ME 233 Spring 2010
Solution to Homework #4

1. (a) Compute the marginal probability density functions

$$\begin{aligned} p_Y(y) &= \int_{-\infty}^{\infty} p_{XY}(x,y)dx = \int_{y/2}^{1} dx = (1-\frac{y}{2}) \quad 0 \le y \le 2 \\ &= \begin{cases} 1-\frac{y}{2}\,, & 0 \le y \le 2 \\ 0\,, & \text{elsewhere} \end{cases} \end{aligned}$$

$$\begin{aligned} p_X(x) &= \int_{-\infty}^{\infty} p_{XY}(x,y)dy = \int_{0}^{2x} dy = 2x \quad 0 \le x \le 1 \\ &= \begin{cases} 2x\,, & 0 \le x \le 1 \\ 0\,, & \text{elsewhere} \end{cases} \end{aligned}$$

(b) Compute the marginal mean m_X.

$$m_X = E\{X\} = \int_{-\infty}^{\infty} x\, p_X(x)dx = \int_0^1 x\,2xdx = 2\int_0^1 x^2dx = \frac{2}{3} = 0.6667$$

(c) Compute the marginal variance of X.

$$\begin{aligned} \Lambda_{XX} &= \int_{-\infty}^{\infty} (x-m_X)^2\, p_X(x)dx = \int_{-\infty}^{\infty} (x^2 - {m_X}^2)\, p_X(x)dx = \int_0^1 (x^2 - \left(\frac{2}{3}\right)^2)\, 2x\, dx \\ &= 2\int_0^1 (x^3 - \frac{4}{9}x)dx = 2\left[\frac{1}{4} - \frac{4}{9}\frac{1}{2}\right] = \left[\frac{1}{2} - \frac{4}{9}\right] = \frac{1}{18} = 0.0556 \end{aligned}$$

(d) Obtain an expression for the conditional probability density function $p_{X|Y}(x|y)$

$$p_{X|Y}(x|y) = \frac{p_{XY}(x,y)}{p_Y(y)} = \begin{cases} \frac{1}{1-\frac{y}{2}}\,, & \frac{y}{2} \le x \le 1 \\ 0\,, & \text{elsewhere} \end{cases}$$

(e) Determine the conditional mean $E\{X|Y=y\}$,

$$E\{X|Y=y\} = \int_{-\infty}^{\infty} x\, p_{X|Y}(x|y)dx = \int_{\frac{y}{2}}^{1} \left[\frac{1}{1-\frac{y}{2}}\right] x\, dx = \frac{1}{2}(1+\frac{y}{2})$$

(f) Determine the conditional mean $E\{X|Y=0.5\}$.

$$E\{X|Y=0.5\} = \frac{1}{2}(1+\frac{1}{4}) = 0.6250$$

(g) Notice that the conditional mean $E\{X|Y\}$ can be thought as a function of the random variable Y. Therefore, it is itself a random variable. Lets introduce the notation

$$m_{X|Y}(Y) = E\{X|Y\} = \int_{-\infty}^{\infty} x\, p_{X|Y}(x|Y)dx$$

Prove that the expected value of the conditional mean $m_{X|Y}(Y)$ is equal to the marginal mean of X.

$$\begin{aligned}
E\{m_{X|Y}(Y)\} &= \int_{-\infty}^{\infty} m_{X|Y}(y)\, p_Y(y)dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, p_{X|Y}(x|y)p_Y(y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, p_{XY}(x,y)dxdy \\
&= \int_{-\infty}^{\infty} x\, p_X(x)dx = E\{X\} = m_X
\end{aligned}$$

and verify this result by computing $E\{m_{X|Y}(Y)\}$ and comparing it to m_X for the example above.

$$\begin{aligned}
E[m_{X|Y}(Y)] &= \int_{-\infty}^{\infty} \frac{1}{2}(1+\frac{y}{2})p_Y(y)dy = \frac{1}{2}\int_0^2 (1-\frac{y^2}{4})dy \\
&= 0.6667 = \frac{2}{3} = m_X
\end{aligned}$$

(h) Compute the variance of the conditional mean $m_{X|Y}(Y)$ for the example above.

$$\begin{aligned}
\Lambda_{m_{X|Y}\, m_{X|Y}} &= \int_{-\infty}^{\infty} (m_{X|Y}(y) - m_X)^2\, p_Y(y)dy = \int_{-\infty}^{\infty} (m_{X|Y}(y)^2 - {m_X}^2)\, p_Y(y)dy \\
&= \int_0^2 \left[\frac{1}{4}\left(1+\frac{y}{2}\right)^2 - \left(\frac{2}{3}\right)^2\right](1-\frac{y}{2})\,dy = 2\int_0^2 \left[\frac{1}{4}\left(1+\frac{y}{2}\right)^2 - \left(\frac{2}{3}\right)^2\right](1-\frac{y}{2})\,d\frac{y}{2} \\
&= \int_0^1 \left[\frac{1}{2}(1+t)^2 - \frac{8}{9}\right](1-t)\,dt = \int_0^1 \left(\frac{t^2}{2} + t - \frac{7}{18}\right)(1-t)\,dt \\
&= \int_0^1 \left(-\frac{t^3}{2} - \frac{t^2}{2} + \frac{25}{18}t - \frac{7}{18}\right)dt = \frac{1}{72}
\end{aligned}$$

Obviously, $\Lambda_{m_{X|Y}\, m_{X|Y}} = \frac{1}{72} < \Lambda_{XX} = \frac{1}{18}$.

(i) With the results from the previous parts, we have:

$$\begin{aligned}
\Lambda_{X|Y X|Y}(Y=y) &= E\{(X - m_{X|Y}(Y))^2|Y=y\} = \int_{-\infty}^{\infty} (x - m_{X|Y}(Y))^2\, p_{X|Y}(x|Y=y)dx \\
&= \int_{\frac{y}{2}}^1 \left[x - \frac{1}{2}\left(1+\frac{y}{2}\right)\right]^2 \frac{1}{1-\frac{y}{2}}\,dx \\
&= \int_{\frac{y}{2}}^1 \left[x^2 - x\left(1+\frac{y}{2}\right) + \frac{1}{4}\left(1+\frac{y}{2}\right)^2\right]\frac{1}{1-\frac{y}{2}}\,dx \\
&= \frac{1}{3}\frac{1-\left(\frac{y}{2}\right)^3}{1-\frac{y}{2}} - \frac{1}{2}\frac{1-\left(\frac{y}{2}\right)^2}{1-\frac{y}{2}}\left(1+\frac{y}{2}\right) + \frac{1}{4}\left(1+\frac{y}{2}\right)^2 \\
&= \frac{1}{3}\left(1+\frac{y}{2}+\frac{y^2}{4}\right) - \frac{1}{2}\left(1+\frac{y}{2}\right)^2 + \frac{1}{4}\left(1+\frac{y}{2}\right)^2 \\
&= \frac{1}{12}\left(1-\frac{y}{2}\right)^2
\end{aligned}$$

Therefore, we have:

$$\Lambda_{X|Y X|Y}(Y) = \frac{1}{12}\left(1-\frac{Y}{2}\right)^2$$

Notice that the conditional variance of X given Y, $\Lambda_{X|Y X|Y}(Y)$ is also a random variable.

(j) Finally, compute the expected value of the the conditional variance of X given Y,

$$\begin{aligned}
E\{\Lambda_{X|YX|Y}(Y)\} &= \int_{-\infty}^{\infty} \Lambda_{X|YX|Y}(y)\, p_Y(y) dy \\
&= \int_{-\infty}^{\infty} \frac{1}{12}\left(1-\frac{y}{2}\right)^2 p_Y(y) dy \\
&= \frac{1}{12}\int_0^2 \left(1-\frac{y}{2}\right)^3 dy = \frac{1}{6}\int_0^1 (1-t)^3\, dt = \frac{1}{24}
\end{aligned}$$

For the example above, we can verify that

$$\begin{aligned}
\Lambda_{XX} &= \frac{1}{18} = \frac{1}{24} + \frac{1}{72} \\
&= \Lambda_{m_{X|Y}\, m_{X|Y}} + E\{\Lambda_{X|YX|Y}(Y)\}\,.
\end{aligned}$$

2. (a) Figure 1 shows the MATLAB estimates of the auto-covariances and cross-covariances of W and Y. As we would expect, $\Lambda_{WW}(j)$ is approximately a unit pulse and $\Lambda_{YY}(j)$ is approximately symmetric. Also, $\Lambda_{YW}(-j) \approx \Lambda_{WY}(j)$ is approximately 0 for positive j, as causality dictates.

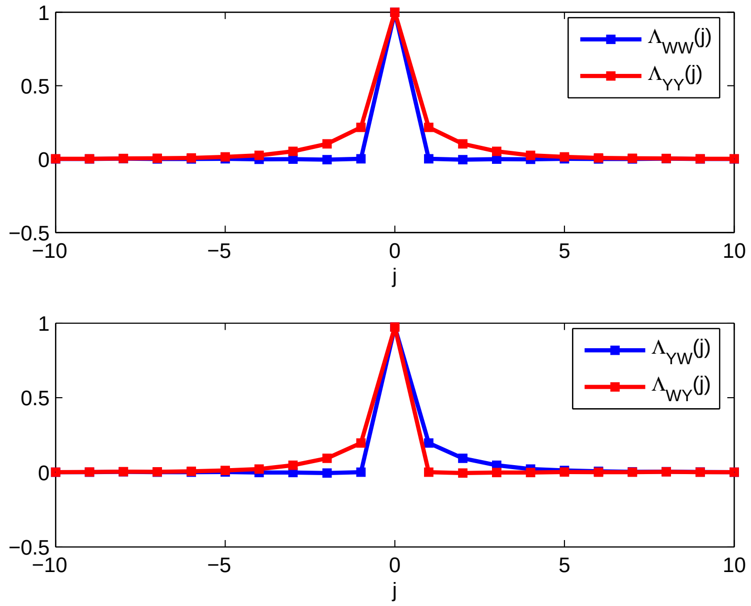

Figure 1: MATLAB estimates of auto-covariances and cross-covariances

(b) To find $\Lambda_{YW}(l)$, it is easiest to first find $\Lambda_{YW}(z)$. Thus, we first note that

$$\begin{aligned}
\Lambda_{YW}(z) &= G(z)\Lambda_{WW}(z) \\
G(z) &= \frac{z-0.3}{z-0.5} \\
\Lambda_{WW}(z) &= \mathcal{Z}\{\delta(l)\} = 1 \\
\Rightarrow \Lambda_{YW}(z) &= \frac{z-0.3}{z-0.5}.
\end{aligned}$$

Now, with the aid of inverse Z-transform tables, we get that

$$\begin{aligned}
\Lambda_{YW}(l) &= \mathcal{Z}^{-1}\left\{\frac{0.4z}{z-0.5} + 0.6\right\} \\
&= \begin{cases} 0.4(0.5)^l + 0.6\delta(l) & l \geq 0 \\ 0 & l < 0 \end{cases}.
\end{aligned}$$

Figure 2 shows that the values of $\Lambda_{YW}(l)$ determined through MATLAB simulation match up well with the values determined above.

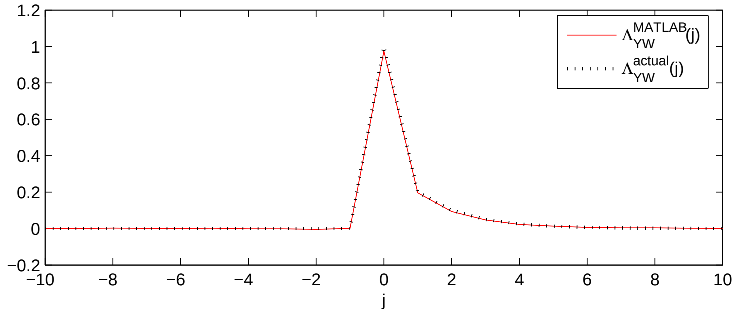

Figure 2: Comparison of MATLAB-determined cross-covariance to actual values

(c) Now that we have $\Lambda_{YW}(l)$, finding $\Lambda_{WY}(l)$ is a trivial matter. Using the property that $\Lambda_{YW}(l) = \Lambda_{WY}(-l)$, we see that

$$\Lambda_{WY}(l) = \begin{cases} 0.4(0.5)^{-l} + 0.6\delta(l) & l \leq 0 \\ 0 & l > 0 \end{cases}.$$

Figure 3 shows that the values of $\Lambda_{WY}(l)$ determined through MATLAB simulation match up well with the values determined above.

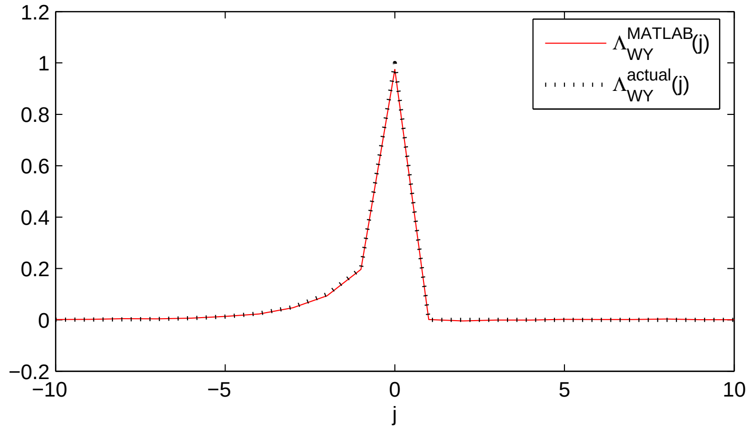

Figure 3: Comparison of MATLAB-determined cross-covariance to actual values

To find $\Lambda_{WY}(z)$, it is easiest to recognize that the following general property applies to any random variables X and U:

$$\begin{aligned} \Lambda_{XU}(z) &= \sum_{l=-\infty}^{\infty} z^{-l}\Lambda_{XU}(l) \\ &= \sum_{l=-\infty}^{\infty} (z^{-1})^{l}\Lambda_{UX}(-l) \\ &= \sum_{l=-\infty}^{\infty} (z^{-1})^{-l}\Lambda_{UX}(l) \\ &= \Lambda_{UX}(z^{-1}). \end{aligned}$$

Applying this property to our system here gives

$$\Lambda_{WY}(z) = \Lambda_{YW}(z^{-1}) = \frac{z^{-1} - 0.3}{z^{-1} - 0.5} = \frac{0.3z - 1}{0.5z - 1}.$$