UNIVERSITY OF CALIFORNIA AT BERKELEY
Department of Mechanical Engineering
ME233 Advanced Control Systems II
Spring 2012

Midterm Examination I

Your Name:

Closed book and closed notes.
Two double-sided sheets (i.e. 4 pages) of handwritten notes on 8.5" × 11" paper are allowed.
Please answer all questions.

Problem:	1	2	3	Total
Max. Grade:	30	40	30	100
Grade:				

Problem 1

Consider the state space system

$$\begin{aligned} X(k+1) &= AX(k) + BW(k) \\ Y(k) &= CX(k) + V(k) \end{aligned}$$

where $X(0), W(0), W(1), \dots, V(0), V(1), \dots$ are independent Gaussian random vectors and

$$\begin{aligned} E\{X(0)\} &= x_0 & E\{(X(0)-x_0)(X(0)-x_0)^T\} &= X_0 \\ E\{W(k)\} &= 0 & E\{W(k+j)W^T(k)\} &= \Sigma_W \delta(j) \\ E\{V(k)\} &= 0 & E\{V(k+j)V^T(k)\} &= \Sigma_V \delta(j) \end{aligned}$$

1. Find $\Lambda_{XY}(k,j)$ and $\Lambda_{YY}(k,j)$ in terms of $\Lambda_{XX}(k,j)$.
2. Find the least squares estimator of $X(0)$ given $Y(0)$ and $Y(1)$.

Problem 2

Consider the discrete-time linear time-invariant system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

where D is square, i.e. the system has the same number of inputs as outputs. In this problem, we are interested in the smallest possible values of the cost functions

$$J_m[N] = \sum_{k=m}^{N-1} 2u^T(k)y(k)$$

over all possible choices of $u(m), \ldots, u(N-1)$ for $m = 0, 1, \ldots, N-1$. Thus, we are interested in solving the optimization problems

$$J_m^o[x_m, N] = \min_{u(m),\ldots,u(N-1)} J_m[N] \quad \text{s.t.} \quad x(m) = x_m$$

for $m = 0, 1, \ldots, N-1$.

Let the sequence of matrices $P_0, P_1, P_2, \ldots$ satisfy the discrete Riccati difference equation

$$P_{k+1} = A^T P_k A - (A^T P_k B + C^T)(B^T P_k B + D + D^T)^{-1}(B^T P_k A + C) \ .$$

with the initial condition $P_0 = 0$. Also assume that this sequence of matrices satisfies the conditions

- $B^T P_k B + D + D^T \succ 0, \quad k = 0, 1, 2, \ldots$
- $\lim_{k\to\infty} P_k = P_\infty$

1. Use dynamic programming to prove that $J_m^o[x_m, N] = x_m^T P_{(N-m)}\, x_m$. You may find it convenient to define $J_N^o[x_N, N] = x_N^T P_0\, x_N = 0$ to facilitate the proof.
2. Prove that $J_0^o[x_0, N] \leq 0$ for all x_0, N.
3. Suppose that $\|x(0)\| \leq \alpha$. Find a value of γ such that

   $$\sum_{k=0}^{\infty} 2u^T(k)y(k) \geq \gamma$$

   regardless of how $u(0), u(1), u(2), \ldots$ are chosen.

Problem 3

In this problem, we consider the discrete-time linear time-invariant system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \tag{1}$$

where $u(k)$ is the control input and $y(k)$ is the output of the system.

1. Find the optimal control policy $u_1^o(k)$ that minimizes the cost function

$$J = y_f^T(N)\, y_f(N) + \sum_{k=0}^{N-1} \left[y_f^T(k)\, y_f(k) + u^T(k) R\, u(k) \right] \tag{2}$$

   for the system (1) where $R \succ 0$, $y_f(k)$ is defined by

$$\begin{aligned} x_f(k+1) &= A_f\, x_f(k) + B_f\, y(k) \\ y_f(k) &= C_f\, x_f(k) + D_f\, y(k) \end{aligned}$$

   and $x_f(0) = 0$.

   Also find the corresponding value of the optimal cost.

2. Suppose we now make the restriction for $k = 0, \ldots, N-1$ that $u(k)$ can only be an explicit function of $x(0), \ldots, x(k)$. Under this restriction, find the optimal control policy $u_2^o(k)$ that minimizes the cost function (2) for the system (1).

   Also find the corresponding value of the optimal cost.

   Hint: The optimal control policy can be written as the output of a state space system that has $x(k)$ as its input.