University of California, Berkeley
Department of Mechanical Engineering
ME 233 Advanced Control Systems II

Spring 2014 **Midterm I (March 04 2014)**

Closed book and closed lecture notes; One 8.5 × 11 handwritten summary sheet allowed;
Write down your name and student ID on all pages that you submit for grading.

1. [15 points] Filtering and estimation:

   (a) Consider a continuous-time system

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t)$$
$$y(t) = \begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + 2v(t)$$

   where $x(0) = 0$, $w(t)$ and $v(t)$ are zero-mean independent random processes with $E\left[w^2(t)\right] = 1$ and $E\left[v^2(t)\right] = 1$. Obtain the variance and spectral density of $y(t)$ at the steady state.

   (b) Consider a discrete-time system

$$x(k+1) = Ax(k) + Bu(k) + Bw(k)$$
$$y(k) = Cx(k) + v(k)$$

   Assume now, that the random processes $w(k)$ and $v(k)$ are independent, Gaussian, but not zero-mean, with the statistical properties: $E[w(k)] = w_o$, $E\left[(w(k) - w_o)^2\right] = W_o$, $E[(w(k) - w_o)(w(k+l) - w_o)] = 0 \ \forall l \neq 0$ ($w(k) - w_o$ is white); and $E[v(k)] = v_o$ $E\left[(v(k) - v_o)^2\right] = V_o$, $E[(v(k) - v_o)(v(k+l) - v_o)] = 0 \ \forall l \neq 0$ ($v(k) - v_o$ is white). Assume the initial state is a Gaussian random vector with mean x_o and covariance X_o. Obtain the Kalman Filter.

2. [15 points] Consider a first-order discrete-time system

$$x(k+1) = ax(k) + bu(k) + w(k)$$

   where a, b, $x(k)$, $u(k)$ and $w(k)$ are all scalars. $w(k)$ is the disturbance term.

   (a) If $w(k)$ is a white Gaussian random process with zero-mean and variance W. Assume $x(k)$ is fully accessible, and $x(0)$ is zero-mean, Gaussian, with variance X_o. Obtain the control law to minimize

$$J = E\left\{\frac{1}{2}Sx(N)^2 + \frac{1}{2}\sum_{j=0}^{N-1}\left[x(j)^2 + Ru^2(j)\right]\right\}, \ R > 0$$

   where the expetation is taken over $x(0)$ and $\{w(j)\}_{j=0}^{N-1}$. Write down also the equation of the optimal cost to go and the Riccati equation.

(b) Now assume $w(k)$ is a **deterministic** disturbance. Assume the value of this disturbance is known in the optimization horizon, i.e. $\{w(j): \ 0 \leq j \leq N-1\}$ is known at time $k = 0$. Consider the optimal control to minimize

$$J = \frac{1}{2} S x(N)^2 + \frac{1}{2} \sum_{j=0}^{N-1} \left[x(j)^2 + R u^2(j) \right], \ R > 0$$

Show, by using dynamic programming, that the optimal control law is

$$u(k) = -\frac{1}{R + b^2 P(k+1)} \{ bP(k+1) a x(k) + b [f(k+1) + P(k+1) w(k)] \}$$

where $P(k)$ and $f(k)$, along with $g(k)$, define the optimal cost to go as

$$J_k^o = \frac{1}{2} P(k) x^2(k) + f(k) x(k) + g(k)$$

3. [15 points] Consider a first-order discrete-time system

$$\begin{aligned} x(k+1) &= a x(k) + w(k), \ |a| < 1 \\ y(k) &= x(k) + v(k) \end{aligned}$$

where $w(k)$ and $v(k)$ are independent white Gaussian random processes with zero mean. Their variances are 1 and V respectively. Instead of a Kalman filter, consider a regular observer design

$$\hat{x}(k+1) = a\hat{x}(k) + \beta (y(k) - \hat{x}(k))$$

Assume β is chosen such that the observer dynamics is stable. Obtain the steady-state variance of the estimation error

$$e(k+1) = x(k+1) - \hat{x}(k+1)$$

as a function of a, β, and V.