Tony Kelman
ME 233 – Homework 5

1.a)

$$Y(k)-0.5Y(k-1)=W(k)-0.3W(k-1),\ E\{Y(0)\}=0,\ E\{Y(0)^2\}=0,\ E\{Y(0)W(k)\}=0$$

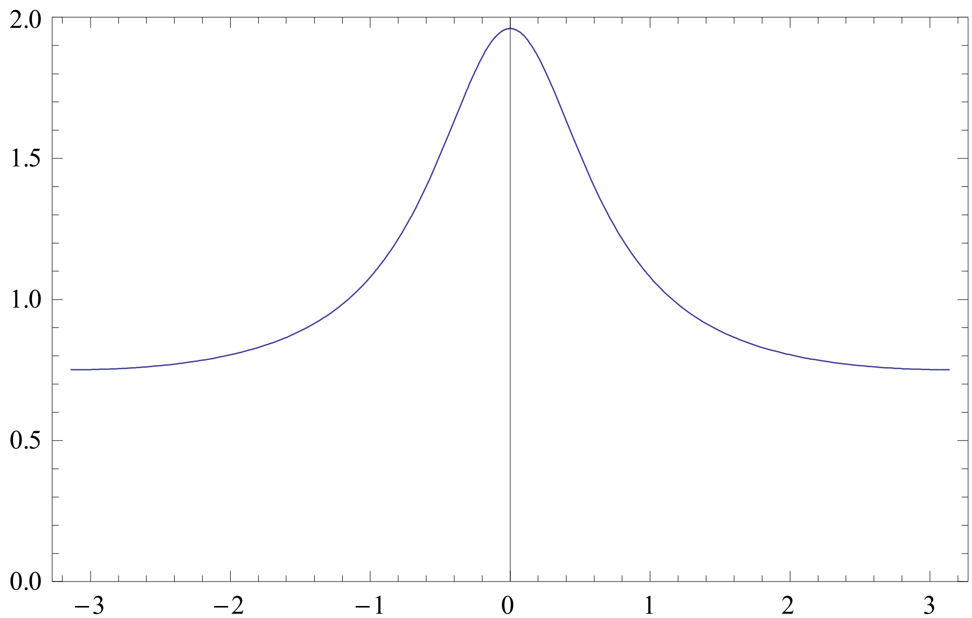
$$m_W=0,\ \Lambda_{WW}(l)=E\{W(k+l)W(k)\}=\delta(l)\ \text{so}\ \Phi_{WW}(\omega)=1$$

$$G(z)=\frac{Y(z)}{W(z)}=\frac{z-0.3}{z-0.5}$$

$$\Phi_{YY}(\omega)=G(e^{j\omega})\Phi_{WW}(\omega)G^T(e^{-j\omega})=\frac{e^{j\omega}-0.3}{e^{j\omega}-0.5}\cdot\frac{e^{-j\omega}-0.3}{e^{-j\omega}-0.5}=\frac{1.09-0.3e^{j\omega}-0.3e^{-j\omega}}{1.25-0.5e^{j\omega}-0.5e^{-j\omega}}$$

$$\Phi_{YY}(\omega)=\frac{0.75-0.3e^{j\omega}-0.3e^{-j\omega}}{1.25-0.5e^{j\omega}-0.5e^{-j\omega}}+\frac{0.34}{1.25-0.5e^{j\omega}-0.5e^{-j\omega}}=0.6+\frac{0.34}{1.25-\cos(\omega)}$$

$\Phi_{YY}(\omega)$ vs ω



1.b)

$$\Phi_{YY}(\omega)=\frac{1.09-0.3e^{j\omega}-0.3e^{-j\omega}}{(e^{j\omega}-0.5)(e^{-j\omega}-0.5)}\ \text{so}\ \Lambda_{YY}(z)=\frac{1.09-0.3z-0.3z^{-1}}{(z-0.5)(z^{-1}-0.5)}$$

Partial fraction $\Lambda_{YY}(z)=\frac{Az}{z-0.5}+B+\frac{Az^{-1}}{z^{-1}-0.5}+B=\frac{2A+2.5B-(0.5A+B)z-(0.5A+B)z^{-1}}{(z-0.5)(z^{-1}-0.5)}$

Matching coefficients, $2A+2.5B=1.09,\ 0.5A+B=0.3$

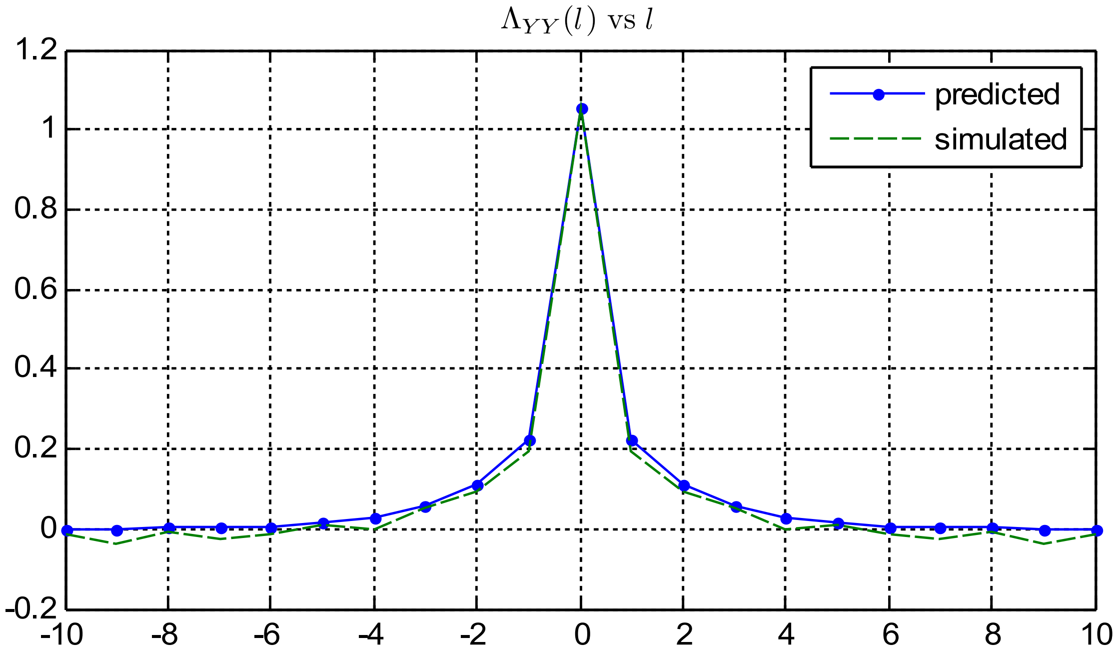
$(2-2.5\cdot 0.5)A=1.09-2.5\cdot 0.3,\ (1-0.25\cdot 2.5)B=0.3-0.25\cdot 1.09$

$A=0.34/0.75=0.4533,\ B=0.0275/0.375=0.0733$

Taking inverse Z transform, $\Lambda_{YY}(l)=\begin{cases} A\cdot 0.5^l & \text{for } l>0 \\ 2A+2B & \text{for } l=0 \\ A\cdot 0.5^{-l} & \text{for } l<0 \end{cases}$

$\Lambda_{YY}(l)=\Lambda_{YY}^C(l)+\Lambda_{YY}^A(l)$, where $\Lambda_{YY}^A(l)=\Lambda_{YY}^C(-l)$ and $\Lambda_{YY}^C(l)=\begin{cases} 0.4533\cdot 0.5^l+0.0733\,\delta(l) & \text{for } l\ge 0 \\ 0 & \text{for } l<0 \end{cases}$

Note that $\Lambda_{YY}(0)\neq 1$ so in the below plot, instead of using the 'coeff' scaling on the Matlab results from homework 4, I used the 'none' option then divided by N



1.c)

$$(Y(k)-0.5\,Y(k-1))\,W(k)=(W(k)-0.3\,W(k-1))\,W(k)$$
$$E\{Y(k)W(k)-0.5\,Y(k-1)W(k)\}=E\{W(k)W(k)-0.3\,W(k-1)W(k)\}$$
$$E\{Y(k)W(k)\}-0.5\,E\{Y(k-1)W(k)\}=E\{W(k)W(k)\}-0.3\,E\{W(k-1)W(k)\}$$
$$\Lambda_{YW}(0)-0.5\,\Lambda_{YW}(-1)=\Lambda_{WW}(0)-0.3\,\Lambda_{WW}(-1)$$

Λ_{YW} is causal so $\Lambda_{YW}(-1)=0$, $\Lambda_{YW}(0)=\delta(0)-0.3\,\delta(-1)=1$

1.d)

$$(Y(k)-0.5\,Y(k-1))\,W(k-1)=(W(k)-0.3\,W(k-1))\,W(k-1)$$
$$E\{Y(k)W(k-1)-0.5\,Y(k-1)W(k-1)\}=E\{W(k)W(k-1)-0.3\,W(k-1)W(k-1)\}$$
$$E\{Y(k)W(k-1)\}-0.5\,E\{Y(k-1)W(k-1)\}=E\{W(k)W(k-1)\}-0.3\,E\{W(k-1)W(k-1)\}$$
$$\Lambda_{YW}(1)-0.5\,\Lambda_{YW}(0)=\Lambda_{WW}(1)-0.3\,\Lambda_{WW}(0)=\delta(1)-0.3\,\delta(0)$$
$$\Lambda_{YW}(1)=\delta(1)-0.3\,\delta(0)+0.5\,\Lambda_{YW}(0)=0-0.3+0.5=0.2$$

1.e)

$$Y(k)^2=(0.5\,Y(k-1)+W(k)-0.3\,W(k-1))^2$$
$$Y(k)Y(k)=0.25\,Y(k-1)Y(k-1)+Y(k-1)W(k)-0.3\,Y(k-1)W(k-1)+W(k)W(k)$$
$$\qquad -0.6\,W(k)W(k-1)+0.09\,W(k-1)W(k-1)$$
$$E\{Y(k)Y(k)\}=0.25\,E\{Y(k-1)Y(k-1)\}+E\{Y(k-1)W(k)\}-0.3\,E\{Y(k-1)W(k-1)\}$$
$$\qquad +E\{W(k)W(k)\}-0.6\,E\{W(k)W(k-1)\}+0.09\,E\{W(k-1)W(k-1)\}$$
$$\Lambda_{YY}(0)=0.25\,\Lambda_{YY}(0)+\Lambda_{YW}(-1)-0.3\,\Lambda_{YW}(0)+\Lambda_{WW}(0)-0.6\,\Lambda_{WW}(1)+0.09\,\Lambda_{WW}(0)$$
$$0.75\,\Lambda_{YY}(0)=0-0.3+\delta(0)-0.6\,\delta(1)+0.09\,\delta(0)=-0.3+1.09$$
$$\Lambda_{YY}(0)=0.79/0.75=1.0533$$

2.

$\{X(k)\}_{-\infty}^{\infty}\in\mathbb{R}^n$ WSS, $\Lambda_{XX}(j)=E\{\tilde{X}(k+j)\,\tilde{X}^T(k)\}$

$$\mathrm{Tr}[\Lambda_{XX}(j)]=E\{\tilde{X}^T(k+j)\,\tilde{X}(k)\}=E\left\{\sum_{i=1}^{n}\tilde{X}_i^{\,T}(k+j)\,\tilde{X}_i(k)\right\}=\sum_{i=1}^{n}E\{\tilde{X}_i^{\,T}(k+j)\,\tilde{X}_i(k)\}=\sum_{i=1}^{n}\Lambda_{X_iX_i}(j)$$

$\mathrm{Trace}[\Lambda_{XX}(0)]=\sum_{i=1}^{n}\Lambda_{X_iX_i}(0)$, and for scalar X_i we have $\Lambda_{X_iX_i}(0)\geq\left|\Lambda_{X_iX_i}(j)\right|$

from $Z_i = \begin{bmatrix} X_i(k) \\ X_i(k+j) \end{bmatrix}$, $\Lambda_{Z_iZ_i}(0) = \begin{bmatrix} \Lambda_{X_iX_i}(0) & \Lambda_{X_iX_i}(j) \\ \Lambda_{X_iX_i}(j) & \Lambda_{X_iX_i}(0) \end{bmatrix} \geq 0$ so $\det \Lambda_{Z_iZ_i}(0) = \Lambda_{X_iX_i}(0)^2 - \Lambda_{X_iX_i}(j)^2 \geq 0$

$\text{Trace}[\Lambda_{XX}(0)] = \sum_{i=1}^{n} \Lambda_{X_iX_i}(0) \geq \sum_{i=1}^{n} |\Lambda_{X_iX_i}(j)|$

By triangle inequality, $\sum_{i=1}^{n} |\Lambda_{X_iX_i}(j)| \geq \left|\sum_{i=1}^{n} \Lambda_{X_iX_i}(j)\right| = |\text{Trace}[\Lambda_{XX}(j)]|$

$\text{Trace}[\Lambda_{XX}(0)] \geq |\text{Trace}[\Lambda_{XX}(j)]|$

3.a)

$\begin{bmatrix} X_1(k+1) \\ X_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.08 & -1 \\ 0.7 & 0.1 \end{bmatrix} \begin{bmatrix} X_1(k) \\ X_2(k) \end{bmatrix} + \begin{bmatrix} 0.34 \\ 0.3 \end{bmatrix} W(k)$, and $Y(k) = [0 \quad 3] X(k) + V(k)$

$E\{X(0)\} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $E\{X(0)X^T(0)\} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$, $m_w = E\{W(k)\} = 10$, $E\{V(k)\} = 0$

$E\left\{\begin{bmatrix} W(k+j) - m_w \\ V(k+j) \end{bmatrix} [(W(k) - m_w) \quad V(k)]\right\} = \begin{bmatrix} \Sigma_{ww} & 0 \\ 0 & \Sigma_{vv} \end{bmatrix} \delta(j) = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \delta(j)$

$E\left\{\begin{bmatrix} W(k) - m_w \\ V(k) \end{bmatrix} X(0)^T\right\} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$E\{X(k+1)\} = A\,E\{X(k)\} + B\,E\{W(k)\}$, $E\{Y(k)\} = C\,E\{X(k)\} + E\{V(k)\}$

$m_x(k+1) = A\,m_x(k) + B\,m_w$, $m_y(k) = C\,m_x(k) + 0$

At steady-state $\bar{m}_x = A\bar{m}_x + B\,m_w$, so $\bar{m}_x = (I - A)^{-1} B\,m_w$ and $\bar{m}_y = C\,\bar{m}_x = C(I - A)^{-1} B\,m_w$

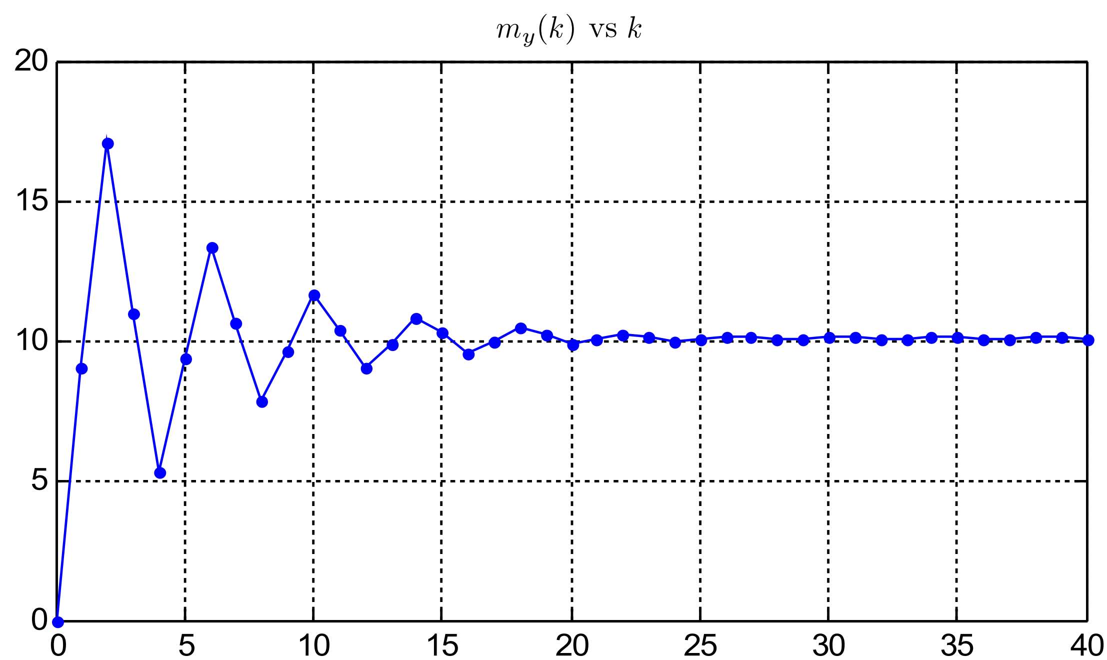
$m_x(k+1) - \bar{m}_x = A\,m_x(k) - A\bar{m}_x + A\bar{m}_x + B\,m_w - \bar{m}_x = A(m_x(k) - \bar{m}_x) - (I - A)\bar{m}_x + B\,m_w$

$(I - A)\bar{m}_x = B\,m_w$, so $m_x(k+1) - \bar{m}_x = A(m_x(k) - \bar{m}_x)$

$m_x(k) - \bar{m}_x = A^k(m_x(0) - \bar{m}_x)$, and since $m_x(0) = E\{X(0)\} = [0 \quad 0]^T$, $m_x(k) = (I - A^k)\bar{m}_x$

$m_y(k) = C\,m_x(k) = C(I - A^k)(I - A)^{-1} B\,m_w$

$\bar{m}_y = [0 \quad 3] \begin{bmatrix} 1.08 & 1 \\ -0.7 & 0.9 \end{bmatrix}^{-1} \begin{bmatrix} 0.34 \\ 0.3 \end{bmatrix} \cdot 10 = \frac{10}{1.672} [0 \quad 3] \begin{bmatrix} 0.9 & -1 \\ 0.7 & 1.08 \end{bmatrix} \begin{bmatrix} 0.34 \\ 0.3 \end{bmatrix} = \frac{10}{1.672} [0 \quad 3] \begin{bmatrix} 0.006 \\ 0.562 \end{bmatrix} = 10.084$



3.b)

$\Lambda_{XX}(k,0)=E\{(X(k)-m_x(k))(X(k)-m_x(k))^T\}$

$\Lambda_{XX}(k+1,0)=E\{(X(k+1)-m_x(k+1))(X(k+1)-m_x(k+1))^T\}$

$\Lambda_{XX}(k+1,0)=E\{(AX(k)+BW(k)-Am_x(k)-Bm_w)(AX(k)+BW(k)-Am_x(k)-Bm_w)^T\}$

$\Lambda_{XX}(k+1,0)=E\{(A(X(k)-m_x(k))+B(W(k)-m_w))(A(X(k)-m_x(k))+B(W(k)-m_w))^T\}$

$\Lambda_{XX}(k+1,0)=A\Lambda_{XX}(k,0)A^T+B\Lambda_{WX}(k,0)A^T+A\Lambda_{XW}(k,0)B^T+B\Lambda_{WW}(k,0)B^T$

$\Lambda_{WX}(k,0)=\Lambda_{XW}^T(k,0)=0$ since $X(k)$ only depends on $W(k-1)$ and earlier

$\Lambda_{XX}(k+1,0)=A\Lambda_{XX}(k,0)A^T+B\Lambda_{WW}(k,0)B^T=A\Lambda_{XX}(k,0)A^T+B\Sigma_{ww}\delta(0)B^T$

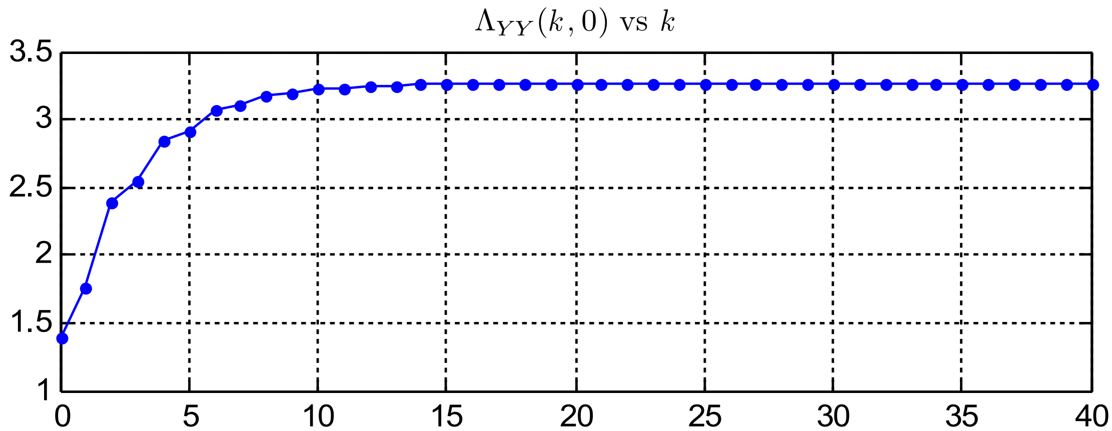
$\Lambda_{YY}(k,0)=E\{(Y(k)-m_y(k))(Y(k)-m_y(k))^T\}$

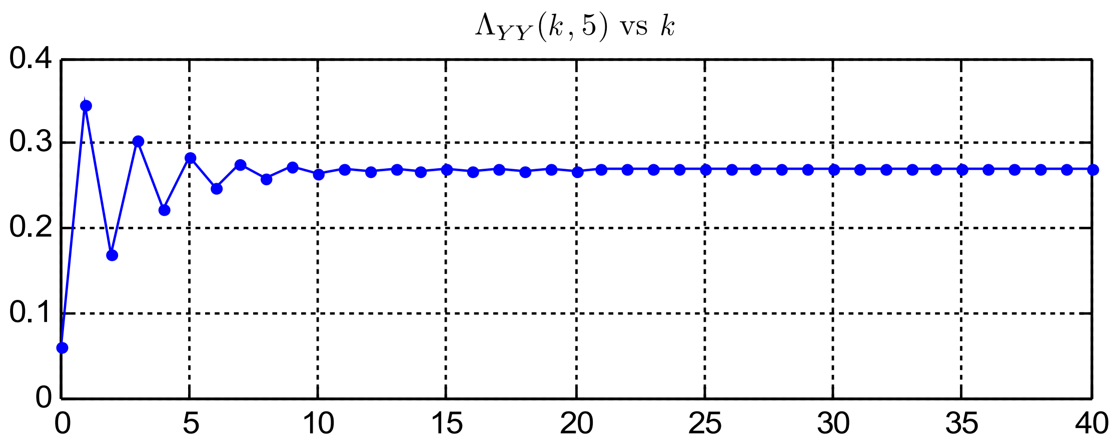
$\Lambda_{YY}(k,0)=E\{(C(X(k)-m_x(k))+V(k))(C(X(k)-m_x(k))+V(k))^T\}$

$\Lambda_{YY}(k,0)=C\Lambda_{XX}(k,0)C^T+\Lambda_{VX}(k,0)C^T+C\Lambda_{XV}(k,0)+\Lambda_{VV}(k,0)$

$X(k)$ doesn't depend on $V(k)$ at all so $\Lambda_{VX}(k,0)=\Lambda_{XV}^T(k,0)=0$

$\Lambda_{YY}(k,0)=C\Lambda_{XX}(k,0)C^T+\Lambda_{VV}(k,0)=C\Lambda_{XX}(k,0)C^T+\Sigma_{vv}\delta(0)$





3.c) (plot above)

$\Lambda_{XX}(k,5)=E\{(X(k+5)-m_x(k+5))(X(k)-m_x(k))^T\}$

$X(k+1)-m_x(k+1)=A(X(k)-m_x(k))+B(W(k)-m_w)$

$$X(k+j)-m_x(k+j)=A^j(X(k)-m_x(k))+\sum_{i=0}^{j-1}A^{j-1-i}B(W(k+i)-m_w)$$

$$\Lambda_{XX}(k,j)=A^j\Lambda_{XX}(k,0)+\sum_{i=0}^{j-1}A^{j-1-i}B\Lambda_{WX}(k,i)$$

$X(k)$ doesn't depend on $w(k+i)$ for $i\geq 0$ so $\Lambda_{WX}(k,i)=\Lambda_{XW}^T(k,-i)=0$ for $i\geq 0$

$\Lambda_{XX}(k,j)=A^j\Lambda_{XX}(k,0)$ for $j\geq 0$

$\Lambda_{XX}(k,-j)=\Lambda_{XX}^T(k,j)$ so in general $\Lambda_{XX}(k,j)=\begin{cases} A^j\Lambda_{XX}(k,0) & \text{for } j\geq 0 \\ \Lambda_{XX}(k,0)(A^{-j})^T & \text{for } j\leq 0\end{cases}$

$\Lambda_{YY}(k,j)=E\{(Y(k+j)-m_y(k+j))(Y(k)-m_y(k))^T\}$

$\Lambda_{YY}(k,j)=E\{(C(X(k+j)-m_x(k+j))+V(k+j))(C(X(k)-m_x(k))+V(k))^T\}$

$\Lambda_{YY}(k,j)=C\Lambda_{XX}(k,j)C^T+\Lambda_{VX}(k,j)C^T+C\Lambda_{XV}(k,j)+\Lambda_{VV}(k,j)$

$X(k\pm j)$ doesn't depend on $V(k)$ at all so $\Lambda_{VX}(k,j)=\Lambda_{XV}^T(k,-j)=0$ for all j

$\Lambda_{YY}(k,j)=C\Lambda_{XX}(k,j)C^T+\Lambda_{VV}(k,j)=C\Lambda_{XX}(k,j)C^T+\Sigma_{vv}\delta(j)$

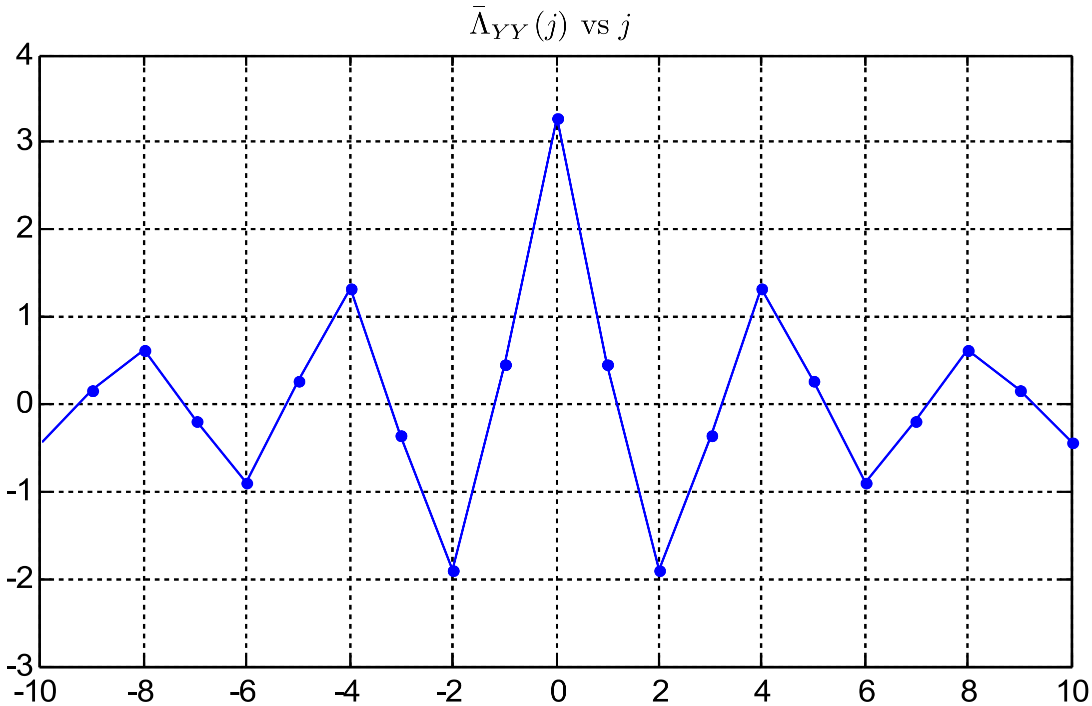
$\Lambda_{YY}(k,j)=\begin{cases} C A^j\Lambda_{XX}(k,0)C^T+\Sigma_{vv}\delta(j) & \text{for } j\geq 0 \\ C\Lambda_{XX}(k,0)(A^{-j})^T C^T+\Sigma_{vv}\delta(j) & \text{for } j\leq 0\end{cases}$

3.d)

At steady state $\bar{\Lambda}_{XX}(0)=A\bar{\Lambda}_{XX}(0)A^T+B\Sigma_{ww}\delta(0)B^T$

from Matlab, $\bar{\Lambda}_{XX}(0)=\text{dlyap}(A,\ B\Sigma_{ww}B^T)=\begin{bmatrix}0.4308 & 0.0276\\ 0.0276 & 0.308\end{bmatrix}$

$\bar{\Lambda}_{YY}(j)=\begin{cases} C A^j\bar{\Lambda}_{XX}(0)C^T+\Sigma_{vv}\delta(j) & \text{for } j\geq 0 \\ C\bar{\Lambda}_{XX}(0)(A^{-j})^T C^T+\Sigma_{vv}\delta(j) & \text{for } j\leq 0\end{cases}$



3.e)

$G(z)=C(zI-A)^{-1}B$ from $W(z)$ to $Y(z)$

Let $U(k)=\begin{bmatrix}W(k)\\ V(k)\end{bmatrix}$, transfer function $H(z)=[G(z)\quad 1]$ from $U(z)$ to $Y(z)$

$$\Lambda_{UU}(j)=\begin{bmatrix}\Sigma_{ww} & 0\\ 0 & \Sigma_{vv}\end{bmatrix}\delta(j),\text{ so }\Phi_{UU}(\omega)=\begin{bmatrix}\Sigma_{ww} & 0\\ 0 & \Sigma_{vv}\end{bmatrix}$$

$$\Phi_{YY}(\omega)=H(\omega)\Phi_{UU}(\omega)H^T(-\omega)=[G(\omega)\quad 1]\begin{bmatrix}\Sigma_{ww} & 0\\ 0 & \Sigma_{vv}\end{bmatrix}\begin{bmatrix}G^T(-\omega)\\ 1\end{bmatrix}=[G(\omega)\quad 1]\begin{bmatrix}\Sigma_{ww}G^T(-\omega)\\ \Sigma_{vv}\end{bmatrix}$$

$$\Phi_{YY}(\omega)=G(\omega)\Sigma_{ww}G^T(-\omega)+\Sigma_{vv}$$

3.f)

$$G(z)=C(zI-A)^{-1}B=[0\quad 3]\begin{bmatrix}z+0.08 & 1\\ -0.7 & z-0.1\end{bmatrix}^{-1}\begin{bmatrix}0.34\\ 0.3\end{bmatrix}$$

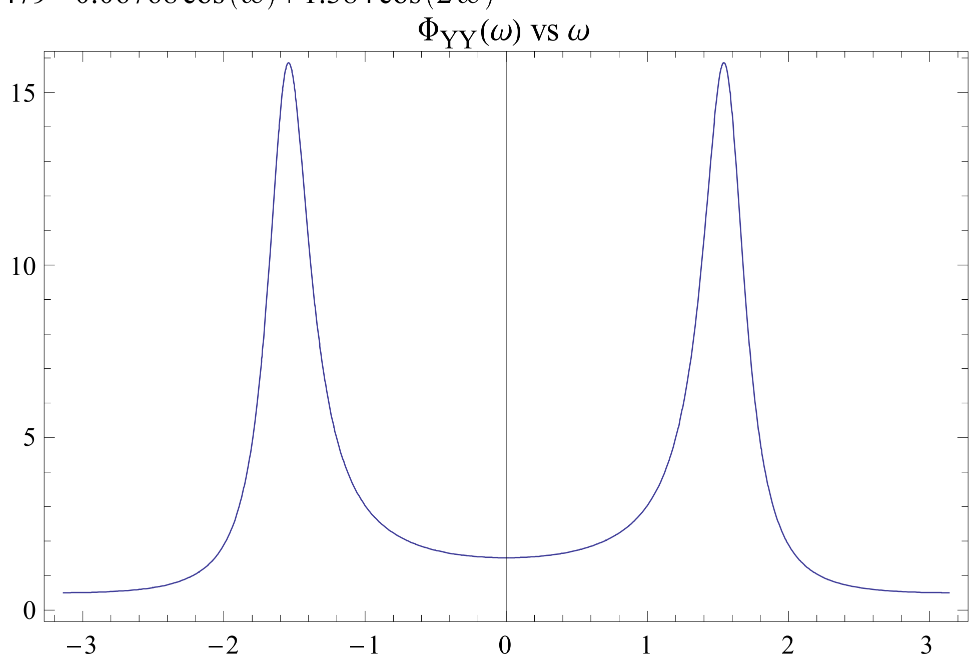
$$G(z)=\frac{1}{z^2-0.02z+0.692}[0\quad 3]\begin{bmatrix}z-0.1 & -1\\ 0.7 & z+0.08\end{bmatrix}\begin{bmatrix}0.34\\ 0.3\end{bmatrix}=\frac{1}{z^2-0.02z+0.692}[0\quad 3]\begin{bmatrix}0.34z-0.334\\ 0.3z+0.262\end{bmatrix}$$

$$G(z)=\frac{0.9z+0.786}{z^2-0.02z+0.692},\ G(\omega)=\frac{0.9e^{j\omega}+0.786}{e^{2j\omega}-0.02e^{j\omega}+0.692}$$

$$\Phi_{YY}(\omega)=\frac{0.9e^{j\omega}+0.786}{e^{2j\omega}-0.02e^{j\omega}+0.692}\cdot\frac{0.9e^{-j\omega}+0.786}{e^{-2j\omega}-0.02e^{-j\omega}+0.692}+0.5$$

$$\Phi_{YY}(\omega)=\frac{1.428+0.7074e^{j\omega}+0.7074e^{-j\omega}}{1.479-0.03384e^{j\omega}-0.03384e^{-j\omega}+0.692e^{2j\omega}+0.692e^{-2j\omega}}+0.5$$

$$\Phi_{YY}(\omega)=\frac{1.428+1.4148\cos(\omega)}{1.479-0.06768\cos(\omega)+1.384\cos(2\omega)}+0.5$$



4.

Stable min phase SISO linear system, SSS zero-mean noise $W(t)$ with $E\{W(t)W(t+\tau)\}=\delta(\tau)$

$$\Phi_{YY}(\omega)=\frac{0.25\omega^2+1}{\omega^4+5\omega^2+1}=G(\omega)\Phi_{WW}(\omega)G^T(-\omega)=G(\omega)G^T(-\omega),\text{ since }\Phi_{WW}(\omega)=1$$

Assume $G(s)=Y(s)/W(s)$ is of the form $G(s)=\dfrac{b_1s+b_2}{s^2+a_1s+a_2}$

$$\frac{0.25\omega^2+1}{\omega^4+5\omega^2+1}=\frac{b_1j\omega+b_2}{-\omega^2+a_1j\omega+a_2}\cdot\frac{-b_1j\omega+b_2}{-\omega^2-a_1j\omega+a_2}=\frac{b_1^2\omega^2+b_2^2}{\omega^4+(a_1^2-2a_2)\omega^2+a_2^2}$$

Matching coefficients, $0.25=b_1^2$, $1=b_2^2$, $5=a_1^2-2a_2$, $1=a_2^2$

$b_1=\pm 0.5$, $b_2=\pm 1$, $a_1=\pm\sqrt{5+2a_2}$, $a_2=\pm 1$

$G(s)$ of this form will have a zero at $s=-b_2/b_1$ and poles at $s=\left(-a_1\pm\sqrt{a_1^2-4a_2}\right)/2$

$G(s)$ stable and minimum phase means all poles and zeros must be in the left half plane.

So b_2 and b_1 must have the same sign, $a_1>0$ and $\sqrt{a_1^2-4a_2}<a_1$, so $a_2>0$

$b_1=\pm 0.5$, $b_2=2b_1$, $a_1=\sqrt{7}$, $a_2=1$

$$G(s)=\pm\frac{0.5s+1}{s^2+\sqrt{7}s+1}$$

5.a)

$$\frac{d}{dt}X(t)=aX(t)+b_w W(t),\ Y(t)=X(t)+V(t),\ E[X(0)]=x_0$$

$W(t)$ and $V(t)$ stationary and independent zero-mean white processes, $E[W(t)]=E[V(t)]=0$

$$E[W(t)W(t+\tau)]=\sigma_W^2\delta(\tau),\ E[V(t)V(t+\tau)]=\sigma_V^2\delta(\tau),\ E[(X(0)-x_0)^2]=\sigma_X^2$$

$$E[W(t)V(t+\tau)]=E[W(t)(X(0)-x_0)]=E[V(t)(X(0)-x_0)]=0$$

$$\frac{d}{dt}\hat{X}(t)=a\hat{X}(t)+L[Y(t)-\hat{X}(t)],\ E[\hat{X}(0)]=0,\ \tilde{X}(t)=X(t)-\hat{X}(t)$$

$$\frac{d}{dt}\tilde{X}(t)=\frac{d}{dt}X(t)-\frac{d}{dt}\hat{X}(t)=aX(t)+b_w W(t)-a\hat{X}(t)-L[Y(t)-\hat{X}(t)]$$

$$\frac{d}{dt}\tilde{X}(t)=aX(t)+b_w W(t)-a\hat{X}(t)-L[X(t)+V(t)-\hat{X}(t)]$$

$$\frac{d}{dt}\tilde{X}(t)=(a-L)(X(t)-\hat{X}(t))+b_w W(t)-LV(t)=(a-L)\tilde{X}(t)+b_w W(t)-LV(t)$$

5.b)

$$\sigma_{\tilde{X}}^2=E[\tilde{X}(t)^2]\text{ so }\frac{d}{dt}\sigma_{\tilde{X}}^2=E[\frac{d}{dt}\tilde{X}(t)^2]=2E[\tilde{X}(t)\frac{d}{dt}\tilde{X}(t)]$$

$$\frac{d}{dt}\sigma_{\tilde{X}}^2=2E[(a-L)\tilde{X}(t)^2+b_w\tilde{X}(t)W(t)-L\tilde{X}(t)V(t)]$$

$$\frac{d}{dt}\sigma_{\tilde{X}}^2=2(a-L)\sigma_{\tilde{X}}^2+E[\tilde{X}(t)(b_w W(t)-LV(t))]$$

$$\tilde{X}(t)=e^{(a-L)t}\tilde{X}(0)+\int_0^t e^{(a-L)(t-\tau)}(b_w W(\tau)-LV(\tau))d\tau$$

$$\frac{d}{dt}\sigma_{\tilde{X}}^2=2(a-L)\sigma_{\tilde{X}}^2+E[(e^{(a-L)t}\tilde{X}(0)+\int_0^t e^{(a-L)(t-\tau)}(b_w W(\tau)-LV(\tau))d\tau)(b_w W(t)-LV(t))]$$

$$\frac{d}{dt}\sigma_{\tilde{X}}^2=2(a-L)\sigma_{\tilde{X}}^2+e^{(a-L)t}(b_w E[\tilde{X}(0)W(t)]-LE[\tilde{X}(0)V(t)])$$

$$+\int_0^t e^{(a-L)(t-\tau)}E[(b_w W(\tau)-LV(\tau))(b_w W(t)-LV(t))]d\tau$$

Assuming that $E[W(t)\hat{X}(0)]=E[V(t)\hat{X}(0)]=0$,

$$E[W(t)\tilde{X}(0)]=E[W(t)(X(0)-\hat{X}(0))]=E[W(t)X(0)]-E[W(t)\hat{X}(0)]$$

$$E[W(t)\tilde{X}(0)]=E[W(t)(X(0)-x_0)+W(t)x_0]-E[W(t)\hat{X}(0)]$$

$$E[\tilde{X}(0)W(t)]=E[W(t)(X(0)-x_0)]+x_0E[W(t)]-0=0$$

Likewise $E[\tilde{X}(0)V(t)]=E[V(t)(X(0)-x_0)]+x_0E[V(t)]-E[V(t)\hat{X}(0)]=0$

$$\frac{d}{dt}\sigma_{\tilde{X}}^2=2(a-L)\sigma_{\tilde{X}}^2+\int_0^t e^{(a-L)(t-\tau)}(b_w^2E[W(\tau)^2]-2b_wLE[W(\tau)V(\tau)]+L^2E[V(t)^2])d\tau$$

$$\frac{d}{dt}\sigma_{\tilde{X}}^2=2(a-L)\sigma_{\tilde{X}}^2+\int_0^t e^{(a-L)(t-\tau)}(b_w^2\sigma_W^2\delta(0)-2b_w L\cdot 0+L^2\sigma_V^2\delta(0))d\tau$$

$$\frac{d}{dt}\sigma_{\tilde{X}}^2=2(a-L)\sigma_{\tilde{X}}^2+(b_w^2\sigma_W^2+L^2\sigma_V^2)\int_0^t e^{(a-L)(t-\tau)}d\tau=2(a-L)\sigma_{\tilde{X}}^2+(b_w^2\sigma_W^2+L^2\sigma_V^2)\frac{e^{(a-L)t}-1}{a-L}$$

At steady-state $\frac{d}{dt}\bar{\sigma}_{\tilde{X}}^2=0$ and as $t\to\infty$, assuming $(a-L)<0$, $0=2(a-L)\bar{\sigma}_{\tilde{X}}^2-\frac{b_w^2\sigma_W^2+L^2\sigma_V^2}{a-L}$

$$\bar{\sigma}_{\tilde{X}}^2=\frac{b_w^2\sigma_W^2+L^2\sigma_V^2}{2(a-L)^2}$$

5.c)

$$\frac{\partial\bar{\sigma}_{\tilde{X}}^2}{\partial L}=\frac{L\sigma_V^2}{(a-L)^2}+\frac{b_w^2\sigma_W^2+L^2\sigma_V^2}{(a-L)^3}=\frac{aL\sigma_V^2+b_w^2\sigma_W^2}{(a-L)^3}$$

$\frac{\partial\bar{\sigma}_{\tilde{X}}^2}{\partial L}=0$ for $L=\frac{-b_w^2\sigma_W^2}{a\sigma_V^2}$

With that value of L, $\bar{\sigma}_{\tilde{X}}^2=\frac{a^2\sigma_V^4 b_w^2\sigma_W^2+b_w^4\sigma_W^4\sigma_V^2}{2(a^2\sigma_V^2+b_w^2\sigma_W^2)^2}=\frac{(a^2\sigma_V^2+b_w^2\sigma_W^2)\sigma_V^2 b_w^2\sigma_W^2}{2(a^2\sigma_V^2+b_w^2\sigma_W^2)^2}=\frac{\sigma_V^2 b_w^2\sigma_W^2}{2(a^2\sigma_V^2+b_w^2\sigma_W^2)}$

6.a)

$$E\{Y(k)\}=0,\ \Lambda_{YY}(j)=E\{Y(k+j)Y(k)\}=\sigma_j$$

$$\hat{y}(k)\,|_{k-1,\cdots,k-n}=E\{Y(k)|y(k-1),\cdots,y(k-n)\}=\sum_{i=1}^n a_i y(k-i)$$

$$E\{(Y(k)-\sum_{i=1}^n a_i Y(k-i))^2\}=E\left\{Y(k)^2-\sum_{i=1}^n 2a_i Y(k)Y(k-i)+\sum_{i=1}^n\left(a_i Y(k-i)\sum_{j=1}^n a_j Y(k-j)\right)\right\}$$

$$E\{(Y(k)-\sum_{i=1}^n a_i Y(k-i))^2\}=\sigma_0-\sum_{i=1}^n 2a_i\sigma_i+\sum_{i=1}^n\sum_{j=1}^n a_i a_j\sigma_{j-i}$$

$$\frac{\partial}{\partial a_i}E\{(Y(k)-\sum_{i=1}^n a_i Y(k-i))^2\}=-2\sigma_i+2a_i\sigma_0+\sum_{j\neq i}2a_j\sigma_{j-i}=-2\sigma_i+\sum_{j=1}^n 2a_j\sigma_{j-i}$$

Setting this derivative to zero we have $\sum_{j=1}^n a_j\sigma_{j-i}=\sigma_i$

Expressing in matrix form, $[\sigma_{1-i}\quad\sigma_{2-i}\quad\cdots\quad\sigma_{n-i}]\begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix}=\sigma_i$

$\sigma_k=\sigma_{-k}$ so combining for all i, $\begin{bmatrix}\sigma_0&\sigma_1&\cdots&\sigma_{n-1}\\\sigma_1&\sigma_0&\cdots&\sigma_{n-2}\\\vdots&\vdots&\ddots&\vdots\\\sigma_{n-1}&\sigma_{n-2}&\cdots&\sigma_0\end{bmatrix}\begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix}=\begin{bmatrix}\sigma_1\\\sigma_2\\\vdots\\\sigma_n\end{bmatrix}$

6.b)

$$\tilde{Y}(k)=Y(k)-\hat{y}(k)\,|_{k-1,\cdots,k-n}=Y(k)-\sum_{i=1}^n a_i y(k-i),\ \sigma_{\tilde{Y}}=E\{\tilde{Y}^2(k)\}$$

$$\sigma_{\tilde{Y}}=E\{\tilde{Y}^2(k)\}=E\{(Y(k)-\sum_{i=1}^n a_i Y(k-i))^2\}=\sigma_0-\sum_{i=1}^n 2a_i\sigma_i+\sum_{i=1}^n\sum_{j=1}^n a_i a_j\sigma_{j-i}$$

$$\sigma_{\tilde{Y}}=\sigma_0-\sum_{i=1}^{n}2a_i\sigma_i+\sum_{i=1}^{n}a_i\left(\sum_{j=1}^{n}a_j\sigma_{j-i}\right)=\sigma_0-\sum_{i=1}^{n}2a_i\sigma_i+\sum_{i=1}^{n}a_i\sigma_i=\sigma_0-\sum_{i=1}^{n}a_i\sigma_i$$

6.c)

$Y(z)=\dfrac{z+0.2}{(z+0.4)(z+0.8)}W(z),\ w(k)$ zero mean unit variance white Gaussian so $\Sigma_{ww}=1$

State-space realization $X(k+1)=\begin{bmatrix}-0.4 & 0.4472\\ 0 & -0.8\end{bmatrix}X(k)+\begin{bmatrix}0\\1\end{bmatrix}W(k),\ Y(k)=\begin{bmatrix}-0.4472 & 1\end{bmatrix}X(k)$

$$\sigma_0=C\,\mathrm{dlyap}(A,BB^T)C^T=4.3417$$

$$\sigma_1=CA\,\mathrm{dlyap}(A,BB^T)C^T=-3.7955$$

$$\sigma_2=CA^2\,\mathrm{dlyap}(A,BB^T)C^T=3.1653$$

$$\begin{bmatrix}a_1\\a_2\end{bmatrix}=\begin{bmatrix}\sigma_0 & \sigma_1\\ \sigma_1 & \sigma_0\end{bmatrix}^{-1}\begin{bmatrix}\sigma_1\\ \sigma_2\end{bmatrix}=\begin{bmatrix}-1.0046\\-0.1492\end{bmatrix}$$

$$\tilde{y}(k)=y(k)-\sum_{i=1}^{n}a_i y(k-i)=y(k)+1.0046\,y(k-1)+0.1492\,y(k-2)$$

$$H(q^{-1})=1+1.0046\,q^{-1}+0.1492\,q^{-2}$$

$$\sigma_{\tilde{y}}=\sigma_0-\sum_{i=1}^{n}a_i\sigma_i=1.0009$$

6.d)

Simulation results for $N=5000,\ M=1000:\sigma_0\approx 4.688,\ \sigma_{\tilde{y}}\approx 1.0119$

```
sys = ss(zpk(-0.2,[-0.4 -0.8],1,-1))
s0 = sys.C*dlyap(sys.A,sys.B*(sys.B'))*(sys.C)'
s1 = sys.C*sys.A*dlyap(sys.A,sys.B*(sys.B'))*(sys.C)'
s2 = sys.C*sys.A^2*dlyap(sys.A,sys.B*(sys.B'))*(sys.C)'
a = inv([s0 s1; s1 s0])*[s1;s2]
syt = s0 - a(1)*s1 - a(2)*s2
N = 5000;
w = randn(N,1);
y = lsim(sys,w,0:N-1);
for i=1:N
    yt(i) = y(i)-a(1)*y(max(i-1,1))-a(2)*y(max(i-2,1));
end
M = 1000;
s0sim = sum(y(M:N).^2)/(N-M)
sytsim = sum(yt(M:N).^2)/(N-M)
```