1. Consider a (first-order) discrete-time system described by

$$x(k+1) = ax(k) + w(k) + c$$

where $x(0)$ and c are random variables, and $w(k)$ is a white random process (sequence) with

$$E[x(0)] = 0, \quad E[x^2(0)] = 4, \quad E[w(k))] = 0, \quad E[w(k)w(j)] = W\delta_{kj},$$
$$E[c] = 0, \quad E[c^2] = C, \quad E[x(0)c] = 0, \quad E[x(0)w(k)] = 0, \quad E[w(k)c] = 0$$

c remains constant once an experiment starts.

 (a) Write equations for computing the variance of $x(k)$. Do not forget to give the initial condition for the equations. You do not have to solve the equations.
 (b) Assume that the system is asymptotically stable. Obtain the variance of $x(k)$ at the steady state.

2. A constant value x is repeatedly measured. The measurement process involves noise effects. Assume that the system is described by

$$y(k) = x + v(k)$$

where $y(k)$ is the k-th measurement, $v(k)$ is the measurement noise, and x and $v(k)$ are Gaussian distributed with

$$E[x] = 0, \quad E[x^2] = X_0, \quad E[v(k)] = 0, \quad E[v(k)v(j)] = V\delta_{kj}, \quad E[x(0)v(k)] = 0$$

 (a) Obtain the least square estimate
 $$\hat{x}(k) = E[x|y(0), y(1), ..., y(k)]$$
 and the estimation error covariance.
 (b) Show that in the limit of X_0 approaching to ∞, i.e. no prior information on x, $\hat{x}\,(k)$ and the estimation error covariance, respectively, are asymptotic to
 $$[y(0) + y(1) + ... + y(k)]/(k+1), \quad V/(k+1)$$
 Solve this problem in two ways. In the first approach, regard that $y(0)$, $y(1)$, $y(2)$, etc as elements of a random vector Y_{vec}: i.e.
 $$Y_{vec} = [y(0), y(1), ..., y(k)]^T$$
 and apply the least square formula (LS-6) once. In this case, check part b for $k = 2$. In the second approach, you consider the measurement equation along with the state equation, $x(k+1) = x(k)$, and apply the Kalman filter equations to the problem.

3. Consider the discrete-time system given by Eq. (KF-52) page KF-11 of the course reader. Use MATLAB to work on the following problems.

 (a) Obtain X_{11}.
 (b) Obtain the steady-state Kalman-filter gain and estimation-error covariance matrix for $r = 0.05$ and 0.5.
 (c) Simulate the time response of the system and the Kalman filter. Use rand('normal') to generate input and measurement noise processes. Compare the system state and estimated state by generating time plots similar to figures on page KF-13. Obtain the estimation-error covariance matrix from time average also.
 (d) Utilize the return difference equality and draw a root locus plot for r varying from 0 to ∞.

4. Show that $e_y(k)$ given by Eq. (KF-51) in the course reader is a white random sequence.