UNIVERSITY OF CALIFORNIA AT BERKELEY
Department of Mechanical Engineering
ME233 Advanced Control Systems II

Midterm Examination I **Spring 2008**

Closed Book and Closed Notes. Two 8.5×11 pages of handwritten notes allowed.

Your Name:

Please answer all questions.

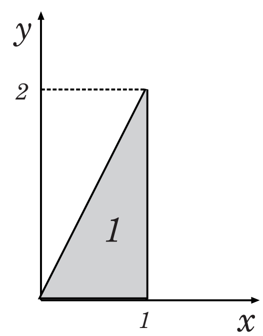
Problem:	1	2	3	Total
Max. Grade:	30	35	35	100
Grade:				

1 Problem

A pair of random variables, X and Y have a joint probability density function (PDF)

$$p_{XY}(x,y) = \begin{cases} 1\,, & 0 \le y \le 2x \qquad 0 \le x \le 1 \\ 0\,, & \text{elsewhere} \end{cases}$$

The figure below shows the support of $p_{XY}(x,y)$



1. Compute the marginal probability density function $p_Y(y) = \int_{-\infty}^{\infty} p_{XY}(x,y)dx$.
2. Compute the marginal mean $m_Y = E\{Y\}$.
3. Obtain an expression for the conditional probability density function $p_{X|y}(x)$, i.e. the conditional PDF of X given the outcome $Y = y$ for $0 \le y \le 2$.
4. Compute the conditional mean $E\{X|Y = 0.5\}$, i.e. the expected value of X given the outcome $Y = 0.5$.

2 Problem

A stochastic system is described as follows

$$\begin{aligned} X(k+1) &= 0.5\,X(k) + 0.5\,W(k) \\ Y(k) &= X(k) + V(k) \end{aligned}$$

where the initial condition $X(0) \sim \mathcal{N}(\hat{x}_o, \sigma^2_{X_o})$ [1] is normal with $\hat{x} \in \mathcal{R}$, and $W(k)$ and $V(k)$ are stationary uncorrelated normal white random sequences that satisfy

$$\begin{bmatrix} W(k) \\ V(k) \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \hat{w} \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2_W & 0 \\ 0 & \sigma^2_V \end{bmatrix} \right), \qquad \begin{aligned} E\{X(0) \begin{bmatrix} \tilde{W}(k) & V(k) \end{bmatrix}\} &= \begin{bmatrix} 0 & 0 \end{bmatrix} \\ \tilde{W}(k) &= W(k) - \hat{w} \end{aligned}$$

Assume the following values

$\hat{x}_o$	σ_{X_o}	$\hat{w}$	σ_W	σ_V
1	2	1	2	0.5

1. Calculate $\hat{x}(2) = E\{X(2)\}$.
2. Calculate $\Lambda_{XX}(2,0) = E\{\tilde{X}(2)\tilde{X}(2)\}$. [2]
3. Calculate $\Lambda_{XX}(0,2) = E\{\tilde{X}(2)\tilde{X}(0)\}$.
4. Calculate $\bar{\Lambda}_{XX}(0) = \lim_{k\to\infty} E\{\tilde{X}^2(k)\}$.
5. Calculate $E\{X(0)|Y(2) = y_2\}$, where $y_2 = 3$.

[1] Remember that if $Z \sim \mathcal{N}(\hat{z}, \Lambda_{ZZ})$ is normal (Gaussian), $E\{Z\} = \hat{z}$ and $E\{(Z-\hat{z})(Z-\hat{z})^T\} = \Lambda_{ZZ}$.

[2] $\tilde{X}(k) = X(k) - E\{X(k)\}$.

3 Problem

Consider the design of an infinite-horizon LQR for the following discrete time LTI system

$$\begin{aligned} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} &= \begin{bmatrix} 1.5 & 1 \\ 1 & 1.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u(k)\,, \qquad x(0) = x_o \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \end{aligned}$$

The transfer function $G(z) = Y(z)/U(z)$ is

$$G(z) = C\,(zI - A)^{-1}\,B = \frac{z + 0.5}{(z - 0.5)\,(z - 2.5)}$$

1. Assume that the cost functional is

$$J = \sum_{k=0}^{\infty} \left\{ y^2(k) + R\,u^2(k) \right\}\,, \qquad R > 0\,.$$

   (a) Determine if there exists a unique stabilizing optimal control law $u^o(k) = -K\,x(k)$ that is the solution to the above problem for $R \in (0, \infty)$.

   (b) Draw the reciprocal root locus and determine:

      (i) The asymptotic values of the eigenvalues of the close loop system for $R \to \infty$.

      (ii) The asymptotic values of the eigenvalues of the close loop system for $R \to 0$.

2. Assume now that the cost functional is

$$J = \sum_{k=0}^{\infty} \left\{ y^2(k) + \sqrt{R}\,y(k)u(k) + R\,u^2(k) \right\}\,, \qquad R > 0\,. \tag{1}$$

   Notice that this cost contains a cross term between the output $y(k)$ and the control action $u(k)$. In order to determine the optimal control law, an auxiliary feedback term in the control law can be used:

$$u(k) = -L\,y(k) + v(k) \tag{2}$$

   where $v(k)$ is considered as the "new" control input and the output injection gain L must be determined so that the problem can be re-casted into a standard LQR problem.

(a) Substitute the control (2) into the cost functional (1) and determine the output injection gain L so that the resulting expression for the cost functional is of the form

$$J = \sum_{k=0}^{\infty} \left\{ \bar{Q}\, y^2(k) + R\, v^2(k) \right\}, \qquad R > 0 \qquad \bar{Q} > 0.$$

(b) Determine if there exists a unique stabilizing optimal control law

$$u^o(k) = -K\, x(k) = -(LC + \bar{K})\, x(k)$$

that is the solution to the above problem for $R \in (0, \infty)$.

Hint: Notice that the following results are true:

- $[A, B]$ is controllable (stabilizable) $\Leftrightarrow$ $[A + BF, B]$ is controllable (stabilizable), for any constant F of appropriate dimensions.

- $[A, C]$ is observable (detectable) $\Leftrightarrow$ $[A + HC, C]$ is observable (detectable), for any constant H of appropriate dimensions.