[1]

a. From the block diagram, we have

$$Y(s) = G_p(s)[D(s) + U(s)], \tag{1}$$

$$U(s) = R^*(s) - Q(s)G_{pn}^{-1}(s)[Y(s) - G_{pn}(s)U(s)]. \tag{2}$$

Plugging (1) into (2) and solving for $U(s)$, we obtain

$$U(s) = \frac{G_{pn}(s)}{G_{pn}(s) + Q(s)[G_p(s) - G_{pn}(s)]} R^*(s) - \frac{Q(s)G_p(s)}{G_{pn}(s) + Q(s)[G_p(s) - G_{pn}(s)]} D(s). \tag{3}$$

Substituting $U(s)$ in (1) by (3) yields

$$Y(s) = \frac{G_p(s)G_{pn}(s)}{G_{pn}(s) + Q(s)[G_p(s) - G_{pn}(s)]} R^*(s) + \frac{G_p(s)G_{pn}(s)[1 - Q(s)]}{G_{pn}(s) + Q(s)[G_p(s) - G_{pn}(s)]} D(s). \tag{4}$$

Therefore,

$$G_{r^*y}(s) = \frac{G_p(s)G_{pn}(s)}{G_{pn}(s) + Q(s)[G_p(s) - G_{pn}(s)]},$$
$$G_{dy}(s) = \frac{G_p(s)G_{pn}(s)[1 - Q(s)]}{G_{pn}(s) + Q(s)[G_p(s) - G_{pn}(s)]}.$$

b. When $Q(j\omega) = 1$, we have $G_{r^*y}(j\omega) = G_{pn}(j\omega)$ and $G_{dy}(j\omega) = 0$. Thus,

$$Y(j\omega) = G_{pn}(j\omega)R^*(j\omega).$$

[2] This is a stationary stochastic control problem with exactly known state.

The optimal feedback control law is the same as the deterministic case and is given by

$$u(k) = -[R + B^T P_s B]^{-1} B^T P_s A x(k) = -[1 + P_s]^{-1} P_s x(k) = -k_{opt} x(k),$$

where $P_s = \dfrac{1+\sqrt{5}}{2}$ is the positive solution of algebraic Riccati equation:

$$P = P - P[1 + P]^{-1} P + 1.$$

Then the optimal feedback gain is given by

$$k_{opt} = \frac{1+\sqrt{5}}{3+\sqrt{5}}.$$

The minimum value of the performance index is obtained from

$$J_{opt} = E[x^2 + u^2] = X + {k_{opt}}^2 X = (1 + {k_{opt}}^2) X,$$

where X is the steady state covariance of

$$x(k+1) = (1 - k_{opt})x(k) + w(k) .$$

Solving

$$X = (1 - k_{opt})^2 X + W$$

for X gives

$$X = \frac{W}{1 - (1 - k_{opt})^2} = \frac{7 + 3\sqrt{5}}{5 + 3\sqrt{5}} W .$$

So

$$J_{opt} = P_s W = \frac{1 + \sqrt{5}}{2} W .$$

The optimal value can also be obtained by using eq. (LQG-30) in the reader without the term, $BK_s Z_s A$, from the Kalman filter.

[3]

a. The cost function in the time domain is given by

$$J = 2\pi \int_0^\infty \left\{ x_f^T(t) x_f(t) + \rho u^T(t) R u(t) \right\} dt ,$$

where $x_f(t)$ is determined by the following equations:

$$\frac{d}{dt} z_1(t) = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} z_1(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} C x(t)$$
$$x_f(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} z_1(t)$$

Denote

$$A_1 = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} C \text{ and } C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

b. The optimal control law is given by

$$U(s) = -[K + K_1 (sI - A_1)^{-1} B_1] X(s) = -KX(s) - K_1 (sI - A_1)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} Y(s) .$$

Suppose $K_1 = \begin{bmatrix} a & b \end{bmatrix}$. Then $K_1 (sI - A_1)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{a + bs}{s^2 + \omega_0^2}$. So the control signal can be expressed as

$$U(s) = -KX(s) - \frac{a + bs}{s^2 + \omega_0^2} Y(s) .$$

Notice that $\frac{a + bs}{s^2 + \omega_0^2}$ is the internal model controller for a sinusoidal signal. Then the control structure for tracking a sinusoidal reference is shown in Fig. 1.

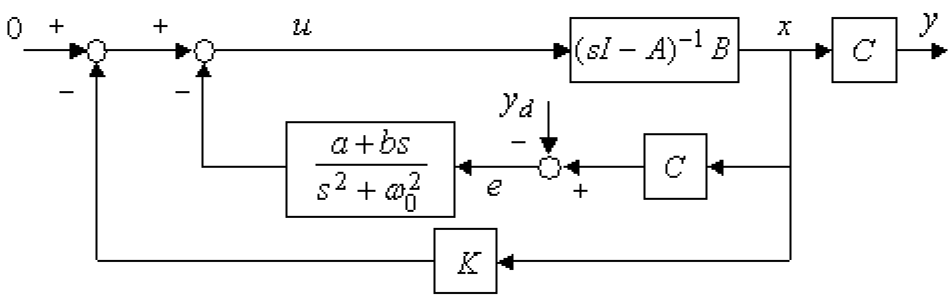

Figure 1 Block diagram of the closed loop system for tracking y_d.

The closed loop transfer function from the reference y_d to the tracking error $e = Cx - y_d = y - y_d$ is given by

$$G_{y_d e}(s) = -\frac{[1+K(sI-A)^{-1}B](s^2+\omega_0^2)}{[1+K(sI-A)^{-1}B](s^2+\omega_0^2)+(a+bs)C(sI-A)^{-1}B}.$$

So the asymptotically perfect tracking is achieved, since the closed loop system is asymptotically stable and $(s^2+\omega_0^2)y_d(t)=0$.

Notice that the term $(s^2+\omega_0^2)$ will not appear in the numerator of the closed loop transfer function from y_d to $y-y_d$, if y_d is injected at the place where 0 is injected in the block diagram.