UNIVERSITY OF CALIFORNIA AT BERKELEY
Department of Mechanical Engineering
ME233 Advanced Control Systems II
Spring 2010

Homework #3

Assigned: Th., Feb. 11
Due: Th., Feb. 18

1. The "Monty Hall" three-door problem asks a contestant to choose one of three doors, hoping to find the one door that conceals a prize. The other two doors conceal duds. After the contestant chooses, Monty Hall (the master of ceremonies of the Let's Make a Deal television show) opens one of the doors the player did not choose to reveal a dud. Then the contestant is permitted to stay with their original choice or switch to the other unopened door. The player's probability of getting the prize if she does not switch is 1/3. Determine the player's probability of getting the prize if she switches.

   Hint: Let the doors be called x, y and x. Let C_x be the event that the car is behind door x and so on. Let H_x be the event that the host opens door x and so on. Assuming that you choose door x, the probability that you win a car if you then switch your choice is given by

   $$P\left((H_z \cap C_y) \cup (H_y \cap C_z)\right)$$

   Notice that, by Baye's rule, $P(H_z \cap C_y) = P(C_y)P(H_z|C_y)$.

2. A product is produced by three different factories: A, B, C. Factory A produces 50% of the total production, factory B 25%, and factory C 25%. In factories A and B, one percent of the items produced are defective, while in factory C 3 percent are defective. Calculate: (a) The probability that a (randomly chosen) defective item comes from factory A and (b) The probability that a (randomly chosen) non-defective item comes from factory C.

   Hint: Let $P_X(X = A)$ be the probability that a randomly chosen item comes from factory A, $P_X(X = B)$ from factory B and $P_X(X = C)$ from factory C. Then $P_X(X = A) = 1/2$ and $P_X(X = B) = P_X(X = C) = 1/4$. Let $P_Y(Y = D)$ be the probability that a randomly chosen item is defective. Thus, $P_{Y/X}(Y = D|X = A) = 1/100$ is the conditional probably that a chosen item is defective, given that it comes from factory A. Likewise, $P_{Y/X}(Y = D|X = B) = 1/100$ and $P_{Y/X}(Y = D|X = C) = 3/100$. Using Baye's formula first calculate the array for the joint probability $P_{Y,X}(Y, X)$. For example,

   $$P_{Y,X}(Y = D, X = A) = P_{Y/X}(Y = D|X = A)\, P_X(X = A) = \frac{1}{100}\,\frac{1}{2}\,.$$

   Calculate the marginal probability $P_Y(Y = D)$ and then use Baye's formula again to determine $P_{X/Y}(X = A|Y = D)$. Similar calculations are done for part (b).

3. Consider three independent random variables, X_1, X_2 and X_3. Each of them is uniformly distributed between 0 and 1. Obtain the probability density function (PDF) for

   (a) $X_1 + X_2$

   (b) $X_1 + X_2 + X_3$.

   Hint: Use the result in lecture 3, which states that, if X and Y are two independent random variables and $Z = X + Y$, then the PDF of Z, $P_{_Z}(z)$, is the convolution of the PDF of X, $p_{_X}(x)$, and the PDF of Y, $p_{_Y}(y)$, i.e.

   $$p_{_Z}(z) = \int_{-\infty}^{\infty} p_{_X}(x)\, p_{_Y}(z - x)\, dx$$

4. Let $X \sim N(m_X, \sigma_X^2)$ (i.e. X is a Gaussian random variable with mean m_X and variance σ_X^2). Then the moment generating function of X (i.e. the Fourier transform of the PDF of X) is

$$P_X(j\omega) \quad = \quad \mathcal{F}\{p_X(\cdot)\} = E\{e^{-j\omega X}\} = \exp\left(j\omega m_X \quad - \frac{\sigma_X^2 \omega^2}{2}\right)$$

Now let $X \sim N(m_X, \sigma_X^2)$ and $Y \sim N(m_y, \sigma_Y^2)$ be independent. Show that if $Z = X + Y$, then $Z \sim N(m_X + m_Y, \sigma_X^2 + \sigma_Y^2)$

5. Let $X \sim N(10, 2)$, $V_1 \sim N(0, 1)$ and $V_2 \sim N(0, 2)$ be independent random variables. Assume, that you are trying to make a measurement of X with two instruments. Let $Y = X + V_1$ be the measurement of X using the first instrument and $Z = X + V_2$ be the measurement of X using the second instrument, where V_1 and V_2 are respectively the measurement noises of the instruments.

   (a) Determine $m_{X|Y=11}$, i.e. the conditional expectation of X given that the measurement with the first instrument yielded $Y = 11$.

   (b) Determine $m_{X|Z=9}$, i.e. the conditional expectation of X given that the measurement with the second instrument yielded $Z = 9$.

   (c) Determine $m_{X|(Y=11,Z=9)}$, i.e. the conditional expectation of X given that the measurement with the first instrument yielded $Y = 11$ and the second measurement yielded $Z = 9$.