ME 233 Spring 2012
Midterm 2 Solutions

Problem 1

This problem is a pole placement, disturbance rejection, tracking control problem. We first define

$$A(q^{-1}) = 1 - 0.7q^{-1} \qquad B(q^{-1}) = 1 - 0.5q^{-1} \qquad \mathrm{d} = 1$$

so that the plant model can be written as

$$A(q^{-1})y(k) = q^{-\mathrm{d}}B(q^{-1})[u(k) + d(k)]$$

Since perfect tracking is desired, we must cancel all plant zeros. This corresponds to factoring $B(q^{-1})$ as $B^s(q^{-1})B^u(q^{-1})$ where

$$B^s(q^{-1}) = 1 - 0.5q^{-1} \qquad B^u(q^{-1}) = 1$$

Since we would like the closed-loop poles to only include poles at the origin and the plant zeros, we choose

$$A_c^{'} = 1$$

As for the disturbance annihilating polynomial, there are two choices that make sense:

$$\begin{aligned} A_{d1}(q^{-1}) &= 1 - q^{-6} \\ A_{d2}(q^{-1}) &= (1 - q^{-2})(1 - q^{-3}) \end{aligned}$$

The first one has the benefit of having fewer terms, whereas the second one has the benefit of being lower order.

To design the controller, we must first solve the Diophantine equation

$$A_c^{'}(q^{-1}) = A(q^{-1})A_d(q^{-1})R^{'}(q^{-1}) + q^{-\mathrm{d}}B^u(q^{-1})S(q^{-1})$$

Since $\mathrm{d} = 1$ and the order of $B^u(q^{-1})$ is zero, we see that $R^{'}(q^{-1})$ must have order 0. In particular, this means that $R^{'}(q^{-1}) = 1$. Plugging this into the Diophantine along with $B^u(q^{-1}) = 1$, we have

$$\begin{aligned} A_c^{'}(q^{-1}) &= A(q^{-1})A_d(q^{-1}) + q^{-1}S(q^{-1}) \\ \Rightarrow \qquad S(q^{-1}) &= q[A_c^{'}(q^{-1}) - A(q^{-1})A_d(q^{-1})] \end{aligned}$$

Therefore if we choose $A_d(q^{-1}) = A_{d1}(q^{-1})$, we obtain $S(q^{-1})$ as

$$S_1(q^{-1}) = 0.7 + q^{-5} - 0.7q^{-6}$$

whereas if we choose $A_d(q^{-1}) = A_{d2}(q^{-1})$, we instead obtain $S(q^{-1})$ as

$$S_2(q^{-1}) = 0.7 + q^{-1} + 0.3q^{-2} - 0.7q^{-3} - q^{-4} + 0.7q^{-5}$$

The feedforward part of the controller is given by $T(q, q^{-1}) = A_c^{'}(q^{-1})q^{+\mathrm{d}} = q$. The control law is thus given by

$$\begin{aligned} B^s(q^{-1})A_{di}(q^{-1})u(k) &= q\, y_d(k) - S_i(q^{-1})y(k) \\ &= y_d(k+1) - S_i(q^{-1})y(k) \end{aligned}$$

where $i \in \{1, 2\}$.

Problem 2

1. We first rewrite the cost function as

$$J = \sum_{k=0}^{\infty} x^T(k)C^TCx(k) = x_0^TC^TCx_0 + \sum_{k=1}^{\infty} x^T(k)C^TCx(k)$$

Plugging in the state dynamics, we have

$$J = x_0^TC^TCx_0 + \sum_{k=0}^{\infty}[Ax(k)+Bu(k)]^TC^TC[Ax(k)+Bu(k)]$$

Defining $\alpha = x_0^TC^TCx_0$, $Q = A^TC^TCA$, $R = B^TC^TCB$, and $S = A^TC^TCB$, we rewrite J as

$$J = \alpha + \sum_{k=0}^{\infty} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \tag{1}$$

Since $C^TC \succeq 0$, we have that

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \begin{bmatrix} A^T \\ B^T \end{bmatrix} C^TC \begin{bmatrix} A & B \end{bmatrix} \succeq 0$$

Also, since $CB \neq 0$ and CB is scalar, we see that $R = (CB)^2 > 0$.

2. Since α does not depend on the choice of the control, minimizing J is equivalent to minimizing $J - \alpha$. Now note that the problem of minimizing J via choice of $u(0), u(1), \ldots$ is a standard infinite-horizon LQR problem. Therefore, under the assumptions that (A, B) is stabilizable and the state space realization $C_J(zI-A)^{-1}B + D_J$ has no transmission zeros on or outside the unit circle, the solution is given by

$$\begin{aligned} u^o(k) &= -Kx(k) \\ K &= [B^TPB+R]^{-1}[B^TPA+S^T] \\ P &= A^TPA + Q - [A^TPB+S][B^TPB+R]^{-1}[B^TPA+S^T] \\ P &\succeq 0 \end{aligned}$$

If the condition that the state space realization $C_J(zI-A)^{-1}B + D_J$ has no transmission zeros on or outside the unit circle were relaxed to the condition that $C_J(zI-A)^{-1}B + D_J$ has no transmission zeros on the unit circle, the condition $P \succeq 0$ should be strengthened to the condition that $A - BK$ is Schur.

3. Plugging in the definitions of Q, R, and S into the DARE from the previous part yields

$$\begin{aligned} P &= A^TPA + A^TC^TCA - [A^TPB + A^TC^TCB][B^TPB + B^TC^TCB]^{-1}[B^TPA + B^TC^TCA] \\ &= A^T(P+C^TC)A - [A^T(P+C^TC)B][B^T(P+C^TC)B]^{-1}[B^T(P+C^TC)A] \end{aligned}$$

Defining $\bar{P} = P + C^TC$, we therefore rewrite the DARE in terms of $\bar{P}$ as

$$\begin{aligned} \bar{P} - C^TC &= A^T\bar{P}A - A^T\bar{P}B(B^T\bar{P}B)^{-1}B^T\bar{P}A \\ \Rightarrow \bar{P} &= A^T\bar{P}A + C^TC - A^T\bar{P}B(B^T\bar{P}B)^{-1}B^T\bar{P}A \end{aligned}$$

It should be noted that $\bar{P} = P + C^TC \succeq P \succeq 0$, i.e. that $\bar{P} \succeq 0$. Similarly to how we rewrote the DARE, we rewrite the optimal control law as

$$\begin{aligned} u^o(k) &= -[B^TPB + B^TC^TCB]^{-1}[B^TPA + B^TC^TCA]x(k) \\ &= -(B^T\bar{P}B)^{-1}B^T\bar{P}Ax(k) \end{aligned}$$

Problem 3

1. If we define $X_f(z) = Q_f(z)X(z)$ where $X(z)$ and $X_f(z)$ are respectively the Z-transforms of $x(k)$ and $x_f(k)$, the power spectral density of $x_f(k)$ is given by

$$\Phi_{X_f X_f}(\omega) = Q_f(e^{j\omega})\Phi_{XX}(\omega)Q_f^*(e^{j\omega})$$

This allows us to rewrite the cost function as

$$\begin{aligned} J &= \text{trace}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\Phi_{X_f X_f}(\omega) + \rho\Phi_{UU}(\omega)\right) d\omega\right] \\ &= \text{trace}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{X_f X_f}(\omega) d\omega + \rho\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{UU}(\omega) d\omega\right] \\ &= \text{trace}\left[E\{x_f(k)x_f^T(k) + \rho E\{u(k)u^T(k)\}\right] \\ &= E\left\{\text{trace}[x_f(k)x_f^T(k) + \rho u(k)u^T(k)]\right\} \\ &= E\{x_f^T(k)x_f(k) + \rho u^T(k)u(k)\} \end{aligned}$$

2. We first define the extended state $x_e(k) = [x^T(k)\ x_f^T(k)]^T$ so that the system and cost function dynamics can be written

$$x_e(k+1) = A_e x_e(k) + B_e u(k) + B_{we} w(k) \tag{2}$$
$$y(k) = C_e x_e(k) + v(k) \tag{3}$$

where

$$A_e = \begin{bmatrix} A & 0 \\ B_f & A_f \end{bmatrix} \qquad B_e = \begin{bmatrix} B \\ 0 \end{bmatrix} \qquad B_{we} = \begin{bmatrix} B_w \\ 0 \end{bmatrix} \qquad C_e = \begin{bmatrix} C & 0 \end{bmatrix}$$

We then define

$$Q = \begin{bmatrix} D_f^T \\ C_f^T \end{bmatrix} \begin{bmatrix} D_f & C_f \end{bmatrix} \qquad R = \rho I$$

so that the cost function can be written as

$$J = E\{x_e^T(k)Qx_e(k) + u^T(k)Ru(k)\} \tag{4}$$

3. Finding the output feedback control that minimizes (4) subject to (2)–(3) is a standard infinite-horizon output feedback LQG optimal control problem. The solution is therefore given by

$$u^o(k) = -K\hat{x}_e(k)$$

where K is the standard deterministic LQR gain given by

$$\begin{aligned} &K = (B_e^T P B_e + R)^{-1} B_e^T P A_e \\ &P = A_e^T P A_e + Q - A_e^T P B_e (B_e^T P B_e + R)^{-1} B_e^T P A_e \\ &\text{such that } A_e - B_e K \text{ is Schur} \end{aligned}$$

and $\hat{x}_e(k)$ is the a posteriori state estimate generated by the stationary Kalman filter

$$\begin{aligned} \hat{x}_e(k) &= \hat{x}_e^o(k) + F[y(k) - C_e\hat{x}^o(k)] \\ \hat{x}_e^o(k+1) &= A_e\hat{x}_e(k) + B_e u(k) \end{aligned}$$

where

$$\begin{aligned} &F = MC_e^T(C_e M C_e^T + V)^{-1} \\ &M = A_e M A_e^T + B_{we} W B_{we}^T - A_e M C_e^T (C_e M C_e^T + V)^{-1} C_e M A_e^T \\ &\text{such that } A_e - A_e F C_e \text{ is Schur} \end{aligned}$$