ME233

Hyperstability and Positivity (Re: Y. D. Landau, Adaptive Control, The Model Reference Approach, Marcel Dekker, 1979)

The hyperstability problem was formulated by Popov for a class of nonlinear feedback systems as depicted in Fig. H-1. In the feedforward path of the nonlinear system, we have a linear time invariant block defined by

$$\begin{array}{ccc} \textit{Continuous Time Case} & : & \textit{Discrete Time Case} \\ \dfrac{dx(t)}{dt} = Ax(t)+Bu(t) & : & x(k+1) = Ax(k)+Bu(k) \\ v(t) = Cx(t)+Du(t) & : & v(k) = Cx(k)+Du(k) \end{array} \tag{H-1}$$

or by the time transfer function

$$G(s) = C(sI-A)^{-1}B+D \quad : \quad G(z) = C(zI-A)^{-1}B+D \tag{H-2}$$

The feedback block is a time varying nonlinear block that satisfies the *Popov inequality*

$$\int_{t_0}^{t_1} w^T(t)v(t)dt \geq -\gamma^2 \textit{ for all } t_1 \geq t_0 \quad : \quad \sum_{k=0}^{k_1} w^T(k)v(k) \geq -\gamma^2 \textit{ for all } k_1 \tag{H-3}$$

Definition (Hyperstability): The feedback system in Fig. H-1 is hyperstable if and only if there exists a positive constant $\delta > 0$ and a positive constant $\gamma > 0$ such that all solutions x (x is the state vector of the linear block) satisfy

$$\|x(t)\| < \delta \cdot [\|x(t_0)\|+\gamma] \textit{ for all } t > t_0 \quad : \quad \|x(k)\| < \delta \cdot [\|x(0)\|+\gamma] \textit{ for all } k > 0 \tag{H-4}$$

for all feedback blocks that satisfy the Popov inequality.

Definition (Asymptotic Hyperstability): The feedback system in Fig. H-1 is asymptotically hyperstable if and only if

i. it is hyperstable, and
ii. for all bounded w satisfying the Popov inequality,

$$\lim_{t\to\infty} x(t) = 0 \quad : \quad \lim_{k\to\infty} x(k) = 0 \tag{H-5}$$

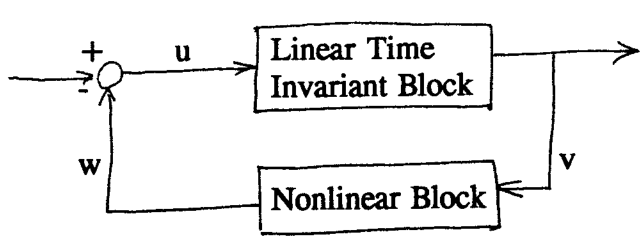

Fig. H-1 Nonlinear Feedback Block

Notice that the hyperstability and asymptotic hyperstability are defined for a class of nonlinear blocks characterized by the Popov inequality. Therefore, further conditions for the stability of the feedback system depend on the characteristics of the linear feedforward block. Hyperstability results are given by the following theorems.

Theorem (Hyperstability): The feedback system in Fig. H-1 is hyperstable if and only if the system described by (H-1) is positive or the transfer function (H-2) is positive real.

Theorem (Asymptotic Hyperstability): The feedback system in Fig. H-1 is asymptotically hyperstable if and only if the transfer function (H-1) is strictly positive real.

Positive systems, positive real and strictly positive real transfer functions are defined below for the SISO case.

Definition (Positive System): The system described by (H-1) is positive if the summation of the input output product can be expressed as

$$\int_{t=t_0}^{t} v(\tau)u(\tau)d\tau = \xi[x(t)] - \xi[x(t_0)] + \int_{t=t_0}^{t} \lambda(x(\tau),u(\tau))d\tau \quad \textit{for all } t:$$

$$\sum_{j=0}^{k-1} v(j)u(j) = \xi[x(k)] - \xi[x(0)] + \sum_{j=0}^{k-1} \lambda(x(j),u(j)) \quad \textit{for all } k$$

where

$$\xi[x] > 0 \quad \textit{for } x \neq 0 \quad \textit{and} \quad \lambda(x,u) \geq 0 \quad \textit{for all } x \textit{ and } u$$

Definition (Positive Real Transfer Function--continuous time case): The transfer function (H-2) is positive real if

i. G(s) is real for real s.

ii. Re{G(s)} > 0 for s with Re{s} > 0.

This definition is stated here to realize where the name, Positive Real TF, comes from. It is not practical to check i. and ii. for every s. Therefore, we normally use the following definition, which is equivalent to the above definition and is more directly related to analysis tools.

Definition (Positive Real Transfer Function): The transfer function (H-2) is positive real if

Continuous time case

i. it does not possess any pole in Re{s} > 0.

ii. any pole on the imaginary axis does not repeat and the associated residue (i.e. the coefficient appearing in the partial fraction expansion) is non-negative, and

iii. $G(j\omega) + G(-j\omega) = 2\,\mathrm{Re}\{G(j\omega)\}$ is non-negative for all real ω's for which $s = j\omega$ is not a pole of G(s).

Discrete time case

i. it does not possess any pole outside of the unit circle on z-plane

ii. any pole on the unit circle does not repeat and the associated residue (i.e. the coefficient appearing in the partial fraction expansion) is non-negative, and

iii. $G(e^{j\omega}) + G(e^{-j\omega}) = 2\mathrm{Re}\{G(e^{j\omega})\}$ is non-negative for all real ω's, $0 \leq \omega \leq \pi$ for which $z = e^{j\omega}$ is not a pole of $G(z)$.

Definition (Strictly Positive Real Transfer Function): The transfer function (H-2) is strictly positive real (SPR) if

Continuous time case

i. it does not possess any pole in $\mathrm{Re}\{s\} \geq 0$, and

ii. $G(j\omega) + G(-j\omega) = 2\,\mathrm{Re}\{G(j\omega)\} > 0$ for all ω's.

Discrete time case

1. it does not possess any pole outside of and on the unit circle on z-plane, and

ii. $G(e^{j\omega}) + G(e^{-j\omega}) = 2\mathrm{Re}\{G(e^{j\omega}) > 0$ for all real ω's, $0 \leq \omega \leq \pi$.

Examples:

1. $G(z) = c$, $c > 0$, is SPR.

2. $G(z) = 1/(z - a)$, $|a| < 1$, is asymptotically stable but is not positive real since $\mathrm{Re}\{G(e^{j\omega}) = (\cos\omega - a)/(1 + a^2 - 2a\cos\omega)$ which is negative for $\omega > \cos^{-1}a$.

3. $G(z) = z/(z - a)$, $|a| < 1$, is asymptotically stable and SPR. Note that $\mathrm{Re}\{G(e^{j\omega}) = (1 - a\cos\omega)/(1 + a^2 - 2a\cos\omega)$ which is positive for all real ω's, $0 \leq \omega \leq \pi$.

4. $G(z) = 1/(1 - a_1z^{-1} - a_2z^{-2} - \dots - a_nz^{-n})$, $|a_1| + |a_2| + \dots + |a_n| < 1$, is SPR.

The hyperstability and asymptotic hyperstability theorems are given in terms of necessary and sufficient conditions. This is a consequence of how (asymptotic) hyperstability is defined. When we utilize these theorems, we normally use the sufficiency portion: for example, given a feedback system with a particular nonlinear feedback block and a linear block, we examine 1. whether the nonlinear block satisfies the Popov inequality and 2. whether the linear block is (strictly) positive real. If the two conditions are positively confirmed, (asymptotic) hyperstability assures (asymptotic) stability in the ordinary sense.

For nonlinear blocks satisfying the Popov inequality, the following properties are known. Proof is straightforward.

Lemma: Suppose that two systems, S1 and S2, both satisfy the Popov inequality. Then, the system obtained as the parallel combination of S1 and S2 satisfies the Popov inequality. Also, the system obtained as the feedback connection of S1 and S2 satisfies the Popov inequality (see Fig. H-2).

The definition of hyperstability and (strictly) positive realness of transfer functions can best be understood by relating them to passive engineering systems. Imagine that the linear block in Fig. H-1 represents a passive system (e.g. see Fig. H-3). If u, v and x are selected properly, uv represent the instantaneous power supplied to the system and $||x(t)||^2$ represents the energy stored in the system. For the system in Fig. H-3, u and v are respectively the force applied to the mass and the velocity of the mass. Letting z_1 and z_2 represent the position and velocity of the mass from the equilibrium state, we define the state variables by

$$x_1 = \sqrt{k/2}z_1 \quad and \quad x_2 = \sqrt{m/2}z_2$$

Then, $||x(t)||^2$ is the sum of the potential energy stored in the spring and the kinetic energy of the mass. Although the Popov inequality is written in terms of w and v, it can be rewritten in terms of u and v noting u = -w: i.e.

$$\int_{t_0}^{t_1} u(t)v(t)dt \le \gamma_0^2 \text{ for all } t_1 \ge t_0$$

This inequality implies that the total energy supplied to the system from outside never exceeds γ_o^2. In this case, the energy stored in the passive system will never exceed the sum of the initial stored energy plus γ_o^2. In equation form,

$$\begin{aligned} & \|x(t_1)\|^2 \le \|x(t_0)\|^2 + \gamma_0^2 \\ \rightarrow \quad & \|x(t_1)\|^2 \le (\|x(t_0)\| + \gamma_0)^2 \\ \rightarrow \quad & \|x(t_1)\| \le \|x(t_0)\| + \gamma_0 \end{aligned}$$

Notice that the last inequality is a special case of (H-4). Namely, the definition of hyperstability requires the linear block behave like a passive system. Of course, the statement in the definition can be used to define passive systems in a broader sense. The state and output equations for the mass spring system are

$$\frac{dx_1}{dt} = \sqrt{\frac{k}{m}}x_2, \quad \frac{dx_2}{dt} = -\sqrt{\frac{k}{m}}x_1 - \frac{k}{m}x_2 + \frac{1}{\sqrt{2m}}u, \quad v = \sqrt{\frac{2}{m}}x_2$$

and the transfer function from u to v is

$$G_{uv}(s) = \frac{s}{ms^2 + bs + k}$$

which is positive real.

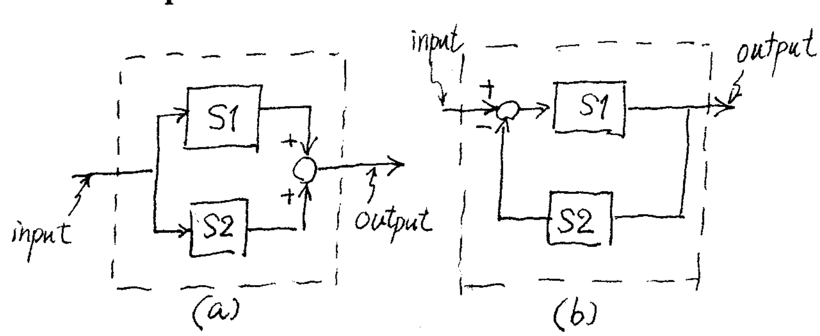

Fig. H-2 (a) Parallel Combination of S1 and S2.
(b) Feedback Connection of S1 and S2.

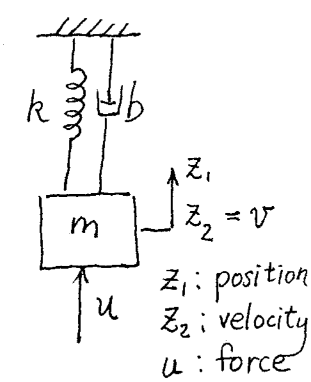

Fig. H-3 Mass-Spring-Dashpot System

Example: Continuous Time MRAS

In this problem, the reference model and adjustable system are respectively given by

$$\begin{aligned} &\textit{Reference model:} && \frac{d^2y}{dt^2}+a_1\frac{dy}{dt}+a_2y = b_1\frac{du}{dt}+b_2u \\ &\textit{Adjustable system:} && \frac{d^2\hat{y}}{dt^2}+\hat{a}_1(t)\frac{d\hat{y}}{dt}+\hat{a}_2(t)\hat{y} = \hat{b}_1(t)\frac{du}{dt}+\hat{b}_2(t)u \end{aligned} \tag{H-6}$$

We assume that u(t) and du(t)/dt is bounded and the reference model is asymptotically stable. Therefore, y(t) is bounded also. We define

$$\begin{aligned} &\textit{Output error signal:} && e = y-\hat{y} \\ &\textit{Adaptation error signal:} && v = c_1\frac{de}{dt}+c_2e \\ &\textit{Parameter vector:} && \theta^T = [a_1,\ a_2,\ b_1,\ b_2] \\ &\textit{Signal vector:} && \phi^T(t) = \left[-\frac{d\,\hat{y}}{dt}\quad -\hat{y}\quad \frac{du}{dt}\quad u\right] \end{aligned} \tag{H-7}$$

We will soon find why the output error signal cannot be used as an adaptation error signal.

Let the parameter adaptation algorithm be given by

$$\frac{d\hat{\theta}(t)}{dt} = F\phi(t)v(t), \quad F > 0 \tag{H-8}$$

Stability analysis by Popov's Hyperstability

From the defining equations for the reference model and adjustable system, we obtain

$$\begin{aligned} &\ddot{e}+a_1\dot{e}+a_2e+(a_1-\hat{a}_1(t))\frac{d\hat{y}}{dt}+(a_2-\hat{a}_2(t))\hat{y} = (b_1-\hat{b}_1(t))\frac{du}{dt}+(b_2-\hat{b}_2(t))u \\ &\rightarrow \left(\frac{d^2}{dt^2}+a_1\frac{d}{dt}+a_2\right)e = -(\hat{\theta}(t)-\theta)^T\phi(t) = -\tilde{\theta}^T(t)\phi(t) \end{aligned} \tag{H-9}$$

The PAA, Eq. (H-8), implies

$$\frac{d\tilde{\theta}(t)}{dt} = F\phi(t)v(t) \tag{H-10}$$

Eqs. (H-9) and (H-10) are put in the following feedback form.

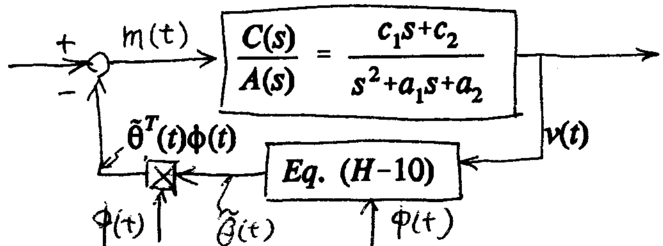

Fig. H-4 Equivalent Feedback System

To apply the Popov (asymptotic) hyperstability theorem, we show that the feedback block satisfies the Popov inequality. Noting the input and output for the nonlinear block (see figure above),

$$\begin{aligned}\int_0^{t_1}(\tilde{\theta}^T(t)\phi(t))v(t)\,dt &= \int_0^{t_1}\tilde{\theta}^T F^{-1}\frac{d\tilde{\theta}(t)}{dt}dt \\ &= \int_0^{t_1}\frac{1}{2}\frac{d}{dt}(\tilde{\theta}^T(t)F^{-1}\tilde{\theta}(t))dt \\ &= \frac{1}{2}\tilde{\theta}^T(t_1)F^{-1}\tilde{\theta}(t_1)-\frac{1}{2}\tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0) \\ &\geq -\frac{1}{2}\tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0)\end{aligned} \qquad \text{(H-11)}$$

Therefore, if the linear block, C(s)/A(s), is strictly positive real, the feedback loop is at least hyperstable. Furthermore, the SPR condition of C(s)/A(s) implies that

$$\int_0^{t_1} m(t)v(t)dt \geq \beta \quad (\beta \in R) \qquad \text{(H-12)}$$

Noting $m(t) = -\tilde{\theta}^T(t)\phi(t)$ and the development in (H-11), (H-12) implies that $||\tilde{\theta}||$ is bounded. Since v(t) is bounded, e(t) and de(t)/dt are both bounded. u(t) is bounded, which implies that y(t) and dy(t)/dt are both bounded. du(t)/dt is bounded. Noting $\hat{y} = e+y$ and $d\hat{y}/dt = de/dt + dy/dt$, $\phi(t)$ is bounded. Therefore, $w(t) = -\tilde{\theta}\,\phi$ is bounded. Now, the asymptotic hyperstability theorem can be applied and the convergence of v(t) to zero is concluded. Of course, this implies that e and de/dt converge to zero also. Normally, we skip details of the above development, and we jump into the asymptotic hyperstability after showing the Popov inequality and the SPR condition of the linear feedforward block.

<u>Stability analysis by Lyapunov's stability</u>

Eq. (H-9) and the adaptation error signal can be written in the state equation form

$$\frac{d}{dt}\begin{bmatrix} e \\ \dot{e} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}\begin{bmatrix} e \\ \dot{e} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} m, \quad m = -\tilde{\theta}^T(t)\phi(t) = -\phi^T(t)\tilde{\theta}(t) \tag{H-13}$$

$$v(t) = [c_2 \quad c_1]\begin{bmatrix} e \\ \dot{e} \end{bmatrix}$$

Define A, B and C by

$$A = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad and \quad C = [c_2 \quad c_1] \tag{H-14}$$

Try a candidate Lyapunov function given by

$$V = \frac{1}{2}\begin{bmatrix} e \\ \dot{e} \end{bmatrix}^T P \begin{bmatrix} e \\ \dot{e} \end{bmatrix} + \frac{1}{2}\tilde{\theta}^T F^{-1}\tilde{\theta} \tag{H-15}$$

Time derivative of V is

$$\dot{V} = \frac{1}{2}\begin{bmatrix} e \\ \dot{e} \end{bmatrix}^T [A^T P + PA]\begin{bmatrix} e \\ \dot{e} \end{bmatrix} + \left(\begin{bmatrix} e \\ \dot{e} \end{bmatrix}^T PBm + [d\tilde{\theta}/dt]^T F^{-1}\tilde{\theta}\right) \tag{H-16}$$

If there exists a symmetric positive definite matrix P such that

$$A^T P + PA = -Q < 0 (i.e. Q \text{ is positive definite}) \quad and \quad PB = C^T \tag{H-17}$$

the first term in the right hand side of Eq. (H-16) is negative and the second term vanishes if PAA is given by Eq. (H-8), i.e. Eq. (H-10) is satisfied. We note that the conditions stated in (H-17) is equivalent to the SPR condition for the transfer function from m to v: i.e.

$$C(sI-A)^{-1}B = \frac{c_1 s + c_2}{s^2 + a_1 s + a_2} \tag{H-18}$$

From the viewpoint of designing PAA, you utilize (H-17) in the following way: 1. pick up a positive definite Q and find P (since A is asymptotically stable, $P > 0$ is assured), and 2. set C to $B^T P$ (recall that the selection of C is a part of the PAA design).

The conditions in (H-17) make V a Lyapunov function. V is quadratic, which implies $\alpha(||x||) \leq V(x) \leq \beta(||x||)$ where α and β are nondecreasing functions of $||x||$ and $\alpha(||x||) \to \infty$ as $||x|| \to \infty$ where $x^T = [e, de/dt, \tilde{\theta}^T]$. This implies that $||x||$ are bounded and $[e, de/dt]^T$ (note: $dV/dt = -[e \;\; de/dt]Q[e \;\; de/dt]^T$ and $Q > 0$) are square integrable (see e.g. Narendra and Annaswamy, Stable Adaptive Systems, Prentice Hall). If $\phi(t)$ is bounded, noting Eq. (H-13) $d/dt[e, de/dt]^T$ is bounded. This implies that $[e \;\; de/dt]^T$ converges to zero (Barbalat lemma; e.g. see Narendra and Annaswamy).

As stated, the conditions in (H-17) are useful in the design of MRAS. Related results are summarized below (e.g. see Landau, Adaptive Control -- The Model Reference Approach, Marcel Dekker, 1979).

Lemma SPR1: The transfer function matrix (H-2) is SPR, and the elements of G(s) are analytic in Re{s} > $-\mu$ if there exists a symmetric positive definite matrix P, a matrix L, a scalar $\mu > 0$ (or a symmetric positive definite matrix Q') and a matrix K such that

$$\begin{aligned} PA+A^TP &= -LL^T-2\mu P = -Q' \\ B^TP+K^TL^T &= C \\ K^TK &= D+D^T \end{aligned}$$

Note: Since Q' and P are positive definite, we can always find a scalar $\mu > 0$ such that $Q' - 2\mu P \geq 0$, which implies the existence of a matrix L such that $LL^T = Q' - 2\mu P$.

Lemma SPR2: The transfer function matrix (H-2) is SPR if there exist a symmetric positive definite matrix P and a symmetric positive definite matrix Q such that the conditions in (H-17) are verified.

Lemma SPR3: The discrete time transfer function (H-2) is SPR if there exist a symmetric positive definite matrix P, a symmetric positive definite matrix Q, and matrices K and L such that

$$\begin{aligned} A^TPA-P &= -LL^T-Q=-Q' \\ B^TPA+K^TL^T &= C \\ K^TK &= D+D^T-B^TPB \end{aligned}$$

Note: For discrete time systems, results analogous to the one in Lemma SPR2 do not exist. In the discrete time case, the presence of D is necessary for SPR condition (recall Examples 2 and 3 on page HS-3).