University of California
Department of Mechanical Engineering

ME233 Advanced Control Systems II   Spring 2013

Final Examination   May 16, 2013 (Th)
5 Problems: start with the one that you are most comfortable with.
Open reader, Open notes; you may also refer to your own summary sheets for midterm exams.

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[1] (15 points) Consider a first-order system

$$x(k+1) = ax(k) + bu(k) + b_w w(k)$$

Suppose we have two sensors:

$$y_1(k) = c_1 x(k) + v_1(k)$$
$$y_2(k) = c_2 x(k) + v_2(k)$$

We have the following conditions: a, b, c_1, c_2, b_w are nonzero and

$$E\left[x(0)\right] = x_0,\ E\left[\left(x(0) - x_0\right)^2\right] = X_0$$

$w(k)$ is white, Gaussian, with zero mean and variance $\sigma_w^2 > 0$;

$v_1(k)$ and $v_2(k)$ are independent Gaussian random processes. $v_1(k)$ is white, $E\left[v_1(k)\right] = 0$, $E\left[v_1(k)^2\right] = \sigma_1^2$; $v_2(k)$ has a bias $E\left[v_2(k)\right] = l_0 > 0$, $E\left[(v_2(k) - l_0)^2\right] = \sigma_2^2$, but $E\left[(v_2(k) - l_0)(v_2(k+\tau) - l_0)\right] = 0$ for any $\tau \neq 0$. Assume also that $\sigma_1^2 > \sigma_2^2 > 0$, namely, the first sensor has no bias but larger variance and the second sensor has a bias but smaller variance. Read all the questions before you start to write down the solutions.

a). [4 points] Suppose that we use only the first sensor, namely, $y(k) = y_1(k)$. Obtain the Kalman filter and its initial conditions to estimate $x(k)$. Use $f_1(k)$ to denote the Kalman filter gain and $m_1(k)$ and $z_1(k)$ to denote the estimation error variances.

b). [7 points] Suppose now that we use both sensors, namely, $y(k) = \left[y_1(k), y_2(k)\right]^T = Cx(k) + v(k)$, where $C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ and $v(k) = \begin{bmatrix} v_1(k) \\ v_2(k) \end{bmatrix}$. Obtain the Kalman filter and its initial conditions. Notice that the second sensor has a bias. Use $f_2(k)$ to denote the Kalman filter gain and $m_2(k)$ and $z_2(k)$ to denote the estimation error variances.

c). [4 points] What are the conditions for the Kalman filters in a) and b) to have steady states? Explain whether they are satisfied or not.

[2] (20 points) Series Model Reference Adaptive Systems (MRAS)

Consider a single-input, single-output system described by

$$y(k+1) = -\sum_{i=1}^{2} a_i y(k+1-i) + \sum_{j=0}^{2} b_j u(k-j), \qquad b_0 > 0 \tag{1}$$

where $|b_1/b_0| + |b_2/b_0| < 1$.

Note that Eq. (1) may be expressed as

$$\begin{aligned} u(k-1) &= [(1+a_1 z^{-1} + a_2 z^{-2})y(k) - (b_1 + b_2 z^{-1})u(k-2)]/b_0 \\ &= (\alpha_0 + \alpha_1 z^{-1} + \alpha_2 z^{-2})y(k) - (\beta_1 + \beta_2 z^{-1})u(k-2) \end{aligned}$$

where $\alpha_i = a_i / b_0$ and $\beta_j = b_j / b_0$.

Consider a series adjustable system that reconstructs the input from the output: i.e. a series adjustable system is an estimate of the system inverse. At time *k*, the reconstructed $u(k-1)$ is

$$\hat{u}(k-1) = \hat{\alpha}_0(k)y(k) + \hat{\alpha}_1(k)y(k-1) + \hat{\alpha}_2(k)y(k-2) - \hat{\beta}_1(k)\hat{u}(k-2) - \hat{\beta}_2(k)\hat{u}(k-3) \tag{2}$$

The parameter adaptation algorithm is given by

$$\begin{aligned} \hat{\alpha}_i(k) &= \hat{\alpha}_i(k-1) + k_{\alpha i} y(k-i)\varepsilon(k), \quad k_{\alpha i} > 0, \quad i = 0,1,2 \\ \hat{\beta}_j(k) &= \hat{\beta}_j(k-1) - k_{\beta i}\hat{u}(k-j-1)\varepsilon(k), \quad k_{\beta j} > 0, \quad j = 1,2 \\ \varepsilon(k) &= u(k-1) - \hat{u}(k-1) \end{aligned} \tag{3}$$

a) [13 points] Prove that the adaptation algorithm is asymptotically stable: i.e. $\lim_{k\to\infty} \varepsilon(k) = 0$.
b) [7 points] Explain that the algorithm (3) is not implementable, and provide an implementable algorithm.

[3] (20 points) Consider a discrete-time system described by

$$Y(z) = \frac{z^{-1}(b_0 + b_1 z^{-1})}{1 + a_1 z^{-1} + a_2 z^{-2}} U(z) \tag{1}$$

where the system parameters are unknown. Note that the input-output difference equation is

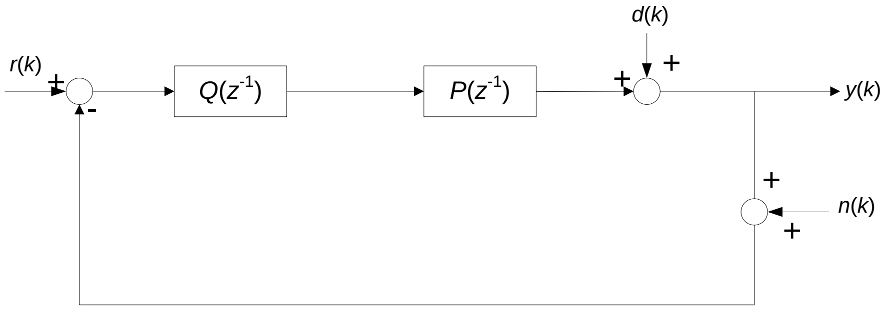
$$y(k) = -a_1 y(k-1) - a_2 y(k-2) + b_0 u(k-1) + b_1 u(k-2) \tag{2}$$

It is known that for a step input, $u(k) = 1$ ($k \geq 0$), the asymptotic final value of $y(k)$ is 2. Utilize this information to express the system in the form

$$w(k) = [a_1 \quad a_2 \quad b_0]\varphi(k)$$

and obtain the least square algorithm to identify the parameter vector $[a_1 \quad a_2 \quad b_0]$. Write down the estimate of b_1 also.

[4] (30 points) Consider the following general control structure



The stable plant is given by

$$P(z^{-1}) = \frac{b_0 z^{-1}(1+z^{-1})}{(1-0.9z^{-1})^2}$$

where $b_0>0$.

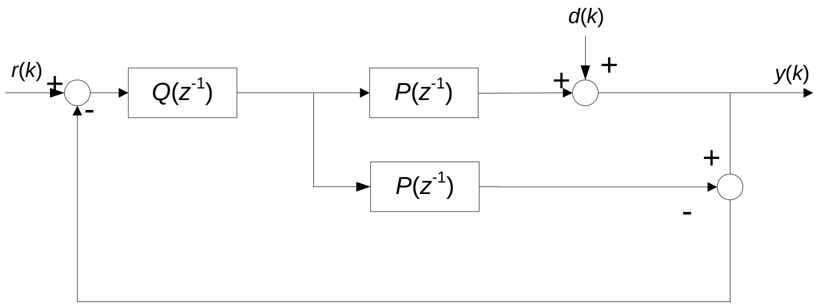
Assume that the disturbance $d(k)$ contains a significant amount of low-frequency components, while the sensor noise $n(k)$ dominates at high frequencies.

a) [4 points] Consider the desired *magnitude response* of $P(z^{-1})Q(z^{-1})$ for attenuation of the disturbance $d(k)$. Circle one option in your answer sheet for each of the design considerations below:

At low frequencies:	large gain	or	small gain
At high frequencies:	large gain	or	small gain

b) [6 points] If in particular, $d(k) = l_0$ (a constant disturbance). Use pole placement to design a controller $Q(z^{-1})=R(z^{-1})/M(z^{-1})$ that provides perfect rejection of $d(k)$. Place two closed-loop poles at 0.8 and the remaining poles at the origin, namely the closed-loop characteristic polynomial is $D(z^{-1}) = (1-0.8z^{-1})^2$. Utilize the Diophantine equation and determine $Q(z^{-1})$. (Hint: you need an I action)

For the remaining part of the problem, consider adding one block $P(z^{-1})$ in the controller structure as shown below:



c) Suppose we want to use this control structure to achieve the same goal of rejecting the low-frequency disturbance $d(k)$.
   1) [6 points] $Q(z^{-1})$ should be stable and approximate the inverse of $P(z^{-1})$ at low frequencies. Explain why these two conditions are required.
   2) [4 points] Use the zero-phase-error-tracking (ZPET) idea to design this inverse-approximation filter $Q(z^{-1})$. Consider $1+z^{-1}$ as uncancellable in the ZPET design.
   3) [4 points] With the designed $Q(z^{-1})$, obtain the complementary sensitivity function. If the desired output is $y_d(k)$, design the reference $r(k)$.

d) [6 points] Now assume there are actually some uncertainties and the actual plant is $\tilde{P}(z^{-1}) = P(z^{-1})[1+\Delta(z^{-1})]$. Assume $\Delta(z^{-1})$ is stable and bounded. Derive the closed-loop robust stability condition.

[5] (15 points) Consider the discrete feedback system sketched below.

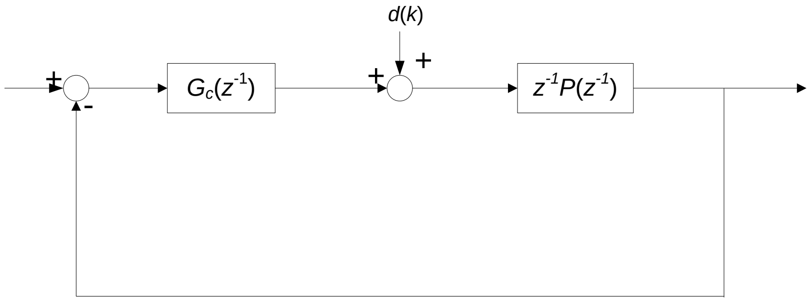


Fig. 1 Discrete Time Feedback System

The controller has been designed to provide the closed-loop stability and a baseline performance. The plant transfer function is written as

$$z^{-1}P(z^{-1}) = z^{-1}\frac{B(z^{-1})}{A(z^{-1})}$$

where $A(z^{-1}) = 1 + a_1 z^{-1} + a_2 z^{-2}$, $B(z^{-1}) = b_0 + b_1 z^{-1} + b_2 z^{-2}$, $b_0 > 0$, and all roots of $B(z^{-1}) = 0$ are inside the unit circle.

To enhance the closed loop performance, the disturbance observer has been added as shown in the following figure.

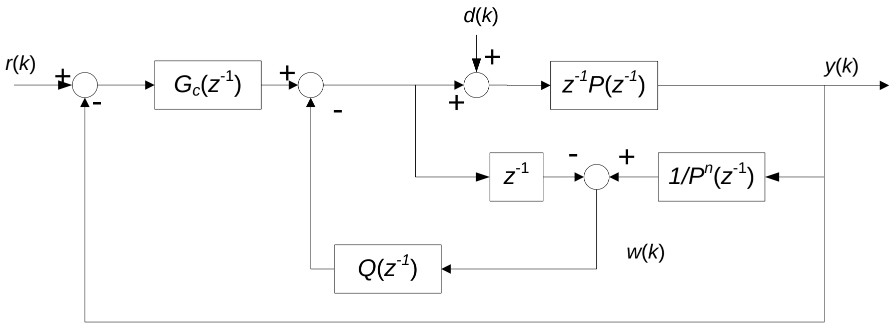


Fig. 2 Discrete Feedback System with Disturbance Observer

In the figure, $P^n(z^{-1})$ is the nominal model of $P(z^{-1})$. For simplicity, assume $P(z^{-1})$ = $P^n(z^{-1})$.

a) [6 points] Show that, when $z^{-1}P(z^{-1}) = z^{-1}P^n(z^{-1})$, the sensitivity function of the closed-loop system with the disturbance observer, is given by

$$S^n_{wDOB}(z^{-1}) = S^n(z^{-1})(1 - z^{-1}Q(z^{-1}))$$

where $S^n(z^{-1})$ is the sensitivity function of the feedback system in Fig. 1.

b) [9 points] Design $Q(z^{-1})$ so that constant disturbances and sinusoidal disturbances with a **known** frequency ω, i.e. $d(k) = d_c + c\sin(\omega k + \varphi)$ where d_c, c and φ are not known, are perfectly rejected.

Extra credit question: do the following only if you have completed [1]-[5] and are comfortable with your answers to the previous questions.

[E1]. (maximum 10 extra points) [Problem 5 continued] Now consider the case where the disturbance frequency ω in 5b) is unknown. Assume $r(k) = 0$. Derive an adaptive approach such that the output $y(k)$ converges asymptotically to zero.

Hint: under the problem assumptions, you can prove that:

$$y(k) = (1 - q^{-1}Q(q^{-1}))q^{-1}P(q^{-1})S^n(q^{-1})d(k)$$

and $w(k) = d(k-1)$, where q^{-1} is the one-step delay operator. Hence

$$y(k) = (1 - q^{-1}Q(q^{-1}))P(q^{-1})S^n(q^{-1})w(k) = (1 - q^{-1}Q(q^{-1}))w_1(k)$$

where we have defined $w_1(k) = P(q^{-1})S^n(q^{-1})w(k)$. You can use

$$y(k) = (1 - q^{-1}Q(q^{-1}))w_1(k)$$

as the model for adaptation.