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ME 233 – Homework 2

1.(a)

$$x(k+1)=A\,x(k)+B\,u(k),\ y(k)=C\,x(k),\ x_0=[1 \quad 0 \quad 1]^T$$

$$J[x(m),m,S,N]=\frac{1}{2}x^T(N)S\,x(N)+\frac{1}{2}\sum_{k=m}^{N-1}\{y^T(k)y(k)+u^T(k)R\,u(k)\}$$

$$A=\begin{bmatrix}1.2 & 0 & 0\\ 0 & 0 & 1\\ 0 & -2 & 0\end{bmatrix},\ B=\begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix},\ C=[10 \quad 0 \quad 0],\ R=\begin{bmatrix}10 & 0\\ 0 & 10\end{bmatrix},\ Q=C^T C=\begin{bmatrix}100 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Characteristic equation of A: $(s-1.2)(s^2+2)=0$, so eigenvalues are 1.2 and $\pm j\sqrt{2}$

All three are outside the unit circle. For $\lambda=1.2$,

$$\text{rank}[\lambda I-A \quad B]=\text{rank}\begin{bmatrix}0 & 0 & 0 & 1 & 0\\ 0 & 1.2 & -1 & 0 & 0\\ 0 & 2 & 1.2 & 0 & 1\end{bmatrix}=3$$

For $\lambda=j\sqrt{2}$, $$\text{rank}[\lambda I-A \quad B]=\text{rank}\begin{bmatrix}j\sqrt{2}-1.2 & 0 & 0 & 1 & 0\\ 0 & j\sqrt{2} & -1 & 0 & 0\\ 0 & 2 & j\sqrt{2} & 0 & 1\end{bmatrix}=3$$

For $\lambda=-j\sqrt{2}$, $$\text{rank}[\lambda I-A \quad B]=\text{rank}\begin{bmatrix}-j\sqrt{2}-1.2 & 0 & 0 & 1 & 0\\ 0 & -j\sqrt{2} & -1 & 0 & 0\\ 0 & 2 & -j\sqrt{2} & 0 & 1\end{bmatrix}=3$$

All (unstable) eigenvalues are controllable by the above PBH test, so $[A,B]$ is stabilizable.

For $\lambda=1.2$, $$\text{null}\begin{bmatrix}\lambda I-A\\ C\end{bmatrix}=3-\text{rank}\begin{bmatrix}0 & 0 & 0\\ 0 & 1.2 & -1\\ 0 & 2 & 1.2\\ 10 & 0 & 0\end{bmatrix}=0$$

For $\lambda=j\sqrt{2}$, $$\text{null}\begin{bmatrix}\lambda I-A\\ C\end{bmatrix}=3-\text{rank}\begin{bmatrix}j\sqrt{2}-1.2 & 0 & 0\\ 0 & j\sqrt{2} & -1\\ 0 & 2 & j\sqrt{2}\\ 10 & 0 & 0\end{bmatrix}=1$$

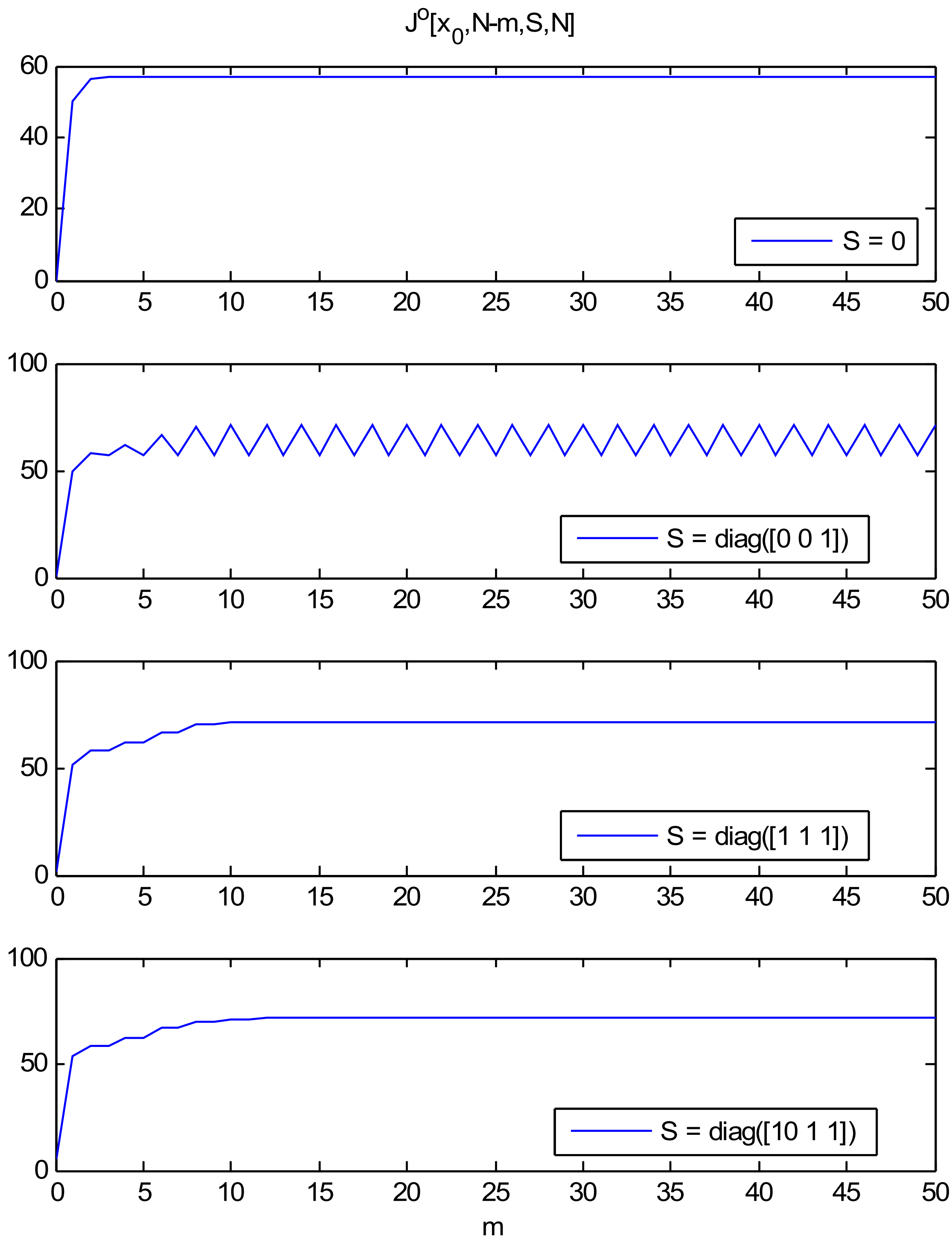
For $\lambda=-j\sqrt{2}$, $$\text{null}\begin{bmatrix}\lambda I-A\\ C\end{bmatrix}=3-\text{rank}\begin{bmatrix}-j\sqrt{2}-1.2 & 0 & 0\\ 0 & -j\sqrt{2} & -1\\ 0 & 2 & -j\sqrt{2}\\ 10 & 0 & 0\end{bmatrix}=1$$

The two complex unstable eigenvalues are unobservable so $[A,C]$ is not detectable.

Riccati equation: $P(k-1)=Q+A^T P(k)A-A^T P(k)B(R+B^T P(k)B)^{-1}B^T P(k)A,\ P(N)=S$

$$J^o[x_0,m,S,N]=\min_{U_{[m,N-1]}} J[x_0,m,S,N]=\frac{1}{2}x_0^T P(m)\,x_0$$

`dare(A,B,Q,R)=` $$P_\infty=\begin{bmatrix}113.2315 & 0 & 0\\ 0 & 30 & 0\\ 0 & 0 & 30\end{bmatrix}$$



For $S=0$, $P(0)=P(1)=\begin{bmatrix} 113.2315 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, (converged to round-off precision)

For $S=\mathrm{Diag}(0,0,1)$, $P(0)=\begin{bmatrix}113.2315 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 30\end{bmatrix}$, $P(1)=\begin{bmatrix}113.2315 & 0 & 0\\ 0 & 30 & 0\\ 0 & 0 & 0\end{bmatrix}$

For $S=\mathrm{Diag}(1,1,1)$, $P(0)=P(1)=\begin{bmatrix}113.2315 & 0 & 0\\ 0 & 30 & 0\\ 0 & 0 & 30\end{bmatrix}$, (converged to $2.3\cdot 10^{-12}$)

For $S=\mathrm{Diag}(10,1,1)$, $P(0)=P(1)=\begin{bmatrix}113.2315 & 0 & 0\\ 0 & 30 & 0\\ 0 & 0 & 30\end{bmatrix}$, (converged to $2.3\cdot 10^{-12}$)

$[A,C]$ is not detectable so the Riccati difference equation did not converge to a unique solution.
The RDE solutions were bounded above, but depended on initial conditions - solution was oscillatory for the case of $S=\mathrm{Diag}(0,0,1)$

1.(b)

$A=\begin{bmatrix}1.2 & 1 & 0\\ 0 & 0 & 1\\ 0 & -2 & 0\end{bmatrix}$, $B=\begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}$, $C=\begin{bmatrix}10 & 0 & 0\end{bmatrix}$, $R=\begin{bmatrix}10 & 0\\ 0 & 10\end{bmatrix}$, $Q=C^T C=\begin{bmatrix}100 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

Same characteristic equation and eigenvalues: $(s-1.2)(s^2+2)=0$, $\lambda=1.2$ and $\pm j\sqrt{2}$

For $\lambda=1.2$, $\mathrm{rank}[\lambda I-A \quad B]=\mathrm{rank}\begin{bmatrix}0 & -1 & 0 & 1 & 0\\ 0 & 1.2 & -1 & 0 & 0\\ 0 & 2 & 1.2 & 0 & 1\end{bmatrix}=3$

For $\lambda=j\sqrt{2}$, $\mathrm{rank}[\lambda I-A \quad B]=\mathrm{rank}\begin{bmatrix}j\sqrt{2}-1.2 & -1 & 0 & 1 & 0\\ 0 & j\sqrt{2} & -1 & 0 & 0\\ 0 & 2 & j\sqrt{2} & 0 & 1\end{bmatrix}=3$

For $\lambda=-j\sqrt{2}$, $\mathrm{rank}[\lambda I-A \quad B]=\mathrm{rank}\begin{bmatrix}-j\sqrt{2}-1.2 & -1 & 0 & 1 & 0\\ 0 & -j\sqrt{2} & -1 & 0 & 0\\ 0 & 2 & -j\sqrt{2} & 0 & 1\end{bmatrix}=3$

All (unstable) eigenvalues are controllable by the above PBH test, so $[A,B]$ is again stabilizable.

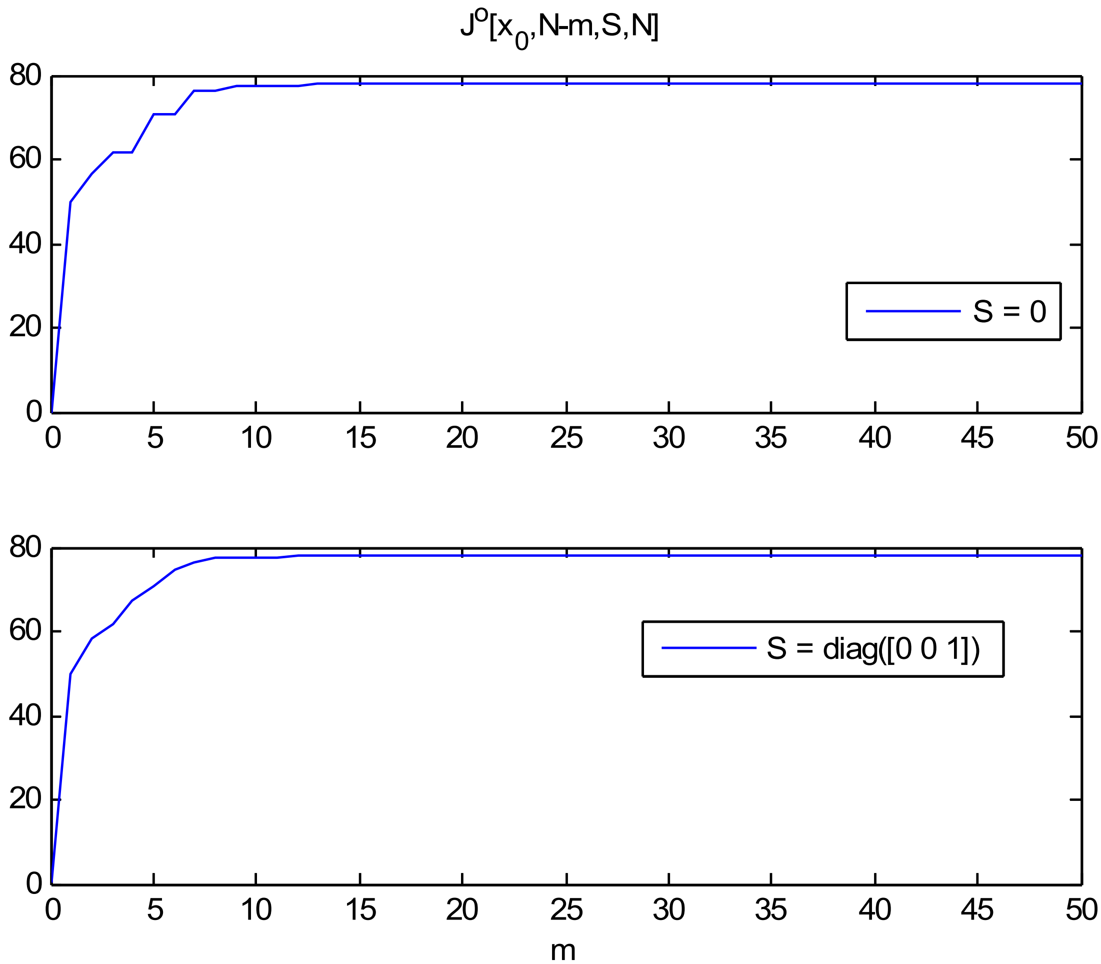
For $\lambda=1.2$, $\mathrm{null}\begin{bmatrix}\lambda I-A\\ C\end{bmatrix}=3-\mathrm{rank}\begin{bmatrix}0 & -1 & 0\\ 0 & 1.2 & -1\\ 0 & 2 & 1.2\\ 10 & 0 & 0\end{bmatrix}=0$

For $\lambda=j\sqrt{2}$, $\mathrm{null}\begin{bmatrix}\lambda I-A\\ C\end{bmatrix}=3-\mathrm{rank}\begin{bmatrix}j\sqrt{2}-1.2 & -1 & 0\\ 0 & j\sqrt{2} & -1\\ 0 & 2 & j\sqrt{2}\\ 10 & 0 & 0\end{bmatrix}=0$

For $\lambda=-j\sqrt{2}$, $\mathrm{null}\begin{bmatrix}\lambda I-A\\ C\end{bmatrix}=3-\mathrm{rank}\begin{bmatrix}-j\sqrt{2}-1.2 & -1 & 0\\ 0 & -j\sqrt{2} & -1\\ 0 & 2 & -j\sqrt{2}\\ 10 & 0 & 0\end{bmatrix}=0$

All (unstable) eigenvalues are observable by the above PBH test, so this $[A,C]$ is detectable.

`dare(A,B,Q,R)=` $P_\infty=\begin{bmatrix}113.2312 & 10.9845 & 1.0685\\ 10.9845 & 41.1428 & 0.6637\\ 1.0685 & 0.6637 & 40.1572\end{bmatrix}$



For $S=0$, $P(0)=P(1)=\begin{bmatrix} 113.2312 & 10.9845 & 1.0685 \\ 10.9845 & 41.1428 & 0.6637 \\ 1.0685 & 0.6637 & 40.1572 \end{bmatrix}$, (converged to round-off precision)

For $S=\text{Diag}(0,0,1)$, $P(0)=P(1)=\begin{bmatrix} 113.2312 & 10.9845 & 1.0685 \\ 10.9845 & 41.1428 & 0.6637 \\ 1.0685 & 0.6637 & 40.1572 \end{bmatrix}$, (converged to 2^{-44})

$[A, C]$ is detectable now so the Riccati difference equation converged to a unique solution.
The convergence dynamics are slightly smoother and faster for $S=\text{Diag}(0,0,1)$ than for $S=0$.

1.(c)

$A=\begin{bmatrix} 0.8 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$, $B=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $C=\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, $R=0.1$, $Q=C^T C=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Diagonal A matrix, so by inspection the eigenvalues are 0.8 and -1. The eigenvalue at -1 is on the unit circle, but since it has multiplicity 2 its Jordan block may not be size 1. It could potentially be unstable (depending on the rank of $\lambda I - A$) so we need to check it.

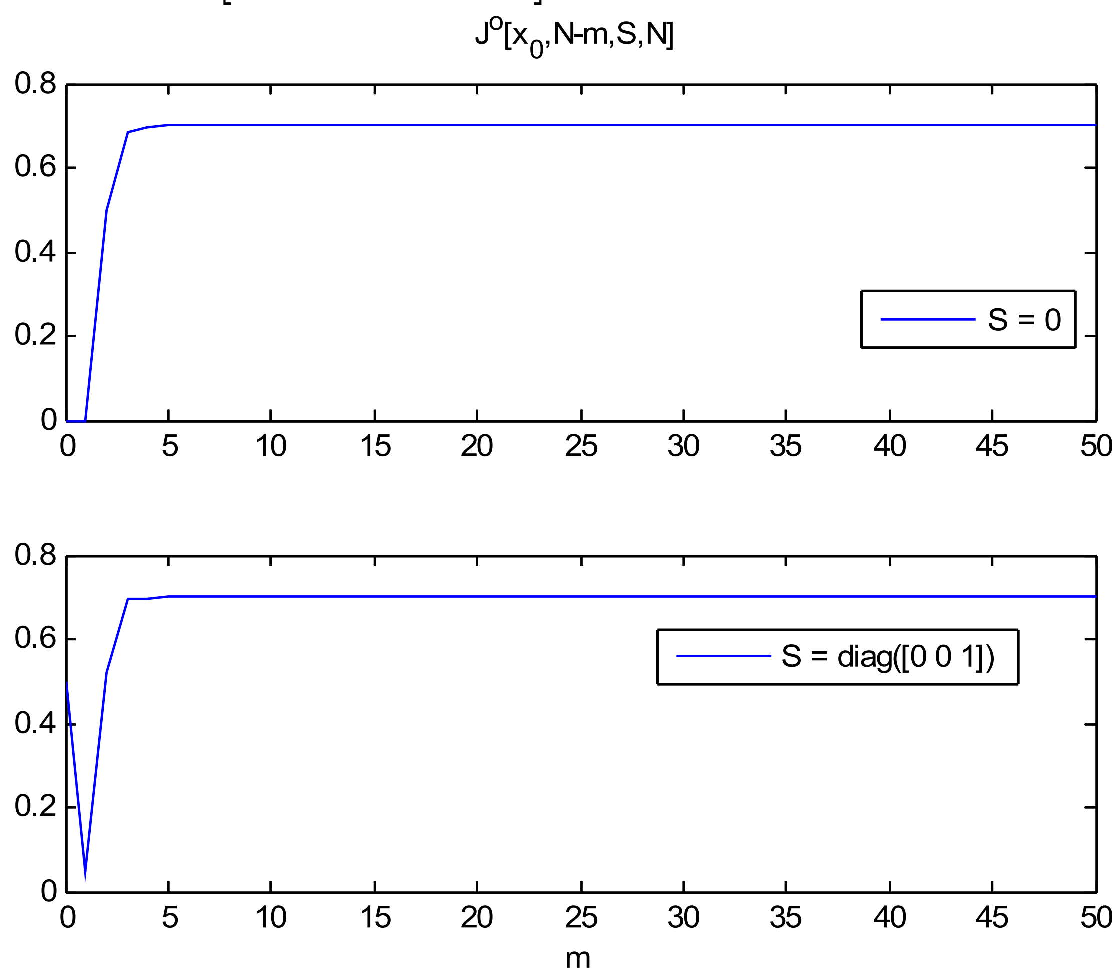
For $\lambda=-1$, $\text{rank}[\lambda I - A \quad B]=\text{rank}\begin{bmatrix} -1.8 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}=3$

All potentially unstable eigenvalues are controllable, so this $[A, B]$ is stabilizable.

For $\lambda=-1$, $\text{null}\begin{bmatrix}\lambda I-A\\ C\end{bmatrix}=3-\text{rank}\begin{bmatrix}-1.8 & -1 & -1\\ 0 & 0 & -1\\ 0 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}=0$

All potentially unstable eigenvalues are observable, so this $[A, C]$ is detectable.

`dare(A,B,Q,R)=` $P_\infty=\begin{bmatrix}0 & 0 & 0\\ 0 & 2.1429 & -1.2246\\ 0 & -1.2246 & 1.3996\end{bmatrix}$



For $S=0$, $P(0)=P(1)=\begin{bmatrix}0 & 0 & 0\\ 0 & 2.1429 & -1.2246\\ 0 & -1.2246 & 1.3996\end{bmatrix}$, (converged to round-off precision)

For $S=\text{Diag}(0,0,1)$, $P(0)=P(1)=\begin{bmatrix}0 & 0 & 0\\ 0 & 2.1429 & -1.2246\\ 0 & -1.2246 & 1.3996\end{bmatrix}$, (converged to round-off precision)

The RDE converged to a unique solution as expected with a stabilizable and detectable system. Other than the initial condition, the convergence dynamics are very similar for these choices of S.

2.

$x(k+1)=Ax(k)+Bu(k),\ x(0)\neq 0$

$J=\frac{1}{2}\sum_{k=0}^{\infty}\{x^T(k)Qx(k)+2x^T(k)Su(k)+u^T(k)Ru(k)\}$

$Q=Q^T\geq 0,\ R=R^T>0,\ \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}\geq 0 \Leftrightarrow Q-SR^{-1}S^T\geq 0$

$u(k)=-K_1x(k)+u_1(k)$

$J=\frac{1}{2}\sum_{k=0}^{\infty}\{x^T(k)Qx(k)+2x^T(k)S(-K_1x(k)+u_1(k))+(-x^T(k)K_1^T+u_1^T(k))R(-K_1x(k)+u_1(k))\}$

$J=\frac{1}{2}\sum_{k=0}^{\infty}\{x^T(k)Qx(k)-2x^T(k)SK_1x(k)+2x^T(k)Su_1(k)+x^T(k)K_1^TRK_1x(k)-x^T(k)K_1^TRu_1(k)$

$-u_1^T(k)RK_1x(k)+u_1^T(k)Ru_1(k)\}$

$x^T(k)K_1^TRu_1(k)$ is a scalar so $x^T(k)K_1^TRu_1(k)=(x^T(k)K_1^TRu_1(k))^T=u_1^T(k)RK_1x(k)$

$J=\frac{1}{2}\sum_{k=0}^{\infty}\{x^T(k)(Q-2SK_1+K_1^TRK_1)x(k)+x^T(k)(2S-2K_1^TR)u_1(k)+u_1^T(k)Ru_1(k)\}$

There are no cross terms between $x(k)$ and $u_1(k)$ if $2S-2K_1^TR=0$

So $K_1^T=SR^{-1}$ or $K_1=(R^{-1})^TS^T=(R^T)^{-1}S^T=R^{-1}S^T$

So this is equivalent to a standard LQR problem with $Q-2SK_1+K_1^TRK_1$ in place of Q

$Q-2SK_1+K_1^TRK_1=Q-2SR^{-1}S^T+SR^{-1}RR^{-1}S^T=Q-SR^{-1}S^T$

3.(a)

$J=\sum_{k=0}^{\infty}\{y^2(k)+Ru^2(k)\}$

$G(z)=C(zI-A)^{-1}B=\frac{z(z+2)}{(z-1)(z+0.5)(z-2)}$

$G(z^{-1})=\frac{z^{-1}(z^{-1}+2)}{(z^{-1}-1)(z^{-1}+0.5)(z^{-1}-2)}=\frac{2z(z+0.5)}{(z-1)(z+2)(z-0.5)}$

$G(z^{-1})^TG(z)=\frac{2z^2(z+2)(z+0.5)}{(z-1)^2(z+0.5)(z-2)(z+2)(z-0.5)}$

Root locus given by $1+\frac{1}{R}G(z^{-1})^TG(z)=0$

$R(z-1)^2(z+0.5)(z-2)(z+2)(z-0.5)=-2z^2(z+2)(z+0.5)$

Cancellations at $z=-0.5$ and $z=-2$ (poles there no matter the value of R)

$R(z-1)^2(z-2)(z-0.5)=-2z^2$

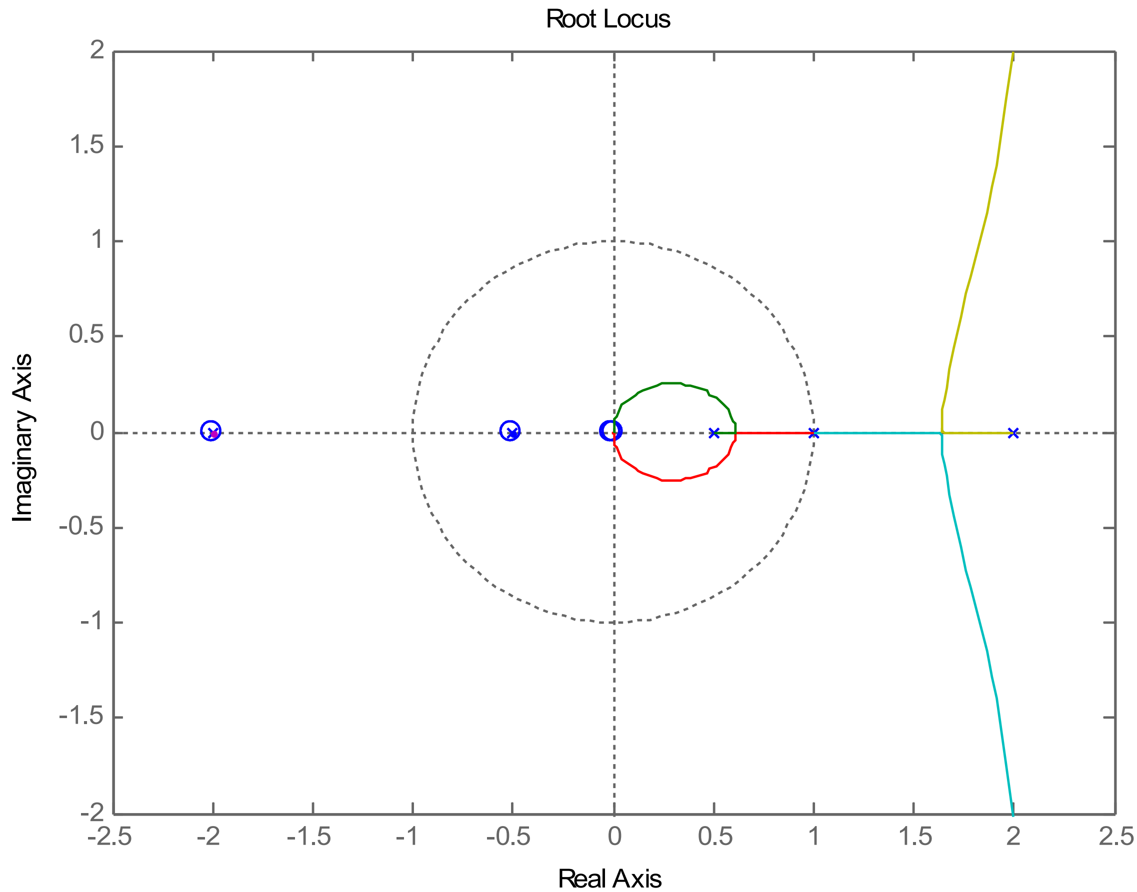
3.(b)

As $R \to 0$, the eigenvalues of $A_c = A - BK$ go to the stable zeros of A and the reciprocals of the unstable zeros: 0 and -0.5

3.(c)

As $R \to \infty$, the eigenvalues of $A_c = A - BK$ go to the stable poles of A and the reciprocals of the unstable poles: 1, -0.5, and 0.5

3.(d)



3.(e)

$R(z-1)^2(z-2)(z-0.5) = -2z^2$

$(z^2 - 2z + 1)(z^2 - 2.5z + 1) = -2z^2/R$

$z^4 - 4.5z^3 + 7z^2 - 4.5z + 1 = -2z^2/R$

For two equal, real, nonzero closed-loop eigenvalues λ_0, $(z-\lambda_0)^2(z-1/\lambda_0)^2 = 0$

$(z^2 - 2\lambda_0 z + \lambda_0^2)(z^2 - 2z/\lambda_0 + 1/\lambda_0^2) = 0$

$z^4 - 2(\lambda_0 + 1/\lambda_0)z^3 + (4 + \lambda_0^2 + 1/\lambda_0^2)z^2 - 2(\lambda_0 + 1/\lambda_0)z + 1 = 0$

$-2(\lambda_0 + 1/\lambda_0) = -4.5$

$\lambda_0^2 - 2.25\lambda_0 + 1 = 0$, so $\lambda_0 = \dfrac{2.25 \pm \sqrt{5.0625 - 4}}{2} = 0.6096$ or 1.64

Stable $\lambda_0 = 0.6096$, and $4 + \lambda_0^2 + 1/\lambda_0^2 = 7 + 2/R_0$

$R_0 = 2/(\lambda_0^2 + 1/\lambda_0^2 - 3) = 32$

3.(f)

$$G(z)=\frac{z(z+2)}{(z-1)(z+0.5)(z-2)}=\frac{z^2+2z}{z^3-2.5z^2+0.5z+1}$$

Controllable canonical form:

$$x(k+1)=\begin{bmatrix}2.5 & -0.5 & -1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}x(k)+\begin{bmatrix}1\\0\\0\end{bmatrix}u(k),\ y(k)=\begin{bmatrix}1 & 2 & 0\end{bmatrix}x(k)$$

3.(g)

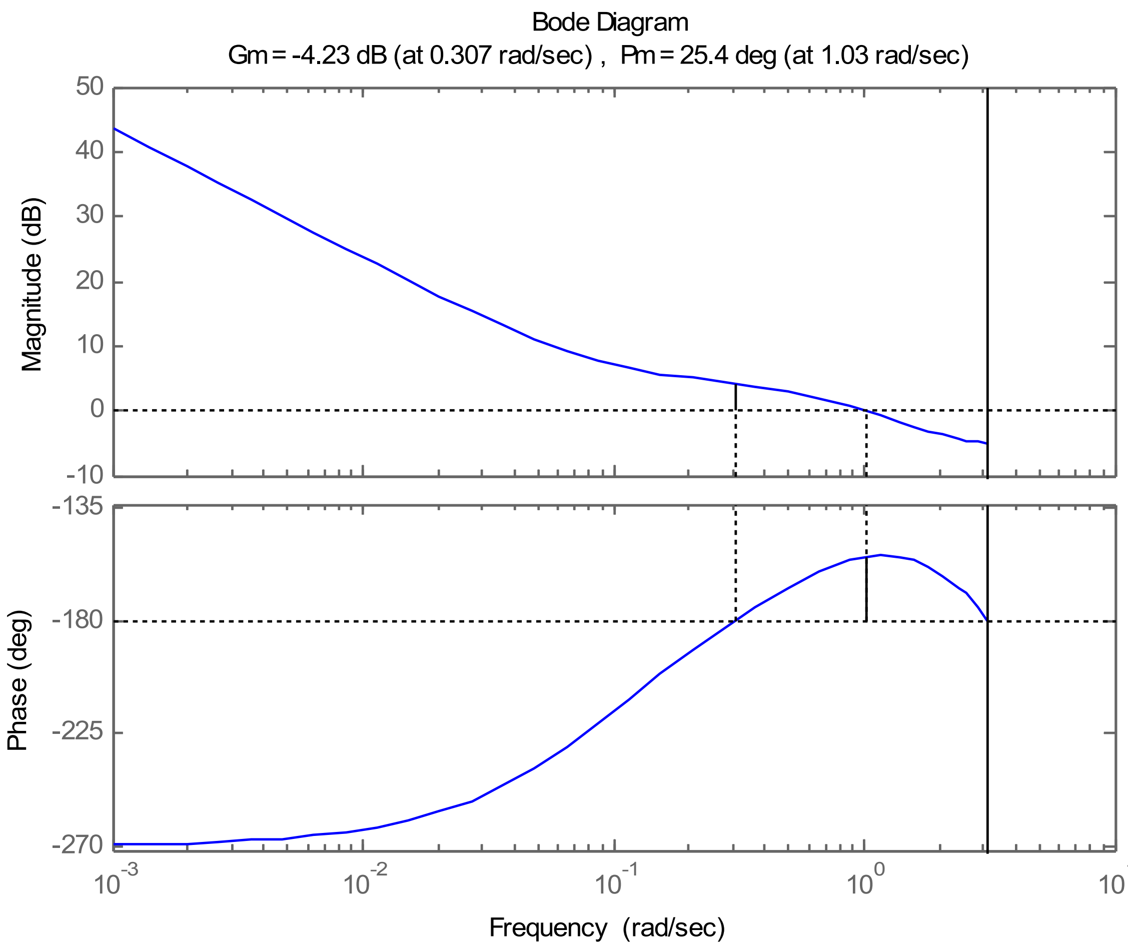
$$P_0=\begin{bmatrix}140.2159 & -43.8617 & -56.9848\\ -43.8617 & 28.3002 & 23.6155\\ -56.9848 & 23.6155 & 26.054\end{bmatrix}$$

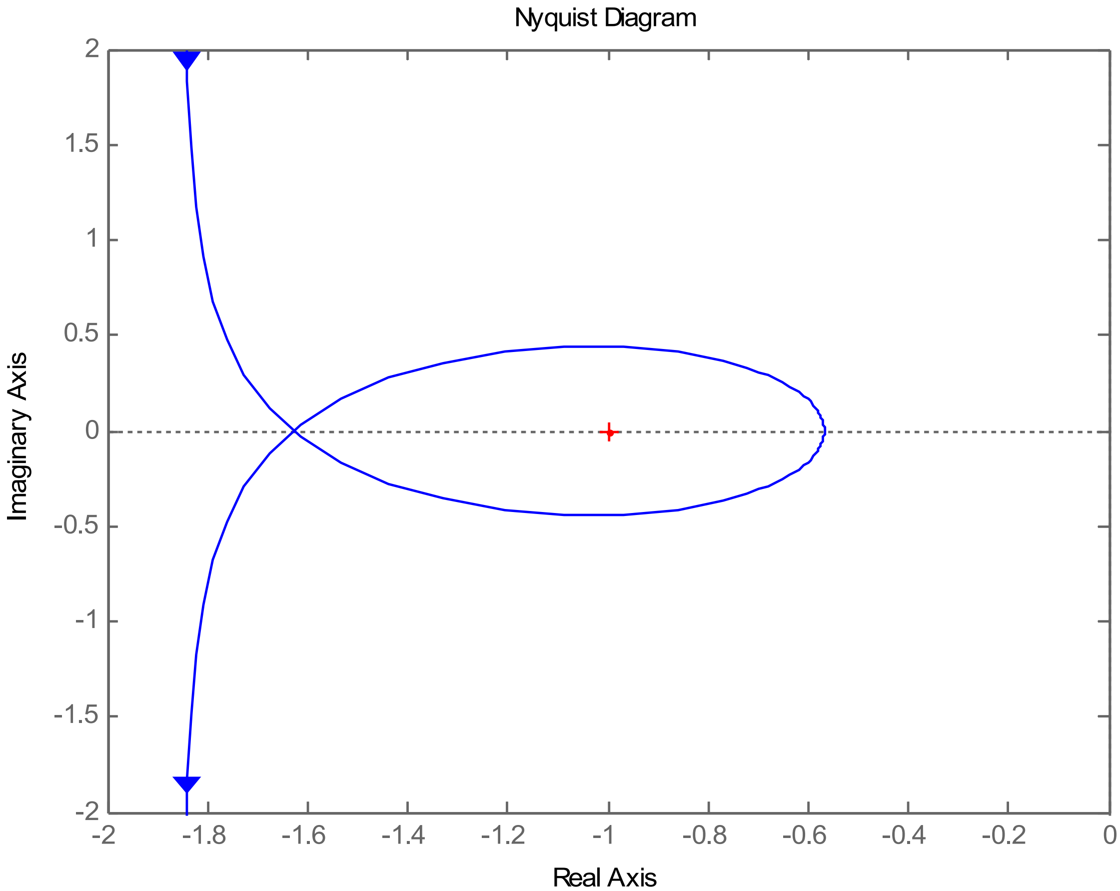
$$K_0=\begin{bmatrix}1.7808 & -0.738 & -0.8142\end{bmatrix}$$

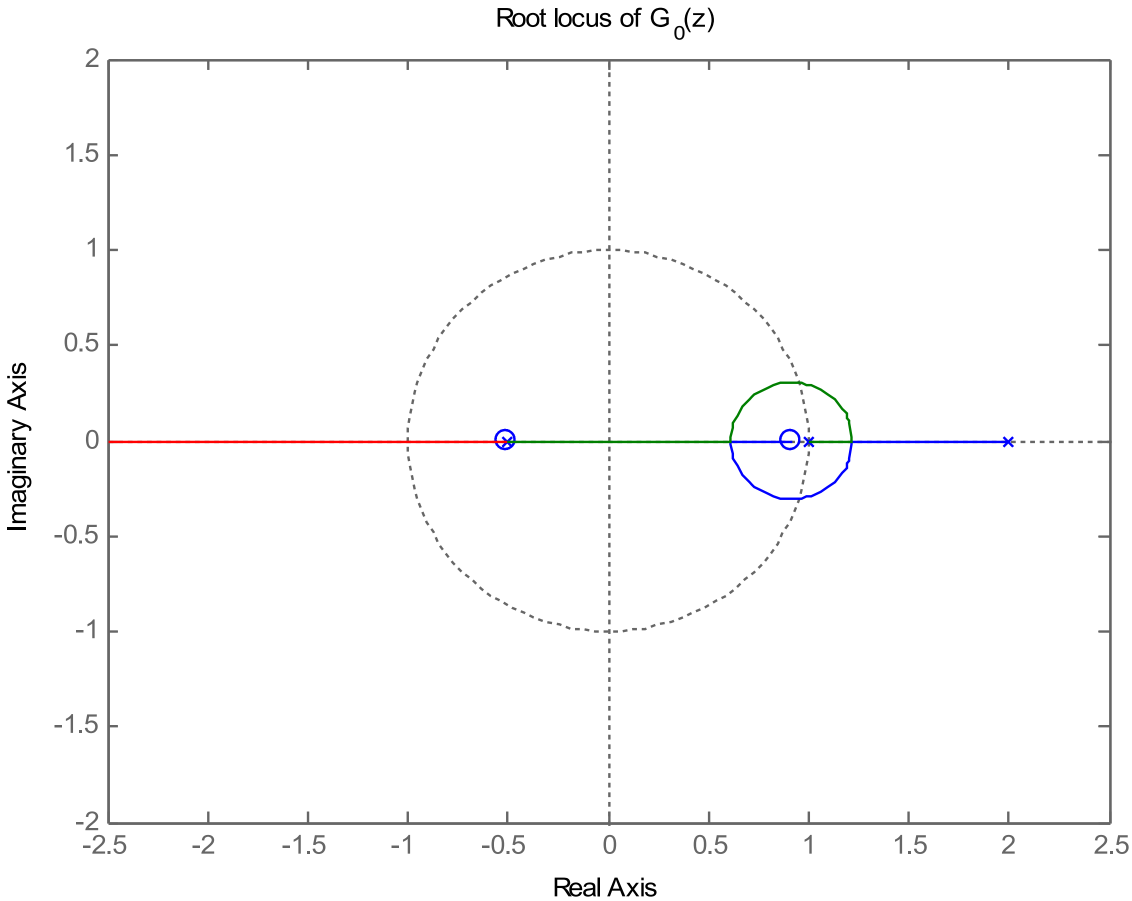
$\lambda=0.6096$, 0.6096, and -0.5

3.(h)

$$G_0(z)=K_0(zI-A)^{-1}B=\frac{1.781z^2-0.738z-0.8142}{z^3-2.5z^2+0.5z+1}$$




Nyquist Diagram
Imaginary Axis
Real Axis
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.8
-1.6
-1.4
-1.2
-1
-0.8
-0.6
-0.4
-0.2
0



3.(i)

$$r=\sqrt{\frac{R_0}{R_0+B^T P_0 B}}=0.4311$$

$$PM>2\sin^{-1}(0.5r)=0.4345\text{ rad}=24.9^{\circ}$$

Stable for $\frac{1}{1+r}<\gamma<\frac{1}{1-r}$

Stable for $0.6988<\gamma<1.7577$

$$20\log_{10}\left(\frac{1}{1+r}\right)=-3.1132\text{ dB},\ 20\log_{10}\left(\frac{1}{1-r}\right)=4.8987\text{ dB}$$

3.(j)

Riccati equation: $P=C^T C+A^T P A-A^T P B(R+B^T P B)^{-1}B^T P A$

Similarity transform: $\bar{A}=T A T^{-1}$, $\bar{B}=T B$, $\bar{C}=C T^{-1}$

Riccati for transformed system: $\bar{P}=\bar{C}^T\bar{C}+\bar{A}^T\bar{P}\bar{A}-\bar{A}^T\bar{P}\bar{B}(R+\bar{B}^T\bar{P}\bar{B})^{-1}\bar{B}^T\bar{P}\bar{A}$

$$\bar{P}=(T^{-1})^T C^T C T^{-1}+(T^{-1})^T A^T T^T\bar{P}T A T^{-1}-(T^{-1})^T A^T T^T\bar{P}T B(R+B^T T^T\bar{P}T B)^{-1}B^T T^T\bar{P}T A T^{-1}$$

$$\bar{P}=(T^T)^{-1}(C^T C+A^T T^T\bar{P}T A-A^T T^T\bar{P}T B(R+B^T T^T\bar{P}T B)^{-1}B^T T^T\bar{P}T A)T^{-1}$$

$$T^T\bar{P}T=C^T C+A^T T^T\bar{P}T A-A^T T^T\bar{P}T B(R+B^T T^T\bar{P}T B)^{-1}B^T T^T\bar{P}T A$$

So $T^T\bar{P}T$ satisfies the same Riccati equation as P, and since we know the solution to the Riccati equation is unique (controllable and observable system), we have $T^T\bar{P}T=P$

Therefore $B^T P B = B^T T^T \bar{P} T B = \bar{B}^T \bar{P} \bar{B}$ and $\sqrt{\dfrac{R}{R + B^T P B}} = \sqrt{\dfrac{R}{R + \bar{B}^T \bar{P} \bar{B}}}$

Guaranteed LQR margins are therefore preserved under a similarity transform.