ME232 Lecture #25

Some Properties of LQ Systems

Return Difference Equality and Robustness of LQ Regulators

Consider the controlled plant

$$\frac{dx(t)}{dt} = Ax(t)+Bu(t) \tag{25-1}$$

where x and u are respectively n- and r-dimensional, and the performance index

$$J = \int_0^\infty x^T(t)Qx(t)+u^T(t)Ru(t)dt, \quad Q = C^TC \tag{25-2}$$

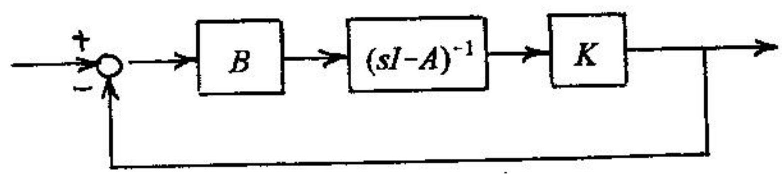
The optimal feedback control law is

$$u(t) = -R^{-1}B^TP_+x(t) = -Kx(t) \tag{25-3}$$

where P_+ is the positive solution of ARE

$$A^TP_++P_+A-P_+BR^{-1}B^TP_++C^TC = 0 \tag{25-4}$$

Notice that the optimal closed loop system can be represented as shown below.



For this system, the following equality holds:

$$[I_r+K\Phi(-s)B]^TR[I_r+K\Phi(s)B] = R+G^T(-s)G(s) \tag{25-5}$$

where

$$\Phi(s) = (sI-A)^{-1}, \quad G(s) = C(sI-A)^{-1}B = C\Phi(s)B \tag{25-6}$$

Notice that $K\Phi(s)B$ is square (rxr), but G(s) is not necessarily square. $I_r + K\Phi(s)B$ is called the return difference function matrix and Eq. (25-5) the return difference equality.

Proof of Eq. (25-5): The steady state Riccati equation yields

$$-A^TP_+-P_+A-C^TC+P_+BR^{-1}B^TP_++sP_+-sP_+ = 0$$
$$\Rightarrow P_+(sI-A)+(-sI-A^T)P_++K^TRK = C^TC$$

Multiplying $B^T(-sI-A^T)^{-1}$ from left and $(sI-A)^{-1}B$ from right to the second equality above and noting RK

$= B^T P_+$, we obtain

$$B^T(-sI-A^T)^{-1}K^T R + RK(sI-A)^{-1}B + B^T(-sI-A^T)^{-1}K^T RK(sI-A)^{-1}B = B^T(-sI-A^T)^{-1}C^T C(sI-A)^{-1}B$$
$$\Rightarrow [I_r + B^T(-sI-A^T)^{-1}K^T]R[I_r + K(sI-A)^{-1}B] = R + B^T(-sI-A^T)^{-1}C^T C(sI-A)^{-1}B$$
$$\Rightarrow Eq.(25\text{-}5).$$

Stability Margins of Scalar LQ Regulators

For single input systems, Eq. (25-5) becomes

$$[1+A(-s)][1+A(s)] = 1+\frac{1}{R}G^T(-s)G(s), \quad A(s) = K\Phi(s)B \tag{25-7}$$

Letting $s = j\omega$ in (25-7),

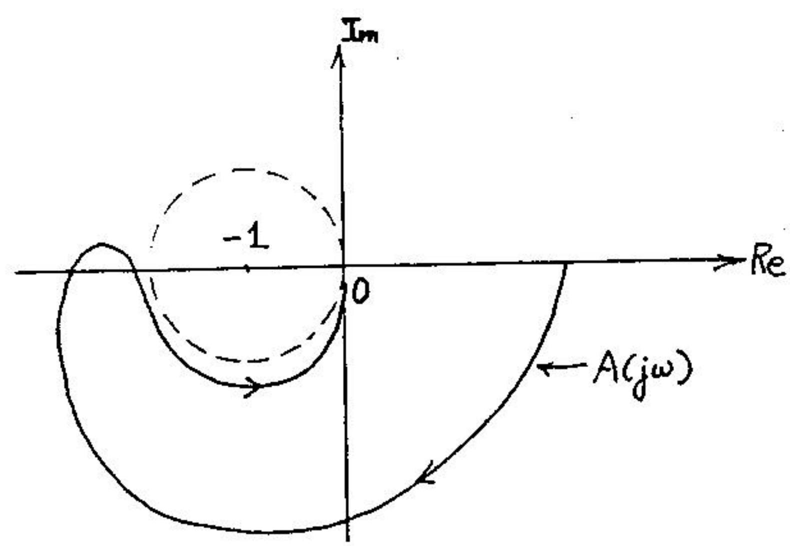
$$\begin{aligned}[1+A(-j\omega)][1+A(j\omega)] &= 1+\frac{1}{R}G^T(-j\omega)G(j\omega) \\ \Rightarrow \|1+A(j\omega)\|^2 &= 1+\frac{1}{R}G^T(-j\omega)G(j\omega)\end{aligned} \tag{25-8}$$

Since $G^T(-j\omega)G(j\omega) \geq 0$, (25-8) implies

$$\|1+A(j\omega)\| \geq 1 \tag{25-9}$$

(25-9) is called the Kalman inequality and provides the robustness properties of the LQ regulator.

Note that $A(j\omega)$ is the frequency response of the LQ loop transfer function (see figure on the previous page). (25-9) implies that the frequency response plot does not enter a unit circle centered on the point (-1, 0) in the complex plane (see figure below).



This figure indicates that the LQ regulator with full state feedback has

i. at least a 60 degree phase margin
ii. infinite gain margin
iii. stability is guaranteed up to a 50 % reduction in the gain

i.-iii. imply that the LQ regulator is robust for gain variations and other variations, which may cause variations in phase shift. However, they do not imply that the LQ regulator is robust for any kind of variations or unmodelled dynamics. Furthermore, these robustness properties will be lost if an observer is inserted in the loop to replace the state feedback control by the estimated state feedback control.

Discrete Time Case

Recall the stationary discrete-time LQ problem and solution in Theorem LQ-5 on page 130. The return difference equality for the discrete-time LQ system is

$$[I_r+K\Phi(z^{-1})B]^T(R+B^TP_+B)[I_r+K\Phi(z)B] = R+G^T(z^{-1})G(z) \tag{25-10}$$

where

$$\Phi(z) = (zI-A)^{-1}, \quad G(z) = C(zI-A)^{-1}B$$

Proof of (25-10): The discrete ARE yields

$$A^TP_+A-P_++C^TC-A^TP_+B[R+B^TP_+B]^{-1}B^TP_+A \pm zA^TP_+ = 0$$
$$\Rightarrow -A^TP_+(zI-A)-(z^{-1}I-A^T)zP_++C^TC-A^TP_+B[R+B^TP_+B]^{-1}B^TP_+A = 0$$

Multiplying $B^T(z^{-1}I-A^T)^{-1}$ from left and $(zI-A)^{-1}B$ from right and rearranging terms, we obtain

$$B^T(z^{-1}I-A^T)^{-1}A^TP_+B+B^TzP_+(zI-A)^{-1}B+B^T(z^{-1}I-A^T)^{-1}A^TP_+B$$
$$[R+B^TP_+B]^{-1}B^TP_+A(zI-A)^{-1}B = B^T(z^{-1}I-A^T)^{-1}C^TC(zI-A)^{-1}B$$
$$\Rightarrow (\textit{add and subtract } B^TP_+A(zI-A)^{-1}B \Rightarrow$$
$$B^T(z^{-1}I-A^T)^{-1}A^TP_+B+B^TP_+A(zI-A)^{-1}B+B^TP_+B$$
$$+B^T(z^{-1}I-A^T)^{-1}A^TP_+B[R+B^TP_+B]^{-1}B^TP_+A(zI-A)^{-1}B = G^T(z^{-1})G(z)$$

Noting $B^TP_+A = [R+B^TP_+B]K$ in the above equality, we obtain (25-10).

For single input systems, Eq. (25-10) becomes

$$[1+A(z^{-1})][1+A(z)]=\frac{1}{R+B^TP_+B}[R+G^T(z^{-1})G(z)] \tag{25-11}$$

where

$$A(z) = K(zI-A)^{-1}B$$

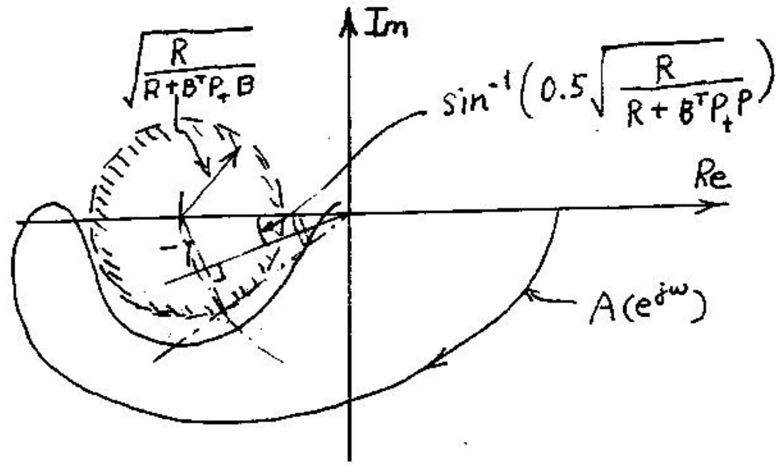
Letting $z = e^{j\omega}$ in (25-11), we obtain

$$[1+A(e^{-j\omega})][1+A(e^{j\omega})] = \frac{1}{R+B^TP_+B}[R+G^T(e^{-j\omega})G(e^{j\omega})]$$

$$\Rightarrow$$

$$\|1+A(e^{j\omega})\| \geq \sqrt{\frac{R}{R+B^TP_+B}}$$

This inequality implies that the frequency response plot of the LQ loop transfer function does not enter a circle with radius $[R/(R+B^TP_+B]^{1/2}$ centered at (-1, 0) on the complex plane (see figure below).



From this figure, the stability robustness of discrete time LQR in terms of the gain and phase margins is

i. Phase margin > $2\sin^{-1}\{0.5[R/(R+B^TP_+B)]^{1/2}\}$
ii. Gain margin > $1/\{1 - [R/(R+B^TP_+B)]^{1/2}\}$
iii. Stability is guaranteed for the % loop gain change

$$\frac{100}{1+\sqrt{R/(R+B^TP_+B)}} < \%\ \textit{loop gain change} < \frac{100}{1-\sqrt{R/(R+B^TP_+B)}}$$

Note that these results are not as good as the continuous time results, which stated a 60 degree PM and infinite GM. In particular, the radius of the circle, $[R/(R+B^TP_+B)]^{1/2}$, becomes smaller as R is decreased, which implies a smaller GM and PM.

A good reference for properties of LQ systems is Optimal Control---Linear Quadratic Methods by Anderson and Moore, Prentice Hall, 1990.

Steady State Property of LQ Systems

Notice that the LQR is a proportional state feedback control system. The controller does not have an integrator. This implies that the steady state of the LQR is not offset error free when the system is subjected to a constant disturbance or a constant reference input. The proportional state feedback control law is a consequence of the formulation of the standard LQ problem; recall that the plant equation did not have a disturbance term. When disturbances or nonzero reference inputs are expected, the standard LQ problem must be modified so that the controller will possess a right structure. This can be accomplished in a variety of ways.

LQI controller (Linear Quadratic Optimal Control with Integrator) for Continuous Time Systems.

The plant is described by

$$\frac{dx(t)}{dt} = Ax(t)+Bu(t), \quad y(t) = Cx(t) \tag{25-12}$$

where

- **x**: n-dimensional state vector
- **u**: m-dimensional input vector
- **y**: m-dimensional output vector

We assume the following conditions:

1. Plant is controllable and observable
2. **A** is nonsingular (i.e. no open loop pole at the origin)
3. $\mathbf{CA^{-1}B}$ is nonsingular

Now consider the quadratic performance index,

$$J = \int_0^\infty [y(t)-r]^T Q_y [y(t)-r] + v^T(t) R v(t) dt \tag{25-13}$$

where $\mathbf{Q_y}$ and **R** are symmetric and positive definite, r is an m-dimensional constant set point vector and **v** is the derivative of **u**, i.e.

$$\mathbf{du(t)/dt = v(t)}$$

We define an (m+n)-dimensional state vector $\tilde{\mathbf{x}} = [\mathbf{e}^T, (\mathrm{d}\mathbf{x}/\mathrm{d}t)^T]^T$ ($\mathbf{e} = \mathbf{y} - \mathbf{r}$). Then the state equation for the enlarged state vector is

$$\frac{d}{dt}\begin{bmatrix} e \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & C \\ 0 & A \end{bmatrix}\begin{bmatrix} e \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} v(t) \Rightarrow \frac{d\tilde{x}(t)}{dt} = \tilde{A}\tilde{x}(t) + \tilde{B}v(t) \tag{25-14}$$

The performance index is written as

$$J = \int_0^{\infty} \tilde{x}^T(t)\begin{bmatrix} Q_y & 0 \\ 0 & 0 \end{bmatrix}\tilde{x}(t) + v^T(t)Rv(t)dt \tag{25-15}$$

Notice that the problem is now reformulated as a standard LQ problem.

The controllability and observability condition required for the Riccati equation to have a stationary positive definite solution must be checked for

$$(\tilde{A},\ \tilde{B}) \quad and \quad (\tilde{A},\ Q_y^{\frac{1}{2}}[I_m \quad 0_n])$$

They are satisfied under Assumptions 1, 2 and 3. You should check them yourself. The feedback control law for v(t) is given by

$$v(t) = -R^{-1}\tilde{B}^T\tilde{P}_+\tilde{x}(t) = -K_e e(t) - K_x\dot{x}(t) \tag{25-16}$$

where K_e, K_x and P_+ are given by

$$\begin{bmatrix} K_e & K_x \end{bmatrix} = \begin{bmatrix} R^{-1}B^T\tilde{P}_{xe} & R^{-1}B^T\tilde{P}_{xx} \end{bmatrix} \qquad \tilde{P}_+ = \begin{bmatrix} \tilde{P}_{ee} & \tilde{P}_{ex} \\ \tilde{P}_{xe} & \tilde{P}_{xx} \end{bmatrix} > 0$$

$$\tilde{A}^T\tilde{P} + \tilde{P}\tilde{A} - \tilde{P}\tilde{B}R^{-1}\tilde{B}^T\tilde{P} + \begin{bmatrix} Q_y & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Integrating Eq. (25-16) from initial time (0) to t, we obtain

$$u(t) = K_e\int_0^t [r - y(\tau)]d\tau - K_x x(t)$$

where x(0) = 0 has been assumed. Notice that we have obtained a PI (Proportional plus Integral) control law.

Note: The main idea in this development is the use of the time derivative of u(t) in the cost functional. In the original LQ formulation, u(t) was penalized, a consequence of which is the convergence of u(t) to zero for the cost functional to remain bounded. By including the time derivative of u(t) in the cost functional, u(t) does not have to converge to zero but to any constant value.

For discrete time systems, the LQI controller can be developed in a similar manner. In particular, include $\Delta u(k)^T R \Delta u(k) = (u(k) - u(k-1))^T R(u(k) - u(k-1))$ in the cost functional instead of $u(k)^T R u(k)$. The enlarged state equation must be written for $e(k) = r - y(k)$ and $\Delta x(k) = x(k) - x(k-1)$.

ME232 Lectures #26

The Closed Loop Eigenvalues of LQ System (Ref. Anderson and Moore, Section 5.2)

The closed loop system is characterized by

$$\frac{dx}{dt} = (A-BK)x, \qquad K = R^{-1}B^TP_+ \tag{26-1}$$

and the closed loop eigenvalues satisfy the closed loop characteristic equation

$$\det[\lambda I_n - A + BK] = 0 \tag{26-2}$$

We note the following relation.

$$\begin{aligned}\det[I_r + K\Phi(s)B] &= \det[I_r + K(sI-A)^{-1}B] \\ &= \det[I_n + (sI-A)^{-1}BK] \\ &= \det(sI-A)^{-1}\det[sI-A+BK] \\ &= \frac{\det[sI-A+BK]}{\det[sI-A]}\end{aligned} \tag{26-3}$$ *

where r is the dimension of the control input vector, u.

From the return difference equality, (25-5), we obtain

$$\det[I_r + K\Phi(-s)B]\det[R]\det[I_r + K\Phi(s)B] = \det[R]\det[I_r + R^{-1}G^T(-s)G(s)] \tag{26-4}$$

where

$$\Phi(s) = (sI-A)^{-1}, \qquad G(s) = C(sI-A)^{-1}B = C\Phi(s)B$$

From Eqs. (26-3) and (26-4), we obtain

$$\beta(s)\beta(-s) = \phi(s)\phi(-s)\det[I_r + R^{-1}G^T(-s)G(s)] \tag{26-5}$$

where

$$\beta(s) = \det[sI-A+BK] \quad and \quad \phi(s) = \det[sI-A]$$

are the closed loop and open loop characteristic polynomials.

Single Input Systems ($R = r$)

For single input systems, Eq. (26-5) is

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* This equality is analogous to $1 + G(s) = 1 + \frac{B(s)}{A(s)} = \frac{A(s)+B(s)}{A(s)}$ where the denominator and numerator of the right hand side is the open loop characteristic polynomial and closed loop characteristic polynomial, respectively.

$$\beta(s)\beta(-s) = \phi(s)\phi(-s)[1+\frac{1}{r}G^T(-s)G(s)], \qquad G(s) = \frac{1}{\phi(s)}\begin{bmatrix}\psi_1(s)\\ \vdots \\ \psi_m(s)\end{bmatrix} \tag{26-6}$$

where m is the number of rows in C. Writing

$$G^T(-s)G(s) = \frac{m(-s)m(s)}{\phi(-s)\phi(s)}$$

Eq. (26-6) is written as

$$\beta(s)\beta(-s) = \phi(s)\phi(-s)+\frac{1}{r}m(s)m(-s) \tag{26-7}$$

Notice that when G(s) is scalar, i.e. C is a row vector, Eq. (26-7) is

$$\beta(s)\beta(-s) = \phi(s)\phi(-s)+\frac{1}{r}\psi(s)\psi(-s) \tag{26-8}$$

where

$$G(s) = C(sI-A)^{-1}B = \psi(s)/\phi(s).$$

For single input systems, Eq. (26-7) or (26-8) can be used to construct the root locus plot for the optimal closed loop system with r varied from 0 to ∞. Root locus plots obtained from these equations are called symmetric root locus plots because of their properties as discussed below.

Symmetric Root Locus

As shown above, the closed loop eigenvalues satisfy

$$\phi(s)\phi(-s)+\frac{1}{r}m(s)m(-s) = 0 \tag{26-9}$$

where $m(s) = \psi(s)$ when C is a row vector. We note that if p_c is a root of (26-9), $-p_c$ is also a root of (26-9). Therefore, the root locus plot (with varying r) is symmetric with respect to the imaginary axis as well as to the real axis, which is the reason for the name, Symmetric Root Locus. Let us write

$$\frac{m(s)}{\phi(s)} = \frac{\alpha\prod_{j=1}^{l}(s-z_{oj})}{\prod_{i=1}^{n}(s-p_{oi})} \tag{26-10}$$

Then, Eq. (26-9) in the standard root locus form is

$$\prod_{i=1}^{n}(s-p_{oi})(s+p_{oi})+(-1)^{n-l}\frac{\alpha^2}{r}\prod_{j=1}^{l}(s-z_{oj})(s+z_{oj}) = 0 \tag{26-11}$$

Now applying the root locus rules, we may conclude the following:

1. As $r \to 0$ (i.e. the loop gain approaches the infinity), $2l$ among 2n roots of (26-11) asymptotically approach zeros (z_{oj}'s) and their negatives ($-z_{oj}$'s) ($j = 1,2,\ldots, l$).

2. As $r \to 0$, remaining $2(n-l)$ roots of (26-11) asymptotically approach straight lines which intersect in the origin and make angles with the positive real axis

$$\frac{k\pi}{n-l},\quad k = 0,\ 1,\ 2,\ \ldots,\ 2n-2l-1 \quad \textit{for } (n-l) \textit{ odd}$$
$$\frac{(k+(1/2))\pi}{n-l},\quad k = 0,\ 1,\ 2,\ \ldots,\ 2n-2l-1 \quad \textit{for } (n-l) \textit{ even}$$

   where the difference between the odd and even cases is due to $(-1)^{n-l}$ factor in (26-11).

3. As $r \to 0$, the $2(n-l)$ faraway roots of (26-11) are asymptotically at a distance

$$(\alpha^2/r)^{1/2(n-l)}$$

   from the origin.

4. As $r \to \infty$, the 2n roots of (26-11) approach the poles (p_{oi}'s) and their negatives ($-p_{oi}$'s) ($i = 1,2,\ldots,n$).

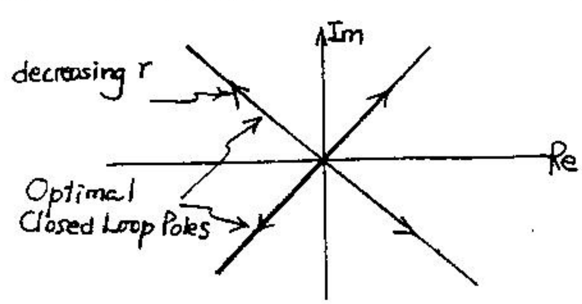
Note 1. The poles of the optimal feedback system are n stable roots that we obtain from the root locus plot.

Note 2. The configuration of roots indicated by 2 is known as a Butterworth configuration of order n-l.

Example 1 (A pure inertia system on page 125.) From the state equation and the quadratic performance index on pages 125 and 126,

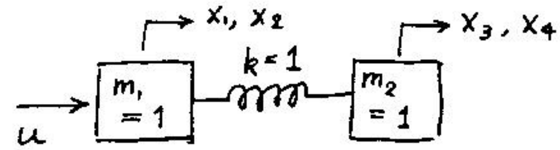
$$m(s)/\phi(s) = 1/s^2$$

In this problem, $n = 2$ and $l = 0$. Therefore the root locus plot shown below is immediately obtained. Notice that the closed loop system has a 0.707 damping, which was found on page 126. In the present case, we have obtained this conclusion without solving the Riccati equation.



Example 2 (A two inertia system)

We consider a two inertia system sketched below.



The state equations and the quadratic performance index for this problem are

$$
\begin{aligned}
dx_1/dt &= x_2 \\
dx_2/dt &= -(x_1-x_3)+u \\
dx_3/dt &= x_4 \\
dx_4/dt &= -(x_3-x_1) \\
y &= x_3 \\
J &= \int_0^\infty (y^2+ru^2)dt
\end{aligned}
$$

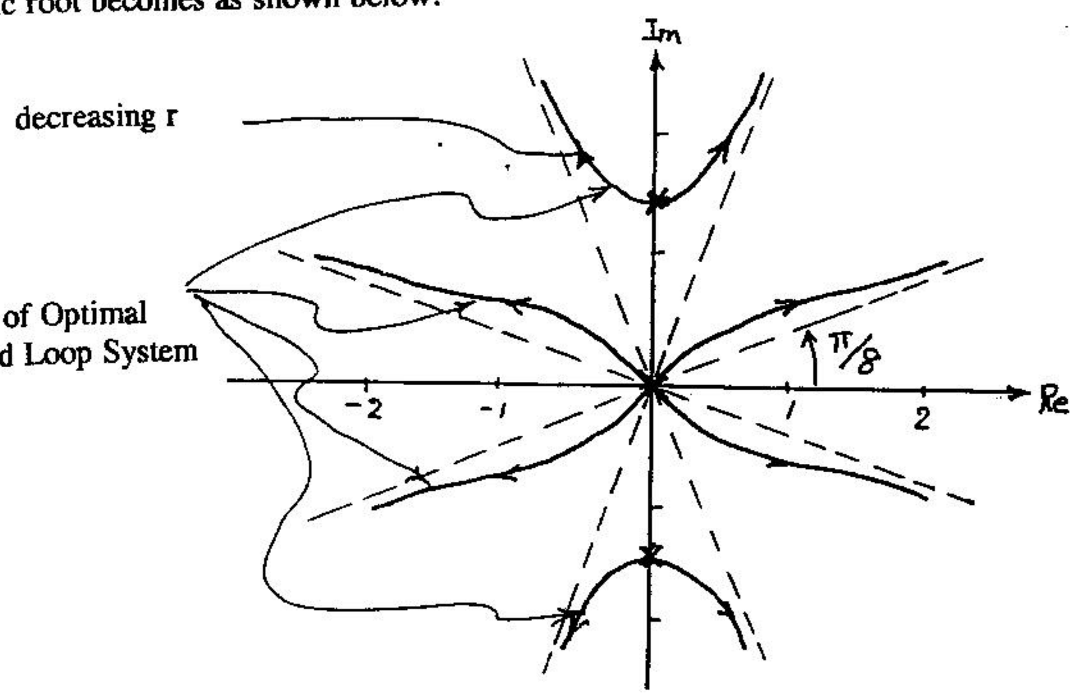
The transfer function G(s) becomes

$$\frac{Y(s)}{U(s)} = \frac{\psi(s)}{\phi(s)} = \frac{1}{s^2(s^2+2)}, \qquad (n=4, \quad l=0)$$

Eq. (26-11) is, therefore,

$$s^4(s^2+2)^2+\frac{1}{r} = 0 \quad or \quad 1+\frac{1}{rs^4(s^2+2)^2} = 0$$

The symmetric root becomes as shown below.



When C is selected so that n-1 zeros of $C\Phi(s)B$ are in the left half side of the complex plane, for $r \to 0$, the optimal feedback gain is asymptotic to

$$K = \pm\frac{1}{\sqrt{r}}C \quad (+\ for\ CB>0 \quad and \quad -\ for\ CB<0) \tag{26-12}$$

This relation is obtained from (25-8). For r (R) → 0, (25-8) holds only if A(jω) grows as fast as $1/\sqrt{r}$: i.e.

$$|A(j\omega)| = |K(j\omega I-A)^{-1}B| \approx \frac{1}{\sqrt{r}}|C(j\omega I-A)^{-1}B|$$

The zeros of A(s) are known to be in the left half side of the complex plane; recall that the relative degree of A(s) is 1 and that A(s) allows the infinite gain margin. (26-12) follows from this observation.

Furthermore, the open loop gain at high frequencies is

$$|K(j\omega I-A)^{-1}B| \approx |\frac{KB}{j\omega}| \approx \frac{|CB|/\sqrt{r}}{\omega}$$

Therefore, the gain crossover frequency of A(jω) is approximately

$$\omega_g = \frac{1}{\sqrt{r}}|CB| \tag{26-13}$$

The asymptotic properties for $r \to 0$ may be utilized in the design. One possibility is to select C so that the zeros of $C\Phi(s)B$ are at the desired closed loop poles. Then let $Q = C^TC$ and r small. This procedure allows you to approximately specify n-1 poles. The remaining pole is on the negative real axis and is pushed toward $-\infty$ as r is decreased. r can be adjusted by Eq. (26-13) to achieve the desired bandwidth.

Asymptotic Behavior (Multi-variable Case)

Let

$$J = \int_0^\infty \{x^T(t)C^TCx(t)+\rho u^T(t)Nu(t)\}dt, \quad R = \rho N \tag{26-14}$$

and consider the two cases: $\rho \to \infty$ and $\rho \to 0$.

$\rho \to \infty$:

The return difference equality is

$$[I_r+K\Phi(-s)B]^T\rho N[I_r+K\Phi(s)B] = \rho N+G^T(-s)G(s)$$

Therefore, for $s = j\omega$ and $\rho \to \infty$,

$$|\det[I_r+K\Phi(j\omega)B]|^2\det[N] = \det[N+\frac{1}{\rho}G^T(-j\omega)G(j\omega]\to\det[N] \tag{26-15}$$

From Eqs. (26-3) and (26-15), for $\rho \to \infty$

$$|\frac{\det[j\omega I-A+BK]}{\det[j\omega I-A]}| \to 1 \quad or \quad \beta(s)\beta(-s) \to \phi(s)\phi(-s) \tag{26-16}$$

This result corresponds to 4 on page 140, which we concluded for single input systems based on the symmetric root locus.

$\rho \to 0$:

In this case, we still have asymptotic properties analogous to 1.-3. for single input systems. However, situation is a little more complicated because, in the multi-input case, the choice of K is not unique for a set of closed loop eigenvalues. In fact, this non-uniqueness aspect allows us to specify not only the closed loop eigenvalues but also the associated eigenvectors to some extent. An interesting case is discussed in Section 6.2 of Anderson and Moore.

The Closed Loop Eigenvalues of LQ Systems (Discrete Time Case)

The discrete time closed loop system is characterized by

$$x(k+1) = (A-BK)x(k)$$

Notice that Eq. (LQ-76) can be used for the discrete time case by replacing s by z. Therefore, from Eq. (26-3) and the return difference equality (25-10) (on page 134)

$$\beta(z)\beta(z^{-1}) = \phi(z)\phi(z^{-1})\frac{\det[R+G^T(z^{-1})G(z)]}{\det[R+B^TP_+B]} \tag{26-17}$$

where

$$\beta(z) = \det[zI-A+BK] \quad and \quad \phi(z) = \det[zI-A]$$

are the closed loop and open loop characteristic polynomials.

Since $\det[R+B^TP_+B] > 0$, the optimal closed loop poles are obtained by solving

$$(z)\phi(z^{-1})\det[R+G^T(z^{-1})G(z)] = 0 \quad or \quad \phi(z)\phi(z^{-1})\det[I_r+R^{-1}G^T(z^{-1})G(z)] = \tag{26-18}$$

The characteristic roots of (26-18) inside the unit circle are the optimal closed loop eigenvalues.

For single input systems, (26-18) can be written as

$$z^n\phi(z^{-1})\phi(z)+\frac{1}{r}z^n m(z^{-1})m(z) = 0 \tag{26-19}$$

where

$$G(z^{-1})^T G(z) = \frac{m(z^{-1})m(z)}{\phi(z^{-1})\phi(z)}, \quad G(z) = \frac{1}{\phi(z)}\begin{bmatrix}\psi_1(z)\\ \vdots \\ \psi_m(z)\end{bmatrix} \tag{26-20}$$

Writing

$$\frac{m(z)}{\phi(z)} = \frac{\alpha\prod_{j=1}^{l}(z-z_{oj})}{\prod_{i=1}^{n}(z-p_{oi})}$$

(26-19) becomes

$$\prod_{i=1}^{n}(z-\frac{1}{p_{oi}})(z-p_{oi})+(-z)^{n-l}\frac{\alpha^2}{r}\frac{\prod_{j=1}^{l}z_{oj}}{\prod_{i=1}^{n}p_{oi}}\prod_{j=1}^{l}(z-\frac{1}{z_{oj}})(z-z_{oj}) = 0 \tag{26-21}$$

Applying the rules of the root locus, we obtain the following results:

1. $r \to \infty$: If all the open loop poles are inside of the unit circle, the eigenvalues of the optimal closed loop system are asymptotic to the open loop eigenvalues as r is increased. Since the roots of $\phi(z) = 0$ and $z^n\phi(z^{-1}) = 0$ have reciprocal relationships, if any open loop eigenvalues are unstable (i.e. outside of the unit circle), then the corresponding eigenvalues of the optimal closed loop system are asymptotic to the reciprocals the unstable open loop eigenvalues.

2. $r \to 0$: When r is small, the eigenvalues of the optimal closed loop system are attracted towards stable open loop zeros and reciprocals of unstable zeros. When the order of m(z) is lower than that of $\phi(z)$ by $n-l$, $n-l$ closed loop eigenvalues are asymptotic to the origin. In particular, if the order of m(z) is zero, all the closed loop eigenvalues are asymptotic to the origin and the closed loop response is asymptotically of finite time settling.