k} \sin^2(\omega\, j) & \sum_{j=1}^{k} \sin(\omega\, j)\sin(\omega\,(j-1)) \\ \sum_{j=1}^{k} \sin(\omega\, j)\sin(\omega\,(j-1)) & \sum_{j=1}^{k} \sin^2(\omega\,(j-1)) \end{bmatrix} \end{aligned}$$

$$C_2 = \frac{1}{2} \begin{bmatrix} 1 & \cos(\omega) \\ \cos(\omega) & 1 \end{bmatrix} > 0 \qquad 0 < \omega < \pi$$

Conditions for PE in FIR Models

Examples: Sinusoid input

Since

$$(1 - 2\cos(\omega) q^{-1} + q^{-2}) u(k) = 0$$

and

$$C_2 = \frac{1}{2} \begin{bmatrix} 1 & \cos(\omega) \\ \cos(\omega) & 1 \end{bmatrix} > 0$$

The pure sinusoid input is PE of order $n = 2$.

Conditions for PE in FIR Models

Examples: Sum of Sinusoids

Consider an input that is a sum of m sinusoids, with m distinct frequencies

$$u(k) = \sum_{i=1}^{m} \sin(\omega_i\, k)\,. \qquad \begin{matrix} 0 < \omega_i < \pi \\ \omega_i \neq w_j \end{matrix}$$

$$\boxed{u(k) \text{ is PE of order } n = 2m.}$$

Conditions for PE in FIR Models

Examples: Random process

Consider a colored random process

$$u(k) = G(q)\, w(k)$$

where $w(k)$ is white noise.

$$\boxed{u(k) \text{ is PE of any order.}}$$

PE in Filtered Signals

Filtered signals:

- $u(k)$ be PE of order n
- Let $$\boxed{v(k) = A(q^{-1})u(k)}$$
- $A(q^{-1})$ is a polynomial of degree $m < n$

$$\boxed{\begin{matrix} v(k) \text{ is PE of order } r. \\ n - m \leq r \leq n \end{matrix}}$$

PE in Filtered Signals

Filtered signals:

- $u(k)$ be PE of order n
- Let $$\boxed{v(k) = \frac{1}{A(q^{-1})}u(k)}$$
- $A(q^{-1})$ is a Schur polynomial

$$\boxed{v(k) \text{ is also PE of order } n.}$$

ARMA Model

Consider the following system

$$A(q^{-1})y(k) = q^{-1}\,B(q^{-1})u(k)$$

Where

- $u(k)$ known ***bounded*** input
- $y(k)$ measured output

ARMA Model

Consider the following system

$$A(q^{-1})y(k) = q^{-1}\,B(q^{-1})u(k)$$

Where

$$A(q^{-1}) = 1 + a_1 q^{-1} + \cdots + a_n q^{-n}$$ **(Schur)**

$$B(q^{-1}) = b_o + b_1 q^{-1} + \cdots + b_m q^{-m}$$

- Orders n and m are ***known***
- a*'s* and b*'s* are ***unknown*** but ***constant*** coefficients

ARMA Model

ARMA model can be written as:

$$\begin{aligned} y(k+1) &= -\sum_{i=1}^{n} a_i\, y(k-i+1) + \sum_{i=0}^{m} b_i\, u(k-i) \\ &= \theta^T \phi(k) \end{aligned}$$

Where:

$$\theta = \left[\begin{array}{ccccc} a_1 & \cdots & a_n & b_o \cdots & b_m \end{array}\right]^T \in \mathcal{R}^{n+m+1}$$

$$\phi(k) = \left[\begin{array}{cccccc} -y(k) & \cdots & -y(k-n) & u(k) & \cdots & u(k-m) \end{array}\right]^T$$

PE in ARMA models

ARMA model:

$$A(q^{-1})y(k) = q^{-1}\,B(q^{-1})u(k)$$

where

$$y(k+1) = \theta^T \phi(k)$$

$$\theta \in \mathcal{R}^{n+m+1} \qquad \phi(k) \in \mathcal{R}^{n+m+1}$$

PE in ARMA models

ARMA model:

$$A(q^{-1})y(k) = q^{-1}\,B(q^{-1})u(k)$$

where

$$y(k+1) \;=\; \theta^T\,\phi(k)$$

$$\theta \in \mathcal{R}^{n+m+1} \qquad\qquad \phi(k) \in \mathcal{R}^{n+m+1}$$

Parameter estimates convergence to the true values if the regressor $\phi(k)$ is PE of order ***n+m+1***

PE in ARMA models

Theorem:

Consider the parameter estimation of the ARMA system using the LS estimation algorithm. If

- $A(q^{-1})$ is Schur
- $A(q^{-1})$ and $B(q^{-1})$ are co-prime
- $\boxed{u(k) \text{ is PE of order } n+m+1}$

Parameter estimates convergence to the true values

PE in ARMA models - Proof

Assume that the parameter error converges:

$$\bar{\theta} = \lim_{k\to\infty} \tilde{\theta}(k)$$

Define: the LS output estimation error by

$$e(k) = \phi(k-1)^T\bar{\theta}$$

Notice that,

$$\begin{aligned} E\{e^2(k)\} &= \bar{\theta}^T\,E\{\phi(j-1)\phi(j-1)^T\}\,\bar{\theta} \\ &= \bar{\theta}^T\,C_n\,\bar{\theta} \end{aligned}$$

PE in ARMA models - Proof

We know that, under no noise assumption

$$E\{e^2(k)\} \;=\; 0$$

Therefore, since

$$E\{e^2(k)\} \;=\; \bar{\theta}^T\,C_n\,\bar{\theta} = 0$$

To prove persistence of excitation, we need to show that

$$E\{e^2(k)\} = 0 \iff \bar{\theta} = 0$$

PE in ARMA models - Proof

Thus, we need to show that

$$\boxed{E\{e^2(k)\} = 0 \iff \bar{\theta} = 0}$$

Notice that,

$$e(k) = q^{-1}\,\bar{B}(q^{-1})\,u(k) - \bar{A}(q^{-1})y(k)\,,$$

Where,

$$\bar{A}(q^{-1}) = A(q^{-1}) - \hat{A}(q^{-1})$$

$$\bar{B}(q^{-1}) = B(q^{-1}) - \hat{B}(q^{-1})$$

PE in ARMA models - Proof

From

$$e(k) = q^{-d}\,\bar{B}(q^{-1})\,u(k) - \bar{A}(q^{-1})y(k)\,,$$

$$y(k) = \frac{q^{-1}B(q^{-1})}{A(q^{-1})}\,u(k)$$

We obtain

$$\begin{aligned} e(k) &= q^{-1}\,\bar{B}(q^{-1})\,u(k) - \bar{A}(q^{-1})\frac{q^{-1}\,B(q^{-1})}{A(q^{-1})}u(k) \\ &= q^{-1}\left[\bar{B}(q^{-1})\,A(q^{-1}) - \bar{A}(q^{-1})\,B(q^{-1})\right]\frac{1}{A(q^{-1})}\,u(k)\,. \end{aligned}$$

PE in ARMA models - Proof

Therefore,

$$e(k) = q^{-1}\underbrace{\left[\bar{B}(q^{-1})\,A(q^{-1}) - \bar{A}(q^{-1})\,B(q^{-1})\right]}_{\text{Polynomial of order } \boldsymbol{n+m}}\underbrace{\frac{1}{A(q^{-1})}\,u(k)}_{v(k)}\,.$$

PE in ARMA models - Proof

$$e(k) = q^{-1}\underbrace{\left[\bar{B}(q^{-1})\,A(q^{-1}) - \bar{A}(q^{-1})\,B(q^{-1})\right]}_{\text{Polynomial of order } \boldsymbol{n+m}}\underbrace{\frac{1}{A(q^{-1})}\,u(k)}_{v(k)}\,.$$

Notice that since, $A(q^{-1})$ is Schur and

$$v(k) = \frac{1}{A(q^{-1})}u(k)$$

$$\boxed{\begin{array}{c} u(k) \text{ is PE of order } n+m+1 \\ \Updownarrow \\ v(k) \text{ is PE of order } n+m+1 \end{array}}$$

PE in ARMA models - Proof

$$e(k) \;=\; q^{-1}\,\underbrace{\left[\bar{B}(q^{-1})\,A(q^{-1}) - \bar{A}(q^{-1})\,B(q^{-1})\right]}_{\text{Polynomial of order } n+m}\,\underbrace{\frac{1}{A(q^{-1})}\,u(k)}_{v(k)}\,.$$

- $v(k)$ is PE of order $n+m+1$
- $e(k)$ is PE of order ≥ 1 **unless**

$$\left[\bar{B}(q^{-1})\,A(q^{-1}) - \bar{A}(q^{-1})\,B(q^{-1})\right] = 0$$

PE in ARMA models - Proof

Thus, we have shown that if

$u(k)$ is PE of order $n+m+1$,

$$\boxed{E\{e^2(k)\} = 0 \iff \left[\bar{B}(q^{-1})\,A(q^{-1}) - \bar{A}(q^{-1})\,B(q^{-1})\right] = 0}$$

We now need to show that

$$\left[\bar{B}(q^{-1})\,A(q^{-1}) - \bar{A}(q^{-1})\,B(q^{-1})\right] = 0 \quad \iff \bar{\theta} = 0$$

PE in ARMA models - Proof

Consider the Diophantine equation

$$\left[\bar{B}(q^{-1})\,A(q^{-1}) - \bar{A}(q^{-1})\,B(q^{-1})\right] = 0$$

- where $A(q^{-1})$ and $B(q^{-1})$ are co-prime

and:

$$\bar{A}(q^{-1}) = A(q^{-1}) - \hat{A}(q^{-1})$$

$$\bar{B}(q^{-1}) = B(q^{-1}) - \hat{B}(q^{-1})$$

PE in ARMA models - Proof

$$\left[\bar{B}(q^{-1})\,A(q^{-1}) - \bar{A}(q^{-1})\,B(q^{-1})\right] = 0$$

This equation can be written as follows:

$$D\,\bar{\theta}^* = 0$$

$$\bar{\theta}^* = \left[\;\bar{b}_o \cdots \;\; \bar{b}_m \;\; -\bar{a}_1 \;\; \cdots \;\; -\bar{a}_n\;\right]^T \in \mathcal{R}^{n+m+1}$$

and:

$$\bar{a}_i = a_i - \hat{a}_i$$

$$\bar{b}_i = b_i - \hat{b}_i$$

PE in ARMA models - Proof

$$D\,\bar{\theta}^{*} = 0$$

$$D = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & b_0 & 0 & 0 & \cdots & 0 \\
a_1 & 1 & 0 & \cdots & 0 & b_1 & b_0 & 0 & \cdots & 0 \\
a_2 & a_1 & 1 & \cdots & 0 & b_2 & b_1 & b_0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n-1} & a_{n-2} & a_{n-3} & \cdots & 1 & b_m & b_{m-1} & b_{m-2} & \cdots & b_o \\
a_n & a_{n-1} & a_{n-2} & \cdots & a_1 & 0 & b_m & b_{m-1} & \cdots & b_1 \\
0 & a_n & a_{n-1} & \cdots & a_2 & 0 & 0 & b_m & \cdots & b_2 \\
0 & 0 & a_n & \cdots & a_3 & 0 & 0 & 0 & \cdots & b_3 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & a_n & 0 & 0 & 0 & \cdots & b_m
\end{bmatrix}$$

m $\qquad\qquad$ $n+1$

PE in ARMA models - Proof

$$D\,\bar{\theta}^{*} = 0$$

$A(q^{-1})$ and $B(q^{-1})$ are co-prime

$\Downarrow$

D is nonsingular and $\bar{\theta}^{*} = 0$

Therefore, when $u(k)$ is PE of order $n + m + 1$

Parameter estimates convergence to the true values