UNIVERSITY OF CALIFORNIA AT BERKELEY
Department of Mechanical Engineering
ME233 Advanced Control Systems II
Spring 2011

Midterm Examination I

Your Name:

Closed book and closed notes.
Two double-sided sheets (i.e. 4 pages) of handwritten notes on 8.5" × 11" paper are allowed.
Please answer all questions.

Problem:	1	2	3	Total
Max. Grade:	30	40	30	100
Grade:				

Problem 1

Let X be an n-dimensional random vector and Y be a m-dimensional random vector with

$$E\{X\} = m_X$$
$$E\{Y\} = m_Y$$
$$E\left\{\begin{bmatrix} X - m_X \\ Y - m_Y \end{bmatrix} \begin{bmatrix} (X - m_X)^T & (Y - m_Y)^T \end{bmatrix}\right\} = \begin{bmatrix} \Lambda_{XX} & \Lambda_{XY} \\ \Lambda_{YX} & \Lambda_{YY} \end{bmatrix} .$$

Now consider the linear estimator of X given by

$$Z = AY + b$$

where $A \in \mathcal{R}^{n \times m}$ and $b \in \mathcal{R}^{n \times 1}$ are the estimator parameters to be designed. Also consider the corresponding estimation error

$$J = X - Z .$$

Denote the mean of J as m_J and the covariance of J as Λ_{JJ}.

1. Find m_J and Λ_{JJ} as functions of the matrix A and the vector b.

2. Under the assumption that Λ_{YY} is invertible, determine values of A and b so that $m_J = 0$ and trace$[\Lambda_{JJ}]$ is minimized.

   Hint: Use the technique of completing the square to express

   $$\Lambda_{JJ} = (A - M_1)M_2(A - M_1)^T + M_3$$

   for some matrices M_1, M_2, and M_3, where $M_2 \succ 0$, and $M_3 = M_3^T$. None of these three matrices should depend on A.

Problem 2

Consider the discrete-time linear time-invariant system

$$\begin{aligned} x(k+1) &= Ax(k) + Bd(k) \\ p(k) &= Cx(k) \end{aligned}$$

and the cost function

$$J_m[x_m, N] = \sum_{k=m}^{N-1} \left\{ p^T(k)p(k) - d^T(k)d(k) \right\} \quad \text{s.t.} \quad x(m) = x_m$$

which is defined for $m = 0, 1, \ldots, N-1$. In this problem, we are interested in the worst-case (i.e. largest) possible value of this cost function over all choices of d, i.e. we are interested in solving the optimization problems

$$J_m^o[x_m, N] = \max_{d(m),\ldots,d(N-1)} J_m[x_m, N]$$

for $m = 0, 1, \ldots, N-1$.

Let the sequence of matrices $P_0, P_1, P_2, \ldots$ satisfy the discrete Riccati difference equation

$$P_{k+1} = A^T P_k A + C^T C - A^T P_k B (B^T P_k B - I)^{-1} B^T P_k A \,.$$

with the initial condition $P_0 = 0$. Also assume that this sequence of matrices satisfies the conditions

- $B^T P_k B - I \prec 0, \quad k = 0, 1, 2, \ldots$
- $\lim_{k \to \infty} P_k = P_\infty$

1. Use dynamic programming to prove that $J_m^o[x_m, N] = x_m^T P_{(N-m)}\, x_m$. You may find it convenient to define $J_N^o[x_N, N] = x_N^T P_0\, x_N = 0$ to facilitate the proof.

2. Let $x(0) = 0$. Show that whenever $\sum_{k=0}^{\infty} d^T(k)d(k)$ is finite and nonzero, then

$$\left(\frac{\sum_{k=0}^{\infty} p^T(k)p(k)}{\sum_{k=0}^{\infty} d^T(k)d(k)} \right) \leq 1 \,.$$

Problem 3

Consider the discrete-time linear time-invariant system

$$x(k+1) = Ax(k) + Bu(k) + B_w w(k) \tag{1}$$

where $u(k)$ is the control input, $x(0)$ is Gaussian, and $w(k)$ is white Gaussian noise. The relevant means and covariances are given by

$$\begin{aligned} E\{x(0)\} &= x_0 & E\{[x(0)-x_0][x(0)-x_0]^T\} &= X_0 \\ E\{w(k)\} &= 0 & E\{w(k+j)w^T(k)\} &= W\delta(j) \\ & & E\{[x(0)-x_0]w^T(k)\} &= 0\ . \end{aligned}$$

In this problem, we will design a control policy $u(k)$ for this system; throughout this problem, $u(k)$ is only allowed to be a function of $x(0), \ldots, x(k)$ and $w(0), \ldots, w(k)$.

1. In this problem, we would like to find the control policy for the system (1) that minimizes the cost function

   $$J_N = E\left\{ x^T(N)Q_f\, x(N) + \sum_{k=0}^{N-1} \left[x^T(k)C^TCx(k) + u^T(k)Ru(k) \right] \right\}$$

   where $Q_f \succeq 0$, $R \succ 0$. This optimal control problem can be reformulated as an optimal <u>state feedback</u> LQG control problem in which the state dynamics have the form

   $$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}(k)u(k) + \bar{B}_w\eta(k) \tag{2}$$

   and the cost function has the form

   $$J_N = E\left\{ \bar{x}^T(N)\bar{Q}_f\, \bar{x}(N) + \sum_{k=0}^{N-1} \left[\bar{x}^T(k)\bar{C}^T\bar{C}\bar{x}(k) + u^T(k)Ru(k) \right] \right\}\ . \tag{3}$$

   Obtain expressions for $\bar{A}$, $\bar{B}$, $\bar{B}_w$, $\bar{C}$, $\bar{Q}_f$, $\bar{x}(k)$, and $\eta(k)$ that correspond to this reformulation. Also show that $\eta(k)$ is uncorrelated with $\bar{x}(0)$ for all $k \geq 0$.

2. Find the optimal state feedback control policy $u^o(k)$ for the system (2) that minimizes the cost function (3). Leave the resulting expressions in terms of $\bar{A}$, $\bar{B}$, $\bar{B}_w$, $\bar{C}$, $\bar{Q}_f$, and $\bar{x}(k)$. You do not need to find the associated optimal cost.

3. Consider the cost function

   $$\begin{aligned} J' &= \lim_{N\to\infty} \left[\frac{1}{N} E\left\{ \bar{x}^T(N)\bar{Q}_f\, \bar{x}(N) + \sum_{k=0}^{N-1} \left[\bar{x}^T(k)\bar{C}^T\bar{C}\bar{x}(k) + u^T(k)Ru(k) \right] \right\} \right] \\ &= E\left\{ \bar{x}^T(k)\bar{C}^T\bar{C}\bar{x}(k) + u^T(k)Ru(k) \right\}\ . \end{aligned} \tag{4}$$

   List a set of conditions that guarantees the existence of a stabilizing state feedback control policy for the system (2) that minimizes the cost function (4). Leave the conditions in terms of $\bar{A}$, $\bar{B}$, $\bar{B}_w$, $\bar{C}$, and $\bar{x}(k)$. You do not need to find an optimal control policy or the corresponding optimal cost.

This material will not be tested in midterm 1 this year