# ME 233 Advance Control II

## Lecture 14

## Frequency-Shaped Linear Quadratic Regulator

(ME233 Class Notes pp.FSLQ1-FSLQ5)



## Outline

- Parseval's theorem
- Frequency shaped LQR cost function
- Implementation



## Infinite Horizon LQR

nth order LTI system:

$$\dot{x}(t) = A\,x(t) + B\,u(t) \qquad x(0) = x_o$$

Find the optimal control:

$$u(t) = -K\,x(t)$$

which minimizes the cost functional:

$$J = \frac{1}{2}\int_0^\infty \left\{x^T Q\,x + \rho\,u^T R\,u\right\} dt$$

$$Q = Q^T \succeq 0 \qquad R = R^T \succ 0 \quad \rho > 0$$



## Parseval's theorem

- Let $f(t) : [0, \infty) \to \mathcal{R}^n$
- Its (symmetric) Fourier transform is defined by

$$F(j\omega) = \mathcal{F}(f(t)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

and

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(j\omega) e^{+j\omega t} d\omega$$

# Parseval's theorem

$$\int_{-\infty}^{\infty} f^T(t) f(t) dt = \int_{-\infty}^{\infty} F^*(j\omega) F(j\omega) d\omega$$

where

$$F(j\omega) = \mathcal{F}(f(t))$$

$F^*(j\omega) = F^T(-j\omega)$ (complex conjugate transpose)



$$\int_{-\infty}^{\infty} f^T(t) f(t) dt = \int_{-\infty}^{\infty} F^*(j\omega) F(j\omega) d\omega$$

Proof:

$$\int_{-\infty}^{\infty} f^T(t) f(t) dt = \int_{-\infty}^{\infty} f^T(t) \overbrace{\left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(j\omega) e^{+j\omega t} d\omega \right)}^{f(t)} dt$$

$$= \int_{-\infty}^{\infty} \underbrace{\left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^T(t) e^{+j\omega t} dt \right)}_{F^T(-j\omega)} F(j\omega) d\omega$$



# Frequency Cost Function

By Parseval's theorem, the cost functional:

$$J = \frac{1}{2} \int_0^{\infty} \left\{ x^T(t)\, Q\, x(t) + \rho\, u^T(t)\, R\, u(t) \right\} dt$$

with $\begin{cases} x(t) = 0 & t < 0 \\ u(t) = 0 & t < 0 \end{cases}$

is equivalent to

$$J = \frac{1}{2} \int_{-\infty}^{\infty} \{ X^*(j\omega)\, Q\, X(j\omega) + \rho\, U^*(j\omega)\, R\, U(j\omega) \} \, dw$$

$$X(j\omega) = \mathcal{F}(x(t)) \qquad U(j\omega) = \mathcal{F}(u(t))$$



# Frequency-Shaped Cost Function

**Key idea:** Make matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$ functions of frequency

$$J = \frac{1}{2} \int_{-\infty}^{\infty} \{ X^*(j\omega)\, Q(j\omega)\, X(j\omega) + \rho\, U^*(j\omega)\, R(j\omega)\, U(j\omega) \} \, d\omega$$

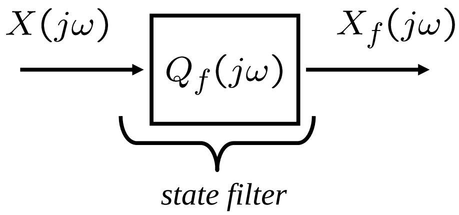
where

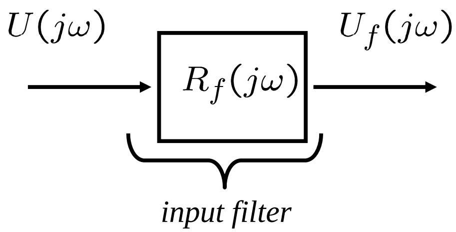
$$Q(j\omega) = Q_f^*(j\omega) Q_f(j\omega) \succeq 0$$

$$R(j\omega) = R_f^*(j\omega) R_f(j\omega) \succ 0$$

# Frequency-Shaped Cost Function

Define the state and input filters






# Frequency-Shaped Cost Function

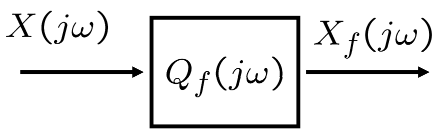
$$
\begin{aligned}
J &= \frac{1}{2}\int_{-\infty}^{\infty} \{X^*(j\omega)\underbrace{Q(j\omega)}_{Q_f^*(j\omega)Q_f(j\omega)}X(j\omega) \\
&\qquad + \rho\, U^*(j\omega)\underbrace{R(j\omega)}_{R_f^*(j\omega)R_f(j\omega)}U(j\omega)\}\, d\omega
\end{aligned}
$$

can be written

$$
\begin{aligned}
J &= \frac{1}{2}\int_{-\infty}^{\infty} \Big\{X_f^*(j\omega)\, X_f(j\omega) \\
&\qquad + \rho\, U_f^*(j\omega)\, U_f(j\omega)\Big\}\, d\omega
\end{aligned}
$$



# Realizing the filters using LTI's




can be realized by

$$
\begin{aligned}
\dot{z}_1(t) &= A_1\, z_1(t) + B_1\, x(t) \\
x_f(t) &= C_1\, z_1(t) + D_1\, x(t)
\end{aligned}
$$

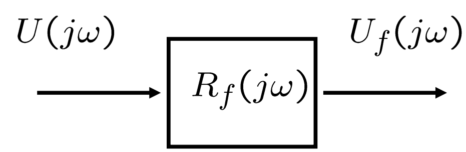
so that

$$Q_f(s) = C_1(sI - A_1)^{-1}B_1 + D_1$$

is causal or strictly causal.



# Realizing the filters using LTI's




can be realized by (with $D_2^T D_2 \succ 0$ )

$$
\begin{aligned}
\dot{z}_2(t) &= A_2\, z_2(t) + B_2\, u(t) \\
u_f(t) &= C_2\, z_2(t) + D_2\, u(t)
\end{aligned}
$$

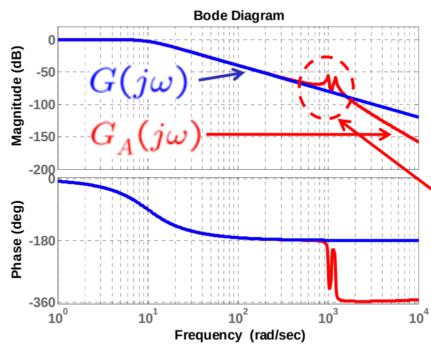
so that

$$R_f(s) = C_2(sI - A_2)^{-1}B_2 + D_2$$

is causal (but not strictly causal).

# Example Hard Disk Drive

Consider a simplified model of a voice coil motor and suspension




$$G_A(s) = G(s)\,[1 + \Delta(s)]$$

uncertainty

nominal model

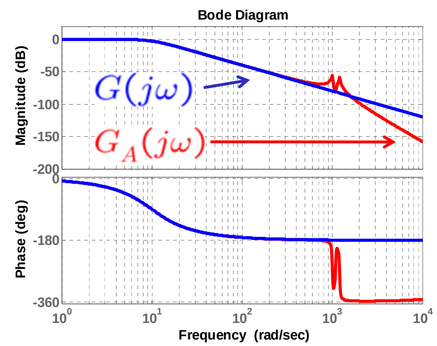
actual plant

high-frequency resonance modes are neglected in the nominal model

nominal model $$G(s) = \frac{100}{s^2 + 14s + 100}$$



# Example Hard Disk Drive




$$G_A(s) = G(s)\,[1 + \Delta(s)]$$

uncertainty

nominal model

actual plant

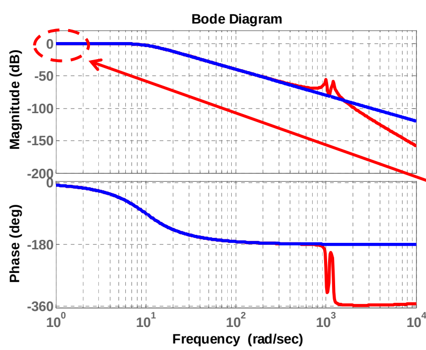
nominal model

$$\frac{d}{dt}\underbrace{\begin{bmatrix} p \\ v \end{bmatrix}}_{x(t)} = \begin{bmatrix} 0 & 1 \\ -100 & -14 \end{bmatrix} \underbrace{\begin{bmatrix} p \\ v \end{bmatrix}}_{x(t)} + \begin{bmatrix} 1 \\ 100 \end{bmatrix} u$$

output is position $$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} p \\ v \end{bmatrix}$$



# Example: Frequency State Weight Q(jω)




$$\frac{d}{dt}\begin{bmatrix} p \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -100 & -14 \end{bmatrix} \underbrace{\begin{bmatrix} p \\ v \end{bmatrix}}_{x(t)} + \begin{bmatrix} 1 \\ 100 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} p \\ v \end{bmatrix}$$

we want zero steady state (i.e. dc) position

set position cost function weight to go to ∞ as ω → 0

**Example**

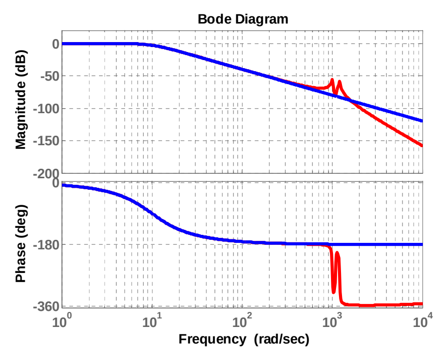
Set weight on $|P(j\omega)|^2$ to $\frac{1}{\omega^2}$

Set weight on $|V(j\omega)|^2$ to 0

$$\underbrace{\begin{bmatrix} P(j\omega) \\ V(j\omega) \end{bmatrix}^*}_{X(j\omega)^*} \underbrace{\begin{bmatrix} \frac{1}{\omega^2} & 0 \\ 0 & 0 \end{bmatrix}}_{Q(j\omega)} \underbrace{\begin{bmatrix} P(j\omega) \\ V(j\omega) \end{bmatrix}}_{X(j\omega)}$$



# Example: Frequency State Weight Q(jω)




$$\frac{d}{dt}\begin{bmatrix} p \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -100 & -14 \end{bmatrix} \underbrace{\begin{bmatrix} p \\ v \end{bmatrix}}_{x(t)} + \begin{bmatrix} 1 \\ 100 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} p \\ v \end{bmatrix}$$

set position weight to go to ∞ as ω → 0

**Example**

$$\underbrace{\begin{bmatrix} P(j\omega) \\ V(j\omega) \end{bmatrix}^*}_{X(j\omega)^*} \underbrace{\begin{bmatrix} \frac{1}{\omega^2} & 0 \\ 0 & 0 \end{bmatrix}}_{Q(j\omega)} \underbrace{\begin{bmatrix} P(j\omega) \\ V(j\omega) \end{bmatrix}}_{X(j\omega)} = \underbrace{\underbrace{\begin{bmatrix} P(j\omega) \\ V(j\omega) \end{bmatrix}^*}_{X(j\omega)^*} \underbrace{\begin{bmatrix} \frac{-1}{j\omega} \\ 0 \end{bmatrix}}_{Q_f(j\omega)^*}}_{X_f(j\omega)^*} \underbrace{\underbrace{\begin{bmatrix} \frac{1}{j\omega} & 0 \end{bmatrix}}_{Q_f(j\omega)} \underbrace{\begin{bmatrix} P(j\omega) \\ V(j\omega) \end{bmatrix}}_{X(j\omega)}}_{X_f(j\omega)}$$

# Example: Frequency State Weight Q(jω)

$$X(j\omega) \longrightarrow \boxed{Q_f(j\omega)} \longrightarrow X_f(j\omega)$$

state space realization

$$\begin{aligned} \dot{z}_1(t) &= A_1\, z_1(t) + B_1\, x(t) \\ x_f(t) &= C_1\, z_1(t) + D_1\, x(t) \end{aligned}$$

**Example**

$$X(f) = \underbrace{\begin{bmatrix} \frac{1}{j\omega} & 0 \end{bmatrix}}_{Q_f(j\omega)} X(j\omega)$$

state space realization

$$Q_f(j\omega) = \begin{bmatrix} \frac{1}{j\omega} & 0 \end{bmatrix} \Rightarrow \left\{ \begin{aligned} \frac{d}{dt} z_1(t) &= \underbrace{0}_{A_1}\, z_1(t) + \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{B_1} x(t) \\ x_f(t) &= \underbrace{1}_{C_1}\, z_1(t) + \underbrace{\begin{bmatrix} 0 & 0 \end{bmatrix}}_{D_1} x(t) \end{aligned} \right.$$



# Example Hard Disk Drive

$$J = \frac{1}{2} \int_{-\infty}^{\infty} \{X^*(j\omega)\, Q(j\omega)\, X(j\omega) + \rho\, U^*(j\omega)\; U(j\omega)\}\, d\omega$$

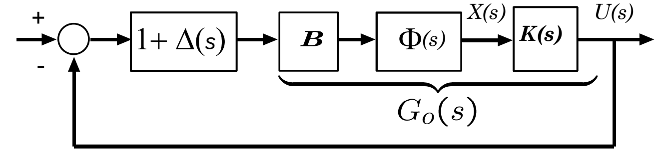
nominal model

weights: $Q(j\omega) = \begin{bmatrix} \frac{1}{\omega^2} & 0 \\ 0 & 0 \end{bmatrix}$

$$\frac{d}{dt}\begin{bmatrix} p \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -100 & -14 \end{bmatrix} \begin{bmatrix} p \\ v \end{bmatrix} + \begin{bmatrix} 1 \\ 100 \end{bmatrix} u$$

$\rho \approx 1.6E-8$

FS-LQR is a dynamic state feedback



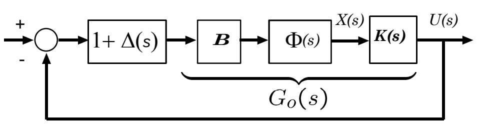

sufficient condition for robustness

$$T(s) = \frac{G_o(s)}{1 + G_o(s)} \qquad |T(j\omega)| \le \frac{1}{|\Delta(j\omega)|}$$



# Example Hard Disk Drive

$$J = \frac{1}{2} \int_{-\infty}^{\infty} \{X^*(j\omega)\, Q(j\omega)\, X(j\omega) + \rho\, U^*(j\omega)\; U(j\omega)\}\, d\omega$$

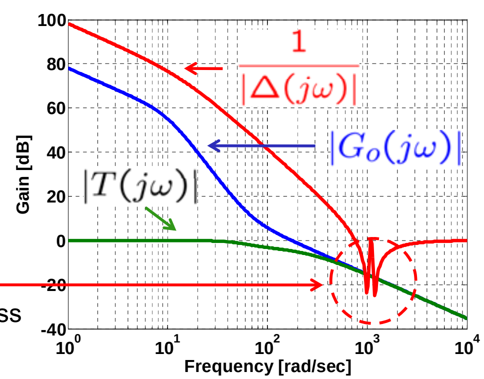



weights: $Q(j\omega) = \begin{bmatrix} \frac{1}{\omega^2} & 0 \\ 0 & 0 \end{bmatrix}$

$\rho \approx 1.6E-8$

$$T(s) = \frac{G_o(s)}{1 + G_o(s)}$$

potential
lack of robustness



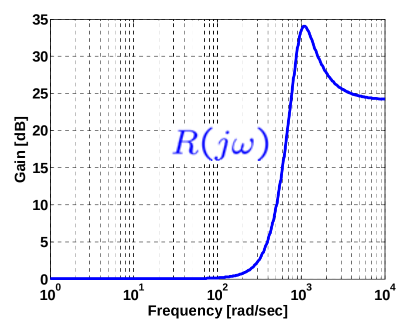


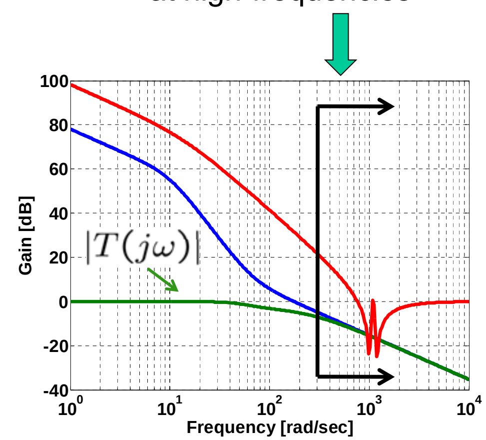

# Example: Frequency Control Weight R(jω)

$$J = \frac{1}{2} \int_{-\infty}^{\infty} \{X^*(j\omega)\, Q(j\omega)\, X(j\omega) + \rho\, U^*(j\omega)\, \underbrace{R(j\omega)}\, U(j\omega)\}\, d\omega$$

**increase** control penalty
at high-frequencies

**Example**

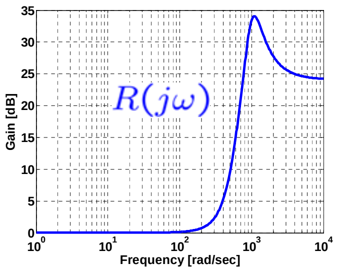


# Example: Frequency Control Weight R(jω)

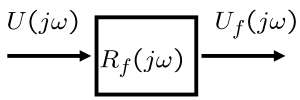
$$J = \frac{1}{2}\int_{-\infty}^{\infty} \{X^*(j\omega)\,Q(j\omega)\,X(j\omega) + \rho\,\underbrace{U^*(j\omega)\,R(j\omega)\,U(j\omega)}_{U_f^*(j\omega)U_f(j\omega)}\}\,d\omega$$

**Example**




$$R(j\omega) = R_f^*(j\omega)\,R_f(j\omega)$$

$$R_f(j\omega) = 4\frac{s^2 + 700s + (500)^2}{s^2 + 600s + (1000)^2}$$




state space realization

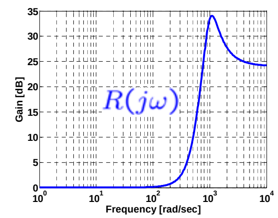
$$\begin{cases} \dfrac{d}{dt}z_2 = \underbrace{\begin{bmatrix} -600 & -980 \\ -100 & 0 \end{bmatrix}}_{A_2} z_2 + \underbrace{\begin{bmatrix} 64 \\ 0 \end{bmatrix}}_{B_2} u \\ u_f = \underbrace{\begin{bmatrix} 6.3 & -46 \end{bmatrix}}_{C_2} z_2 + \underbrace{4}_{D_2}\, u \end{cases}$$

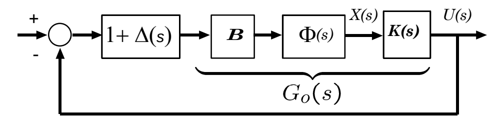


# Example: Frequency Control Weight R(jω)

$$J = \frac{1}{2}\int_{-\infty}^{\infty} \{X^*(j\omega)\,Q(j\omega)\,X(j\omega) + \rho\,U^*(j\omega)\,R(j\omega)\,U(j\omega)\}\,d\omega$$

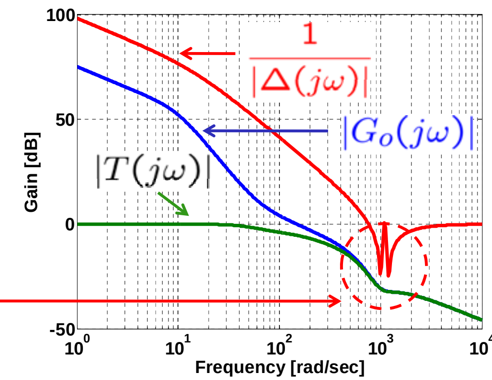
**Example**




$$Q(j\omega) = \begin{bmatrix} \frac{1}{\omega^2} & 0 \\ 0 & 0 \end{bmatrix}$$

$\rho \approx 1.6E-8$




sufficient robustness condition is satisfied



# Cost Function Realization

$$J = \frac{1}{2}\int_{-\infty}^{\infty} \{X^*(j\omega)\,Q(j\omega)\,X(j\omega) + \rho\,U^*(j\omega)\,R(j\omega)\,U(j\omega)\}\,d\omega$$

is equivalent to

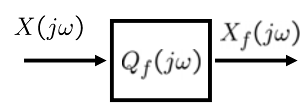
$$J \;=\; \frac{1}{2}\int_0^{\infty} \left\{x_f^T(t)\,x_f(t) \;+\; \rho\,u_f^T(t)\;u_f(t)\right\}\,dt$$

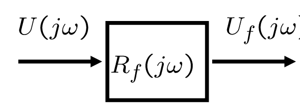


# Cost Function Realization

$$J \;=\; \frac{1}{2}\int_0^{\infty} \left\{x_f^T(t)\,x_f(t) \;+\; \rho\,u_f^T(t)\;u_f(t)\right\}\,dt$$

we know that,




state space realization

$$\begin{aligned} \dot{z}_1(t) &= A_1\,z_1(t) + B_1\,x(t) \\ x_f(t) &= C_1\,z_1(t) + D_1\,x(t) \end{aligned}$$

state space realization

$$\begin{aligned} \dot{z}_2(t) &= A_2\,z_2(t) + B_2\,u(t) \\ u_f(t) &= C_2\,z_2(t) + D_2\,u(t) \end{aligned}$$

# Cost Function Realization

$$J = \frac{1}{2}\int_0^\infty \left\{ x_f^T(t)\, x_f(t) \;+\; \rho\, u_f^T(t)\; u_f(t) \right\} dt$$

$$\begin{aligned} \dot{z}_1(t) &= A_1\, z_1(t) + B_1\, x(t) \\ x_f(t) &= C_1\, z_1(t) + D_1\, x(t) \end{aligned} \qquad \begin{aligned} \dot{z}_2(t) &= A_2\, z_2(t) + B_2\, u(t) \\ u_f(t) &= C_2\, z_2(t) + D_2\, u(t) \end{aligned}$$

Plus: $\dot{x}(t) = A\, x(t) + B\, u(t)$

define extended state $x_e(t) = \begin{bmatrix} x(t) \\ z_1(t) \\ z_2(t) \end{bmatrix}$



# Cost Function Realization

$$J = \frac{1}{2}\int_0^\infty \left\{ x_f^T(t)\, x_f(t) \;+\; \rho\, u_f^T(t)\; u_f(t) \right\} dt$$

We can combine state equations and output as follows:

$$\frac{d}{dt}\begin{bmatrix} x \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ B_1 & A_1 & 0 \\ 0 & 0 & A_2 \end{bmatrix} \begin{bmatrix} x \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ B_2 \end{bmatrix} u$$

$$\begin{bmatrix} x_f \\ u_f \end{bmatrix} = \begin{bmatrix} D_1 & C_1 & 0 \\ 0 & 0 & C_2 \end{bmatrix} \begin{bmatrix} x \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ D_2 \end{bmatrix} u$$



# Extended System Dynamics

$$\frac{d}{dt}\underbrace{\begin{bmatrix} x \\ z_1 \\ z_2 \end{bmatrix}}_{x_e} = \underbrace{\begin{bmatrix} A & 0 & 0 \\ B_1 & A_1 & 0 \\ 0 & 0 & A_2 \end{bmatrix}}_{A_e} \underbrace{\begin{bmatrix} x \\ z_1 \\ z_2 \end{bmatrix}}_{x_e} + \underbrace{\begin{bmatrix} B \\ 0 \\ B_2 \end{bmatrix}}_{B_e} u$$

$$\dot{x}_e(t) = A_e\, x_e(t) + B_e\, u(t)$$



# Extended System Cost

$$J = \frac{1}{2}\int_0^\infty \left\{ x_f^T(t)\, x_f(t) \;+\; u_{ff}^T(t)\; u_{ff}(t) \right\} dt$$

$$\begin{bmatrix} x_f \\ u_{ff} \end{bmatrix} = \underbrace{\begin{bmatrix} D_1 & C_1 & 0 \\ 0 & 0 & \sqrt{\rho}C_2 \end{bmatrix}}_{C_e} \underbrace{\begin{bmatrix} x \\ z_1 \\ z_2 \end{bmatrix}}_{x_e} + \underbrace{\begin{bmatrix} 0 \\ \sqrt{\rho}D_2 \end{bmatrix}}_{D_e} u$$

results in

$$J = \frac{1}{2}\int_0^\infty \left\{ x_e^T\, C_e^T C_e\, x_e + 2\; x_e^T\, C_e^T D_e\, u \;+\; u^T\, D_e^T D_e\, u \right\} dt$$

# Extended System Cost

$$J = \frac{1}{2}\int_0^\infty \left\{ x_e^T \underbrace{C_e^T C_e}_{Q_e} x_e + 2\, x_e^T \underbrace{C_e^T D_e}_{N_e} u + u^T \underbrace{D_e^T D_e}_{R_e} u \right\} dt$$

where

$$Q_e = \begin{bmatrix} D_1^T & 0 \\ C_1^T & 0 \\ 0 & \sqrt{\rho} C_2^T \end{bmatrix} \begin{bmatrix} D_1 & C_1 & 0 \\ 0 & 0 & \sqrt{\rho} C_2 \end{bmatrix}$$

$$N_e = \begin{bmatrix} D_1^T & 0 \\ C_1^T & 0 \\ 0 & \sqrt{\rho} C_2^T \end{bmatrix} \begin{bmatrix} 0 \\ \sqrt{\rho} D_2 \end{bmatrix} \qquad R_e = \begin{bmatrix} 0 & \sqrt{\rho} D_2^T \end{bmatrix} \begin{bmatrix} 0 \\ \sqrt{\rho} D_2 \end{bmatrix}$$



# Extended System Cost

$$J = \frac{1}{2}\int_0^\infty \left\{ x_e^T Q_e x_e + 2\, x_e^T N_e u + u^T R_e u \right\} dt$$

where

$$Q_e = \begin{bmatrix} D_1^T D_1 & D_1^T C_1 & 0 \\ C_1^T D_1 & C_1^T C_1 & 0 \\ 0 & 0 & \rho C_2^T C_2 \end{bmatrix} \qquad N_e = \begin{bmatrix} 0 \\ 0 \\ \rho C_2^T D_2 \end{bmatrix}$$

$$R_e = \rho D_2^T D_2 \succ 0$$



# Extended System LQR

Given the extended dynamics

$$\dot{x}_e(t) = A_e x_e(t) + B_e u(t)$$

Find the optimal control:

$$u(t) = -K_e x_e(t)$$

which minimizes the cost extended functional:

$$J = \frac{1}{2}\int_0^\infty \left\{ x_e^T Q_e x_e + 2\, x_e^T N_e u + u^T R_e u \right\} dt$$



# Extended LQR Solution

$$\dot{x}_e(t) = A_e x_e(t) + B_e u(t)$$

$$J = \frac{1}{2}\int_0^\infty \left\{ x_e^T \underbrace{C_e^T C_e}_{Q_e} x_e + 2\, x_e^T N_e u + \rho u^T D_2^T D_2 u \right\} dt$$

where

$$\rho D_2^T D_2 \succ 0 \qquad C_e = \begin{bmatrix} D_1 & C_1 & 0 \\ 0 & 0 & \sqrt{\rho} C_2 \end{bmatrix} = \begin{bmatrix} & C_q & \\ 0 & 0 & \sqrt{\rho} C_2 \end{bmatrix}$$

Then

$[A_e, B_e]$ is stabilizable

$[A_e - B_e R_e^{-1} N_e^T, C_q]$ is detectable

$\Longleftrightarrow$

There exists a stabilizing optimal control shown in the next page

# Extended LQR Solution

Optimal Control:

$$u(t) \;=\; -K_e\, x_e(t)$$

where

$$K_e = R_e^{-1}\left[B_e^T P_e + N_e^T\right]$$

and

$$P_e A_e + A_e^T P_e + Q_e$$
$$-\left[B_e^T P_e + N_e^T\right]^T R_e^{-1}\left[B_e^T P_e + N_e^T\right] = 0$$



# Implementation

- Control

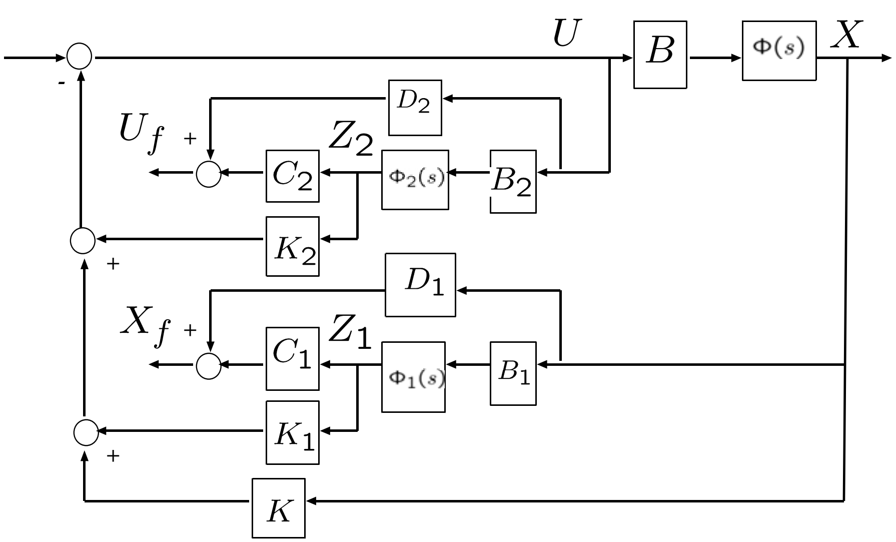
$$u(t) \;=\; -K_e\, x_e(t)$$

$$u(t) \;=\; -\begin{bmatrix} K & K_1 & K_2 \end{bmatrix}\begin{bmatrix} x(t) \\ z_1(t) \\ z_2(t) \end{bmatrix}$$

$$u(t) \;=\; -K\,x(t) - K_1\,z_1(t) - K_2\,z_2(t)$$

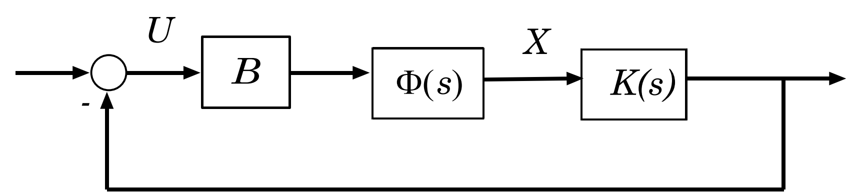


# Block Diagram






# Equivalent Block Diagram




$$K(s) = \left[I + K_2\Phi_2(s)B_2\right]^{-1}\left[K + K_1\Phi_1(s)B_1\right]$$

# FSLQR with reference input

- For simplicity, lets assume a scalar output

$$y(t) = Cx(t) \qquad\qquad y \in \mathcal{R}$$

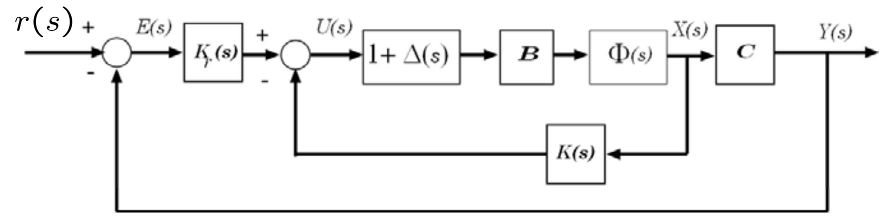
- Assume that we want to design a FSLQR that will achieve asymptotic output convergence to a reference input

$$e(t) = r(t) - y(t)$$

$$\lim_{t \to \infty} e(t) = 0$$



# FSLQR with reference input




- Assume that the reference input $\boldsymbol{r(s)}$ satisfies

$$r(s) = \frac{\bar{B}_r(s)}{A_r(s)}$$

- Where $\boldsymbol{A_r(s)}$ has root in the imaginary axis



# Reference input examples

- Assume that $\boldsymbol{r(t) = r_o}$

$$r(s) = \frac{1}{s} r_o \quad \Longrightarrow \quad A_r(s) = s$$

- Assume that $r(t) = r_o \sin(\omega_r t)$

$$r(s) = \frac{\omega_r^2}{s^2 + \omega_r^2} r_o \quad \Longrightarrow \quad A_r(s) = s^2 + \omega_r^2$$



# FSLQR with reference input

- Define the reference frequency weight

$$Q_R(j\omega) = Q_r^*(j\omega) Q_r(j\omega) \succeq 0$$

- Where

$$Q_r(s) = \frac{B_r(s)}{A_r(s)}$$

$\boldsymbol{A_r(s)}$ is the denominator of $\boldsymbol{r(s)}$

$$r(s) = \frac{\bar{B}_r(s)}{A_r(s)}$$

# Frequency-Shaped Cost Function

$$J = \frac{1}{2}\int_{-\infty}^{\infty} \{X^*(j\omega)\, Q(j\omega)\, X(j\omega)$$

$$+ \rho\; U^*(j\omega)\, R(j\omega)\, U(j\omega)\}\; d\omega$$

- with

$$Q(j\omega) = \underbrace{C^T Q_r^*(j\omega) Q_r(j\omega) C}_{\text{used for achieving } \lim_{t\to\infty} e(t) = 0} + Q_f^*(j\omega) Q_f(j\omega)$$



# Frequency-Shaped Cost Function

$$J = \frac{1}{2}\int_{-\infty}^{\infty} \{X^*(j\omega)\, Q(j\omega)\, X(j\omega)$$

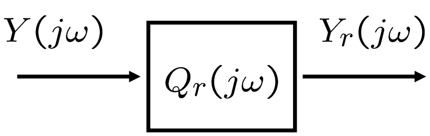
$$+ \rho\; U^*(j\omega)\, R(j\omega)\, U(j\omega)\}\; d\omega$$

$$J = \frac{1}{2}\int_{-\infty}^{\infty} \left\{Y_r^*(j\omega)\, Y_r(j\omega) + X_f^*(j\omega)\, X_f(j\omega)\right.$$

$$\left. + \;\rho U_f^*(j\omega)\; U_f(j\omega)\right\}\; d\omega$$



# Realizing the filters using LTI's




can be realized by

$$\begin{aligned} \dot{z}_r(t) &= A_r\, z_r(t) + B_r\, y(t) \\ x_r(t) &= C_r\, z_r(t) + D_r y(t) \end{aligned}$$

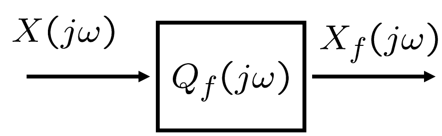
such that

$$Q_r(s) = C_r(sI - A_r)^{-1} B_r + D_r \;= \frac{B_r(s)}{A_r(s)}$$

denominator of ***r(s)***



# Realizing the filters using LTI's




can be realized by

$$\begin{aligned} \dot{z}_1(t) &= A_1\, z_1(t) + B_1\, x(t) \\ x_f(t) &= C_1\, z_1(t) + D_1\, x(t) \end{aligned}$$

such that

$$Q_f(s) = C_1(sI - A_1)^{-1} B_1 + D_1$$

# Realizing the filters using LTI's

$$U(j\omega) \longrightarrow \boxed{R_f(j\omega)} \longrightarrow U_f(j\omega)$$

can be realized by

$$\begin{aligned} \dot{z}_2(t) &= A_2\, z_2(t) + B_2\, u(t) \\ u_f(t) &= C_2\, z_2(t) + D_2\, u(t) \end{aligned}$$

such that

$$R_f(s) = C_2(sI - A_2)^{-1}B_2 + D_2$$



# Cost Function Realization

$$J = \frac{1}{2}\int_{-\infty}^{\infty} \Big\{ y_r^T(t)\, y_r(t) + x_f^T(t)\, x_f(t) + \rho\, u_f^T(t)\; u_f(t) \Big\}\, dt$$

where,

$$\frac{d}{dt}\begin{bmatrix} x \\ z_r \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} A & 0 & 0 & 0 \\ B_r C & A_r & 0 & 0 \\ B_1 & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_2 \end{bmatrix} \begin{bmatrix} x \\ z_r \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ 0 \\ B_2 \end{bmatrix} u$$

$$\begin{bmatrix} y_r \\ x_f \\ u_f \end{bmatrix} = \begin{bmatrix} D_r C & C_r & 0 & 0 \\ D_1 & 0 & C_1 & 0 \\ 0 & 0 & 0 & C_2 \end{bmatrix} \begin{bmatrix} x \\ z_r \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ D_2 \end{bmatrix} u$$



# Extended System Dynamics

$$\frac{d}{dt}\underbrace{\begin{bmatrix} x \\ z_r \\ z_1 \\ z_2 \end{bmatrix}}_{x_e} = \underbrace{\begin{bmatrix} A & 0 & 0 & 0 \\ B_r C & A_r & 0 & 0 \\ B_1 & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_2 \end{bmatrix}}_{A_e} \underbrace{\begin{bmatrix} x \\ z_r \\ z_1 \\ z_2 \end{bmatrix}}_{x_e} + \underbrace{\begin{bmatrix} B \\ 0 \\ 0 \\ B_2 \end{bmatrix}}_{B_e} u$$

$$\dot{x}_e(t) = A_e\, x_e(t) + B_e\, u(t)$$



# Extended System Cost

$$J = \frac{1}{2}\int_{-\infty}^{\infty} \Big\{ x_e^T\, Q_e\, x_e + 2\; x_e^T\, N_e\, u + \; u^T\, R_e\, u \Big\}\, dt$$

$$Q_e = \begin{bmatrix} C^T D_r^T & D_1^T & 0 \\ C_r^T & 0 & 0 \\ 0 & C_1^T & 0 \\ 0 & 0 & \sqrt{\rho} C_2^T \end{bmatrix} \begin{bmatrix} D_r C & C_r & 0 & 0 \\ D_1 & 0 & C_1 & 0 \\ 0 & 0 & 0 & \sqrt{\rho} C_2 \end{bmatrix}$$

$$N_e = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \rho C_2^T D_2 \end{bmatrix} \qquad R_e = \rho D_2^T D2$$

# Extended LQR Solution

$$\dot{x}_e(t) = A_e\, x_e(t) + B_e\, u(t)$$

$$J = \frac{1}{2}\int_0^\infty \left\{ x_e^T \underbrace{C_e^T C_e}_{Q_e} x_e + 2\; x_e^T N_e\, u + \;\rho u^T D_2^T D_2\, u \right\} dt$$

where

$$\rho\, D_2^T D_2 \succ 0 \qquad C_e = \begin{bmatrix} D_r C & C_r & 0 & 0 \\ D_1 & 0 & C_1 & 0 \\ 0 & 0 & 0 & \sqrt{\rho} C_2 \end{bmatrix} = \begin{bmatrix} & C_q & & \\ 0 & 0 & 0 & \sqrt{\rho} C_2 \end{bmatrix}$$

Then

$[A_e, B_e]$ is stabilizable

$[A_e - B_e R_e^{-1} N_e^T, C_q]$ is detectable

$\Longleftrightarrow$

There exists a stabilizing optimal control shown in the next page



# Extended LQR Solution

Optimal Control Gain:

$$K_e = R_e^{-1}\left[B_e^T P_e + N_e^T\right]$$

where

$$P_e A_e + A_e^T P_e + Q_e$$

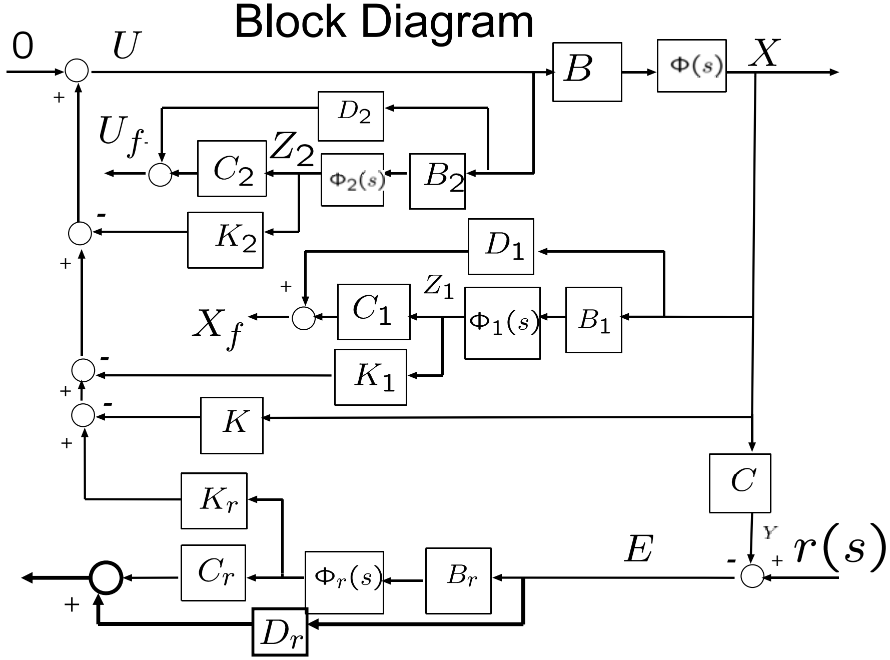
$$-\left[B_e^T P_e + N_e^T\right]^T R_e^{-1}\left[B_e^T P_e + N_e^T\right] = 0$$

and

$$K_e = \begin{bmatrix} K & K_r & K_1 & K_2 \end{bmatrix}$$

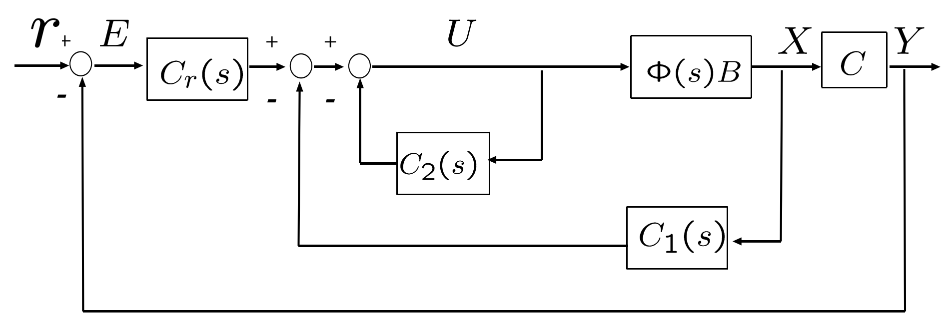


# Block Diagram






# FSLQR with reference input – Block Diagram

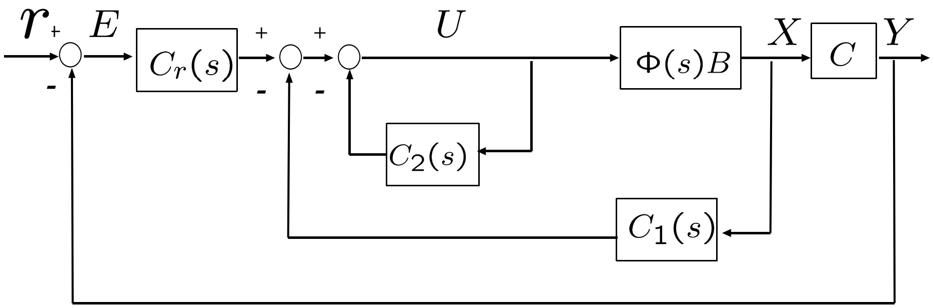



where

$$C_r(s) = K_r \Phi_r(s) B_r \qquad C_2(s) = K_2 \Phi_2(s) B_2$$

$$C_1(s) = K + K_1 \Phi_1(s) B_1$$

# Block Diagram

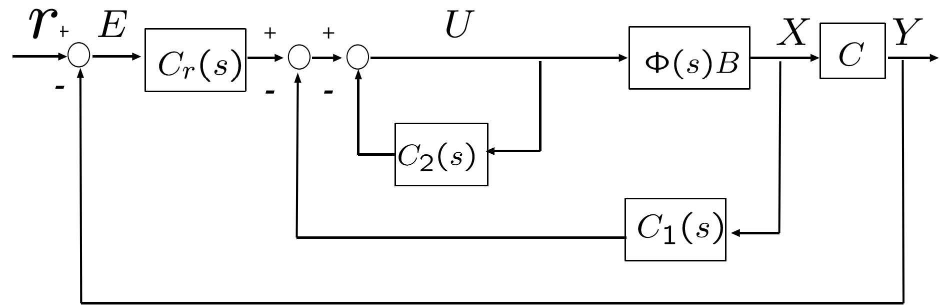



Remember that the poles of $C_r(s)$ are $\boldsymbol{A_r(s)}$ , and

$$r(s) = \frac{B_r(s)}{A_r(s)}$$



# SISO

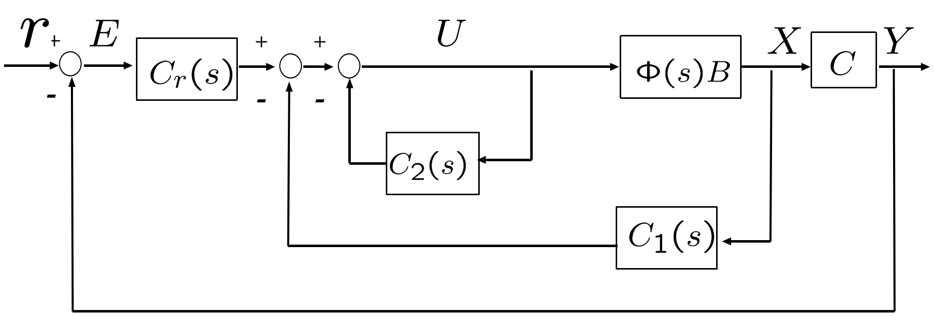



The close loop dynamics from $\boldsymbol{r(s)}$ to $\boldsymbol{E(s)}$ will be of the form

$$E(s) = \frac{B_c'(s)A_r(s)}{A_c(s)}\, r(s) \qquad r(s) = \frac{B_r(s)}{A_r(s)}$$



# SISO



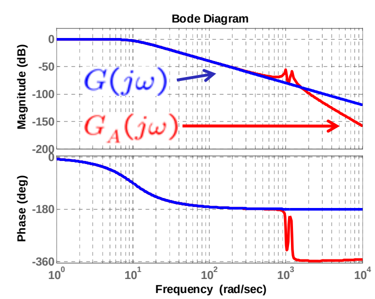

Therefore, since $\boldsymbol{A_c(s)}$ is Hurwitz,

$$E(s) = \frac{B_c'(s)B_r(s)}{A_c(s)} \qquad \lim_{s\to 0} sE(s) = 0 \qquad \lim_{t\to\infty} e(t) = 0$$



# Example Hard Disk Drive

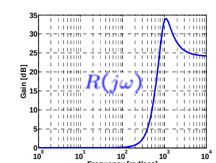
$$J = \frac{1}{2}\int_{-\infty}^{\infty} \{Y^*(j\omega)Q_r(j\omega)\,Y(j\omega) + \rho\, u^*(j\omega)R(j\omega)u(j\omega)\}\, d\omega$$



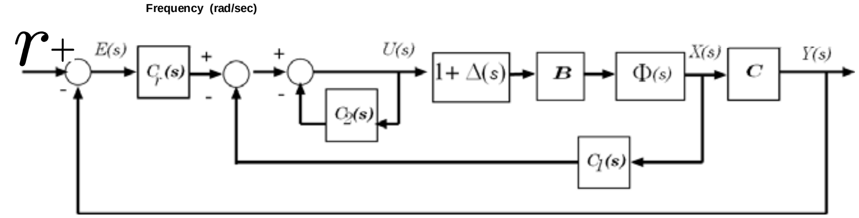

Cost weights:

$$Q_r(j\omega) = \frac{1}{\omega^2}$$

$$\rho \approx 1.6E-8$$

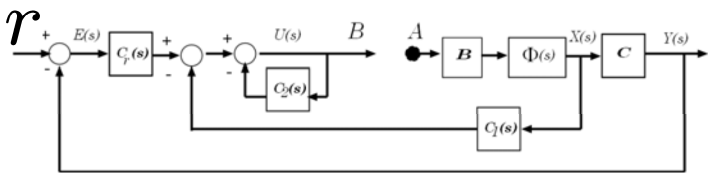



$$r(s) = \frac{r_o}{s}$$ reference input



## Example Hard Disk Drive – Robustness Analysis

$$J = \frac{1}{2}\int_{-\infty}^{\infty} \{Y^*(j\omega)Q_r(j\omega)\,Y(j\omega) + \rho\, u^*(j\omega)R(j\omega)u(j\omega)\}\,d\omega$$



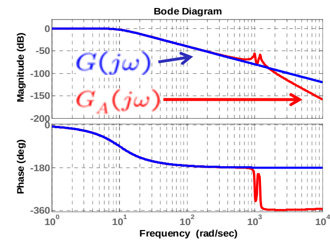

$$G_o(s) = \frac{A(s)}{B(s)} \quad \Rightarrow \quad G_o(s) = \frac{[C_r(s)C + C_1(s)]\;\Phi(s)B}{1 + C_2(s)}$$

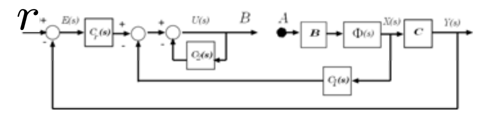
$$T(s) = \frac{G_o(s)}{1 + G_o(s)}$$



## Example Hard Disk Drive

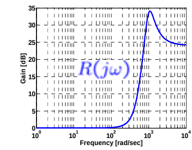
$$J = \frac{1}{2}\int_{-\infty}^{\infty} \{Y^*(j\omega)Q_r(j\omega)\,Y(j\omega) + \rho\, u^*(j\omega)R(j\omega)u(j\omega)\}\,d\omega$$

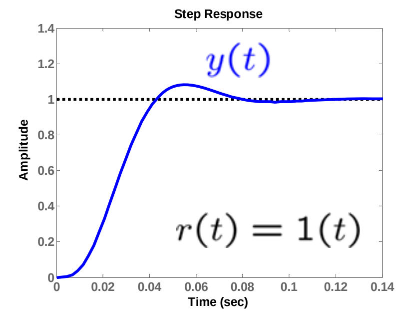


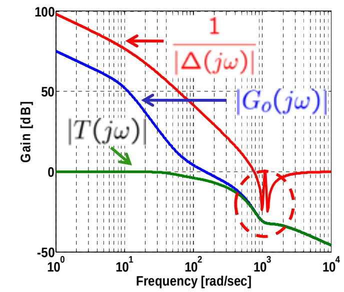



$$Q_r(j\omega) = \frac{1}{\omega^2}$$

$$\rho \approx 1.6E-8$$

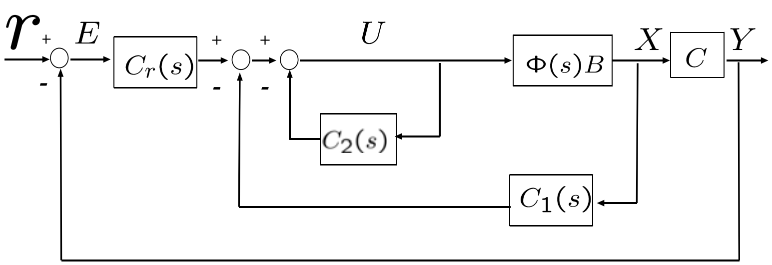





## FSLQR with reference input – State Estimation



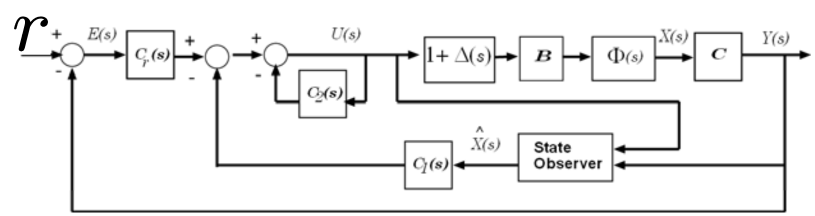

Assume that :

- state ***x(t)*** is not measurable
- only output ***y(t)*** is available

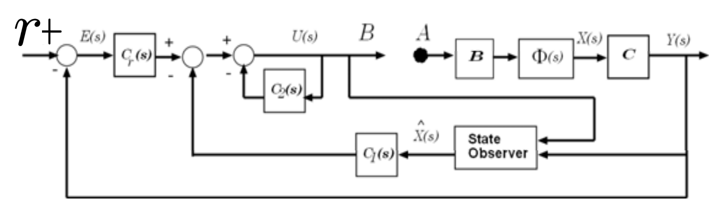
⇒ Use state observer



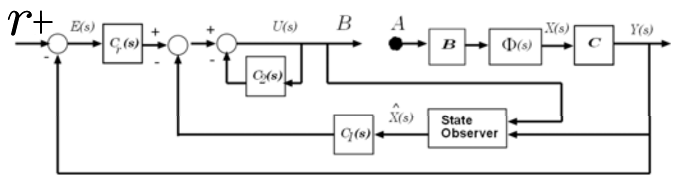
## FSLQR with reference input – State Estimation




Robustness analysis:



# Loop Transfer Recovery




Assume that $G(s) = C\Phi(s)B$ is square and has no unstable zeros

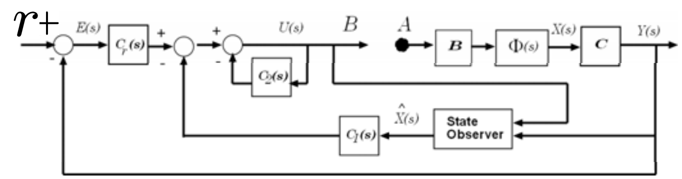
observer:
$$\begin{cases} \dfrac{d}{dt}\hat{x}(t) = A\,\hat{x}(t) + B\,u(t) + L\,\tilde{y}(t) \\ \tilde{y}(t) = y(t) - C\,\hat{x}(t) \end{cases}$$

observer gain
$$\begin{cases} L = \dfrac{1}{\mu} M_\mu C^T N^{-1} \quad N = N^T \succ 0 \quad \mu > 0 \\ AM_\mu + M_\mu A^T + BB^T - \dfrac{1}{\mu} M_\mu C^T N^{-1} C M_\mu = 0 \end{cases}$$



# Loop Transfer Recovery

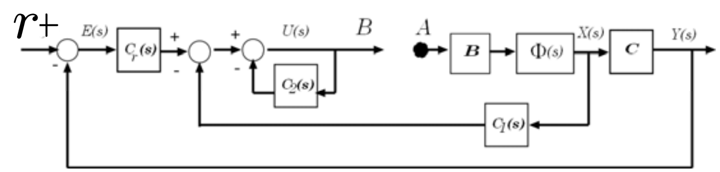
Assume that $G(s) = C\Phi(s)B$ is square and has no unstable zeros




$$L = \frac{1}{\mu} M_\mu C^T N^{-1} \quad N = N^T \succ 0 \quad \mu > 0$$
$$AM_\mu + M_\mu A^T + BB^T - \frac{1}{\mu} M_\mu C^T N^{-1} C M_\mu = 0$$

*Make it approximate (point-wise in s)*

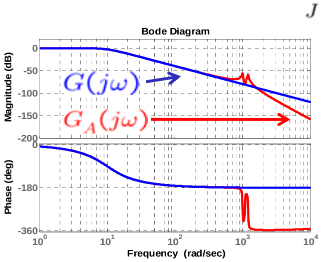
$\mu \longrightarrow 0$

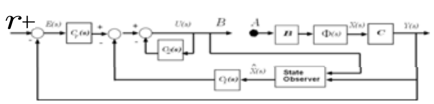




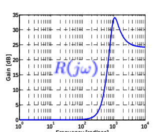

# Example Hard Disk Drive

$$J = \frac{1}{2}\int_{-\infty}^{\infty} \{Y^*(j\omega) Q_r(j\omega) Y(j\omega) + \rho\, u^*(j\omega) R(j\omega) u(j\omega)\}\, d\omega$$



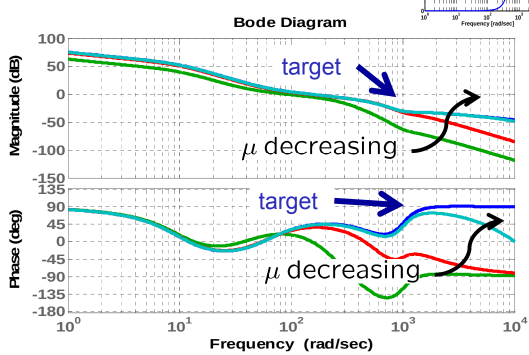
$$Q_r(j\omega) = \frac{1}{\omega^2} \qquad \rho \approx 1.6E-8$$




Bode plots of

$$G_o(s) = \frac{A(s)}{B(s)}$$

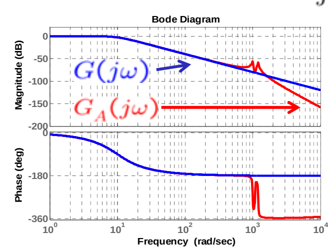
$\mu$ decreases

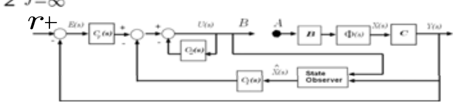





# Example Hard Disk Drive

$$J = \frac{1}{2}\int_{-\infty}^{\infty} \{Y^*(j\omega) Q_r(j\omega) Y(j\omega) + \rho\, u^*(j\omega) R(j\omega) u(j\omega)\}\, d\omega$$



$$Q_r(j\omega) = \frac{1}{\omega^2} \qquad \rho \approx 1.6E-8$$