ME 233 Advance Control II

Lecture 16

Deterministic Input/Output Approach to SISO Discrete Time Systems

Repetitive Control

Deterministic SISO ARMA models

SISO ARMA model

$$A(q^{-1})\,y(k) = q^{-\mathrm{d}}\,B(q^{-1})\,[\,u(k) + d(k)]$$

Where all inputs and outputs are scalars:

- $u(k)$ control input
- $d(k)$ is a periodic disturbance of period N
- $y(k)$ output

Repetitive control assumptions

$$A(q^{-1})\,y(k) = q^{-\mathrm{d}}\,B(q^{-1})\,[\,u(k) + d(k)]$$

Both the disturbance and the reference model output are periodic sequences,

$$\left[1 - q^{-N}\right]\,d(k) \;=\; 0$$

$$\left[1 - q^{-N}\right]\,y_d(k) \;=\; 0$$

where N is a ***known*** and large number

Deterministic SISO ARMA models

$$A(q^{-1})\,y(k) = q^{-\mathrm{d}}\,B(q^{-1})\,[\,u(k) + d(k)]$$

Where polynomials:

$$A(q^{-1}) \;=\; 1 + a_1 q^{-1} + \cdots + a_n q^{-n}$$

$$B(q^{-1}) \;=\; b_o + b_1 q^{-1} + \cdots + b_m q^{-m}$$

are co-prime and d is the ***known*** pure time delay

Deterministic SISO ARMA models

The zero polynomial:

$$B^*(q) = q^m \, B(q^{-1}) = 0$$

has

- m_u zeros which we do not wish to cancel.
- m_s zeros inside the unite circle (asymptotically stable) which we wish to cancel.

$$B(q^{-1}) = B^s(q^{-1}) \, B^u(q^{-1})$$

$B^{s*}(q) = q^{m_s} B^s(q^{-1})$ is Schur

$B^{u*}(q) = q^{m_u} B^u(q^{-1})$ zeros which we **do not** wish to cancel

Deterministic SISO ARMA models

The zero polynomial:

$$B(q^{-1}) = B^s(q^{-1}) \, B^u(q^{-1})$$

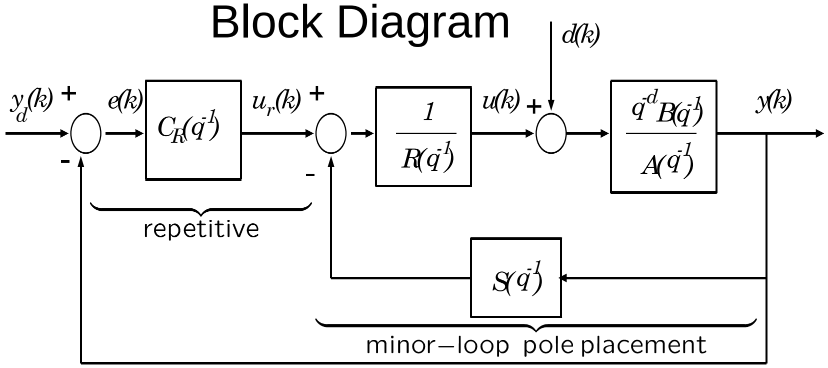
Without loss of generality, we will assume that

$$B^s(q^{-1}) = 1 \, + \cdots + b^s_{m_s} \, q^{-m_s}$$

$$B^u(q^{-1}) = b_o + \cdots + b^u_{m_u} \, q^{-m_u}$$

i.e. the polynomial $B^s(q^{-1})$ is monic

Block Diagram



Control strategy: We design the controller in two stages

1. **Minor-loop pole placement:** Place minor-loop poles, (that will be cancelled later)
2. **Repetitive compensator:** Reject periodic disturbance
   Follow periodic reference

Control Objectives

1. Minor-loop Pole Placement: The poles of the minor-loop system are placed at specific locations in the complex plane. **They will be cancelled later.**

- **Minor-loop pole polynomial:**

$$\boxed{A_c(q^{-1}) = B^s(q^{-1}) \, A_c^{'}(q^{-1})}$$

Where:

- $B^s(q^{-1})$ cancelable plant zeros
- $A_c^{'}(q^{-1})$ monic Schur polynomial chosen by the designer

$$A_c^{'}(q^{-1}) = 1 + a_{c1}^{'} q^{-1} + \cdots + a^{'}_{c_{n_c^{'}}} q^{-n_c^{'}}$$

Control Objectives

2. Tracking: The output sequence $y(k)$ must asymptotically follow a ***reference*** sequence $y_d(k)$ which is periodic

$$\left[1 - q^{-N}\right] y_d(k) = 0$$

- **Error signal:**

$$\boxed{e(k) = y_d(k) - y(k)}$$

3) Disturbance rejection: The closed loop system must reject a class of deterministic disturbances which satisfy

$$\left[1 - q^{-N}\right] d(k) = 0$$

Step1: Minor-loop pole placement

Diophantine equation: Obtain polynomials $R(q^{-1}),\ S(q^{-1})$ which satisfy:

$$\boxed{A_c(q^{-1}) = A(q^{-1})\,\underline{R(q^{-1})} + q^{-\mathrm{d}}B(q^{-1})\,\underline{S(q^{-1})}}$$

Close loop poles *Plant poles* *plant zeros*

$$R(q^{-1}) = R^{'}(q^{-1})\,\underline{B^s(q^{-1})}$$
$$A_c(q^{-1}) = \underline{B^s(q^{-1})}\,A_c^{'}(q^{-1})$$

We will factor out the $B^s(q^{-1})$ *polynomial next*

The disturbance annihilating polynomial has not been included

Minor-loop pole placement

Diophantine equation: Obtain polynomials $\underline{R^{'}(q^{-1})},\ \underline{S(q^{-1})}$ which satisfy:

$$\boxed{A_c^{'}(q^{-1}) = A(q^{-1})\,\underline{R^{'}(q^{-1})} + q^{-\mathrm{d}}B^u(q^{-1})\,\underline{S(q^{-1})}}$$

Close loop poles *Plant poles* *Unstable plant zeros*

$$R(q^{-1}) = R^{'}(q^{-1})\,B^s(q^{-1})$$
$$A_c(q^{-1}) = B^s(q^{-1})\,A_c^{'}(q^{-1})$$

The disturbance annihilating polynomial has not been included

Diophantine equation

$$A_c^{'}(q^{-1}) = A(q^{-1})\,R^{'}(q^{-1}) + q^{-\mathrm{d}}B^u(q^{-1})\,S(q^{-1})$$

Solution:

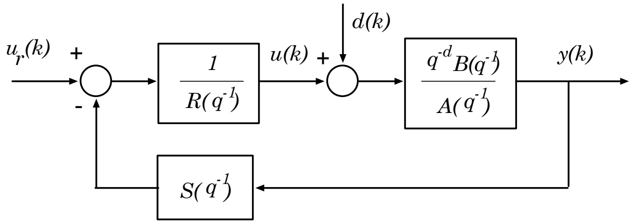
$$R^{'}(q^{-1}) = 1 + r_1^{'}q^{-1} + \cdots + r_{n_r^{'}}\,q^{-n_r^{'}}$$

$$S(q^{-1}) = s_o + s_1 q^{-1} + \cdots + s_{n_s}q^{-n_s}$$

$$n_r^{'} = d + m_u - 1$$

$$n_s = \max\{\,n-1\,,\,n_c^{'} - d - m_u\,\}$$

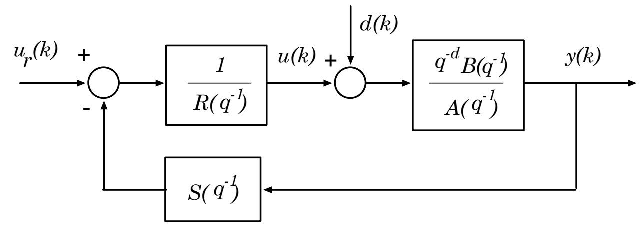
Minor-loop pole placement



$$u(k) = \frac{1}{R(q^{-1})}\left[u_r(k) - S(q^{-1})y(k)\right]$$

Minor-loop pole placement

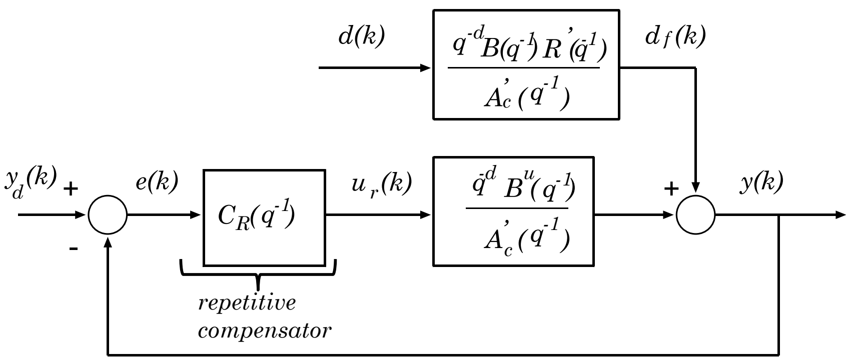
Close loop dynamics



$$y(k) \;=\; \frac{q^{-\mathrm{d}}B^u(q^{-1})}{A_c'(q^{-1})}\,u_r(k) + \underbrace{\frac{q^{-\mathrm{d}}B(q^{-1})R'(q^{-1})}{A_c'(q^{-1})}\,d(k)}_{d_f(k)}$$

filtered repetitive disturbance

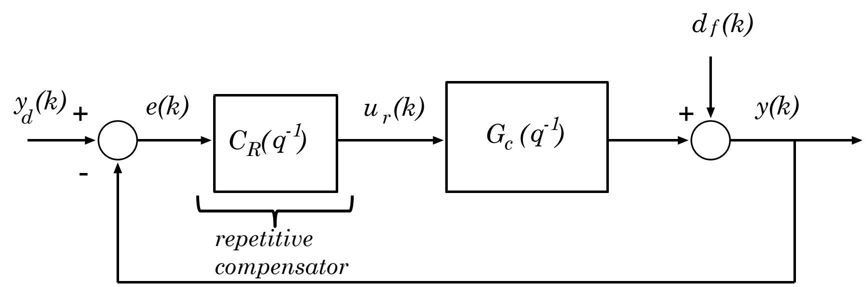
Equivalent Block Diagram



Notice that $d_f(k)$ is still a periodic disturbance

$$\left[1 - q^{-N}\right]\,y_d(k) \;=\; 0 \qquad\qquad \left[1 - q^{-N}\right]\,d_f(k) \;=\; 0$$

Equivalent Block Diagram

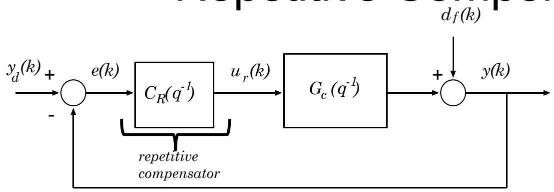


where

$$G_c(q^{-1}) \;=\; \frac{q^{-\mathrm{d}}B^u(q^{-1})}{A_c'(q^{-1})}$$

$$\left[1 - q^{-N}\right]\,y_d(k) \;=\; 0 \qquad\qquad \left[1 - q^{-N}\right]\,d_f(k) \;=\; 0$$

Repetitive Compensator



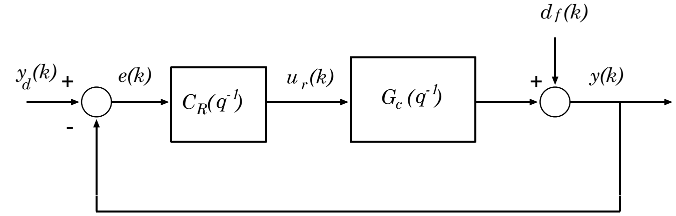
$$G_c(q^{-1}) \;=\; \frac{q^{-\mathrm{d}} B^u(q^{-1})}{A_c'(q^{-1})}$$

Repetitive compensator strategy:

1. Cancel stable poles and delay $A_c'(q^{-1})\;\; q^{-\mathrm{d}}$
2. Zero-phase error compensation for $B^u(q^{-1})$
3. Include annihilating polynomial in the denominator $1 - q^{-N}$

$$C_R(q^{-1}) \;=\; \frac{k_r}{b}\left[\frac{q^{-N}}{1-q^{-N}}\right]\left[q^d A_c'(q^{-1})\, B^u(q)\right]$$

Repetitive Compensator



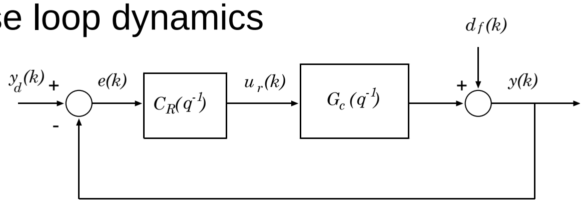
Repetitive compensator :

$$\boxed{C_R(q^{-1}) \;=\; \frac{k_r}{b}\left[\frac{q^{-N}}{1-q^{-N}}\right]\left[q^d A_c'(q^{-1})\, B^u(q)\right]}$$

$$(N \geq d + m_u)$$

Repetitive Compensator

- Close loop dynamics



$$e(k) \;=\; \frac{1}{1 + C_R(q^{-1}) G_c(q^{-1})}\left[y_d(k) - d_f(k)\right]$$

$$G_c(q^{-1}) \;=\; \frac{q^{-\mathrm{d}} B^u(q^{-1})}{A_c'(q^{-1})}$$

$$C_R(q^{-1}) \;=\; \frac{k_r}{b}\left[\frac{q^{-N}}{1-q^{-N}}\right]\left[q^d A_c'(q^{-1})\, B^u(q)\right]$$

Repetitive Controller

Close loop dynamics: doing a bit of algebra, we obtain,

$$\boxed{e(k) = \frac{q^N - 1}{\bar{A}_c^*(q)}\left[y_d(k) - d_f(k)\right]}$$

Where the close loop poles are the roots of

$$\boxed{\bar{A}_c^*(q) = \left(q^N - 1\right) + \frac{k_r}{b} B^u(q)\, B^u(q^{-1})}$$

Repetitive Controller

since,

$$(q^N - 1)\left(y_d(k) - d_f(k)\right) = 0$$

we obtain

$$\boxed{\bar{A}_c^*(q)e(k) = 0}$$

Where the close loop poles are the roots of

$$\boxed{\bar{A}_c^*(q) = \left(q^N - 1\right) + \frac{k_r}{b} B^u(q)\, B^u(q^{-1})}$$

Repetitive Controller

Theorem

The tracking error $e(k) \to 0$ if the gains $k_r,\, b$ are selected as follows:

1. $b > \max_{\omega \in [0,\pi]} |B^u(e^{j\omega})|^2$

2. $2 > k_r > 0$

Close loop poles for minimum phase zeros

Consider now the case when there are **no unstable zeros**, i.e.

$B^u(q^{-1}) = b_o$ and $\dfrac{B^u(q)\, B^u(q^{-1})}{b} = 1$

Therefore,

$$\boxed{\left(q^N - 1\right) + k_r = 0 \quad \Rightarrow \quad q^N = 1 - k_r}$$

and all close loop are asymptotically stable for:

$$\boxed{2 > k_r > 0}$$

Close loop poles for minimum phase zeros

For the case when the are no unstable zeros,

$\boxed{q^N = 1 - k_r}$ and $\boxed{2 > k_r > 0}$

we have N asymptotically stable close loop poles at:

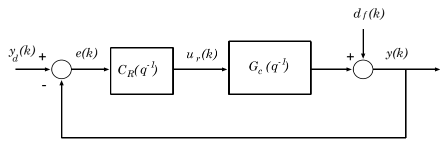
$$\lambda_i = |1 - k_r|^{\frac{1}{N}}\, e^{j\,\frac{2\pi i}{N}}\; i = 0, \pm 1, \pm 2, \cdots \text{ for } \quad 0 < k_r \leq 1$$

or

$$\lambda_i = |1 - k_r|^{\frac{1}{N}}\, e^{j\,\frac{\pi i}{N}}\; i = 0, \pm 1, \pm 2, \cdots \text{ for } \quad 1 < k_r < 2$$

Repetitive control example $\begin{matrix}(B^u(q^{-1}) = b_o)\\(d=1)\end{matrix}$

Assume that $\boxed{N = 4}$

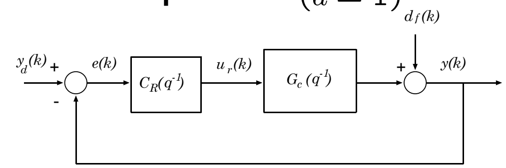


$$G_c(q^{-1}) \;=\; \frac{q^{-\mathrm{d}}B^u(q^{-1})}{A_c'(q^{-1})} = \frac{q^{-1}}{A_c'(q^{-1})}$$

$$C_R(q^{-1}) \;=\; \frac{k_r}{b}\, q^{-(N-d)}\, \frac{A_c'(q^{-1})\, B^u(q)}{1 - q^{-N}} = k_r q^{-3} \frac{A_c'(q^{-1})}{1 - q^{-4}}$$

$$G_c(q^{-1})C_R(q^{-1}) \;=\; k_r \frac{q^{-4}}{1 - q^{-4}} = k_r \frac{1}{q^4 - 1}$$

Repetitive control example $\begin{matrix}(B^u(q^{-1}) = 1)\\(d=1)\end{matrix}$

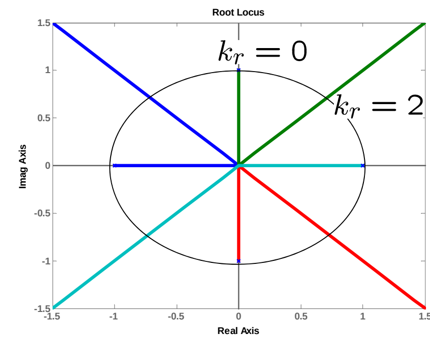


Open loop TF

$$G_c(q^{-1})C_R(q^{-1}) \;=\; k_r \frac{q^{-4}}{1 - q^{-4}} = k_r \frac{1}{q^4 - 1}$$

Close loop poles:

$$\boxed{1 + k_r \frac{1}{z^4 - 1} = 0}$$



Repetitive controller close loop poles

The close loop poles are the roots of

$$\boxed{\left(q^N - 1\right) + k_r \frac{B^u(q)\, B^u(q^{-1})}{b} = 0}$$

Lets first select the constant b such that

$$\left|\frac{B^u(z)\, B^u(z^{-1})}{b}\right|_{z=e^{j\omega}} < 1$$

Thus,

$$\boxed{b > \max_{\omega \in [0,\pi]} |B^u(e^{j\omega})|^2}$$

Close loop poles for non-minimum phase zeros

Consider now the general case when the are unstable zeros

$B^u(q^{-1}) \neq b_o$ but $\left|\frac{B^u(z)\, B^u(z^{-1})}{b}\right|_{z=e^{j\omega}} < 1$

Therefore $\bar{A}_c^*(z) = 0$ is equivalent to

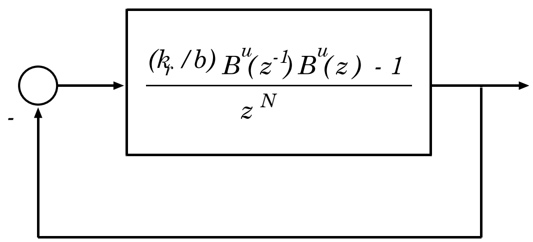
$$z^N - 1 + k_r \frac{B^u(z)\, B^u(z^{-1})}{b} = 0$$

$$\boxed{1 + \frac{\frac{k_r}{b} B^u(z)\, B^u(z^{-1}) - 1}{z^N} = 0}$$

Close loop poles for non-minimum phase zeros

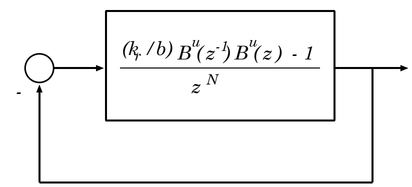
Therefore $\bar{A}_c^*(z) = 0$ is equivalent to

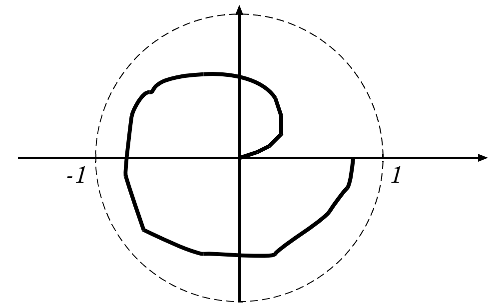
$$1 + \frac{\frac{k_r}{b} B^u(z)\, B^u(z^{-1}) - 1}{z^N} = 0$$



Close loop poles for non-minimum phase zeros

By Nyquist's theorem, the close loop system is asymptotically stable if there are no encirclements around –1.





This is guaranteed if

$$\left| \frac{\frac{k_r}{b} B^u(e^{j\omega})\, B^u(e^{-j\omega}) - 1}{e^{j\omega N}} \right| < 1 \quad \text{for } \omega \in [0, \pi]$$

Close loop poles for non-minimum phase zeros

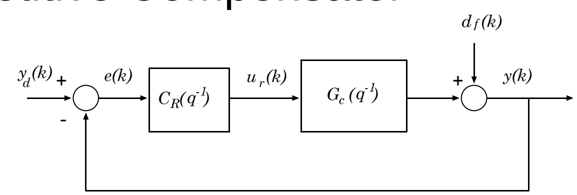
Since, $|e^{j\omega N}| = 1$ and $\left| \frac{B^u(e^{j\omega})\, B^u(e^{j\omega})}{b} \right| < 1$

$$\boxed{2 > k_r > 0} \quad \Rightarrow \quad \left| \frac{k_r}{b} B^u(e^{j\omega})\, B^u(e^{-j\omega}) - 1 \right| < 1$$

and

$$\boxed{\left| \frac{\frac{k_r}{b} B^u(e^{j\omega})\, B^u(e^{-j\omega}) - 1}{e^{j\omega N}} \right| < 1}$$

Repetitive Compensator



Repetitive compensator:

$$\boxed{C_R(q^{-1}) \;=\; \frac{k_r}{b}\, q^{-(N-d)}\, \frac{A_c'(q^{-1})\, B^u(q)}{\underline{1 - q^{-N}}}}$$

The controller has N open-loop poles in the unit circle

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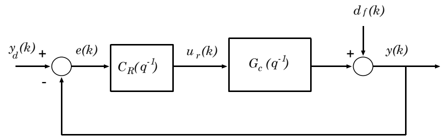
Repetitive control example $\begin{matrix}(B^u(q^{-1}) = b_o) \\ (d = 1)\end{matrix}$

Assume that $\boxed{N = 4}$



$$G_c(q^{-1}) \;=\; \frac{q^{-\mathrm{d}} B^u(q^{-1})}{A'_c(q^{-1})} = \frac{q^{-1}}{A'_c(q^{-1})}$$

$$C_R(q^{-1}) \;=\; \frac{k_r}{b}\, q^{-(N-d)}\, \frac{A'_c(q^{-1})\, B^u(q)}{1 - q^{-N}} = k_r q^{-3} \frac{A'_c(q^{-1})}{1 - q^{-4}}$$

$$G_c(q^{-1}) C_R(q^{-1}) \;=\; k_r \frac{q^{-4}}{1 - q^{-4}} = k_r \frac{1}{q^4 - 1}$$

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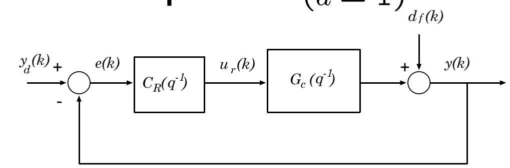
Repetitive control example $\begin{matrix}(B^u(q^{-1}) = 1) \\ (d = 1)\end{matrix}$

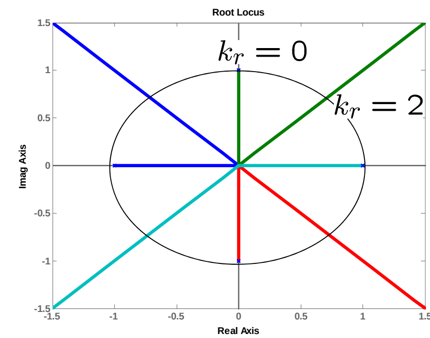


Open loop TF

$$G_c(q^{-1}) C_R(q^{-1}) \;=\; k_r \frac{q^{-4}}{1 - q^{-4}} = k_r \frac{1}{q^4 - 1}$$

Close loop poles:

$$\boxed{1 + k_r \frac{1}{z^4 - 1} = 0}$$



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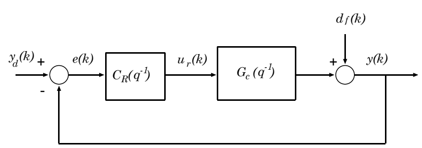
Repetitive control, inexact cancellation

Assume that $\boxed{N = 4}$



Plant:

$$G_c(q^{-1}) \;=\; \frac{q^{-1}}{A'_c(q^{-1})} = \frac{q^{-1}}{\bar{A}'_c(q^{-1})} \frac{0.8\, q^{-1}}{1 - 0.2 q^{-1}}$$

But, unmodeled dynamics is not cancelled

$$C_R(q^{-1}) = k_r q^{-3} \frac{\bar{A}'_c(q^{-1})}{1 - q^{-4}}$$

therefore,

$$G_c(q^{-1}) C_R(q^{-1}) \;=\; \frac{0.8\, k_r}{(q - 0.2)\,(q^4 - 1)}$$

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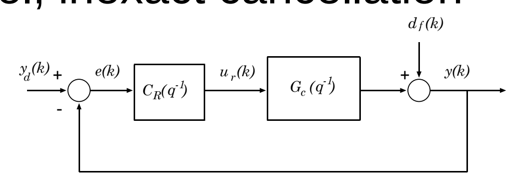
Repetitive control, inexact cancellation

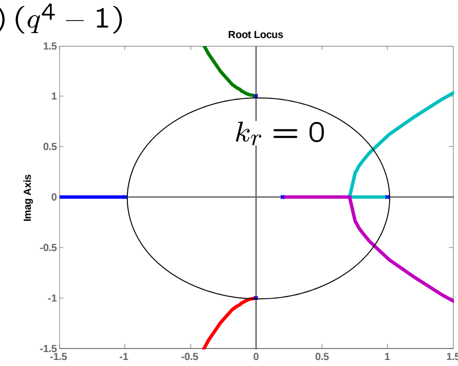


Open loop TF

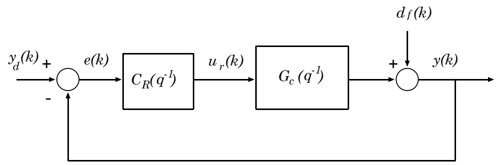
$$G_c(q^{-1}) C_R(q^{-1}) \;=\; \frac{0.8\, k_r}{(q - 0.2)\,(q^4 - 1)}$$

Close loop poles:

$$\boxed{1 + k_r \frac{0.8}{(z - 0.2)(z^4 - 1)} = 0}$$



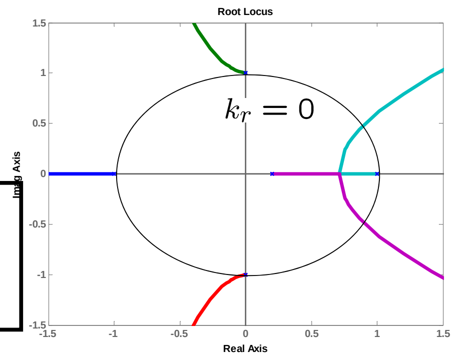
Repetitive control, inexact cancellation



Repetitive control is not robust to unmodeled dynamics

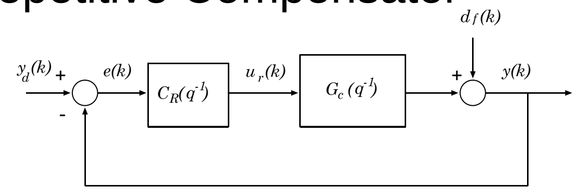
Close loop poles:

$$1 + k_r \frac{0.8}{(z-0.2)(z^4-1)} = 0$$



Robust Repetitive Compensator

Add Q-filter



$$C_R(q^{-1}) \;=\; \frac{k_r}{b}\, q^{-(N-d)}\, \frac{A_c'(q^{-1})\, B^u(q)}{1 - \underline{Q(q,q^{-1})}\, q^{-N}}$$

$Q(q,q^{-1})$ moving average filter with zero-phase shift characteristics

Controller's N open-loop poles are no longer in the unit circle

Robust Repetitive Compensator

$Q(q,q^{-1})$ moving average filter with zero-phase shift characteristics

$$Q(q,q^{-1}) = \frac{\gamma_p q^p + \cdots \gamma_1 q + \gamma_o + \gamma_1 q^{-1} + \cdots \gamma_{p-1} q^{-(p-1)} + \gamma_p q^{-p}}{2\gamma_p + 2\gamma_{p-1} \cdots 2\gamma_1 + \gamma_o}$$

$$N > p \qquad \gamma_o > \gamma_1 > \cdots > \gamma_p > 0$$

$Q(q,q^{-1})$ has unit DC gain and gain decreases as frequency increases

Robust Repetitive Compensator

$$C_R(q^{-1}) \;=\; \frac{k_r}{b}\, q^{-(N-d)}\, \frac{A_c'(q^{-1})\, B^u(q)}{1 - \underline{Q(q,q^{-1})}\, q^{-N}}$$

Notice that the disturbance *d(k)* is no longer completely annihilated, since

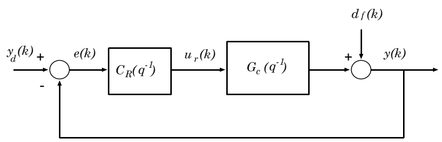
$$\left[1 - Q(q,q^{-1})\, q^{-N}\right] d(k) \neq 0$$

However, with a proper choice of Q filter,

$$\left| \left[1 - Q(q,q^{-1})\, q^{-N}\right] d(k) \right| << |d(k)|$$

Robust Rep. control, inexact cancellation

Assume that $\boxed{N = 4}$



Plant:

$$G_c(q^{-1}) = \frac{q^{-1}}{A_c'(q^{-1})} = \frac{q^{-1}}{\bar{A}_c'(q^{-1})}\,\frac{0.8\,q^{-1}}{1-0.2q^{-1}}$$

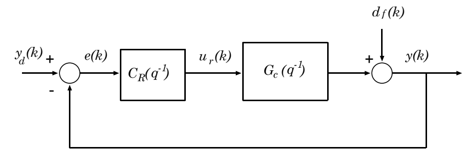
But, unmodeled dynamics is not cancelled

$$C_R(q^{-1}) = k_r q^{-3}\frac{\bar{A}_c'(q^{-1})}{1-Q(q,q^{-1})\,q^{-4}}$$

where,

$$Q(q,q^{-1}) = \frac{q+4+q^{-1}}{6}$$

Robust Rep. control, inexact cancellation



Close loop poles:

$$1 + k_r\frac{2.4\,z}{(z-0.2)(6\,z^5 - z^2 - 4\,z - 1)} = 0$$

Close loop system is asymptotically stable for a finite range of k_r