ME 233 Advance Control II

Lecture 1
Introduction to Probability Theory

(ME233 Class Notes pp. PR1-PR3)

Outline

- Sample Space and Events
- Probability function
- Discrete Random Variables
- Probability mass function, expectation and variance

Sample Space and Events

Assume:

- We do an experiment many times.
  - Each time we do an experiment we call that a ***trial***
- The outcome of the experiment may be different at each trial.

ω_i : The i^{th} possible outcome of the experiment

Sample Space and Events

Sample Space Ω **:**

The space which contains all possible outcomes of an experiment.

$$\Omega = \{\omega_1, \omega_2, \cdots, w_n\}$$

ω_i **:** The i^{th} possible outcome of the experiment

Each outcome is an element of Ω

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Example: Dice

Experiment:

A situation whose ***outcome*** depends on chance

- throwing a die once

Sample Space Ω

The set of <u>all possible</u> ***outcomes*** of an experiment

Ω = {⚀, ⚁, ⚂, ⚃, ⚄, ⚅}

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Events

Event S_j :

Is a subset of the union of the sample space Ω and the empty set ϕ

If a sample space has n outcomes:

$$\Omega = \{\omega_1, \omega_2, \cdots, w_n\}$$

There are 2^n events:

$$\mathcal{S} = \{S_1, \cdots, S_{2^n}\}$$

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Probability - events

Experiment: throwing a die once

Ω = {⚀, ⚁, ⚂, ⚃, ⚄, ⚅}

Outcomes: *elements of the sample space* S

Events: *Are <u>subsets</u> of the sample space* S

An event occurs if any of the outcomes in that event occurs.

Empty subsets are ***null*** *or* ***impossible events***

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Probability - events

Experiment: throwing a die once

Ω = {⚀, ⚁, ⚂, ⚃, ⚄, ⚅}

<u>Some events:</u>

- The event E of observing an even number of dots:

  E = {⚁, ⚃, ⚅}

- The event O of observing an odd number of dots:

  O = {⚀, ⚂, ⚄}

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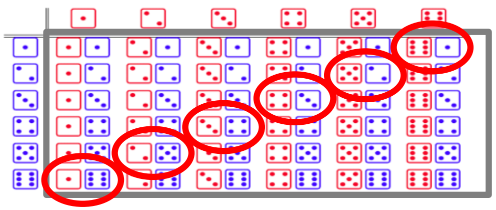
Example: throwing a pair of dice
(one red and one blue)

– the sample space has ***36*** outcomes:

$\Omega =$



- *The event L of obtaining the number **7** is*

$$L = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$$

L occurs if any of the outcomes in L occurs.

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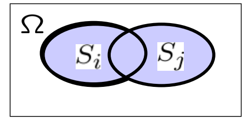
Union, Complement and Intersection

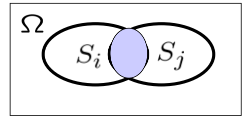
- Union of two events:

$$S_i \cup S_j$$



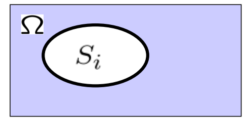
- Intersection of two events:

$$S_i \cap S_j$$



- Complement of an event:

$$\backslash S_i = S_i^c$$



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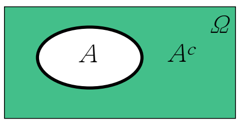
Complement

- The **complement** of an event A, denoted by A^c, is the set of outcomes that are not in A
- A^c occurring means that A *does not occur*

$$A^c = \{\omega \mid \omega \in \Omega \text{ and } \omega \notin A\}$$



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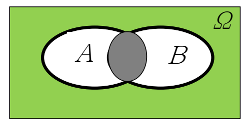
Intersection of two events

- The **intersection** of two events A and B, denoted by $A \cap B$, is the set of outcomes that are in A, **and** B.
- *If* the event $A \cap B$ *occurs, then* ***both*** A and B *occur*



- *Events* A and B are mutually exclusive if they cannot both occur at the same time, i.e. if

$$A \cap B = \emptyset$$

Example of Intersection of two events

Experiment: throwing of a dice once

$$\Omega = \{⚀, ⚁, ⚂, ⚃, ⚄, ⚅\}$$

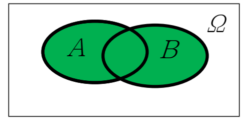
- Events E and O are mutually exclusive

$$E = \{⚁, ⚃, ⚅\} \qquad O = \{⚀, ⚂, ⚄\}$$

$$E \bigcap O = \varnothing$$

Union of two events

- The **union** of two events A and B, denoted by $A \cup B$, is the set of outcomes that are in A, or B, or both
- *If the event* $A \cup B$ *occurs, then either* A *or* B *or both occur*



Probability function

We now consider the probability that a certain event occurs.

Recall: An event occurs if any of the outcomes in that event occurs.

The probability of event A will be denoted by

$$P(A)$$

Probability

A number between 0 and 1, inclusive, that indicates **how likely an event is to occur**.

- An event with probability of 0 is a **null event.**
- An event with probability of 1 is a **certain event.**

- Probability of event A is denoted as $P(A)$.
- The closer $P(A)$ to 1, the more likely is A to happen.

Intuitive Notion of Probability

The probability of event A is

$$P(A) = \frac{\text{Possible outcomes associated with } A}{\text{Total possible outcomes}}$$

(Assumes each outcome is equally likely)

$$0 \leq P(A) \leq 1$$

Assigning Probability - Frequentist approach

- An experiment is repeated n times under essentially identical conditions
- if the event A occurs m times and n is large

$$P(A) \approx \frac{m}{n}$$

Dice example

Experiment: throwing a **fair** die once

Ω = {⚀, ⚁, ⚂, ⚃, ⚄, ⚅} $\Omega = \{1,2,3,4,5,6\}$

- *P(Ω) = 1*
- *P(1) = 1/6, P(3) =1/6, P(6) = 1/6*
- *P(even number) = 3/6 = 1/2*
- *P(odd number) = 3/6 = 1/2*

Probability Space

The probability space is the triple:

$$(\Omega, \mathcal{S}, P)$$

Where

- Ω is the sample space
- $\mathcal{S}$ the set of all possible events
- $P : \mathcal{S} \to [0, 1]$ is the probability function

Probability function

Probability function: $P : \mathcal{S} \rightarrow [0,1]$

Satisfies 3 axioms:

1. $P(S_i) \geq 0, \qquad \forall S_i \in \mathcal{S}$

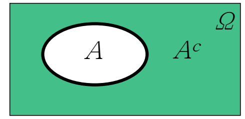
2. $P(\Omega) = 1$

3. $P(S_i \cup S_j) = P(S_i) + P(S_j)$ if $S_i \cap S_j = \emptyset$ where $S_i, S_j \in \mathcal{S}$

Complement

- The **complement** of an event A, denoted by A^c, is the set of outcomes that are not in A
- A^c occurring means that A *does not occur*

$$A^c = \{\omega \mid \omega \in \Omega \text{ and } \omega \notin A\}$$



$$\boxed{P(A^c) = 1 - P(A)}$$

Independent Events

- Two events are **independent** if

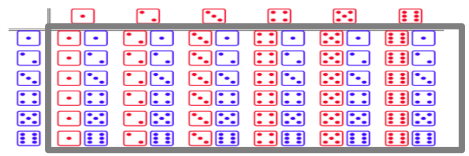
$$\boxed{P(A \cap B) = P(A) \times P(B)}$$

- Intuitively, two events are independent if the events do not influence each other:
  - Event A occurring does not affect the chances of B occurring, and vice versa.

Example of independence

Experiment: throwing a pair of dice (one red and one blue)

$\Omega =$



36 possible outcomes

The probability of throwing a red 1 and a blue 5 is

$$P(1 \cap 5) = 1/36$$

$$= 1/6 \times 1/6 = P(1) \times P(5)$$

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Law of Union

- Recall: If A and B are mutually exclusive

$$\boxed{P(A \cup B) = P(A) + P(B)}$$

- If A and B are not mutually exclusive

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

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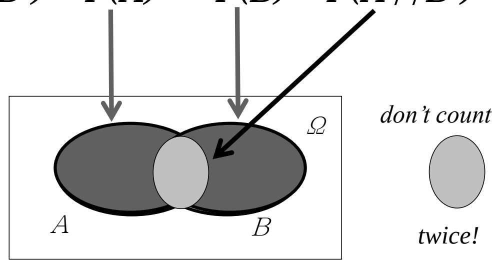
Law of Union

- If A and B are not mutually exclusive

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



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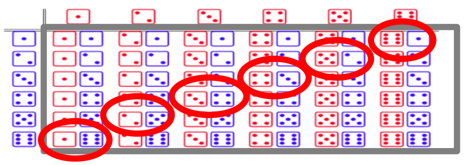
Example

Experiment: throwing a pair of dice (one red and one blue)

$\Omega =$



- $P(L)$ = *the probability of obtaining a* ***7***

$$L = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$$

$$P(L) = P(1,6) + P(2,5) + P(3,4) + P(4,3) + P(5,2) + P(6,1)$$

$$P(L) = 6/36 = 1/6$$

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Joint Probability

Let A and B be two events

$$P(A \cap B)$$

is often called the ***joint probability*** of A and B

$$P(A) \qquad\qquad P(B)$$

are often called the ***marginal probabilities*** of A and B

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Conditional Probability

Let A and B be two events and $P(B) \neq 0$

The conditional probability of event A given that event B has occurred is

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

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Bayes' Rule

Let A and B be two events

$$\boxed{\begin{aligned} P(A|B)P(B) &= P(B|A)P(A) \\ &= P(A \cap B) \end{aligned}}$$

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Independence

The following are equivalent:

1. A and B *are* ***independent***
2. $P(A \cap B) = P(A)\,P(B)$
3. $P(A|B) = P(A)$
4. $P(B|A) = P(B)$

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Array of Probabilities

Let C and D be two chance experiments.

Set of disjoint events associated with C

$$\mathcal{C} = \{C_1, C_2, \cdots C_m\}$$

Set of disjoint events associated with D

$$\mathcal{D} = \{D_1, D_2, \cdots D_n\}$$

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Array of Probabilities

We can construct:

C \ D	Event D_1	Event D_2	$\cdots$	Event D_n	Marginal Probabilities
Event C_1	$P(C_1 \cap D_1)$	$P(C_1 \cap D_2)$	$\cdots$	$P(C_1 \cap D_n)$	$P(C_1) = \sum_{i=1}^{n} P(C_1 \cap D_i)$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
Event C_m	$P(C_m \cap D_1)$	$P(C_m \cap D_1)$	$\cdots$	$P(C_m \cap D_n)$	$P(C_m) = \sum_{i=1}^{n} P(C_m \cap D_i)$
Marginal Probabilities	$P(D_1) = \sum_{i=1}^{m} P(C_i \cap D_1)$	$P(D_2) = \sum_{i=1}^{m} P(C_i \cap D_2)$	$\cdots$	$P(D_n) = \sum_{i=1}^{m} P(C_i \cap D_n)$	$Sum = 1$

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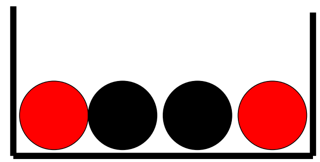
Example:

There are 4 balls in one jar, 2 balls are red and two balls are black.

- A person can remove a ball from the jar two times, without seeing the balls inside the jar.



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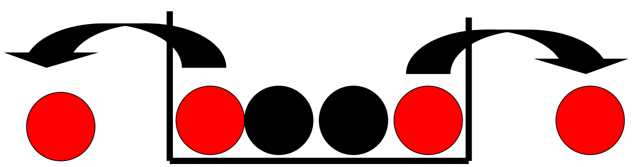
Example:

What is the probability of removing a red ball after having removing a red ball the first time?

To answer this question, lets build the table of probabilities.



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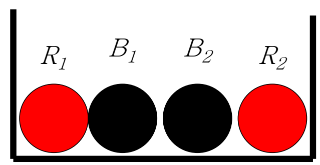
Example:

What is the probability of removing a red ball after having removing a red ball the first time?

To answer this question, lets build the table of probabilities.

Labels:



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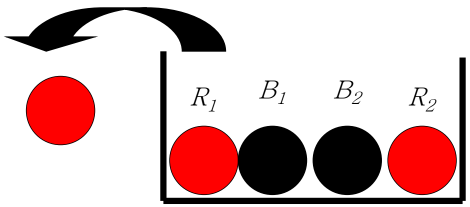
Example:

Probability of picking R_1 the first time?

$$P(R_1) = 1/4$$



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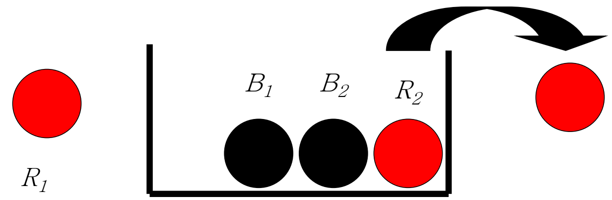
Example:

Probability of picking R_2 with only 3 balls left?

$$P(R_2) = 1/3 \qquad \text{(second time)}$$



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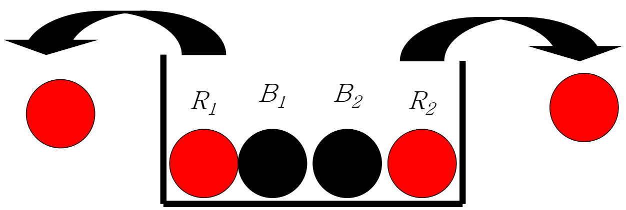
Example:

Probability of picking R_1 the first time and R_2 the second time?

$$P(R_1 \cap R_2) = 1/4 \times 1/3 = 1/12$$



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Example: Array of Probabilities

1 pick \ *2 pick*	R_1	R_2	B_1	B_2	Marginal Probabilities
R_1	0	1/12	1/12	1/12	1/4
R_2	1/12	0	1/12	1/12	1/4
B_1	1/12	1/12	0	1/12	1/4
B_2	1/12	1/12	1/12	0	1/4
Marginal Probabilities	1/4	1/4	1/4	1/4	$Sum = 1$

Probability of pick red balls consecutively

Probability of event *A*: picking <u>a red ball</u> the first time and <u>a red ball</u> the second time?

- Event B: Picking R_1 first and R_2 second
- Event C: Picking R_2 first and R_1 second

<u>Mutually exclusive</u> *events*

$$P(A) = P(B \cup C)$$
$$= P(B) + P(C)$$
$$= \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

Example: Array of Probabilities

1 pick \ 2 pick	*Red*	*Black*	Marginal Probabilities
Red	*1/6*	*1/3*	1/2
Black	*1/3*	*1/6*	1/2
Marginal Probabilities	1/2	1/2	$Sum = 1$

Example:

What is the probability of picking a red ball the second time after having picked a red ball the first time?

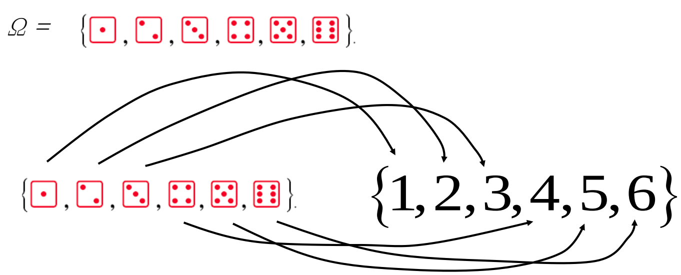
$$P(Red_2|Red_1) = \frac{P(Red_2 \cap Red_1)}{P(Red_1)}$$

$$\boxed{P(Red_2|Red_1) = \frac{1/6}{1/2} = \frac{1}{3}}$$

Discrete random variable

Given a sample Space Ω, a random variable X is a function that assigns to each outcome a unique numerical value.

- <u>Example:</u> throwing of a die once 



Discrete random variable

- Example: throwing of a die once 

$\Omega = \{$  $\}$

- In this case, the random variable X only takes discrete values

$$x_i \in \{1,2,3,4,5,6\}$$

- The discrete random variable X is defined by the **probability mass function**

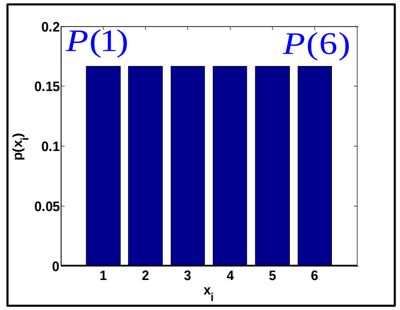
$$P(x_i) = P(X = x_i)$$

the probability that, after throwing a die, X will be equal to x_i

Discrete random variable

- For a **fair die,** the probability mass function of the random variable X is

$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$$



the probability mass function satisfies:

$$\sum_{i=1}^{6} P(x_i) = 1$$

Expected value

- For a discrete random variable X taking on the N possible values

$$x_1,\ x_2,\ x_3,\ \ldots\ ,\ x_k,\ \ldots,\ x_N$$

the **expected value** or **mean** of X is defined by

$$\boxed{E[X] = m_x = \hat{x} = \sum_{k=1}^{N} x_k P(x_k)}$$

$$E[X] = x_1 P(x_1) + x_2 P(x_2) + \cdots + x_N P(x_N)$$

Expected value of a function

- For a discrete random variable X taking on the N possible values

$$x_1,\ x_2,\ x_3,\ \ldots\ ,\ x_k,\ \ldots,\ x_N$$

and the real-valued function f

the **expected value** or **mean** of $Y=f(X)$ is defined by

$$\boxed{E[Y] = E[f(X)] = \sum_{k=1}^{N} f(x_k) P(x_k)}$$

$$E[Y] = f(x_1)P(x_1) + \cdots + f(x_N)P(x_N)$$

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Variance and standard deviation

- For a discrete random variable X taking on the N possible values

  $x_1, x_2, x_3, \ldots, x_k, \ldots, x_N$ and a mean $m_X = \hat{x}$

the **variance** of X is defined by

$$\boxed{E[(X - m_X)^2] = \sigma_X{}^2 = \sum_{k=1}^{N} (x_k - m_X)^2 P(x_k)}$$

where $\boldsymbol{\sigma_X}$ is the standard deviation of X

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Cumulative Distribution Function

- The cumulative distribution function (CDF) for a discrete random variable X is

$$F_X(x) = P(X \le x)$$

Find index k *such that* $\quad x_k \le x < x_{k+1}$

$$F_X(x) = \sum_{j=1}^{k} P(x_j)$$

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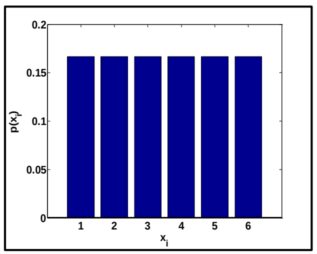
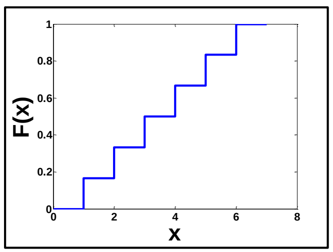
Cumulative Distribution Function

- The cumulative distribution function (CDF) for a discrete random variable X is

$$F_X(x) = \sum_{j=1}^{k} P(x_j) \qquad x_k \le x < x_{k+1}$$





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Sum of two uniform independent random variables

- Let X and Y be 2 independent random variables with constant probability mass function
- Let $Z = X + Y$
- The probability mass function of Z will not be constant

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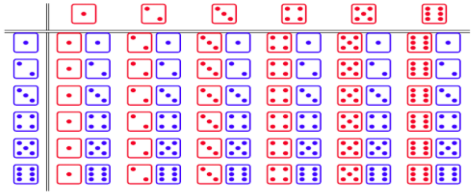
Throwing two fair dice

Experiment: throwing a pair of **fair** dice (red and blue)

$\Omega =$ 

– the sample space has ***36*** outcomes:

– each outcome has a ***1/36*** probability of occurring

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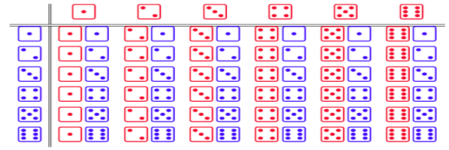
Throwing two fair dice

$\Omega =$ 

- Define the random variable Z associated with the **event** of observing the total number of dots on both dice after each throw

$Z = k$ when the throw results in the number k

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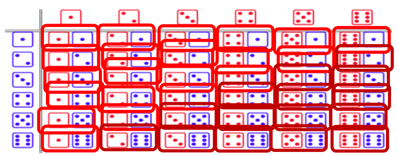
Throwing two fair dice

$\Omega =$ 

number of outcomes ***36***

Z only takes discrete values

$$Z_i \in \{2,3,4,5,6,7,8,9,10,11,12\}$$

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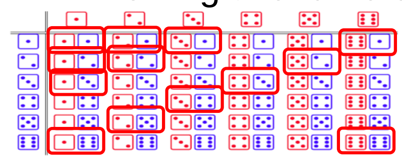
Throwing two fair dice

$\Omega =$ 

probability of each outcome ***1/36***

we now estimate:

$Z=2 \rightarrow \; P(2)=1/36$ $\qquad$ $Z=7 \rightarrow \; P(7)=6/36$

$Z=3 \rightarrow \; P(3)=2/36$ $\qquad$ $Z=12 \rightarrow \; P(12)=1/36$

$Z=4 \rightarrow \; P(4)=3/36$

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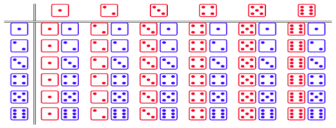
Throwing two fair dice

$\Omega =$ 

The **probability mass function** is

$P(2) = 1/36$ $P(5) = 4/36$ $P(8) = 5/36$ $P(11) = 2/36$

$P(3) = 2/36$ $P(6) = 5/36$ $P(9) = 4/36$ $P(12) = 1/36$

$P(4) = 3/36$ $P(7) = 6/36$ $P(10) = 3/36$

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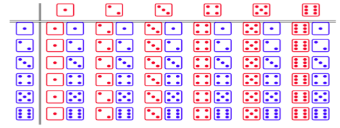
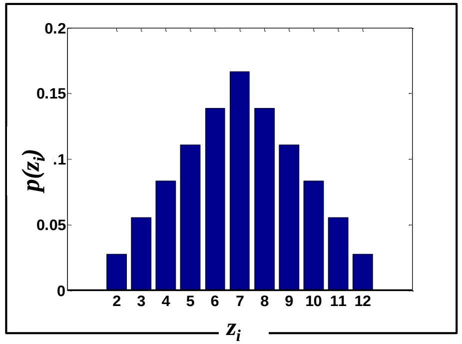
Throwing two fair dice

$\Omega =$ 



the probability mass function satisfies:

$$\sum_{k=2}^{12} P(k) = 1$$