ME 233 Advance Control II

Discrete Time Kalman Filter

(ME233 Class Notes pp.KF1-KF6)

Wiener Filtering

Norbert Wiener:

- Well-known as the founder of cybernetics, a field he developed in the 1970s that emphasized the modeling of humans as communication and control systems.
- In 1942 he did significant work in the use of time series for military applications; an example of which would be the prediction of the location of enemy planes at the next time step.
- His work in filtering, prediction and smoothing came about in 1949. Wiener filtering is solved for Gaussian time series and under certain assumptions, stationary time series.

Rudy Kalman:

- First major contribution was the introduction of the self-tuning regulator in adaptive control.
- Between 1959 and 1964 he wrote a series of seminal papers:
  - First, the new approach to the filtering problem, known today as Kalman Filtering
  - In the meantime, the all pervasive concept of controllability and its dual, the concept of observability, were formulated.
- By combining the filtering and the control ideas, the first systematic theory for control synthesis, known today as the Linear-Quadratic-Gaussian or LQG theory, resulted.

Deterministic - state feedback

State variable feedback:

$$x(k+1) = A\,x(k) + B\,u(k)$$

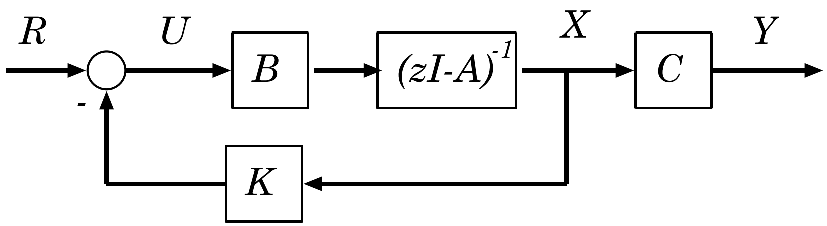
$$u(k) = -K\,x(k) + r(k)$$

With fictitious reference input ***r(k)***

$$r(k) = r_o = 0$$

Deterministic - state feedback

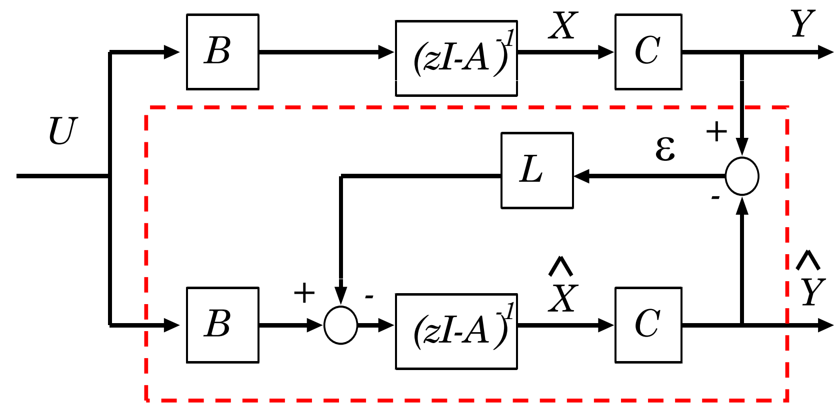
- State Variable Feedback



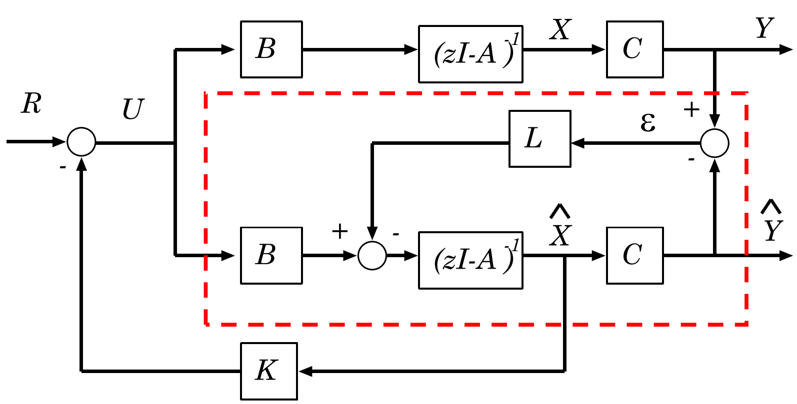
- What happens if the state is not directly measurable – only the output $y(k)$?

Deterministic– state estimation

- State observer

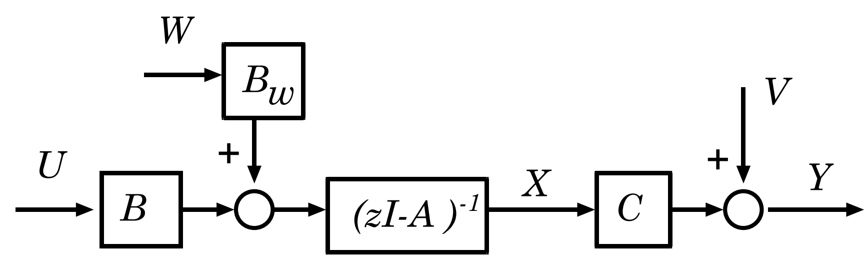


Deterministic– state observer feedback



Stochastic State Estimation

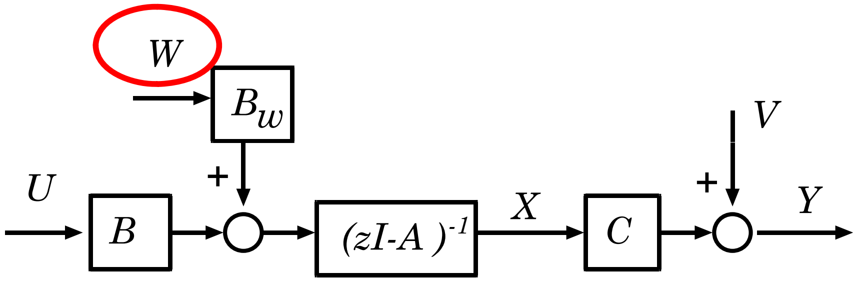
System is now contaminated by noise



Two random disturbances

Stochastic State Estimation

System is now contaminated by noise

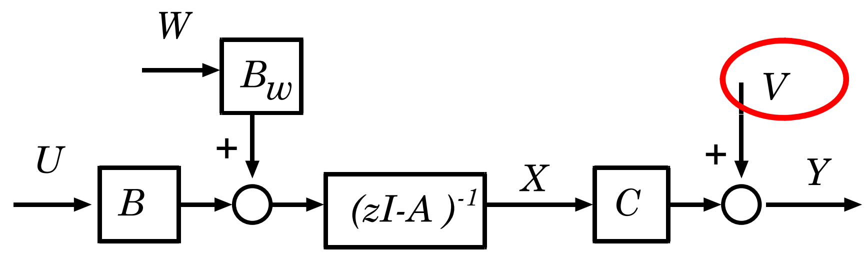


- Input noise $\boldsymbol{w(k)}$ - contaminates the state

$\Rightarrow$ $\boldsymbol{x(k)}$ is now a random sequence

Stochastic State Estimation

System is now contaminated by noise



- Measurement noise $\boldsymbol{v(k)}$ - contaminates the output $\boldsymbol{y(k)}$

Stochastic state model

$$
\begin{aligned}
x(k+1) &= A\,x(k) + B\,u(k) + B_w\,w(k) \\
y(k) &= C\,x(k) + v(k)
\end{aligned}
$$

Where:

- $u(k)$ **known control input**
- $w(k)$ input noise
- $v(k)$ measurement noise

Initial Conditions

- $x(0)$ is Gaussian with **known** marginal mean and covariance:

$$E\{x(0)\} = \hat{x}(0) = x_o$$

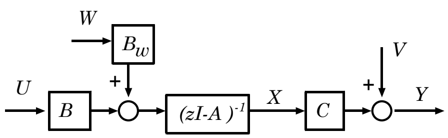
$$\Lambda_{xx}(0,0) = E\{\tilde{x}(0)\tilde{x}^T(0)\} = X_o$$

$$\tilde{x}(0) = x(0) - \hat{x}(0)$$

Noises

$w(k)$ and $v(k)$ are:

- **<u>Gaussian</u> zero mean white noises**
- independent from each other and from $x(0)$
- but not necessarily stationary



Noises

$$\Lambda_{ww}(k,l) = E\{w(k+l)w^T(k)\} = W(k)\,\delta(l)$$

$$\Lambda_{vv}(k,l) = E\{v(k+l)v^T(k)\} = V(k)\,\delta(l)$$

$$\Lambda_{wv}(k,l) = \Lambda_{vw}^T(k,l) = 0$$

$$E\{\tilde{x}(0)w^T(k)\} = 0 \qquad E\{\tilde{x}(0)v^T(k)\} = 0$$

Output Measurements

$y(k)$ is the measured output, which is also a stochastic variable.

- set of available measurements at the instant j

$$Y_j = \{y(0),\, y(1),\, \cdots,\, y(j)\}$$

each sample of y(k) is an outcome!

Notation so far …

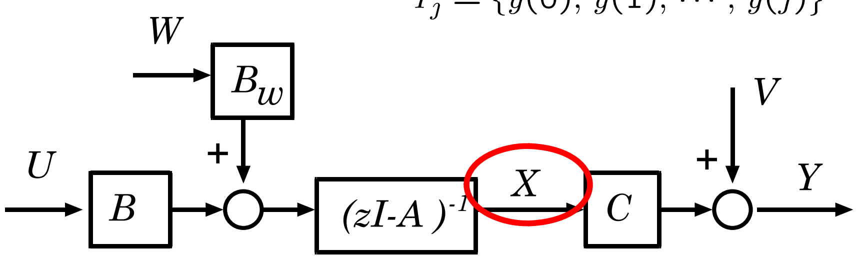
- Initial state marginal mean: x_o
- Initial state marginal covariance: X_o
- Input noise covariance : $W(k)$
- Measurement noise covariance: $V(k)$
- Set of $\boldsymbol{j}$ output measurements: Y_j

  $\{y(0),\, y(1),\, \cdots,\, y(j)\}$

Kalman Filter Objective

Obtain the **best state estimate** given available measurements

$$Y_j = \{y(0),\, y(1),\, \cdots,\, y(j)\}$$



Conditional state estimation problem

Conditional state estimation

$$\hat{x}(k|j) = E\{x(k)|Y_j\}$$

Conditional state estimate

given the set of available measurements:

$$Y_j = \{y(0),\, y(1),\, \cdots,\, y(j)\}$$

Conditional state estimation

$$\hat{x}(k|j) = E\{x(k)|Y_j\}$$

$$Y_j = \{y(0),\, y(1),\, \cdots,\, y(j)\}$$

Notice that when:

- $k = j$ this is a filtering problem ⇦ ***our focus***
- $k > j$ this is a prediction problem
- $k < j$ this a smoothing problem

A-priori state estimates

$$\hat{x}^o(k) = \hat{x}(k|k-1)$$

Conditional state estimate
given the set of available measurements: Y_{k-1}

$$\{y(0),\, y(1),\, \cdots,\, y(k-1)\}$$

A-priori state estimation error:

$$\tilde{x}^o(k) = \tilde{x}(k|k-1) = x(k) - \hat{x}^o(k)$$

A-posteriori state estimates

$$\hat{x}(k) = \hat{x}(k|k)$$

Conditional state estimate
given the set of available measurements: Y_k

$\{y(0),\, y(1),\, \cdots,\, y(k)\}$

A-posteriori state estimation error:

$$\tilde{x}(k) = \tilde{x}(k|k) = x(k) - \hat{x}(k)$$

State Estimate Covariances

A-priori estimation error covariance:

$$\begin{aligned} M(k) &= E\{\tilde{x}^o(k)\tilde{x}^{oT}(k)\} \\ &= E\{\tilde{x}(k|k-1)\tilde{x}^T(k|k-1)\} \end{aligned}$$

A-posteriori estimation error covariance:

$$\begin{aligned} Z(k) &= E\{\tilde{x}(k)\tilde{x}^T(k)\} \\ &= E\{\tilde{x}(k|k)\tilde{x}^T(k|k)\} \end{aligned}$$

State Estimate Covariances

Notice that:

$$\text{trace}\, Z(k) \;\leq\; \text{trace}\, M(k)$$

A-posteriori *A-priori*

$$E\{|\tilde{x}(k)|^2\} \leq E\{|\tilde{x}^o(k)|^2\}$$

Initial Conditions for a-priori estimate

Notice that:

$$\hat{x}^o(0) = \hat{x}(0|-1)$$

a-priori state estimate without knowing ***y(0)***

$$\hat{x}^o(0) = \hat{x}(0|-1) = \underbrace{E\{x(0)\} = \hat{x}(0)}_{\text{marginal estimation}} = x_o$$

Initial Conditions for a-priori estimate

Notice that:

$$M(0) \;=\; E\{\tilde{x}^o(0)\tilde{x}^{oT}(0)\}$$

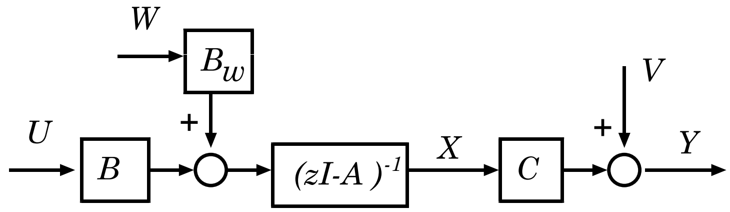
$$= E\{\tilde{x}(0)\tilde{x}^T(0)\} = \underbrace{\Lambda_{xx}(0,0)}_{\text{marginal covariance}}$$

$$=\; X_o$$

Kalman Filter Solution

Given:

- I.C.: $\hat{x}^o(0) = x_o \qquad M(0) \;=\; X_o$
- Noise covariance intensities: $W(k) \quad V(k)$

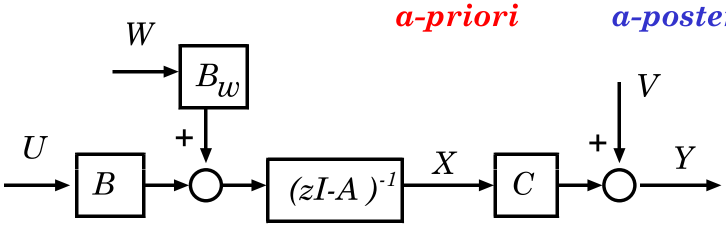


Kalman Filter Solution

Recursively find:

- State estimates: $\hat{x}^o(k)$ $\hat{x}(k)$
- State covariances $M(k)$ $Z(k)$

a-priori ***a-posteriori***



Kalman Filter Solution

Remember:

- Conditional state estimates:

$\hat{x}^o(k) = \hat{x}(k|k-1)$ a-priori, before *y(k)*

$\hat{x}(k) = \hat{x}(k|k)$ a-posteriori, after *y(k)*

Kalman Filter Solution

Remember:

- noises are Gaussian, zero-mean and uncorrelated with each other and the initial state:

$$\Lambda_{ww}(k,l) = W(k)\,\delta(l)$$

$$\Lambda_{vv}(k,l) = V(k)\,\delta(l)$$

$$\Lambda_{wv}(k,l) = 0$$

$$\Lambda_{wx}(0,k) = \Lambda_{vx}(0,k) = 0$$

Kalman Filter Solution: *k = 0*

- **Before** measurement *y(0):*

$$\hat{x}^o(0) = \hat{x}(0|-1) = E\{x(0)\} = x_o$$

(given)

$$\tilde{x}^o(0) = x(0) - x_o$$

$$\begin{aligned} M(0) &= \Lambda_{x(0)x(0)} \\ &= E\{\tilde{x}^o(0)\tilde{x}^{oT}(0)\} \\ &= X_o \quad \textit{(given)} \end{aligned}$$

Kalman Filter Solution: *k = 0*

- A-priori output estimate:

$$\begin{aligned} \hat{y}^o(0) &= E\{y(0)\} = E\{C\,x(0) + v(0)\} \\ &= C\,\hat{x}^o(0) = Cx_o \end{aligned}$$

$$(x_o = E\{x(0)\} \neq x(0))$$

A-priori output estimation error (***KF residual***)

$$\tilde{y}^o(0) = C\,\tilde{x}^o(0) + v(0)$$

Kalman Filter Solution: *k = 0*

Review of the results so far:

$$\left.\begin{aligned} \hat{x}^o(0) &= x_o \\ \tilde{y}^o(0) &= y(0) - C\,\hat{x}^o(0) \\ M(0) &= X_o \end{aligned}\right\} \textbf{\textit{a-priori}}$$

$$\left.\begin{aligned} \hat{x}(0) &= \\ Z(0) &= \end{aligned}\right\} \textbf{\textit{a-posteriori}}$$

Kalman Filter Solution: *k = 0*

- **After** measurement *y(0)*:

Calculate a-posteriori state estimate using the conditional estimation formula for Gaussians:

$$\widehat{x}(0) = \widehat{x}(0|0) = E\{x(0)|y(0)\}$$

$$= \widehat{x}^o(0) + \Lambda_{x(0)y(0)}\,\Lambda^{-1}_{y(0)y(0)}\,\tilde{y}^o(0)$$

$$\widehat{x}(0) = \widehat{x}^o(0) + \Lambda_{x(0)y(0)}\,\Lambda^{-1}_{y(0)y(0)}\,\tilde{y}^o(0)$$

Calculate:

$$\begin{aligned}\Lambda_{x(0)y(0)} &= E\{\tilde{x}^o(0)\tilde{y}^{oT}(0)\}\\ &= E\{\tilde{x}^o(0)\,[C\,\tilde{x}^o(0) + v(0)]^T\}\\ &\qquad (E\{\tilde{x}^o(0)v^T(0)\} = 0)\\ &= \underbrace{E\{\tilde{x}^o(0)\tilde{x}^{oT}(0)\}}_{M(0)}\,C^T\\ &= M(0)\,C^T\end{aligned}$$

$$\widehat{x}(0) = \widehat{x}^o(0) + M(0)C^T\,\Lambda^{-1}_{y(0)y(0)}\,\tilde{y}^o(0)$$

Calculate:

$$\begin{aligned}\Lambda_{y(0)y(0)} &= E\{\tilde{y}^o(0)\tilde{y}^{oT}(0)\}\\ &= E\{[C\,\tilde{x}^o(0) + v(0)]\,[C\,\tilde{x}^o(0) + v(0)]^T\}\\ &\qquad (E\{\tilde{x}^o(0)v^T(0)\} = 0)\\ &= C\,\underbrace{E\{\tilde{x}^o(0)\tilde{x}^{oT}(0)\}}_{M(0)}\,C^T + \underbrace{E\{v(0)v^T(0)\}}_{V(0)}\\ &= C\,M(0)\,C^T + V(0)\end{aligned}$$

Kalman Filter Solution: *k = 0*

- **a-posteriori state** estimate:

$$\widehat{x}(0) = \widehat{x}^o(0) + \underbrace{\Lambda_{x(0)y(0)}}_{M(0)C^T}\,\underbrace{\Lambda^{-1}_{y(0)y(0)}}_{[CM(0)C^T + V(0)]^{-1}}\,\tilde{y}^o(0)$$

$$\boxed{\begin{aligned}\widehat{x}(0) &= \widehat{x}^o(0) + M(0)C^T\left[C\,M(0)C^T + V(0)\right]^{-1}\tilde{y}^o(0)\\ \tilde{y}^o(0) &= y(0) - C\,\widehat{x}^o(0) \qquad \widehat{x}^o(0) = x_o\end{aligned}}$$

Kalman Filter Solution: *k = 0*

Review of the results so far:

$$\left.\begin{aligned}\hat{x}^o(0) &= x_o \\ \tilde{y}^o(0) &= y(0) - C\,\hat{x}^o(0) \\ M(0) &= X_o\end{aligned}\right\}\ \textbf{\textit{a-priori}}$$

$$\hat{x}(0) = \hat{x}^o(0) + M(0)C^T\left[C\,M(0)C^T + V(0)\right]^{-1}\tilde{y}^o(0)$$

$$Z(0) = E\{\tilde{x}(0)\tilde{x}^T(0)\} = \Lambda_{\tilde{x}(0)\tilde{x}(0)}$$

Kalman Filter Solution: *k = 0*

- **A-posteriori state** estimation error:

$$\tilde{x}(0) = x(0) - \hat{x}(0)$$

- **A-posteriori state** estimation covariance:

$$Z(0) = E\{\tilde{x}(0)\tilde{x}^T(0)\} = \Lambda_{\tilde{x}(0)\tilde{x}(0)}$$

Kalman Filter Solution: *k = 0*

- **a-posteriori state** estimation covariance:

$$Z(0) = E\{\tilde{x}(0)\tilde{x}^T(0)\} = \Lambda_{\tilde{x}(0)\tilde{x}(0)}$$

- Use least squares result:

$$\Lambda_{\tilde{x}(0)\tilde{x}(0)} = \underbrace{\Lambda_{x(0)x(0)}}_{M(0)} - \underbrace{\Lambda_{x(0)y(0)}}_{M(0)C^T}\underbrace{\Lambda^{-1}_{y(0)y(0)}}_{[CM(0)C^T + V(0)]^{-1}}\Lambda_{y(0)x(0)}$$

$$\boxed{Z(0) = M(0) - M(0)C^T\left[CM(0)C^T + V(0)\right]^{-1}CM(0)}$$

Kalman Filter Solution: *k = 0*

Review of the results so far:

$$\hat{x}^o(0) = x_o$$

$$\tilde{y}^o(0) = y(0) - C\,\hat{x}^o(0)$$

$$M(0) = X_o$$

$$\hat{x}(0) = \hat{x}^o(0) + M(0)C^T\left[C\,M(0)C^T + V(0)\right]^{-1}\tilde{y}^o(0)$$

$$Z(0) = M(0) - M(0)C^T\left[CM(0)C^T + V(0)\right]^{-1}CM(0)$$

Kalman Filter Solution: ***k = 1***

Before measurement *y(1)*:

- Determine a-priori state estimate $\hat{x}^o(1)$

$$\hat{x}^o(1) = \hat{x}(1|0) = E\{x(1)|y(0)\}$$

- Determine a-priori state estimation error covariance

$$M(1) = E\{\tilde{x}^o(1)\tilde{x}^{oT}(1)\}$$

Kalman Filter Solution: ***k = 1***

A-priori state estimate:

$$\hat{x}^o(1) = \hat{x}(1|0) = E\{x(1)|y(0)\}$$

- **Use state equation and take conditional expectations:**

$$\begin{aligned} x(1) &= A\,x(0) + B\,u(0) + B_w\,w(0) \\ \hat{x}(1|0) &= A\,\hat{x}(0|0) + B\,u(0) \end{aligned}$$

$$\boxed{\hat{x}^o(1) = A\,\hat{x}(0) + B\,u(0)}$$

Kalman Filter Solution: ***k = 1***

A-priori state estimation error:

$$\tilde{x}^o(1) = x(1) - \hat{x}^o(1)$$

- **Use state equation:**

$$\begin{aligned} x(1) &= A\,x(0) + B\,u(0) + B_w\,w(0) \\ \hat{x}^o(1) &= A\,\hat{x}(0) + B\,u(0) \end{aligned}$$

$$\boxed{\tilde{x}^o(1) = A\,\tilde{x}(0) + B_w\,w(0)}$$

Kalman Filter Solution: ***k = 1***

A-priori state estimation error covariance:

$$M(1) = E\{\tilde{x}^o(1)\tilde{x}^{oT}(1)\}$$

- **Use:**

$$\tilde{x}^o(1) = A\,\tilde{x}(0) + B_w\,w(0) \qquad E\{\tilde{x}(0)w(0)\} = 0$$

$$\begin{aligned} \underbrace{E\{\tilde{x}^o(1)\tilde{x}^{oT}(1)\}}_{M(1)} &= A\,\underbrace{E\{\tilde{x}(0)\tilde{x}^T(0)\}}_{Z(0)}A^T \\ &= +B_w\,\underbrace{E\{w(0)w^T(0)\}}_{W(0)}B_w^T \end{aligned}$$

Kalman Filter Solution: *k = 1*

A-priori state estimation error covariance:

$$M(1) = E\{\tilde{x}^o(1)\tilde{x}^{oT}(1)\}$$

$$\boxed{M(1) \ = \ A\,Z(0)A^T + B_w W(0) B_w^T}$$

Kalman Filter Solution: *k = 1*

- **Before** measurement *y(1):*

$$\begin{aligned}\hat{y}^o(1) \ &= \ E\{y(1)|y(0)\} \\ &= E\{C\,x(1) + v(1)|y(0)\} \\ &= \ C\,E\{x(1)|y(0)\} \\ &= \ C\hat{x}^o(1)\end{aligned}$$

Kalman Filter Solution: *k = 1*

Before measurement *y(1):*

$$\hat{y}^o(1) \ = \ C\,\hat{x}^o(1)$$

A-priori output estimation error $\tilde{y}^o(1)$

$$\tilde{y}^o(1) \ = \ y(1) - \hat{y}^o(1)$$

$$\tilde{y}^o(1) \ = \ y(1) - C\,\hat{x}^o(1)$$

Kalman Filter Solution: *k = 1*

Review of the results so far:

$$\left.\begin{aligned}\hat{x}^o(1) \ &= \ A\,\hat{x}(0) + B\,u(0) \\ \tilde{y}^o(1) \ &= \ y(1) - C\,\hat{x}^o(1) \\ M(1) \ &= \ A\,Z(0)A^T + B_w W(0) B_w^T\end{aligned}\right\} \textit{a-priori}$$

$$\left.\begin{aligned}\hat{x}(1) \ &= \\ Z(1) \ &= \end{aligned}\right\} \textit{a-posteriori}$$

Kalman Filter Solution: *k = 1*

Before measurement *y(1)*:

$$\hat{y}^o(1) = C\,\hat{x}^o(1)$$

A-priori output estimation error $\tilde{y}^o(1)$

$$\begin{aligned}\tilde{y}^o(1) &= y(1) - \hat{y}^o(1) \\ &= Cx(1) + v(1) - C\,\hat{x}^o(1) \\ &= C\,\tilde{x}^o(1) + v(1)\end{aligned}$$

Kalman Filter Solution: *k = 1*

IMPORTANT: Property 1 of least squares estimation:

$$\hat{y}^o(1) = E\{y(1)|y(0)\}$$

- **The a-priori output estimation error** $\tilde{y}^o(1)$ **is uncorrelated with** $y(0)$

$$\boxed{E\{y(0)\tilde{y}^{oT}(1)\} = 0}$$

Kalman Filter Solution: *k = 1*

- Calculate a-posteriori state estimate using the conditional estimation formula for Gaussians:
- **Use recursive least squares estimation property 3:**

$$\begin{aligned}\hat{x}(1) &= \hat{x}(1|1) \\ &= E\{x(1)|y(0),\, y(1)\} \\ &= E\{x(1)|y(0),\, \tilde{y}^o(1)\}\end{aligned}$$

Least Squares Estimation : Property 3

Estimate $\boldsymbol{X}$ given outcomes: $\boldsymbol{Y = y}$ and $\boldsymbol{Z = z}$

$$\hat{x}_{|yz} = \hat{x}_{|y} + E\{\tilde{X}_{|Y}|\tilde{z}_{|y}\}$$

In this case: $\boldsymbol{X = x(1)}$ $\boldsymbol{Y = y(0)}$ $\boldsymbol{Z = y(1)}$

$$\hat{x}(1)_{|y(0)y(1)} = \hat{x}(1)_{|y(0)} + E\{\tilde{x}(1)_{|y(0)}|\tilde{y}(1)_{|y(0)}\}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\hat{x}(1) = \hat{x}^0(1) + E\{\tilde{x}^o(1)|\tilde{y}^o(1)\}$$

Kalman Filter Solution: *k = 1*

- **Use recursive least squares estimation:**

$$\begin{aligned}\hat{x}(1) &= E\{x(1)|y(0),\, \tilde{y}^o(1)\} \\ &= \underbrace{E\{x(1)|y(0)\}}_{\hat{x}^o(1)} + E\{\tilde{x}^o(1)|\tilde{y}^o(1)\} \\ &= \hat{x}^o(1) + E\{\tilde{x}^o(1)|\tilde{y}^o(1)\}\end{aligned}$$

Kalman Filter Solution: *k = 1*

- Calculate a-posteriori state estimate using the conditional estimation formula for Gaussians:

$$\hat{x}(1) = \hat{x}^o(1) + E\{\tilde{x}^o(1)|\tilde{y}^o(1)\}$$

$$E\{\tilde{x}^o(1)|\tilde{y}^o(1)\} = \Lambda_{\tilde{x}^o(1)\tilde{y}^o(1)}\Lambda^{-1}_{\tilde{y}^o(1)\tilde{y}^o(1)}\,\tilde{y}^o(1)$$

Kalman Filter Solution: *k = 1*

- Calculate a-posteriori state estimate using the conditional estimation formula for Gaussians:

$$\hat{x}(1) = \hat{x}^o(1) + E\{\tilde{x}^o(1)|\tilde{y}^o(1)\}$$

$$\hat{x}(1) = \hat{x}^o(1) + \Lambda_{\tilde{x}^o(1)\tilde{y}^o(1)}\Lambda^{-1}_{\tilde{y}^o(1)\tilde{y}^o(1)}\,\tilde{y}^o(1)$$

$$\hat{x}(1) = \hat{x}^o(1) + \Lambda_{\tilde{x}^o(1)\tilde{y}^o(1)}\Lambda^{-1}_{\tilde{y}^o(1)\tilde{y}^o(1)}\,\tilde{y}^o(1)$$

- Calculate $\Lambda_{\tilde{x}^o(1)\tilde{y}^o(1)}$

$$\begin{aligned}\Lambda_{\tilde{x}^o(1)\tilde{y}^o(1)} &= E\{\tilde{x}^o(1)\tilde{y}^{oT}(1)\} \\ &= E\{\tilde{x}^o(1)\,[C\,\tilde{x}^o(1) + v(1)]^T\} \\ &= \underbrace{E\{\tilde{x}^o(1)\tilde{x}^{oT}(1)\}}_{M(1)}\,C^T \\ &= M(1)\,C^T\end{aligned}$$

$$\hat{x}(1) = \hat{x}^o(1) + M(1)\,C^T \Lambda^{-1}_{\tilde{y}^o(1)\tilde{y}^o(1)}\,\tilde{y}^o(1)$$

- Calculate $\Lambda_{\tilde{y}^o(1)\tilde{y}^o(1)}$

$$\begin{aligned}
\Lambda_{\tilde{y}^o(1)\tilde{y}^o(1)} &= E\{\tilde{y}^o(1)\tilde{y}^{oT}(1)\} \\
&= E\{[C\,\tilde{x}^o(1) + v(1)]\,[C\,\tilde{x}^o(1) + v(1)]^T\} \\
&\qquad\qquad (E\{\tilde{x}(1)v(1)\} = 0) \\
&= C\,\underbrace{E\{\tilde{x}^o(1)\tilde{x}^{oT}(1)\}}_{M(1)}\,C^T + \underbrace{E\{v(1)v^T(1)\}}_{V(1)} \\
&= C\,M(1)\,C^T + V(1)
\end{aligned}$$

Kalman Filter Solution: ***k = 1***

- **a-posteriori state** estimate:

$$\hat{x}(1) = \hat{x}^o(1) + \underbrace{\Lambda_{\tilde{x}^o(1)\tilde{y}^o(1)}}_{M(1)C^T}\underbrace{\Lambda^{-1}_{\tilde{y}^o(1)\tilde{y}^o(1)}}_{[CM(1)C^T+V(1)]^{-1}}\tilde{y}^o(1)$$

$$\boxed{\begin{aligned}
\hat{x}(1) &= \hat{x}^o(1) + M(1)C^T\left[C\,M(1)C^T + V(1)\right]^{-1}\tilde{y}^o(1) \\
\tilde{y}^o(1) &= y(1) - C\,\hat{x}^o(1)
\end{aligned}}$$

Kalman Filter Solution: ***k = 1***

Review of the results so far:

$$\tilde{x}^o(1) = A\,\tilde{x}(0) + B_w\,w(0)$$

$$\tilde{y}^o(1) = y(1) - C\,\hat{x}^o(1)$$

$$M(1) = A\,Z(0)A^T + B_w W(0) B_w^T$$

$$\hat{x}(1) = \hat{x}^o(1) + M(1)C^T\left[C\,M(1)C^T + V(1)\right]^{-1}\tilde{y}^o(1)$$

$$Z(1) =$$

Kalman Filter Solution: ***k = 1***

- **A-posteriori state** estimation error:

$$\tilde{x}(1) = x(1) - \hat{x}(1)$$

- **A-posteriori state** estimation error covariance:

$$Z(1) = E\{\tilde{x}(1)\tilde{x}^T(1)\} = \Lambda_{\tilde{x}(1)\tilde{x}(1)}$$

Kalman Filter Solution: ***k = 1***

- **a-posteriori state** estimation covariance:

$$Z(1) \;=\; E\{\tilde{x}(1)\tilde{x}^T(1)\} = \Lambda_{\tilde{x}(1)\tilde{x}(1)}$$

- Use least squares result:

$$\underbrace{\Lambda_{\tilde{x}(1)\tilde{x}(1)}}_{Z(1)} = \underbrace{\Lambda_{\tilde{x}^o(1)\tilde{x}^o(1)}}_{M(1)} - \underbrace{\Lambda_{\tilde{x}^o(1)\tilde{y}^o(1)}}_{M(1)C^T}\underbrace{\Lambda^{-1}_{\tilde{y}^o(1)\tilde{y}^o(1)}}_{[CM(1)C^T+V(1)]^{-1}}\Lambda_{\tilde{y}^o(1)\tilde{x}^o(1)}$$

$$\boxed{Z(1) \;=\; M(1) - M(1)C^T\left[CM(1)C^T + V(1)\right]^{-1} CM(1)}$$

Kalman Filter Solution: ***k = 1***

Review:

$$\hat{x}^o(1) \;=\; A\,\hat{x}(0) + B\,u(0)$$
$$\tilde{y}^o(1) \;=\; y(1) - C\,\hat{x}^o(1)$$
$$M(1) \;=\; A\,Z(0)A^T + B_w W(0) B_w^T$$

$$\hat{x}(1) \;=\; \hat{x}^o(1) + M(1)C^T\left[C\,M(1)C^T + V(1)\right]^{-1}\tilde{y}^o(1)$$

$$Z(1) \;=\; M(1) - M(1)C^T\left[CM(1)C^T + V(1)\right]^{-1} CM(1)$$

Equations are entirely recursive!

Kalman Filter Solution

1) Compute a-priori output estimation error residual:

$$\tilde{y}^o(k) \;=\; y(k) - C\,\hat{x}^o(k) \qquad\qquad \hat{x}^o(0) = x_o$$

2) Compute a-posteriori state estimate:

$$\hat{x}(k) \;=\; \hat{x}^o(k) + M(k)C^T\left[C\,M(k)C^T + V(k)\right]^{-1}\tilde{y}^o(k)$$

$$M(0) \;=\; X_o$$

3) Update a-priori state estimate:

$$\hat{x}^o(k+1) \;=\; A\,\hat{x}(k) + B\,u(k)$$

Kalman Filter Solution

4) Compute a-posteriori state estimation error covariance:

$$Z(k) \;=\; M(k) - M(k)C^T\left[CM(k)C^T + V(k)\right]^{-1} CM(k)$$

$$M(0) \;=\; X_o$$

5) Update a-priori state estimation error covariance:

$$M(k+1) \;=\; A\,Z(k)A^T + B_w W(k) B_w^T$$