Several Fundamental Properties of Schur Complements

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Notation: $\bullet$ represents entries which follow from symmetry; $M \succ 0$ (resp. $M \succeq 0$) denotes that M is a positive definite (resp. positive semi-definite) matrix.

If M_{22} is invertible, we define the notation

$$\mathcal{S}\left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right] := M_{11} - M_{12}M_{22}^{-1}M_{21}$$

The matrix $M_{11} - M_{12}M_{22}^{-1}M_{21}$ is called the Schur complement of M_{22} in the matrix $\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$.

Proposition 1 (Recursive Determinant Computation)**.** *If M_{22} is invertible, then*

$$\det\left(\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\right) = \det\left(\mathcal{S}\left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right]\right)\det(M_{22})$$

Proof. Note that

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\begin{bmatrix} I & 0 \\ -M_{22}^{-1}M_{21} & I \end{bmatrix} = \begin{bmatrix} M_{11} - M_{12}M_{22}^{-1}M_{21} & M_{12} \\ 0 & M_{22} \end{bmatrix}.$$

Taking the determinant of both sides of this equation completes the proof. ■

Example 2. *Consider the matrix*

$$P := \begin{bmatrix} I & L \\ R & I \end{bmatrix}.$$

From Proposition 1, we see that $\det(P) = \det(I - LR)$*. Also note that*

$$\det(P) = \det\left(\begin{bmatrix} I & L \\ R & I \end{bmatrix}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} I & L \\ R & I \end{bmatrix}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\right) = \det\left(\begin{bmatrix} I & R \\ L & I \end{bmatrix}\right).$$

Thus, applying Proposition 1 again, we see that $\det(P) = \det(I - RL)$*. Putting these two expressions for* $\det(P)$ *together, we see that* $\det(I - LR) = \det(I - RL)$*.* □

Proposition 3 (Basic Algebraic Properties)**.** *If M_{22} is invertible, then the following hold:*

1. $L\left(\mathcal{S}\left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right]\right)R = \mathcal{S}\left(\left[\begin{array}{c|c} L & 0 \\ \hline 0 & I \end{array}\right]\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\left[\begin{array}{c|c} R & 0 \\ \hline 0 & I \end{array}\right]\right)$
2. $\mathcal{S}\left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right] = \mathcal{S}\left(\left[\begin{array}{c|c} I & 0 \\ \hline 0 & T_L \end{array}\right]\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\left[\begin{array}{c|c} I & 0 \\ \hline 0 & T_R \end{array}\right]\right)$ *whenever T_L and T_R are invertible*
3. $\mathcal{S}\left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right] = \mathcal{S}\left(\left[\begin{array}{c|c} I & Q_L \\ \hline 0 & I \end{array}\right]\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\left[\begin{array}{c|c} I & 0 \\ \hline Q_R & I \end{array}\right]\right)$

4. $\alpha\left(\mathcal{S}\left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22}\end{array}\right]\right) = \mathcal{S}\left(\left[\begin{array}{c|c} \alpha M_{11} & \alpha M_{12} \\ \hline \alpha M_{21} & \alpha M_{22}\end{array}\right]\right)$ *whenever* $\alpha \in \mathbb{C}\backslash\{0\}$

Proof. These four statements are respectively equivalent to

$$\begin{aligned} L(M_{11} - M_{12}M_{22}^{-1}M_{21})R &= LM_{11}R - (LM_{12})M_{22}^{-1}(M_{21}R) \\ M_{11} - M_{12}M_{22}^{-1}M_{21} &= M_{11} - (M_{12}T_R)(T_LM_{22}T_R)^{-1}(T_LM_{21}) \\ M_{11} - M_{12}M_{22}^{-1}M_{21} &= \begin{bmatrix} I & Q_L \end{bmatrix}\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}\end{bmatrix}\begin{bmatrix} I \\ Q_R\end{bmatrix} \\ &\quad - (M_{12} + Q_LM_{22})M_{22}^{-1}(M_{21} + M_{22}Q_R) \\ \alpha(M_{11} - M_{12}M_{22}^{-1}M_{21}) &= \alpha M_{11} - (\alpha M_{12})(\alpha M_{22})^{-1}(\alpha M_{21}) \end{aligned}$$

all of which trivially hold. ■

Example 4. *Consider the discrete algebraic Riccati equation (DARE)*

$$P = \mathcal{S}\left[\begin{array}{c|c} A^TPA + Q & A^TPB + S \\ \hline B^TPA + S^T & B^TPB + R\end{array}\right] \tag{1}$$

and suppose that R is invertible. It is often of interest in controller and filter design to determine whether or not this equation has a solution such that Λ has all of its eigenvalues inside the open unit disk where

$$\Lambda := \mathcal{S}\left[\begin{array}{c|c} A & B \\ \hline B^TPA + S^T & B^TPB + R\end{array}\right].$$

Such a solution, if it exists, is called a stabilizing solution of the DARE. Using the statement (3) of Proposition 3 with $Q_L^T = Q_R = -R^{-1}S^T$, we see with some algebra that

$$\mathcal{S}\left[\begin{array}{c|c} A^TPA + Q & A^TPB + S \\ \hline B^TPA + S^T & B^TPB + R\end{array}\right] = \mathcal{S}\left[\begin{array}{c|c} \hat{A}^TP\hat{A} + \hat{Q} & \hat{A}^TPB \\ \hline B^TP\hat{A} & B^TPB + R\end{array}\right]$$

where $\hat{A} := A - BR^{-1}S^T$ and $\hat{Q} := Q - SR^{-1}S^T$. Moreover, applying statement (3) of Proposition 3 again with $Q_L = 0$ and $Q_R = -R^{-1}S^T$, we see that

$$\mathcal{S}\left[\begin{array}{c|c} A & B \\ \hline B^TPA + S^T & B^TPB + R\end{array}\right] = \mathcal{S}\left[\begin{array}{c|c} \hat{A} & B \\ \hline B^TP\hat{A} & B^TPB + R\end{array}\right]$$

Therefore, the stabilizing solutions of the DARE (1) *are equivalent to the stabilizing solutions of the DARE*

$$P = \mathcal{S}\left[\begin{array}{c|c} \hat{A}^TP\hat{A} + \hat{Q} & \hat{A}^TPB \\ \hline B^TP\hat{A} & B^TPB + R\end{array}\right].$$

Therefore, when R is invertible, we can transform any DARE with $S \neq 0$ into an equivalent DARE (in terms of its stabilizing solutions) for which $S = 0$. □

Proposition 5 (Iterative Schur Complements)**.** *Let M_{33} be invertible and define*

$$\begin{bmatrix} \hat{M}_{11} & \hat{M}_{12} \\ \hat{M}_{21} & \hat{M}_{22}\end{bmatrix} := \mathcal{S}\left[\begin{array}{cc|c} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ \hline M_{31} & M_{32} & M_{33}\end{array}\right].$$

Then

$$\mathcal{S}\left[\begin{array}{c|cc} M_{11} & M_{12} & M_{13} \\ \hline M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33}\end{array}\right] = \mathcal{S}\left[\begin{array}{c|c} \hat{M}_{11} & \hat{M}_{12} \\ \hline \hat{M}_{21} & \hat{M}_{22}\end{array}\right] \tag{2}$$

and the existence of the inverse on either side of the equation is equivalent to the existence of the inverse on the other side of the equation.

Proof. First note that $\hat{M}_{ij} = M_{ij} - M_{i3}M_{33}^{-1}M_{3j}$ for $i, j \in \{1, 2\}$. Applying statement (2) of Proposition 3 with

$$T_L = \begin{bmatrix} I & -M_{23}M_{33}^{-1} \\ 0 & I \end{bmatrix} \qquad\qquad T_R = \begin{bmatrix} I & 0 \\ -M_{33}^{-1}M_{32} & I \end{bmatrix}$$

yields

$$\begin{aligned}
\mathcal{S}\left[\begin{array}{c|cc} M_{11} & M_{12} & M_{13} \\ \hline M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{array}\right] &= \mathcal{S}\left[\begin{array}{c|cc} M_{11} & \hat{M}_{12} & M_{13} \\ \hline \hat{M}_{21} & \hat{M}_{22} & 0 \\ M_{31} & 0 & M_{33} \end{array}\right] \\
&= M_{11} - \begin{bmatrix} \hat{M}_{12} & M_{13} \end{bmatrix} \begin{bmatrix} \hat{M}_{22}^{-1} & 0 \\ 0 & M_{33}^{-1} \end{bmatrix} \begin{bmatrix} \hat{M}_{21} \\ M_{31} \end{bmatrix} \\
&= \hat{M}_{11} - \hat{M}_{12}\hat{M}_{22}^{-1}\hat{M}_{21}
\end{aligned}$$

which completes the proof. ■

In the previous proposition, we showed that the Schur complement of $\begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix}$ in the matrix

$$\tilde{M} := \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

is the same as the Schur complement of $\hat{M}_{22}$ in the matrix

$$\hat{M} := \begin{bmatrix} \hat{M}_{11} & \hat{M}_{12} \\ \hat{M}_{21} & \hat{M}_{22} \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} - \begin{bmatrix} M_{13} \\ M_{23} \end{bmatrix} M_{33}^{-1} \begin{bmatrix} M_{31} & M_{32} \end{bmatrix} .$$

Note that $\hat{M}$ is obtained by taking the Schur complement of M_{33} in $\tilde{M}$. This means that we can evaluate expressions of the form in the left-hand side of (2) by taking successive Schur complements. In particular, we first take the Schur complement of M_{33} in $\tilde{M}$ to obtain $\hat{M}$ and then take the Schur complement of $\hat{M}_{22}$ in $\hat{M}$ to yield the left-hand side of (2).

Example 6. *Suppose that M_{22} is invertible and define*

$$\begin{bmatrix} \hat{M}_{11} & \hat{M}_{13} \\ \hat{M}_{31} & \hat{M}_{33} \end{bmatrix} := \mathcal{S}\left[\begin{array}{cc|c} M_{11} & M_{13} & M_{12} \\ M_{31} & M_{33} & M_{32} \\ \hline M_{21} & M_{23} & M_{22} \end{array}\right].$$

Note that $\hat{M}_{ij} = M_{ij} - M_{i2}M_{22}^{-1}M_{2j}$ for $i, j \in \{1, 3\}$. Using statement (2) of Proposition 3 to permute rows and columns, we obtain that

$$\mathcal{S}\left[\begin{array}{c|cc} M_{11} & M_{12} & M_{13} \\ \hline M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{array}\right] = \mathcal{S}\left[\begin{array}{c|cc} M_{11} & M_{13} & M_{12} \\ \hline M_{31} & M_{33} & M_{32} \\ M_{21} & M_{23} & M_{22} \end{array}\right].$$

Applying Proposition 5 to the right-hand side of this equation then yields

$$\mathcal{S}\left[\begin{array}{c|cc} M_{11} & M_{12} & M_{13} \\ \hline M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{array}\right] = \mathcal{S}\left[\begin{array}{c|c} \hat{M}_{11} & \hat{M}_{13} \\ \hline \hat{M}_{31} & \hat{M}_{33} \end{array}\right].$$

□

Example 7. *Suppose that M_{23} is invertible and define*

$$\begin{bmatrix} \hat{M}_{11} & \hat{M}_{12} \\ \hat{M}_{31} & \hat{M}_{32} \end{bmatrix} := \mathcal{S} \left[\begin{array}{cc|c} M_{11} & M_{12} & M_{13} \\ M_{31} & M_{32} & M_{33} \\ \hline M_{21} & M_{22} & M_{23} \end{array} \right].$$

Using statement (2) of Proposition 3 to swap the second and third rows, we obtain that

$$\mathcal{S} \left[\begin{array}{c|cc} M_{11} & M_{12} & M_{13} \\ \hline M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{array} \right] = \mathcal{S} \left[\begin{array}{c|cc} M_{11} & M_{12} & M_{13} \\ \hline M_{31} & M_{32} & M_{33} \\ M_{21} & M_{22} & M_{23} \end{array} \right].$$

Applying Proposition 5 to the right-hand side of this equation then yields

$$\mathcal{S} \left[\begin{array}{c|cc} M_{11} & M_{12} & M_{13} \\ \hline M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{array} \right] = \mathcal{S} \left[\begin{array}{c|c} \hat{M}_{11} & \hat{M}_{12} \\ \hline \hat{M}_{31} & \hat{M}_{32} \end{array} \right].$$

□

Example 8 (Matrix Inversion Lemma)**.** *Suppose A and D are invertible. Note that, by Proposition 5,*

$$(A + BDC)^{-1} = -\mathcal{S} \left[\begin{array}{c|c} 0 & I \\ \hline I & A + BDC \end{array} \right] = -\mathcal{S} \left[\begin{array}{c|cc} 0 & I & 0 \\ \hline I & A & B \\ 0 & C & -D^{-1} \end{array} \right].$$

Using the results of Example 7, we see that

$$\begin{aligned} (A + BDC)^{-1} &= -\mathcal{S} \left(\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & -D^{-1} \end{array} \right] - \left[\begin{array}{c} I \\ \hline C \end{array} \right] A^{-1} \left[\begin{array}{c|c} I & B \end{array} \right] \right) \\ &= -\left[-A^{-1} - A^{-1}B(-D^{-1} - CA^{-1}B)^{-1}CA^{-1} \right] \\ &= A^{-1} - A^{-1}B(D^{-1} + CA^{-1}B)^{-1}CA^{-1}. \end{aligned}$$

□

Example 9. *In this example, we find the inverse of the matrix $\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ under the assumption that M_{22} is invertible. Note that, by the definition of $\mathcal{S}$ and Proposition 5,*

$$\begin{aligned} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}^{-1} &= \mathcal{S} \left[\begin{array}{cc|cc} 0 & 0 & -I & 0 \\ 0 & 0 & 0 & -I \\ \hline I & 0 & M_{11} & M_{12} \\ 0 & I & M_{21} & M_{22} \end{array} \right] = \mathcal{S} \left[\begin{array}{cc|c} 0 & 0 & -I \\ 0 & M_{22}^{-1} & M_{22}^{-1}M_{21} \\ \hline I & -M_{12}M_{22}^{-1} & M_{11} - M_{12}M_{22}^{-1}M_{21} \end{array} \right] \\ &= \begin{bmatrix} 0 & 0 \\ 0 & M_{22}^{-1} \end{bmatrix} + \begin{bmatrix} I \\ -M_{22}^{-1}M_{21} \end{bmatrix} (M_{11} - M_{12}M_{22}^{-1}M_{21})^{-1} \begin{bmatrix} I & -M_{12}M_{22}^{-1} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ -M_{22}^{-1}M_{21} & I \end{bmatrix} \begin{bmatrix} (M_{11} - M_{12}M_{22}^{-1}M_{21})^{-1} & 0 \\ 0 & M_{22}^{-1} \end{bmatrix} \begin{bmatrix} I & -M_{12}M_{22}^{-1} \\ 0 & I \end{bmatrix}. \end{aligned}$$

□

For $M = M^*$, Define $\nu_+(M)$, $\nu_0(M)$, and $\nu_-(M)$ respectively to be the number of positive, zero, and negative eigenvalues of M (counted with multiplicity). Define the inertia of M to be the ordered triple

$$\mathcal{N}(M) := (\nu_+(M), \nu_0(M), \nu_-(M)).$$

The basic result for the inertia of Hermitian matrices is Sylvester's law of inertia, which states that

$$\mathcal{N}(M) = \mathcal{N}(X^*MX)$$

whenever X is nonsingular (e.g. [1]).

Proposition 10 (Recursive Inertia Computation)**.** *If M_{11} and M_{22} are Hermitian and M_{22} is invertible, then*

$$\mathcal{N}\left(\begin{bmatrix} M_{11} & \bullet \\ M_{21} & M_{22} \end{bmatrix}\right) = \mathcal{N}\left(\mathcal{S}\left[\begin{array}{c|c} M_{11} & \bullet \\ \hline M_{21} & M_{22} \end{array}\right]\right) + \mathcal{N}(M_{22})$$

Proof. Choosing $X = \begin{bmatrix} I & 0 \\ -M_{22}^{-1}M_{21} & I \end{bmatrix}$, we see that

$$\begin{aligned} \mathcal{N}\left(\begin{bmatrix} M_{11} & \bullet \\ M_{21} & M_{22} \end{bmatrix}\right) &= \mathcal{N}\left(X^T \begin{bmatrix} M_{11} & \bullet \\ M_{21} & M_{22} \end{bmatrix} X\right) \\ &= \mathcal{N}\left(\begin{bmatrix} M_{11} - M_{21}^* M_{22}^{-1} M_{21} & 0 \\ 0 & M_{22} \end{bmatrix}\right) \\ &= \mathcal{N}(M_{11} - M_{21}^* M_{22}^{-1} M_{21}) + \mathcal{N}(M_{22}). \end{aligned}$$ ■

We now apply these methods to quadratic optimizations.

Proposition 11 (Quadratic Optimization)**.** *Define*

$$J(x,y) := \begin{bmatrix} x \\ y \end{bmatrix}^* \begin{bmatrix} M_{11} & \bullet \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \qquad J_o(x) := x^* \left(\mathcal{S}\left[\begin{array}{c|c} M_{11} & \bullet \\ \hline M_{21} & M_{22} \end{array}\right]\right) x$$

Where M_{11} and M_{22} are Hermitian. The optimization problem $\inf_y J(x,y)$ (resp. $\sup_y$) has a unique optimizer if and only if $M_{22} \succ 0$ (resp. $\prec$). In this case, $\inf_y J(x,y) = J_o(x)$ (resp. $\sup_y$) and the optimizer is given by

$$y_o = \left(\mathcal{S}\left[\begin{array}{c|c} 0 & I \\ \hline M_{21} & M_{22} \end{array}\right]\right) x.$$

Proof. First note that if M_{22} is singular, then the optimization problems are either unbounded or have multiple optimizers. Now assume that M_{22} is invertible. Note that $J(x,y) = x^*(M_{11} - M_{21}^* M_{22}^{-1} M_{21})x + (y - y_o)^* M_{22}(y - y_o)$. Therefore, $J(x, y_o) = J_o(x)$. Moreover, if $M_{22} \succ 0$ (resp. $\prec$) we see that $J(x,y) > J_o(x,y)$ (resp. $<$) when $y \neq y_o$. ■

Proposition 12 (Quadratic Optimization Involving Schur Complements)**.** *Let M_{33} be invertible and define*

$$J(x,y) := \begin{bmatrix} x \\ y \end{bmatrix}^* \left(\mathcal{S}\left[\begin{array}{cc|c} M_{11} & \bullet & \bullet \\ M_{21} & M_{22} & \bullet \\ \hline M_{31} & M_{32} & M_{33} \end{array}\right]\right) \begin{bmatrix} x \\ y \end{bmatrix}$$

$$J_o(x) := x^* \left(\mathcal{S}\left[\begin{array}{c|cc} M_{11} & \bullet & \bullet \\ \hline M_{21} & M_{22} & \bullet \\ M_{31} & M_{32} & M_{33} \end{array}\right]\right) x$$

where M_{ii} is Hermitian for $i = 1,2,3$. The optimization problem $\inf_y J(x,y)$ (resp. $\sup_y J(x,y)$) has a unique optimizer if and only if

$$\mathcal{S}\left[\begin{array}{c|c} M_{22} & \bullet \\ \hline M_{32} & M_{33} \end{array}\right] \succ 0$$

(resp. $\prec$). In this case, $\inf_y J(x,y) = J_o(x)$ (resp. $\sup_y$) and the optimizer is given by

$$y_o = \left(\mathcal{S}\left[\begin{array}{c|cc} 0 & I & 0 \\ \hline M_{21} & M_{22} & M_{32}^* \\ M_{31} & M_{32} & M_{33} \end{array}\right]\right) x.$$

Proof. Define $\hat{M}_{ij} := M_{ij} - M_{i3}M_{33}^{-1}M_{3j}$ for $i, j \in \{1, 2\}$. Since

$$J(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}^* \begin{bmatrix} \hat{M}_{11} & \bullet \\ \hat{M}_{21} & \hat{M}_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

we apply Proposition 11 to see that the optimization problem $\inf_y J(x, y)$ (resp. $\sup_y J(x, y)$) has a unique optimizer if and only if $\hat{M}_{22} \succ 0$ (resp. $\prec$); the optimal cost and the optimizer are respectively given by $x^*(\hat{M}_{11} - \hat{M}_{21}^* \hat{M}_{22}^{-1} \hat{M}_{21})x$ and $-\hat{M}_{22}^{-1}\hat{M}_{21}x$. By Proposition 5, these are respectively equal to $J_o(x)$ and y_o. ∎

Example 13. *Define*

$$J(x,y,z) := \begin{bmatrix} x \\ y \\ z \end{bmatrix}^* \begin{bmatrix} M_{11} & \bullet & \bullet \\ M_{21} & M_{22} & \bullet \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where M_{ii} is Hermitian for $i = 1, 2, 3$. Consider the optimization problem

$$\sup_y \inf_z J(x, y, z).$$

By Proposition 11, the inner optimization problem has a unique optimizer if and only if $M_{33} \succ 0$; the optimal cost and optimizer are respectively given by

$$\hat{J}(x,y) := \begin{bmatrix} x \\ y \end{bmatrix}^* \left(\mathcal{S} \left[\begin{array}{cc|c} M_{11} & \bullet & \bullet \\ M_{21} & M_{22} & \bullet \\ \hline M_{31} & M_{32} & M_{33} \end{array} \right] \right) \begin{bmatrix} x \\ y \end{bmatrix}$$

$$z^o := \left(\mathcal{S} \left[\begin{array}{cc|c} 0 & 0 & I \\ \hline M_{31} & M_{32} & M_{33} \end{array} \right] \right) \begin{bmatrix} x \\ y \end{bmatrix}.$$

By Proposition 12, the outer optimization problem has a unique optimizer if and only if $\mathcal{S} \left[\begin{array}{c|c} M_{22} & \bullet \\ \hline M_{32} & M_{33} \end{array} \right] \prec 0$*; the optimal cost and optimizer are respectively given by*

$$J^o(x,y) := x^* \left(\mathcal{S} \left[\begin{array}{c|cc} M_{11} & \bullet & \bullet \\ \hline M_{21} & M_{22} & \bullet \\ M_{31} & M_{32} & M_{33} \end{array} \right] \right) x$$

$$y^o := \left(\mathcal{S} \left[\begin{array}{c|cc} 0 & I & 0 \\ \hline M_{21} & M_{22} & M_{32}^* \\ M_{31} & M_{32} & M_{33} \end{array} \right] \right) x.$$

□

References

[1] Roger A. Horn and Charles R. Johnson. *Matrix Analysis*. Cambridge University Press, New York, 1985.