ME233 Advance Control II
Lecture 2

Review of
ME 232 Lectures 25 & 26

Linear Quadratic Regulators (LQR)
PART II

(ME232 Class Notes pp. 135-137)

Outline

Previous lecture :

- Dynamic programming
- Solution of finite-horizon LQR

This Lecture: Review ME232 results on

- Infinite horizon LQR (steady state)
  - Stability margins
  - Reciprocal root locus

Infinite Horizon LQ regulator

LTI system:

$$x(k+1) = A\,x(k) + B\,u(k) \qquad x(0) = x_o$$

LQR that minimizes the cost:

$$J[x_o] = \frac{1}{2}\sum_{k=0}^{\infty}\left\{x^T(k)\,Q\,x(k) + u^T(k)\,R\,u(k)\right\}$$

$$Q = C^T C \succeq 0 \qquad R \succ 0$$

Infinite Horizon (IH) LQ regulator

Assume that ***[A,B]*** stabilizable and ***[A,C]*** detectable,

- Optimal, asymptotically stable, close loop system

$$x(k+1) = [A - B\,K]\;x(k) \qquad x(0) = x_o$$

$$K = \left[R + B^T P B\right]^{-1} B^T P A$$

Algebraic Riccati Equation (ARE)

$$P = Q + A^T P A - A^T P B\left[R + B^T P B\right]^{-1} B^T P A$$

Infinite Horizon LQ Regulator

Lets analyze the stability and robustness properties of the closed loop system:

$$x(k+1) \;=\; A\,x(k) + B\,u(k)$$

$$u(k) \;=\; -K\,x(k) + v(k)$$

With fictitious reference input ***v(k)***

$$v(k) \;=\; v_o \;= 0$$

Infinite Horizon LQ Regulator

Use the Z-transform:

$$X(z) \;=\; (zI - A)^{-1} B\,U(z)$$

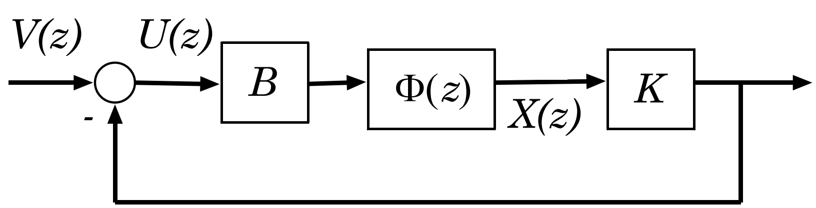
$$U(z) \;=\; -K\,U(z) + V(z)$$

Define

$$\Phi(z) = (zI - A)^{-1}$$

Infinite Horizon LQ Regulator

Close loop system block diagram:



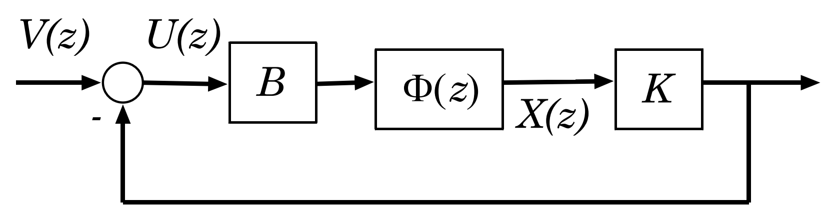
$$\Phi(z) = (zI - A)^{-1}$$

$$X(z) \;=\; \Phi(z) B\,U(z)$$

$$U(z) \;=\; V(z) - K\,X(z)$$

Infinite Horizon LQ Regulator

Close loop system block diagram:

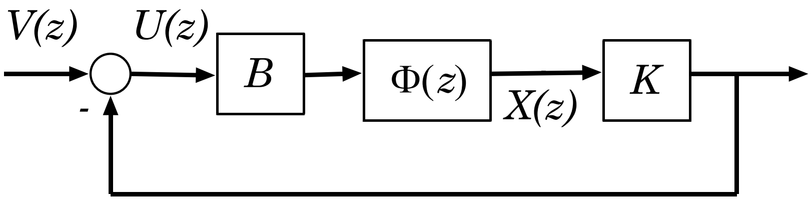


Open loop transfer function:

$$G_o(z) = K\Phi(z)B$$

Infinite Horizon LQ Regulator

Close loop system block diagram:

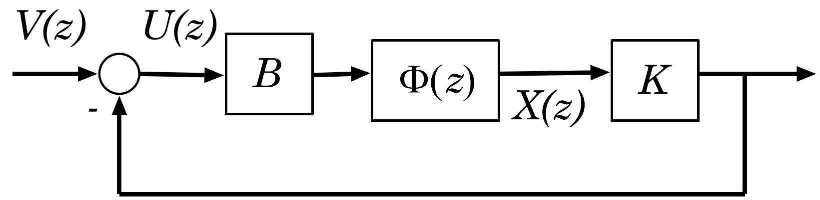


Close loop sensitivity transfer function
(from *V(z)* to *U(z))* :

$$S(z) = [I + K\Phi(z)B]^{-1}$$

Infinite Horizon LQ Regulator

Close loop system block diagram:

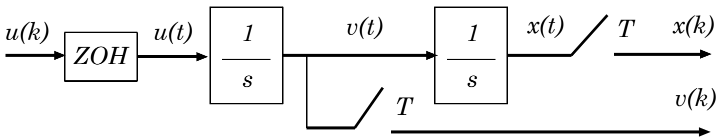


For Single Input Systems $u(k) \in \mathcal{R}$

$$S(z) = \frac{1}{1 + K\Phi(z)B}$$

Example – Double Integrator

Double integrator with ZOH and sampling time T =1:

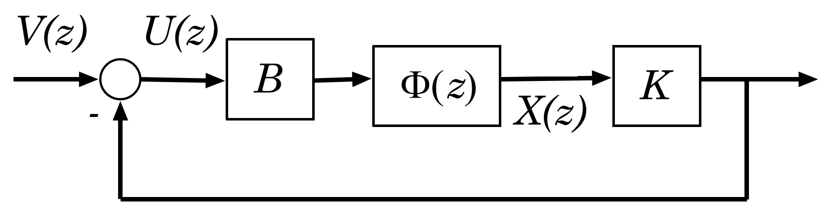


$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} u(k)$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u(k)$$

Example – Double Integrator

Close loop system block diagram:

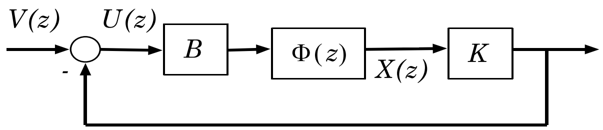


$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \qquad J = \frac{1}{2} \sum_{k=0}^{\infty} \left\{ y^2(k) + R\,u^2(k) \right\}$$

Example – Double Integrator

Close loop system block diagram:



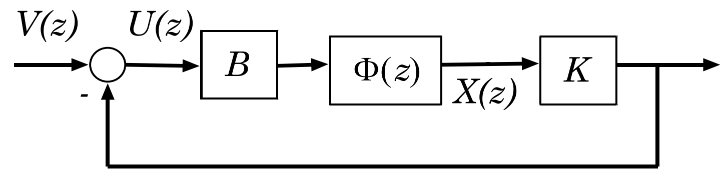
For $R = 10$ we obtained $K = \begin{bmatrix} 0.21 & 0.65 \end{bmatrix}$

Open loop transfer function:

$$G_o(z) = K\Phi(z)B = \begin{bmatrix} 0.21 & 0.61 \end{bmatrix} \begin{bmatrix} (z-1) & -1 \\ 0 & (z-1) \end{bmatrix}^{-1} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

$$= \frac{0.76\,z - 0.55}{(z-1)^2}$$

Example – Double Integrator

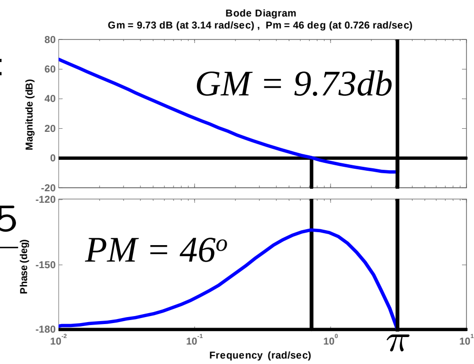


$R = 10$

Open loop transfer function:

$$G_o(j\omega) = K\Phi(j\omega)B$$

$$= \frac{0.76\,e^{j\omega} - 0.55}{(e^{j\omega}-1)^2}$$

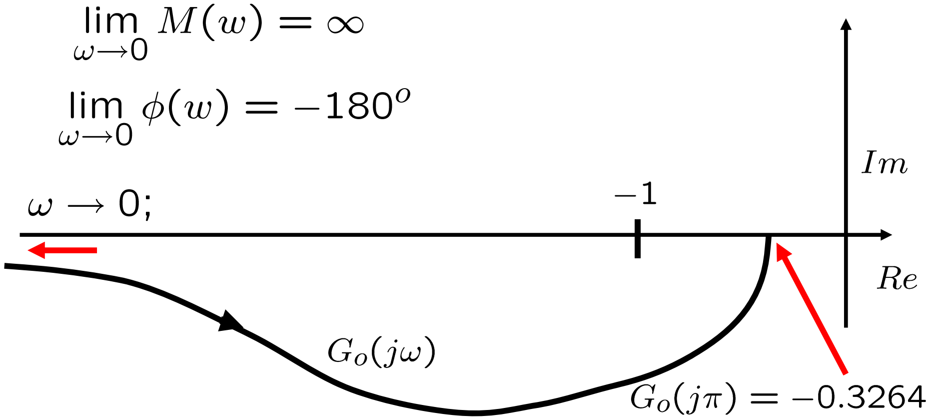


Example – Nyquist plot

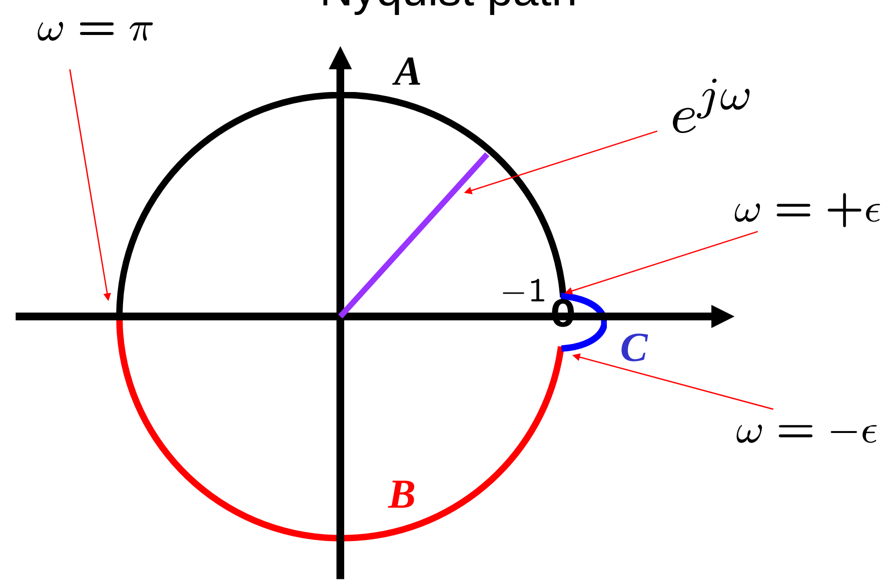
$$G_o(j\omega) = \frac{0.76\,e^{j\omega} - 0.55}{(e^{j\omega}-1)^2} = M(w)\,e^{j\phi(w)}$$

$$\lim_{\omega \to 0} M(w) = \infty$$

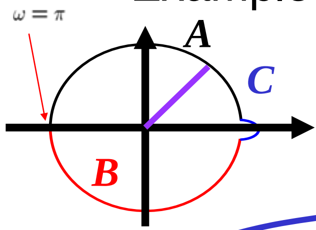
$$\lim_{\omega \to 0} \phi(w) = -180^o$$



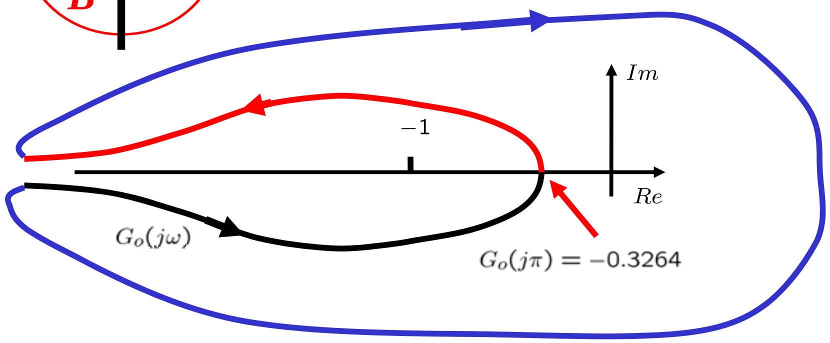
Nyquist path



Example – Nyquist Theorem

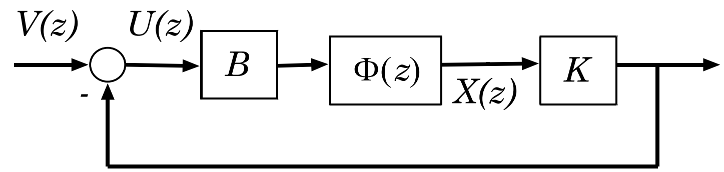


$$G_o(j\omega) = \frac{0.76\, e^{j\omega} - 0.55}{(e^{j\omega} - 1)^2}$$



$$GM_{db} = 20\log_{10}(1/0.326) = 9.74db$$

Example – Double Integrator

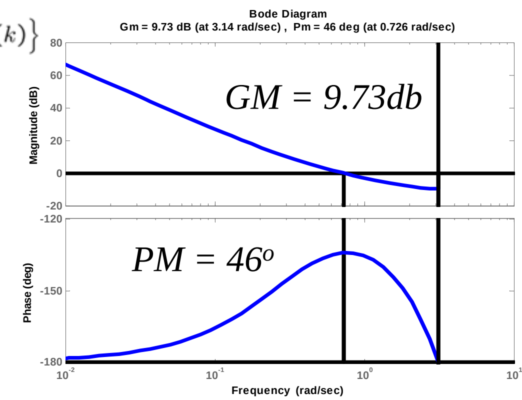


$$J[x_o] = \frac{1}{2}\sum_{k=0}^{\infty}\left\{y^2(k) + u^T(k)\,R\,u(k)\right\}$$

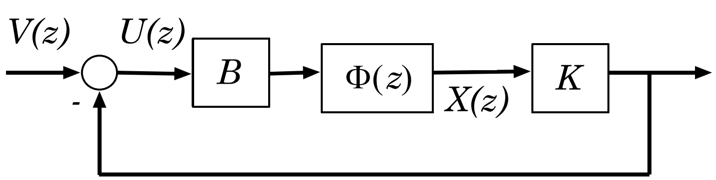
$$R = 10$$

Bode Plot

$$G_o(j\omega) = \frac{0.76\, e^{j\omega} - 0.55}{(e^{j\omega} - 1)^2}$$



Example – Double Integrator

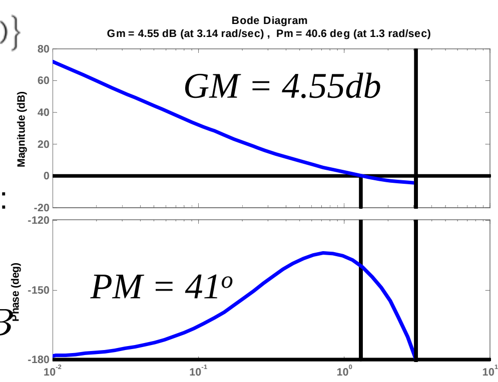


$$J[x_o] = \frac{1}{2}\sum_{k=0}^{\infty}\left\{y^2(k) + u^T(k)\,R\,u(k)\right\}$$

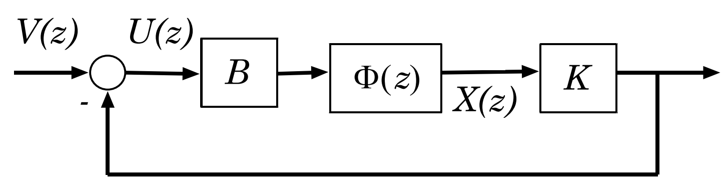
$$R = 0.1$$

Open loop transfer function:

$$G_o(j\omega) = K\Phi(j\omega)B$$



Example – Double Integrator

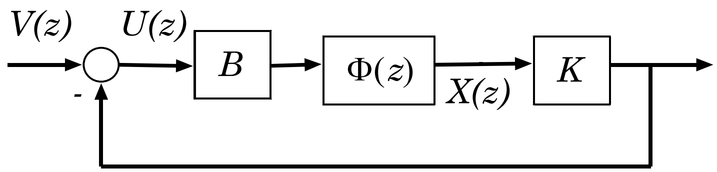


$$R = 10$$

Sensitivity transfer function:

$$S(z) = \frac{1}{1 + K\Phi(z)B} = \frac{z^2 - 2z + 1}{z^2 - 1.24z + 0.45}$$

Example – Double Integrator

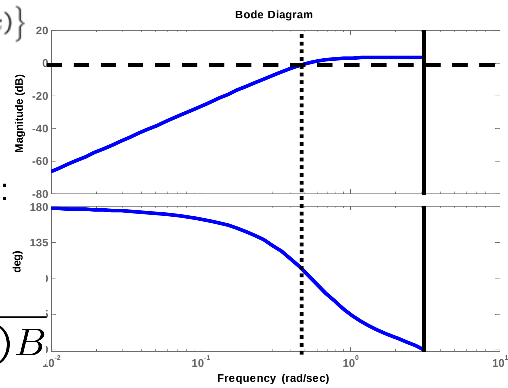


$$J[x_o] = \frac{1}{2}\sum_{k=0}^{\infty}\left\{y^2(k) + u^T(k)\,R\,u(k)\right\}$$

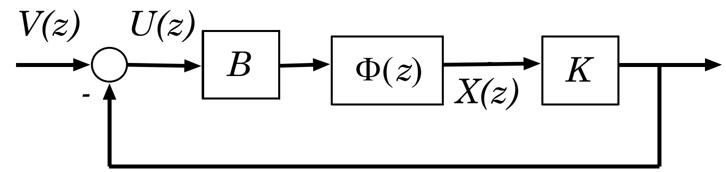
$R = 10$

Sensitivity transfer function:

$$S(j\omega) = \frac{1}{1 + K\Phi(j\omega)B}$$



Example – Double Integrator

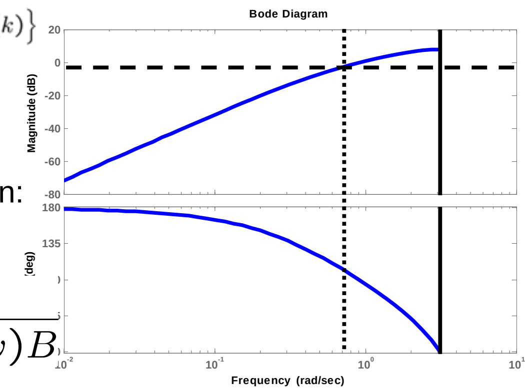


$$J[x_o] = \frac{1}{2}\sum_{k=0}^{\infty}\left\{y^2(k) + u^T(k)\,R\,u(k)\right\}$$

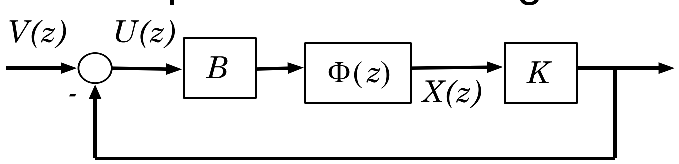
$R = 0.1$

Open loop transfer function:

$$S(j\omega) = \frac{1}{1 + K\Phi(j\omega)B}$$



Example – Double Integrator

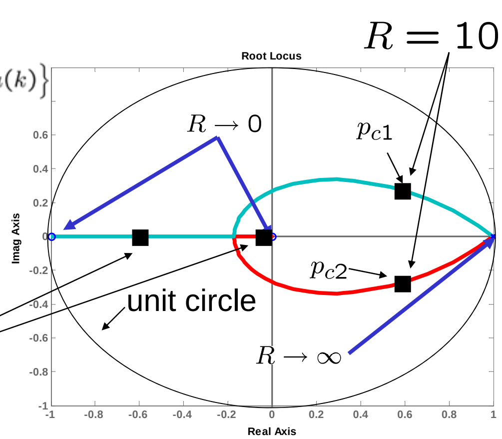


$$J[x_o] = \frac{1}{2}\sum_{k=0}^{\infty}\left\{y^2(k) + u^T(k)\,R\,u(k)\right\}$$

Close loop poles for varying values of

$R \in (\infty, 0)$

$R = 0.1$

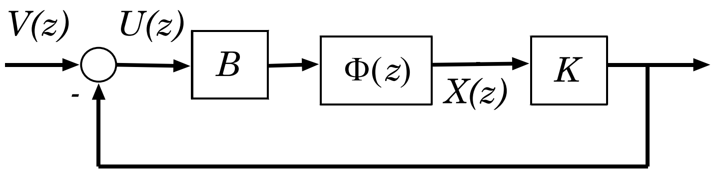


Stability and Robustness of LQR

- For Single Input LQR systems, $(\,u(k) \in \mathcal{R}\,)$
- Guaranteed open loop frequency response gain and phase margins can be determined in close form.
- Locus of the LQR close loop poles as a function of varying $R \in (\infty, 0)$ can be easily plotted

LQR Return difference equality

Return difference for LQR



Open loop transfer function:

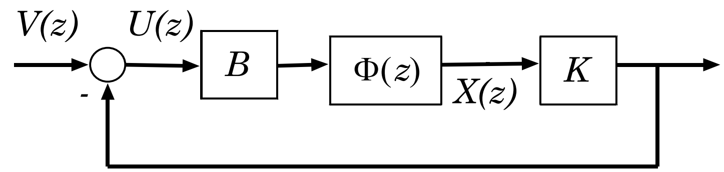
$$G_o(z) = K\Phi(z)B$$

Close loop sensitivity transfer function ($V(z)$ to $U(z)$)

$$S(z) = [I + K\Phi(z)B]^{-1} = [I + G_o(z)]^{-1}$$

$$S(z) = [\text{return difference}]^{-1}$$

Return difference for LQR



Open loop transfer function:

$$G_o(z) = K\Phi(z)B$$

Return difference: $[I + K\Phi(z)B] = [I + G_o(z)]$

Return difference for LQR

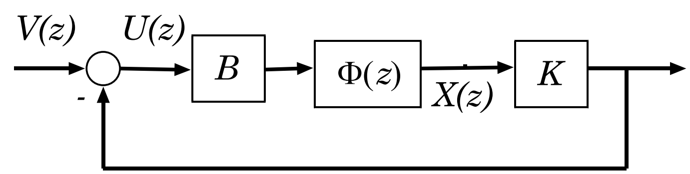
Close loop poles are the zeros of the return difference

Open loop poles are the poles of the return difference

$$\begin{aligned}\text{Det}[I + G_o(z)] &= \text{Det}[I + K\Phi(z)B] \\ &= \text{Det}[I + BK\Phi(z)] \\ &= \text{Det}[\Phi^{-1}(z) + BK]\text{Det}\Phi(z) \\ &= \text{Det}[zI - A + BK]\text{Det}[zI - A]^{-1}\end{aligned}$$

$$\boxed{\text{Det}[I + G_o(z)] = \frac{\text{Det}[zI - A + BK]}{\text{Det}[zI - A]}}$$

Output weighting in LQ cost



Open loop transfer function

$$G_o(z) = K\Phi(z)B$$

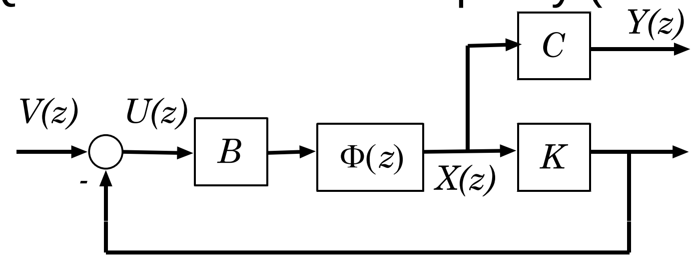
LQ cost:

$$J = \frac{1}{2}\sum_{k=0}^{\infty}\left\{\underline{y^2(k)} + Ru^2(k)\right\}$$

Open loop transfer function from $U(z)$ to $Y(z)$:

$$G(z) = C\Phi(z)B$$

LQ Return Difference Equality (RDE)



Return difference equality (see ME232 class notes):

$$\boxed{[I + G_o(z^{-1})]^T\,[R + B^T PB]\,[I + G_o(z)] = R + G^T(z^{-1})\,G(z)}$$

Open loop transfer function:

$$G_o(z) = K\Phi(z)B$$

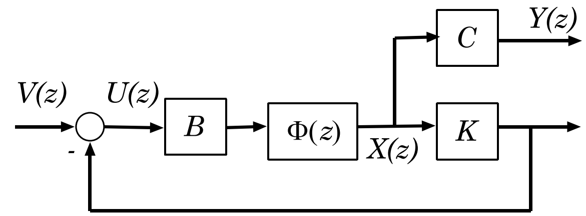
TF from *U(z)* to *Y(z)*:

$$G(z) = C\Phi(z)B$$

RDE for Single Input Systems

When:

$$u(k) \in \mathcal{R}$$



$$\boxed{(1 + G_o(z^{-1})\,)(1 + G_o(z)\,) = \frac{R}{R + B^T PB}\left[1 + \frac{1}{R}G(z^{-1})^T\,G(z)\right]}$$

Open loop transfer function:

$$G_o(z) = K\Phi(z)B$$

TF from *U(z)* to *Y(z)*:

$$G(z) = C\Phi(z)B$$

Return Difference Frequency Response

$$(1 + G_o(z^{-1})\,)(1 + G_o(z)\,) = \frac{R}{R + B^T PB}\left[1 + \frac{1}{R}G(z^{-1})^T\,G(z)\right]$$

Set $z = e^{j\omega}$:

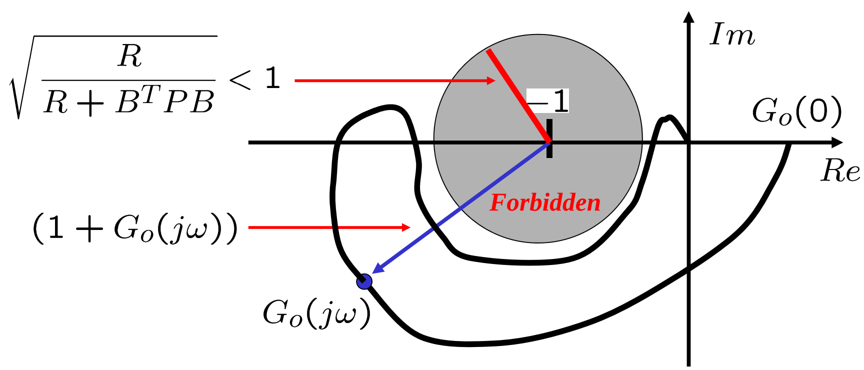
$$\underbrace{(1 + G_o(e^{-j\omega})\,)(1 + G_o(e^{j\omega})\,)}_{|(1+G_o(e^{j\omega}))|^2} = \frac{R}{R + B^T PB}\underbrace{\left[1 + \frac{1}{R}\underbrace{G(e^{-j\omega})^T\,G(e^{j\omega})}_{|G(e^{j\omega})|^2}\right]}_{\geq 1}$$

$$\boxed{|(1 + G_o(e^{j\omega}))|^2 \geq \frac{R}{R + B^T PB}}$$

Stability Margins of Single Input LQR

$$\boxed{|(1 + G_o(e^{j\omega}))| \geq \sqrt{\frac{R}{R + B^T PB}}}$$

Nyquist plot of $G_o(j\omega)$



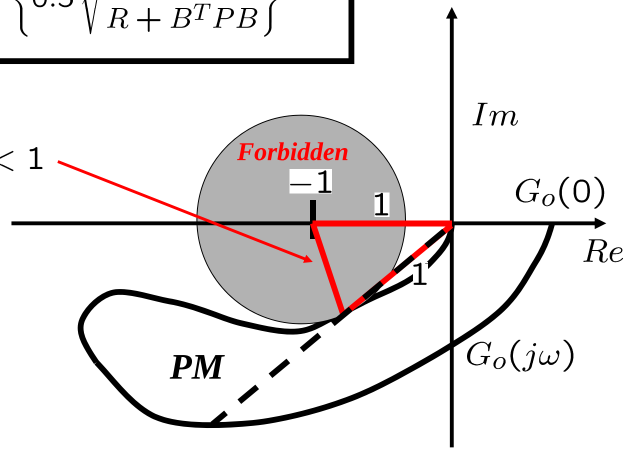
Phase Margin of Single Input LQR

Worst possible PM:

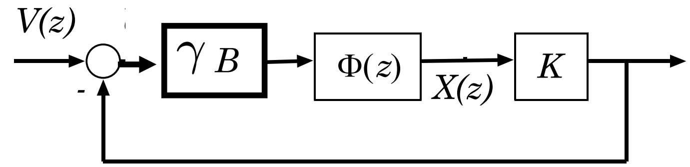
$$\boxed{PM \geq 2\sin^{-1}\left\{0.5\sqrt{\frac{R}{R+B^TPB}}\right\}}$$

Nyquist plot of $G_o(j\omega)$

$$\sqrt{\frac{R}{R+B^TPB}} < 1$$



Loop Gain γ Margins

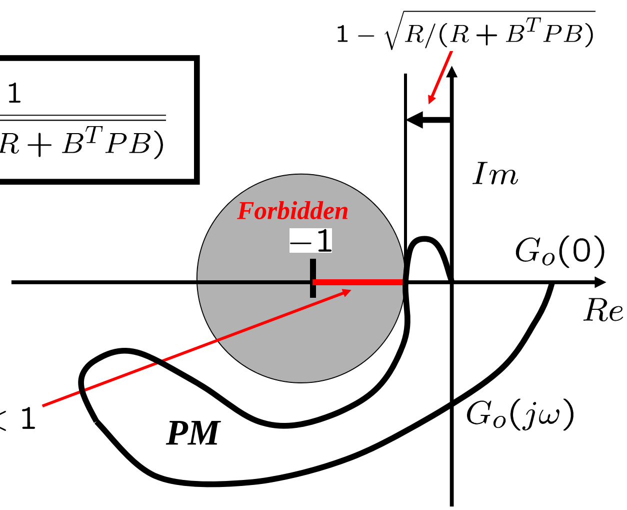


- Control system is designed for $\gamma = 1$
- How big (or small) can γ be before the system becomes unstable?

Gain Margin – one possibility

System is guaranteed to be asymptotically stable when

$$\boxed{\gamma < \frac{1}{1-\sqrt{R/(R+B^TPB)}}}$$

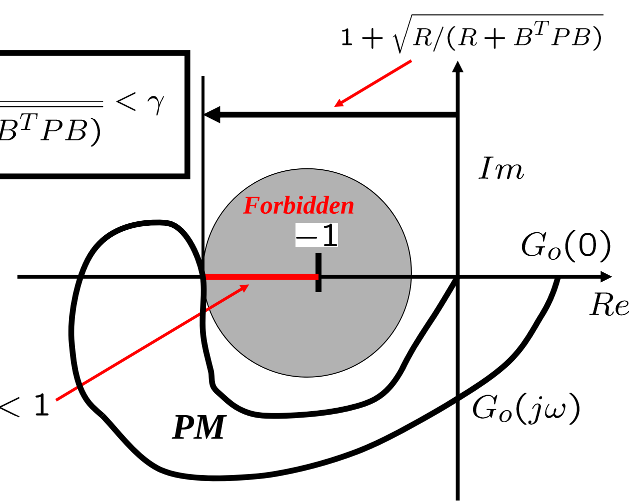


$$\sqrt{\frac{R}{R+B^TPB}} < 1$$

Gain Margin – another possibility

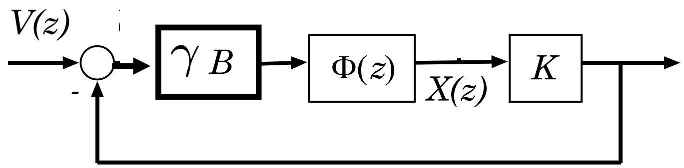
System is guaranteed to be asymptotically stable when

$$\boxed{\frac{1}{1+\sqrt{R/(R+B^TPB)}} < \gamma}$$



$$\sqrt{\frac{R}{R+B^TPB}} < 1$$

Loop Gain γ Margins

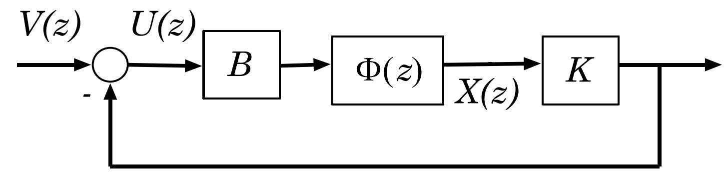


- Control system is designed for $\gamma = 1$

- System is ***guaranteed*** to remain asymptotically stable for

$$\frac{1}{1+\sqrt{R/(R+B^TPB)}} < \gamma < \frac{1}{1-\sqrt{R/(R+B^TPB)}}$$

Example – Double Integrator



For $R = 10$ we obtained $K = \begin{bmatrix} 0.21 & 0.65 \end{bmatrix}$

$$P = \begin{bmatrix} 3 & 3.16 \\ 3.16 & 8.1 \end{bmatrix}$$

Open loop transfer function:

$$G_o(z) = K\Phi(z)B = \frac{0.76\,z - 0.55}{(z-1)^2}$$

Example – Double Integrator

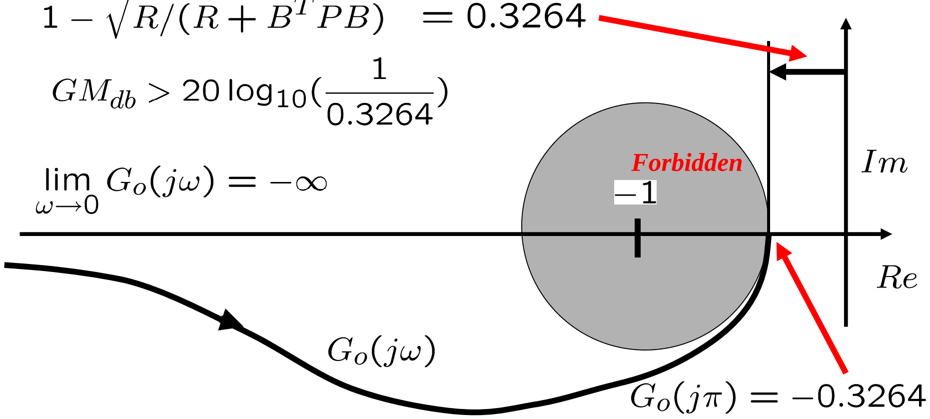
$$\boxed{G_o(j\omega) = \frac{0.76\,e^{j\omega} - 0.55}{(e^{j\omega}-1)^2}}$$

$$\boxed{GM_{db} > 9.726\text{ db}}$$

$$1-\sqrt{R/(R+B^TPB)} = 0.3264$$

$$GM_{db} > 20\log_{10}\left(\frac{1}{0.3264}\right)$$

$$\lim_{\omega\to 0} G_o(j\omega) = -\infty$$



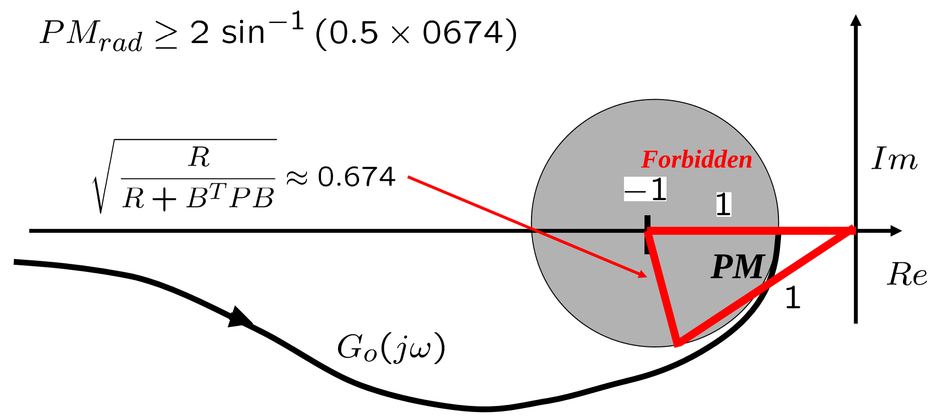
Example – Double Integrator

$$\boxed{G_o(j\omega) = \frac{0.76\,e^{j\omega} - 0.55}{(e^{j\omega}-1)^2}}$$

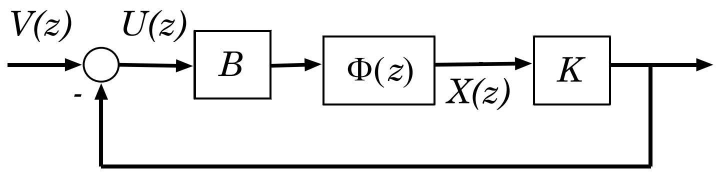
$$\boxed{PM_{deg} > 39.3^o}$$

$$PM_{rad} \geq 2\sin^{-1}(0.5 \times 0674)$$

$$\sqrt{\frac{R}{R+B^TPB}} \approx 0.674$$



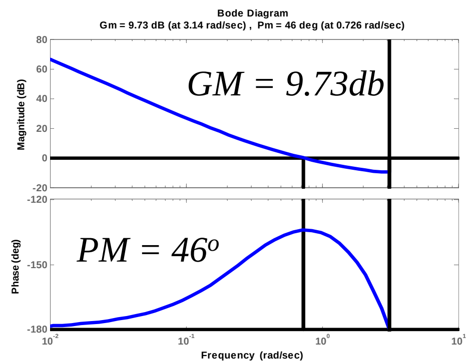
Example – Double Integrator



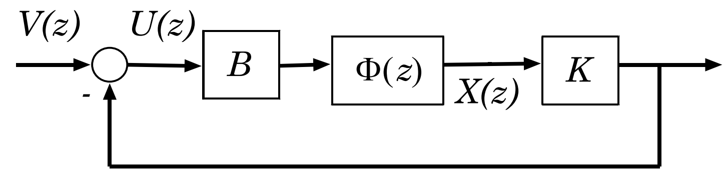
$$G_o(z) = \frac{0.76\,z - 0.55}{(z-1)^2}$$

Open loop transfer function:

$$R = 10$$



Poles of an LQR

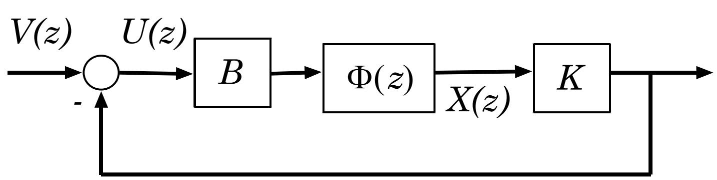


Close loop poles are the zeros of the return difference

Open loop poles are the poles of the return difference

$$\boxed{\text{Det}[I + G_o(z)] = \frac{\text{Det}[zI - A + BK]}{\text{Det}[zI - A]}}$$

Poles of an LQR

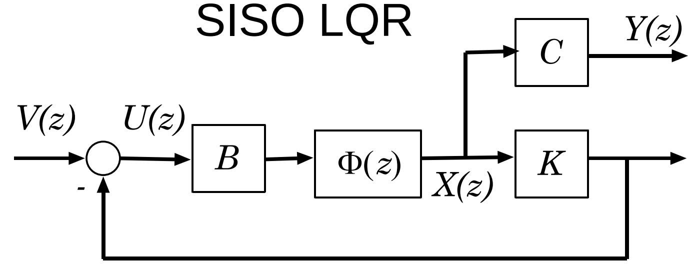


Open loop polynomial: $A(z) = \text{Det}[zI - A]$

Close loop polynomial: $A_c(z) = \text{Det}[zI - A + BK]$

$$\boxed{\text{Det}[I + G_o(z)] = \frac{\text{Det}[zI - A + BK]}{\text{Det}[zI - A]}}$$

SISO LQR



Open loop transfer function from $U(z)$ to $Y(z)$:

$$G(z) = C\Phi(z)B$$

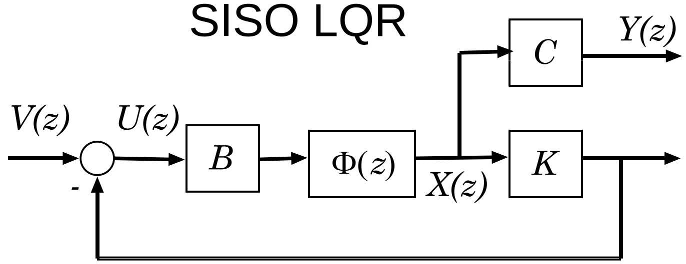
when

$$u(k) \in \mathcal{R}$$

$$y(k) \in \mathcal{R}$$

$$G(z) = \frac{\bar{B}(z)}{A(z)}$$

SISO LQR



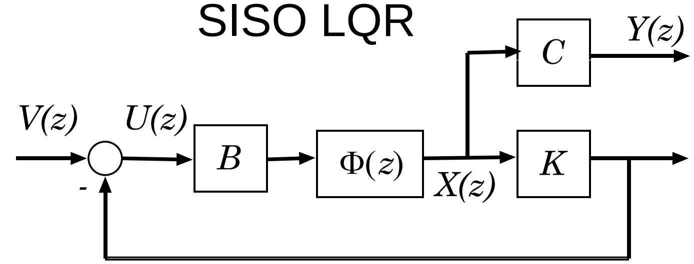
Open loop poles: $A(z) = 0$

Close loop poles: $A_c(z) = 0$

Open loop plant zeros: $\bar{B}(z) = 0$

TF from U → Y $$G(z) = C\Phi(z)B = \frac{\bar{B}(z)}{A(z)}$$

SISO LQR



Open loop polynomial: $A(z) = z^n + a_1 z^{n-1} + \cdots + a_0$

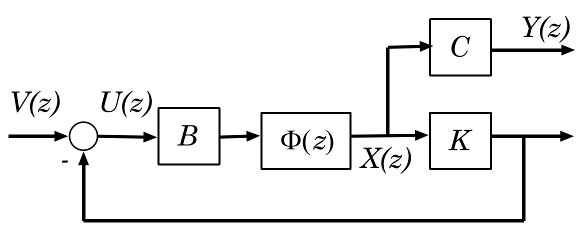
Close loop polynomial: $A_c(z) = z^n + a_{c1} z^{n-1} + \cdots + a_{c0}$

Open loop plant zero polynomial: $\bar{B}(z) = \bar{b}_m (z^m + b_1 z^{m-1} + \cdots + b_0)$

TF from U → Y $$G(z) = C\Phi(z)B = \frac{\bar{B}(z)}{A(z)}$$

SISO Return Difference Equality (RDE)

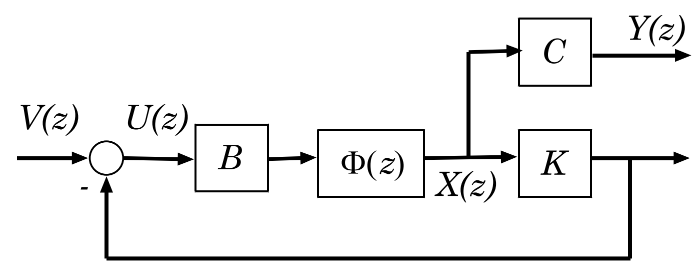
$$\boxed{\begin{array}{c} u(k) \in \mathcal{R} \\ y(k) \in \mathcal{R} \end{array}}$$



$$\underbrace{(1+G_o(z^{-1}))}_{\frac{A_c(z^{-1})}{A(z^{-1})}}\underbrace{(1+G_o(z))}_{\frac{A_c(z)}{A(z)}} = \underbrace{\frac{R}{R+B^TPB}}\left[1+\frac{1}{R}\underbrace{G(z^{-1})}_{\frac{\bar{B}(z^{-1})}{A(z^{-1})}}\,\underbrace{G(z)}_{\frac{\bar{B}(z)}{A(z)}}\right]$$

$\gamma > 0$ for $R \in (0, \infty)$

SISO Return Difference Equality (RDE)



$$\frac{A_c(z^{-1})A_c(z)}{A(z^{-1})A(z)} = \gamma\left[1 + \frac{1}{R}\frac{\bar{B}(z^{-1})\bar{B}(z)}{A(z^{-1})A(z)}\right]$$

Basis for the Reciprocal root locus technique

RDE Left hand side:

2n zeros of the transfer function:

$$\frac{A_c(z^{-1})A_c(z)}{A(z^{-1})A(z)} = 0$$

n close loop poles: $A_c(z) = (z - p_{c1})\cdots(z - p_{cn})$

$$\underbrace{\boldsymbol{n} \text{ zeros of: } A_c(z^{-1})}_{\boldsymbol{n} \textit{ reciprocals of close loop poles}} = \left(z - \frac{1}{p_{c1}}\right)\cdots\left(z - \frac{1}{p_{cn}}\right)\frac{a_{co}}{z^n}$$

$$a_{co} = (-1)^n p_{c1}\, p_{c1} \cdots p_{cn}$$

RDE Left hand side:

2n zeros of the transfer function:

$$\frac{A_c(z^{-1})A_c(z)}{A(z^{-1})A(z)} = 0$$

n close loop poles: $p_{c1},\ p_{c2},\ \cdots p_{cn}$

n reciprocals of close loop poles: $\frac{1}{p_{c1}},\ \frac{1}{p_{c2}},\ \cdots \frac{1}{p_{cn}}$

$$\boxed{|p_{ci}| < 1 \qquad \left|\frac{1}{p_{ci}}\right| > 1 \qquad R \in (0, \infty)}$$

LQ Reciprocal Root Locus

$$\frac{A_c(z^{-1})A_c(z)}{A(z^{-1})A(z)} = \gamma\left[1 + \frac{1}{R}\frac{\bar{B}(z^{-1})\bar{B}(z)}{A(z^{-1})A(z)}\right]$$

$$\boxed{\frac{\prod_{i=1}^n(z-p_{ci})\prod_{i=1}^n(z-\frac{1}{p_{ci}})}{\prod_{i=1}^n(z-p_{oi})\prod_{i=1}^n(z-\frac{1}{p_{oi}})} = \beta\left[1 + \frac{b_o}{a_o}\frac{\bar{b}_m^2}{R}\frac{z^{n-m}\prod_{i=1}^m(z-z_{oi})\prod_{i=1}^m(z-\frac{1}{z_{oi}})}{\prod_{i=1}^n(z-p_{oi})\prod_{i=1}^n(z-\frac{1}{p_{oi}})}\right]}$$

$\beta = \left(\frac{a_o}{a_{co}}\right)\frac{R}{R + B^T P B}$ Is a constant, which does not affect the Reciprocal root locus

LQ Reciprocal Root Locus

$$\frac{\prod_{i=1}^n(z-p_{ci})\prod_{i=1}^n(z-\frac{1}{p_{ci}})}{\prod_{i=1}^n(z-p_{oi})\prod_{i=1}^n(z-\frac{1}{p_{oi}})} = \beta\left[1 + \frac{b_o}{a_o}\frac{\bar{b}_m^2}{R}\frac{z^{n-m}\prod_{i=1}^m(z-z_{oi})\prod_{i=1}^m(z-\frac{1}{z_{oi}})}{\prod_{i=1}^n(z-p_{oi})\prod_{i=1}^n(z-\frac{1}{p_{oi}})}\right]$$

$\left|\frac{1}{p_{ci}}\right| < 1$ inverses of close loop eigenvalues (**always unstable**)

$|p_{ci}| < 1$ closed loop eigenvalues **(always asymptotically stable)**

LQ Reciprocal Root Locus

$$\frac{\prod_{i=1}^n (z-p_{ci})\prod_{i=1}^n (z-\frac{1}{p_{ci}})}{\prod_{i=1}^n (z-p_{oi})\prod_{i=1}^n (z-\frac{1}{p_{oi}})} = \beta\left[1+\frac{b_o}{a_o}\frac{\bar{b}_m^2}{R}\frac{z^{n-m}\prod_{i=1}^m (z-z_{oi})\prod_{i=1}^m (z-\frac{1}{z_{oi}})}{\prod_{i=1}^n (z-p_{oi})\prod_{i=1}^n (z-\frac{1}{p_{oi}})}\right]$$

open loop eigenvalues

Inverses of open loop eigenvalues

LQ Reciprocal Root Locus

$$\frac{\prod_{i=1}^n (z-p_{ci})\prod_{i=1}^n (z-\frac{1}{p_{ci}})}{\prod_{i=1}^n (z-p_{oi})\prod_{i=1}^n (z-\frac{1}{p_{oi}})} = \beta\left[1+\frac{b_o}{a_o}\frac{\bar{b}_m^2}{R}\frac{z^{n-m}\prod_{i=1}^m (z-z_{oi})\prod_{i=1}^m (z-\frac{1}{z_{oi}})}{\prod_{i=1}^n (z-p_{oi})\prod_{i=1}^n (z-\frac{1}{p_{oi}})}\right]$$

n-m zeros at the origin

zeros of $\bar{B}(z)$

Inverses of zeros of $\bar{B}(z)$

LQ Reciprocal Root Locus

$$\frac{\prod_{i=1}^n (z-p_{ci})\prod_{i=1}^n (z-\frac{1}{p_{ci}})}{\prod_{i=1}^n (z-p_{oi})\prod_{i=1}^n (z-\frac{1}{p_{oi}})} = \beta\left[1+\frac{b_o}{a_o}\frac{\bar{b}_m^2}{R}\frac{z^{n-m}\prod_{i=1}^m (z-z_{oi})\prod_{i=1}^m (z-\frac{1}{z_{oi}})}{\prod_{i=1}^n (z-p_{oi})\prod_{i=1}^n (z-\frac{1}{p_{oi}})}\right]$$

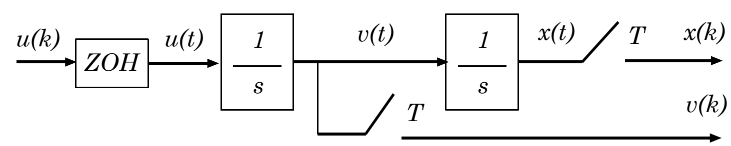
$$\bar{B}(z) = \bar{b}_m\,(z^n + \cdots + b_o) \qquad A(z) = z^n + \cdots + a_o,$$

$\frac{b_o}{a_o} > 0 \quad \Rightarrow$ negative feedback

$\frac{b_o}{a_o} < 0 \quad \Rightarrow$ positive feedback

Example – Double Integrator

Double integrator with ZOH and sampling time T=1:

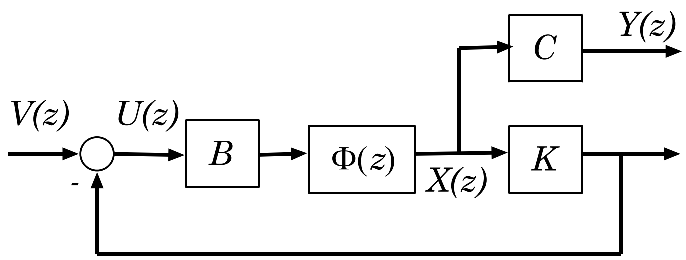


$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)$$

$$J = \frac{1}{2}\sum_{k=0}^{\infty}\left\{y^2(k) + R\,u^2(k)\right\} \qquad R > 0$$

Example – Double Integrator



$$G(z) = C\Phi(z)B$$

$$G(z) = \frac{\bar{B}(z)}{A(z)} = \frac{\frac{1}{2}(z+1)}{(z-1)^2} = \frac{\frac{1}{2}(z+1)}{(z^2-2z+1)}$$

$$\frac{b_o}{a_o} = \frac{1}{1}$$

Example – Double Integrator

$$\frac{\bar{B}(z)}{A(z)} = \frac{\frac{1}{2}(z+1)}{(z^2-2z+1)} \quad \left\{ \begin{array}{rcll} \bar{b}_m & = & \frac{1}{2} & \\ a_o & = & 1 & b_o = 1 \\ n & = & 2 & m = 1 \\ n-m & = & 1 & \\ \frac{a_o}{b_o} & = & 1 & \end{array} \right.$$

$$1 + \frac{1}{R}\,\frac{\bar{B}(z^{-1})\bar{B}(z)}{A(z^{-1})A(z)} = 0$$

$$1 + \frac{(1/2)^2}{R}\left(\frac{1}{1}\right)\frac{z(z+1)(z+1)}{(z-1)(z-1)(z-1)(z-1)} = 0$$

Example – Double Integrator

$$1 + \frac{1}{R}\,\frac{\bar{B}(z^{-1})\bar{B}(z)}{A(z^{-1})A(z)} = 0$$

$$1 + \frac{1}{R}\,\frac{\overbrace{\tfrac{1}{2}(z^{-1}+1)}^{\bar{B}(z^{-1})}\;\overbrace{\tfrac{1}{2}(z+1)}^{\bar{B}(z)}}{\underbrace{(z^{-1}-1)^2}_{A(z^{-1})}\;\underbrace{(z-1)^2}_{A(z)}} = 0$$

Example – Double Integrator

$$1 + \frac{1}{R}\,\frac{\frac{1}{2}(z^{-1}+1)\;\;\frac{1}{2}(z+1)}{(z^{-1}-1)^2\;\;(z-1)^2} = 0$$

$$1 + \frac{1}{R}\,\frac{\frac{1}{4}\,z^{-1}(z+1)\;\;(z+1)}{(z^{-1}-1)^2\;\;(z-1)^2} = 0$$

$$1 + \frac{1}{R}\,\frac{\frac{1}{4}\,z^{-1}(z+1)\;\;(z+1)}{z^{-2}\,(z-1)^2\;\;(z-1)^2} = 0$$

Example – Double Integrator

$$1 + \frac{1}{R}\,\frac{\bar{B}(z^{-1})\bar{B}(z)}{A(z^{-1})A(z)} = 0$$

1 open loop zero @ 0 ***2 open loop zeros @ -1***

$$1 + \frac{0.25}{R}\;\frac{z\,(z+1)^2}{(z-1)^4} = 0$$

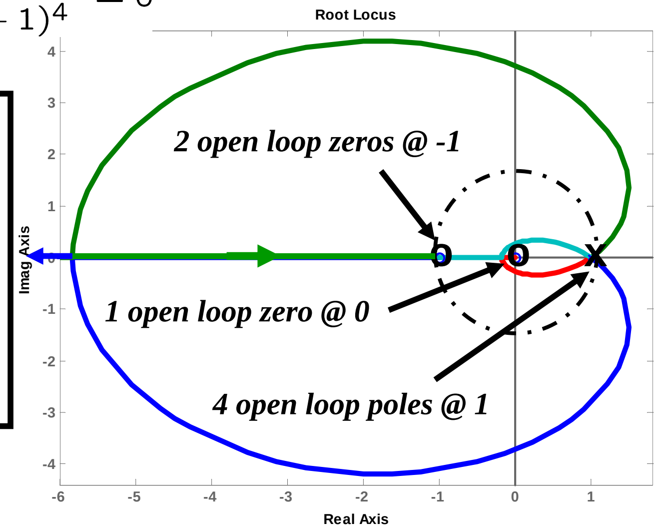
4 open loop poles @ 1

Example – Double Integrator

$$1 + \frac{0.25}{R}\,\frac{z\,(z+1)^2}{(z-1)^4} = 0$$

LQR poles always inside unit circle

$R \in (0, \infty)$

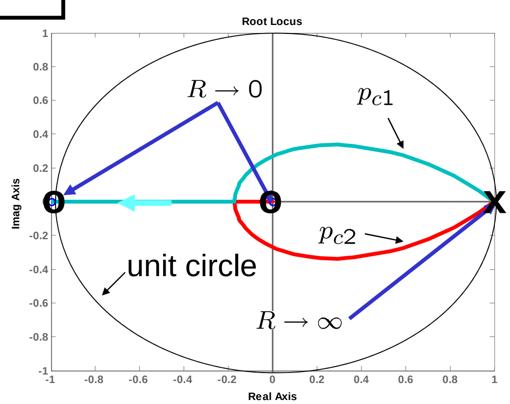


LQR Close Loop poles

$$\boxed{1 + \frac{0.25}{R}\,\frac{z\,(z+1)^2}{(z-1)^4} = 0}$$

$$R \to \infty \;\Rightarrow\; \begin{cases} p_{c1} \to 1 \\ p_{c2} \to 1 \end{cases}$$

$$R \to 0 \;\Rightarrow\; \begin{cases} p_{c1} \to -1 \\ p_{c2} \to 0 \end{cases}$$



LQR Close Loop poles reciprocals

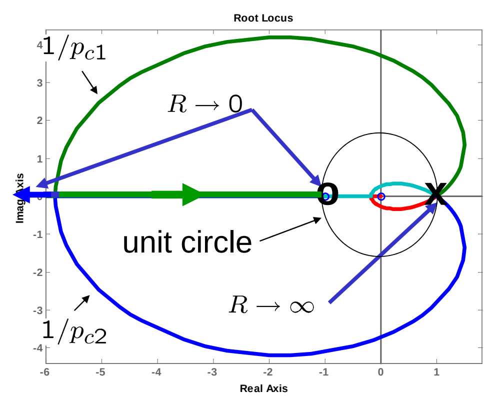
$$\boxed{1 + \frac{0.25}{R}\,\frac{z\,(z+1)^2}{(z-1)^4} = 0}$$

$$R \to \infty \;\Rightarrow\; \begin{cases} 1/p_{c1} \to 1 \\ 1/p_{c2} \to 1 \end{cases}$$

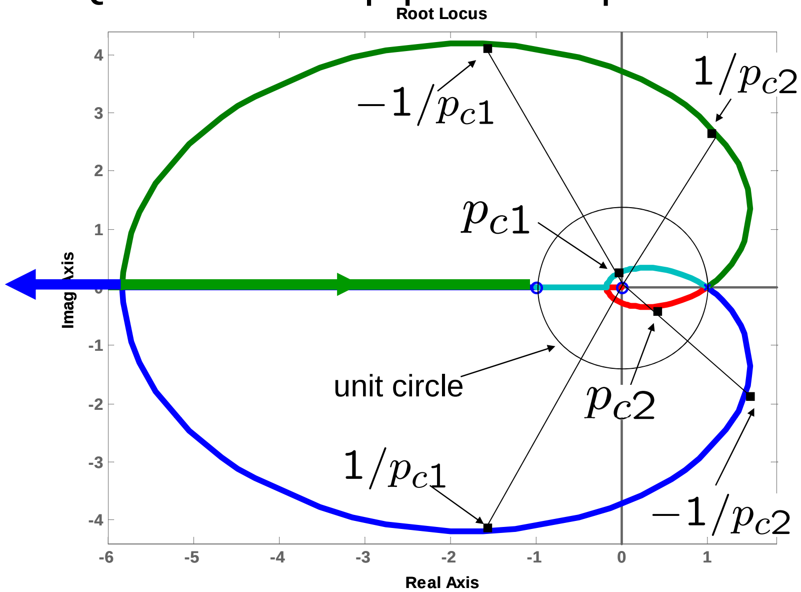
$$R \to 0 \;\Rightarrow$$

$$\begin{cases} 1/p_{c1} \to -1 \\ 1/p_{c2} \to -\infty \end{cases}$$

always unstable



LQR Close Loop poles reciprocals



Summary

- Convergence of LQR as horizon $N \to \infty$
  - $[A\ B]$ stabilizable
  - $[A\ C]$ detectable
- Infinite horizon LQR
- Solution of algebraic Riccati equation
- Close loop system is asymptotically stable
- Return difference equality
  - Guaranteed gain and phase margins of LQR
  - Reciprocal root locus (LQR close loop poles)

Additional Material

- More details on plotting the LQR Reciprocal root locus
- LQR Reciprocal Root Locus with open loop eigenvalues and zeros at the origin

Roots and their reciprocals

Consider the polynomial:

$$A(z) = z^n + a_1 z^{n-1} + \cdots + a_0$$

$$= (z - p_1)(z - p_2) \cdots (z - p_n)$$

$$A(z) = z^n - \left(\sum_{i=1}^{n} p_i \right) z^{n-1} + \cdots + (-1)^n \prod_{i=1}^{n} p_i$$

$$a_o = (-1)^n \prod_{i=1}^{n} p_i$$

Roots and their reciprocals

Consider now the transfer function:

$$A(z) = z^{-n} + a_1 z^{-(n-1)} + \cdots + a_0$$

$$= (z^{-1} - p_1)\,(z^{-1} - p_2)\,\cdots\,(z^{-1} - p_n)$$

$$= \frac{a_0}{z^n}\,(z - \frac{1}{p_1})\,(z - \frac{1}{p_2})\,\cdots\,(z - \frac{1}{p_n})$$

$$a_o = (-1)^n \prod_{i=1}^{n} p_i$$

Zeros of $A(z^{-1})$ ***are reciprocals of roots of*** $A(z)$

RDE Left hand side:

$$\frac{A_c(z^{-1})A_c(z)}{A(z^{-1})A(z)} = 0$$

$$\frac{A_c(z)}{A(z)} = \frac{z^n + a_{c\,n-1} z^{n-1} + \cdots + a_{co}}{z^n + a_{n-1} z^{n-1} + \cdots + a_o}$$

$$= \frac{(z - p_{c1})\,(z - p_{c2})\,\cdot\,(z - p_{cn})}{(z - p_{o1})\,(z - p_{o2})\,\cdot\,(z - p_{on})}$$

RDE Right hand side:

$$\frac{A_c(z^{-1})A_c(z)}{A(z^{-1})A(z)} = 0$$

$$\frac{A_c(z^{-1})}{A(z^{-1})} = \frac{z^{-n} + a_{c\,n-1} z^{-n+1} + \cdots + a_c}{z^{-n} + a_{n-1} z^{-n+1} + \cdots + a_o}$$

$$= \frac{a_{co}\, z^{-n}\,(z - \frac{1}{p_{c1}})\,(z - \frac{1}{p_{c2}})\,\cdot\,(z - \frac{1}{p_{cn}})}{a_o\, z^{-n}\,(z - \frac{1}{p_{o1}})\,(z - \frac{1}{p_{o2}})\,\cdot\,(z - \frac{1}{p_{on}})}$$

RDE Right hand side:

$$1 + \frac{1}{R}\,\frac{\bar{B}(z^{-1})\bar{B}(z)}{A(z^{-1})A(z)} = 0$$

$$\frac{\bar{B}(z)}{A(z)} = \frac{\bar{b}_m z^m + \bar{b}_{m-1} z^{m-1} + \cdots + \bar{b}_o}{z^n + a_{n-1} z^{n-1} + \cdots + a_o}$$

$$= \frac{\bar{b}_m\,(z - z_{o1})\,(z - z_{o2})\,\cdot\,(z - z_{om})}{(z - p_{o1})\,(z - p_{o2})\,\cdot\,(z - p_{on})}$$

RDE Right hand side:

$$1+\frac{1}{R}\frac{\bar{B}(z^{-1})\bar{B}(z)}{A(z^{-1})A(z)}=0$$

$$\frac{\bar{B}(z^{-1})}{A(z^{-1})}=\frac{\bar{b}_m z^{-m}+\bar{b}_{m-1}z^{-m+1}+\cdots+\bar{b}_o}{z^{-n}+a_{n-1}z^{-n+1}+\cdots+a_o}$$

$$=\frac{\bar{b}_m\, b_o\, z^{-m}\,(z-\frac{1}{z_{o1}})\,(z-\frac{1}{z_{o2}})\,\cdot(z-\frac{1}{z_{om}})}{a_o\, z^{-n}\,(z-\frac{1}{p_{o1}})\,(z-\frac{1}{p_{o2}})\,\cdot(z-\frac{1}{p_{on}})}$$

RDE Right hand side:

$$1+\frac{1}{R}\frac{\bar{B}(z^{-1})\bar{B}(z)}{A(z^{-1})A(z)}=0$$

$$1+\frac{\bar{b}_m^2}{R}\,\frac{b_o}{a_o}\,z^{n-m}\,\frac{\prod_{i=1}^{m}(z-z_{oi})\,\prod_{i=1}^{m}(z-\frac{1}{z_{oi}})}{\prod_{i=1}^{n}(z-p_{oi})\,\prod_{i=1}^{n}(z-\frac{1}{p_{oi}})}=0$$

LQ Reciprocal Root Locus

$$\frac{A_c(z^{-1})A_c(z)}{A(z^{-1})A(z)}=\gamma\left[1+\frac{1}{R}\frac{\bar{B}(z^{-1})\bar{B}(z)}{A(z^{-1})A(z)}\right]$$

$$\boxed{\frac{\prod_{i=1}^{n}(z-p_{ci})\,\prod_{i=1}^{n}(z-\frac{1}{p_{ci}})}{\prod_{i=1}^{n}(z-p_{oi})\,\prod_{i=1}^{n}(z-\frac{1}{p_{oi}})}=\beta\left[1+\frac{b_o}{a_o}\frac{\bar{b}_m^2}{R}\,\frac{z^{n-m}\prod_{i=1}^{m}(z-z_{oi})\,\prod_{i=1}^{m}(z-\frac{1}{z_{oi}})}{\prod_{i=1}^{n}(z-p_{oi})\,\prod_{i=1}^{n}(z-\frac{1}{p_{oi}})}\right]}$$

$$\beta=\left(\frac{a_o}{a_{co}}\right)\frac{R}{R+B^TPB}$$

Is a constant, which does not affect the Reciprocal root locus

LQ Reciprocal Root Locus

$$\frac{\prod_{i=1}^{n}(z-p_{ci})\,\prod_{i=1}^{n}(z-\frac{1}{p_{ci}})}{\prod_{i=1}^{n}(z-p_{oi})\,\prod_{i=1}^{n}(z-\frac{1}{p_{oi}})}=\beta\left[1+\frac{b_o}{a_o}\frac{\bar{b}_m^2}{R}\,\frac{z^{n-m}\prod_{i=1}^{m}(z-z_{oi})\,\prod_{i=1}^{m}(z-\frac{1}{z_{oi}})}{\prod_{i=1}^{n}(z-p_{oi})\,\prod_{i=1}^{n}(z-\frac{1}{p_{oi}})}\right]$$

$|\frac{1}{p_{ci}}|<1$ inverses of close loop eigenvalues (**always unstable**)

$|p_{ci}|<1$ closed loop eigenvalues **(always asymptotically stable)**

LQ Reciprocal Root Locus

$$\frac{\prod_{i=1}^n (z-p_{ci}) \prod_{i=1}^n (z-\frac{1}{p_{ci}})}{\prod_{i=1}^n (z-p_{oi}) \prod_{i=1}^n (z-\frac{1}{p_{oi}})} = \beta \left[1 + \frac{b_o}{a_o}\frac{\bar{b}_m^2}{R} \frac{z^{n-m}\prod_{i=1}^m (z-z_{oi})\prod_{i=1}^m (z-\frac{1}{z_{oi}})}{\prod_{i=1}^n (z-p_{oi})\prod_{i=1}^n (z-\frac{1}{p_{oi}})}\right]$$

open loop eigenvalues

Inverses of open loop eigenvalues

LQ Reciprocal Root Locus

$$\frac{\prod_{i=1}^n (z-p_{ci}) \prod_{i=1}^n (z-\frac{1}{p_{ci}})}{\prod_{i=1}^n (z-p_{oi}) \prod_{i=1}^n (z-\frac{1}{p_{oi}})} = \beta \left[1 + \frac{b_o}{a_o}\frac{\bar{b}_m^2}{R} \frac{z^{n-m}\prod_{i=1}^m (z-z_{oi})\prod_{i=1}^m (z-\frac{1}{z_{oi}})}{\prod_{i=1}^n (z-p_{oi})\prod_{i=1}^n (z-\frac{1}{p_{oi}})}\right]$$

n-m zeros at the origin

zeros of $\bar{B}(z)$

Inverses of zeros of $\bar{B}(z)$

LQ Reciprocal Root Locus

$$\frac{\prod_{i=1}^n (z-p_{ci}) \prod_{i=1}^n (z-\frac{1}{p_{ci}})}{\prod_{i=1}^n (z-p_{oi}) \prod_{i=1}^n (z-\frac{1}{p_{oi}})} = \beta \left[1 + \frac{b_o}{a_o}\frac{\bar{b}_m^2}{R} \frac{z^{n-m}\prod_{i=1}^m (z-z_{oi})\prod_{i=1}^m (z-\frac{1}{z_{oi}})}{\prod_{i=1}^n (z-p_{oi})\prod_{i=1}^n (z-\frac{1}{p_{oi}})}\right]$$

$$\bar{B}(z) = \bar{b}_m (z^n + \cdots + b_o) \qquad A(z) = z^n + \cdots + a_o,$$

$\frac{b_o}{a_o} > 0 \Rightarrow$ negative feedback

$\frac{b_o}{a_o} < 0 \Rightarrow$ positive feedback

LQ Reciprocal Root Locus

$$\frac{\prod_{i=1}^n (z-p_{ci}) \prod_{i=1}^n (z-\frac{1}{p_{ci}})}{\prod_{i=1}^n (z-p_{oi}) \prod_{i=1}^n (z-\frac{1}{p_{oi}})} = \beta \left[1 + \frac{b_o}{a_o}\boxed{\frac{\bar{b}_m^2}{R}} \frac{z^{n-m}\prod_{i=1}^m (z-z_{oi})\prod_{i=1}^m (z-\frac{1}{z_{oi}})}{\prod_{i=1}^n (z-p_{oi})\prod_{i=1}^n (z-\frac{1}{p_{oi}})}\right]$$

$$R \to \infty \Rightarrow p_{ci} \to \text{Stable} \begin{cases} p_{oi} \\ \text{or} \\ 1/p_{oi} \end{cases}$$

$$R \to \infty \Rightarrow 1/p_{ci} \to \text{Unstable} \begin{cases} p_{oi} \\ \text{or} \\ 1/p_{oi} \end{cases}$$

LQ Reciprocal Root Locus

$$\frac{\prod_{i=1}^n (z-p_{ci}) \prod_{i=1}^n \left(z-\frac{1}{p_{ci}}\right)}{\prod_{i=1}^n (z-p_{oi}) \prod_{i=1}^n \left(z-\frac{1}{p_{oi}}\right)} = \beta \left[1 + \frac{b_o}{a_o}\boxed{\frac{\bar{b}_m^2}{R}}\, \frac{z^{n-m}\prod_{i=1}^m (z-z_{oi})\prod_{i=1}^m \left(z-\frac{1}{z_{oi}}\right)}{\prod_{i=1}^n (z-p_{oi})\prod_{i=1}^n \left(z-\frac{1}{p_{oi}}\right)}\right]$$

$R \to 0 \;\Rightarrow\; p_{ci} \to \text{Stable} \begin{cases} z_{oi} \\ \text{or} \\ 1/z_{oi} \end{cases}$ $\quad \boxed{i = 1, \cdots, m}$

$R \to 0 \;\Rightarrow\; p_{ci} \to 0$ $\quad \boxed{i = m+1, \cdots, n}$

LQ Reciprocal Root Locus

$$\frac{\prod_{i=1}^n (z-p_{ci}) \prod_{i=1}^n \left(z-\frac{1}{p_{ci}}\right)}{\prod_{i=1}^n (z-p_{oi}) \prod_{i=1}^n \left(z-\frac{1}{p_{oi}}\right)} = \beta \left[1 + \frac{b_o}{a_o}\boxed{\frac{\bar{b}_m^2}{R}}\, \frac{z^{n-m}\prod_{i=1}^m (z-z_{oi})\prod_{i=1}^m \left(z-\frac{1}{z_{oi}}\right)}{\prod_{i=1}^n (z-p_{oi})\prod_{i=1}^n \left(z-\frac{1}{p_{oi}}\right)}\right]$$

$R \to 0 \;\Rightarrow\; 1/p_{ci} \to \text{Unstable} \begin{cases} z_{oi} \\ \text{or} \\ 1/z_{oi} \end{cases}$ $\quad \boxed{i = 1, \cdots, m}$

$R \to 0 \;\Rightarrow\; |1/p_{ci}| \to \infty$ $\quad \boxed{i = m+1, \cdots, n}$

LQ Reciprocal Root Locus

$$\frac{\prod_{i=1}^n (z-p_{ci}) \prod_{i=1}^n \left(z-\frac{1}{p_{ci}}\right)}{\prod_{i=1}^n (z-p_{oi}) \prod_{i=1}^n \left(z-\frac{1}{p_{oi}}\right)} = \beta \left[1 + \boxed{\frac{b_o}{a_o}}\boxed{\frac{\bar{b}_m^2}{R}}\, \frac{z^{n-m}\prod_{i=1}^m (z-z_{oi})\prod_{i=1}^m \left(z-\frac{1}{z_{oi}}\right)}{\prod_{i=1}^n (z-p_{oi})\prod_{i=1}^n \left(z-\frac{1}{p_{oi}}\right)}\right]$$

$$\bar{B}(z) = \bar{b}_m\,(z^n + \cdots + b_o) \qquad A(z) = z^n + \cdots + a_o,$$

$R \to 0 \;\Rightarrow\; |1/p_{ci}| \to \infty$ $\quad \boxed{\frac{b_o}{a_o} > 0}$

$\frac{(2q+1)\,\pi}{n-m} \qquad q = 0,\, 1,\, \cdots,\, (n-m)-1$

(negative feedback RL rules)

LQ Reciprocal Root Locus

$$\frac{\prod_{i=1}^n (z-p_{ci}) \prod_{i=1}^n \left(z-\frac{1}{p_{ci}}\right)}{\prod_{i=1}^n (z-p_{oi}) \prod_{i=1}^n \left(z-\frac{1}{p_{oi}}\right)} = \beta \left[1 + \boxed{\frac{b_o}{a_o}}\boxed{\frac{\bar{b}_m^2}{R}}\, \frac{z^{n-m}\prod_{i=1}^m (z-z_{oi})\prod_{i=1}^m \left(z-\frac{1}{z_{oi}}\right)}{\prod_{i=1}^n (z-p_{oi})\prod_{i=1}^n \left(z-\frac{1}{p_{oi}}\right)}\right]$$

$$\bar{B}(z) = \bar{b}_m\,(z^n + \cdots + b_o) \qquad A(z) = z^n + \cdots + a_o,$$

$R \to 0 \;\Rightarrow\; |1/p_{ci}| \to \infty$ $\quad \boxed{\frac{b_o}{a_o} < 0}$

$\frac{(2q)\,\pi}{n-m} \qquad q = 0,\, 1,\, \cdots,\, (n-m)-1$

(positive feedback RL rules)

LQR Reciprocal Root Locus with open loop eigenvalues and zeros at the origin

Consider a nth order LTI system:

$$x(k+1) = A\,x(k) + B\,u(k) \qquad x(0) = x_o$$

Under the optimal control which minimizes

$$J = \frac{1}{2}\sum_{k=0}^{\infty}\left\{y^T(k)\,y(k) + u^T(k)\,R\,u(k)\right\}$$

$$y(k) = C\,x(k)$$

Assume that

$$\boxed{\det(A) = 0}$$

LQR Reciprocal Root Locus with open loop eigenvalues and zeros at the origin

Notice that optimal control law is given by:

$$u(k) = -K\,x(k)$$

where:

$$\begin{aligned} K &= \left[R + B^T P B\right]^{-1} B^T P A \\ &= K' A \end{aligned}$$

As a consequence, the close loop matrix A_c will also be singular:

$$A_c = A - BK = (I - BK')\,A$$

LQR Reciprocal Root Locus with open loop eigenvalues and zeros at the origin

Therefore if,

$$A(z) = \det(zI - A) = z^r\,A'(z)$$

$$A'(z) = z^{n-r} + \cdots + a_o', \qquad a_o' \neq 0$$

then:

$$A_c(z) = \det(zI - A + BK) = z^{r_c}\,A_c'(z)$$

$$A_c'(z) = z^{n-r_c} + \cdots + a_{co}',$$

$$r_c \geq 1$$

This implies that at least one open loop eigenvalue at the origin is invariant under LQR control.

LQR Reciprocal Root Locus with open loop eigenvalues and zeros at the origin

Lets also assume that there can be zeros at the origin. Thus,

$$\begin{aligned} \bar{B}(z) &= z^p\,\bar{B}'(z) \\ \bar{B}'(z) &= \bar{b}_m\,(z^{m-p} + \cdots + b_o), \\ &= \bar{b}_m \prod_{i=1}^{m-p}(z - z_{oi}) \\ b_o &= \prod_{i=1}^{m-p} z_{oi} \neq 0 \end{aligned}$$

Reciprocal Root Locus with open loop eigenvalues and zeros at the origin

To determine r_c and to plot the remaining n-r_c close loop eigenvalues, we utilize:

$$A_c(z^{-1})\,A_c(z) = \gamma\left[A(z^{-1})\,A(z) + \frac{1}{R}\bar{B}(z^{-1})\;\bar{B}(z)\right]$$

where:

$$A(z) = z^{-r}\,A^{'}(z),\; A^{'}(z) = \prod_{i=1}^{n-r}(z-p_{oi}),\; a_o = \prod_{i=1}^{n-r}(-p_{oi})$$

$$A_c(z) = z^{-r_c}\,A_c^{'}(z),\; A_c^{'}(z) = \prod_{i=1}^{n-r_c}(z-p_{ci}),\; a_{co} = \prod_{i=1}^{n-r_c}(-p_{ci})$$

LQR Reciprocal Root Locus with open loop eigenvalues and zeros at the origin

Thus, from

$$A_c(z^{-1})\,A_c(z) = \gamma\left[A(z^{-1})\,A(z) + \frac{1}{R}\bar{B}(z^{-1})\;\bar{B}(z)\right]$$

we obtain

$$z^{-(n-r_c)}\prod_{i=1}^{n-r_c}(z-p_{ci})(z-\frac{1}{p_{ci}}) = \beta\left[z^{-(n-r)}\prod_{i=1}^{n-r}(z-p_{oi})(z-\frac{1}{p_{oi}}) + \frac{\bar{b}_m^2}{R}\frac{b_o}{a_o}z^{-(m-p)}\prod_{i=1}^{m-p}(z-z_{oi})(z-\frac{1}{z_{oi}})\right]$$

where

$$r_c = n - \max\left[(n-r),\,(m-p)\right]$$

LQR Reciprocal Root Locus with open loop eigenvalues and zeros at the origin

Case 1: $(n-r) \geq (m-p) \;\Rightarrow\; r_c = r$

There are r close loop eigenvalues at the origin and

The remaining n-r close loop eigenvalues are plotted using:

$$\frac{\prod_{i=1}^{n-r}(z-p_{ci})(z-\frac{1}{p_{ci}})}{\prod_{i=1}^{n-r}(z-p_{oi})(z-\frac{1}{p_{oi}})} = \beta\left[1+\frac{\bar{b}_m^2}{R}\frac{b_o}{a_o}\frac{z^{[(n-r)-(m-p)]}\prod_{i=1}^{m-p}(z-z_{oi})(z-\frac{1}{z_{oi}})}{\prod_{i=1}^{n-r}(z-p_{oi})(z-\frac{1}{p_{oi}})}\right]$$

$$\beta = \left(\frac{a_o}{a_{co}}\right)\frac{R}{R+B^TPB}$$

LQR Reciprocal Root Locus with open loop eigenvalues and zeros at the origin

Case 2: $(m-p) > (n-r) \;\Rightarrow r_c = n-(m-p)$

There are $r_c < r$ close loop eigenvalues at the origin and

the remaining m - p close loop eigenvalues are plotted using:

$$\frac{\prod_{i=1}^{m-p}(z-p_{ci})(z-\frac{1}{p_{ci}})}{\prod_{i=1}^{m-p}(z-z_{oi})(z-\frac{1}{z_{oi}})} = \alpha\left[1+\frac{R}{\bar{b}_m^2}\frac{a_o}{b_o}\frac{z^{[(m-p)-(n-r)]}\prod_{i=1}^{n-r}(z-p_{oi})(z-\frac{1}{p_{oi}})}{\prod_{i=1}^{m-p}(z-z_{oi})(z-\frac{1}{z_{oi}})}\right]$$

$$\alpha = \left(\frac{b_o}{a_{co}}\frac{\bar{b}_m^2}{R}\right)\frac{R}{R+B^TPB}$$

Notice that R is in the numerator and the zeros are in the denominator