[1] 1. The algebraic Riccati equation for the stationary LQ problem:

$$A^T PA - P + A^T PB\left[R + B^T PB\right]^{-1} B^T PA + C^T C = 0 \text{, with}$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 2 \end{bmatrix}.$$

The positive definite solution of the equation is

$$P_+ = \begin{bmatrix} 1 & 2 \\ 2 & \frac{1}{2}\left[5 - R + \sqrt{R^2 + 10R + 9}\right] \end{bmatrix}.$$

2. When $R = 0$, $P_+ = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and the optimal feedback control law is $u(k) = -\begin{bmatrix} 0 & \frac{1}{2} \end{bmatrix} x(k)$.

The resulting closed loop state equation is

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix} x(k), \text{ i.e. } \begin{cases} x_1(k+1) = x_2(k) \\ x_2(k+1) = -\frac{1}{2} x_2(k) \end{cases}.$$

Suppose the initial condition is $x_1(0) = x_1$ and $x_2(0) = x_2$. Then the output of the optimal closed loop system is

$$y(k) = \begin{cases} x_1 + 2x_2, & \text{for } k = 0 \\ 0, & \text{for } k > 0 \end{cases}$$

3. The open loop transfer function from $u(k)$ to $y(k)$ is given by

$$G(z) = C(zI - A)^{-1} B = \frac{2z + 1}{z^2}.$$

So $G(z^{-1}) = \dfrac{2z^{-1} + 1}{z^{-2}} = z(z+2)$. The symmetric root locus is determined by the equation:

$$1 + \frac{1}{R} \frac{2z(z + \frac{1}{2})(z+2)}{z^2} = 0.$$

The root locus is shown in Fig. 1 below.

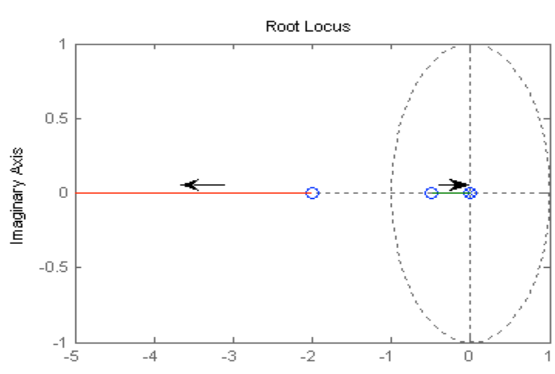
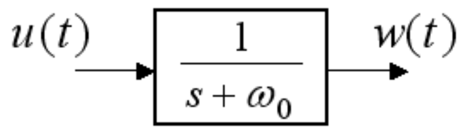


Fig. 1 Root locus plot. Arrows point in the direction of increasing R.

[2] We can factorize the spectral density of $w(t)$ as

$$\Phi_{ww}(\omega) = \frac{1}{(j\omega + \omega_0)(-j\omega + \omega_0)}.$$

So the colored noise $w(t)$ can be considered as the output of the stable linear system ($\omega_0 \geq 0$):



where $u(t)$ is white with zero mean and unit variance. Then

$$\frac{dw(t)}{dt} = -\omega_0 w(t) + u(t).$$

Combining $w(t)$ and $x(t)$, we have the following augmented system

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & \omega_0 \end{bmatrix}\begin{bmatrix} x(t) \\ w(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).$$

Then at the steady state, the covariance matrix, X_{ss}, of $\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}$ satisfies the Lyapunov equation:

$$\begin{bmatrix} a & b \\ 0 & \omega_0 \end{bmatrix} X_{ss} + X_{ss}\begin{bmatrix} a & b \\ 0 & \omega_0 \end{bmatrix}^T = -\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}.$$

The solution is

$$X_{ss} = \begin{bmatrix} \dfrac{b^2}{2a\omega_0(a-\omega_0)} & \dfrac{b}{2\omega_0(\omega_0 - a)} \\ \dfrac{b}{2\omega_0(\omega_0 - a)} & \dfrac{1}{2\omega_0} \end{bmatrix}.$$

Notice that $x(t)$ is zero mean at the steady state. Therefore, $E[x^2(t)]$ is the (1, 1) element of X_{ss}:

$$E[x^2(t)] = \frac{b^2}{2a\omega_0(a-\omega_0)}.$$

[3] The least square estimate of x given y is $E[x \mid y] = E(x) + X_{xy}X_{yy}^{-1}[y - E(y)]$, where

$$E[x] = 0,\ E[y] = 0,\ X_{xy} = E[xy^T] = XC^T + S,\ X_{yy} = E[yy^T] = CXC^T + CS + S^TC^T + V.$$

Thus, $E[x \mid y] = (XC^T + S)(CXC^T + CS + S^TC^T + V)^{-1} y$. The estimation error covariance matrix is given by

$$X_{\tilde{x}\tilde{x}} = X_{xx} - X_{xy}X_{yy}^{-1}X_{yx} = X - (XC^T + S)(CXC^T + CS + S^TC^T + V)^{-1}(CX + S^T).$$

[4] From the ordinary Kalman filter, we can compute $\hat{x}(k+1\,|\,k)$. Denote $Y_k = [y(0)\ \dots\ y(k)]^T$.

We have

$$\begin{aligned}
&\hat{x}(k+2\,|\,k) = E[x(k+2)\,|\,Y_k] = E[(Ax(k+1) + Bu(k+1) + B_w w(k+1))\,|\,Y_k] \\
&= E\big[Ax(k+1)\,|\,Y_k\big] + E\big[Bu(k+1)\,|\,Y_k\big] + E\big[B_w w(k+1)\,|\,Y_k\big] \\
&= AE\big[x(k+1)\,|\,Y_k\big] + Bu(k+1) \\
&= A\hat{x}(k+1\,|\,k) + Bu(k+1)
\end{aligned}$$

Similarly, we can get

$$\hat{x}(k+3\,|\,k) = A\hat{x}(k+2\,|\,k) + Bu(k+2) = A^2\hat{x}(k+1\,|\,k) + ABu(k+1) + Bu(k+2).$$

We also know from the Kalman filter that

$$M(k+1) = E\big[(x(k+1) - \hat{x}(k+1\,|\,k))(x(k+1) - \hat{x}(k+1\,|\,k))^T\big].$$

From the expression of the estimation error,

$$\begin{aligned}
&\tilde{x}(k+3\,|\,k) = x(k+3) - \hat{x}(k+3\,|\,k) \\
&\quad = [A^2 x(k+1) + ABu(k+1) + Bu(k+2) + AB_w w(k+1) + B_w w(k+2)] \\
&\quad\quad - [A^2\hat{x}(k+1\,|\,k) + ABu(k+1) + Bu(k+2)] \\
&\quad = A^2\tilde{x}(k+1\,|\,k) + AB_w w(k+1) + B_w w(k+2),
\end{aligned}$$

we can obtain the estimation covariance matrix associated with $\hat{x}(k+3\,|\,k)$:

$$\begin{aligned}
&E[\tilde{x}(k+3\,|\,k)\tilde{x}^T(k+3\,|\,k)] = A^2 E[\tilde{x}(k+1\,|\,k)\tilde{x}^T(k+1\,|\,k)](A^T)^2 + AB_w W B_w^T A^T + B_w W B_w^T \\
&= A^2 M(k+1)(A^T)^2 + AB_w W B_w^T A^T + B_w W B_w^T
\end{aligned}$$