ME 233 – Advanced Control II

Lecture 18

Minimum Variance Regulator

Richard Conway

UC Berkeley

April 18, 2012

Outline

Outline

Model Form

We consider a state space model of the form

$$\begin{aligned} x(k+1) &= \hat{A}x(k) + \hat{B}u(k) + \hat{B}_w w(k) \\ y(k) &= \hat{C}x(k) + v(k) \end{aligned}$$

where

- $u(k)$ is the **scalar** control signal
- $y(k)$ is the **scalar** measurement signal
- $w(k)$ is the input noise
  (white, zero-mean, $E\{w(k)w^T(k)\} = W$)
- $v(k)$ is the measurement noise
  (white, zero-mean, $E\{v(k)v^T(k)\} = V$)
- $E\{w(k)v^T(k)\} = 0$

Stationary Kalman Filter V2 (Review)

The optimal state estimator is given by

$$\hat{x}^o(k+1) = \hat{A}\hat{x}^o(k) + \hat{B}u(k) + \hat{L}\tilde{y}^o(k)$$
$$\tilde{y}^o(k) = y(k) - \hat{C}\hat{x}^o(k)$$

where

$$\hat{L} = \hat{A}M\hat{C}^T[\hat{C}M\hat{C}^T + V]^{-1}$$
$$M = \hat{A}M\hat{A}^T + \hat{B}_w W \hat{B}_w^T - \hat{A}M\hat{C}^T[\hat{C}M\hat{C}^T + V]^{-1}\hat{C}M\hat{A}^T$$
$$\hat{A} - \hat{L}\hat{C} \text{ is Schur}$$

Stationary Kalman Filter V2 (Review)

The optimal state estimator is given by

$$\hat{x}^o(k+1) = \hat{A}\hat{x}^o(k) + \hat{B}u(k) + \hat{L}\tilde{y}^o(k)$$
$$\tilde{y}^o(k) = y(k) - \hat{C}\hat{x}^o(k)$$

where

$$\hat{L} = \hat{A}M\hat{C}^T[\hat{C}M\hat{C}^T + V]^{-1}$$
$$M = \hat{A}M\hat{A}^T + \hat{B}_w W \hat{B}_w^T - \hat{A}M\hat{C}^T[\hat{C}M\hat{C}^T + V]^{-1}\hat{C}M\hat{A}^T$$
$$\hat{A} - \hat{L}\hat{C} \text{ is Schur}$$

Also, the signal $\tilde{y}^o(k)$ is zero-mean, white, and has covariance $\hat{C}M\hat{C}^T + V$.

Alternate Model Form

Using the Kalman Filter V2, we can write

$$\hat{x}^o(k+1) = \hat{A}\hat{x}^o(k) + \hat{B}u(k) + \hat{L}\epsilon(k)$$
$$y(k) = \hat{C}\hat{x}^o(k) + \epsilon(k)$$

where $\epsilon(k) = \tilde{y}^o(k)$.

Alternate Model Form

Using the Kalman Filter V2, we can write

$$\hat{x}^o(k+1) = \hat{A}\hat{x}^o(k) + \hat{B}u(k) + \hat{L}\epsilon(k)$$
$$y(k) = \hat{C}\hat{x}^o(k) + \epsilon(k)$$

where $\epsilon(k) = \tilde{y}^o(k)$.

As a transfer function, this is

$$\begin{aligned} Y(z) = {} & [\hat{C}(zI - \hat{A})^{-1}\hat{B}]U(z) \\ & + [1 + \hat{C}(zI - \hat{A})^{-1}\hat{L}]E(z) \end{aligned}$$

Alternate Model Form

Using the Kalman Filter V2, we can write

$$\hat{x}^o(k+1) = \hat{A}\hat{x}^o(k) + \hat{B}u(k) + \hat{L}\epsilon(k)$$
$$y(k) = \hat{C}\hat{x}^o(k) + \epsilon(k)$$

where $\epsilon(k) = \tilde{y}^o(k)$.

As a transfer function, this is

$$\begin{aligned} Y(z) &= [\hat{C}(zI - \hat{A})^{-1}\hat{B}]U(z) \\ &\quad + [1 + \hat{C}(zI - \hat{A})^{-1}\hat{L}]E(z) \end{aligned}$$

Recall that $1 + \hat{C}(zI - \hat{A})^{-1}\hat{L} = \dfrac{\det[zI - (\hat{A} - \hat{L}\hat{C})]}{\det[zI - \hat{A}]}$

Alternate Transfer Function Model

From the previous slide, we have that

$$Y(z) = \frac{\bar{B}(z)}{\bar{A}(z)}U(z) + \frac{\bar{C}(z)}{\bar{A}(z)}E(z)$$

Alternate Transfer Function Model

From the previous slide, we have that

$$Y(z) = \frac{\bar{B}(z)}{\bar{A}(z)}U(z) + \frac{\bar{C}(z)}{\bar{A}(z)}E(z)$$

where

$$\begin{aligned}
\bar{A}(z) &= z^n + a_1 z^{n-1} + \cdots + a_n & &= \det[zI - \hat{A}] \\
\bar{C}(z) &= z^n + c_1 z^{n-1} + \cdots + c_n & &= \det[zI - (\hat{A} - \hat{L}\hat{C})] \\
\bar{B}(z) &= b_0 z^m + \cdots + b_m
\end{aligned}$$

Alternate Transfer Function Model

From the previous slide, we have that

$$Y(z) = \frac{\bar{B}(z)}{\bar{A}(z)} U(z) + \frac{\bar{C}(z)}{\bar{A}(z)} E(z)$$

where

$$\begin{aligned}
\bar{A}(z) &= z^n + a_1 z^{n-1} + \cdots + a_n &&= \det[zI - \hat{A}] \\
\bar{C}(z) &= z^n + c_1 z^{n-1} + \cdots + c_n &&= \det[zI - (\hat{A} - \hat{L}\hat{C})] \\
\bar{B}(z) &= b_0 z^m + \cdots + b_m
\end{aligned}$$

Since $\hat{A} - \hat{L}\hat{C}$ is Schur, the polynomial $\bar{C}(z)$ is Schur

Polynomials in q^{-1}

We now define $\mathrm{d} = n - m$ and the polynomials

$$
\begin{aligned}
A(z^{-1}) = z^{-n}\bar{A}(z) \qquad &= 1 + a_1 z^{-1} + \cdots + a_n z^{-n} \\
C(z^{-1}) = z^{-n}\bar{C}(z) \qquad &= 1 + c_1 z^{-1} + \cdots + c_n z^{-n} \\
B(z^{-1}) = z^{-m}\bar{B}(z) \qquad &= b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}
\end{aligned}
$$

Polynomials in q^{-1}

We now define $\mathrm{d} = n - m$ and the polynomials

$$\begin{aligned}
A(z^{-1}) &= z^{-n}\bar{A}(z) &&= 1 + a_1 z^{-1} + \cdots + a_n z^{-n} \\
C(z^{-1}) &= z^{-n}\bar{C}(z) &&= 1 + c_1 z^{-1} + \cdots + c_n z^{-n} \\
B(z^{-1}) &= z^{-m}\bar{B}(z) &&= b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}
\end{aligned}$$

so that we can write the transfer function from the previous slide as

$$Y(z) = \frac{z^{-\mathrm{d}} B(z^{-1})}{A(z^{-1})} U(z) + \frac{C(z^{-1})}{A(z^{-1})} E(z)$$

Polynomials in q^{-1}

We now define $\mathrm{d} = n - m$ and the polynomials

$$
\begin{aligned}
A(z^{-1}) &= z^{-n}\bar{A}(z) &&= 1 + a_1 z^{-1} + \cdots + a_n z^{-n} \\
C(z^{-1}) &= z^{-n}\bar{C}(z) &&= 1 + c_1 z^{-1} + \cdots + c_n z^{-n} \\
B(z^{-1}) &= z^{-m}\bar{B}(z) &&= b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}
\end{aligned}
$$

so that we can write the transfer function from the previous slide as

$$
Y(z) = \frac{z^{-\mathrm{d}} B(z^{-1})}{A(z^{-1})} U(z) + \frac{C(z^{-1})}{A(z^{-1})} E(z)
$$

Note in particular that $C(z^{-1})$ is an anti-Schur polynomial of z^{-1}

ARMAX Plant Model

We have now transformed the original state space plant model to

$$A(q^{-1})y(k) = q^{-\mathrm{d}}B(q^{-1})u(k) + C(q^{-1})\epsilon(k)$$

where $C(q^{-1})$ is an anti-Schur polynomial of q^{-1} and $\epsilon(k)$ is zero-mean white noise with covariance $\hat{C}M\hat{C}^T + V$

ARMAX Plant Model

We have now transformed the original state space plant model to

$$A(q^{-1})y(k) = q^{-\mathrm{d}}B(q^{-1})u(k) + C(q^{-1})\epsilon(k)$$

where $C(q^{-1})$ is an anti-Schur polynomial of q^{-1} and $\epsilon(k)$ is zero-mean white noise with covariance $\hat{C}M\hat{C}^T + V$

This type of model is called an <u>ARMAX</u> model because it is an ARMA model with an eXogenous input.

Outline

Minimum Variance Regulator (MVR) Problem

Given the ARMAX model

$$A(q^{-1})y(k) = q^{-\mathrm{d}}B(q^{-1})u(k) + C(q^{-1})\epsilon(k)$$

where

- $C(q^{-1})$ is an anti-Schur polynomial of q^{-1}
- $B(q^{-1})$ has no zeros on the unit circle
- $\epsilon(k)$ is zero-mean white noise
- The plant has no unstable pole-zero cancelations, i.e. the polynomials $A(q^{-1})$ and $B(q^{-1})$ have no common zeros such that $|q^{-1}| < 1$

Minimum Variance Regulator (MVR) Problem

Given the ARMAX model

$$A(q^{-1})y(k) = q^{-\mathrm{d}}B(q^{-1})u(k) + C(q^{-1})\epsilon(k)$$

where

- $C(q^{-1})$ is an anti-Schur polynomial of q^{-1}
- $B(q^{-1})$ has no zeros on the unit circle
- $\epsilon(k)$ is zero-mean white noise
- The plant has no unstable pole-zero cancelations, i.e. the polynomials $A(q^{-1})$ and $B(q^{-1})$ have no common zeros such that $|q^{-1}| < 1$

find the stabilizing feedback control law that minimizes the output variance $E\{y^2(k)\}$

Outline

Factorization of B and $\bar{B}$

In general, the polynomial $\bar{B}(q) = q^m B(q^{-1})$ has

- m_s zeros strictly inside the unit circle (stable plant zeros)
- m_u zeros strictly outside the unit circle (unstable plant zeros)

Factorization of B and $\bar{B}$

In general, the polynomial $\bar{B}(q) = q^m B(q^{-1})$ has

- m_s zeros strictly inside the unit circle (stable plant zeros)
- m_u zeros strictly outside the unit circle (unstable plant zeros)

Perform the factorization

$$B(q^{-1}) = B^s(q^{-1}) B^u(q^{-1})$$

where

- $\bar{B}^s(q) = q^{m_s} B^s(q^{-1})$ has its zeros inside the unit circle (These are the stable plant zeros)
- $\bar{B}^u(q) = q^{m_u} B^u(q^{-1})$ has its zeros outside the unit circle (These are the unstable plant zeros)
- $\bar{B}^u(0) = 1$

Minimum Variance Regulator (MVR) Solution

- The optimal control $u_*(k)$ is given by

$$B^s(q^{-1})R(q^{-1})u_*(k) = -S(q^{-1})y(k)$$

where $R(q^{-1})$ and $S(q^{-1})$ are found by solving the Diophantine equation

$$C(q^{-1})\bar{B}^u(q^{-1}) = A(q^{-1})R(q^{-1}) + q^{-\mathrm{d}}B^u(q^{-1})S(q^{-1})$$

where

$$R(q^{-1}) = 1 + r_1 q^{-1} + \cdots + r_{n_r} q^{-n_r}$$
$$S(q^{-1}) = s_0 + s_1 q^{-1} + \cdots + s_{n_s} q^{-n_s}$$

and $n_r = m_u + \mathrm{d} - 1$ and $n_s = n - 1$

Minimum Variance Regulator (MVR) Solution

- The optimal cost is

$$E\{y^2(k)\} = E\{\epsilon_f^2(k)\}$$

where $\epsilon_f(k)$ is defined in terms of $\epsilon(k)$ by the ARMA model

$$\bar{B}^u(q^{-1})\epsilon_f(k) = R(q^{-1})\epsilon(k)$$

Constructing the MVR

1. Find $\hat{L}$ using a stationary Kalman filter design
2. Construct $C(q^{-1}) = q^{-n} \det[qI - (\hat{A} - \hat{L}\hat{C})]$
3. Factor $B(q^{-1}) = B^s(q^{-1})B^u(q^{-1})$ as described previously (don't forget that $\bar{B}^u(0) = 1$)
4. Solve the Diophantine equation

$$C(q^{-1})\bar{B}^u(q^{-1}) = A(q^{-1})R(q^{-1}) + q^{-\mathrm{d}}B^u(q^{-1})S(q^{-1})$$

5. Form the optimal controller

$$B^s(q^{-1})R(q^{-1})u_*(k) = -S(q^{-1})y(k)$$

Solution Comments

- Be careful with $B^u(q^{-1})$, $\bar{B}^u(q)$, and $\bar{B}^u(q^{-1})$

Solution Comments

- Be careful with $B^u(q^{-1})$, $\bar{B}^u(q)$, and $\bar{B}^u(q^{-1})$
  - $B^u(q^{-1})$ is a Schur polynomial in q^{-1}

Solution Comments

- Be careful with $B^u(q^{-1})$, $\bar{B}^u(q)$, and $\bar{B}^u(q^{-1})$
  - $B^u(q^{-1})$ is a Schur polynomial in q^{-1}
  - $\bar{B}^u(q)$ is an anti-Schur polynomial in q

Solution Comments

- Be careful with $B^u(q^{-1})$, $\bar{B}^u(q)$, and $\bar{B}^u(q^{-1})$
  - $B^u(q^{-1})$ is a Schur polynomial in q^{-1}
  - $\bar{B}^u(q)$ is an anti-Schur polynomial in q
  - $\bar{B}^u(q^{-1})$ is an anti-Schur polynomial in q^{-1}

Solution Comments

- Be careful with $B^u(q^{-1})$, $\bar{B}^u(q)$, and $\bar{B}^u(q^{-1})$
  - $B^u(q^{-1})$ is a Schur polynomial in q^{-1}
  - $\bar{B}^u(q)$ is an anti-Schur polynomial in q
  - $\bar{B}^u(q^{-1})$ is an anti-Schur polynomial in q^{-1}
- Note that the Diophantine equation involves both $B^u(q^{-1})$ and $\bar{B}^u(q^{-1})$.

Solution Comments

- Be careful with $B^u(q^{-1})$, $\bar{B}^u(q)$, and $\bar{B}^u(q^{-1})$
  - $B^u(q^{-1})$ is a Schur polynomial in q^{-1}
  - $\bar{B}^u(q)$ is an anti-Schur polynomial in q
  - $\bar{B}^u(q^{-1})$ is an anti-Schur polynomial in q^{-1}
- Note that the Diophantine equation involves both $B^u(q^{-1})$ and $\bar{B}^u(q^{-1})$.
- Since $\bar{B}^u(q^{-1})$ is anti-Schur, the operator $\dfrac{R(q^{-1})}{\bar{B}^u(q^{-1})}$ is BIBO.

Solution Comments

- Be careful with $B^u(q^{-1})$, $\bar{B}^u(q)$, and $\bar{B}^u(q^{-1})$
  - $B^u(q^{-1})$ is a Schur polynomial in q^{-1}
  - $\bar{B}^u(q)$ is an anti-Schur polynomial in q
  - $\bar{B}^u(q^{-1})$ is an anti-Schur polynomial in q^{-1}
- Note that the Diophantine equation involves both $B^u(q^{-1})$ and $\bar{B}^u(q^{-1})$.
- Since $\bar{B}^u(q^{-1})$ is anti-Schur, the operator $\dfrac{R(q^{-1})}{\bar{B}^u(q^{-1})}$ is BIBO.

$$\Rightarrow \quad \epsilon_f(k) = \frac{R(q^{-1})}{\bar{B}^u(q^{-1})}\epsilon(k) \text{ has bounded covariance}$$

Special Case: $B(q^{-1})$ is anti-Schur

When $B(q^{-1})$ is anti-Schur, we have

- $B^s(q^{-1}) = B(q^{-1})$
- $B^u(q^{-1}) = \bar{B}^u(q) = \bar{B}^u(q^{-1}) = 1$

Special Case: $B(q^{-1})$ is anti-Schur

When $B(q^{-1})$ is anti-Schur, we have

- $B^s(q^{-1}) = B(q^{-1})$
- $B^u(q^{-1}) = \bar{B}^u(q) = \bar{B}^u(q^{-1}) = 1$
- Expressing $R(q^{-1}) = 1 + r_1 q^{-1} + \cdots + r_{n_r} q^{-n_r}$, the optimal cost is

$$\begin{aligned}
E\{y^2(k)\} &= E\{[R(q^{-1})\epsilon(k)]^2\} \\
&= E\{[\epsilon(k) + r_1\epsilon(k-1) + \cdots + r_{n_r}\epsilon(k-n_r)]^2\} \\
&= E\{\epsilon^2(k)\} + r_1^2 E\{\epsilon^2(k-1)\} + \cdots + r_{n_r}^2 E\{\epsilon^2(k-n_r)\} \\
&= (1 + r_1^2 + \cdots + r_{n_r}^2) E\{\epsilon^2(k)\}
\end{aligned}$$

Special Case: $B(q^{-1})$ is anti-Schur

When $B(q^{-1})$ is anti-Schur, we have

- $B^s(q^{-1}) = B(q^{-1})$
- $B^u(q^{-1}) = \bar{B}^u(q) = \bar{B}^u(q^{-1}) = 1$
- Expressing $R(q^{-1}) = 1 + r_1 q^{-1} + \cdots + r_{n_r} q^{-n_r}$, the optimal cost is

$$\begin{aligned} E\{y^2(k)\} &= E\{[R(q^{-1})\epsilon(k)]^2\} \\ &= E\{[\epsilon(k) + r_1\epsilon(k-1) + \cdots + r_{n_r}\epsilon(k-n_r)]^2\} \\ &= E\{\epsilon^2(k)\} + r_1^2 E\{\epsilon^2(k-1)\} + \cdots + r_{n_r}^2 E\{\epsilon^2(k-n_r)\} \\ &= (1 + r_1^2 + \cdots + r_{n_r}^2) E\{\epsilon^2(k)\} \end{aligned}$$

Therefore

$$\boxed{E\{y^2(k)\} = (1 + r_1^2 + \cdots + r_{n_r}^2)(\hat{C} M \hat{C}^T + V)}$$

Outline

Proof Methodology

The proof will be done in 4 parts:

1. Rewrite the system dynamics in a more convenient form

Proof Methodology

The proof will be done in 4 parts:

1. Rewrite the system dynamics in a more convenient form
2. Prove that $E\{z(k-d)\epsilon_f(k)\} = 0$, where $z(k)$ is a sequence to be defined in subsequent slides

Proof Methodology

The proof will be done in 4 parts:

1. Rewrite the system dynamics in a more convenient form
2. Prove that $E\{z(k-d)\epsilon_f(k)\} = 0$, where $z(k)$ is a sequence to be defined in subsequent slides
3. Prove optimality of proposed control scheme

Proof Methodology

The proof will be done in 4 parts:

1. Rewrite the system dynamics in a more convenient form
2. Prove that $E\{z(k-d)\epsilon_f(k)\} = 0$, where $z(k)$ is a sequence to be defined in subsequent slides
3. Prove optimality of proposed control scheme
4. Verify closed-loop stability

Proof Methodology

The proof will be done in 4 parts:

1. Rewrite the system dynamics in a more convenient form
2. Prove that $E\{z(k-d)\epsilon_f(k)\} = 0$, where $z(k)$ is a sequence to be defined in subsequent slides
3. Prove optimality of proposed control scheme
4. Verify closed-loop stability

Comments on the notation in this proof:

- Capital letters always denote polynomials; lower case letters denote sequences (except d and q)
- Dependency of polynomials on q^{-1} will be omitted
  e.g. $\bar{B}^u$ will refer to $\bar{B}^u(q^{-1})$
- Dependency of sequences on k will be omitted
  e.g. y will refer to $y(k)$

Part 1: Rewrite Dynamics

The plant dynamics are

$$Ay = q^{-\mathrm{d}}Bu + C\epsilon$$

and the Diophantine equation gives

$$[RA]y = [C - q^{-\mathrm{d}}S]y$$

Part 1: Rewrite Dynamics

The plant dynamics are

$$Ay = q^{-\mathrm{d}}Bu + C\epsilon$$

and the Diophantine equation gives

$$[RA]y = [C - q^{-\mathrm{d}}S]y$$

Combining these two equations gives

$$R[q^{-\mathrm{d}}Bu + C\epsilon] = [C - q^{-\mathrm{d}}S]y$$

Part 1: Rewrite Dynamics

The plant dynamics are

$$Ay = q^{-\mathrm{d}}Bu + C\epsilon$$

and the Diophantine equation gives

$$[RA]y = [C - q^{-\mathrm{d}}S]y$$

Combining these two equations gives

$$R[q^{-\mathrm{d}}Bu + C\epsilon] = [C - q^{-\mathrm{d}}S]y$$

$$\Rightarrow \quad Cy - q^{-\mathrm{d}}(Sy + BRu) - CR\epsilon = 0$$

Part 1: Rewrite Dynamics

From the previous slide:

$$Cy - q^{-\mathrm{d}}(Sy + BRu) - CR\epsilon = 0$$

- Define $z(k)$ in terms of $y(k)$ and $u(k)$ using

$$Cz = Sy + BRu$$

(note that we are not necessarily using the optimal control)

Part 1: Rewrite Dynamics

From the previous slide:

$$Cy - q^{-\mathrm{d}}(Sy + BRu) - CR\epsilon = 0$$

- Define $z(k)$ in terms of $y(k)$ and $u(k)$ using

$$Cz = Sy + BRu$$

  (note that we are not necessarily using the optimal control)
- Define $\epsilon_f = R\epsilon$

Part 1: Rewrite Dynamics

From the previous slide:

$$Cy - q^{-\mathrm{d}}(Sy + BRu) - CR\epsilon = 0$$

- Define $z(k)$ in terms of $y(k)$ and $u(k)$ using

  $$Cz = Sy + BRu$$

  (note that we are not necessarily using the optimal control)
- Define $\epsilon_f = R\epsilon$
- From the top equation,

$$Cy - q^{-\mathrm{d}}Cz - C\epsilon_f = 0 \qquad \Rightarrow \qquad C(y - q^{-\mathrm{d}}z - \epsilon_f) = 0$$

Part 1: Rewrite Dynamics

From the previous slide:

$$Cy - q^{-\mathrm{d}}(Sy + BRu) - CR\epsilon = 0$$

- Define $z(k)$ in terms of $y(k)$ and $u(k)$ using

  $$Cz = Sy + BRu$$

  (note that we are not necessarily using the optimal control)
- Define $\epsilon_f = R\epsilon$
- From the top equation,

  $$Cy - q^{-\mathrm{d}}Cz - C\epsilon_f = 0 \qquad \Rightarrow \qquad C(y - q^{-\mathrm{d}}z - \epsilon_f) = 0$$

  Since C is anti-Schur, we have $y - q^{-\mathrm{d}}z - \epsilon_f \longrightarrow 0$

Part 1: Rewrite Dynamics

From the previous slide:

$$Cy - q^{-\mathrm{d}}(Sy + BRu) - CR\epsilon = 0$$

- Define $z(k)$ in terms of $y(k)$ and $u(k)$ using

$$Cz = Sy + BRu$$

  (note that we are not necessarily using the optimal control)
- Define $\epsilon_f = R\epsilon$
- From the top equation,

$$Cy - q^{-\mathrm{d}}Cz - C\epsilon_f = 0 \qquad \Rightarrow \qquad C(y - q^{-\mathrm{d}}z - \epsilon_f) = 0$$

  Since C is anti-Schur, we have $y - q^{-\mathrm{d}}z - \epsilon_f \longrightarrow 0$

$$\boxed{y(k) = z(k - \mathrm{d}) + \epsilon_f(k)}$$

Part 2: $E\{z(k-d)\epsilon_f(k)\} = 0$

- Since $\epsilon(k) = y(k) - E\{y(k)|y(k-1), y(k-2), \ldots\}$, we use least squares property 1 to see that

$$E\{y(k-\ell)\epsilon(k+p)\}, \qquad \forall \ell > 0, p \geq 0$$

Part 2: $E\{z(k-d)\epsilon_f(k)\} = 0$

- Since $\epsilon(k) = y(k) - E\{y(k)|y(k-1), y(k-2), \ldots\}$, we use least squares property 1 to see that

$$E\{y(k-\ell)\epsilon(k+p)\}, \qquad \forall \ell > 0, p \geq 0$$

- $\epsilon_f(k+\mathrm{d}-1) = \epsilon(k+\mathrm{d}-1) + r_1\epsilon(k+\mathrm{d}-2) + \cdots + r_{\mathrm{d}-1}\epsilon(k)$

$$\Rightarrow \quad E\{y(k-\ell)\epsilon_f(k+\mathrm{d}-1)\} = 0 \qquad \forall \ell > 0$$

Part 2: $E\{z(k-d)\epsilon_f(k)\} = 0$

- Since $\epsilon(k) = y(k) - E\{y(k)|y(k-1), y(k-2), \ldots\}$, we use least squares property 1 to see that

$$E\{y(k-\ell)\epsilon(k+p)\}, \qquad \forall \ell > 0, p \geq 0$$

- $\epsilon_f(k+\mathrm{d}-1) = \epsilon(k+\mathrm{d}-1) + r_1\epsilon(k+\mathrm{d}-2) + \cdots + r_{\mathrm{d}-1}\epsilon(k)$

$$\Rightarrow \quad E\{y(k-\ell)\epsilon_f(k+\mathrm{d}-1)\} = 0 \qquad \forall \ell > 0$$

- Since $u(k)$ is a function of $y(k), y(k-1), \ldots$

$$E\{u(k-\ell)\epsilon_f(k+\mathrm{d}-1)\} = 0 \qquad \forall \ell > 0$$

Part 2: $E\{z(k-d)\epsilon_f(k)\} = 0$

- Since $\epsilon(k) = y(k) - E\{y(k)|y(k-1), y(k-2), \ldots\}$, we use least squares property 1 to see that

$$E\{y(k-\ell)\epsilon(k+p)\}, \qquad \forall \ell > 0, p \geq 0$$

- $\epsilon_f(k+\mathrm{d}-1) = \epsilon(k+\mathrm{d}-1) + r_1\epsilon(k+\mathrm{d}-2) + \cdots + r_{\mathrm{d}-1}\epsilon(k)$

$$\Rightarrow \quad E\{y(k-\ell)\epsilon_f(k+\mathrm{d}-1)\} = 0 \qquad \forall \ell > 0$$

- Since $u(k)$ is a function of $y(k), y(k-1), \ldots$

$$E\{u(k-\ell)\epsilon_f(k+\mathrm{d}-1)\} = 0 \qquad \forall \ell > 0$$

- Since $z(k)$ is a function of $y(k), y(k-1), \ldots$ and $u(k), u(k-1), \ldots$

$$E\{z(k-\ell)\epsilon_f(k+\mathrm{d}-1)\} = 0 \qquad \forall \ell > 0$$

Part 2: $E\{z(k-d)\epsilon_f(k)\} = 0$

- Since $\epsilon(k) = y(k) - E\{y(k)|y(k-1), y(k-2), \ldots\}$, we use least squares property 1 to see that

$$E\{y(k-\ell)\epsilon(k+p)\}, \qquad \forall \ell > 0, p \geq 0$$

- $\epsilon_f(k+\mathrm{d}-1) = \epsilon(k+\mathrm{d}-1) + r_1\epsilon(k+\mathrm{d}-2) + \cdots + r_{\mathrm{d}-1}\epsilon(k)$

$$\Rightarrow \quad E\{y(k-\ell)\epsilon_f(k+\mathrm{d}-1)\} = 0 \qquad \forall \ell > 0$$

- Since $u(k)$ is a function of $y(k), y(k-1), \ldots$

$$E\{u(k-\ell)\epsilon_f(k+\mathrm{d}-1)\} = 0 \qquad \forall \ell > 0$$

- Since $z(k)$ is a function of $y(k), y(k-1), \ldots$ and $u(k), u(k-1), \ldots$

$$E\{z(k-\ell)\epsilon_f(k+\mathrm{d}-1)\} = 0 \qquad \forall \ell > 0$$

- Choosing $\ell = 1$ completes part 2

Part 3: Optimal Control

Recall that

$$y(k) = z(k - \mathrm{d}) + \epsilon_f(k)$$

$$E\{z(k-d)\epsilon_f(k)\} = 0$$

Part 3: Optimal Control

Recall that

$$y(k) = z(k - \mathrm{d}) + \epsilon_f(k)$$

$$E\{z(k-d)\epsilon_f(k)\} = 0$$

Therefore,

$$E\{y^2(k)\} = E\{z^2(k-d)\} + E\{\epsilon_f^2(k)\}$$

Part 3: Optimal Control

Recall that

$$y(k) = z(k - \mathrm{d}) + \epsilon_f(k)$$

$$E\{z(k-d)\epsilon_f(k)\} = 0$$

Therefore,

$$E\{y^2(k)\} = E\{z^2(k-d)\} + E\{\epsilon_f^2(k)\}$$

Notes:

- At this point, we have only assumed that $u(k)$ stabilizes the system (so that the relevant covariances are bounded)

Part 3: Optimal Control

Recall that

$$y(k) = z(k - \mathrm{d}) + \epsilon_f(k)$$

$$E\{z(k-d)\epsilon_f(k)\} = 0$$

Therefore,

$$E\{y^2(k)\} = E\{z^2(k-d)\} + E\{\epsilon_f^2(k)\}$$

Notes:

- At this point, we have only assumed that $u(k)$ stabilizes the system (so that the relevant covariances are bounded)
- $E\{\epsilon_f^2(k)\}$ does not depend on the choice of the control law

Part 3: Optimal Control

Recall that

$$y(k) = z(k - \mathrm{d}) + \epsilon_f(k)$$

$$E\{z(k-d)\epsilon_f(k)\} = 0$$

Therefore,

$$E\{y^2(k)\} = E\{z^2(k-d)\} + E\{\epsilon_f^2(k)\}$$

Notes:

- At this point, we have only assumed that $u(k)$ stabilizes the system (so that the relevant covariances are bounded)
- $E\{\epsilon_f^2(k)\}$ does not depend on the choice of the control law
- $\Rightarrow$ We would like to choose u to minimize $E\{z^2\}$

Part 3: Optimal Control

Recall that

$$y(k) = z(k - \mathrm{d}) + \epsilon_f(k)$$

$$E\{z(k-d)\epsilon_f(k)\} = 0$$

Therefore,

$$E\{y^2(k)\} = E\{z^2(k-d)\} + E\{\epsilon_f^2(k)\}$$

Notes:

- At this point, we have only assumed that $u(k)$ stabilizes the system (so that the relevant covariances are bounded)
- $E\{\epsilon_f^2(k)\}$ does not depend on the choice of the control law
- $\Rightarrow$ We would like to choose u to minimize $E\{z^2\}$
- If we can make $E\{z^2\} = 0$, the control must be optimal

Part 3: Optimal Control

Goal: choose u so that $E\{z^2\} = 0$

Part 3: Optimal Control

Goal: choose u so that $E\{z^2\} = 0$

- If we apply the control signal $u_*(k)$ defined by

$$BRu_* = -Sy$$

Part 3: Optimal Control

Goal: choose u so that $E\{z^2\} = 0$

- If we apply the control signal $u_*(k)$ defined by

$$BRu_* = -Sy$$

we have

$$Cz = BRu_* + Sy = 0$$

Part 3: Optimal Control

Goal: choose u so that $E\{z^2\} = 0$

- If we apply the control signal $u_*(k)$ defined by

$$BRu_* = -Sy$$

  we have

$$Cz = BRu_* + Sy = 0$$

- $C(q^{-1})$ is an anti-Schur polynomial $\Rightarrow$ $z(k) \longrightarrow 0$.

Part 3: Optimal Control

Goal: choose u so that $E\{z^2\} = 0$

- If we apply the control signal $u_*(k)$ defined by

$$BRu_* = -Sy$$

  we have

$$Cz = BRu_* + Sy = 0$$

- $C(q^{-1})$ is an anti-Schur polynomial $\Rightarrow$ $z(k) \longrightarrow 0$.
- For this control signal, $E\{z^2\} = 0$, which means that $u_*(k)$ is optimal, provided that the closed-loop system is stable

Part 3: Optimal Control

Goal: choose u so that $E\{z^2\} = 0$

- If we apply the control signal $u_*(k)$ defined by

$$BRu_* = -Sy$$

  we have

$$Cz = BRu_* + Sy = 0$$

- $C(q^{-1})$ is an anti-Schur polynomial $\Rightarrow$ $z(k) \longrightarrow 0$.
- For this control signal, $E\{z^2\} = 0$, which means that $u_*(k)$ is optimal, provided that the closed-loop system is stable
- Also note that $E\{y^2\} = E\{\epsilon_f^2\}$, provided that the closed-loop system is stable

Part 4: Closed-loop stability

From the plant dynamics and feedback law, we have

$$\begin{bmatrix} A & -q^{-\mathrm{d}}B \\ S & BR \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C\epsilon \\ 0 \end{bmatrix}$$

Part 4: Closed-loop stability

From the plant dynamics and feedback law, we have

$$\begin{bmatrix} A & -q^{-\mathrm{d}}B \\ S & BR \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C\epsilon \\ 0 \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \begin{bmatrix} y \\ u \end{bmatrix} &= \frac{1}{BAR + q^{-\mathrm{d}}BS} \begin{bmatrix} BR & q^{-\mathrm{d}}B \\ -S & A \end{bmatrix} \begin{bmatrix} C\epsilon \\ 0 \end{bmatrix} \\ &= \frac{1}{B(AR + q^{-\mathrm{d}}S)} \begin{bmatrix} BRC\epsilon \\ -SC\epsilon \end{bmatrix} = \frac{1}{BC} \begin{bmatrix} BRC\epsilon \\ -SC\epsilon \end{bmatrix} \end{aligned}$$

Part 4: Closed-loop stability

From the plant dynamics and feedback law, we have

$$\begin{bmatrix} A & -q^{-\mathrm{d}}B \\ S & BR \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C\epsilon \\ 0 \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \begin{bmatrix} y \\ u \end{bmatrix} &= \frac{1}{BAR + q^{-\mathrm{d}}BS} \begin{bmatrix} BR & q^{-\mathrm{d}}B \\ -S & A \end{bmatrix} \begin{bmatrix} C\epsilon \\ 0 \end{bmatrix} \\ &= \frac{1}{B(AR + q^{-\mathrm{d}}S)} \begin{bmatrix} BRC\epsilon \\ -SC\epsilon \end{bmatrix} = \frac{1}{BC} \begin{bmatrix} BRC\epsilon \\ -SC\epsilon \end{bmatrix} \end{aligned}$$

Since $C(q^{-1})B(q^{-1})$ is an anti-Schur polynomial of q^{-1}, the closed-loop system is stable ■

Outline

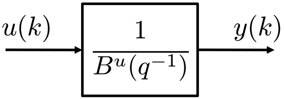
A-causal but BIBO Systems

Recall that the polynomial $B^u(q^{-1})$ is Schur

A-causal but BIBO Systems

Recall that the polynomial $B^u(q^{-1})$ is Schur

The AR model $B^u(q^{-1})y(k) = u(k)$ corresponds to the block diagram



A-causal but BIBO Systems

Recall that the polynomial $B^u(q^{-1})$ is Schur

The AR model $B^u(q^{-1})y(k) = u(k)$ corresponds to the block diagram

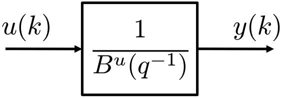
$$\xrightarrow{u(k)} \boxed{\frac{1}{B^u(q^{-1})}} \xrightarrow{y(k)}$$

We can interpret the operator $\dfrac{1}{B^u(q^{-1})}$ in two ways:

A-causal but BIBO Systems

Recall that the polynomial $B^u(q^{-1})$ is Schur

The AR model $B^u(q^{-1})y(k) = u(k)$ corresponds to the block diagram



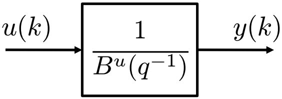
We can interpret the operator $\dfrac{1}{B^u(q^{-1})}$ in two ways:

1. Causal, but unstable

A-causal but BIBO Systems

Recall that the polynomial $B^u(q^{-1})$ is Schur

The AR model $B^u(q^{-1})y(k) = u(k)$ corresponds to the block diagram



We can interpret the operator $\dfrac{1}{B^u(q^{-1})}$ in two ways:

1. Causal, but unstable
2. A-causal, but BIBO

Interpretation 1: Causal, but Unstable

We are considering the AR model $B^u(q^{-1})y(k) = u(k)$ where

$$B^u(q^{-1}) = 1 + b_1^u q^{-1} + \cdots + b_{m_u}^u q^{-m_u}$$

Interpretation 1: Causal, but Unstable

We are considering the AR model $B^u(q^{-1})y(k) = u(k)$ where

$$B^u(q^{-1}) = 1 + b_1^u q^{-1} + \cdots + b_{m_u}^u q^{-m_u}$$

$$\longrightarrow \qquad (1 + b_1^u q^{-1} + \cdots + b_{m_u}^u q^{-m_u})y(k) = u(k)$$

Interpretation 1: Causal, but Unstable

We are considering the AR model $B^u(q^{-1})y(k) = u(k)$ where

$$B^u(q^{-1}) = 1 + b_1^u q^{-1} + \cdots + b_{m_u}^u q^{-m_u}$$

$$\longrightarrow \qquad (1 + b_1^u q^{-1} + \cdots + b_{m_u}^u q^{-m_u})y(k) = u(k)$$

Interpreting the AR model as causal, but unstable corresponds to

$$\begin{aligned} y(k) &= u(k) - [b_1^u q^{-1} + \cdots + b_{m_u}^u q^{-m_u}]y(k) \\ &= u(k) - b_1^u\, y(k-1) - \cdots - b_{m_u}^u\, y(k-m_u) \end{aligned}$$

Interpretation 1: Causal, but Unstable

We are considering the AR model $B^u(q^{-1})y(k) = u(k)$ where

$$B^u(q^{-1}) = 1 + b_1^u q^{-1} + \cdots + b_{m_u}^u q^{-m_u}$$

$$\longrightarrow \qquad (1 + b_1^u q^{-1} + \cdots + b_{m_u}^u q^{-m_u})y(k) = u(k)$$

Interpreting the AR model as causal, but unstable corresponds to

$$\begin{aligned} y(k) &= u(k) - [b_1^u q^{-1} + \cdots + b_{m_u}^u q^{-m_u}]y(k) \\ &= u(k) - b_1^u\, y(k-1) - \cdots - b_{m_u}^u\, y(k-m_u) \end{aligned}$$

$\longrightarrow \qquad y(k)$ is a function of $u(k), u(k-1), u(k-2), \ldots$

Interpretation 2: A-causal, but BIBO

We are considering the AR model

$$(1 + b_1^u q^{-1} + \cdots + b_{m_u}^u q^{-m_u}) y(k) = u(k)$$

Interpretation 2: A-causal, but BIBO

We are considering the AR model

$$(1 + b_1^u q^{-1} + \cdots + b_{m_u}^u q^{-m_u}) y(k) = u(k)$$

Interpreting the AR model as a-causal, but BIBO corresponds to

$$b_{m_u}^u q^{-m_u} y(k) = u(k) - [1 + b_1^u q^{-1} + \cdots + b_{m_u-1}^u q^{-m_u+1}] y(k)$$

Interpretation 2: A-causal, but BIBO

We are considering the AR model

$$(1 + b_1^u q^{-1} + \cdots + b_{m_u}^u q^{-m_u})y(k) = u(k)$$

Interpreting the AR model as a-causal, but BIBO corresponds to

$$b_{m_u}^u q^{-m_u} y(k) = u(k) - [1 + b_1^u q^{-1} + \cdots + b_{m_u-1}^u q^{-m_u+1}]y(k)$$

$$\Rightarrow b_{m_u}^u y(k) = q^{m_u} u(k) - [q^{m_u} + b_1^u q^{m_u-1} + \cdots + b_{m_u-1}^u q]y(k)$$

$$\Rightarrow y(k) = \frac{1}{b_{m_u}}[u(k+m_u) - y(k+m_u) - b_1^u y(k+m_u-1)$$

$$- \cdots - b_{m_u-1}^u y(k+1)]$$

Interpretation 2: A-causal, but BIBO

We are considering the AR model

$$(1 + b_1^u q^{-1} + \cdots + b_{m_u}^u q^{-m_u}) y(k) = u(k)$$

Interpreting the AR model as a-causal, but BIBO corresponds to

$$b_{m_u}^u q^{-m_u} y(k) = u(k) - [1 + b_1^u q^{-1} + \cdots + b_{m_u-1}^u q^{-m_u+1}] y(k)$$

$$\Rightarrow b_{m_u}^u y(k) = q^{m_u} u(k) - [q^{m_u} + b_1^u q^{m_u-1} + \cdots + b_{m_u-1}^u q] y(k)$$

$$\Rightarrow y(k) = \frac{1}{b_{m_u}} [u(k+m_u) - y(k+m_u) - b_1^u y(k+m_u-1)$$

$$- \cdots - b_{m_u-1}^u y(k+1)]$$

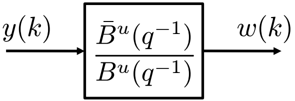
$$\boxed{y(k) \text{ is a function of } u(k+m_u), u(k+m_u+1), u(k+m_u+2), \ldots}$$

A-causal but BIBO All-Pass Filter

Let $w(k)$ be the output of the a-causal, but BIBO ARMAX model

$$B^u(q^{-1})w(k) = \bar{B}^u(q^{-1})y(k)$$

This corresponds to the block diagram



A-causal but BIBO All-Pass Filter

Let $w(k)$ be the output of the a-causal, but BIBO ARMAX model

$$B^u(q^{-1})w(k) = \bar{B}^u(q^{-1})y(k)$$

This corresponds to the block diagram

$$y(k) \longrightarrow \boxed{\frac{\bar{B}^u(q^{-1})}{B^u(q^{-1})}} \longrightarrow w(k)$$

Claim:

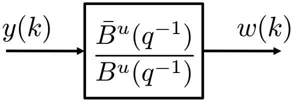
$$\left|\frac{\bar{B}^u(e^{-j\omega})}{B^u(e^{-j\omega})}\right| = 1 \qquad \forall \omega \in [0, 2\pi]$$

A-causal but BIBO All-Pass Filter

Let $w(k)$ be the output of the a-causal, but BIBO ARMAX model

$$B^u(q^{-1})w(k) = \bar{B}^u(q^{-1})y(k)$$

This corresponds to the block diagram



Claim:

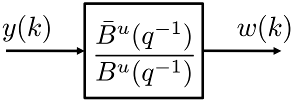
$$\left|\frac{\bar{B}^u(e^{-j\omega})}{B^u(e^{-j\omega})}\right| = 1 \qquad \forall \omega \in [0, 2\pi]$$

Proof:

$$\bar{B}^u(q) = q^{m_u} B^u(q^{-1}) \quad \Rightarrow \quad \bar{B}^u(q^{-1}) = q^{-m_u} B^u(q)$$

$$\Rightarrow \quad |\bar{B}^u(e^{-j\omega})| = |e^{-j\omega m_u} B^u(e^{j\omega})| = |B^u(e^{j\omega})| = |B^u(e^{-j\omega})| \qquad \blacksquare$$

A-causal but BIBO All-Pass Filter



The power spectral density of $w(k)$ is

$$\Phi_{WW}(\omega) = \left| \frac{\bar{B}^u(e^{-j\omega})}{B^u(e^{-j\omega})} \right|^2 \Phi_{YY}(\omega) = \Phi_{YY}(\omega)$$

A-causal but BIBO All-Pass Filter

$$y(k) \longrightarrow \boxed{\frac{\bar{B}^u(q^{-1})}{B^u(q^{-1})}} \longrightarrow w(k)$$

The power spectral density of $w(k)$ is

$$\Phi_{WW}(\omega) = \left| \frac{\bar{B}^u(e^{-j\omega})}{B^u(e^{-j\omega})} \right|^2 \Phi_{YY}(\omega) = \Phi_{YY}(\omega)$$

Therefore

$$\Lambda_{WW}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_{WW}(\omega) d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_{YY}(\omega) d\omega = \Lambda_{YY}(0)$$

A-causal but BIBO All-Pass Filter

$$y(k) \longrightarrow \boxed{\frac{\bar{B}^u(q^{-1})}{B^u(q^{-1})}} \longrightarrow w(k)$$

The power spectral density of $w(k)$ is

$$\Phi_{WW}(\omega) = \left| \frac{\bar{B}^u(e^{-j\omega})}{B^u(e^{-j\omega})} \right|^2 \Phi_{YY}(\omega) = \Phi_{YY}(\omega)$$

Therefore

$$\Lambda_{WW}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_{WW}(\omega) d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_{YY}(\omega) d\omega = \Lambda_{YY}(0)$$

$$\boxed{E\{w^2(k)\} = E\{y^2(k)\}}$$

Outline

Proof Methodology

The proof will be done in 4 parts:

1. Rewrite the system dynamics in a more convenient form

Proof Methodology

The proof will be done in 4 parts:

1. Rewrite the system dynamics in a more convenient form
2. Prove that $E\{z(k-d)\epsilon_f(k)\} = 0$, where $z(k)$ is a sequence to be defined in subsequent slides

Proof Methodology

The proof will be done in 4 parts:

1. Rewrite the system dynamics in a more convenient form
2. Prove that $E\{z(k-d)\epsilon_f(k)\} = 0$, where $z(k)$ is a sequence to be defined in subsequent slides
3. Prove optimality of proposed control scheme

Proof Methodology

The proof will be done in 4 parts:

1. Rewrite the system dynamics in a more convenient form
2. Prove that $E\{z(k-d)\epsilon_f(k)\} = 0$, where $z(k)$ is a sequence to be defined in subsequent slides
3. Prove optimality of proposed control scheme
4. Verify closed-loop stability

Proof Methodology

The proof will be done in 4 parts:

1. Rewrite the system dynamics in a more convenient form
2. Prove that $E\{z(k-d)\epsilon_f(k)\} = 0$, where $z(k)$ is a sequence to be defined in subsequent slides
3. Prove optimality of proposed control scheme
4. Verify closed-loop stability

Comments on the notation in this proof:

- Capital letters always denote polynomials; lower case letters denote sequences (except d and q)
- Dependency of polynomials on q^{-1} will be omitted e.g. $\bar{B}^u$ will refer to $\bar{B}^u(q^{-1})$
- Dependency of sequences on k will be omitted e.g. y will refer to $y(k)$

Part 1: Rewrite Dynamics

The plant dynamics are

$$Ay = q^{-\mathrm{d}}Bu + C\epsilon$$

and the Diophantine equation gives

$$[RA]y = [C\bar{B}^u - q^{-\mathrm{d}}B^uS]y$$

Part 1: Rewrite Dynamics

The plant dynamics are

$$Ay = q^{-\mathrm{d}}Bu + C\epsilon$$

and the Diophantine equation gives

$$[RA]y = [C\bar{B}^u - q^{-\mathrm{d}}B^u S]y$$

Combining these two equations gives

$$R[q^{-\mathrm{d}}Bu + C\epsilon] = [C\bar{B}^u - q^{-\mathrm{d}}B^u S]y$$

Part 1: Rewrite Dynamics

The plant dynamics are

$$Ay = q^{-\mathrm{d}} Bu + C\epsilon$$

and the Diophantine equation gives

$$[RA]y = [C\bar{B}^u - q^{-\mathrm{d}} B^u S]y$$

Combining these two equations gives

$$R[q^{-\mathrm{d}} Bu + C\epsilon] = [C\bar{B}^u - q^{-\mathrm{d}} B^u S]y$$

$$\Rightarrow \quad C\bar{B}^u y - q^{-\mathrm{d}}(B^u S y + BRu) - CR\epsilon = 0$$

Part 1: Rewrite Dynamics

The plant dynamics are

$$Ay = q^{-\mathrm{d}}Bu + C\epsilon$$

and the Diophantine equation gives

$$[RA]y = [C\bar{B}^u - q^{-\mathrm{d}}B^uS]y$$

Combining these two equations gives

$$R[q^{-\mathrm{d}}Bu + C\epsilon] = [C\bar{B}^u - q^{-\mathrm{d}}B^uS]y$$

$$\Rightarrow \quad C\bar{B}^u y - q^{-\mathrm{d}}(B^uSy + BRu) - CR\epsilon = 0$$

Factoring B^u out of the term in parentheses yields

$$\Rightarrow \quad C\bar{B}^u y - q^{-\mathrm{d}}B^u(Sy + B^sRu) - CR\epsilon = 0$$

Part 1: Rewrite Dynamics

From the previous slide:

$$C\bar{B}^u y - q^{-\mathrm{d}} B^u (Sy + B^s R u) - CR\epsilon = 0$$

- Define $z(k)$ in terms of $y(k)$ and $u(k)$ using

$$Cz = Sy + B^s R u$$

(note that we are not necessarily using the optimal control)

Part 1: Rewrite Dynamics

From the previous slide:

$$C\bar{B}^u y - q^{-\mathrm{d}} B^u (Sy + B^s R u) - CR\epsilon = 0$$

- Define $z(k)$ in terms of $y(k)$ and $u(k)$ using

  $$Cz = Sy + B^s R u$$

  (note that we are not necessarily using the optimal control)
- Define $\bar{\epsilon}_f$ and w by

  $$B^u \bar{\epsilon}_f = R\epsilon \qquad\qquad B^u w = \bar{B}^u y$$

  We interpret these relationships as a-causal, but BIBO

Part 1: Rewrite Dynamics

From the previous slide:

$$C\bar{B}^u y - q^{-\mathrm{d}} B^u (Sy + B^s Ru) - CR\epsilon = 0$$

- Define $z(k)$ in terms of $y(k)$ and $u(k)$ using

$$Cz = Sy + B^s Ru$$

  (note that we are not necessarily using the optimal control)

- Define $\bar{\epsilon}_f$ and w by

$$B^u \bar{\epsilon}_f = R\epsilon \qquad\qquad B^u w = \bar{B}^u y$$

  We interpret these relationships as a-causal, but BIBO

- From the top equation,

$$CB^u w - q^{-\mathrm{d}} CB^u z - CB^u \bar{\epsilon}_f = 0$$

$$\Rightarrow CB^u (w - q^{-\mathrm{d}} z - \bar{\epsilon}_f) = 0$$

Part 1: Rewrite Dynamics

So far, we know that

$$CB^u(w - q^{-\mathrm{d}} z - \bar{\epsilon}_f) = 0$$

- Since C is anti-Schur, we have $B^u(w - q^{-\mathrm{d}} z - \bar{\epsilon}_f) \longrightarrow 0$

Part 1: Rewrite Dynamics

So far, we know that

$$CB^u(w - q^{-\mathrm{d}}z - \bar{\epsilon}_f) = 0$$

- Since C is anti-Schur, we have $B^u(w - q^{-\mathrm{d}}z - \bar{\epsilon}_f) \longrightarrow 0$
- If we regard $\dfrac{1}{B^u(q^{-1})}$ as a-causal but BIBO, we have $w - q^{-\mathrm{d}}z - \bar{\epsilon}_f \longrightarrow 0$

Part 1: Rewrite Dynamics

So far, we know that

$$CB^u(w - q^{-\mathrm{d}}z - \bar{\epsilon}_f) = 0$$

- Since C is anti-Schur, we have $B^u(w - q^{-\mathrm{d}}z - \bar{\epsilon}_f) \longrightarrow 0$
- If we regard $\dfrac{1}{B^u(q^{-1})}$ as a-causal but BIBO, we have $w - q^{-\mathrm{d}}z - \bar{\epsilon}_f \longrightarrow 0$
- We have now obtained

$$\boxed{w(k) = z(k - \mathrm{d}) + \bar{\epsilon}_f(k)}$$

Part 1: Rewrite Dynamics

So far, we know that

$$CB^u(w - q^{-\mathrm{d}}z - \bar{\epsilon}_f) = 0$$

- Since C is anti-Schur, we have $B^u(w - q^{-\mathrm{d}}z - \bar{\epsilon}_f) \longrightarrow 0$
- If we regard $\dfrac{1}{B^u(q^{-1})}$ as a-causal but BIBO, we have $w - q^{-\mathrm{d}}z - \bar{\epsilon}_f \longrightarrow 0$
- We have now obtained

$$\boxed{w(k) = z(k - \mathrm{d}) + \bar{\epsilon}_f(k)}$$

- Also note that, because $w(k) = \dfrac{\bar{B}^u(q^{-1})}{B^u(q^{-1})}y(k)$

$$\boxed{E\{w^2(k)\} = E\{y^2(k)\}}$$

Part 2: $E\{z(k-d)\bar{\epsilon}_f(k)\} = 0$

- Since $\epsilon(k) = y(k) - E\{y(k)|y(k-1), y(k-2), \ldots\}$, we use least squares property 1 to see that

$$E\{y(k-\ell)\epsilon(k+p)\}, \qquad \forall \ell > 0, p \geq 0$$

Part 2: $E\{z(k-d)\bar{\epsilon}_f(k)\} = 0$

- Since $\epsilon(k) = y(k) - E\{y(k)|y(k-1), y(k-2), \ldots\}$, we use least squares property 1 to see that

$$E\{y(k-\ell)\epsilon(k+p)\}, \qquad \forall \ell > 0, p \geq 0$$

- Defining $\epsilon_r = R\epsilon$, we have

$$\begin{aligned} \epsilon_r(k+n_r) &= \epsilon(k+n_r) + r_1\epsilon(k+n_r-1) + \cdots + r_{n_r}\epsilon(k) \\ \Rightarrow \quad & E\{y(k-\ell)\epsilon_r(k+n_r+p)\} = 0 \qquad \forall \ell > 0, p \geq 0 \end{aligned}$$

Part 2: $E\{z(k-d)\bar{\epsilon}_f(k)\} = 0$

- Since $\epsilon(k) = y(k) - E\{y(k)|y(k-1), y(k-2), \ldots\}$, we use least squares property 1 to see that

$$E\{y(k-\ell)\epsilon(k+p)\}, \qquad \forall \ell > 0, p \geq 0$$

- Defining $\epsilon_r = R\epsilon$, we have

$$\epsilon_r(k+n_r) = \epsilon(k+n_r) + r_1\epsilon(k+n_r-1) + \cdots + r_{n_r}\epsilon(k)$$
$$\Rightarrow \quad E\{y(k-\ell)\epsilon_r(k+n_r+p)\} = 0 \qquad \forall \ell > 0, p \geq 0$$

- Regarding the relationship $B^u\bar{\epsilon}_f = \epsilon_r$ as a-causal but BIBO, and noting that $n_r = m_u + d - 1$, we see that $\bar{\epsilon}_f(k + \mathrm{d} - 1)$ is a function of $\epsilon_r(k+n_r)$, $\epsilon_r(k+n_r+1), \cdots$, which implies

Part 2: $E\{z(k-d)\bar{\epsilon}_f(k)\} = 0$

- Since $\epsilon(k) = y(k) - E\{y(k)|y(k-1), y(k-2), \ldots\}$, we use least squares property 1 to see that

$$E\{y(k-\ell)\epsilon(k+p)\}, \qquad \forall \ell > 0, p \geq 0$$

- Defining $\epsilon_r = R\epsilon$, we have

$$\epsilon_r(k+n_r) = \epsilon(k+n_r) + r_1\epsilon(k+n_r-1) + \cdots + r_{n_r}\epsilon(k)$$
$$\Rightarrow \quad E\{y(k-\ell)\epsilon_r(k+n_r+p)\} = 0 \qquad \forall \ell > 0, p \geq 0$$

- Regarding the relationship $B^u\bar{\epsilon}_f = \epsilon_r$ as a-causal but BIBO, and noting that $n_r = m_u + d - 1$, we see that $\bar{\epsilon}_f(k+\mathrm{d}-1)$ is a function of $\epsilon_r(k+n_r)$, $\epsilon_r(k+n_r+1), \cdots$, which implies

$$E\{y(k-\ell)\bar{\epsilon}_f(k+\mathrm{d}-1)\} = 0 \qquad \forall \ell > 0$$

Part 2: $E\{z(k-d)\bar{\epsilon}_f(k)\} = 0$

- Since $u(k)$ is a function of $y(k), y(k-1), \ldots$

$$E\{u(k-\ell)\bar{\epsilon}_f(k+\mathrm{d}-1)\} = 0 \qquad \forall \ell > 0$$

Part 2: $E\{z(k-d)\bar{\epsilon}_f(k)\} = 0$

- Since $u(k)$ is a function of $y(k), y(k-1), \dots$

$$E\{u(k-\ell)\bar{\epsilon}_f(k+\mathrm{d}-1)\} = 0 \qquad \forall \ell > 0$$

- Since $z(k)$ is a function of $y(k), y(k-1), \dots$ and $u(k), u(k-1), \dots$

$$E\{z(k-\ell)\bar{\epsilon}_f(k+\mathrm{d}-1)\} = 0 \qquad \forall \ell > 0$$

Part 2: $E\{z(k-d)\bar{\epsilon}_f(k)\} = 0$

- Since $u(k)$ is a function of $y(k), y(k-1), \ldots$

$$E\{u(k-\ell)\bar{\epsilon}_f(k+\mathrm{d}-1)\} = 0 \qquad \forall \ell > 0$$

- Since $z(k)$ is a function of $y(k), y(k-1), \ldots$ and $u(k), u(k-1), \ldots$

$$E\{z(k-\ell)\bar{\epsilon}_f(k+\mathrm{d}-1)\} = 0 \qquad \forall \ell > 0$$

- Choosing $\ell = 1$ yields

$$E\{z(k-d)\bar{\epsilon}_f(k)\} = 0$$

Part 3: Optimal Control

So far, we know

$$w(k) = z(k - \mathrm{d}) + \bar{\epsilon}_f(k)$$

$$E\{z(k-d)\bar{\epsilon}_f(k)\} = 0$$

$$E\{y^2(k)\} = E\{w^2(k)\}$$

Part 3: Optimal Control

So far, we know

$$w(k) = z(k - \mathrm{d}) + \bar{\epsilon}_f(k)$$

$$E\{z(k-d)\bar{\epsilon}_f(k)\} = 0$$

$$E\{y^2(k)\} = E\{w^2(k)\}$$

$$\Rightarrow E\{y^2(k)\} = E\{z^2(k-d)\} + E\{\bar{\epsilon}_f^2(k)\}$$

Part 3: Optimal Control

So far, we know

$$w(k) = z(k - \mathrm{d}) + \bar{\epsilon}_f(k)$$

$$E\{z(k-d)\bar{\epsilon}_f(k)\} = 0$$

$$E\{y^2(k)\} = E\{w^2(k)\}$$

$$\Rightarrow E\{y^2(k)\} = E\{z^2(k-d)\} + E\{\bar{\epsilon}_f^2(k)\}$$

Notes:

- At this point, we have only assumed that $u(k)$ stabilizes the system (so that the relevant covariances are bounded)

Part 3: Optimal Control

So far, we know

$$w(k) = z(k - \mathrm{d}) + \bar{\epsilon}_f(k)$$

$$E\{z(k-d)\bar{\epsilon}_f(k)\} = 0$$

$$E\{y^2(k)\} = E\{w^2(k)\}$$

$$\Rightarrow E\{y^2(k)\} = E\{z^2(k-d)\} + E\{\bar{\epsilon}_f^2(k)\}$$

Notes:

- At this point, we have only assumed that $u(k)$ stabilizes the system (so that the relevant covariances are bounded)
- The value of $E\{\bar{\epsilon}_f^2(k)\}$ does not depend on the control law

Part 3: Optimal Control

So far, we know

$$w(k) = z(k - \mathrm{d}) + \bar{\epsilon}_f(k)$$

$$E\{z(k-d)\bar{\epsilon}_f(k)\} = 0$$

$$E\{y^2(k)\} = E\{w^2(k)\}$$

$$\Rightarrow E\{y^2(k)\} = E\{z^2(k-d)\} + E\{\bar{\epsilon}_f^2(k)\}$$

Notes:

- At this point, we have only assumed that $u(k)$ stabilizes the system (so that the relevant covariances are bounded)
- The value of $E\{\bar{\epsilon}_f^2(k)\}$ does not depend on the control law
- $\Rightarrow$ We would like to choose u to minimize $E\{z^2\}$

Part 3: Optimal Control

So far, we know

$$w(k) = z(k-\mathrm{d}) + \bar{\epsilon}_f(k)$$

$$E\{z(k-d)\bar{\epsilon}_f(k)\} = 0$$

$$E\{y^2(k)\} = E\{w^2(k)\}$$

$$\Rightarrow E\{y^2(k)\} = E\{z^2(k-d)\} + E\{\bar{\epsilon}_f^2(k)\}$$

Notes:

- At this point, we have only assumed that $u(k)$ stabilizes the system (so that the relevant covariances are bounded)
- The value of $E\{\bar{\epsilon}_f^2(k)\}$ does not depend on the control law
- $\Rightarrow$ We would like to choose u to minimize $E\{z^2\}$
- If we can make $E\{z^2\} = 0$, the control must be optimal

Part 3: Optimal Control

Goal: choose u so that $E\{z^2\} = 0$

Part 3: Optimal Control

Goal: choose u so that $E\{z^2\} = 0$

- If we apply the control signal $u_*(k)$ defined by

$$B^s R u_* = -Sy$$

Part 3: Optimal Control

Goal: choose u so that $E\{z^2\} = 0$

- If we apply the control signal $u_*(k)$ defined by

$$B^s R u_* = -Sy$$

  we have

$$Cz = B^s R u_* + Sy = 0$$

Part 3: Optimal Control

Goal: choose u so that $E\{z^2\} = 0$

- If we apply the control signal $u_*(k)$ defined by

$$B^s R u_* = -Sy$$

  we have

$$Cz = B^s R u_* + Sy = 0$$

- $C(q^{-1})$ is an anti-Schur polynomial $\Rightarrow$ $z(k) \longrightarrow 0$.

Part 3: Optimal Control

Goal: choose u so that $E\{z^2\} = 0$

- If we apply the control signal $u_*(k)$ defined by

$$B^s R u_* = -Sy$$

  we have

$$Cz = B^s R u_* + Sy = 0$$

- $C(q^{-1})$ is an anti-Schur polynomial $\Rightarrow$ $z(k) \longrightarrow 0$.
- For this control signal, $E\{z^2\} = 0$, which means that $u_*(k)$ is optimal, provided that the closed-loop system is stable.

Part 3: Optimal Control

Goal: choose u so that $E\{z^2\} = 0$

- If we apply the control signal $u_*(k)$ defined by

  $$B^s R u_* = -Sy$$

  we have

  $$Cz = B^s R u_* + Sy = 0$$

- $C(q^{-1})$ is an anti-Schur polynomial $\Rightarrow$ $z(k) \longrightarrow 0$.
- For this control signal, $E\{z^2\} = 0$, which means that $u_*(k)$ is optimal, provided that the closed-loop system is stable.
- Also note that $E\{y^2\} = E\{\bar{\epsilon}_f^2\}$, provided that the closed-loop system is stable

Part 3: Optimal Control

- Provided that the closed-loop system is stable, we have $E\{y^2\} = E\{\bar{\epsilon}_f^2\}$ where $\bar{\epsilon}_f$ is generated by the BIBO a-causal ARMA model $B^u \bar{\epsilon}_f = R\epsilon$

Part 3: Optimal Control

- Provided that the closed-loop system is stable, we have $E\{y^2\} = E\{\bar{\epsilon}_f^2\}$ where $\bar{\epsilon}_f$ is generated by the BIBO a-causal ARMA model $B^u \bar{\epsilon}_f = R\epsilon$
- We can instead express the optimal cost in terms of a BIBO causal ARMA model as

$$\boxed{E\{y^2\} = E\{\epsilon_f^2\} \text{ where } \epsilon_f \text{ is the output of } \bar{B}^u \epsilon_f = R\epsilon}$$

Part 3: Optimal Control

- Provided that the closed-loop system is stable, we have $E\{y^2\} = E\{\bar{\epsilon}_f^2\}$ where $\bar{\epsilon}_f$ is generated by the BIBO a-causal ARMA model $B^u \bar{\epsilon}_f = R\epsilon$
- We can instead express the optimal cost in terms of a BIBO causal ARMA model as

$$\boxed{E\{y^2\} = E\{\epsilon_f^2\} \text{ where } \epsilon_f \text{ is the output of } \bar{B}^u \epsilon_f = R\epsilon}$$

  (Remember that $\bar{B}^u$ refers to $\bar{B}^u(q^{-1})$)
- To see this, note that since ϵ_f is the output of the a-causal but BIBO ARMA model $B^u \bar{\epsilon}_f = \bar{B}^u \epsilon_f$ and the operator $\left(\dfrac{\bar{B}^u}{B^u}\right)$ is an a-causal all-pass filter, we have that $E\{\epsilon_f^2\} = E\{\bar{\epsilon}_f^2\}$

Part 4: Closed-loop stability

From the plant dynamics and feedback law, we have

$$\begin{bmatrix} A & -q^{-\mathrm{d}}B \\ S & B^s R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C\epsilon \\ 0 \end{bmatrix}$$

Part 4: Closed-loop stability

From the plant dynamics and feedback law, we have

$$\begin{bmatrix} A & -q^{-\mathrm{d}}B \\ S & B^s R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C\epsilon \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y \\ u \end{bmatrix} = \frac{1}{B^s AR + q^{-\mathrm{d}}BS} \begin{bmatrix} B^s R & q^{-\mathrm{d}}B \\ -S & A \end{bmatrix} \begin{bmatrix} C\epsilon \\ 0 \end{bmatrix}$$

$$= \frac{1}{B^s(AR + q^{-\mathrm{d}}B^u S)} \begin{bmatrix} B^s RC\epsilon \\ -SC\epsilon \end{bmatrix} = \frac{1}{C\bar{B}^u B^s} \begin{bmatrix} B^s RC\epsilon \\ -SC\epsilon \end{bmatrix}$$

Part 4: Closed-loop stability

From the plant dynamics and feedback law, we have

$$\begin{bmatrix} A & -q^{-\mathrm{d}}B \\ S & B^s R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C\epsilon \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y \\ u \end{bmatrix} = \frac{1}{B^s AR + q^{-\mathrm{d}}BS} \begin{bmatrix} B^s R & q^{-\mathrm{d}}B \\ -S & A \end{bmatrix} \begin{bmatrix} C\epsilon \\ 0 \end{bmatrix}$$

$$= \frac{1}{B^s(AR + q^{-\mathrm{d}}B^u S)} \begin{bmatrix} B^s RC\epsilon \\ -SC\epsilon \end{bmatrix} = \frac{1}{C\bar{B}^u B^s} \begin{bmatrix} B^s RC\epsilon \\ -SC\epsilon \end{bmatrix}$$

Since $C(q^{-1})\bar{B}^u(q^{-1})B^s(q^{-1})$ is an anti-Schur polynomial of q^{-1}, the closed-loop system is stable ■