UNIVERSITY OF CALIFORNIA AT BERKELEY
Department of Mechanical Engineering
ME233 Advanced Control Systems II
Spring 2010

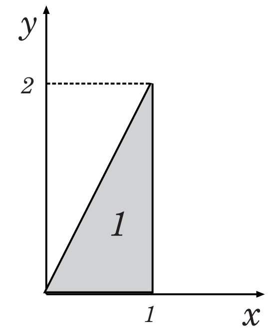
Homework #4

Assigned: Th., Feb. 18
Due: Th., Feb. 25

1) A pair of random variables, X and Y have a joint probability density function (PDF)

$$p_{XY}(x,y) = \begin{cases} 1\,, & 0 \le y \le 2x \quad 0 \le x \le 1 \\ 0\,, & \text{elsewhere} \end{cases}$$

The adjacent figure shows the support of $p_{XY}(x,y)$



(a) Compute the marginal probability density functions

$$p_Y(y) = \int_{-\infty}^{\infty} p_{XY}(x,y)dx \quad p_X(x) = \int_{-\infty}^{\infty} p_{XY}(x,y)dy$$

(b) Compute the marginal mean $m_X = E\{X\} = \int_{-\infty}^{\infty} x\, p_X(x)dx$.

(c) Compute the marginal variance of X.

$$\Lambda_{XX} = \int_{-\infty}^{\infty} (x - m_X)^2\, p_X(x)dx$$

(d) Obtain an expression for the conditional probability density function $p_{X|Y}(x|y)$, i.e. the conditional PDF of X given the outcome $Y = y$ for $0 \le y \le 2$, where

$$p_{X|Y}(x|y) \quad = \quad \frac{p_{XY}(x,y)}{p_Y(y)}$$

(e) Determine the conditional mean $E\{X|Y = y\}$, i.e. the expected value of X given the outcome $Y = y$ for $0 \le y \le 2$.

(f) Determine the conditional mean $E\{X|Y = 0.5\}$.

(g) Notice that the conditional mean $E\{X|Y\}$ can be thought as a function of the random variable Y. Therefore, it is itself a random variable. Lets introduce the notation

$$m_{X|Y}(Y) \quad = \quad E\{X|Y\} = \int_{-\infty}^{\infty} x\, p_{X|Y}(x|Y)dx$$

Prove that the expected value of the conditional mean $m_{X|Y}(Y)$ is equal to the marginal mean of X, i.e.

$$E\{m_{X|Y}(Y)\} = \int_{-\infty}^{\infty} m_{X|Y}(y)\, p_Y(y)dy = m_X = \int_{-\infty}^{\infty} x\, p_X(x)dx$$

and verify this result by computing $E\{m_{X|Y}(Y)\}$ and comparing it to m_X for the example above.

(h) Compute the variance of the conditional mean $m_{X|Y}(Y)$ for the example above.

$$\Lambda_{m_{X|Y}\, m_{X|Y}} = \int_{-\infty}^{\infty} (m_{X|Y}(y) - m_X)^2\, p_Y(y) dy$$

you should find that $\Lambda_{m_{X|Y}\, m_{X|Y}} < \Lambda_{XX}$.

(i) Obtain an expression for the conditional variance of X given Y

$$\Lambda_{X|Y X|Y}(Y) = E\{(X - m_{X|Y}(Y))^2 | Y\} = \int_{-\infty}^{\infty} (x - m_{X|Y}(Y))^2\, p_{X|Y}(x|Y) dx$$

for the example above. Notice that the conditional variance of X given Y, $\Lambda_{X|Y X|Y}(Y)$ is also a random variable.

(j) Finally, compute the expected value of the the conditional variance of X given Y,

$$E\{\Lambda_{X|Y X|Y}(Y)\} = \int_{-\infty}^{\infty} \Lambda_{X|Y X|Y}(y)\, p_Y(y) dy$$

for the example above and verify that

$$\Lambda_{XX} = \Lambda_{m_{X|Y}\, m_{X|Y}} + E\{\Lambda_{X|Y X|Y}(Y)\}\,.$$

We will prove this last result in lecture shortly.

2) Consider the stochastic system

$$Y(k) - 0.5Y(k-1) = W(k) - 0.3W(k-1) \tag{1}$$

where $W(k)$ is a Wide Sense stationary (WSS) zero mean white random signal with unit variance, i.e.

$$m_W = 0 \qquad \Lambda_{WW}(l) = E\{W(k+l)W(k)\} = \delta(l)$$

and $\delta(l)$ is the unit pulse function.

(a) Do a matlab simulation of the response of this system for one sample sequence $w(k)$:

(i) Generate the sample sequence $w(k)$ using `w = randn(N,1);` , where `N` is a large number (e.g. 5000).

(ii) Generate the sample output sequence $y(k)$ using `[y,k] = lsim(sys1,w,k);`. Notice that the vector `k` must be defined.

(iii) Generate and plot the the estimates of the covariances and cross-covariances $\Lambda_{WW}(j)$, $\Lambda_{WY}(j)$, $\Lambda_{YW}(j)$, $\Lambda_{YY}(j)$, for $j = \{-10, -9, \cdots, 0, \cdots 10\}$ using the matlab command `xcov` (e.g. `cov_wy = xcov(w,y,10,'coeff');`. (Read the help on `xcov` to understand what the argument `'coeff'` does) [1].

[1] The matlab function `xcov` is part of the signal processing toolbox. Those of you who do not have access to this toolbox can use a similar function `me233_autocov`, by Richard Conway, and can be downloaded from the ME233 bspace web site.

(b) Determine the cross-covariance (cross-correlation) function

$$\Lambda_{YW}(l) = E\{Y(k+l)W(k)\}$$

and $\Lambda_{YW}(z) = \sum_{l=-\infty}^{\infty} z^{-l}\Lambda_{YW}(l)$. Plot $\Lambda_{YW}(l)$ for $l = \{-10, -9, \cdots, 0, \cdots 10\}$ and compare your results with those obtained by matlab. Notice that $\Lambda_{YW}(l)$ is a casual sequence, i.e. $\Lambda_{YW}(l) = 0$ for $l < 0$ and all the poles of $\Lambda_{YW}(z)$ will be inside the unit circle.

(c) Determine the cross-covariance (cross-correlation) function

$$\Lambda_{WY}(l) = E\{W(k+l)Y(k)\}$$

and $\Lambda_{WY}(z) = \sum_{l=-\infty}^{\infty} z^{-l}\Lambda_{WY}(l)$. Plot $\Lambda_{WY}(l)$ for $l = \{-10, -9, \cdots, 0, \cdots 10\}$ and compare your results with those obtained by matlab. Notice that $\Lambda_{WY}(l)$ is an anti-casual sequence, i.e. $\Lambda_{WY}(l) = 0$ for $l > 0$ and all the poles of $\Lambda_{WY}(z)$ will be outside the unit circle.

(we will continue this problem on the next homework assignment)