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ME 233 Advanced Control II

Lecture 12
Kalman Filters Stationary Properties and LQR-KF Duality

(ME233 Class Notes pp.KF1-KF6)

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Summary

- Stationary Kalman filters (KF):
  - KF algebraic Riccati equation
  - Convergence properties
- Kalman filter / LQR duality
- KF return difference equality
  - Reciprocal root locus
  - Guaranteed robustness margins

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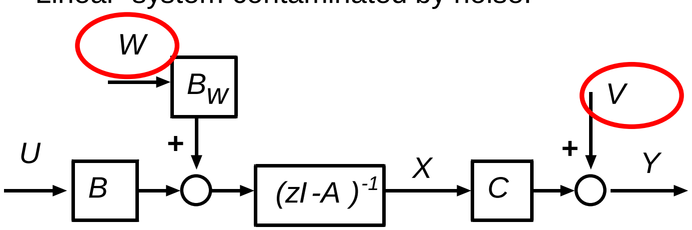
Stochastic State Estimation

Linear system contaminated by noise:



Two random disturbances:

- Input noise $w(k)$ - contaminates the state $x(k)$
- Measurement noise $v(k)$ - contaminates the output $y(k)$

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Stochastic state model

State estimation of LTI system:

$$x(k+1) = A\,x(k) + B\,u(k) + B_w\,w(k)$$
$$y(k) = C\,x(k) + v(k)$$

Where:

- $u(k)$ **known control input**
- $w(k)$ Gaussian, uncorrelated, zero mean, input noise
- $v(k)$ Gaussian, uncorrelated, zero mean, meas. noise
- $x(0)$ Gaussian

Assumptions (review)

- Initial conditions:

$$E\{x(0)\} = x_o \qquad E\{\tilde{x}^o(0)\tilde{x}^{oT}(0)\} = X_o$$

- Noise properties:

$$\left.\begin{aligned} &E\{w(k)\} = 0 \\ &E\{v(k)\} = 0 \\ &E\{w(k+l)w^T(k)\} = W(k)\,\delta(l) \\ &E\{v(k+l)v^T(k)\} = V(k)\,\delta(l) \\ &E\{w(k+l)v^T(k)\} = 0 \end{aligned}\right\}$$

Zero-mean Gaussian uncorrelated noises

$$E\{\tilde{x}^o(0)w^T(k)\} = 0 \qquad E\{\tilde{x}^o(0)v^T(k)\} = 0$$

Kalman Filter Solution V-1 (review)

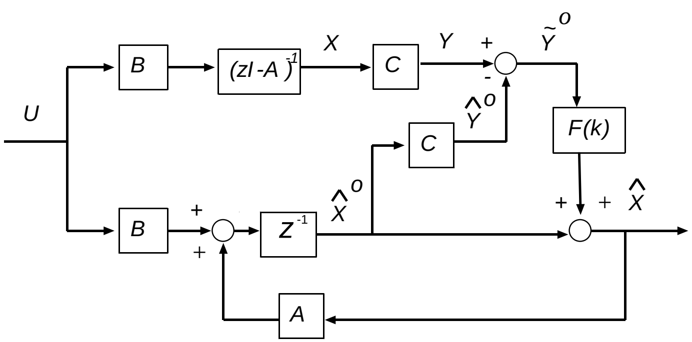
A-posteriori state observer structure:

$$\begin{aligned} \hat{x}(k) &= \hat{x}^o(k) + F(k)\,\tilde{y}^o(k) \\ \hat{x}^o(k+1) &= A\,\hat{x}(k) + B\,u(k) \\ \tilde{y}^o(k) &= y(k) - C\,\hat{x}^o(k) \end{aligned}$$

$$\begin{aligned} F(k) &= M(k)C^T\left[C\,M(k)C^T + V(k)\right]^{-1} \\ M(k+1) &= AM(k)A^T + B_wW(k)B_w^T \\ &\quad - AM(k)C^T\left[CM(k)C^T + V(k)\right]^{-1}CM(k)A^T \end{aligned}$$

Kalman Filter Solution V-1 (review)

- A-posteriori estimator as output



Kalman Filter Solution V-2 (review)

A-priori state observer structure:

$$\begin{aligned} \hat{x}^o(k+1) &= A\,\hat{x}^o(k) + B\,u(k) + L(k)\,\tilde{y}^o(k) \\ \tilde{y}^o(k) &= y(k) - C\,\hat{x}^o(k) \end{aligned}$$

$$\begin{aligned} L(k) &= A\,M(k)C^T\left[C\,M(k)C^T + V(k)\right]^{-1} \\ M(k+1) &= AM(k)A^T + B_wW(k)B_w^T \\ &\quad - AM(k)C^T\left[CM(k)C^T + V(k)\right]^{-1}CM(k)A^T \\ M(0) &= X_o \end{aligned}$$

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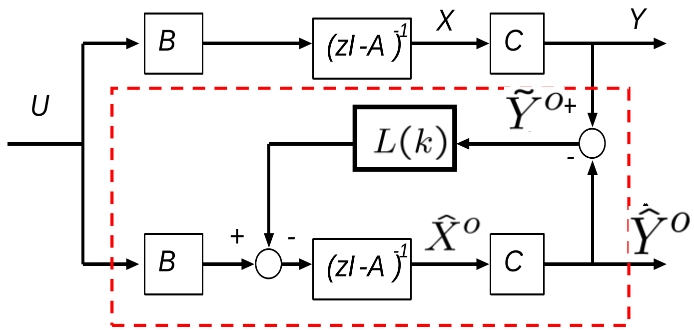
Kalman Filter Solution V-2 (review)

- Same structure as deterministic a-priori observer



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Kalman Filter State Space (review)

$$\hat{x}^o(k+1) = [A - L(k)C]\hat{x}^o(k) + \begin{bmatrix} B & L(k) \end{bmatrix} \begin{bmatrix} u(k) \\ y(k) \end{bmatrix}$$

$$\hat{x}(k) = [I - F(k)C]\hat{x}^o(k) + \begin{bmatrix} 0 & F(k) \end{bmatrix} \begin{bmatrix} u(k) \\ y(k) \end{bmatrix}$$

$$\begin{aligned} F(k) &= M(k)C^T \left[C\,M(k)C^T + V(k)\right]^{-1} \\ L(k) &= AM(k)C^T \left[C\,M(k)C^T + V(k)\right]^{-1} \\ M(k+1) &= AM(k)A^T + B_w W(k) B_w^T \\ &\quad - AM(k)C^T \left[CM(k)C^T + V(k)\right]^{-1} CM(k)A^T \end{aligned}$$

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Kalman Filter (KF) Properties (review)

The KF a-priori output error (*a-priori output residual*)

$$\tilde{y}^o(k) \;=\; y(k) - C\,\hat{x}^o(k)$$

is often called the ***innovation***

it contains only the "new information" in $y(k)$

Moreover,

$$\Lambda_{\tilde{y}^o\tilde{y}^o}(k,j) = [CM(k)C^T + V(k)]\delta(j)$$

i.e. $\tilde{y}^o(k)$ is an uncorrelated RVS

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KF as an innovations filter (review)

For the figure on the next slide, we will assume without loss of generality that the control input is zero, i.e.

$$u(k) = 0 \qquad k = 0,\, 1,\, \cdots$$

- **Plant:**

$$\begin{aligned} x(k+1) &= A\,x(k) + B_w\,w(k) \\ y(k) &= C\,x(k) + v(k) \end{aligned}$$

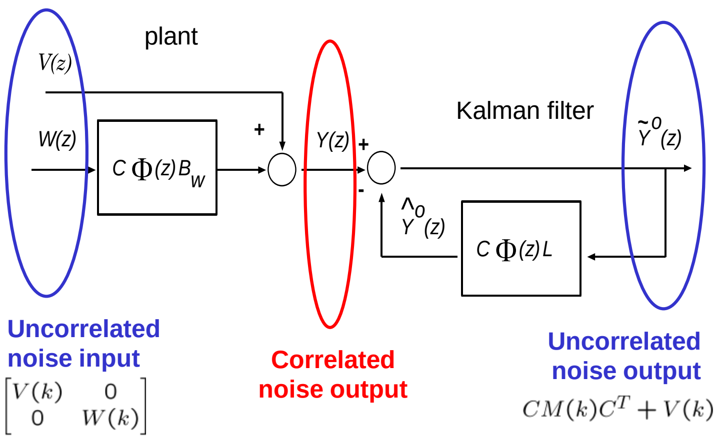
- **Kalman filter V-2:**

$$\hat{x}^o(k+1) = A\hat{x}^o(k) + L(k)\tilde{y}^o(k)$$

$$\hat{y}^o(k) = C\hat{x}^o(k)$$

KF as an innovations filter (review)

$$\Phi(z) = (zI - A)^{-1}$$



Kalman Filter (KF) Properties (review)

- The KF is a linear time varying estimator.
- The KF is the **optimal state estimator** when the input and measurement noises are Gaussian.
- The KF is still the **optimal *linear* state estimator** even when the input and measurement noises are **not** Gaussian.
- The KF covariance Riccati equation is iterated in a forward manner, rather than in a backwards manner as in the LQR.

$$M(0) \rightarrow M(k)$$

Steady State Kalman Filter

- Assume now that we want to estimate the state under zero-mean, stationary input and output Gaussian white noise, I.e.

$$x(k+1) = Ax(k) + Bu(k) + B_w w(k)$$
$$y(k) = Cx(k) + v(k)$$

$$\left.\begin{aligned} E\{w(k)\} &= 0 \\ E\{v(k)\} &= 0 \\ E\{w(k+l)w^T(k)\} &= W\,\delta(l) \\ E\{v(k+l)v^T(k)\} &= V\,\delta(l) \\ E\{w(k+l)v^T(k)\} &= 0 \end{aligned}\right\}$$

WSS Gaussian Noise

A priori estimation error dynamics

$$\tilde{x}^o(k+1) = [A - L(k)C]\tilde{x}^o(k) + B_w w(k) - L(k)v(k)$$

Proof:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + B_w w(k) \\ \hat{x}^o(k+1) = A\hat{x}^o(k) + Bu(k) + L(k)\tilde{y}^o(k) \end{cases}$$

Subtracting equations gives

$$\tilde{x}^o(k+1) = A\tilde{x}^o(k) + B_w w(k) - L(k)\tilde{y}^o(k)$$

$$C\tilde{x}^o(k) + v(k)$$

∎

Steady state Kalman filter, question 1

1) When does there exist a **BOUNDED limiting** solution

$$M_\infty$$

to the Riccati Eq.

$$\begin{aligned} M(k+1) = {} & AM(k)A^T + B_w W B_w^T \\ & - AM(k)C^T[CM(k)C^T + V]^{-1}CM(k)A^T \end{aligned}$$

for each choice of $M(0) \succeq 0$?

Steady state Kalman filter, question 2

2) When does there exist a **UNIQUE limiting** solution

$$M_\infty$$

to the Riccati Eq.

$$\begin{aligned} M(k+1) = {} & AM(k)A^T + B_w W B_w^T \\ & - AM(k)C^T[CM(k)C^T + V]^{-1}CM(k)A^T \end{aligned}$$

regardless of the choice of $M(0) \succeq 0$?

Steady state Kalman filter, question 3

3) When does the **limiting** solution

$$M_\infty$$

to the Riccati Eq.

yield **asymptotically stable** estimation error dynamics?

$$A_c = A - L_\infty C$$ is Schur
(all eigenvalues inside unit circle)

$$L_\infty = AM_\infty C^T \left[CM_\infty C^T + V\right]^{-1}$$

Detectability Assumption

We are only interested in the case where the estimation error dynamics are asymptotically stable

If *(C,A)* is not detectable, then there does not exist a estimator that results is asymptotically stable estimation error dynamics

➡ For the stationary Kalman filter, we always assume that *(C,A)* is detectable

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Theorem 1 : Existence of a bounded $\boldsymbol{M}_\infty$

Let (C,A) be detectable
(unobservable modes are asymptotically stable)

Then, for $M(0) = X_0 = 0$ as $k \to \infty$
the solution of the Riccati Eq.

$$M(k+1) = AM(k)A^T + B_w W B_w^T - AM(k)C^T[CM(k)C^T + V]^{-1}CM(k)A^T$$

converges to a **BOUNDED limiting** solution $\boldsymbol{M}_\infty$
that satisfies the algebraic Riccati equation (DARE):

$$M_\infty = AM_\infty A^T + B_w W B_w^T - AM_\infty C^T[CM_\infty C^T + V]^{-1}CM_\infty A^T$$

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Theorem 1 : Notes

- Theorem 1 only guarantees the existence of a bounded solution $\boldsymbol{M}_\infty$ to the algebraic Riccati Equation
$$M_\infty = AM_\infty A^T + B_w W B_w^T - AM_\infty C^T[CM_\infty C^T + V]^{-1}CM_\infty A^T$$
- The solution may not be unique.
- Different initial conditions $M(0) = X_0$ may result in different limiting solutions $\boldsymbol{M}_\infty$ or may yield no limiting solution at all!

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Theorem 2 : Existence and uniqueness of a positive definite asymptotic stabilizing solution

If *(C,A)* is detectable and *(A,$B_w W^{1/2}$)* is controllable

1) There exists a unique, bounded solution $M_\infty \succ 0$ to the DARE
$$M_\infty = AM_\infty A^T + B_w W B_w^T - AM_\infty C^T[CM_\infty C^T + V]^{-1}CM_\infty A^T$$
2) The estimation error dynamics are **asymptotically stable**
$$\tilde{x}^o(k+1) = [A - L_\infty C]\tilde{x}^o(k) + B_w w(k) - L_\infty v(k)$$
$$L_\infty = AM_\infty C^T[CM_\infty C^T + V]^{-1}$$

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Theorem 3 : Existence of a stabilizing solution

If *(C,A)* is detectable and *(A,$B_w W^{1/2}$)* is stabilizable

1) There exists a unique, bounded solution $M_\infty \succeq 0$ to the DARE
$$M_\infty = AM_\infty A^T + B_w W B_w^T - AM_\infty C^T[CM_\infty C^T + V]^{-1}CM_\infty A^T$$
2) The estimation error dynamics are **asymptotically stable**
$$\tilde{x}^o(k+1) = [A - L_\infty C]\tilde{x}^o(k) + B_w w(k) - L_\infty v(k)$$
$$L_\infty = AM_\infty C^T[CM_\infty C^T + V]^{-1}$$

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Theorem 4: A different approach

The discrete algebraic Riccati equation (DARE) has a solution for which $A - L_\infty C$ is Schur

<u>if and only if</u>

(C, A) is detectable and the matrix pair $(A,\, B_w W^{1/2})$ has no uncontrollable modes on the unit circle.

$$L_\infty = AM_\infty C^T [CM_\infty C^T + V]^{-1}$$

$$\begin{aligned} M_\infty = {} & AM_\infty A^T + B_w W B_w^T \\ & - AM_\infty C^T [CM_\infty C^T + V]^{-1} CM_\infty A^T \end{aligned}$$

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Kalman Filter Solution V-1

A-posteriori state observer structure:

$$\begin{aligned} \widehat{x}(k) &= \widehat{x}^o(k) + F\widetilde{y}^o(k) \\ \widehat{x}^o(k+1) &= A\,\widehat{x}(k) + B\,u(k) \\ \widetilde{y}^o(k) &= y(k) - C\,\widehat{x}^o(k) \end{aligned}$$

$$\begin{aligned} F &= MC^T \left[C\,MC^T + V\right]^{-1} \\ M &= AMA^T + B_w W B_w^T \\ & \quad - AMC^T (CMC^T + V)^{-1} CMA^T \end{aligned}$$

$A - AFC$ is Schur

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Kalman Filter Solution V-2

A-priori state observer structure:

$$\begin{aligned} \widehat{x}^o(k+1) &= A\widehat{x}^o(k) + Bu(k) + L\widetilde{y}^o(k) \\ \widetilde{y}^o(k) &= y(k) - C\,\widehat{x}^o(k) \end{aligned}$$

$$\begin{aligned} L &= AMC^T \left[C\,MC^T + V\right]^{-1} \\ M &= AMA^T + B_w W B_w^T \\ & \quad - AMC^T (CMC^T + V)^{-1} CMA^T \end{aligned}$$

$A - LC$ is Schur

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Kalman Filter State Space

$$\widehat{x}^o(k+1) = [A - LC]\widehat{x}^o(k) + \begin{bmatrix} B & L \end{bmatrix} \begin{bmatrix} u(k) \\ y(k) \end{bmatrix}$$

$$\widehat{x}(k) = [I - FC]\widehat{x}^o(k) + \begin{bmatrix} 0 & F \end{bmatrix} \begin{bmatrix} u(k) \\ y(k) \end{bmatrix}$$

$$\begin{aligned} F &= MC^T \left[C\,MC^T + V\right]^{-1} \\ L &= AMC^T \left[CMC^T + V\right]^{-1} \\ M &= AMA^T + B_w W B_w^T \\ & \quad - AMC^T (CMC^T + V)^{-1} CMA^T \end{aligned}$$

$A - LC$ is Schur

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Kalman Filter (KF) Properties

The KF a-priori output error (*a-priori output residual*)

$$\tilde{y}^o(k) \;=\; y(k) - C\,\hat{x}^o(k)$$

is often called the ***innovation***

it contains only the "new information" in $y(k)$

Moreover,

$$\Lambda_{\tilde{y}^o\tilde{y}^o}(j) = [CMC^T + V]\delta(j)$$

i.e. $\tilde{y}^o(k)$ is white

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KF as an innovations filter

For the figure on the next slide, we will assume without loss of generality that the control input is zero, i.e.

$$u(k) = 0 \qquad k = 0,\, 1,\, \cdots$$

- **Plant:**

$$x(k+1) \;=\; A\,x(k) + B_w\,w(k)$$
$$y(k) \;=\; C\,x(k) + v(k)$$

- **Kalman filter V-2:**

$$\hat{x}^o(k+1) = A\hat{x}^o(k) + L\tilde{y}^o(k)$$
$$\hat{y}^o(k) = C\hat{x}^o(k)$$

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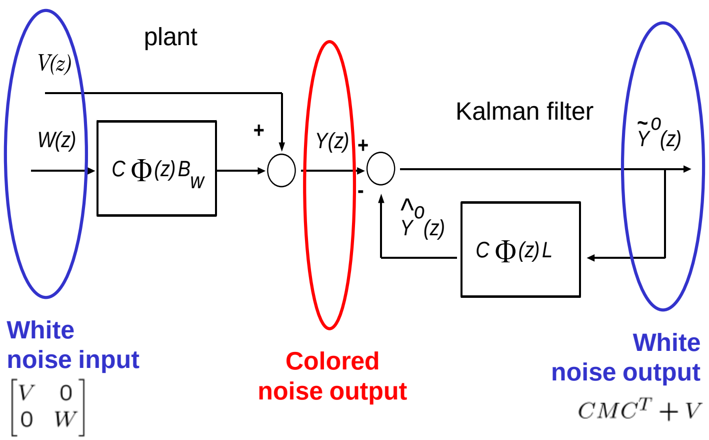
KF as an innovations (whitening) filter

$$\Phi(z) = (zI - A)^{-1}$$



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Kalman Filter & LQR Duality

Recall Steady state LQR:

$$x(k+1) \;=\; A\,x(k) + B\,u(k)$$
$$u(k) = -K\,x(k) + r(k)$$

$$J = \sum_{k=0}^{\infty} \left\{ x^T(k)\, C_Q^T\, C_Q\, x(k) + u^T(k)\, R\, u(k) \right\}$$

$$Q = C_Q^T\, C_Q \geq 0 \qquad\qquad R = R^T > 0$$

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Note:

We need to distinguish between:

- **LQR:** state cost weight $Q = C_Q^T C_Q \geq 0$

$$J = \sum_{k=0}^{\infty} \left\{ x^T(k)\, C_Q^T C_Q\, x(k) + u^T(k)\, R\, u(k) \right\}$$

- **KF:** output matrix C

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) + B_w\,w(k) \\ y(k) &= C\,x(k) + v(k) \end{aligned}$$

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Kalman Filter & LQR Duality

Infinite-horizon LQR Closed-loop dynamics:

$$x(k+1) = (A - B\,K)\,x(k) + B\,r(k)$$

$$K = \left[R + B^T P B\right]^{-1} B^T P A$$

$$\begin{aligned} A^T P A - P &= -C_Q^T C_Q \\ &\quad + A^T P B \left[B^T P B + R\right]^{-1} B^T P A \end{aligned}$$

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Kalman Filter & LQR Duality

Steady State KF Estimation error dynamics

$$\tilde{x}^o(k+1) = (A - LC)\,\tilde{x}^o(k) + B_w\,w(k) - Lv(k)$$

$$L = A\,MC^T \left[C\,MC^T + V\right]^{-1}$$

$$\begin{aligned} AMA^T - M &= -B_w W B_w^T \\ &\quad + AMC^T \left[CMC^T + V\right]^{-1} CMA^T \end{aligned}$$

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Kalman Filter & LQR Duality

Let's compare the DAREs:

$$\begin{aligned} A^T P A - P &= -C_Q^T C_Q \\ &\quad + A^T P B \left[B^T P B + R\right]^{-1} B^T P A \end{aligned} \qquad \boxed{LQR}$$

$$\begin{aligned} AMA^T - M &= -B_w W B_w^T \\ &\quad + AMC^T \left[CMC^T + V\right]^{-1} CMA^T \end{aligned} \qquad \boxed{KF}$$

$$\boxed{P \Rightarrow M}$$

Kalman Filter & LQR Duality

Let's compare the AREs:

LQR

$$A^T P A - P = -C_Q^T C_Q + A^T P B\left[B^T P B + R\right]^{-1} B^T P A$$

KF

$$A M A^T - M = -B_w W B_w^T + A M C^T\left[C M C^T + V\right]^{-1} C M A^T$$

$$\boxed{C_Q^T \Rightarrow B_w W^{1/2} = B_w'}$$

Kalman Filter & LQR Duality

Let's compare the AREs:

LQR

$$A^T P A - P = -C_Q^T C_Q + A^T P B\left[B^T P B + R\right]^{-1} B^T P A$$

KF

$$A M A^T - M = -B_w W B_w^T + A M C^T\left[C M C^T + V\right]^{-1} C M A^T$$

$$\boxed{A \Rightarrow A^T}$$

Kalman Filter & LQR Duality

Let's compare the AREs:

LQR

$$A^T P A - P = -C_Q^T C_Q + A^T P B\left[B^T P B + R\right]^{-1} B^T P A$$

KF

$$A M A^T - M = -B_w W B_w^T + A M C^T\left[C M C^T + V\right]^{-1} C M A^T$$

$$\boxed{B \Rightarrow C^T}$$

Kalman Filter & LQR Duality

Let's compare the AREs:

LQR

$$A^T P A - P = -C_Q^T C_Q + A^T P B\left[B^T P B + R\right]^{-1} B^T P A$$

KF

$$A M A^T - M = -B_w W B_w^T + A M C^T\left[C M C^T + V\right]^{-1} C M A^T$$

$$\boxed{R \Rightarrow V}$$

Kalman Filter & LQR Duality

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Let's compare the Feedback gains:

$$K = \left[R + B^T P B\right]^{-1} B^T P A \qquad \boxed{LQR}$$

$$L^T = \left[V + C M C^T\right]^{-1} C M A^T \qquad \boxed{KF}$$

$$P \Rightarrow M \qquad B \Rightarrow C^T \quad A \Rightarrow A^T \qquad R \Rightarrow V$$

Kalman Filter & LQR Duality

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Let's compare the Feedback gains:

$$K^T = A P B \left[R + B^T P B\right]^{-1} \qquad \boxed{LQR}$$

$$L \;=\; A M C^T \left[V + C M C^T\right]^{-1} \qquad \boxed{KF}$$

$$\boxed{K^T \;\Rightarrow\; L}$$

Kalman Filter & LQR Duality

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Comparing ARE's and feedback gains, we obtain the following duality

$\xrightarrow{duality}$

LQR	KF
P	M
A	A^T
B	C^T
R	V
$C_Q{}^T$	$B'_w = B_w W^{1/2}$
K	L^T
$(A-BK)$	$(A-LC)^T$

Kalman Filter & LQR Duality

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$\xrightarrow{duality}$

LQR	KF
P	M
A	A^T
B	C^T
R	V
$C_Q{}^T$	$B'_w = B_w W^{1/2}$
K	L^T
$(A-BK)$	$(A-LC)^T$

$$A^T P A - P + C_Q^T C_Q - A^T P B \left[B^T P B + R\right]^{-1} B^T P A = 0$$

$$A M A^T - M + B'_w B'^T_w - A M C^T \left[C M C^T + V\right]^{-1} C M A^T = 0$$

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Kalman Filter & LQR Duality

duality →

LQR	KF
P	M
A	A^T
B	C^T
R	V
C_Q^T	$B'_w = B_w W^{1/2}$
K	L^T
$(A-BK)$	$(A-LC)^T$

$$K = \left[B^T P B + R\right]^{-1} B^T P A$$

$$L^T = \left[C M C^T + V\right]^{-1} C M A^T$$

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Kalman Filter & LQR Duality

- It is possible to use duality to prove theorems 1-4 for stationary Kalman filters from the corresponding theorems from the infinite horizon LQR
- The following slides give an outline of how to do this
- The main idea is to design an infinite horizon LQR for a fictitious system

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Theorems 1-4 proof methodology

- Consider the LQR problem:

$$\bar{x}(k+1) = A^T \bar{x}(k) + C^T \bar{u}(k)$$

$$J = \bar{x}^T(N)\, X_0\, \bar{x}(N) + \sum_{k=0}^{N-1} \left\{\bar{x}^T(k)\, B_w W B_w^T\, \bar{x}(k) + \bar{u}^T(k)\, V\, \bar{u}(k)\right\}$$

- Solution:

$$\bar{u}(k) = -[C\bar{P}(k+1)C^T + V]^{-1} C\bar{P}(k+1)A^T \bar{x}(k)$$

$$\begin{aligned}\bar{P}(k-1) &= A\bar{P}(k)A^T + B_w W B_w^T \\ &\quad - A\bar{P}(k)C^T[C\bar{P}(k)C^T + V]^{-1} C\bar{P}(k)A^T\end{aligned}$$

$$\bar{P}(N) = X_0 \quad = M(0)$$

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Theorems 1-4 proof methodology

- The solution of the Riccati equation

$$\begin{aligned}\bar{P}(k-1) &= A\bar{P}(k)A^T + B_w W B_w^T \\ &\quad - A\bar{P}(k)C^T[C\bar{P}(k)C^T + V]^{-1} C\bar{P}(k)A^T \\ \bar{P}(N) &= X_0 = M(0)\end{aligned}$$

  is $\bar{P}(N-k) = M(k)$

- Use LQR convergence results for $\bar{P}(0)$ as $N \to \infty$ to yield convergence results for $\bar{M}(N)$ as $N \to \infty$

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Theorems 1-4 proof methodology

- Other key ideas in proofs
  - (A^T,C^T) stabilizable iff (C,A) detectable
  - Unobservable modes of $((B_w W^{1/2})^T, A^T)$ are the uncontrollable modes of $(A, B_w W^{1/2})$
  - A^T-$C^T L^T$ is Schur iff A-LC is Schur

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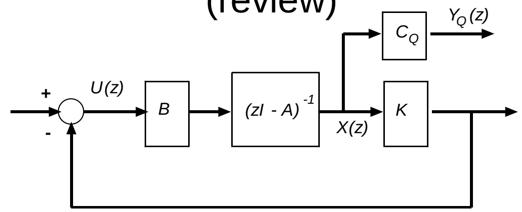
Return difference equality for LQR (review)



$$[I+G_o(z^{-1})]^T\,[R+B^TPB]\,[I+G_o(z)] = R + G_Q^T(z^{-1})\,G_Q(z)$$

Open loop transfer function:

$$G_o(z) = K\Phi(z)B$$

TF from $U(z)$ to $Y_Q(z)$:

$$G_Q(z) = C_Q\Phi(z)B$$

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Return difference equality for LQR (review)

Substituting, $G_o(z) = K\Phi(z)B$ $\quad$ $G_Q(z) = C_Q\Phi(z)B$

into

$$[I+G_o(z^{-1})]^T\,[R+B^TPB]\,[I+G_o(z)] = R + G_Q^T(z^{-1})\,G_Q(z)$$

We obtain,

$$\left[I+K\Phi(z^{-1})B\right]^T\,\left[B^TPB+R\right][I+K\Phi(z)B] =$$

$$R + \left[C_Q\Phi(z^{-1})B\right]^T\,\left[C_Q\Phi(z)B\right]$$

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Kalman Filter & LQR Duality

$$[I+K\Phi(z)B]^T\,\left[B^TPB+R\right]\,\left[I+K\Phi(z^{-1})B\right] =$$

$$R + \left[C_Q\Phi(z)B\right]^T\,\left[C_Q\Phi(z^{-1})B\right]$$

LQR	KF
P	M
A	A^T
B	C^T

LQR	KF
R	V
C_Q^T	$B'_w = B_w W^{1/2}$
K	L^T

$$\left[I+L^T\Phi^T(z)C^T\right]^T\,\left[CMC^T+V\right]\,\left[I+L^T\Phi^T(z^{-1})C^T\right] =$$

$$V + \left[B_w'^T\Phi^T(z)C^T\right]^T\,\left[B_w'^T\Phi^T(z^{-1})C^T\right]$$

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KF return difference equality

From,

$$\left[I + L^T\Phi^T(z)C^T\right]^T \left[CMC^T + V\right] \left[I + L^T\Phi^T(z^{-1})C^T\right] =$$
$$V + \left[B_w'^T\Phi^T(z)C^T\right]^T \left[B_w'^T\Phi^T(z^{-1})C^T\right]$$

we perform transpose operations and notice that:

$$B_w' B_w'^T = B_w W B_w^T$$

This gives the desired result:

$$\boxed{\begin{aligned}&[I + C\Phi(z)L] \left[C\,MC^T + V\right] \left[I + C\Phi(z^{-1})L\right]^T = \\ &\qquad V + [C\Phi(z)B_w]\; W \left[C\Phi(z^{-1})B_w\right]^T\end{aligned}}$$

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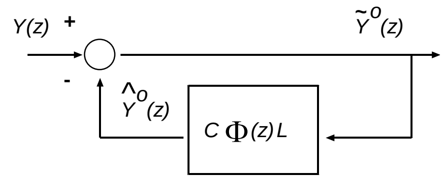
Kalman filter closed-loop eigenvalues

- A-priori KF (for $u(k) = 0$)

$$\begin{aligned}\hat{x}^o(k+1) &= A\,\hat{x}^o(k) + L\,\tilde{y}^o(k)\\ \hat{y}^o(k) &= C\hat{x}^o(k)\end{aligned}$$

$$\tilde{y}^o(k) = y(k) - C\,\hat{x}^o(k)$$



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Kalman filter closed-loop eigenvalues

$$\hat{x}^o(k+1) = \underbrace{(A - LC)}_{A_c}\,\hat{x}^o(k) + L\,y(k)$$

- KF closed-loop eigenvalues

$$\begin{aligned}\hat{C}(z) &= \det\{(zI - A_c)\} = 0\\ &= \det\{(zI - A + LC)\} = 0\end{aligned}$$

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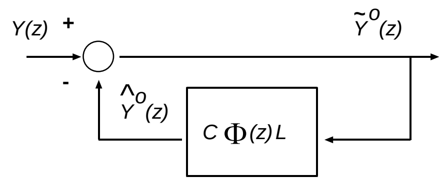
Kalman filter return difference



$$\tilde{Y}^o(z) = [I + C\Phi(z)L]^{-1}\,Y(z)$$

Return difference: $[I + C\Phi(z)L]$

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Kalman filter return difference

- Similar to the LQR case, we have that

$$\det\{[I + C\Phi(z)L]\} = \frac{\widehat{C}(z)}{\widehat{A}(z)}$$

- KF closed-loop eigenvalues

$$\widehat{C}(z) = \det\{(zI - A + LC)\} = 0$$

- KF open-loop eigenvalues

$$\widehat{A}(z) = \det\{(zI - A)\} = 0$$

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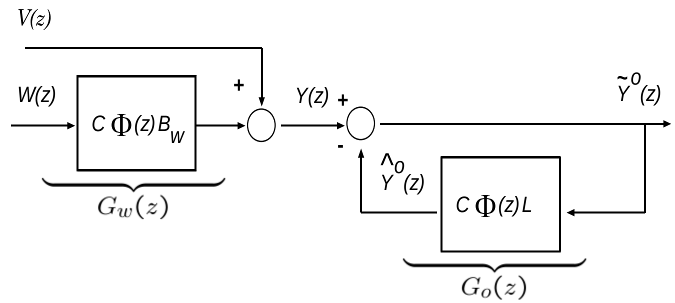
KF return difference equality



$$[I + G_o(z)]\left[C\,MC^T + V\right]\left[I + G_o(z^{-1})\right]^T =$$

$$V + G_w(z)\,W G_w^T(z^{-1})$$

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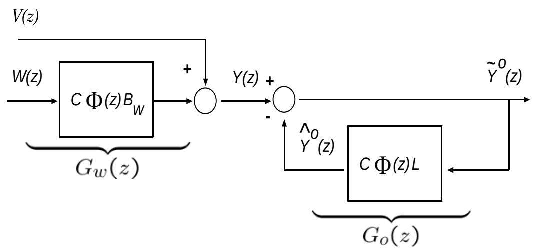
KF return difference equality (SISO)

Assume that both, $w(k) \in \mathcal{R}$ and $y(k),\ v(k) \in \mathcal{R}$



$$\boxed{[1 + G_o(z)][1 + G_o(z^{-1})] = \gamma\,[1 + \frac{W}{V} G_w(z) G_w(z^{-1})]}$$

$$\gamma = \frac{V}{V + C\,MC^T}$$

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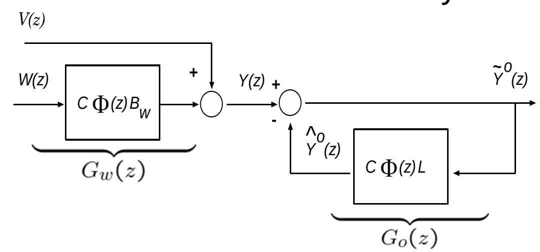
KF root locus for SISO Systems



$$[1 + G_o(z)] = [1 + C\Phi(z)L] = \frac{\widehat{C}(z)}{\widehat{A}(z)}$$ ⟵ *c.l. poles* / ⟵ *o.l. poles*

$$G_w(z) = C\Phi(z)B_w = \frac{\widehat{B}_w(z)}{\widehat{A}(z)}$$ ⟵ *o.l. zeros* / ⟵ *o.l. poles*

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KF root locus for SISO Systems

$$\boxed{\frac{\widehat{C}(z^{-1})\widehat{C}(z)}{\widehat{A}(z^{-1})\widehat{A}(z)} = \gamma\left[1+\rho\,\frac{\widehat{B}_w(z^{-1})\widehat{B}_w(z)}{\widehat{A}(z^{-1})\widehat{A}(z)}\right]}$$

$$\boxed{\rho = \frac{W}{V} \geq 0}$$

input noise intensity / *measurement noise intensity*

$$\gamma = \frac{V}{V + C\,MC^T} > 0, \quad \text{for} \quad V \in (0, \infty)$$

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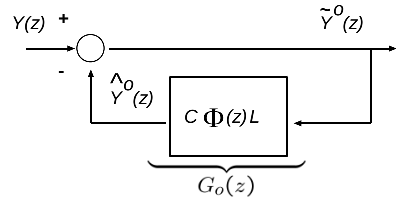
KF Loop phase margins (SISO)



Utilizing LQR-KF duality, $|(1+G_o(e^{j\omega}))| \geq \sqrt{\frac{V}{V+CMC^T}}$

Therefore, a lower bound to the phase margin of $G_o(e^{j\omega})$ is:

$$\boxed{PM \geq 2\sin^{-1}\left\{0.5\sqrt{\frac{V}{V+CMC^T}}\right\}}$$

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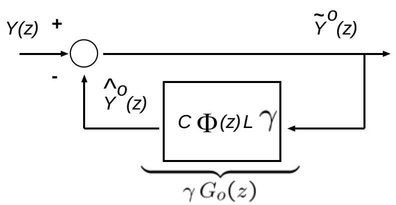
KF Loop gain margins (SISO)



Estimator was designed for $\gamma = 1$

Estimator is ***guaranteed*** to remain asymptotically stable for

$$\boxed{\frac{1}{1+\sqrt{V/(V+CMC^T)}} < \gamma < \frac{1}{1-\sqrt{V/V+CMC^T)}}}$$

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Summary

- Stationary Kalman filters (KF):
  - KF algebraic Riccati equation
  - Convergence properties
- Kalman filter / LQR duality
- KF return difference equality
  - Reciprocal root locus
  - Guaranteed robustness margins