[1] 1. I_4 is positive definite and $P_+ = I_4$ solves the algebraic Riccati equation

$$A^T PA - P + A^T PB\left[R + B^T PB\right]^{-1} B^T PA + C^T C = 0.$$

The optimal control law is

$$u(k) = -[R + B^T P_+ B]^{-1} B^T P_+ Ax(k) = 0,$$

which means that the open loop control is optimal. To see why this makes sense, let's suppose the initial control of the system is $x(0) = \begin{bmatrix} x_1(0) & x_2(0) & x_3(0) & x_4(0) \end{bmatrix}^T$. Then the output of the system with control $u(k)$ becomes

$$y(0) = x_1(0),\ y(1) = x_2(0),\ y(2) = x_3(0),\ y(3) = x_4(0), \text{ and } y(k) = u(k-4), \text{for } k > 4.$$

It is clear that any nonzero $u(k)$ will make the performance index larger. So the open loop control is optimal.

2. The transfer function from $u(k)$ to $y(k)$ for the system,

$$x(k+1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \varepsilon & 0 & 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(k), \qquad y(k) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x(k) = x_1(k),$$

is given by

$$G(z) = \frac{1}{z^4 - \varepsilon}.$$

This can be obtained by noticing that the state space model is in the controllable canonical form or by the following steps:

$$\begin{cases} x_1(k+1) = x_2(k) \\ x_2(k+1) = x_3(k) \\ x_3(k+1) = x_4(k) \\ x_4(k+1) = \varepsilon x_1(k) + u(k) \\ y(k) = x_1(k) \end{cases}$$

$$\Rightarrow\ y(k) = x_2(k-1) = x_3(k-2) = x_4(k-3) = \varepsilon x_1(k-4) + u(k-4) = \varepsilon y_1(k-4) + u(k-4)$$

$$\Rightarrow\ Y(z) = \varepsilon z^{-4} Y(z) + z^{-4} U(z)$$

$$\Rightarrow\ \frac{Y(z)}{U(z)} = \frac{z^{-4}}{1 - \varepsilon z^{-4}} = \frac{1}{z^4 - \varepsilon}$$

Hence the return difference equality is written as

$$1 + \frac{1}{R} \frac{1}{z^4 - \varepsilon} \cdot \frac{1}{z^{-4} - \varepsilon} = 0,$$

or equivalently,

$$1 - \frac{1}{R} \frac{z^4}{(z^4 - \varepsilon)(z^4 - 1/\varepsilon)} = 0.$$

There are 4 open loop zeros at the origin, 4 poles at $\varepsilon^{1/4}$, $j\varepsilon^{1/4}$, $-\varepsilon^{1/4}$, and $-j\varepsilon^{1/4}$, and 4 poles at $\varepsilon^{-1/4}$, $j\varepsilon^{-1/4}$, $-\varepsilon^{-1/4}$, and $-j\varepsilon^{-1/4}$. The symmetric root locus for $\varepsilon = 0.01$ is shown in Fig. 1.

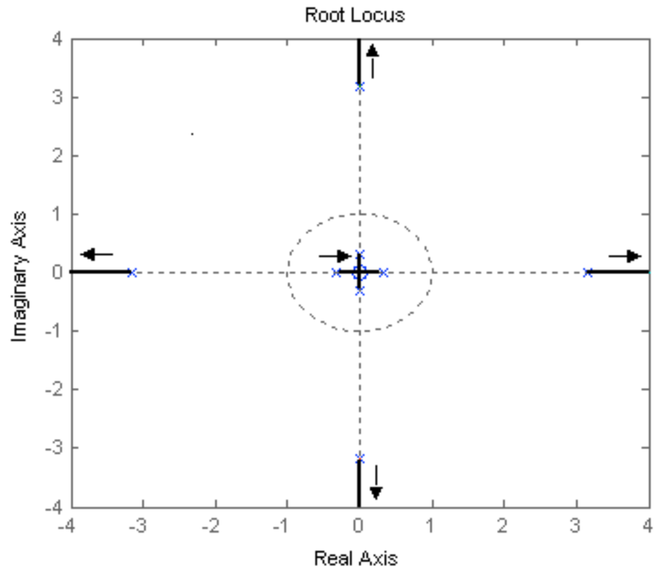

Fig. 1. The symmetric root locus. The arrows point in the direction of decreasing R.

When $\varepsilon \to 0$, the four closed loop poles inside the unit circle all go to the origin and the four symmetric poles go to infinity in the directions of the positive x-axis, the positive y-axis, the negative x-axis, and the negative y-axis, respectively.

[2] 1. The expectations of $x(k)$ and $x_w(k)$ are propogated in the following way:

$$E[x(k+1)] = AE[x(k)] + BC_w E[x_w(k)]$$
$$E[x_w(k+1)] = A_w E[x_w(k)]$$

Since $E[x(k_0)] = 0$ and $E[x_w(k_0)] = 0$, we have $E[x(k)] = 0$ and $E[x_w(k)] = 0$ for any $k \geq k_0$.

Method 1: The covariance matrix of $x(k+1)$ is given by

$$\begin{aligned}
X_{xx}(k+1) &:= E[x(k+1)x^T(k+1)] \\
&= E\{[Ax(k) + Bw(k)][Ax(k) + Bw(k)]^T\} \\
&= E\{[Ax(k) + BC_w x_w(k)][Ax(k) + BC_w x_w(k)]^T\} \\
&= AE\{x(k)x^T(k)\}A^T + AE\{x(k)x_w^T(k)\}C_w^T B^T \\
&\quad + BC_w E\{x_w(k)x^T(k)\}A^T + BC_w E\{x_w(k)x_w^T(k)\}C_w^T B^T \\
&= AX_{xx}(k)A^T + AX_{xx_w}(k)C_w^T B^T + BC_w X_{x_w x}(k)A^T + BC_w X_{x_w x_w}(k)C_w^T B^T
\end{aligned}$$

where $X_{xx_w}(k) = E\{x(k)x_w^T(k)\}$, $X_{x_w x}(k) = E\{x_w(k)x^T(k)\} = X_{xx_w}^T(k)$, and $X_{x_w x_w}(k) = E\{x_w(k)x_w^T(k)\}$.

So we need to figure how $X_{xx_w}(k)$ and $X_{x_w x_w}(k)$ propagate.

For $X_{xx_w}(k)$,

$$
\begin{aligned}
X_{xx_w}(k+1) &:= E[x(k+1)x_w^T(k+1)] \\
&= E\{[Ax(k)+BC_w x_w(k)][A_w x_w(k)+B_w n(k)]^T\} \\
&= AX_{xx_w}(k)A_w^T + BC_w X_{x_w x_w}(k)A_w^T
\end{aligned}
$$

Notice that we have made use of $E[x(k)n(k)]=0$ and $E[x_w(k)n(k)]=0$ for any $k \ge k_0$.

For $X_{x_w x_w}(k)$,

$$
\begin{aligned}
X_{x_w x_w}(k+1) &:= E[x_w(k+1)x_w^T(k+1)] \\
&= E\{[A_w x_w(k)+B_w n(k)][A_w x_w(k)+B_w n(k)]^T\} \\
&= A_w X_{x_w x_w}(k)A_w^T + B_w W B_w^T
\end{aligned}
$$

In conclusion, the covariance matrix of $x(k)$ for any k can be obtained from the following set of equations

$$X_{xx}(k+1) = AX_{xx}(k)A^T + AX_{xx_w}(k)C_w^T B^T + BC_w X_{xx_w}^T(k)A^T + BC_w X_{x_w x_w}(k)C_w^T B^T$$

$$X_{xx_w}(k+1) = AX_{xx_w}(k)A_w^T + BC_w X_{x_w x_w}(k)A_w^T$$

$$X_{x_w x_w}(k+1) = A_w X_{x_w x_w}(k)A_w^T + B_w W B_w^T$$

with initial conditions:

$$X_{xx}(0) = X_o,\ \ X_{xx_w}(0) = 0,\ \text{and}\ \ X_{x_w x_w}(0) = X_{wo}.$$

Method 2: Consider the augmented system:

$$\begin{bmatrix} x(k+1) \\ x_w(k+1) \end{bmatrix} = \begin{bmatrix} A & BC_w \\ 0 & A_w \end{bmatrix}\begin{bmatrix} x(k) \\ x_w(k) \end{bmatrix} + \begin{bmatrix} 0 \\ B_w \end{bmatrix} n(k).$$

Since $n(k)$ is white and $\begin{bmatrix} A & BC_w \\ 0 & A_w \end{bmatrix}$ is stable, we can use equation (PR-50) to get

$$E\left\{\begin{bmatrix} x(k+1) \\ x_w(k+1) \end{bmatrix}\begin{bmatrix} x(k+1) \\ x_w(k+1) \end{bmatrix}^T\right\} = \begin{bmatrix} A & BC_w \\ 0 & A_w \end{bmatrix} E\left\{\begin{bmatrix} x(k) \\ x_w(k) \end{bmatrix}\begin{bmatrix} x(k) \\ x_w(k) \end{bmatrix}^T\right\}\begin{bmatrix} A & BC_w \\ 0 & A_w \end{bmatrix}^T + \begin{bmatrix} 0 \\ B_w \end{bmatrix} W \begin{bmatrix} 0 \\ B_w \end{bmatrix}^T. \quad (1)$$

Notice that $E\left\{\begin{bmatrix} x(k+1) \\ x_w(k+1) \end{bmatrix}\begin{bmatrix} x(k+1) \\ x_w(k+1) \end{bmatrix}^T\right\} = \begin{bmatrix} E[x(k+1)x^T(k+1)] & E[x(k+1)x_w^T(k+1)] \\ E[x_w(k+1)x^T(k+1)] & E[x_w(k+1)x_w^T(k+1)] \end{bmatrix}$ and

$E\left\{\begin{bmatrix} x(k) \\ x_w(k) \end{bmatrix}\begin{bmatrix} x(k) \\ x_w(k) \end{bmatrix}^T\right\} = \begin{bmatrix} E[x(k)x^T(k)] & E[x(k)x_w^T(k)] \\ E[x_w(k)x^T(k)] & E[x_w(k)x_w^T(k)] \end{bmatrix}$. If we use the same definitions of

$X_{xx}(k)$, $X_{xx_w}(k)$ and $X_{x_w x_w}(k)$ and multiply the matrices together in (1), we have

$$\begin{bmatrix} X_{xx}(k+1) & X_{xx_w}(k+1) \\ X^T_{xx_w}(k+1) & X_{x_w x_w}(k+1) \end{bmatrix} =$$

$$\begin{bmatrix} AX_{xx}(k)A^T + AX_{xx_w}(k)C_w^T B^T + BC_w X^T_{xx_w}(k)A^T + BC_w X_{x_w x_w}(k)C_w^T B^T & AX_{xx_w}(k)A_w^T + BC_w X_{x_w x_w}(k)A_w^T \\ [AX_{xx_w}(k)A_w^T + BC_w X_{x_w x_w}(k)A_w^T]^T & A_w X_{x_w x_w}(k)A_w^T + B_w W B_w^T \end{bmatrix}$$

which gives same propagation functions for $X_{xx}(k)$, $X_{xx_w}(k)$ and $X_{x_w x_w}(k)$.

Instead, we can just obtain equation (1) as the propagation function for $E\left\{\begin{bmatrix} x(k) \\ x_w(k) \end{bmatrix}\begin{bmatrix} x(k) \\ x_w(k) \end{bmatrix}^T\right\}$ with initial condition $E\left\{\begin{bmatrix} x(0) \\ x_w(0) \end{bmatrix}\begin{bmatrix} x(0) \\ x_w(0) \end{bmatrix}^T\right\} = \begin{bmatrix} X_0 & 0 \\ 0 & X_w \end{bmatrix}$. Then the covariance matrix of $x(k)$, $E[x(k)x^T(k)]$, is just the (1, 1) element of $E\left\{\begin{bmatrix} x(k) \\ x_w(k) \end{bmatrix}\begin{bmatrix} x(k) \\ x_w(k) \end{bmatrix}^T\right\}$.

2. **Method 1:** The transfer function from $n(k)$ to $w(k)$ is given by

$$G_{nw}(z) = C_w(zI - A_w)^{-1}B_w.$$

Then the spectral density of $w(k)$ is

$$\Phi_{ww}(\omega) = G_{nw}(e^{-j\omega})\Phi_{nn}(\omega)G^T_{nw}(e^{j\omega}) = C_w(e^{-j\omega}I - A_w)^{-1}B_w W B_w^T(e^{j\omega}I - A_w)^{-T}C_w^T.$$

The transfer matrix from $w(k)$ to $x(k)$ is given by

$$G_{wx}(z) = (zI - A)^{-1}B.$$

So the spectral density of $x(k)$ is

$$\begin{aligned}\Phi_{xx}(\omega) &= G_{wx}(e^{-j\omega})\Phi_{ww}(\omega)G^T_{wx}(e^{j\omega}) \\ &= (e^{-j\omega}I - A)^{-1}BC_w(e^{-j\omega}I - A_w)^{-1}B_w W B_w^T(e^{j\omega}I - A_w)^{-T}C_w^T B^T(e^{j\omega}I - A)^{-T}\end{aligned}$$

Method 2: Start from the augmented state equations:

$$\begin{bmatrix} x(k+1) \\ x_w(k+1) \end{bmatrix} = \begin{bmatrix} A & BC_w \\ 0 & A_w \end{bmatrix}\begin{bmatrix} x(k) \\ x_w(k) \end{bmatrix} + \begin{bmatrix} 0 \\ B_w \end{bmatrix} n(k).$$

Notice that $x(k) = [I \quad 0]\begin{bmatrix} x(k) \\ x_w(k) \end{bmatrix}$. The transfer function from $n(k)$ to $x(k)$ is given by

$$G(s) = [I \quad 0]\left(sI - \begin{bmatrix} A & BC_w \\ 0 & A_w \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 \\ B_w \end{bmatrix} = [I \quad 0]\begin{bmatrix} sI - A & -BC_w \\ 0 & sI - A_w \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ B_w \end{bmatrix}.$$

Since $\begin{bmatrix} D & E \\ 0 & F \end{bmatrix}^{-1} = \begin{bmatrix} D^{-1} & -D^{-1}EF^{-1} \\ 0 & F^{-1} \end{bmatrix}$ for any matrix E and invertible matrices D and F,

$$\begin{bmatrix} sI-A & -BC_w \\ 0 & sI-A_w \end{bmatrix}^{-1} = \begin{bmatrix} (sI-A)^{-1} & -(sI-A)^{-1}BC_w(sI-A_w)^{-1} \\ 0 & (sI-A_w)^{-1} \end{bmatrix}.$$

Then we have

$$G(s) = (sI-A)^{-1}BC_w(sI-A_w)^{-1}B_w,$$

which gives us the same result as using Method 1.

[3] Denote the vector $Y_k = [y(0) \quad \cdots \quad y(k)]^T$.

The best estimate of $x(k+2)-x(k+1)$ based on Y_k is the conditional mean:

$$\begin{aligned}
\hat{x} &= E\{[x(k+2)-x(k+1)] \mid Y_k\} \\
&= E\{x(k+2) \mid Y_k\} - E\{x(k+1) \mid Y_k\} \\
&= E\{[Ax(k+1)+w(k+1) \mid Y_k\} - E\{x(k+1) \mid Y_k\} \\
&= (A-I)E\{x(k+1) \mid Y_k\} + E\{w(k+1) \mid Y_k\} \\
&= (A-I)\hat{x}(k+1 \mid k)
\end{aligned}$$

The one step predictor $\hat{x}(k+1 \mid k) = E\{x(k+1) \mid Y_k\}$ is found by the Kalman filter:

$$\hat{x}(k+1 \mid k) = (A - AF_sC)\hat{x}(k \mid k-1) + AF_s y(k),$$

where

$$F_s = \frac{M_s}{M_s+V} \text{ and } M_s = A^2M_s + W - \frac{A^2M_s^2}{M_s+V}.$$

The initial condition is $\hat{x}(0 \mid -1) = 0$.

The estimation error variance is given by

$$\begin{aligned}
&E\{[x(k+2)-x(k+1)-(A-I)\hat{x}(k+1 \mid k)]^2\} \\
&= E\{[Ax(k+1)+w(k+1)-x(k+1)-(A-I)\hat{x}(k+1 \mid k)]^2\} \\
&= E\{[(A-I)(x(k+1)-\hat{x}(k+1 \mid k))+w(k+1)]^2\} \\
&= (A-I)^2E\{[x(k+1)-\hat{x}(k+1 \mid k)]^2\} + E\{[w(k+1)]^2\} \\
&\quad + 2(A-I)E\{[x(k+1)-\hat{x}(k+1 \mid k)]w(k+1)\} \\
&= (A-I)^2M_s + W
\end{aligned}$$

Notice that we have set $E\{[x(k+1)-\hat{x}(k+1 \mid k)]w(k+1)\} = 0$, since $x(k+1)-\hat{x}(k+1 \mid k)$ depends on $w(0)$, ..., $w(k)$, which are independent of $w(k+1)$.