# ME 233 – Advanced Control II

# Lecture 25

# Stability Analysis of a Direct Adaptive Control System

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May 11, 2012

# Outline

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# Deterministic SISO ARMA model

SISO ARMA plant model:

$$A(q^{-1})y(k) = q^{-\mathrm{d}}B(q^{-1})u(k)$$

where $y(k)$ and $u(k)$ are scalar

- $u(k)$ is the control input
- $y(k)$ is the output
- $\mathrm{d}$ is the pure time delay
- no disturbance

# Model assumptions

SISO ARMA plant model:

$$A(q^{-1})y(k) = q^{-\mathrm{d}}B(q^{-1})u(k)$$

where $y(k)$ and $u(k)$ are scalar

- The polynomials

$$A(q^{-1}) = 1 + a_1q^{-1} + \cdots + a_nq^{-n}$$
$$B(q^{-1}) = b_0 + b_1q^{-1} + \cdots + b_mq^{-m}$$

  are co-prime

- $B(q^{-1})$ is anti-Schur
- $m$, $n$, and $\mathrm{d}$ are known
- $0 < b_{mino} \leq b_0$, where $b_{mino}$ is known

## Control Objectives

1. **Pole Placement:** The poles of the closed-loop system must be placed at specific locations in the complex plane

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1. **Pole Placement:** The poles of the closed-loop system must be placed at specific locations in the complex plane

   Closed-loop polynomial:

   $$A_c(q^{-1}) = B(q^{-1})A_c^{'}(q^{-1})$$

   where $A_c^{'}(q^{-1})$ is an anti-Schur polynomial chosen by the designer:

   $$A_c^{'}(q^{-1}) = 1 + a_{c1}^{'}q^{-1} + \cdots + a_{c(n_c^{'})}^{'}q^{-(n_c^{'})}$$

# Control Objectives

2. **Tracking:** The output sequence $y(k)$ must follow an arbitrary bounded reference sequence $y_d(k)$, which is known

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$y_d(k)$ is generated by the reference model

$$A_c'(q^{-1})y_d(k) = q^{-\mathrm{d}}B_m(q^{-1})u_d(k)$$

where

- $u_d(k)$ is a known <u>bounded</u> reference input control input sequence
- $B_m(q^{-1})$ is chosen by the designer

# Control Objectives

2. **Tracking:** The output sequence $y(k)$ must follow an arbitrary bounded reference sequence $y_d(k)$, which is known

$y_d(k)$ is generated by the reference model

$$A'_c(q^{-1})y_d(k) = q^{-\mathrm{d}}B_m(q^{-1})u_d(k)$$

where

- $u_d(k)$ is a known bounded reference input control input sequence
- $B_m(q^{-1})$ is chosen by the designer

Note that $A'_c(q^{-1})$ comes from the pole placement and the reference model delay is the same as the plant delay

# Reformulated plant dynamics

Using the solution of the Diophantine equation

$$A_c'(q^{-1}) = A(q^{-1})R'(q^{-1}) + q^{-\mathrm{d}}S(q^{-1})$$

we rewrite the plant dynamics as

$$A_c'(q^{-1})y(k) = q^{-\mathrm{d}}\Big[R(q^{-1})u(k) + S(q^{-1})y(k)\Big]$$

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where $R(q^{-1}) = R^{'}(q^{-1})B(q^{-1})$ and

$$R(q^{-1}) = r_0 + r_1 q^{-1} + \cdots + r_{n_r} q^{-n_r}$$
$$S(q^{-1}) = s_0 + s_1 q^{-1} + \cdots + s_{n_s} q^{-n_s}$$

$$n_r = m + \mathrm{d} - 1 \qquad\qquad n_s = \max\{n-1, n_c^{'} - \mathrm{d}\}$$

# Reformulated plant dynamics

So far, we know that

$$A_c'(q^{-1})y(k) = q^{-\mathrm{d}}\Big[R(q^{-1})u(k) + S(q^{-1})y(k)\Big]$$

$$R(q^{-1}) = r_0 + r_1 q^{-1} + \cdots + r_{n_r} q^{-n_r}$$

$$S(q^{-1}) = s_0 + s_1 q^{-1} + \cdots + s_{n_s} q^{-n_s}$$

Defining $\eta(k) = A_c'(q^{-1})y(k)$ and

$$\phi(k) = \begin{bmatrix} y(k) & \cdots & y(k-n_s) & u(k) & \cdots & u(k-n_r) \end{bmatrix}^T$$

$$\theta_c = \begin{bmatrix} s_0 & \cdots & s_{n_s} & r_0 & \cdots & r_{n_r} \end{bmatrix}^T$$

we rewrite the plant dynamics as

$$\boxed{\eta(k) = \phi^T(k-\mathrm{d})\theta_c}$$

# Direct adaptive control approach

The plant dynamics are written as

$$\eta(k) = \phi^T(k - \mathrm{d})\theta_c$$

- $\eta(k)$ is the known "filtered output"
- $\phi(k)$ is the known regressor vector
- $\theta_c$ is the <u>unknown</u> parameter vector

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- $\theta_c$ is the <u>unknown</u> parameter vector

  $\Rightarrow$ we use RLS to estimate $\theta_c$

# Tracking control objective

We would like to achieve

$$\lim_{k\to\infty}\{y(k) - y_d(k)\} = 0$$

# Tracking control objective

We would like to achieve

$$\lim_{k\to\infty}\{y(k)-y_d(k)\}=0$$

Since $A_c'(q^{-1})$ is anti-Schur this is equivalent to

$$\begin{aligned}0 &= \lim_{k\to\infty}\{A_c'(q^{-1})[y(k)-y_d(k)]\}\\ &= \lim_{k\to\infty}\{\eta(k)-\eta_d(k)\}\end{aligned}$$

where $\eta_d(k)=A_c'(q^{-1})y_d(k)=q^{-\mathrm{d}}B_m(q^{-1})u_d(k)=r(k-\mathrm{d})$.

# List of error signals

Parameter estimation error:

$$\tilde{\theta}_c(k) = \theta_c - \hat{\theta}_c(k)$$

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Filtered output estimation errors:

$$\begin{aligned}
e^o(k) &= \eta(k) - \phi^T(k-\mathrm{d})\hat{\theta}_c(k-1) && \text{a-priori} \\
&= \phi^T(k-\mathrm{d})\tilde{\theta}_c(k-1) \\
e(k) &= \eta(k) - \phi^T(k-\mathrm{d})\hat{\theta}_c(k) && \text{a-posteriori} \\
&= \phi^T(k-\mathrm{d})\tilde{\theta}_c(k)
\end{aligned}$$

## List of error signals

Parameter estimation error:

$$\tilde{\theta}_c(k) = \theta_c - \hat{\theta}_c(k)$$

Filtered output estimation errors:

$$\begin{aligned} e^o(k) &= \eta(k) - \phi^T(k-\mathrm{d})\hat{\theta}_c(k-1) && \text{a-priori} \\ &= \phi^T(k-\mathrm{d})\tilde{\theta}_c(k-1) \\ e(k) &= \eta(k) - \phi^T(k-\mathrm{d})\hat{\theta}_c(k) && \text{a-posteriori} \\ &= \phi^T(k-\mathrm{d})\tilde{\theta}_c(k) \end{aligned}$$

Filtered output tracking error:

$$\epsilon(k) = \eta(k) - \eta_d(k)$$

# Direct adaptive control

1. $\eta(k+1) = A_c'(q^{-1})y(k+1)$
2. $\phi(k-\mathrm{d}+1) = \begin{bmatrix} y(k-\mathrm{d}+1) \\ \vdots \\ y(k-\mathrm{d}+1-n_s) \\ u(k-\mathrm{d}+1) \\ \vdots \\ u(k-\mathrm{d}+1-n_r) \end{bmatrix}$
3. $e^o(k+1) = \eta(k+1) - \phi^T(k-\mathrm{d}+1)\hat{\theta}_c(k)$
4. $e(k+1) = \dfrac{\lambda_1(k)}{\lambda_1(k) + \phi^T(k-\mathrm{d}+1)F(k)\phi(k-\mathrm{d}+1)} e^o(k+1)$
5. $\hat{\theta}_c^o(k+1) = \hat{\theta}_c(k) + \dfrac{1}{\lambda_1(k)} F(k)\phi(k-\mathrm{d}+1)e(k+1)$

# Direct adaptive control

6. Form $\hat{\theta}_c(k+1)$:

$$\hat{s}_i(k+1) = \hat{s}_i^o(k+1), \quad i = 0, \ldots, n_s$$

$$\hat{r}_i(k+1) = \hat{r}_i^o(k+1), \quad i = 1, \ldots, n_r$$

$$\hat{r}_0(k+1) = \max\{b_{mino}, \hat{r}_0^o(k+1)\} \qquad \text{parameter projection}$$

7. $$F(k+1) = \frac{1}{\lambda_1(k)}\left[F(k) - \lambda_2(k)\frac{F(k)\phi(k-\mathrm{d}+1)\phi^T(k-\mathrm{d}+1)F(k)}{\lambda_1(k)+\lambda_2(k)\phi^T(k-\mathrm{d}+1)F(k)\phi(k-\mathrm{d}+1)}\right]$$

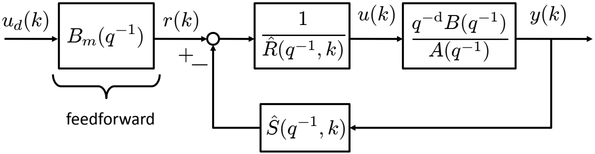
where $\lambda_1(k)$ and $\lambda_2(k)$ are chosen so that

$$0 < \underline{\lambda}_1 \leq \lambda_1(k) \leq 1 \qquad 0 \leq \lambda_2(k) \leq \overline{\lambda}_2 < 2$$

and $0 < K_{min} \leq \lambda_{min}(F(k)) \leq \lambda_{max}(F(k)) \leq K_{max} < \infty$

# Direct adaptive control

8. Apply control




$$\hat{R}(q^{-1},k)u(k) = B_m(q^{-1})u_d(k) - \hat{S}(q^{-1},k)y(k)$$

where

$$\hat{R}(q^{-1},k) = \hat{r}_0(k) + \hat{r}_1(k)q^{-1} + \cdots + \hat{r}_{n_r}(k)q^{-n_r}$$
$$\hat{S}(q^{-1},k) = \hat{s}_0(k) + \hat{s}_1(k)q^{-1} + \cdots + \hat{s}_{n_s}(k)q^{-n_s}$$

# Outline

# Stability theorem

Using the direct adaptive control approach just outlined, the tracking error converges to zero, i.e.

$$\boxed{\lim_{k \to \infty} \epsilon(k) = 0}$$

Moreover, $u(k)$ remains bounded, $e(k) \longrightarrow 0$, and $e^o(k) \longrightarrow 0$.

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Note that the theorem does not require:

- a-priori knowledge that the control input sequence $u(k)$ is bounded
- the polynomial $A(q^{-1})$ is anti-Schur
- any sort of persistence of excitation condition

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Note that the theorem <u>does not</u> require:

- a-priori knowledge that the control input sequence $u(k)$ is bounded
- the polynomial $A(q^{-1})$ is anti-Schur
- any sort of persistence of excitation condition

The theorem <u>does not</u> state that the parameter estimates converge to the true values

# Outline

# Outline of stability theorem proof

1. Use hyperstability theory to show that

$$\lim_{k \to \infty} e(k) = 0$$

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2. Prove the limits

$$\lim_{k\to\infty} \|\hat{\theta}_c(k) - \hat{\theta}_c(k-1)\| = 0$$

$$\lim_{k\to\infty} \frac{[\lambda_1(k-1)e^o(k)]^2}{\lambda_1(k-1) + \phi^T(k-\mathrm{d})F(k-1)\phi(k-\mathrm{d})} = 0$$

$$\lim_{k\to\infty} \frac{[\lambda_1(k-1)\epsilon(k)]^2}{\lambda_1(k-1) + \phi^T(k-\mathrm{d})F(k-1)\phi(k-\mathrm{d})} = 0$$

# Outline of stability theorem proof

3. Prove that there exist $C_1 \geq 0, \; C_2 \geq 0$ such that

$$\|\phi(k - \mathrm{d})\| \leq C_1 + C_2 \max_{j \in \{0,\ldots,k\}} |\epsilon(j)|$$

# Outline of stability theorem proof

3. Prove that there exist $C_1 \geq 0,\ C_2 \geq 0$ such that

$$\|\phi(k-\mathrm{d})\| \leq C_1 + C_2 \max_{j\in\{0,\ldots,k\}} |\epsilon(j)|$$

4. Prove Goodwin's technical lemma, which states that $\|\phi(k)\|$ remains bounded and

$$\lim_{k\to\infty} \epsilon(k) = 0$$

5. Prove that

$$\lim_{k\to\infty} e^o(k) = 0$$

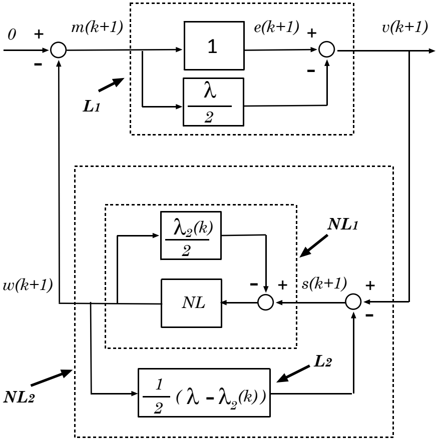
# Outline

# Stability theorem proof, part 1

- We want to show that $e(k) \to 0$
- Simplification: neglect parameter projection
- We will use hyperstability, as in Lecture 21

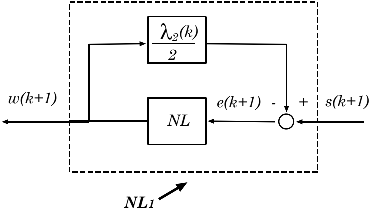
# Stability theorem proof, part 1

As in Lecture 21, the estimation error dynamics can be expressed using the block diagram



# Stability theorem proof, part 1

We will now show that $NL_1$ is P-class:




$$w(k) = -\phi^T(k - \mathrm{d})\tilde{\theta}_c(k)$$

# Stability theorem proof, part 1

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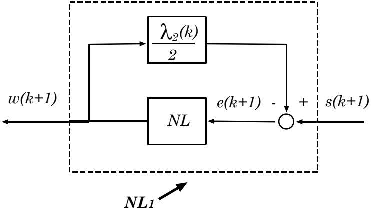


$$w(k) = -\phi^T(k-\mathrm{d})\tilde{\theta}_c(k)$$

Note that $e(k) = s(k) - \dfrac{\lambda_2(k-1)}{2}w(k)$, which implies that

$$s(k) = \frac{\lambda_2(k-1)}{2}w(k) + e(k)$$

# Stability theorem proof, part 1

$$
\begin{aligned}
2w(k)s(k) &= w(k)\left[\lambda_2(k-1)w(k) + 2e(k)\right] \\
&= \lambda_2(k-1)\tilde{\theta}_c^T(k)\phi(k-\mathrm{d})\phi^T(k-\mathrm{d})\tilde{\theta}_c(k) \\
&\quad - 2\tilde{\theta}_c^T(k)[\phi(k-\mathrm{d})e(k)] \\
&= \tilde{\theta}_c^T(k)\Big[\lambda_2(k-1)\phi(k-\mathrm{d})\phi^T(k-\mathrm{d})\Big]\tilde{\theta}_c(k) \\
&\quad - 2\tilde{\theta}_c^T(k)\left[\lambda_1(k)F^{-1}(k-1)\Big(\tilde{\theta}_c(k-1) - \tilde{\theta}_c(k)\Big)\right]
\end{aligned}
$$

# Stability theorem proof, part 1

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\begin{aligned}
2w(k)s(k) &= w(k)\left[\lambda_2(k-1)w(k) + 2e(k)\right] \\
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&\quad - 2\tilde{\theta}_c^T(k)[\phi(k-\mathrm{d})e(k)] \\
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\end{aligned}
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Define $\Delta\theta_c(k) = \hat{\theta}(k) - \hat{\theta}(k-1) = \tilde{\theta}_c(k-1) - \tilde{\theta}_c(k)$

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&\quad - 2\tilde{\theta}_c^T(k)[\phi(k-\mathrm{d})e(k)] \\
&= \tilde{\theta}_c^T(k)\Big[\lambda_2(k-1)\phi(k-\mathrm{d})\phi^T(k-\mathrm{d})\Big]\tilde{\theta}_c(k) \\
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\end{aligned}
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Define $\Delta\theta_c(k) = \hat{\theta}(k) - \hat{\theta}(k-1) = \tilde{\theta}_c(k-1) - \tilde{\theta}_c(k)$

$$
\begin{aligned}
2w(k)s(k) &= \tilde{\theta}_c^T(k)\Big[F^{-1}(k) - \lambda_1(k)F^{-1}(k-1)\Big]\tilde{\theta}_c(k) \\
&\quad - 2\lambda_1(k)\tilde{\theta}_c^T(k)F^{-1}(k-1)\Delta\theta_c(k)
\end{aligned}
$$

# Stability theorem proof, part 1

$$
\begin{aligned}
2w(k)s(k) = {} & \tilde{\theta}_c^T(k)\Big[F^{-1}(k) - \lambda_1(k)F^{-1}(k-1)\Big]\tilde{\theta}_c(k) \\
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\end{aligned}
$$

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$$
\begin{aligned}
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& +2\tilde{\theta}_c^T(k)F^{-1}(k-1)\Delta\theta_c(k)\Big] \\
= {} & \tilde{\theta}_c^T(k)F^{-1}(k)\tilde{\theta}_c(k) \\
& - \lambda_1(k)\Big[\Big(\tilde{\theta}_c(k) + \Delta\theta_c(k)\Big)^T F^{-1}(k-1)\Big(\tilde{\theta}_c(k) + \Delta\theta_c(k)\Big) \\
& - \Delta\theta_c^T(k)F^{-1}(k-1)\Delta\theta_c(k)\Big]
\end{aligned}
$$

# Stability theorem proof, part 1

$$2w(k)s(k) = \tilde{\theta}_c^T(k)\Big[F^{-1}(k) - \lambda_1(k)F^{-1}(k-1)\Big]\tilde{\theta}_c(k) - 2\lambda_1(k)\tilde{\theta}_c^T(k)F^{-1}(k-1)\Delta\theta_c(k)$$

---

$$\begin{aligned}
2w(k)s(k) &= \tilde{\theta}_c^T(k)F^{-1}(k)\tilde{\theta}_c(k) - \lambda_1(k)\Big[\tilde{\theta}_c^T(k)F^{-1}(k-1)\tilde{\theta}_c(k) \\
&\quad +2\tilde{\theta}_c^T(k)F^{-1}(k-1)\Delta\theta_c(k)\Big] \\
&= \tilde{\theta}_c^T(k)F^{-1}(k)\tilde{\theta}_c(k) \\
&\quad - \lambda_1(k)\Big[\Big(\tilde{\theta}_c(k) + \Delta\theta_c(k)\Big)^T F^{-1}(k-1)\Big(\tilde{\theta}_c(k) + \Delta\theta_c(k)\Big) \\
&\quad - \Delta\theta_c^T(k)F^{-1}(k-1)\Delta\theta_c(k)\Big]
\end{aligned}$$

Note that $\tilde{\theta}_c(k) + \Delta\theta_c(k) = \tilde{\theta}_c(k-1)$

# Stability theorem proof, part 1

From the previous slide,

$$2w(k)s(k) = \tilde{\theta}_c^T(k)F^{-1}(k)\tilde{\theta}_c(k) - \lambda_1(k)\tilde{\theta}_c^T(k-1)F^{-1}(k-1)\tilde{\theta}_c(k-1)$$
$$+ \lambda_1(k)\Delta\theta_c^T(k)F^{-1}(k-1)\Delta\theta_c(k)$$

# Stability theorem proof, part 1

From the previous slide,

$$2w(k)s(k) = \tilde{\theta}_c^T(k)F^{-1}(k)\tilde{\theta}_c(k) - \lambda_1(k)\tilde{\theta}_c^T(k-1)F^{-1}(k-1)\tilde{\theta}_c(k-1)$$
$$+ \lambda_1(k)\Delta\theta_c^T(k)F^{-1}(k-1)\Delta\theta_c(k)$$

Since $\lambda_1(k) \leq 1$ and $F(k-1) \succ 0$, this implies that

$$2w(k)s(k) \geq \left[\tilde{\theta}_c^T(k)F^{-1}(k)\tilde{\theta}_c(k) - \tilde{\theta}_c^T(k-1)F^{-1}(k-1)\tilde{\theta}_c(k-1)\right]$$

# Stability theorem proof, part 1

$$2w(k)s(k) \geq \left[\tilde{\theta}_c^T(k)F^{-1}(k)\tilde{\theta}_c(k) - \tilde{\theta}_c^T(k-1)F^{-1}(k-1)\tilde{\theta}_c(k-1)\right]$$

---

Therefore

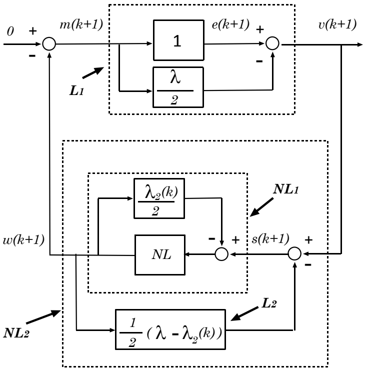
$$\begin{aligned}
\Rightarrow \sum_{j=0}^{k} w(j)s(j) &\geq \frac{1}{2}\sum_{j=0}^{k}\Bigg[\tilde{\theta}_c^T(j)F^{-1}(j)\tilde{\theta}_c(j) \\
&\qquad - \tilde{\theta}_c^T(j-1)F^{-1}(j-1)\tilde{\theta}_c(j-1)\Bigg] \\
&= \frac{1}{2}\left[\tilde{\theta}_c^T(k)F^{-1}(k)\tilde{\theta}_c(k) - \tilde{\theta}_c^T(-1)F^{-1}(-1)\tilde{\theta}_c(-1)\right] \\
&\geq -\frac{1}{2}\tilde{\theta}_c^T(-1)F^{-1}(-1)\tilde{\theta}_c(-1)
\end{aligned}$$

# Stability theorem proof, part 1

We have shown that $NL_1$ is P-class

Using the same arguments as in Lecture 21 (including the asymptotic hyperstability theorem), this yields

$$\boxed{\lim_{k\to\infty} e(k) = 0}$$



# Outline

# Stability theorem proof, part 2

We want to prove the limits

$$\lim_{k\to\infty} \|\hat{\theta}_c(k) - \hat{\theta}_c(k-1)\| = 0$$

$$\lim_{k\to\infty} \frac{[\lambda_1(k-1)e^o(k)]^2}{\lambda_1(k-1) + \phi^T(k-\mathrm{d})F(k-1)\phi(k-\mathrm{d})} = 0$$

$$\lim_{k\to\infty} \frac{[\lambda_1(k-1)\epsilon(k)]^2}{\lambda_1(k-1) + \phi^T(k-\mathrm{d})F(k-1)\phi(k-\mathrm{d})} = 0$$

# Stability theorem proof, part 2

We know that $1 - \lambda/2$ is SPR,

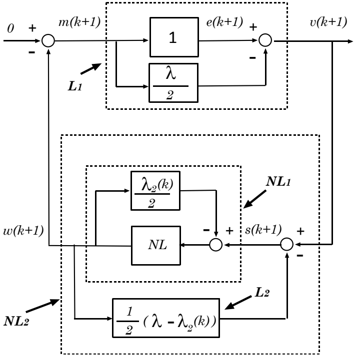
which implies that it is P-class

This implies that there exists $\bar{\gamma} \in \mathcal{R}$ such that

$$-\bar{\gamma}^2 \leq \sum_{j=0}^{k} m(j)v(j)$$

$$= -\sum_{j=0}^{k} w(j)\Big[s(j)$$

$$+ \frac{1}{2}(\lambda - \lambda_2(j-1))w(j)\Big]$$



# Stability theorem proof, part 2

Because $\lambda - \lambda_2(j-1) \geq 0,\ j = -1, 0, 1, \ldots$, we have

$$
\begin{aligned}
-\bar{\gamma}^2 &\leq -\sum_{j=0}^{k} w(j)\Big[s(j) + \frac{1}{2}(\lambda - \lambda_2(j-1))w(j)\Big] \\
&\leq -\sum_{j=0}^{k} w(j)s(j)
\end{aligned}
$$

# Stability theorem proof, part 2

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&\leq -\sum_{j=0}^{k} w(j)s(j)
\end{aligned}$$

which implies that

$$\sum_{j=0}^{k} w(j)s(j) \leq \bar{\gamma}^2$$

# Stability theorem proof, part 2

From part 1 of the stability theorem proof,

$$
\begin{aligned}
2w(k)s(k) = {} & \tilde{\theta}_c^T(k)F^{-1}(k)\tilde{\theta}_c(k) - \lambda_1(k)\tilde{\theta}_c^T(k-1)F^{-1}(k-1)\tilde{\theta}_c(k-1) \\
& + \lambda_1(k)\Delta\theta_c^T(k)F^{-1}(k-1)\Delta\theta_c(k)
\end{aligned}
$$

# Stability theorem proof, part 2

From part 1 of the stability theorem proof,

$$
\begin{aligned}
2w(k)s(k) = {} & \tilde{\theta}_c^T(k)F^{-1}(k)\tilde{\theta}_c(k) - \lambda_1(k)\tilde{\theta}_c^T(k-1)F^{-1}(k-1)\tilde{\theta}_c(k-1) \\
& + \lambda_1(k)\Delta\theta_c^T(k)F^{-1}(k-1)\Delta\theta_c(k)
\end{aligned}
$$

Since $0 < \underline{\lambda}_1 \leq \lambda_1(k) \leq 1$ and $F(k-1) \succ 0$, this implies that

$$
\begin{aligned}
2w(k)s(k) \geq {} & \left[\tilde{\theta}_c^T(k)F^{-1}(k)\tilde{\theta}_c(k) - \tilde{\theta}_c^T(k-1)F^{-1}(k-1)\tilde{\theta}_c(k-1)\right] \\
& + \underline{\lambda}_1\Delta\theta_c^T(k)F^{-1}(k-1)\Delta\theta_c(k)
\end{aligned}
$$

# Stability theorem proof, part 2

$$2w(k)s(k) \geq \left[\tilde{\theta}_c^T(k)F^{-1}(k)\tilde{\theta}_c(k) - \tilde{\theta}_c^T(k-1)F^{-1}(k-1)\tilde{\theta}_c(k-1)\right] + \underline{\lambda}_1 \Delta\theta_c^T(k)F^{-1}(k-1)\Delta\theta_c(k)$$

---

which implies that

$$\begin{aligned} 2\bar{\gamma}^2 &\geq 2\sum_{j=0}^{k} w(j)s(j) \\ &\geq \sum_{j=0}^{k}\left[\tilde{\theta}_c^T(j)F^{-1}(j)\tilde{\theta}_c(j) - \tilde{\theta}_c^T(j-1)F^{-1}(j-1)\tilde{\theta}_c(j-1)\right] \\ &\quad + \sum_{j=0}^{k}\underline{\lambda}_1 \Delta\theta_c^T(j)F^{-1}(j-1)\Delta\theta_c(j) \end{aligned}$$

# Stability theorem proof, part 2

$$2\bar{\gamma}^2 \geq \sum_{j=0}^{k} \left[ \tilde{\theta}_c^T(j) F^{-1}(j) \tilde{\theta}_c(j) - \tilde{\theta}_c^T(j-1) F^{-1}(j-1) \tilde{\theta}_c(j-1) \right]$$
$$+ \sum_{j=0}^{k} \underline{\lambda}_1 \Delta\theta_c^T(j) F^{-1}(j-1) \Delta\theta_c(j)$$

---

$$2\bar{\gamma}^2 = \tilde{\theta}_c^T(k) F^{-1}(k) \tilde{\theta}_c(k) - \tilde{\theta}_c^T(-1) F^{-1}(-1) \tilde{\theta}_c(-1)$$
$$+ \underline{\lambda}_1 \sum_{j=0}^{k} \Delta\theta_c^T(j) F^{-1}(j-1) \Delta\theta_c(j)$$
$$\geq -\tilde{\theta}_c^T(-1) F^{-1}(-1) \tilde{\theta}_c(-1) + \underline{\lambda}_1 \sum_{j=0}^{k} \Delta\theta_c^T(j) F^{-1}(j-1) \Delta\theta_c(j)$$

# Stability theorem proof, part 2

Thus, we know that

$$\sum_{j=0}^{k} \Delta\theta_c^T(j) F^{-1}(j-1) \Delta\theta_c(j) \leq \frac{1}{\underline{\lambda}_1} \left[ 2\bar{\gamma}^2 + \tilde{\theta}_c^T(-1) F^{-1}(-1) \tilde{\theta}_c(-1) \right]$$

# Stability theorem proof, part 2

Thus, we know that

$$\sum_{j=0}^{k} \Delta\theta_c^T(j) F^{-1}(j-1) \Delta\theta_c(j) \le \frac{1}{\underline{\lambda}_1} \left[ 2\bar{\gamma}^2 + \tilde{\theta}_c^T(-1) F^{-1}(-1) \tilde{\theta}_c(-1) \right]$$

Since $F^{-1}(k) \succ 0 \ \forall k$, this implies that

$$\lim_{k\to\infty} \Delta\theta_c^T(k) F^{-1}(k-1) \Delta\theta_c(k) = 0$$

## Stability theorem proof, part 2

Thus, we know that

$$\sum_{j=0}^{k} \Delta\theta_c^T(j) F^{-1}(j-1) \Delta\theta_c(j) \leq \frac{1}{\underline{\lambda}_1} \left[ 2\bar{\gamma}^2 + \tilde{\theta}_c^T(-1) F^{-1}(-1) \tilde{\theta}_c(-1) \right]$$

Since $F^{-1}(k) \succ 0 \; \forall k$, this implies that

$$\lim_{k \to \infty} \Delta\theta_c^T(k) F^{-1}(k-1) \Delta\theta_c(k) = 0$$

Since $\lambda_{min}(F^{-1}(k-1)) = \dfrac{1}{\lambda_{max}(F(k-1))} \geq \dfrac{1}{K_{max}} > 0$, this implies that

$$\boxed{\lim_{k \to \infty} \|\Delta\theta_c(k)\| = 0}$$

# Stability theorem proof, part 2

Substituting the parameter update equation

$$\Delta\theta_c(k) = F(k-1)\phi(k-\mathrm{d})e(k)$$

into

$$\lim_{k\to\infty} \Delta\theta_c^T(k)F^{-1}(k-1)\Delta\theta_c(k) = 0$$

we obtain

$$\lim_{k\to\infty} \phi^T(k-\mathrm{d})F(k-1)\phi(k-\mathrm{d})e^2(k) = 0$$

## Stability theorem proof, part 2

Substituting the parameter update equation

$$\Delta\theta_c(k) = F(k-1)\phi(k-\mathrm{d})e(k)$$

into

$$\lim_{k\to\infty} \Delta\theta_c^T(k)F^{-1}(k-1)\Delta\theta_c(k) = 0$$

we obtain

$$\lim_{k\to\infty} \phi^T(k-\mathrm{d})F(k-1)\phi(k-\mathrm{d})e^2(k) = 0$$

Adding the equation $\lim_{k\to\infty} \lambda_1(k-1)e^2(k) = 0$ to this equation yields

$$\lim_{k\to\infty} [\lambda_1(k-1) + \phi^T(k-\mathrm{d})F(k-1)\phi(k-\mathrm{d})]e^2(k) = 0$$

# Stability theorem proof, part 2

We know that

$$\lim_{k\to\infty}[\lambda_1(k-1)+\phi^T(k-\mathrm{d})F(k-1)\phi(k-\mathrm{d})]e^2(k)=0$$

Since $e(k)=\dfrac{\lambda_1(k-1)e^o(k)}{\lambda_1(k-1)+\phi^T(k-\mathrm{d})F(k-1)\phi(k-\mathrm{d})}$, we have

$$\boxed{\lim_{k\to\infty}\frac{[\lambda_1(k-1)e^o(k)]^2}{\lambda_1(k-1)+\phi^T(k-\mathrm{d})F(k-1)\phi(k-\mathrm{d})}=0}$$

# Stability theorem proof, part 2

Recall that $\eta_d(k) = r(k - \mathrm{d})$ and the control is given by

$$\hat{R}(q^{-1}, k)u(k) = r(k) - \hat{S}(q^{-1}, k)y(k)$$

## Stability theorem proof, part 2

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$$\hat{R}(q^{-1}, k)u(k) = r(k) - \hat{S}(q^{-1}, k)y(k)$$

We therefore see that

$$\begin{aligned}\eta_d(k + \mathrm{d}) &= r(k) = \hat{R}(q^{-1}, k)u(k) + \hat{S}(q^{-1}, k)y(k) \\ &= \phi^T(k)\hat{\theta}_c(k)\end{aligned}$$

# Stability theorem proof, part 2

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We therefore see that

$$\begin{aligned}\eta_d(k + \mathrm{d}) = r(k) &= \hat{R}(q^{-1}, k)u(k) + \hat{S}(q^{-1}, k)y(k) \\ &= \phi^T(k)\hat{\theta}_c(k)\end{aligned}$$

which allows us to say that

$$\begin{aligned}\epsilon(k) = \eta(k) - \eta_d(k) &= \phi^T(k - \mathrm{d})\tilde{\theta}_c(k - \mathrm{d}) \\ &= \phi^T(k - \mathrm{d})\tilde{\theta}_c(k - 1) + \phi^T(k - \mathrm{d})\left[\tilde{\theta}_c(k - \mathrm{d}) - \tilde{\theta}_c(k - 1)\right] \\ &= e^o(k) + \phi^T(k - \mathrm{d})\left[\tilde{\theta}_c(k - \mathrm{d}) - \tilde{\theta}_c(k - 1)\right]\end{aligned}$$

## Stability theorem proof, part 2

For convenience, define

$$\zeta(k) = \frac{\lambda_1(k-1) + \phi^T(k-\mathrm{d})F(k-1)\phi(k-\mathrm{d})}{\lambda_1^2(k-1)}$$

In this notation, we know that $\lim_{k\to\infty} \dfrac{[e^o(k)]^2}{\zeta(k)} = 0$

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In this notation, we know that $\lim_{k\to\infty} \frac{[e^o(k)]^2}{\zeta(k)} = 0$

Since $0 < \lambda_1(k) \leq 1$ and $0 < K_{min} \leq \lambda_{min}(F(k))\ \forall k$, we have

$$\zeta(k) > \phi^T(k-\mathrm{d})F(k-1)\phi(k-\mathrm{d}) \geq K_{min}\|\phi(k-\mathrm{d})\|^2 \geq 0$$

## Stability theorem proof, part 2

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Since $0 < \lambda_1(k) \leq 1$ and $0 < K_{min} \leq \lambda_{min}(F(k)) \; \forall k$, we have

$$\zeta(k) > \phi^T(k-\mathrm{d})F(k-1)\phi(k-\mathrm{d}) \geq K_{min}\|\phi(k-\mathrm{d})\|^2 \geq 0$$

$$\Rightarrow \frac{\|\phi(k-\mathrm{d})\|^2}{\zeta(k)} < \frac{1}{K_{min}}$$

# Stability theorem proof, part 2

By the Cauchy-Schwarz inequality,

$$\left| \frac{\phi^T(k-\mathrm{d})\left[\tilde{\theta}_c(k-\mathrm{d})-\tilde{\theta}_c(k-1)\right]}{\sqrt{\zeta(k)}} \right|$$

$$\leq \frac{\|\phi(k-\mathrm{d})\|}{\sqrt{\zeta(k)}}\|\tilde{\theta}_c(k-\mathrm{d})-\tilde{\theta}_c(k-1)\|$$

$$\leq \frac{1}{\sqrt{K_{min}}}\|\tilde{\theta}_c(k-\mathrm{d})-\tilde{\theta}_c(k-1)\|$$

# Stability theorem proof, part 2

By the Cauchy-Schwarz inequality,

$$\begin{aligned}
\left| \frac{\phi^T(k-\mathrm{d})\left[\tilde{\theta}_c(k-\mathrm{d}) - \tilde{\theta}_c(k-1)\right]}{\sqrt{\zeta(k)}} \right| \\
&\le \frac{\|\phi(k-\mathrm{d})\|}{\sqrt{\zeta(k)}} \|\tilde{\theta}_c(k-\mathrm{d}) - \tilde{\theta}_c(k-1)\| \\
&\le \frac{1}{\sqrt{K_{min}}} \|\tilde{\theta}_c(k-\mathrm{d}) - \tilde{\theta}_c(k-1)\|
\end{aligned}$$

The right-hand side of this inequality converges to zero because $\|\tilde{\theta}_c(k-\mathrm{d}) - \tilde{\theta}_c(k-1)\|$ converges to zero.

## Stability theorem proof, part 2

By the Cauchy-Schwarz inequality,

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&\left| \frac{\phi^T(k-\mathrm{d})\left[\tilde{\theta}_c(k-\mathrm{d}) - \tilde{\theta}_c(k-1)\right]}{\sqrt{\zeta(k)}} \right| \\
&\qquad \le \frac{\|\phi(k-\mathrm{d})\|}{\sqrt{\zeta(k)}} \|\tilde{\theta}_c(k-\mathrm{d}) - \tilde{\theta}_c(k-1)\| \\
&\qquad \le \frac{1}{\sqrt{K_{min}}} \|\tilde{\theta}_c(k-\mathrm{d}) - \tilde{\theta}_c(k-1)\|
\end{aligned}$$

The right-hand side of this inequality converges to zero because $\|\tilde{\theta}_c(k-\mathrm{d}) - \tilde{\theta}_c(k-1)\|$ converges to zero.

Therefore

$$\lim_{k\to\infty} \frac{\phi^T(k-\mathrm{d})\left[\tilde{\theta}_c(k-\mathrm{d}) - \tilde{\theta}_c(k-1)\right]}{\sqrt{\zeta(k)}} = 0$$

# Stability theorem proof, part 2

Since $\epsilon(k) = e^o(k) + \phi^T(k - \mathrm{d})\left[\tilde{\theta}_c(k - \mathrm{d}) - \tilde{\theta}_c(k-1)\right]$, we have

$$\begin{aligned}
\lim_{k\to\infty} \frac{\epsilon(k)}{\sqrt{\zeta(k)}} &= \lim_{k\to\infty} \frac{e^o(k)}{\sqrt{\zeta(k)}} \\
&\quad + \lim_{k\to\infty} \frac{\phi^T(k - \mathrm{d})\left[\tilde{\theta}_c(k - \mathrm{d}) - \tilde{\theta}_c(k-1)\right]}{\sqrt{\zeta(k)}} \\
&= 0 + 0
\end{aligned}$$

# Stability theorem proof, part 2

Since $\epsilon(k) = e^o(k) + \phi^T(k-\mathrm{d})\left[\tilde{\theta}_c(k-\mathrm{d}) - \tilde{\theta}_c(k-1)\right]$, we have

$$\begin{aligned}
\lim_{k\to\infty} \frac{\epsilon(k)}{\sqrt{\zeta(k)}} &= \lim_{k\to\infty} \frac{e^o(k)}{\sqrt{\zeta(k)}} \\
&\quad + \lim_{k\to\infty} \frac{\phi^T(k-\mathrm{d})\left[\tilde{\theta}_c(k-\mathrm{d}) - \tilde{\theta}_c(k-1)\right]}{\sqrt{\zeta(k)}} \\
&= 0 + 0
\end{aligned}$$

Therefore

$$\boxed{\lim_{k\to\infty} \frac{[\lambda_1(k-1)\epsilon(k)]^2}{\lambda_1(k-1) + \phi^T(k-\mathrm{d})F(k-1)\phi(k-\mathrm{d})} = 0}$$

# Outline

# Stability theorem proof, part 3

We want to prove that there exist $C_1 \geq 0,\ C_2 \geq 0$ such that

$$\|\phi(k - \mathrm{d})\| \leq C_1 + C_2 \max_{j \in \{0,\ldots,k\}} |\epsilon(j)|$$

# Stability theorem proof, part 3

We want to prove that there exist $C_1 \geq 0,\ C_2 \geq 0$ such that

$$\|\phi(k-\mathrm{d})\| \leq C_1 + C_2 \max_{j \in \{0,\dots,k\}} |\epsilon(j)|$$

We have the relationships

$$y(k) = \frac{q^{-\mathrm{d}}B(q^{-1})}{A(q^{-1})}u(k) = \frac{B(q^{-1})}{A(q^{-1})}u(k-\mathrm{d})$$

$$\eta(k) = A_c^{'}(q^{-1})y(k)$$

$$\epsilon(k) = \eta(k) - \eta_d(k)$$

which define $\epsilon(k)$ from $u(k)$ and $\eta_d(k)$.

# Stability theorem proof, part 3

We want to prove that there exist $C_1 \geq 0,\ C_2 \geq 0$ such that

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$$y(k) = \frac{q^{-\mathrm{d}}B(q^{-1})}{A(q^{-1})}u(k) = \frac{B(q^{-1})}{A(q^{-1})}u(k-\mathrm{d})$$

$$\eta(k) = A_c^{'}(q^{-1})y(k)$$

$$\epsilon(k) = \eta(k) - \eta_d(k)$$

which define $\epsilon(k)$ from $u(k)$ and $\eta_d(k)$.

We now invert these relationships, i.e. we reconstruct $u(k)$ from $\epsilon(k)$ and $\eta_d(k)$

# Stability theorem proof, part 3

The inverted relationships are

$$u(k - \mathrm{d}) = \frac{A(q^{-1})}{B(q^{-1})} y(k)$$

$$y(k) = \frac{1}{A'_c(q^{-1})} \eta(k)$$

$$\eta(k) = \epsilon(k) + \eta_d(k)$$

# Stability theorem proof, part 3

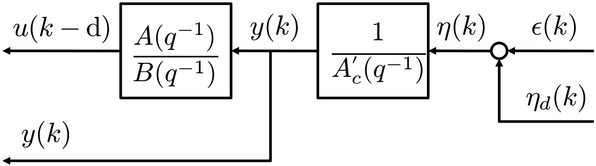
The inverted relationships are

$$u(k-\mathrm{d}) = \frac{A(q^{-1})}{B(q^{-1})} y(k)$$

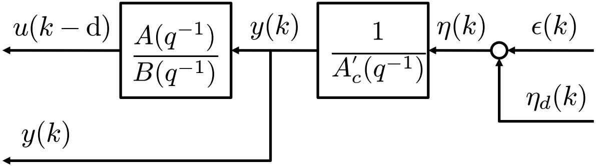
$$y(k) = \frac{1}{A_c'(q^{-1})} \eta(k)$$

$$\eta(k) = \epsilon(k) + \eta_d(k)$$

These relationships are shown in the block diagram



# Stability theorem proof, part 3

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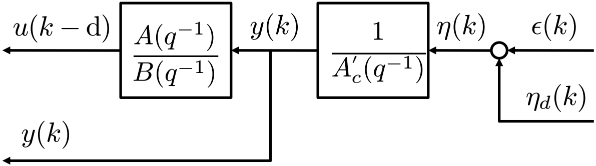
Since $A_c'(q^{-1})$ and $B(q^{-1})$ are anti-Schur, both blocks in the block diagram are causal and BIBO

# Stability theorem proof, part 3




---

Since $A_c'(q^{-1})$ and $B(q^{-1})$ are anti-Schur, both blocks in the block diagram are causal and BIBO

Therefore, we can choose nonnegative $\bar{C}_{1u}$, $C_{2u}$, $\bar{C}_{1y}$, and $C_{2y}$ such that

$$|u(k-\mathrm{d})| \leq \bar{C}_{1u} + C_{2u} \max_{j \leq k} |\eta(j)|$$

$$|y(k)| \leq \bar{C}_{1y} + C_{2y} \max_{j \leq k} |\eta(j)|$$

# Stability theorem proof, part 3

$$|u(k-\mathrm{d})| \leq \bar{C}_{1u} + C_{2u} \max_{j \leq k} |\eta(j)|$$

$$|y(k)| \leq \bar{C}_{1y} + C_{2y} \max_{j \leq k} |\eta(j)|$$

---

Assuming that $|\eta_d(k)| \leq \bar{\eta}_d$, the triangle inequality tells us that

$$|\eta(j)| \leq |\eta_d(k)| + |\epsilon(k)| \leq \bar{\eta}_d + |\epsilon(k)|$$

# Stability theorem proof, part 3

$$|u(k-\mathrm{d})| \leq \bar{C}_{1u} + C_{2u} \max_{j \leq k} |\eta(j)|$$

$$|y(k)| \leq \bar{C}_{1y} + C_{2y} \max_{j \leq k} |\eta(j)|$$

---

Assuming that $|\eta_d(k)| \leq \bar{\eta}_d$, the triangle inequality tells us that

$$|\eta(j)| \leq |\eta_d(k)| + |\epsilon(k)| \leq \bar{\eta}_d + |\epsilon(k)|$$

Defining $C_{1u} = \bar{C}_{1u} + C_{2u}\bar{\eta}_d$ and $C_{1y} = \bar{C}_{1y} + C_{2y}\bar{\eta}_d$ we have

$$|u(k-\mathrm{d})| \leq C_{1u} + C_{2u} \max_{j \leq k} |\epsilon(j)|$$

$$|y(k)| \leq C_{1y} + C_{2y} \max_{j \leq k} |\epsilon(j)|$$

# Stability theorem proof, part 3

$$|u(k-\mathrm{d})| \leq C_{1u} + C_{2u} \max_{j \leq k} |\epsilon(j)|$$

$$|y(k)| \leq C_{1y} + C_{2y} \max_{j \leq k} |\epsilon(j)|$$

---

Since $\max_{j \leq k-\ell} |\epsilon(j)| \leq \max_{j \leq k} |\epsilon(j)|$ for $\ell \geq 0$, we have

$$|u(k-\mathrm{d}-\ell)| \leq C_{1u} + C_{2u} \max_{j \leq k} |\epsilon(j)|$$

$$|y(k-\mathrm{d}-\ell)| \leq C_{1y} + C_{2y} \max_{j \leq k} |\epsilon(j)|$$

for all $\ell \geq 0$

# Stability theorem proof, part 3

$$|u(k-\mathrm{d}-\ell)| \le C_{1u} + C_{2u} \max_{j \le k} |\epsilon(j)|$$

$$|y(k-\mathrm{d}-\ell)| \le C_{1y} + C_{2y} \max_{j \le k} |\epsilon(j)|$$

---

Using the triangle inequality, we have

$$\begin{aligned}
\|\phi(k-\mathrm{d})\| &\le \sum_{j=0}^{n_s} |y(k-\mathrm{d}-j)| + \sum_{i=0}^{n_r} |u(k-\mathrm{d}-i)| \\
&\le \sum_{j=0}^{n_s} \left( C_{1y} + C_{2y} \max_{\ell \le k} |\epsilon(\ell)| \right) + \sum_{i=0}^{n_r} \left( C_{1u} + C_{2u} \max_{\ell \le k} |\epsilon(\ell)| \right)
\end{aligned}$$

# Stability theorem proof, part 3

$$|u(k-\mathrm{d}-\ell)| \le C_{1u} + C_{2u} \max_{j \le k} |\epsilon(j)|$$

$$|y(k-\mathrm{d}-\ell)| \le C_{1y} + C_{2y} \max_{j \le k} |\epsilon(j)|$$

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Using the triangle inequality, we have

$$\begin{aligned} \|\phi(k-\mathrm{d})\| &\le \sum_{j=0}^{n_s} |y(k-\mathrm{d}-j)| + \sum_{i=0}^{n_r} |u(k-\mathrm{d}-i)| \\ &\le \sum_{j=0}^{n_s} \left( C_{1y} + C_{2y} \max_{\ell \le k} |\epsilon(\ell)| \right) + \sum_{i=0}^{n_r} \left( C_{1u} + C_{2u} \max_{\ell \le k} |\epsilon(\ell)| \right) \end{aligned}$$

Therefore

$$\boxed{\begin{aligned} \|\phi(k-\mathrm{d})\| \le {} & [(n_s+1)C_{1y} + (n_r+1)C_{1u}] \\ & + [(n_s+1)C_{2y} + (n_r+1)C_{2u}] \max_{j \le k} |\epsilon(j)| \end{aligned}}$$

# Outline

# Stability theorem proof, part 4

We want to prove Goodwin's technical lemma, which states that $\|\phi(k)\|$ remains bounded and

$$\lim_{k\to\infty} \epsilon(k) = 0$$

# Stability theorem proof, part 4

We want to prove Goodwin's technical lemma, which states that $\|\phi(k)\|$ remains bounded and

$$\lim_{k \to \infty} \epsilon(k) = 0$$

This proof will be done in three steps:

1. Show that $\epsilon(k)$ remains bounded
2. Show that $\|\phi(k)\|$ remains bounded
3. Show that $\epsilon(k) \longrightarrow 0$

# Stability theorem proof, part 4, step 1 ($\epsilon(k)$ bounded)

Recall from part 2 that

$$\lim_{k\to\infty} \frac{[\lambda_1(k-1)\epsilon(k)]^2}{\lambda_1(k-1) + \phi^T(k-\mathrm{d})F(k-1)\phi(k-\mathrm{d})} = 0$$

# Stability theorem proof, part 4, step 1 ($\epsilon(k)$ bounded)

Recall from part 2 that

$$\lim_{k\to\infty} \frac{[\lambda_1(k-1)\epsilon(k)]^2}{\lambda_1(k-1)+\phi^T(k-\mathrm{d})F(k-1)\phi(k-\mathrm{d})} = 0$$

Since $0 < \underline{\lambda}_1 \leq \lambda_1(k) \leq 1$

and $0 < \lambda_{min}(F(k-1)) \leq \lambda_{max}(F(k-1)) \leq K_{max}$

we have

$$\left|\frac{[\lambda_1(k-1)\epsilon(k)]^2}{\lambda_1(k-1)+\phi^T(k-\mathrm{d})F(k-1)\phi(k-\mathrm{d})}\right| \geq \frac{\underline{\lambda}_1^2\epsilon^2(k)}{1+K_{max}\|\phi(k-\mathrm{d})\|^2} > 0$$

# Stability theorem proof, part 4, step 1 ($\epsilon(k)$ bounded)

$$\left|\frac{[\lambda_1(k-1)\epsilon(k)]^2}{\lambda_1(k-1)+\phi^T(k-\mathrm{d})F(k-1)\phi(k-\mathrm{d})}\right| \geq \frac{\underline{\lambda}_1^2\epsilon^2(k)}{1+K_{max}\|\phi(k-\mathrm{d})\|^2} > 0$$

---

For convenience, we define $\bar{\epsilon}(k) \;\; \max_{j\leq k}|\epsilon(j)|$

# Stability theorem proof, part 4, step 1 ($\epsilon(k)$ bounded)

$$\left|\frac{[\lambda_1(k-1)\epsilon(k)]^2}{\lambda_1(k-1)+\phi^T(k-\mathrm{d})F(k-1)\phi(k-\mathrm{d})}\right| \geq \frac{\underline{\lambda}_1^2\epsilon^2(k)}{1+K_{max}\|\phi(k-\mathrm{d})\|^2} > 0$$

---

For convenience, we define $\overline{\epsilon}(k) \;\; \max_{j\leq k}|\epsilon(j)|$

From part 3, we have that $\|\phi(k-\mathrm{d})\|^2 \leq [C_1 + C_2\overline{\epsilon}(k)]^2$, which implies that

$$\left|\frac{[\lambda_1(k-1)\epsilon(k)]^2}{\lambda_1(k-1)+\phi^T(k-\mathrm{d})F(k-1)\phi(k-\mathrm{d})}\right| \geq \frac{\underline{\lambda}_1^2\epsilon^2(k)}{1+K_{max}[C_1+C_2\overline{\epsilon}(k)]^2} > 0$$

# Stability theorem proof, part 4, step 1 ($\epsilon(k)$ bounded)

$$\left|\frac{[\lambda_1(k-1)\epsilon(k)]^2}{\lambda_1(k-1)+\phi^T(k-\mathrm{d})F(k-1)\phi(k-\mathrm{d})}\right| \geq \frac{\underline{\lambda}_1^2\epsilon^2(k)}{1+K_{max}[C_1+C_2\overline{\epsilon}(k)]^2} > 0$$

---

Since

$$\lim_{k\to\infty}\frac{[\lambda_1(k-1)\epsilon(k)]^2}{\lambda_1(k-1)+\phi^T(k-\mathrm{d})F(k-1)\phi(k-\mathrm{d})} = 0$$

we have

$$\lim_{k\to\infty}\frac{\underline{\lambda}_1^2\epsilon^2(k)}{1+K_{max}[C_1+C_2\overline{\epsilon}(k)]^2} = 0$$

# Stability theorem proof, part 4, step 1 ($\epsilon(k)$ bounded)

$$\lim_{k\to\infty} \frac{\underline{\lambda}_1^2 \epsilon^2(k)}{1 + K_{max}[C_1 + C_2\overline{\epsilon}(k)]^2} = 0$$

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Whenever $|\epsilon(k)| = \overline{\epsilon}(k) \geq 1$, we have

$$\begin{aligned}
0 &< \frac{1 + K_{max}[C_1 + C_2\overline{\epsilon}(k)]^2}{\underline{\lambda}_1^2 \overline{\epsilon}^2(k)} \\
&= \frac{1 + K_{max}C_1^2}{\underline{\lambda}_1^2 \overline{\epsilon}^2(k)} + \frac{2K_{max}C_1C_2}{\underline{\lambda}_1^2 \overline{\epsilon}(k)} + \frac{K_{max}C_2^2}{\underline{\lambda}_1^2} \\
&\leq \frac{1}{\underline{\lambda}_1^2}[1 + K_{max}C_1^2 + 2K_{max}C_1C_2 + K_{max}C_2^2]
\end{aligned}$$

# Stability theorem proof, part 4, step 1 ($\epsilon(k)$ bounded)

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&\leq \frac{1}{\underline{\lambda}_1^2}[1 + K_{max}C_1^2 + 2K_{max}C_1C_2 + K_{max}C_2^2]
\end{aligned}$$

This implies that whenever $|\epsilon(k)| = \overline{\epsilon}(k) \geq 1$, we have

$$\frac{\underline{\lambda}_1^2 \epsilon^2(k)}{1 + K_{max}[C_1 + C_2\overline{\epsilon}(k)]^2} \geq \frac{\underline{\lambda}_1^2}{1 + K_{max}[C_1 + C_2]^2} > 0$$

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Since

$$\lim_{k\to\infty} \frac{\underline{\lambda}_1^2 \epsilon^2(k)}{1 + K_{max}[C_1 + C_2\overline{\epsilon}(k)]^2} = 0$$

there can only be a finite number of values of $k$ such that

$|\epsilon(k)| = \overline{\epsilon}(k) = \max_{j\leq k} |\epsilon(j)| \geq 1.$

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Therefore,

$$\boxed{\epsilon(k) \text{ remains bounded}}$$

# Stability theorem proof, part 4, step 2 ($\phi(k)$ bounded)

Recall from part 3 that

$$\|\phi(k-\mathrm{d})\| \leq C_1 + C_2 \max_{j \leq k} |\epsilon(j)|$$

# Stability theorem proof, part 4, step 2 ($\phi(k)$ bounded)

Recall from part 3 that

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Since $\epsilon(k)$ remains bounded, we immediately see that

$$\boxed{\phi(k) \text{ remains bounded}}$$

# Stability theorem proof, part 4, step 3 ($\epsilon(k) \longrightarrow 0$)

Recall from part 2 that

$$\lim_{k \to \infty} \frac{\epsilon^2(k)}{\zeta(k)} = 0$$

where

$$\zeta(k) = \frac{\lambda_1(k-1) + \phi^T(k-\mathrm{d})F(k-1)\phi(k-\mathrm{d})}{\lambda_1^2(k-1)}$$

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Therefore, if we can show that $\zeta(k)$ remains bounded, it must be true that $\epsilon(k) \longrightarrow 0$

# Stability theorem proof, part 4, step 3 ($\epsilon(k) \longrightarrow 0$)

Since $0 < \underline{\lambda}_1 \leq \lambda_1(k) \leq 1$

and $0 < \lambda_{min}(F(k-1)) \leq \lambda_{max}(F(k-1)) \leq K_{max}$

we have

$$
\begin{aligned}
|\zeta(k)| &= \left| \frac{\lambda_1(k-1) + \phi^T(k-\mathrm{d})F(k-1)\phi(k-\mathrm{d})}{\lambda_1^2(k-1)} \right| \\
&\leq \frac{1 + K_{max}\|\phi(k-\mathrm{d})\|^2}{\underline{\lambda}_1^2}
\end{aligned}
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Since the right-hand side is bounded, we see that $\zeta(k)$ remains bounded.

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\end{aligned}
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Since the right-hand side is bounded, we see that $\zeta(k)$ remains bounded.

Therefore

$$\boxed{\lim_{k\to\infty} \epsilon(k) = 0}$$

# Outline

# Stability theorem proof, part 5

Recall from part 2 that

$$\lim_{k\to\infty} \frac{[e^o(k)]^2}{\zeta(k)} = 0$$

where

$$\zeta(k) = \frac{\lambda_1(k-1) + \phi^T(k-\mathrm{d})F(k-1)\phi(k-\mathrm{d})}{\lambda_1^2(k-1)}$$

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We have already shown that $\zeta(k)$ is bounded

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We have already shown that $\zeta(k)$ is bounded

Therefore

$$\boxed{\lim_{k\to\infty} e^o(k) = 0}$$