

ME 233 – Advanced Control II

Lecture 25

Stability Analysis of a
Direct Adaptive Control System

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Outline

Review of direct adaptive control

Stability theorem

Stability theorem proof

Part 1

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Outline

Review of direct adaptive control

Stability theorem

Stability theorem proof

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Deterministic SISO ARMA model

SISO ARMA plant model:

$$A(q^{-1})y(k) = q^{-d}B(q^{-1})u(k)$$

where $y(k)$ and $u(k)$ are scalar

- ▶ $u(k)$ is the control input
- ▶ $y(k)$ is the output
- ▶ d is the pure time delay
- ▶ no disturbance

Model assumptions

SISO ARMA plant model:

$$A(q^{-1})y(k) = q^{-d}B(q^{-1})u(k)$$

where $y(k)$ and $u(k)$ are scalar

- ▶ The polynomials

$$A(q^{-1}) = 1 + a_1q^{-1} + \cdots + a_nq^{-n}$$

$$B(q^{-1}) = b_0 + b_1q^{-1} + \cdots + b_mq^{-m}$$

are co-prime

- ▶ $B(q^{-1})$ is anti-Schur
- ▶ m , n , and d are known
- ▶ $0 < b_{\min o} \leq b_0$, where $b_{\min o}$ is known

Control Objectives

1. **Pole Placement:** The poles of the closed-loop system must be placed at specific locations in the complex plane

Closed-loop polynomial:

$$A_c(q^{-1}) = B(q^{-1})A'_c(q^{-1})$$

where $A'_c(q^{-1})$ is an anti-Schur polynomial chosen by the designer:

$$A'_c(q^{-1}) = 1 + a'_{c1}q^{-1} + \cdots + a'_{c(n'_c)}q^{-(n'_c)}$$

Control Objectives

2. **Tracking:** The output sequence $y(k)$ must follow an arbitrary bounded reference sequence $y_d(k)$, which is known

$y_d(k)$ is generated by the reference model

$$A'_c(q^{-1})y_d(k) = q^{-d}B_m(q^{-1})u_d(k)$$

where

- ▶ $u_d(k)$ is a known bounded reference input control input sequence
- ▶ $B_m(q^{-1})$ is chosen by the designer

Note that $A'_c(q^{-1})$ comes from the pole placement and the reference model delay is the same as the plant delay

Reformulated plant dynamics

Using the solution of the Diophantine equation

$$A'_c(q^{-1}) = A(q^{-1})R'(q^{-1}) + q^{-d}S(q^{-1})$$

we rewrite the plant dynamics as

$$A'_c(q^{-1})y(k) = q^{-d} \left[R(q^{-1})u(k) + S(q^{-1})y(k) \right]$$

where $R(q^{-1}) = R'(q^{-1})B(q^{-1})$ and

$$R(q^{-1}) = r_0 + r_1q^{-1} + \cdots + r_{n_r}q^{-n_r}$$

$$S(q^{-1}) = s_0 + s_1q^{-1} + \cdots + s_{n_s}q^{-n_s}$$

$$n_r = m + d - 1 \qquad n_s = \max\{n - 1, n'_c - d\}$$

Reformulated plant dynamics

So far, we know that

$$A'_c(q^{-1})y(k) = q^{-d} \left[R(q^{-1})u(k) + S(q^{-1})y(k) \right]$$

$$R(q^{-1}) = r_0 + r_1 q^{-1} + \dots + r_{n_r} q^{-n_r}$$

$$S(q^{-1}) = s_0 + s_1 q^{-1} + \dots + s_{n_s} q^{-n_s}$$

Defining $\eta(k) = A'_c(q^{-1})y(k)$ and

$$\phi(k) = \begin{bmatrix} y(k) & \dots & y(k - n_s) & u(k) & \dots & u(k - n_r) \end{bmatrix}^T$$

$$\theta_c = \begin{bmatrix} s_0 & \dots & s_{n_s} & r_0 & \dots & r_{n_r} \end{bmatrix}^T$$

we rewrite the plant dynamics as

$$\boxed{\eta(k) = \phi^T(k - d)\theta_c}$$

Direct adaptive control approach

The plant dynamics are written as

$$\eta(k) = \phi^T(k - d)\theta_c$$

- ▶ $\eta(k)$ is the known “filtered output”
- ▶ $\phi(k)$ is the known regressor vector
- ▶ θ_c is the unknown parameter vector
⇒ we use RLS to estimate θ_c

Tracking control objective

We would like to achieve

$$\lim_{k \rightarrow \infty} \{y(k) - y_d(k)\} = 0$$

Since $A'_c(q^{-1})$ is anti-Schur this is equivalent to

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} \{A'_c(q^{-1})[y(k) - y_d(k)]\} \\ &= \lim_{k \rightarrow \infty} \{\eta(k) - \eta_d(k)\} \end{aligned}$$

where $\eta_d(k) = A'_c(q^{-1})y_d(k) = q^{-d}B_m(q^{-1})u_d(k) = r(k - d)$.

List of error signals

Parameter estimation error:

$$\tilde{\theta}_c(k) = \theta_c - \hat{\theta}_c(k)$$

Filtered output estimation errors:

$$\begin{aligned} e^o(k) &= \eta(k) - \phi^T(k-d)\hat{\theta}_c(k-1) && \text{a-priori} \\ &= \phi^T(k-d)\tilde{\theta}_c(k-1) \end{aligned}$$

$$\begin{aligned} e(k) &= \eta(k) - \phi^T(k-d)\hat{\theta}_c(k) && \text{a-posteriori} \\ &= \phi^T(k-d)\tilde{\theta}_c(k) \end{aligned}$$

Filtered output tracking error:

$$\epsilon(k) = \eta(k) - \eta_d(k)$$

Direct adaptive control

1. $\eta(k+1) = A'_c(q^{-1})y(k+1)$
2. $\phi(k-d+1) = \begin{bmatrix} y(k-d+1) \\ \vdots \\ y(k-d+1-n_s) \\ u(k-d+1) \\ \vdots \\ u(k-d+1-n_r) \end{bmatrix}$
3. $e^o(k+1) = \eta(k+1) - \phi^T(k-d+1)\hat{\theta}_c(k)$
4. $e(k+1) = \frac{\lambda_1(k)}{\lambda_1(k) + \phi^T(k-d+1)F(k)\phi(k-d+1)}e^o(k+1)$
5. $\hat{\theta}_c^o(k+1) = \hat{\theta}_c(k) + \frac{1}{\lambda_1(k)}F(k)\phi(k-d+1)e(k+1)$

Direct adaptive control

6. Form $\hat{\theta}_c(k+1)$:

$$\hat{s}_i(k+1) = \hat{s}_i^o(k+1), \quad i = 0, \dots, n_s$$

$$\hat{r}_i(k+1) = \hat{r}_i^o(k+1), \quad i = 1, \dots, n_r$$

$$\hat{r}_0(k+1) = \max\{b_{\min o}, \hat{r}_0^o(k+1)\} \quad \text{parameter projection}$$

$$7. F(k+1) = \frac{1}{\lambda_1(k)} \left[F(k) - \lambda_2(k) \frac{F(k)\phi(k-d+1)\phi^T(k-d+1)F(k)}{\lambda_1(k) + \lambda_2(k)\phi^T(k-d+1)F(k)\phi(k-d+1)} \right]$$

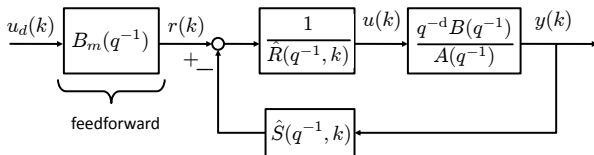
where $\lambda_1(k)$ and $\lambda_2(k)$ are chosen so that

$$0 < \underline{\lambda}_1 \leq \lambda_1(k) \leq 1 \quad 0 \leq \lambda_2(k) \leq \bar{\lambda}_2 < 2$$

$$\text{and } 0 < K_{\min} \leq \lambda_{\min}(F(k)) \leq \lambda_{\max}(F(k)) \leq K_{\max} < \infty$$

Direct adaptive control

8. Apply control



$$\hat{R}(q^{-1}, k)u(k) = B_m(q^{-1})u_d(k) - \hat{S}(q^{-1}, k)y(k)$$

where

$$\hat{R}(q^{-1}, k) = \hat{r}_0(k) + \hat{r}_1(k)q^{-1} + \cdots + \hat{r}_{n_r}(k)q^{-n_r}$$

$$\hat{S}(q^{-1}, k) = \hat{s}_0(k) + \hat{s}_1(k)q^{-1} + \cdots + \hat{s}_{n_s}(k)q^{-n_s}$$

Outline

Review of direct adaptive control

Stability theorem

Stability theorem proof

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Stability theorem

Using the direct adaptive control approach just outlined, the tracking error converges to zero, i.e.

$$\lim_{k \rightarrow \infty} \epsilon(k) = 0$$

Moreover, $u(k)$ remains bounded, $e(k) \rightarrow 0$, and $e^o(k) \rightarrow 0$.

Note that the theorem does not require:

- ▶ a-priori knowledge that the control input sequence $u(k)$ is bounded
- ▶ the polynomial $A(q^{-1})$ is anti-Schur
- ▶ any sort of persistence of excitation condition

The theorem does not state that the parameter estimates converge to the true values

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Outline of stability theorem proof

1. Use hyperstability theory to show that

$$\lim_{k \rightarrow \infty} e(k) = 0$$

2. Prove the limits

$$\lim_{k \rightarrow \infty} \|\hat{\theta}_c(k) - \hat{\theta}_c(k-1)\| = 0$$

$$\lim_{k \rightarrow \infty} \frac{[\lambda_1(k-1)e^o(k)]^2}{\lambda_1(k-1) + \phi^T(k-d)F(k-1)\phi(k-d)} = 0$$

$$\lim_{k \rightarrow \infty} \frac{[\lambda_1(k-1)\epsilon(k)]^2}{\lambda_1(k-1) + \phi^T(k-d)F(k-1)\phi(k-d)} = 0$$

Outline of stability theorem proof

3. Prove that there exist $C_1 \geq 0$, $C_2 \geq 0$ such that

$$\|\phi(k - d)\| \leq C_1 + C_2 \max_{j \in \{0, \dots, k\}} |\epsilon(j)|$$

4. Prove Goodwin's technical lemma, which states that $\|\phi(k)\|$ remains bounded and

$$\lim_{k \rightarrow \infty} \epsilon(k) = 0$$

5. Prove that

$$\lim_{k \rightarrow \infty} e^o(k) = 0$$

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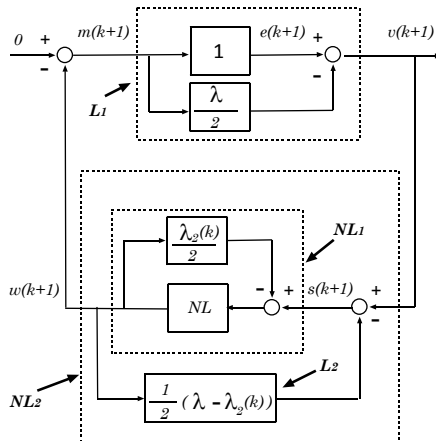
Part 5

Stability theorem proof, part 1

- ▶ We want to show that $e(k) \rightarrow 0$
- ▶ Simplification: neglect parameter projection
- ▶ We will use hyperstability, as in Lecture 21

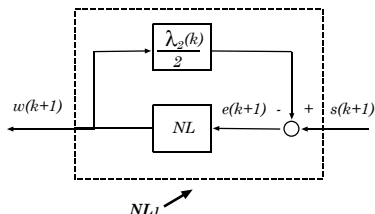
Stability theorem proof, part 1

As in Lecture 21, the estimation error dynamics can be expressed using the block diagram



Stability theorem proof, part 1

We will now show that NL_1 is P-class:



$$w(k) = -\phi^T(k-d)\tilde{\theta}_c(k)$$

Note that $e(k) = s(k) - \frac{\lambda_2(k-1)}{2}w(k)$, which implies that

$$s(k) = \frac{\lambda_2(k-1)}{2}w(k) + e(k)$$

Stability theorem proof, part 1

$$\begin{aligned}2w(k)s(k) &= w(k) [\lambda_2(k-1)w(k) + 2e(k)] \\&= \lambda_2(k-1)\tilde{\theta}_c^T(k)\phi(k-d)\phi^T(k-d)\tilde{\theta}_c(k) \\&\quad - 2\tilde{\theta}_c^T(k)[\phi(k-d)e(k)] \\&= \tilde{\theta}_c^T(k) \left[\lambda_2(k-1)\phi(k-d)\phi^T(k-d) \right] \tilde{\theta}_c(k) \\&\quad - 2\tilde{\theta}_c^T(k) \left[\lambda_1(k)F^{-1}(k-1) \left(\tilde{\theta}_c(k-1) - \tilde{\theta}_c(k) \right) \right]\end{aligned}$$

Define $\Delta\theta_c(k) = \hat{\theta}(k) - \hat{\theta}(k-1) = \tilde{\theta}_c(k-1) - \tilde{\theta}_c(k)$

$$\begin{aligned}2w(k)s(k) &= \tilde{\theta}_c^T(k) \left[F^{-1}(k) - \lambda_1(k)F^{-1}(k-1) \right] \tilde{\theta}_c(k) \\&\quad - 2\lambda_1(k)\tilde{\theta}_c^T(k)F^{-1}(k-1)\Delta\theta_c(k)\end{aligned}$$

Stability theorem proof, part 1

$$2w(k)s(k) = \tilde{\theta}_c^T(k) \left[F^{-1}(k) - \lambda_1(k)F^{-1}(k-1) \right] \tilde{\theta}_c(k) \\ - 2\lambda_1(k)\tilde{\theta}_c^T(k)F^{-1}(k-1)\Delta\theta_c(k)$$

$$2w(k)s(k) = \tilde{\theta}_c^T(k)F^{-1}(k)\tilde{\theta}_c(k) - \lambda_1(k) \left[\tilde{\theta}_c^T(k)F^{-1}(k-1)\tilde{\theta}_c(k) \right. \\ \left. + 2\tilde{\theta}_c^T(k)F^{-1}(k-1)\Delta\theta_c(k) \right] \\ = \tilde{\theta}_c^T(k)F^{-1}(k)\tilde{\theta}_c(k) \\ - \lambda_1(k) \left[\left(\tilde{\theta}_c(k) + \Delta\theta_c(k) \right)^T F^{-1}(k-1) \left(\tilde{\theta}_c(k) + \Delta\theta_c(k) \right) \right. \\ \left. - \Delta\theta_c^T(k)F^{-1}(k-1)\Delta\theta_c(k) \right]$$

Note that $\tilde{\theta}_c(k) + \Delta\theta_c(k) = \tilde{\theta}_c(k-1)$

Stability theorem proof, part 1

From the previous slide,

$$\begin{aligned} 2w(k)s(k) &= \tilde{\theta}_c^T(k)F^{-1}(k)\tilde{\theta}_c(k) - \lambda_1(k)\tilde{\theta}_c^T(k-1)F^{-1}(k-1)\tilde{\theta}_c(k-1) \\ &\quad + \lambda_1(k)\Delta\theta_c^T(k)F^{-1}(k-1)\Delta\theta_c(k) \end{aligned}$$

Since $\lambda_1(k) \leq 1$ and $F(k-1) \succ 0$, this implies that

$$2w(k)s(k) \geq \left[\tilde{\theta}_c^T(k)F^{-1}(k)\tilde{\theta}_c(k) - \tilde{\theta}_c^T(k-1)F^{-1}(k-1)\tilde{\theta}_c(k-1) \right]$$

Stability theorem proof, part 1

$$2w(k)s(k) \geq \left[\tilde{\theta}_c^T(k)F^{-1}(k)\tilde{\theta}_c(k) - \tilde{\theta}_c^T(k-1)F^{-1}(k-1)\tilde{\theta}_c(k-1) \right]$$

Therefore

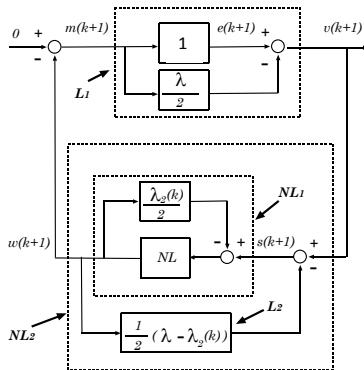
$$\begin{aligned} \Rightarrow \sum_{j=0}^k w(j)s(j) &\geq \frac{1}{2} \sum_{j=0}^k \left[\tilde{\theta}_c^T(j)F^{-1}(j)\tilde{\theta}_c(j) \right. \\ &\quad \left. - \tilde{\theta}_c^T(j-1)F^{-1}(j-1)\tilde{\theta}_c(j-1) \right] \\ &= \frac{1}{2} \left[\tilde{\theta}_c^T(k)F^{-1}(k)\tilde{\theta}_c(k) - \tilde{\theta}_c^T(-1)F^{-1}(-1)\tilde{\theta}_c(-1) \right] \\ &\geq -\frac{1}{2} \tilde{\theta}_c^T(-1)F^{-1}(-1)\tilde{\theta}_c(-1) \end{aligned}$$

Stability theorem proof, part 1

We have shown that NL_1 is
P-class

Using the same arguments as in
Lecture 21 (including the
asymptotic hyperstability
theorem), this yields

$$\lim_{k \rightarrow \infty} e(k) = 0$$



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Stability theorem proof, part 2

We want to prove the limits

$$\lim_{k \rightarrow \infty} \|\hat{\theta}_c(k) - \hat{\theta}_c(k-1)\| = 0$$

$$\lim_{k \rightarrow \infty} \frac{[\lambda_1(k-1)e^o(k)]^2}{\lambda_1(k-1) + \phi^T(k-d)F(k-1)\phi(k-d)} = 0$$

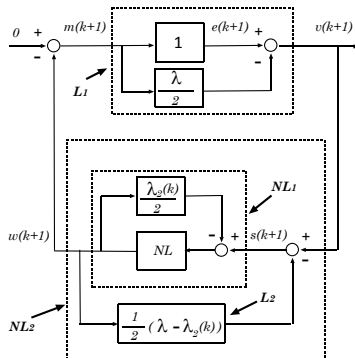
$$\lim_{k \rightarrow \infty} \frac{[\lambda_1(k-1)\epsilon(k)]^2}{\lambda_1(k-1) + \phi^T(k-d)F(k-1)\phi(k-d)} = 0$$

Stability theorem proof, part 2

We know that $1 - \lambda/2$ is SPR,
which implies that it is P-class

This implies that there exists $\bar{\gamma} \in \mathcal{R}$
such that

$$\begin{aligned} -\bar{\gamma}^2 &\leq \sum_{j=0}^k m(j)v(j) \\ &= -\sum_{j=0}^k w(j) \left[s(j) \right. \\ &\quad \left. + \frac{1}{2}(\lambda - \lambda_2(j-1))w(j) \right] \end{aligned}$$



Stability theorem proof, part 2

Because $\lambda - \lambda_2(j-1) \geq 0$, $j = -1, 0, 1, \dots$, we have

$$\begin{aligned} -\bar{\gamma}^2 &\leq -\sum_{j=0}^k w(j) \left[s(j) + \frac{1}{2}(\lambda - \lambda_2(j-1))w(j) \right] \\ &\leq -\sum_{j=0}^k w(j)s(j) \end{aligned}$$

which implies that

$$\sum_{j=0}^k w(j)s(j) \leq \bar{\gamma}^2$$

Stability theorem proof, part 2

From part 1 of the stability theorem proof,

$$\begin{aligned} 2w(k)s(k) &= \tilde{\theta}_c^T(k)F^{-1}(k)\tilde{\theta}_c(k) - \lambda_1(k)\tilde{\theta}_c^T(k-1)F^{-1}(k-1)\tilde{\theta}_c(k-1) \\ &\quad + \lambda_1(k)\Delta\theta_c^T(k)F^{-1}(k-1)\Delta\theta_c(k) \end{aligned}$$

Since $0 < \underline{\lambda}_1 \leq \lambda_1(k) \leq 1$ and $F(k-1) \succ 0$, this implies that

$$\begin{aligned} 2w(k)s(k) &\geq \left[\tilde{\theta}_c^T(k)F^{-1}(k)\tilde{\theta}_c(k) - \tilde{\theta}_c^T(k-1)F^{-1}(k-1)\tilde{\theta}_c(k-1) \right] \\ &\quad + \underline{\lambda}_1\Delta\theta_c^T(k)F^{-1}(k-1)\Delta\theta_c(k) \end{aligned}$$

Stability theorem proof, part 2

$$2w(k)s(k) \geq \left[\tilde{\theta}_c^T(k)F^{-1}(k)\tilde{\theta}_c(k) - \tilde{\theta}_c^T(k-1)F^{-1}(k-1)\tilde{\theta}_c(k-1) \right] \\ + \underline{\lambda}_1 \Delta \theta_c^T(k)F^{-1}(k-1)\Delta \theta_c(k)$$

which implies that

$$2\bar{\gamma}^2 \geq 2 \sum_{j=0}^k w(j)s(j) \\ \geq \sum_{j=0}^k \left[\tilde{\theta}_c^T(j)F^{-1}(j)\tilde{\theta}_c(j) - \tilde{\theta}_c^T(j-1)F^{-1}(j-1)\tilde{\theta}_c(j-1) \right] \\ + \sum_{j=0}^k \underline{\lambda}_1 \Delta \theta_c^T(j)F^{-1}(j-1)\Delta \theta_c(j)$$

Stability theorem proof, part 2

$$\begin{aligned} 2\bar{\gamma}^2 &\geq \sum_{j=0}^k \left[\tilde{\theta}_c^T(j) F^{-1}(j) \tilde{\theta}_c(j) - \tilde{\theta}_c^T(j-1) F^{-1}(j-1) \tilde{\theta}_c(j-1) \right] \\ &\quad + \sum_{j=0}^k \underline{\lambda}_1 \Delta \theta_c^T(j) F^{-1}(j-1) \Delta \theta_c(j) \end{aligned}$$

$$\begin{aligned} 2\bar{\gamma}^2 &= \tilde{\theta}_c^T(k) F^{-1}(k) \tilde{\theta}_c(k) - \tilde{\theta}_c^T(-1) F^{-1}(-1) \tilde{\theta}_c(-1) \\ &\quad + \underline{\lambda}_1 \sum_{j=0}^k \Delta \theta_c^T(j) F^{-1}(j-1) \Delta \theta_c(j) \\ &\geq -\tilde{\theta}_c^T(-1) F^{-1}(-1) \tilde{\theta}_c(-1) + \underline{\lambda}_1 \sum_{j=0}^k \Delta \theta_c^T(j) F^{-1}(j-1) \Delta \theta_c(j) \end{aligned}$$

Stability theorem proof, part 2

Thus, we know that

$$\sum_{j=0}^k \Delta\theta_c^T(j) F^{-1}(j-1) \Delta\theta_c(j) \leq \frac{1}{\underline{\lambda}_1} \left[2\bar{\gamma}^2 + \tilde{\theta}_c^T(-1) F^{-1}(-1) \tilde{\theta}_c(-1) \right]$$

Since $F^{-1}(k) \succ 0 \forall k$, this implies that

$$\lim_{k \rightarrow \infty} \Delta\theta_c^T(k) F^{-1}(k-1) \Delta\theta_c(k) = 0$$

Since $\lambda_{\min}(F^{-1}(k-1)) = \frac{1}{\lambda_{\max}(F(k-1))} \geq \frac{1}{K_{\max}} > 0$, this implies that

$$\boxed{\lim_{k \rightarrow \infty} \|\Delta\theta_c(k)\| = 0}$$

Stability theorem proof, part 2

Substituting the parameter update equation

$$\Delta\theta_c(k) = F(k-1)\phi(k-d)e(k)$$

into

$$\lim_{k \rightarrow \infty} \Delta\theta_c^T(k) F^{-1}(k-1) \Delta\theta_c(k) = 0$$

we obtain

$$\lim_{k \rightarrow \infty} \phi^T(k-d) F(k-1) \phi(k-d) e^2(k) = 0$$

Adding the equation $\lim_{k \rightarrow \infty} \lambda_1(k-1) e^2(k) = 0$ to this equation yields

$$\lim_{k \rightarrow \infty} [\lambda_1(k-1) + \phi^T(k-d) F(k-1) \phi(k-d)] e^2(k) = 0$$

Stability theorem proof, part 2

We know that

$$\lim_{k \rightarrow \infty} [\lambda_1(k-1) + \phi^T(k-d)F(k-1)\phi(k-d)]e^2(k) = 0$$

Since $e(k) = \frac{\lambda_1(k-1)e^o(k)}{\lambda_1(k-1) + \phi^T(k-d)F(k-1)\phi(k-d)}$, we have

$$\lim_{k \rightarrow \infty} \frac{[\lambda_1(k-1)e^o(k)]^2}{\lambda_1(k-1) + \phi^T(k-d)F(k-1)\phi(k-d)} = 0$$

Stability theorem proof, part 2

Recall that $\eta_d(k) = r(k - d)$ and the control is given by

$$\hat{R}(q^{-1}, k)u(k) = r(k) - \hat{S}(q^{-1}, k)y(k)$$

We therefore see that

$$\begin{aligned}\eta_d(k + d) &= r(k) = \hat{R}(q^{-1}, k)u(k) + \hat{S}(q^{-1}, k)y(k) \\ &= \phi^T(k)\hat{\theta}_c(k)\end{aligned}$$

which allows us to say that

$$\begin{aligned}\epsilon(k) &= \eta(k) - \eta_d(k) = \phi^T(k - d)\tilde{\theta}_c(k - d) \\ &= \phi^T(k - d)\tilde{\theta}_c(k - 1) + \phi^T(k - d)[\tilde{\theta}_c(k - d) - \tilde{\theta}_c(k - 1)] \\ &= e^o(k) + \phi^T(k - d)[\tilde{\theta}_c(k - d) - \tilde{\theta}_c(k - 1)]\end{aligned}$$

Stability theorem proof, part 2

For convenience, define

$$\zeta(k) = \frac{\lambda_1(k-1) + \phi^T(k-d)F(k-1)\phi(k-d)}{\lambda_1^2(k-1)}$$

In this notation, we know that $\lim_{k \rightarrow \infty} \frac{[e^o(k)]^2}{\zeta(k)} = 0$

Since $0 < \lambda_1(k) \leq 1$ and $0 < K_{min} \leq \lambda_{min}(F(k)) \forall k$, we have

$$\zeta(k) > \phi^T(k-d)F(k-1)\phi(k-d) \geq K_{min}\|\phi(k-d)\|^2 \geq 0$$

$$\Rightarrow \frac{\|\phi(k-d)\|^2}{\zeta(k)} < \frac{1}{K_{min}}$$

Stability theorem proof, part 2

By the Cauchy-Schwarz inequality,

$$\begin{aligned} & \left| \frac{\phi^T(k-d) [\tilde{\theta}_c(k-d) - \tilde{\theta}_c(k-1)]}{\sqrt{\zeta(k)}} \right| \\ & \leq \frac{\|\phi(k-d)\|}{\sqrt{\zeta(k)}} \|\tilde{\theta}_c(k-d) - \tilde{\theta}_c(k-1)\| \\ & \leq \frac{1}{\sqrt{K_{\min}}} \|\tilde{\theta}_c(k-d) - \tilde{\theta}_c(k-1)\| \end{aligned}$$

The right-hand side of this inequality converges to zero because $\|\tilde{\theta}_c(k-d) - \tilde{\theta}_c(k-1)\|$ converges to zero.

Therefore

$$\lim_{k \rightarrow \infty} \frac{\phi^T(k-d) [\tilde{\theta}_c(k-d) - \tilde{\theta}_c(k-1)]}{\sqrt{\zeta(k)}} = 0$$

Stability theorem proof, part 2

Since $\epsilon(k) = e^o(k) + \phi^T(k-d) [\tilde{\theta}_c(k-d) - \tilde{\theta}_c(k-1)]$, we have

$$\begin{aligned}\lim_{k \rightarrow \infty} \frac{\epsilon(k)}{\sqrt{\zeta(k)}} &= \lim_{k \rightarrow \infty} \frac{e^o(k)}{\sqrt{\zeta(k)}} \\ &\quad + \lim_{k \rightarrow \infty} \frac{\phi^T(k-d) [\tilde{\theta}_c(k-d) - \tilde{\theta}_c(k-1)]}{\sqrt{\zeta(k)}} \\ &= 0 + 0\end{aligned}$$

Therefore

$$\boxed{\lim_{k \rightarrow \infty} \frac{[\lambda_1(k-1)\epsilon(k)]^2}{\lambda_1(k-1) + \phi^T(k-d)F(k-1)\phi(k-d)} = 0}$$

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Stability theorem proof, part 3

We want to prove that there exist $C_1 \geq 0$, $C_2 \geq 0$ such that

$$\|\phi(k-d)\| \leq C_1 + C_2 \max_{j \in \{0, \dots, k\}} |\epsilon(j)|$$

We have the relationships

$$y(k) = \frac{q^{-d}B(q^{-1})}{A(q^{-1})}u(k) = \frac{B(q^{-1})}{A(q^{-1})}u(k-d)$$

$$\eta(k) = A_c'(q^{-1})y(k)$$

$$\epsilon(k) = \eta(k) - \eta_d(k)$$

which define $\epsilon(k)$ from $u(k)$ and $\eta_d(k)$.

We now invert these relationships, i.e. we reconstruct $u(k)$ from $\epsilon(k)$ and $\eta_d(k)$

Stability theorem proof, part 3

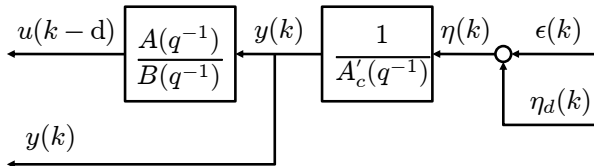
The inverted relationships are

$$u(k - d) = \frac{A(q^{-1})}{B(q^{-1})}y(k)$$

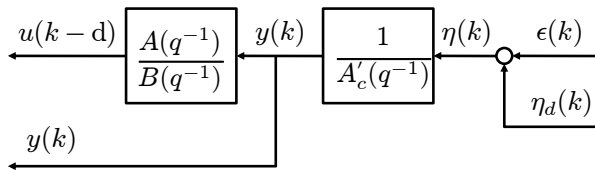
$$y(k) = \frac{1}{A'_c(q^{-1})}\eta(k)$$

$$\eta(k) = \epsilon(k) + \eta_d(k)$$

These relationships are shown in the block diagram



Stability theorem proof, part 3



Since $A'_c(q^{-1})$ and $B(q^{-1})$ are anti-Schur, both blocks in the block diagram are causal and BIBO

Therefore, we can choose nonnegative \bar{C}_{1u} , C_{2u} , \bar{C}_{1y} , and C_{2y} such that

$$|u(k-d)| \leq \bar{C}_{1u} + C_{2u} \max_{j \leq k} |\eta(j)|$$

$$|y(k)| \leq \bar{C}_{1y} + C_{2y} \max_{j \leq k} |\eta(j)|$$

Stability theorem proof, part 3

$$|u(k-d)| \leq \bar{C}_{1u} + C_{2u} \max_{j \leq k} |\eta(j)|$$

$$|y(k)| \leq \bar{C}_{1y} + C_{2y} \max_{j \leq k} |\eta(j)|$$

Assuming that $|\eta_d(k)| \leq \bar{\eta}_d$, the triangle inequality tells us that

$$|\eta(j)| \leq |\eta_d(k)| + |\epsilon(k)| \leq \bar{\eta}_d + |\epsilon(k)|$$

Defining $C_{1u} = \bar{C}_{1u} + C_{2u}\bar{\eta}_d$ and $C_{1y} = \bar{C}_{1y} + C_{2y}\bar{\eta}_d$ we have

$$|u(k-d)| \leq C_{1u} + C_{2u} \max_{j \leq k} |\epsilon(j)|$$

$$|y(k)| \leq C_{1y} + C_{2y} \max_{j \leq k} |\epsilon(j)|$$

Stability theorem proof, part 3

$$|u(k - d)| \leq C_{1u} + C_{2u} \max_{j \leq k} |\epsilon(j)|$$

$$|y(k)| \leq C_{1y} + C_{2y} \max_{j \leq k} |\epsilon(j)|$$

Since $\max_{j \leq k-\ell} |\epsilon(j)| \leq \max_{j \leq k} |\epsilon(j)|$ for $\ell \geq 0$, we have

$$|u(k - d - \ell)| \leq C_{1u} + C_{2u} \max_{j \leq k} |\epsilon(j)|$$

$$|y(k - d - \ell)| \leq C_{1y} + C_{2y} \max_{j \leq k} |\epsilon(j)|$$

for all $\ell \geq 0$

Stability theorem proof, part 3

$$|u(k - d - \ell)| \leq C_{1u} + C_{2u} \max_{j \leq k} |\epsilon(j)|$$

$$|y(k - d - \ell)| \leq C_{1y} + C_{2y} \max_{j \leq k} |\epsilon(j)|$$

Using the triangle inequality, we have

$$\begin{aligned} \|\phi(k - d)\| &\leq \sum_{j=0}^{n_s} |y(k - d - j)| + \sum_{i=0}^{n_r} |u(k - d - i)| \\ &\leq \sum_{j=0}^{n_s} \left(C_{1y} + C_{2y} \max_{\ell \leq k} |\epsilon(\ell)| \right) + \sum_{i=0}^{n_r} \left(C_{1u} + C_{2u} \max_{\ell \leq k} |\epsilon(\ell)| \right) \end{aligned}$$

Therefore

$$\begin{aligned} \|\phi(k - d)\| &\leq [(n_s + 1)C_{1y} + (n_r + 1)C_{1u}] \\ &\quad + [(n_s + 1)C_{2y} + (n_r + 1)C_{2u}] \max_{j \leq k} |\epsilon(j)| \end{aligned}$$

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Stability theorem proof, part 4

We want to prove Goodwin's technical lemma, which states that $\|\phi(k)\|$ remains bounded and

$$\lim_{k \rightarrow \infty} \epsilon(k) = 0$$

This proof will be done in three steps:

1. Show that $\epsilon(k)$ remains bounded
2. Show that $\|\phi(k)\|$ remains bounded
3. Show that $\epsilon(k) \rightarrow 0$

Stability theorem proof, part 4, step 1 ($\epsilon(k)$ bounded)

Recall from part 2 that

$$\lim_{k \rightarrow \infty} \frac{[\lambda_1(k-1)\epsilon(k)]^2}{\lambda_1(k-1) + \phi^T(k-d)F(k-1)\phi(k-d)} = 0$$

Since $0 < \underline{\lambda}_1 \leq \lambda_1(k) \leq 1$

and $0 < \lambda_{\min}(F(k-1)) \leq \lambda_{\max}(F(k-1)) \leq K_{\max}$

we have

$$\begin{aligned} \left| \frac{[\lambda_1(k-1)\epsilon(k)]^2}{\lambda_1(k-1) + \phi^T(k-d)F(k-1)\phi(k-d)} \right| \\ \geq \frac{\underline{\lambda}_1^2 \epsilon^2(k)}{1 + K_{\max} \|\phi(k-d)\|^2} > 0 \end{aligned}$$

Stability theorem proof, part 4, step 1 ($\epsilon(k)$ bounded)

$$\left| \frac{[\lambda_1(k-1)\epsilon(k)]^2}{\lambda_1(k-1) + \phi^T(k-d)F(k-1)\phi(k-d)} \right| \geq \frac{\underline{\lambda}_1^2 \epsilon^2(k)}{1 + K_{max} \|\phi(k-d)\|^2} > 0$$

For convenience, we define $\bar{\epsilon}(k) = \max_{j \leq k} |\epsilon(j)|$

From part 3, we have that $\|\phi(k-d)\|^2 \leq [C_1 + C_2 \bar{\epsilon}(k)]^2$, which implies that

$$\left| \frac{[\lambda_1(k-1)\epsilon(k)]^2}{\lambda_1(k-1) + \phi^T(k-d)F(k-1)\phi(k-d)} \right| \geq \frac{\underline{\lambda}_1^2 \epsilon^2(k)}{1 + K_{max} [C_1 + C_2 \bar{\epsilon}(k)]^2} > 0$$

Stability theorem proof, part 4, step 1 ($\epsilon(k)$ bounded)

$$\left| \frac{[\lambda_1(k-1)\epsilon(k)]^2}{\lambda_1(k-1) + \phi^T(k-d)F(k-1)\phi(k-d)} \right| \geq \frac{\underline{\lambda}_1^2 \epsilon^2(k)}{1 + K_{max}[C_1 + C_2 \bar{\epsilon}(k)]^2} > 0$$

Since

$$\lim_{k \rightarrow \infty} \frac{[\lambda_1(k-1)\epsilon(k)]^2}{\lambda_1(k-1) + \phi^T(k-d)F(k-1)\phi(k-d)} = 0$$

we have

$$\lim_{k \rightarrow \infty} \frac{\underline{\lambda}_1^2 \epsilon^2(k)}{1 + K_{max}[C_1 + C_2 \bar{\epsilon}(k)]^2} = 0$$

Stability theorem proof, part 4, step 1 ($\epsilon(k)$ bounded)

$$\lim_{k \rightarrow \infty} \frac{\lambda_1^2 \epsilon^2(k)}{1 + K_{max}[C_1 + C_2 \bar{\epsilon}(k)]^2} = 0$$

Whenever $|\epsilon(k)| = \bar{\epsilon}(k) \geq 1$, we have

$$\begin{aligned} 0 &< \frac{1 + K_{max}[C_1 + C_2 \bar{\epsilon}(k)]^2}{\lambda_1^2 \bar{\epsilon}^2(k)} \\ &= \frac{1 + K_{max}C_1^2}{\lambda_1^2 \bar{\epsilon}^2(k)} + \frac{2K_{max}C_1C_2}{\lambda_1^2 \bar{\epsilon}(k)} + \frac{K_{max}C_2^2}{\lambda_1^2} \\ &\leq \frac{1}{\lambda_1^2} [1 + K_{max}C_1^2 + 2K_{max}C_1C_2 + K_{max}C_2^2] \end{aligned}$$

This implies that whenever $|\epsilon(k)| = \bar{\epsilon}(k) \geq 1$, we have

$$\frac{\lambda_1^2 \epsilon^2(k)}{1 + K_{max}[C_1 + C_2 \bar{\epsilon}(k)]^2} \geq \frac{\lambda_1^2}{1 + K_{max}[C_1 + C_2]^2} > 0$$

Stability theorem proof, part 4, step 1 ($\epsilon(k)$ bounded)

Whenever $|\epsilon(k)| = \bar{\epsilon}(k) \geq 1$, we have

$$\frac{\lambda_1^2 \epsilon^2(k)}{1 + K_{max}[C_1 + C_2 \bar{\epsilon}(k)]^2} \geq \frac{\lambda_1^2}{1 + K_{max}[C_1 + C_2]^2} > 0$$

Since

$$\lim_{k \rightarrow \infty} \frac{\lambda_1^2 \epsilon^2(k)}{1 + K_{max}[C_1 + C_2 \bar{\epsilon}(k)]^2} = 0$$

there can only be a finite number of values of k such that

$$|\epsilon(k)| = \bar{\epsilon}(k) = \max_{j \leq k} |\epsilon(j)| \geq 1.$$

Therefore,

$\epsilon(k)$ remains bounded

Stability theorem proof, part 4, step 2 ($\phi(k)$ bounded)

Recall from part 3 that

$$\|\phi(k - d)\| \leq C_1 + C_2 \max_{j \leq k} |\epsilon(j)|$$

Since $\epsilon(k)$ remains bounded, we immediately see that

$\phi(k)$ remains bounded

Stability theorem proof, part 4, step 3 ($\epsilon(k) \rightarrow 0$)

Recall from part 2 that

$$\lim_{k \rightarrow \infty} \frac{\epsilon^2(k)}{\zeta(k)} = 0$$

where

$$\zeta(k) = \frac{\lambda_1(k-1) + \phi^T(k-d)F(k-1)\phi(k-d)}{\lambda_1^2(k-1)}$$

Therefore, if we can show that $\zeta(k)$ remains bounded, it must be true that $\epsilon(k) \rightarrow 0$

Stability theorem proof, part 4, step 3 ($\epsilon(k) \rightarrow 0$)

Since $0 < \underline{\lambda}_1 \leq \lambda_1(k) \leq 1$

and $0 < \lambda_{\min}(F(k-1)) \leq \lambda_{\max}(F(k-1)) \leq K_{\max}$

we have

$$\begin{aligned} |\zeta(k)| &= \left| \frac{\lambda_1(k-1) + \phi^T(k-d)F(k-1)\phi(k-d)}{\lambda_1^2(k-1)} \right| \\ &\leq \frac{1 + K_{\max} \|\phi(k-d)\|^2}{\underline{\lambda}_1^2} \end{aligned}$$

Since the right-hand side is bounded, we see that $\zeta(k)$ remains bounded.

Therefore

$$\lim_{k \rightarrow \infty} \epsilon(k) = 0$$

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Stability theorem proof, part 5

Recall from part 2 that

$$\lim_{k \rightarrow \infty} \frac{[e^o(k)]^2}{\zeta(k)} = 0$$

where

$$\zeta(k) = \frac{\lambda_1(k-1) + \phi^T(k-d)F(k-1)\phi(k-d)}{\lambda_1^2(k-1)}$$

We have already shown that $\zeta(k)$ is bounded

Therefore

$$\boxed{\lim_{k \rightarrow \infty} e^o(k) = 0}$$