UNIVERSITY OF CALIFORNIA AT BERKELEY
Department of Mechanical Engineering
ME233 Advanced Control Systems II
Spring 2010

Homework #5

Assigned: Th., Feb. 25
Due: Th., March 4

1) Consider again the stochastic system

$$Y(k) - 0.5Y(k-1) = W(k) - 0.3W(k-1) \quad E\{Y(0)\} = 0 \quad E\{Y(0)^2\} = 0 \quad E\{Y(0)W(k)\} = 0 \quad (1)$$

where $W(k)$ is a Wide Sense stationary (WSS) zero mean white random signal with unit variance, i.e.

$$m_W = 0 \qquad \Lambda_{WW}(l) = E\{W(k+l)W(k)\} = \delta(l)$$

and $\delta(l)$ is the unit pulse function.

(a) Determine the spectral density function

$$\Phi_{YY}(\omega) = \Lambda_{YY}(z)|_{z=e^{j\omega}}$$

of the random sequence $Y(k)$ and plot $\Phi_{YY}(\omega)$ as a function of $\omega \in [-\pi, \pi]$.

(b) Determine the auto-covariance (auto-correlation) function

$$\Lambda_{YY}(l) = E\{Y(k+l)Y(k)\}$$

Plot $\Lambda_{YY}(l)$ for $l = \{-10, -9, \cdots, 0, \cdots 10\}$ and compare your results with those obtained by matlab. Notice that $\Lambda_{WY}(z)$ will have poles outside and inside the unit circle. Thus, $\Lambda_{YY}(l)$ will be the sum of a causal sequence and anti-casual sequence, i.e.

$$\Lambda_{YY}(l) = \Lambda_{YY}^{C}(l) + \Lambda_{YY}^{A}(l)$$

where $\Lambda_{YY}^{C}(l) = 0$ for $l < 0$ and $\Lambda_{YY}^{A}(l) = 0$ for $l > 0$.

(c) Compute $\Lambda_{YW}(0)$ utilizing equation (1).

Hint: Multiply both sides of Eq. (1) by $W(k)$ and take expectations.

(d) Compute $\Lambda_{YW}(1)$ utilizing equation (1).

Hint: Multiply both sides of Eq. (1) by $W(k-1)$ and take expectations.

(e) Compute $\Lambda_{YY}(0)$ utilizing equation (1).

Hint: From Eq. (1) we have

$$Y(k) = 0.5Y(k-1) + W(k) - 0.3W(k-1) \quad (2)$$

Square both sides of Eq. (2) and take expectations.

2) Let $\{X(k)\}_{-\infty}^{\infty} \in \mathcal{R}^n$ be a WSS random sequence and let

$$\Lambda_{XX}(j) = E\left\{\tilde{X}(k+j)\tilde{X}^T(k)\right\} .$$

Show that the following inequality holds.

$$\text{Trace}\,[\Lambda_{XX}(0)] \geq |\text{Trace}\,[\Lambda_{XX}(j)]|$$

3) Consider a second order discrete time system described by

$$\begin{bmatrix} X_1(k+1) \\ X_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.08 & -1 \\ 0.7 & 0.1 \end{bmatrix} \begin{bmatrix} X_1(k) \\ X_2(k) \end{bmatrix} + \begin{bmatrix} 0.34 \\ 0.3 \end{bmatrix} W(k)$$

$$Y(k) = \begin{bmatrix} 0 & 3 \end{bmatrix} X(k) + V(k)$$

where

- $E\{X(0)\} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $E\{X(0)X^T(0)\} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$
- $W(k)$ and $V(k)$ are uncorrelated Gaussian white noises that satisfy:

  $m_w = E\{W(k)\} = 10$, $E\{V(k)\} = 0$,

  $$E\left\{\begin{bmatrix} W(k+j) - m_w \\ V(k+j) \end{bmatrix} \begin{bmatrix} (W(k) - m_w) & V(k) \end{bmatrix}\right\} = \begin{bmatrix} \Sigma_{ww} & 0 \\ 0 & \Sigma_{vv} \end{bmatrix} \delta(j) = \begin{bmatrix} 1 & 0 \\ 0 & .5 \end{bmatrix} \delta(j)$$

  $$E\left\{\begin{bmatrix} W(k) - m_w \\ V(k) \end{bmatrix} X(0)^T\right\} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(a) Use matlab to plot $m_y(k) = E\{Y(k)\}$ for $k = 0, 1, \cdots$, until $m_y(k)$ reaches its steady state value $\bar{m}_y$.

(b) Using matlab, compute $\Lambda_{XX}(k,0) = E\{(X(k) - m_x(k))(X(k) - m_x(k))^T\}$ utilizing the covariance propagation Lyapunov equation and plot $\Lambda_{YY}(k,0) = E\{(Y(k) - m_y(k))^2\}$ for $k = 0, 1, \cdots$, until $\Lambda_{XX}(k,0)$ reaches its steady state value $\bar{\Lambda}_{YY}(0)$.

(c) Using matlab to compute $\Lambda_{XX}(k,5) = E\{(X(k+5) - m_x(k+5))(X(k) - m_x(k))^T\}$, plot $\Lambda_{YY}(k,5) = E\{(Y(k+5) - m_y(k+5))(Y(k) - m_y(k))\}$ for $k = 0, 1, \cdots$, until it reaches its steady state value $\bar{\Lambda}_{YY}(5)$.

(d) Use the matlab function `dlyap` to compute $\bar{\Lambda}_{XX}(0)$ and then plot the steady state covariances $\bar{\Lambda}_{YY}(j)$ for $j = \{-10, -9, \cdots, 0, 1, \cdots 10\}$.

(e) Let $G(z) = C(zI - A)^{-1}B$, be the transfer function from $W(z)$ to $Y(z)$. Obtain an expression for the (steady state) output spectral density, $\Phi_{YY}(\omega)$ in terms of $G(\omega)$, Σ_{ww} and Σ_{vv}.

(f) Use matlab to plot the output spectral density, $\Phi_{YY}(\omega)$, for $\omega \in [-\pi, \pi]$.

4) A stable minimum phase linear system is excited by a SSS zero-mean white noise, $W(t)$ with $E\{W(t)W(t+\tau)\} = \delta(\tau)$. The spectral density of the output is $Y(t)$ is

$$\Phi_{YY}(\omega) = \frac{0.25\omega^2 + 1}{\omega^4 + 5\omega^2 + 1}$$

Determine the SISO transfer function $G(s) = Y(s)/W(s)$.

5) Consider a first order continuous time plant described by

$$\begin{aligned} \frac{d}{dt}X(t) &= a\,X(t) + b_w\,W(t) \qquad E[X(0)] = x_o \\ Y(t) &= X(t) + V(t) \end{aligned}$$

where $W(t)$ and $V(t)$ are stationary and independent zero mean white random processes with

$$E[W(t)W(t+\tau)] = \sigma_W^2\,\delta(\tau)\,, \qquad E[V(t)V(t+\tau)] = \sigma_V^2\,\delta(\tau)\,, \qquad E[(X(0)-x_o)^2] = \sigma_X^2$$
$$E[W(t)V(t+\tau)] = E[W(t)(X(0)-x_o)] = E[V(t)(X(0)-x_o)] = 0$$

and $a < 0$, $b_w > 0$, $\sigma_W^2 > 0$, $\sigma_V^2 > 0$ and $\sigma_X^2 > 0$ are constants.

Consider now a linear state estimator of the form

$$\frac{d}{dt}\hat{X}(t) = a\,\hat{X}(t) + L[Y(t) - \hat{X}(t)] \qquad E[\hat{X}(0)] = 0$$

and define the state estimation error

$$\tilde{X}(t) = X(t) - \hat{X}(t)$$

(a) Obtain the estimation error dynamic equation.

(b) Assuming that $(a - L) < 0$, obtain an expression of the steady state estimation error variance,

$$\bar{\sigma}_{\tilde{X}}^2 = \lim_{t\to\infty} E[\tilde{X}(t)^2]$$

in terms of the constants a, b_w, $\sigma_W^2 > 0$ and $\sigma_V^2 > 0$ and L.

(c) Find the value of L which minimizes the steady state estimation error variance, and the value of the resulting steady state minimum estimation error variance $\bar{\sigma}_{\tilde{X}}^2$. You have obtained the solution for the minimum variance *linear* steady state estimator for this system.

6) Let $\{Y(k)\}_{k=-\infty}^{\infty}$ be a scalar, stationary, Gaussian sequence with

$$E\{Y(k)\} = 0\,, \qquad \Lambda_{YY}(j) = E\{Y(k+j)Y(k)\} = \sigma_j\,.$$

where $\sigma_k = \sigma_{-k}$ [1]

We want to obtain the least squares estimate of $Y(k)$, given a fix number of prior measurements (i.e. outcomes) of Y. For a fix integer n, define

$$\hat{y}(k)|_{k-1,\cdots,k-n} = E\{Y(k)|y(k-1),\cdots y(k-n)\}$$

Because $Y(k),\cdots Y(k-n)$ are jointly Gaussian and zero mean, $\hat{y}(k)|_{k-1,\cdots,k-n}$ must be a linear combination of the prior measurements $y(k-1),\cdots y(k-n)$ (why?), i.e.

$$\hat{y}(k)|_{k-1,\cdots,k-n} = \sum_{i=1}^{n} a_i y(k-i)\,. \tag{3}$$

[1] Notice that, in this problem, I am using σ_k to denote the variance and not the standard deviation.

(a) We want to obtain an expression for the coefficients $a_i's$. These coefficients can be obtained by minimizing

$$E\{(Y(k) - \sum_{i=1}^{n} a_i Y(k-i))^2\} \tag{4}$$

and, as a consequence, they satisfy the following equation

$$\begin{bmatrix} \sigma_0 & \sigma_1 & \cdots & \sigma_{n-1} \\ \sigma_1 & \sigma_0 & \cdots & \sigma_{n-2} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \sigma_{n-1} & \sigma_{n-2} & \cdots & \sigma_0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_n \end{bmatrix} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \cdot \\ \cdot \\ \sigma_n \end{bmatrix} \tag{5}$$

Equation (5) is called the Yule-Walker equation. [2] Derive Eq. (5).

(b) Let

$$\tilde{Y}(k) = Y(k) - \hat{y}(k)|_{k-1,\cdots,k-n} \tag{6}$$

and define $\sigma_{\tilde{Y}} = E\{\tilde{Y}^2(k)\}$ Show that

$$\sigma_{\tilde{Y}} = \sigma_o - \sum_{i=1}^{n} a_i \sigma_i.$$

Notice that $\tilde{y}(k)$ satisfies

$$\tilde{y}(k) = H(q^{-1})\, y(k), \tag{7}$$

where $H(q^{-1})$ is a n-order polynomial in q^{-1} and q^{-1} is the one step delay operator, i.e. $y(k-1) = q^{-1}y(k)$. It can be shown that all the roots of the polynomial $q^n\, H(q^{-1})$ are inside the unit circle (you are not asked to show this).

(c) Let

$$Y(z) = \frac{z + 0.2}{(z+0.4)(z+0.8)} W(z) \tag{8}$$

where $w(k)$ is a zero mean, unit variance and white Gaussian random sequence. Determine the filter $H(q^{-1})$ in Eq. (7) when $n = 2$ and calculate σ_0 and $\sigma_{\tilde{y}}$. [3]

(d) Do a matlab simulation to generate the random sequences $y(k)$ in Eq. (8) $\hat{y}(k)|_{k-1,k-2}$ in Eq. (3) and $\epsilon(k)$ in Eq. (6) for $k = \{0, 1, \cdots N\}$, where N is large number . Numerically calculate σ_0 and σ_ϵ using time averaging, e.g.

$$\sigma_\epsilon \approx \frac{1}{(N-M)} \sum_{k=M}^{N} \epsilon(k)^2\,,$$

where $M < N$ is sufficiently large so that $y(k)$ and $\epsilon(k)$ are approximately stationary for $k \geq M$ (i.e. transient effects are minimized).

[2] A very efficient recursive algorithm for calculating the coefficients $a_i's$ exists. It is known as the Levinson algorithm.

[3] You are encouraged to use the matlab function `dlyap` to determine the required covariances, using a state space realization of the transfer function in Eq. (8).