UNIVERSITY OF CALIFORNIA AT BERKELEY
Department of Mechanical Engineering
ME233 Advanced Control Systems II
Spring 2012

Homework #3

Assigned: Feb. 9 (Th)
Due: Feb. 16 (Th)

1. In this problem, we revisit problem 5 of homework 2.

   A random variable X is repeatedly measured, but the measurements are noisy. Assume that the measurement process can be described by

   $$Y(k) = X + V(k)$$

   where $X, V(0), V(1), V(2), \ldots$ are jointly Gaussian random variables with

   $$\begin{aligned} E\{X\} &= 0 & E\{X^2\} &= X_0 \\ E\{V(k)\} &= 0 & E\{V(k+j)V(k)\} &= \Sigma_V \delta(j) \\ E\{XV(k)\} &= 0\,. \end{aligned}$$

   Let $y(k)$ be the k-th measurement (i.e. outcome of $Y(k)$) and let $\bar{y}(k) = \{y(0), \ldots, y(k)\}$. Using a Kalman filter, obtain the least squares estimate of X given the $k+1$ measurements $y(0), \ldots, y(k)$ and the corresponding estimation error covariance, i.e. find $\hat{x}_{|\bar{y}(k)}$ and $\Lambda_{X_{|\bar{y}(k)} X_{|\bar{y}(k)}}$. You may leave your expressions for $\hat{x}_{|\bar{y}(k)}$ and $\Lambda_{X_{|\bar{y}(k)} X_{|\bar{y}(k)}}$ in a recursive form.

2. Consider the system

   $$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.08 & -1 \\ 0.7 & 0.1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.34 \\ 0.3 \end{bmatrix} (u(k) + w(k))$$
   $$y(k) = \begin{bmatrix} 0 & 3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + v(k)$$

   where $u(k)$ is a deterministic input to be defined subsequently and

   - $E\{X(0)\} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $E\{X(0)X^T(0)\} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$
   - $X(0)$ is Gaussian and $W(k)$ and $V(k)$ are white Gaussian sequences
   - $m_w = E\{W(k)\} = 10, \quad E\{V(k)\} = 0$
   - $E\left\{ \begin{bmatrix} W(k+j) - m_w \\ V(k+j) \end{bmatrix} \begin{bmatrix} (W(k) - m_w) & V(k) \end{bmatrix} \right\} = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix} \delta(j) = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \delta(j)$
   - $E\left\{ \begin{bmatrix} W(k) - m_w \\ V(k) \end{bmatrix} X^T(0) \right\} = 0$

   A Kalman filter is used to estimate the state of the system using the measurement sequence $y(k)$.

(a) Find the steady state values of the following time varying matrices and scalars:

- The a priori state estimation error covariance $M(k)$
- The a posteriori state estimation error covariance $Z(k)$
- The a priori output estimation error covariance

$$\Lambda_{\tilde{y}^o\tilde{y}^o}(k,0) = E\{|\tilde{y}^o(k)|^2\} = CM(k)C^T + V \ .$$

- The Kalman filter gains $L(k)$ and $F(k)$

You should find the steady state values of these quantities by recursively computing their values forwards in time until they converge to their respective steady state values.

(b) Plot the response of $\Lambda_{\tilde{y}^o\tilde{y}^o}(k,0)$.

3. **Kalman filter with correlated input and measurement noise:**

Consider the discrete-time system given by

$$x(k+1) = Ax(k) + Bu(k) + w(k) \tag{1}$$
$$y(k) = Cx(k) + v(k) \tag{2}$$

where $E\{x(0)\} = x_o$, $E\{(x(0)-x_o)(x(0)-x_o)^T\} = X_o$ and

$$E\left\{\begin{bmatrix} w(k) \\ v(k) \end{bmatrix} \begin{bmatrix} w(j)^T & v(j)^T \end{bmatrix}\right\} = \begin{bmatrix} W & S \\ S^T & V \end{bmatrix} \delta(k-j)$$

where $V \in \mathbb{R}^{m\times m}$ is positive definite matrix. The a-priori Kalman filter for this system can be written as

$$\hat{x}^o(k+1) = A\hat{x}^o(k) + Bu(k) + L(k)[y(k) - C\hat{x}^o(k)] \tag{3}$$
$$L(k) = [AM(k)C^T + S][CM(k)C^T + V]^{-1} \tag{4}$$
$$M(k+1) = AM(k)A^T + W - [AM(k)C^T + S][CM(k)C^T + V]^{-1}[CM(k)A^T + S^T] \tag{5}$$

with initial conditions $\hat{x}^o(0) = x_o$ and $M(0) = X_o$.

Derive Eqs. (3)–(5) using previously-derived results in Kalman filtering and noticing that Eqs. (1)–(2) can be written as

$$x(k+1) = A'x(k) + Bu(k) + w'(k) + SV^{-1}y(k)\,,$$

where $A' = A - SV^{-1}C$ and

$$E\left\{\begin{bmatrix} w'(k) \\ v(k) \end{bmatrix} \begin{bmatrix} w'(j)^T & v(j)^T \end{bmatrix}\right\} = \begin{bmatrix} W' & 0 \\ 0 & V \end{bmatrix} \delta(k-j), \qquad W' = W - SV^{-1}S^T \ .$$