ME 233 Spring 2012
Midterm 1 Solutions

Problem 1

1. For convenience, define $\tilde{X}(k) = X(k) - m_X(k)$ and $\tilde{Y}(k) = Y(k) - m_Y(k)$. Since $m_W(k) = 0$ and $m_V(k) = 0$, we have

$$\begin{aligned} m_X(k+1) &= Am_X(k) \\ m_Y(k) &= Cm_X(k) \end{aligned}$$

As a result, we have

$$\begin{aligned} \tilde{X}(k+1) &= A\tilde{X}(k) + BW(k) \\ \tilde{Y}(k) &= C\tilde{X}(k) + V(k) \end{aligned}$$

Using the definition of Λ, we have

$$\begin{aligned} \Lambda_{XY}(k,j) &= E\{\tilde{X}(k+j)\tilde{Y}^T(k)\} \\ &= E\{\tilde{X}(k+j)[C\tilde{X}(k) + V(k)]^T\} \\ &= E\{\tilde{X}(k+j)\tilde{X}^T(k)\}C^T + E\{\tilde{X}(k+j)V^T(k)\} \\ &= \Lambda_{XX}(k,j)C^T \end{aligned}$$

Similarly, we have

$$\begin{aligned} \Lambda_{YY}(k,j) &= E\{\tilde{Y}(k+j)\tilde{Y}^T(k)\} \\ &= E\{[C\tilde{X}(k+j) + V(k+j)][C\tilde{X}(k) + V(k)]^T\} \\ &= CE\{\tilde{X}(k+j)\tilde{X}^T(k)\}C^T + CE\{\tilde{X}(k+j)V^T(k)\} + E\{V(k+j)\tilde{X}^T(k)\} + E\{V(k+j)V^T(k)\} \\ &= C\Lambda_{XX}(k,j)C^T + \Sigma_V\delta(j) \end{aligned}$$

2. We begin by defining

$$Z = \begin{bmatrix} Y(0) \\ Y(1) \end{bmatrix}$$

We thus have

$$m_Z = \begin{bmatrix} m_Y(0) \\ m_Y(1) \end{bmatrix} = \begin{bmatrix} Cm_X(0) \\ Cm_X(1) \end{bmatrix} = \begin{bmatrix} Cx_0 \\ CAx_0 \end{bmatrix}$$

and, using the results from the first part,

$$\begin{aligned} \Lambda_{ZZ} &= E\left\{\begin{bmatrix} \tilde{Y}(0)\tilde{Y}^T(0) & \tilde{Y}(0)\tilde{Y}^T(1) \\ \tilde{Y}(1)\tilde{Y}^T(0) & \tilde{Y}(1)\tilde{Y}^T(1) \end{bmatrix}\right\} = \begin{bmatrix} C\Lambda_{XX}(0,0)C^T + \Sigma_V & C\Lambda_{XX}(1,-1)C \\ C\Lambda_{XX}(0,1)C^T & C\Lambda_{XX}(1,0)C^T + \Sigma_V \end{bmatrix} \\ &= \begin{bmatrix} CX_0C^T + V & CX_0A^TC \\ CAX_0C^T & C[AX_0A^T + B\Sigma_W B^T]C + \Sigma_V \end{bmatrix} \\ \Lambda_{X(0)Z} &= E\left\{\begin{bmatrix} \tilde{X}(0)\tilde{Y}^T(0) & \tilde{X}(0)\tilde{Y}^T(1) \end{bmatrix}\right\} = \begin{bmatrix} \Lambda_{XX}(0,0)C^T & \Lambda_{XX}(1,-1)C^T \end{bmatrix} \\ &= \begin{bmatrix} X_0C^T & X_0A^TC^T \end{bmatrix} \end{aligned}$$

We now use standard least squares results to say that the least squares estimator of $X(0)$ given $Y(0)$ and $Y(1)$ is

$$\begin{aligned} E\{X(0)|Y(0),Y(1)\} &= E\{X(0)|Z\} = m_X(0) + \Lambda_{XZ}\Lambda_{ZZ}^{-1}(Z - m_Z) \\ &= x_0 + \begin{bmatrix} X_0C^T & X_0A^TC^T \end{bmatrix} \begin{bmatrix} CX_0C^T + V & CX_0A^TC \\ CAX_0C^T & C[AX_0A^T + B\Sigma_W B^T]C + \Sigma_V \end{bmatrix}^{-1} \begin{bmatrix} Y(0) - Cx_0 \\ Y(1) - CAx_0 \end{bmatrix} \end{aligned}$$

Problem 2

1. The Bellman equation for this optimization problem is

$$\begin{aligned} J_m^o[x_m, N] &= \min_{u(m)} \Big[2u^T(m)y(m) + J_{m+1}^o[Ax_m + Bu(m), N] \Big] \\ &= \min_{u(m)} \Big[u^T(m)y(m) + y^T(m)u(m) + J_{m+1}^o[Ax_m + Bu(m), N] \Big] \end{aligned}$$

By the induction hypothesis, we have

$$J_{m+1}^o[Ax_m + Bu(m), N] = [Ax_m + Bu(m)]^T P_{(N-m-1)} [Ax_m + Bu(m)]$$

Substituting the expressions for $y(m)$ and J_{m+1}^o into the Bellman equation, we have

$$\begin{aligned} J_m^o[x_m, N] = \min_{u(m)} \Big[& u^T(m)[Cx_m + Du(m)] + [Cx_m + Du(m)]^T u(m) \\ & + [Ax_m + Bu(m)]^T P_{(N-m-1)}[Ax_m + Bu(m)] \Big] \\ = \min_{u(m)} \Big[& x_m^T (A^T P_{(N-m-1)} A) x_m + u^T(m)(B^T P_{(N-m-1)} A + C) x_m \\ & + x_m^T (A^T P_{(N-m-1)} B + C^T) u(m) + u^T(m)(B^T P_{(N-m-1)} B + D + D^T) u(m) \Big] \end{aligned}$$

Performing the minimization yields

$$\begin{aligned} u^o(m) &= -(B^T P_{(N-m-1)} B + D + D^T)^{-1}(B^T P_{(N-m-1)} A + C) x_m \\ J_m^o[x_m, N] &= x_m^T \Big[A^T P_{(N-m-1)} A \\ &\quad - (A^T P_{(N-m-1)} B + C^T)(B^T P_{(N-m-1)} B + D + D^T)^{-1}(B^T P_{(N-m-1)} A + C) \Big] x_m \\ &= x_m^T P_{(N-m)} x_m \end{aligned}$$

2. There are two ways to do this problem. The first way is to notice that when $u(k) = 0,\ \forall k$, we have $J_0[N] = 0$. Therefore, it must hold that

$$\begin{aligned} J_0^o[x_0, N] &= \min_{u(0),\ldots,u(N-1)} J_0[N] \qquad \text{s.t.} \qquad x(0) = x_0 \\ &\le J_0[N]\Big|_{x(0)=x_0, u(k)=0, \forall k} = 0 \end{aligned}$$

The second method for solving this problem involves showing that $P_k \preceq 0,\ \forall k$ by using induction. To show this, we first note that the base case is trivially satisfied because $P_0 = 0 \preceq 0$. We now show that if $P_{(k-1)} \preceq 0$, then $P_k \preceq 0$. Note that

$$\begin{aligned} & B^T P_{(k-1)} B + D + D^T \succ 0 \\ \Rightarrow \qquad & -(A^T P_{(k-1)} B + C^T)(B^T P_{(k-1)} B + D + D^T)^{-1}(B^T P_{(k-1)} A + C) \preceq 0 \end{aligned}$$

By the induction hypothesis, we know that $P_{(k-1)} \preceq 0$, which implies that $A^T P_{(k-1)} A \preceq 0$, which in turn implies that $P_k \preceq 0$ because it is the sum of two negative semi-definite matrices. Using the fact that $P_k \preceq 0,\ \forall k$, we now conclude that

$$J_0^o[x_0, N] = x_0^T P_N x_0 \preceq 0, \qquad \forall x_0, N$$

3.

$$\begin{aligned}\sum_{k=0}^{\infty} 2u^T(k)y(k) &\geq \min_{u(0),u(1),\dots} \left[\sum_{k=0}^{\infty} 2u^T(k)y(k)\right] \\ &= J_0^o[x_0,\infty] = x_0^T P_\infty x_0 \\ &\geq \lambda_{min}(P_\infty)\|x_0\|^T\end{aligned}$$

From the previous part, we know that $P_\infty \preceq 0$, which implies that $\lambda_{min}(P_\infty) \leq 0$. Therefore,

$$\sum_{k=0}^{\infty} 2u^T(k)y(k) \geq \alpha^2 \lambda_{min}(P_\infty)$$

regardless of how $u(0), u(1), \dots$ are chosen.

Problem 3

1. We first define the extended state $x_e(k) = [x^T(k)\ x_f^T(k)]^T$ so that the system and cost function dynamics can be written

$$x_e(k+1) = A_e x_e(k) + B_e u(k) \tag{1}$$
$$y_f(k) = C_e x_e(k) \tag{2}$$

where

$$A_e = \begin{bmatrix} A & 0 \\ B_f C & A_f \end{bmatrix} \qquad B_e = \begin{bmatrix} B \\ 0 \end{bmatrix} \qquad C_e = \begin{bmatrix} D_f C & C_f \end{bmatrix} \qquad x_e(0) = \begin{bmatrix} x(0) \\ 0 \end{bmatrix}$$

With this, the cost function can now be written as

$$J = x_e^T(N) C_e^T C_e x_e(N) + \sum_{k=0}^{N-1} \left[x_e^T(k) C_e^T C_e x_e(k) + u^T(k) R u(k)\right] \tag{3}$$

We now note that minimizing (3) subject to (1)–(2) is a standard LQR problem. We can therefore immediately write down the solution:

$$\begin{aligned}u_1^o(k) &= -K(k+1)x_e(k) \\ K(k) &= [B_e^T P(k) B_e + R]^{-1} B_e^T P(k) A_e \\ P(k-1) &= A_e^T P(k) A_e + C_e^T C_e - A_e^T P(k) B_e [B_e^T P(k) B_e + R]^{-1} B_e^T P(k) A_e \\ P(N) &= C_e^T C_e\end{aligned}$$

The corresponding optimal cost is

$$J^o = x_e^T(0) P(0) x_e(0)$$

2. We first partition $K(k) = [K_x(k)\ K_f(k)]$ so that the optimal control law from the previous part can be written as

$$u_1^o(k) = -K_x(k+1)x(k) - K_f(k+1)x_f(k)$$

Now we note that the value of $x_f(k)$ can be constructed from $x(0), \dots, x(k-1)$ using the recursive relationship

$$x_f(k+1) = A_f x_f(k) + B_f C x(k), \qquad x_f(0) = 0$$

With this, we see that $u_1^o(k)$ can be regarded as a function of $x(0), \dots, x(k)$. In particular,

$$\begin{aligned}x_f(k+1) &= A_f x_f(k) + B_f C x(k), \qquad x_f(0) = 0 \\ u_2^o(k) &= -K_f(k+1)x_f(k) - K_x(k+1)x(k)\end{aligned}$$

Note that the optimal control law is expressed as the output of a state space model with input $x(k)$. Since $u_1^o(k) = u_2^o(k)$, the optimal cost is the same as in the previous part, i.e. $J^o = x_e^T(0)P(0)x_e(0)$.