ME 233 Advance Control II

Lecture 6
Random Processes

(ME233 Class Notes pp. PR6-PR13)

Random Process

A random processes is a ***continuous*** function of time

$$X(\cdot) : \mathcal{R} \rightarrow \mathcal{R}$$

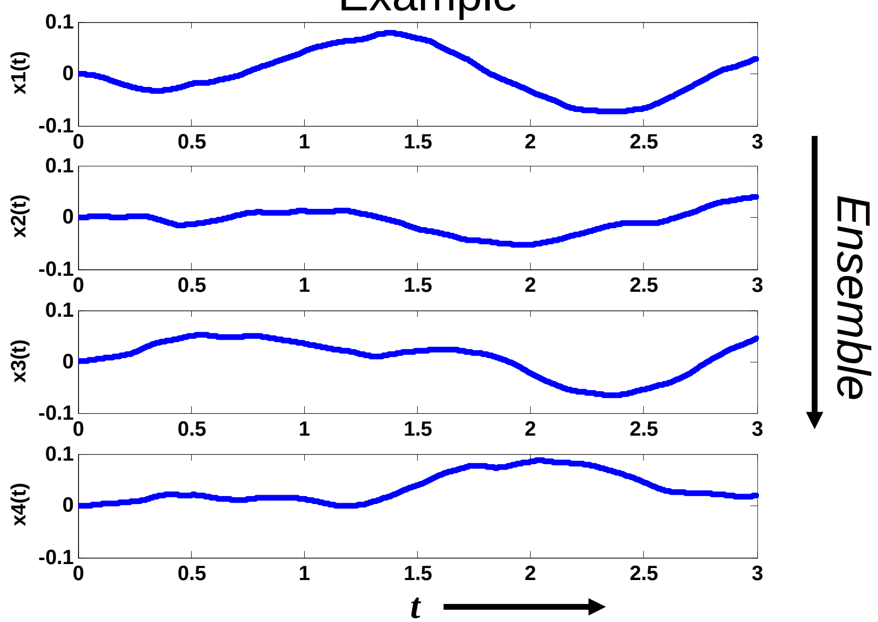
Such that for any time t_o ,

$$X(t_o)$$

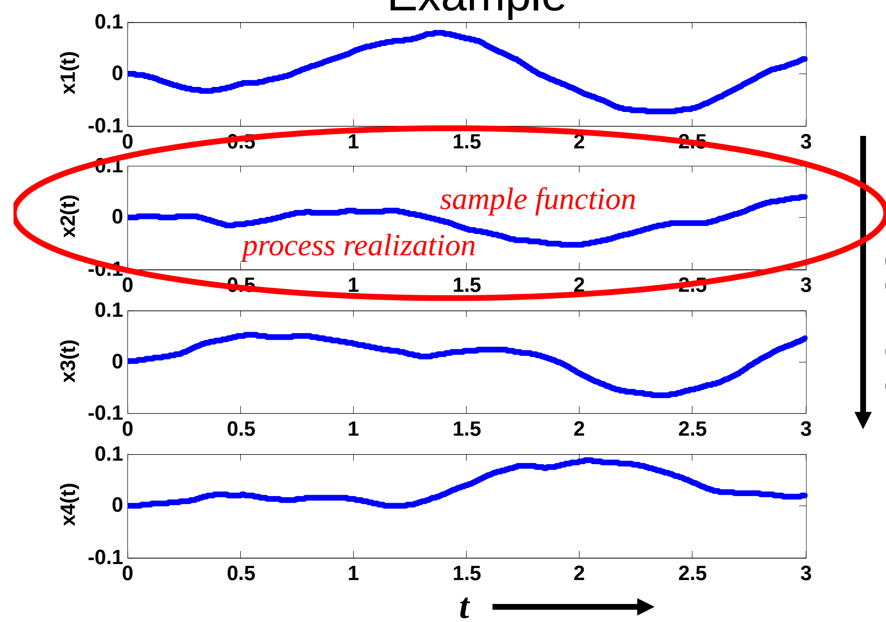
Is a random variable defined over the same probability space

$$(\Omega,\, \mathcal{S},\, Pr)$$

Example



Example



Random process

Let $X(t)$ be a random process

Let $\{t_1, t_2, \cdots, t_N\}$ be a collection of times

$$p_{X(t_1), X(t_2), \cdots, X(t_N)}(x_{t_1}, x_{t_2}, \cdots, x_{t_N})$$

is the join PDF of

$$\{X(t_1), X(t_2), \cdots, X(t_N)\}$$

This is often a huge amount of redundant information

2nd order statistics

Let $X(t)$ be a random vector process

Expected value or mean of *X(t),*

$$E\{X(t)\} = m_X(t)$$

Auto-covariance function:

$$\Lambda_{XX}(t, \underline{\tau}) =$$

$$E\left\{[X(t+\underline{\tau}) - m_X(t+\underline{\tau})]\,[X(t) - m_X(t)]^T\right\}$$

Auto-covariance function

Define: $\tilde{X}(t) = X(t) - m_X(t)$

$$\boxed{\Lambda_{XX}(t, \tau) = E\left\{\tilde{X}(t+\tau)\tilde{X}^T(t)\right\}}$$

$$\Lambda_{XX}(t+\tau) = E\left\{\begin{bmatrix} \tilde{X}_1(t+\tau) \\ \vdots \\ \tilde{X}_n(t+\tau) \end{bmatrix} \begin{bmatrix} \tilde{X}_1(t) & \cdots & \tilde{X}_n(t) \end{bmatrix}\right\}$$

Strict Sense Stationary random sequence

A random process $X(t)$

is **Strict Sense Stationary (SSS)** if the joint probability, is invariant with time

$$P(X(t_1) \le x_{t_1}, \cdots, X(t_N) \le x_{t_N}) \quad =$$

$$P(X(t_1 + \underline{T}) \le x_{t_1}, \cdots, X(t_N + \underline{T}) \le x_{t_N})$$

for any time shift T,

Ergodicity

A **Strict Sense Stationary** random process

$$X(t)$$

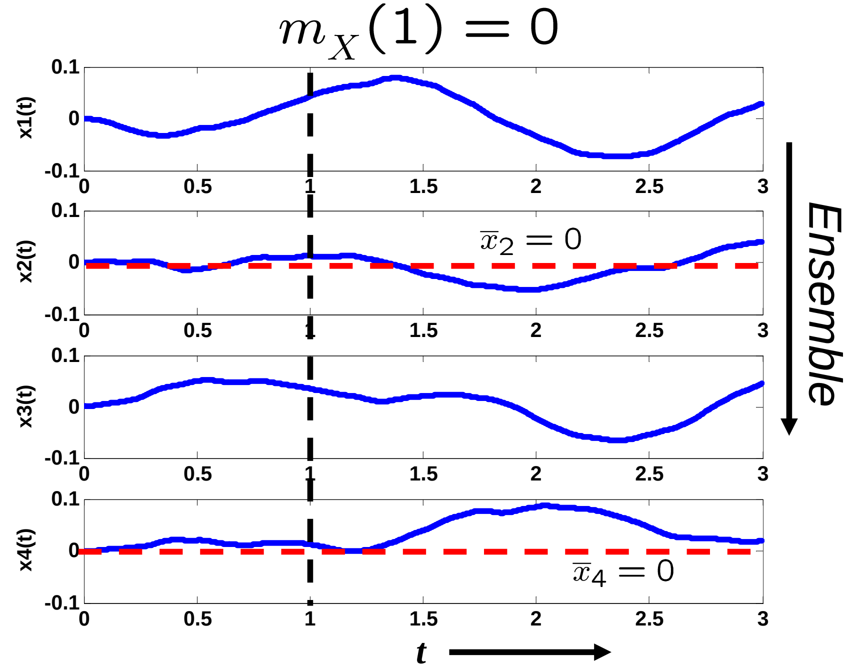
is **ergodic** if we can recover an ensemble average from the time average of any realization:

$$E\{X(t)\} = m_X$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-T}^{T} x(t)dt$$

with probability 1
(almost surely)

$$= \bar{x}$$



Wide Sense Stationarity

A random sequence

is **Wide Sense Stationary (WSS)** if:

1) Its mean is time invariant

$$E\{X(t)\} = m_X$$

$$\text{SSS} \Rightarrow \text{WSS}$$

Wide Sense Stationarity

A random sequence

is **Wide Sense Stationary (WSS)** if:

2) Its covariance only depends on the correlation shift $\boldsymbol{\tau}$

$$\Lambda_{XX}(t,\tau) = \Lambda_{XX}(t+T,\tau)$$

$$\text{SSS} \Rightarrow \text{WSS}$$

Wide Sense Stationarity

A random sequence

is **Wide Sense Stationary (WSS)** if:

2) Its covariance only depends on the correlation shift $\boldsymbol{\tau}$

$$E\left\{\tilde{X}(t+\tau)\tilde{X}^T(t)\right\} = E\left\{\tilde{X}(t)\tilde{X}^T(t-\tau)\right\}$$

$$\text{SSS} \;\Rightarrow\; \text{WSS}$$

Wide Sense Stationarity

The auto-covariance function can be defined only as a function of the correlation time shift $\boldsymbol{\tau}$

$$\Lambda_{XX}(\underline{\tau}) = E\left\{\tilde{X}(t+\underline{\tau})\tilde{X}^T(t)\right\}$$

Notice that:

$$\boxed{\begin{gathered}\Lambda_{XX}(\tau) = \Lambda_{XX}^T(-\tau) \\ \text{trace}\{\Lambda_{XX}(0)\} \geq |\text{trace}\{\Lambda_{XX}(\tau)\}|\end{gathered}}$$

Cross-covariance function

Let $X(t) \in \mathcal{R}^n$ and $Y(t) \in \mathcal{R}^m$ be two **WSS** random vector sequences

The cross-covariance function:

$$\Lambda_{XY}(\tau) = E\left\{\tilde{X}(t+\tau)\tilde{Y}^T(t)\right\}$$

for ***any*** time t

Cross-covariance function

$$\Lambda_{XY}(\tau) = E\left\{\tilde{X}(t+\tau)\tilde{Y}^T(t)\right\}$$

$$\Lambda_{XY}(\tau) = \Lambda_{YX}^T(-\tau)$$

Power Spectral Density Function

For WSS random process, the power spectral density function is the Fourier transform of the auto-covariance function:

$$
\begin{aligned}
\Phi_{XX}(\omega) &= \mathcal{F}\{\Lambda_{XX}(\tau)\} \\
&= \int_{-\infty}^{\infty} \Lambda_{XX}(\tau) e^{-j\omega\tau} d\tau
\end{aligned}
$$

Power Spectral Density Function

Since,

$$
\begin{aligned}
\Lambda_{XX}(\tau) &= \mathcal{F}^{-1}\{\Phi_{XX}(\omega)\} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega\tau} \Phi_{XX}(\omega)\, d\omega
\end{aligned}
$$

$$
\Lambda_{XX}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{XX}(\omega)\, d\omega
$$

White noise

A WSS random process $W(t) \in \mathcal{R}$ **is white** if:

$$
\boxed{\Lambda_{WW}(t) = \sigma_W^2\, \delta(t)}
$$

Where $\delta(t)$ is the **Dirac delta impulse**

white noise is zero mean if $E\{W(t)\} = 0$

White noise

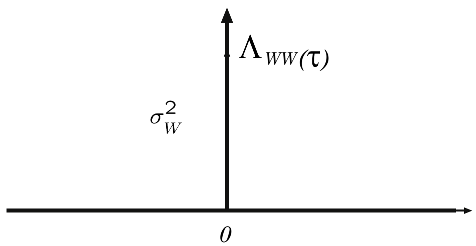
The power spectral density function for white noise is:

$$
\boxed{\Phi_{WW}(w) = \sigma_W^2}
$$

Proof:

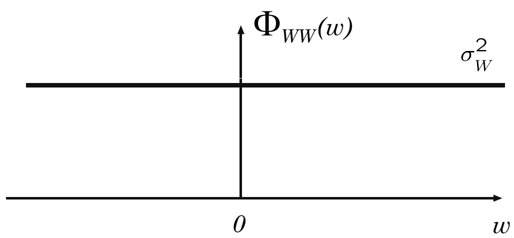
$$
\begin{aligned}
\Phi_{WW}(\omega) &= \int_{-\infty}^{\infty} \Lambda_{WW}(\tau) e^{-j\omega\tau} d\tau \\
&= \sigma_W^2 \int_{-\infty}^{\infty} \cancelto{1}{e^{-j\omega\tau}}\, \delta(\tau)\, d\tau \\
&= \sigma_W^2
\end{aligned}
$$

White noise



$$\Lambda_{WW}(\tau) = \sigma_W^2\,\delta(\tau)$$

$$\Phi_{WW}(w) = \sigma_W^2$$



Infinite bandwidth

White noise vector process

A **WSS** random vector sequence $W(t) \in \mathcal{R}^n$ **is white** if:

$$\Lambda_{WW}(\tau) = \Sigma_{WW}\,\delta(\tau)$$

where

$$\Sigma_{WW} = \Sigma_{WW}^T \geq 0$$

and $\delta(t)$ is the Dirac delta impulse

MIMO Linear Time Invariant Systems

Let $G(t) \in \mathcal{R}^{p\times m}$

be the impulse response of an LTI SISO system with transfer function

$$G(s) = \mathcal{L}\{G(t)\} = \int_{-\infty}^{\infty} e^{-st}\,G(t)\,dt$$

MIMO Linear Time Invariant Systems

Let $U(t) \in \mathcal{R}^m$ be WSS

Then the forced response (zero initial state)

$$Y(t) = \int_{-\infty}^{\infty} G(\tau)U(t-\tau)d\tau$$

$Y(t) \in \mathcal{R}^p$ is also WSS

MIMO Linear Time Invariant Systems

We will assume that

- The WSS random process $U(t)$ is zero mean, I.e.

$$E\{U(t)\} = m_U = 0$$

Thus, the output random process is also zero mean

$$E\{Y(t)\} = m_Y = 0$$

MIMO Linear Time Invariant Systems

Let $U(t)$ be WSS

If

$$U(t) \longrightarrow \boxed{G(t)} \longrightarrow Y(t)$$

Then:

$$\Lambda_{UU}(\tau) \longrightarrow \boxed{G(\tau)} \longrightarrow \Lambda_{YU}(\tau) = E\{\tilde{Y}(t+\tau)\tilde{U}^T(t)\}$$

MIMO Linear Time Invariant Systems

Let $U(t)$ be a WSS random process

$$\Lambda_{UU}(s) \longrightarrow \boxed{G(s)} \longrightarrow \Lambda_{YU}(s)$$

$$\Phi_{UU}(w) \longrightarrow \boxed{G(w)} \longrightarrow \Phi_{YU}(w)$$

$\Phi_{UU}(w) = \Lambda_{UU}(s)|_{s=j\omega}$ $\Phi_{YU}(w) = \Lambda_{YU}(s)|_{s=j\omega}$

MIMO Linear Time Invariant Systems

Let $U(t)$ be a WSS vector random process

If $$Y(t) = \int_{-\infty}^{\infty} G(\tau) U(t-\tau) d\tau$$

Then:

$$\boxed{\Lambda_{YU}(\tau) = \int_{-\infty}^{\infty} G(\eta)\, \Lambda_{UU}(\tau-\eta)\, d\eta}$$

$$\boxed{\Phi_{YU}(w) = G(w)\, \Phi_{UU}(w)}$$

MIMO Linear Time Invariant Systems

$$\boxed{\Lambda_{YU}(\tau) = \int_{-\infty}^{\infty} G(\eta)\,\Lambda_{UU}(\tau-\eta)\,d\eta}$$

Proof: $Y(t) = \int_{-\infty}^{\infty} G(\tau)U(t-\tau)d\tau \qquad (m_U = 0)$

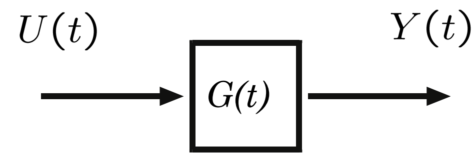
Then:

$$\begin{aligned}
\Lambda_{YU}(\tau) &= E\{Y(t+\tau)U^T(t)\} \\
&= E\left\{\left[\int_{-\infty}^{\infty} G(\eta)\,U(t+\tau-\eta)d\eta\right]U^T(t)\right\} \\
&= \int_{-\infty}^{\infty} G(\eta)\,E\left\{U(t+\tau-\eta)U^T(t)\right\}d\eta \\
&= \int_{-\infty}^{\infty} G(\eta)\,\Lambda_{UU}(\tau-\eta)\,d\eta
\end{aligned}$$

MIMO Linear Time Invariant Systems

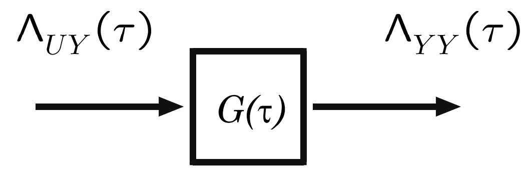
Let $U(t)$ be WSS

If



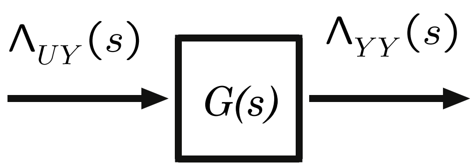
Then:

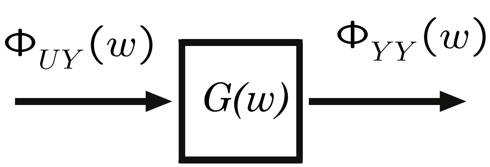
$E\{\tilde{U}(t+\tau)\tilde{Y}^T(t)\} =$



MIMO Linear Time Invariant Systems

Let $U(t)$ be a WSS random process





$\Phi_{UY}(w) = \Lambda_{UY}(s)|_{s=j\omega} \qquad \Phi_{YY}(w) = \Lambda_{YY}(s)|_{s=j\omega}$

MIMO Linear Time Invariant Systems

$$\boxed{\Phi_{UY}(w) = \Phi_{YU}^T(-w)}$$

Proof: Remember that $\Lambda_{UY}(\tau) = \Lambda_{YU}^T(-\tau)$

$$\begin{aligned}
\Phi_{UY}(\omega) &= \int_{-\infty}^{\infty} \Lambda_{UY}(\tau)e^{-j\omega\tau}d\tau \\
&= \int_{-\infty}^{\infty} \Lambda_{YU}^T(-\tau)e^{-j\omega\tau}d\tau = \int_{-\infty}^{\infty} \Lambda_{YU}^T(\tau)e^{j\omega\tau}d\tau \\
&= \Phi_{YU}^T(-\omega)
\end{aligned}$$

MIMO Linear Time Invariant Systems

Let $U(t)$ be WSS

If
$$Y(t) = \int_{-\infty}^{\infty} G(\tau) U(t-\tau) d\tau$$

Then:

$$\boxed{\Phi_{YY}(\omega) = G(\omega)\, \Phi_{UU}(\omega)\, G^T(-\omega)}$$

MIMO Linear Time Invariant Systems

Proof: Use
$$\Phi_{YY}(w) = G(w)\, \Phi_{UY}(w)$$
$$\Phi_{YU}(w) = G(w)\, \Phi_{UU}(w)$$
$$\Phi_{UY}(w) = \Phi^T_{YU}(-w)$$

then

$$\Phi_{UY}(w) = \underbrace{\Phi^T_{UU}(-w)}_{\Phi_{UU}(w)}\, G^T(-w)$$

and

$$\boxed{\Phi_{YY}(\omega) = G(\omega)\, \Phi_{UU}(\omega)\, G^T(-\omega)}$$

White noise driven state space systems

Consider a LTI system driven by white noise:

$$\frac{d}{dt} X(t) = A\, X(t) + B\, W(t)$$

$$Y(t) = C\, X(t)$$

$X(t) \in \mathcal{R}^n$

$W(t) \in \mathcal{R}^p$

$Y(t) \in \mathcal{R}^m$

White noise driven state space systems

$$\begin{aligned} \frac{d}{dt} X(t) &= A\, X(t) + B\, W(t) \\ Y(t) &= C\, X(t) \end{aligned}$$

Assume that $W(t)$ is white, but not stationary

$$m_W(t) = E\{W(t)\}$$

$$\Lambda_{WW}(t,\tau) = \Sigma_{WW}(t)\, \delta(\tau)$$

White noise driven state space systems

$$\frac{d}{dt}X(t) = A\,X(t) + B\,W(t)$$
$$Y(t) = C\,X(t)$$

Assume state Initial Conditions (IC):

$$m_X(0) = E\{X(0)\}$$

$$\Lambda_{XX}(0,0) = E\left\{\tilde{X}(0)\tilde{X}^T(0)\right\}$$

$$E\left\{\tilde{X}(0)\tilde{W}^T(t)\right\} = 0$$

White noise driven state space systems

$$\frac{d}{dt}X(t) = A\,X(t) + B\,W(t)$$

$$Y(t) = C\,X(t)$$

Taking expectations on the equations above, we obtain:

$$\frac{d}{dt}m_X(t) = A\,m_X(t) + B\,m_W(t)$$

$$m_Y(t) = C\,m_X(t)$$

White noise driven state space systems

Subtracting the means,

$$\frac{d}{dt}\tilde{X}(t) = A\,\tilde{X}(t) + B\,\tilde{W}(t)$$

$$\tilde{Y}(t) = C\,\tilde{X}(t)$$

$$m_{\tilde{W}}(t) = 0 \qquad m_{\tilde{X}}(t) = 0 \qquad m_{\tilde{Y}}(t) = 0$$

White noise driven covariance propagation

$$\boxed{\begin{aligned}\frac{d}{dt}\Lambda_{XX}(t,0) &= A\,\Lambda_{XX}(t,0) + \Lambda_{XX}(t,0)\,A^T \\ &+ B\,\Sigma_{WW}(t)\,B^T\end{aligned}}$$

with

$$\Lambda_{XX}(t,0) = E\left\{\tilde{X}(t)\tilde{X}^T(t)\right\}$$

$$\Lambda_{WW}(t,0) = E\left\{\tilde{W}(t)\tilde{W}^T(t)\right\} = \Sigma_{WW}(t)$$

White noise driven **covariance propagation**

Also,

$$\boxed{\Lambda_{XX}(t,\tau) \;=\; e^{A\tau}\,\Lambda_{XX}(t,0) \qquad \tau \geq 0}$$

where:

$$\Lambda_{XX}(t,\tau) \;=\; E\left\{\tilde{X}(t+\tau)\tilde{X}^T(t)\right\}$$

White noise driven **covariance propagation**

Also,

$$\boxed{\Lambda_{XX}(t,-\tau) \;=\; \Lambda_{XX}(t,0)\,e^{A^T\tau} \quad \tau \geq 0}$$

where:

$$\Lambda_{XX}(t,\tau) \;=\; E\left\{\tilde{X}(t+\tau)\tilde{X}^T(t)\right\}$$

Stationary covariance equation

For *W(t)* WSS,

$$m_W(t) \;=\; m_W$$

$$\Lambda_{WW}(t,0) = \Sigma_{WW}$$

and A Hurwitz,

$$\bar{\Lambda}_{XX}(\tau) = \lim_{t\to\infty} E\{\tilde{X}(t+\tau)\tilde{X}^T(t)\}$$

Stationary covariance equation

For *W(t)* WSS, and A Hurwitz,

$$\bar{\Lambda}_{XX}(\tau) = \lim_{t\to\infty} E\{\tilde{X}(t+\tau)\tilde{X}^T(t)\}$$

Satisfies:

$$\boxed{\begin{aligned} &A\,\bar{\Lambda}_{XX}(0) + \bar{\Lambda}_{XX}(0)\,A^T = -B\,\Sigma_{WW}\,B^T \\ &\bar{\Lambda}_{XX}(\tau) = e^{A\tau}\,\bar{\Lambda}_{XX}(0) \qquad \tau \geq 0 \end{aligned}}$$

The next section contains some Proofs of the CT results

Please go over them by yourselves…

Proof of continuous time results – Method 1

We first proof that:

$$\frac{d}{dt}\Lambda_{XX}(t,0) = A\,\Lambda_{XX}(t,0) + \Lambda_{XX}(t,0)\,A^T + B\,\Sigma_{WW}(t)\,B^T$$

By starting from the Discrete Time (DT) results

Proof of continuous time results – Method 1

Approximate the state equation ODE

$$\frac{d}{dt}X(t) = A\,X(t) + B\,W(t)$$

using the Euler numerical integration method.

$$\frac{d}{dt}X(t) \approx \frac{1}{\Delta t}\{X((k+1)\Delta t) - X(k\Delta t)\}$$

- We have to be careful in dealing with white noise $W(t)$

Approximate $W(t)$

1. Define $W(k)$ as the ***time average*** of $W(t)$

$$W(k) \approx \frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t} W(t)dt$$

Similarly, taking expectations

$$m_W(k) \approx \frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t} m_W(t)dt$$

Approximate $\Lambda_{WW}(k,0)$ for ***W(t)*** white

$$\Lambda_{WW}(k,0) = E\{\tilde{W}(k)\tilde{W}^T(k)\}$$

$$\approx E\left\{\underbrace{\left(\frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t}\tilde{W}(t)dt\right)}_{\approx \tilde{W}(k)}\underbrace{\left(\frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t}\tilde{W}^T(\tau)d\tau\right)}_{\approx \tilde{W}^T(k)}\right\}$$

Approximate $\Lambda_{WW}(k,0)$ for ***W(t)*** white

$$\Lambda_{WW}(k,0) = E\{\tilde{W}(k)\tilde{W}^T(k)\}$$

$$\approx \frac{1}{(\Delta t)^2}\int_{k\Delta t}^{(k+1)\Delta t}\int_{k\Delta t}^{(k+1)\Delta t}\underbrace{E\left\{\tilde{W}(t)\tilde{W}^T(\tau)\right\}}_{\Sigma_{WW}(\tau)\underbrace{\delta(t-\tau)}_{\textit{Dirac impulse}}}d\tau dt$$

since for ***W(t)*** white

$$E\left\{\tilde{W}(t)\tilde{W}^T(\tau)\right\} = E\left\{\tilde{W}(\tau+t-\tau)\tilde{W}^T(\tau)\right\} = \Sigma_{WW}(\tau)\delta(t-\tau)$$

Approximate $\Lambda_{WW}(k,0)$ for ***W(t)*** white

$$\Lambda_{WW}(k,0) = E\{\tilde{W}(k)\tilde{W}^T(k)\}$$

$$\approx \frac{1}{(\Delta t)^2}\int_{k\Delta t}^{(k+1)\Delta t}\underbrace{\left[\int_{k\Delta t}^{(k+1)\Delta t}\Sigma_{WW}(\tau)\delta(t-\tau)d\tau\right]}_{\Sigma_{WW}(t)}dt$$

$$\approx \frac{1}{(\Delta t)^2}\int_{k\Delta t}^{(k+1)\Delta t}\Sigma_{WW}(t)dt$$

Approximate $\Lambda_{WW}(k,0)$ for ***W(t)*** white

$$\Lambda_{WW}(k,0) = E\{\tilde{W}(k)\tilde{W}^T(k)\}$$

$$\approx \frac{1}{(\Delta t)^2}\int_{k\Delta t}^{(k+1)\Delta t}\Sigma_{WW}(t)dt$$

$$\approx \frac{1}{(\Delta t)}\underbrace{\left[\frac{1}{(\Delta t)}\int_{k\Delta t}^{(k+1)\Delta t}\Sigma_{WW}(t)dt\right]}_{\Sigma_{WW}(k)}$$

Approximate $\Lambda_{WW}(k,0)$ for ***W(t)*** white

$$\Lambda_{WW}(k,0) \;\approx\; \frac{1}{\Delta t}\,\Sigma_{WW}(k)$$

Where $\Sigma_{WW}(k)$ is the ***time average*** of $\Sigma_{WW}(t)$

$$\Sigma_{WW}(k) \;=\; \frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t} \Sigma_{WW}(\tau)d\tau$$

Numerical Integration

The state equation

$$\frac{d}{dt}X(t) \;=\; A\,X(t) + B\,W(t)$$

By the discrete time state equation

$$X(k+1) \;\approx\; \underbrace{[I+\Delta t\,A]}_{A_d}\,X(k) + \underbrace{B\,\Delta t}_{B_d}\,W(k)$$

where

$$W(k) \approx \frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t} W(t)dt$$

Proof of continuous time results – Method 1

1. Obtain DT state equations by approximating the CT state equation solution:

$$\frac{d}{dt}X(t) \;=\; A\,X(t) + B\,W(t)$$

$$\frac{d}{dt}X(t) \;\approx\; \frac{1}{\Delta t}\{X((k+1)\Delta t) - X(k\Delta t)\}$$

Thus,

$$\boxed{X(k+1) \;\approx\; \underbrace{[I+\Delta t\,A]}_{A_d}\,X(k) + \underbrace{B\,\Delta t}_{B_d}\,W(k)}$$

where

$$W(k) \approx \frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t} W(t)dt$$

Proof of continuous time results – M1

2. Obtain the CT covariance propagation equation from from the DT covariance propagation, using the approximated DT state equation:

$$\Lambda_{XX}(k+1,0) \;\approx\; A_d\,\Lambda_{XX}(k,0)\,A_d^T + B_d\,\frac{1}{\Delta t}\Sigma_{WW}(k)\,B_d^T$$

$$\approx (I+\Delta t\,A)\,\Lambda_{XX}(k,0)\;(I+\Delta t\,A)^T + \Delta t\,B\,\Sigma_{WW}(k)\,B^T$$

$$\approx \Lambda_{XX}(k,0) + \Delta t\,A\Lambda_{XX}(k,0) + \Delta t\,\Lambda_{XX}(k,0)A^T$$

$$+(\Delta t)^2\,A\Lambda_{XX}(k,0)A^T + \Delta t\,B\,\Sigma_{WW}(k)\,B^T$$

Proof of continuous time results – M1

3. Take the limit as $\Delta t \to 0$ of

$$\frac{\Lambda_{XX}((k+1)\Delta t,0)-\Lambda_{XX}(k\Delta t,0)}{\Delta t} \approx$$
$$A\Lambda_{XX}(k\Delta t,0)+\Lambda_{XX}(k\Delta t,0)\,A^T+B\,\Sigma_{WW}(k)\,B^T$$
$$+\Delta t\,A\Lambda_{XX}(k\Delta t,0)\,A^T$$

and noticing that

$$\begin{aligned}\lim_{\Delta t\to 0}\Sigma_{WW}(k) &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t}\Sigma_{WW}(t)dt\\ &= \Sigma_{WW}(t)\end{aligned}$$

Proof of continuous time results – M1

3. Take the limit as $\Delta t \to 0$ of

$$\cancelto{\frac{d}{dt}\Lambda_{WW}(t,0)}{\frac{\Lambda_{XX}((k+1)\Delta t,0)-\Lambda_{XX}(k\Delta t,0)}{\Delta t}} \approx$$
$$A\Lambda_{XX}(\cancelto{t}{k\Delta t},0)+\Lambda_{XX}(\cancelto{t}{k\Delta t},0)\,A^T+B\,\cancelto{\Sigma_{WW}(t)}{\Sigma_{WW}(k)}\,B^T$$
$$+\cancelto{0}{\Delta t}\,A\Lambda_{XX}(k\Delta t,0)\,A^T$$

Thus,

$$\begin{aligned}\frac{d}{dt}\Lambda_{XX}(t,0) &= A\,\Lambda_{XX}(t,0)+\Lambda_{XX}(t,0)\,A^T\\ &\quad +B\,\Sigma_{WW}(t)\,B^T\end{aligned}$$

Proof of continuous time results – Method 2

We now proof that:

$$\begin{aligned}\frac{d}{dt}\Lambda_{XX}(t,0) &= A\,\Lambda_{XX}(t,0)+\Lambda_{XX}(t,0)\,A^T\\ &+ B\,\Sigma_{WW}(t)\,B^T\end{aligned}$$

Directly from continuous time (CT) results

Proof of continuous time results – M2

1) Lets calculate $\frac{d}{dt}\Lambda_{XX}(t,0)$ using

$$\dot{\tilde{X}}(t) = A\,\tilde{X}(t)+B\,\tilde{W}(t)$$

$$\begin{aligned}\frac{d}{dt}\Lambda_{XX}(t,0) &= \frac{d}{dt}E\{\tilde{X}(t)\tilde{X}^T(t)\}\\ &= E\{\underbrace{\dot{\tilde{X}}(t)}_{A\tilde{X}(t)+B\tilde{W}(t)}\tilde{X}^T(t)\}+E\{\tilde{X}(t)\underbrace{\dot{\tilde{X}}^T(t)}_{\tilde{X}^T(t)A^T+\tilde{W}^T(t)B^T}\}\\ &= A\Lambda_{XX}(t,0)+\Lambda_{XX}(t,0)A^T\\ &\quad +B\,E\{\tilde{W}(t)\tilde{X}^T(t)\}+E\{\tilde{X}(t)\tilde{W}^T(t)\}B^T\end{aligned}$$

Proof of continuous time results – M2

2) We now need to calculate

$$B\,E\{\tilde W(t)\tilde X^T(t)\} + E\{\tilde X(t)\tilde W^T(t)\}B^T$$

using

$$\tilde X(t) = e^{At}\,\tilde X(0) + \int_0^t e^{A(t-\tau)} B\,\tilde W(\tau)d\tau$$

$$\begin{aligned} B\,E\{\tilde W(t)\tilde X^T(t)\} &= B\,E\{\tilde W(t)\tilde X(0)\}e^{A^T t} \\ &= +B\int_0^t E\{\tilde W(t)\tilde W(\tau)\}B^T e^{A^T(t-\tau)}d\tau \end{aligned}$$

Proof of continuous time results – M2

2) We now need to calculate $B\,E\{\tilde W(t)\tilde X^T(t)\}$ using

$$\tilde X(t) = e^{At}\,\tilde X(0) + \int_0^t e^{A(t-\tau)} B\,\tilde W(\tau)d\tau$$

$$\begin{aligned} B\,E\{\tilde W(t)\tilde X^T(t)\} &= B\,\underbrace{E\{\tilde W(t)\tilde X(0)\}}_{=o}\,e^{A^T t} \\ &= +B\int_0^t \underbrace{E\{\tilde W(t)\tilde W^T(\tau)\}}_{\Sigma_{WW}(\tau)\delta(t-\tau)}B^T e^{A^T(t-\tau)}d\tau \\ &= B\int_0^t \Sigma_{WW}(\tau)\delta(t-\tau)B^T e^{A^T(t-\tau)}d\tau \end{aligned}$$

(notice that the Dirac impulse occurs at the edge t)

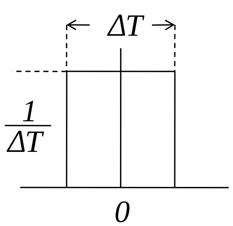
Proof of continuous time results – M2

2) Continuing,

$$\begin{aligned} B\,E\{\tilde W(t)\tilde X^T(t)\} &= B\int_0^t \Sigma_{WW}(\tau)\delta(t-\tau)B^T e^{A^T(t-\tau)}d\tau \\ &= B\int_0^t \Sigma_{WW}(t-\eta)\delta(\eta)B^T e^{A^T\eta}d\eta \end{aligned}$$

(make integral symmetrical w/r *0*)

$$\begin{aligned} &= \frac{1}{2}B\int_{-t}^t \Sigma_{WW}(t-\eta)\delta(\eta)B^T e^{A^T\eta}d\eta \\ &= \frac{1}{2}B\,\Sigma_{WW}(t)B^T \end{aligned}$$



Proof of continuous time results – M2

2) A similar calculation for $E\{\tilde X(t)\tilde W^T(t)\}\,B^T$ yields

$$\begin{aligned} E\{\tilde X(t)\tilde W^T(t)\}B^T &= e^{At}\,\underbrace{E\{\tilde X(0)\tilde W^T(t)\}}_{=o}\,B^T \\ &= +\int_0^t e^{A(t-\tau)}B\,\underbrace{E\{\tilde W(\tau)\tilde W^T(t)\}}_{\Sigma_{WW}(t)\delta(\tau-t)}\,d\tau\,B^T \\ &= \int_0^t e^{A(t-\tau)}B\Sigma_{WW}(t)\delta(\tau-t)d\tau\,B^T \end{aligned}$$

(notice that the Dirac impulse occurs at the edge t)

Proof of continuous time results – M2

2) Continuing,

$$
\begin{aligned}
E\{\tilde{X}(t)\tilde{W}^T(t)\}B^T &= \int_0^t e^{A(t-\tau)} B\Sigma_{WW}(t)\delta(\tau - t)d\tau \, B^T \\
&= \int_{-t}^0 e^{-A\eta} B\Sigma_{WW}(t)\delta(\eta)d\eta \, B^T
\end{aligned}
$$

(make integral symmetrical w/r *0*)

$$
\begin{aligned}
&= \frac{1}{2}\int_{-t}^t e^{-A\eta} B\Sigma_{WW}(t)\delta(\eta)d\eta \, B^T \\
&= \frac{1}{2} B\,\Sigma_{WW}(t)\,B^T
\end{aligned}
$$

Proof of continuous time results – M2

2) Thus

$$
\boxed{B\,E\{\tilde{W}(t)\tilde{X}^T(t)\} + E\{\tilde{X}(t)\tilde{W}^T(t)\}B^T = B\,\Sigma_{WW}(t)\,B^T}
$$

and

$$
\begin{aligned}
\frac{d}{dt}\Lambda_{XX}(t,0) &= A\,\Lambda_{XX}(t,0) + \Lambda_{XX}(t,0)\,A^T \\
&\quad + B\,\Sigma_{WW}(t)\,B^T
\end{aligned}
$$

Proof of continuous time results – M2

Now we proof that:

$$
\Lambda_{XX}(t,\tau) = e^{A\tau}\,\Lambda_{XX}(t,0) \qquad \tau \geq 0
$$

Notice that:

$$
\tilde{X}(t+\tau) = e^{A\tau}\,\tilde{X}(t) + \int_t^{t+\tau} e^{A(t+\tau-\eta)}\,B\,\tilde{W}(\eta)d\eta
$$

where,

$$
\begin{aligned}
\tilde{X}(t) &= X(t) - m_X(t) \\
\tilde{W}(t) &= W(t) - m_W(t)
\end{aligned}
$$

Proof of continuous time results – M2

Therefore,

$$
\begin{aligned}
\Lambda_{XX}(t,\tau) &= E\{\tilde{X}(t+\tau)\tilde{X}^T(t)\} \\
&= e^{A\tau}\,\underbrace{E\{\tilde{X}(t)\tilde{X}^T(t)\}}_{\Lambda_{WW}(t,0)} \\
&\quad + \cancelto{0}{\int_t^{t+\tau} e^{A(t+\tau-\eta)}\,B\,E\{\tilde{W}(\eta)\tilde{X}^T(t)\}\,d\eta}
\end{aligned}
$$

Notice that $\tilde{W}(\eta)$ and $\tilde{X}(t)$ are uncorrelated for $\eta > t$

$$
E\{\tilde{W}(\eta)\tilde{X}^T(t)\} = \begin{cases} \frac{1}{2}\Sigma_{WW}(t)\,B^T & \eta = t \\ 0 & \eta > t \end{cases}
$$

Proof of continuous time results – M2

Thus,

$$\Lambda_{XX}(t,\tau) \;=\; e^{A\tau}\Lambda_{XX}(t,0) \qquad \tau \geq 0$$