# ME 233 Spring 2011
# Solution to Homework #1

1. (a) First we will define

$$\begin{array}{lcl} P(A) & - & \text{probability that a randomly chosen item comes from factory A} \\ P(B) & - & \text{probability that a randomly chosen item comes from factory B} \\ P(C) & - & \text{probability that a randomly chosen item comes from factory C} \\ P(D) & - & \text{probability that a randomly chosen item is defective} \\ P(N) & - & \text{probability that a randomly chosen item is not defective} \end{array}$$

With these definitions, we can state our given information as

$$\begin{aligned} P(A) &= \frac{1}{4} \\ P(B) &= \frac{1}{2} \\ P(C) &= \frac{1}{4} \\ P(N|A) &= \frac{98}{100} \\ P(N|B) &= \frac{98}{100} \\ P(N|C) &= \frac{99}{100} \end{aligned}$$

Using Bayes' Rule, we can say

$$\begin{aligned} P(A \cap N) &= P(N|A)P(A) = \frac{98}{100} \times \frac{1}{4} \\ P(B \cap N) &= P(N|B)P(B) = \frac{98}{100} \times \frac{1}{2} \\ P(C \cap N) &= P(N|C)P(C) = \frac{99}{100} \times \frac{1}{4} \end{aligned}$$

With this data, we can now construct the array for the joint probability shown in Table 1. To construct the last entry in the 'N' column, we add all of the elements above it. To construct the 'D' column, we subtract the 'N' column from the 'Marginal Probabilities' column.
Thanks to Table 1, we see that our desired result is the marginal probability of the 'D' column:

$$P(D) = \frac{7}{400}$$

(b) Using Bayes' Rule, our desired result is given by

$$P(C|N) = \frac{P(C \cap N)}{P(N)} = \frac{99}{393}$$

| | N | D | Marginal Probabilities |
|---|---|---|---|
| A | $\frac{98}{100} \times \frac{1}{4} = \frac{49}{200}$ | $\frac{1}{4} - \frac{49}{200} = \frac{1}{200}$ | $\frac{1}{4}$ |
| B | $\frac{98}{100} \times \frac{1}{2} = \frac{49}{100}$ | $\frac{1}{2} - \frac{49}{100} = \frac{1}{100}$ | $\frac{1}{2}$ |
| C | $\frac{99}{100} \times \frac{1}{4} = \frac{99}{400}$ | $\frac{1}{4} - \frac{99}{400} = \frac{1}{400}$ | $\frac{1}{4}$ |
| Marginal Probabilities | $\frac{49}{200} + \frac{49}{100} + \frac{99}{400} = \frac{393}{400}$ | $\frac{1}{200} + \frac{1}{100} + \frac{1}{400} = \frac{7}{400}$ | 1 |

Table 1: Array of joint probability

2. In this problem, we will denote the three doors as $x$, $y$, and $z$. Without loss of generality, we will assume that the contestant originally picked door $x$. We now define $C_i$ to be the event that the car is behind door $i$ and $H_j$ to be the event that the host opens door $j$. With this in mind, note that the mutually exclusive events $C_x$, $C_y \cap H_z$, and $C_z \cap H_y$ cover the sample space, i.e.

$$1 = P(C_x) + P(C_y \cap H_z) + P(C_z \cap H_y).$$

Given that the contestant switches her guess, the probability that she will win is given by $P((C_y \cap H_z) \cup (C_z \cap H_y))$. Since the event $C_y \cap H_z$ is disjoint from the event $C_z \cap H_y$, we can say that

$$\begin{aligned} P(\text{win}|\text{she switches}) &= P((C_y \cap H_z) \cup (C_z \cap H_y)) \\ &= P(C_y \cap H_z) + P(C_z \cap H_y) \\ &= 1 - P(C_x) = \frac{2}{3}. \end{aligned}$$

3. Let

$$\begin{aligned} A_i &= \text{the } i\text{th child is a boy and he is born on Tuesday, } i = 1, 2 \\ B &= \text{the two children are boys} \end{aligned}$$

Then,

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) = \frac{14 + 14 - 1}{196} = \frac{27}{196}$$

Thus, $A_i \cap B$ denotes that the two children are boys and the $i$th boy is born on Tuesday. We have

$$\begin{aligned} P(A_i \cap B) &= \frac{7}{196}, \quad i = 1, 2 \\ \Rightarrow P((A_1 \cup A_2) \cap B) &= P((A_1 \cap B) \cup (A_2 \cap B)) \\ &= P(A_1 \cap B) + P(A_2 \cap B) - P((A_1 \cap B) \cap (A_2 \cap B)) \\ &= P(A_1 \cap B) + P(A_2 \cap B) - P(A_1 \cap A_2) = \frac{7 + 7 - 1}{196} = \frac{13}{196} \end{aligned}$$

As a result, we have

$$P(B \,|A_1 \cup A_2\,) = \frac{P((A_1 \cup A_2) \cap B)}{P(A_1 \cup A_2)} = \frac{13}{27}$$

4. (a) First we will define $Y = X_1 + X_2$. Now, since $X_1$ and $X_2$ are independent, we can apply the property that the PDF of $Y$ is the convolution of the PDF of $X_1$ and the PDF of $X_2$:

$$p_Y(y) = \int_{-\infty}^{\infty} p_{X_1}(x_1) p_{X_2}(y - x_1) dx_1$$

Note that the PDF of $X_1$ only takes on the values of 1 and 0. Thus, we only need to integrate $p_{X_2}(y-x_1)$ over the regions where $p_{X_1}(x_1)$ has a value of 1. Thus,

$$p_Y(y) = \int_0^1 p_{X_2}(y-x_1)dx_1$$

Now note that the following conditions hold

$$\begin{aligned} & p_{X_2}(y-x_1) = 1 \\ \Leftrightarrow \quad & 0 \le y - x_1 \le 1 \\ \Leftrightarrow \quad & y-1 \le x_1 \le y \end{aligned}$$

Thus, we get

$$\begin{aligned} p_Y(y) &= \begin{cases} \int_0^y dx_1 & \text{for } 0 \le y \le 1 \\ \int_{y-1}^1 dx_1 & \text{for } 1 \le y \le 2 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} y & \text{for } 0 \le y \le 1 \\ 2-y & \text{for } 1 \le y \le 2 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(b) First we define $Z = X_1 + X_2 + X_3 = Y + X_3$. Following a similar procedure as before,

$$\begin{aligned} p_Z(z) &= \int_{-\infty}^{\infty} p_{X_3}(x_3)p_Y(z-x_3)dx_3 \\ &= \int_0^1 p_Y(z-x_3)dx_3 \\ p_Y(z-x_3) &= \begin{cases} z-x_3 & \text{for } z-1 \le x_3 \le z \\ 2-z+x_3 & \text{for } z-2 \le x_3 \le z-1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Thus, we get that

$$\begin{aligned} p_Z(z) &= \begin{cases} \int_0^z (z-x_3)dx_3 & \text{for } 0 \le z \le 1 \\ \int_0^{z-1}(2-z+x_3)dx_3 + \int_{z-1}^1 (z-x_3)dx_3 & \text{for } 1 \le z \le 2 \\ \int_{z-2}^1 (2-z+x_3)dx_3 & \text{for } 2 \le z \le 3 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{2}z^2 & \text{for } 0 \le z \le 1 \\ -z^2+3z-\frac{3}{2} & \text{for } 1 \le z \le 2 \\ \frac{1}{2}z^2 - 3z + \frac{9}{2} & \text{for } 2 \le z \le 3 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Figure 1 shows the PDFs of $X_1$, $Y$, and $Z$. Notice that each time an extra variable is added onto the random variable being looked at, the mean moves to the right, the maximum value moves to the right, and, to compensate, the maximum value of the PDF starts to drop. Also, the plot of the PDF starts to look more and more like a Gaussian distribution.

5. Since $X$ and $Y$ are Gaussian random variables, their moment generating functions are given by

$$P_X(j\omega) = \mathcal{F}\{p_X(\cdot)\} = \exp\left(-j\omega m_X - \frac{\sigma_X^2\omega^2}{2}\right)$$
$$P_Y(j\omega) = \mathcal{F}\{p_Y(\cdot)\} = \exp\left(-j\omega m_Y - \frac{\sigma_Y^2\omega^2}{2}\right)$$

Since $Z = X + Y$ is the sum of two independent random variables, we can say that

$$P_Z(j\omega) = \mathcal{F}\{p_Z(\cdot)\} = \mathcal{F}\{p_X(\cdot)\}\,\mathcal{F}\{p_Y(\cdot)\}$$

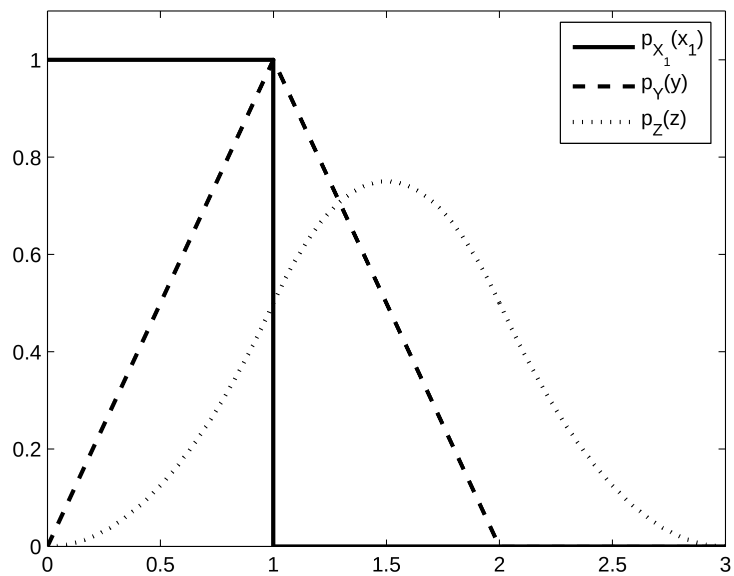


Figure 1: PDFs for $X_1$, $Y$, and $Z$

Substituting our expressions for the moment generating functions of $X$ and $Y$ then gives

$$\begin{aligned} P_Z(j\omega) &= \exp\left\{\left(-j\omega m_X - \frac{\sigma_X^2\omega^2}{2}\right) + \left(-j\omega m_Y - \frac{\sigma_Y^2\omega^2}{2}\right)\right\} \\ &= \exp\left\{-j\omega\left(m_X + m_Y\right) - \frac{\left(\sigma_X^2 + \sigma_Y^2\right)\omega^2}{2}\right\} \end{aligned}$$

Note that this is the moment generating function of a Gaussian random variable with mean $m_X + m_Y$ and variance $\sigma_X^2 + \sigma_Y^2$. Therefore, $Z \sim N(m_X + m_Y, \sigma_X^2 + \sigma_Y^2)$.