UNIVERSITY OF CALIFORNIA AT BERKELEY
Department of Mechanical Engineering
ME233 Advanced Control Systems II

Infinite Horizon Linear Quadratic Regulator Properties

In this handout we will present fairly complete proofs of two theorems concerning the existence, uniqueness and stability of infinite horizon linear quadratic regulators for discrete time systems. The proofs presented here are a combination of results from similar proofs contained in Chapter 2.4 and of (Lewis, 1986) and Chapter 2.1 of (Anderson and Moore, 1977). The sampling index convention and most symbols in this handout follow those in the ME233 class notes.

1 Finite Horizon Linear Quadratic Regulator

Consider the Linear Time Invariant (LTI) n-th order discrete time system

$$x(k+1) \quad = \quad A\,x(k) + B\,u(k) \tag{1}$$

where $x \in \mathcal{R}^n$, and $u(k) \in \mathcal{R}^m$.

The optimal control that minimizes the finite horizon cost functional

$$J[x(0)] \quad = \quad \frac{1}{2}\,x^T(N)\,S\,x(N) + \frac{1}{2}\sum_{k=0}^{N-1}\left\{x^T(k)\,C^T\,C\;x(k) + u^T(k)\,R\,u(k)\right\}\,, \tag{2}$$

where $S = S^T \succeq 0$, $R = R^T \succ 0$, and $C^T C = Q \succeq 0$, is given by

$$u^o(k) = -K(k+1)\,x(k) \tag{3}$$

and the time varying gain matrix $K(k)$ is computed recursively (backwards) by the following Joseph stabilized Riccati equation

$$\begin{aligned} K(k) &= \left[R + B^T P(k) B\right]^{-1} B^T P(k) A \\ P(k-1) &= C^T\,C + K^T(k)\,R\,K(k) + (A - BK(k))^T P(k)(A - BK(k)) \end{aligned} \tag{4}$$

with boundary condition $P(N) = S$.

2 Sufficient conditions for the existence of a steady state solution to the infinite horizon Riccati equation (Terminal cost matrix $S = 0$)

Theorem *1*
Let the pair $[A, B]$ be stabilizable. Then, as $N \to \infty$, for $P(N) = S = 0$, The Riccati Eq. (4) converges to a bounded steady state solution P. Furthermore, $P \succeq 0$ is a solution of the algebraic Riccati equation

$$K \quad = \quad \left[R + B^T P B\right]^{-1} B^T P A \tag{5}$$

$$P \quad = \quad C^T\,C + K^T\,R\,K + (A - BK)^T P(A - BK)\,. \tag{6}$$

Proof:

Since $[A, B]$ is stabilizable, there exists a constant feedback gain matrix L such that, utilizing the control law

$$u(k) = -L\, x(k), \tag{7}$$

results in an asymptotically stable close loop system

$$x_{_L}(k+1) = (A - B\,L) x_{_L}(k) \qquad x_{_L}(0) = x_o \tag{8}$$

and $x_{_L}(k)$ converges to 0 geometrically. That is, the state of the asymptotically stable close loop system (27) satisfies

$$\sum_{k=0}^{\infty} \|x_{_L}(k)\|^2 < \infty \tag{9}$$

where $\|v\| = \sqrt{v^T\, v}$ for any vector $v \in \mathcal{R}^n$. The proof of this result is in the ME232 class notes and also in section 5.

The cost (2) associated with the (suboptimal) control law (7) at the instant m, state $x_{_L}(m) = x_{_{Lm}}$, final state cost weight S and finite horizon N is given by

$$J_{_L}[x_{_{Lm}}, m, S, N] \;=\; \frac{1}{2}\, x_{_L}^T(N)\, S\, x_{_L}(N) + \frac{1}{2} \sum_{k=m}^{N-1} x_{_L}^T(k)\, \left\{ C^T\, C\, + L^T\, R\, L \right\}\, x_{_L}(k)\,, \tag{10}$$

For any bounded initial condition $x_{_L}(0)$, $x_{_L}(k)$ converges to 0 geometrically (matrix $A_{_{Lc}} = [A - BL]$ is Schur). Therefore, as $N \to \infty$,

$$\lim_{N\to\infty} J_L[x_{Lm}, m, S, N] = J_L[x_{Lm}] < \infty$$

for all $0 \le m < \infty$.

This can be shown as follows. First notice that, since $\lim_{N\to\infty} x_{_L}(N) = 0$, as $N \to \infty$, the final cost term in Eq. (10) is zero. From the cost Eq. (10) and utilizing (28), we obtain

$$\begin{aligned}
\lim_{N\to\infty} J_{_L}[x_{_L}(0), m, S, N] &\le \lim_{N\to\infty} J_{_L}[x_{_L}(0), 0, S, N] = \lim_{N\to\infty} J_{_L}[x_{_L}(0), 0, 0, N] \\
\lim_{N\to\infty} J_{_L}[x_{_L}(0), 0, 0, N] &= \frac{1}{2} \sum_{k=0}^{\infty} x_{_L}^T(k)\, \left\{ C^T\, C\, + L^T\, R\, L \right\}\, x_{_L}(k) \\
&\le \frac{1}{2} \left[\sigma_{\max} \left(C^T\, C\, + L^T\, R\, L \right) \right] \sum_{k=0}^{\infty} \|x_{_L}(k)\|^2 < \infty\,.
\end{aligned}$$

Referring to the cost Eq. (2), since $Q \succeq 0$ and $R \succ 0$, for any $0 \le m < N$,

$$J_{_L}[x_{_L}, m, 0, N+1] \ge J_{_L}[x_{_L}, m, 0, N] \tag{11}$$

This result is shown in Section 4.

Thus, for any arbitrary x_{L} and $0 \leq m < N$, $J_{L}[x_{L}, m, 0, N]$ is a nondecreasing sequence of N. Moreover, since $\lim_{N\to\infty} J_{L}[x_{L}(0), m, 0, N] < \infty$, the the sequence has to converge, i.e.

$$\lim_{N\to\infty} J_{L}[x_{L}, m, 0, N] = J_{L}[x_{L}]$$

Consider now the cost (2) associated with the **optimal** control law (3) at the instant m, state $x(m) = x_m$, final state cost weight S and finite horizon N, which is given by

$$\begin{aligned} J^o[x_m, m, S, N] &= \frac{1}{2}\, x^T(N)\, S\, x(N) + \frac{1}{2} \sum_{k=m}^{N-1} x^T(k) \left\{ C^T\, C\, + K^T(k+1)\, R\, K(k+1) \right\} x(k)\,, \\ &= x_m^T\, P(m, S, N)\, x_m \end{aligned} \tag{12}$$

where $K(k)$ and $P(m, S, N)$ are computed recursively (backwards) by

$$\begin{aligned} K(k) &= \left[R + B^T P(k, S, N) B\right]^{-1} B^T P(k, S, N) A \\ P(k-1, S, N) &= C^T\, C + K^T(k)\, R\, K(k) + (A - BK(k))^T P(k, S, N))(A - BK(k)) \end{aligned} \tag{13}$$

with boundary condition $P(N, S, N) = S \succeq 0$.

Based on the *principle of optimality*, we can obtain the following three results:

1. For any arbitrary $x_m \in \mathcal{R}^n$, $0 \leq m < N$, $S \succeq 0$ and $N > 0$, the optimal cost has to be less than or equal to the cost incurred by any other control. Therefore,

$$0 \leq J^o[x_m, m, S, N] \leq J_{L}[x_m, m, S, N] < \infty$$

2. For any arbitrary x_m and $0 \leq m < N$, $N > 0$ and $S = 0$, the optimal cost increases as the time horizon increases:

$$J^o[x_m, m, 0, N+1] \geq J^o[x_m, m, 0, N] \tag{14}$$

This result is shown in Section 4.

3. From the above two results we can conclude that $J^o[x_m, m, 0, N]$ is a nondecreasing sequence of N, which is bounded above by the sequence $J_{L}[x_m, m, 0, N+1]$, which converges. Therefore, the sequence$J^o[x_m, m, 0, N]$ has to converge

$$\lim_{N\to\infty} J^o[x_m, m, 0, N] = J^o[x_m, S = 0] < \infty\,. \tag{15}$$

Using the above three results, we now show that the matrix $P(m, 0, N)$ in Eq. (12), which is computed backwards by the Riccati equation (13) must also converge as $N \to \infty$. To do so, we re-write Eq. (14) in terms of $P(m, 0, N)$ and $P(m, 0, N+1)$.

$$\begin{aligned} \frac{1}{2}\, x_m^T\, P(m, 0, N+1)\, x_m - \frac{1}{2}\, x_m^T\, P(m, 0, N)\, x_m &= \frac{1}{2}\, x_m^T\, \left[P(m, 0, N+1) - \, P(m, 0, N)\right]\, x_m \\ &= \frac{1}{2}\, x_m^T\, \Delta P(m, 0, N)\, x_m \geq 0 \end{aligned}$$

Since x_m is arbitrary, this in turn implies that

$$P(m,0,N+1) = P(m,0,N) + \Delta P(m,0,N)\,, \qquad \Delta P(m,0,N) \succeq 0 \tag{16}$$

Now, by (15), for any arbitrary $x_m \in \mathcal{R}^n$,

$$\frac{1}{2} x_m^T \left\{ \lim_{N\to\infty} P(m,0,N) \right\} x_m = J^o[x_m, S=0] < \infty \tag{17}$$

Eqs. (16) and (17) together with the fact that $x_m \in \mathcal{R}^n$ is arbitrary imply that

$$\lim_{N\to\infty} P(m,0,N) = \bar{P} \tag{18}$$

This can be proven as follows. Since $x_m \in \mathcal{R}^n$ is arbitrary, choose $x_m = e_i$, where e_i is a vector with zeros for all entries except for the ith element, which is one. Therefore, for any diagonal element of $P(m,0,N) \succeq 0$ we obtain

$$P_{ii}(m,0,N+1) = P_{ii}(m,0,N) + \Delta P_{ii}(m,0,N)\,, \qquad \Delta P_{ii}(m,0,N) \geq 0$$

$$\lim_{N\to\infty} P_{ii}(m,0,N) = \bar{P}_{ii} < \infty$$

Therefore, all diagonal elements of $P(m,0,N) \succeq 0$ are nondecreasing sequences of N, which are bounded and must posses a limit as $N \to \infty$. For the off-diagonal element $P_{ij}(m,0,N)$, the existence of limit $\bar{P}_{ij}$ can be proven by observing that

$$2P_{ij}(m,0,N) = (e_i + e_j)^T P(m,0,N)\,(e_i + e_j) - P_{ii}(m,0,N) - P_{jj}(m,0,N) \tag{19}$$

The existence of the limit $\bar{P}_{ij}$ in the left hand side of (19) is guaranteed by the fact that term in the right hand side (19) of has a limit as $N \to \infty$.

3 Sufficient conditions for the existence of a unique positive definite steady state solution to the infinite horizon Riccati equation and a stabilizing optimal control law

The next theorem provides necessary and sufficient conditions for the existence of a unique and positive definite steady state solution to the Riccati equation. Before discussing this result, we will first provide two lemmas that will be useful in our discussion.

Lemma *1*

The following three statements are equivalent:

1. The pair $[A_c, \bar{C}]$ is observable and A_c is Schur (i.e. all of its eigenvalues are inside the unit circle).
2. The infinite horizon observability grammian

   $$P = \sum_{k=0}^{\infty} A_c^{T^k}\, \bar{C}^T\, \bar{C}\, A_c^k \succ 0$$

   is bounded and positive definite.
3. $P \succ 0$ is the solution of the following Lyapunov equation

   $$A_c^T P A_c - P = -\bar{C}^T \bar{C}$$

The proof of the lemma can be found in the ME232 class notes.

Lemma *2*

Observability is invariant under static gain output feedback injection.

Proof: L et the pair $[A_c, \bar{C}]$ be observable. Therefore, any initial state $x(0)$ of the n-th order LTI system

$$\begin{aligned} x(k+1) &= A_c x(k) \\ \bar{y}(k) &= \bar{C} x(k) \end{aligned}$$

can be reconstructed from

$$\bar{Y}_{n-1} = \begin{bmatrix} \bar{y}(0) \\ \vdots \\ \bar{y}(n-1) \end{bmatrix} = \bar{O}\, x(0) = \begin{bmatrix} \bar{C} \\ \bar{C} A_c \\ \vdots \\ \bar{C} A_c^{n-1} \end{bmatrix} x(0)$$

since the observability matrix $\bar{O}$ must be rank n.

Consider now the system under a static output injection:

$$\begin{aligned} x(k+1) &= A_c x(k) + L\bar{y}(k) \\ \bar{y}(k) &= \bar{C} x(k)\,, \end{aligned}$$

for any constant matrix L of appropriate dimensions.
It can be shown using induction that

$$\bar{y}(k) - \bar{C} \sum_{j=0}^{k-1} A_c^{k-1-j}\, L\, \bar{y}(j) = \bar{C}\, A_c^k\, x(0)$$

Therefore, any initial state $x(0)$ can be reconstructed from $\bar{Y}_n$ as follows

$$\bar{Z}_n = \begin{bmatrix} \bar{y}(0) \\ \bar{y}(1) - L\bar{y}(0) \\ \vdots \\ \bar{y}(n-1) - \sum_{j=0}^{n-2} A_c^{n-2-j}\, L\, \bar{y}(j) \end{bmatrix} = \begin{bmatrix} \bar{C} \\ \bar{C} A_c \\ \vdots \\ \bar{C} A_c^{n-1} \end{bmatrix} x(0)$$

Thus, $[A_c, \bar{C}]$ is observable if and only if $[A_c - L\bar{C}, \bar{C}]$ is observable for any matrix L of appropriate dimensions. QED.

Theorem *2*

Let the pair $[A, C]$ be observable. Then, the following three statements are equivalent:

1. $[A, B]$ is stabilizable.
2. For any $P(N) = S$, as $N \to \infty$, there exists a unique positive definite bounded steady state solution P of (4), which is also the unique solution of the algebraic Riccati equation

$$K = \left[R + B^T P B\right]^{-1} B^T P A \tag{20}$$

$$P = C^T C + K^T R K + (A - BK)^T P (A - BK)\,. \tag{21}$$

3. The close loop matrix

$$A_c = A - B\,K \tag{22}$$

is Schur, that is all of its eigenvalues are inside the unit circle, where K is given by Eq. (20).

Proof: **1) $\Rightarrow$ 3) and 2):** Let the pair $[A, C]$ be observable and the pair $[A, B]$ be stabilizable, and consider the case when the terminal cost gain $S = 0$ and define the optimal infinite horizon cost

$$\begin{aligned} J^o[x(0)] &= \frac{1}{2} \min_{U_{(0,\infty)}} \sum_{k=0}^{\infty} \left\{ x^T(k)\, C^T\, C\; x(k) + u^T(k)\, R\, u(k) \right\}\,, \\ &= \frac{1}{2} x^T(0)\, P\, x(0) \end{aligned} \tag{23}$$

where

$$U_{(0,\infty)} = \{u(0),\, u(1),\, \cdots,\, \}$$

is the set of all available control sequences $u(k)$'s for $0 \le k < \infty$.

By Theorem 1, $J^o[x(0)]$ is bounded and the Riccati equation (4) when $P(N) = S = 0$ converges to a bounded steady solution $P \succeq 0$ that satisfies the algebraic Riccati equation (5), which is repeated here for convenience.

$$\begin{aligned} K &= \left[R + B^T P B \right]^{-1} B^T P A \\ P &= C^T\, C + K^T\, R\, K + (A - BK)^T P (A - BK)\,. \end{aligned}$$

Defining the matrices

$$A_c = A - BK \qquad C^T C = Q \succeq 0 \qquad D^T D = R \succ 0$$

and

$$\bar{C} = \begin{bmatrix} C \\ DK \end{bmatrix},$$

the optimal closed loop system and cost can be re-written as follows.

$$\begin{aligned} x(k+1) &= A_c\, x(k) \\ \bar{y}(k) &= \bar{C}\, x(k) \end{aligned}$$

$$J^o[x(0)] = \frac{1}{2} \sum_{k=0}^{\infty} \bar{y}^T(k)\, \bar{y}(k) = x^T(0) P x(0)$$

Notice that the pair $[A, C]$ is observable implies that the pair $[A, \bar{C}]$ is also observable. Moreover, notice that because $D^T D = R \succ 0$, which in turn implies that D is invertible,

$$A_c = A - B\,K = A - \begin{bmatrix} 0 & BD^{-1} \end{bmatrix} \bar{C}\,.$$

Therefore, by Lemma 2 (observability invariance under output injection), the pair $[A,\bar{C}]$ is observable if and only if the pair $[A-BK,\bar{C}]=[A_c,\bar{C}]$ is observable.

Notice now that, for any arbitrary bounded initial condition $x(0)$, the optimal cost can be further expressed as follows

$$J^o[x(0)] \;=\; x^T(0)\left[\sum_{k=0}^{\infty} A_c^{T^k}\,\bar{C}^T\,\bar{C}\,A_c^k\right]x(0)=x^T(0)Px(0)<\infty$$

Therefore, the bounded matrix P satisfies

$$P=\sum_{k=0}^{\infty} A_c^{T^k}\,\bar{C}^T\,\bar{C}\,A_c^k \tag{24}$$

(i.e. it must be at least positive semi-definite and bounded), and it also satisfies the algebraic Riccati equation (5), which can be re-written as

$$A_c^T\,PA_c-P=-\bar{C}^T\bar{C}\,. \tag{25}$$

Since the pair $[A_c,\bar{C}]$ is observable, P as given in Eq. (24) must be positive definite. Therefore, since P is also bounded, Eq. (24) and (25) imply that A_c must be Schur (i.e. all its eigenvalues are inside the unit circle).

Now, since A_c is Schur, the close loop system is asymptotically stable. Therefore,

$$\lim_{N\to\infty} x^T(N)\,Sx(N)=0$$

for any bounded matrix S. This implies that the optimal control that minimizes the cost (23) also minimizes the cost

$$\begin{aligned} J^o[x(0)] \;&=\; \frac{1}{2}\min_{U_{(m,\infty)}}\left\{x^T(N)Sx(N)+\frac{1}{2}\sum_{k=0}^{\infty}\left\{x^T(k)\,C^T\,C\;x(k)+u^T(k)\,R\,u(k)\right\}\right\} \\ &=\; \frac{1}{2}x^T(0)\,P\,x(0) \end{aligned}$$

Therefore, the solution $P\succ 0$ of the algebraic Riccati Eq. (5) must be unique and is the steady state solution of the Riccati Eq. (4) for any $P(N)=S=S^T\succeq 0$ when $N\to\infty$. Q.E.D.

Proof: 3) $\Rightarrow$ 1) and 2):
Let the matrix matrix $A_c=A-BK$ be Schur, where K satisfies the algebraic Riccati equation (5). Then the pair $[A,B]$ must be stabilizable. Since A_c is Schur and the pair $[A_c,\bar{C}]$ is observable, then the bounded and positive definite matrix $P\succ 0$ must be the unique solution of the Lyapunov equation (25) and is the steady state solution of the Riccati Eq. (4) for any $P(N)=S=S^T\succeq 0$ when $N\to\infty$. Q.E.D.

Proof: 2) $\Rightarrow$ 3) and 1):
$[A,C]$ observable implies that $[A_c,\bar{C}]$ is also observable. Therefore, the fact that the bounded $P\succ 0$ is the solution of the algebraic Riccati equation (20) implies that (25) is a Lyapunov equation and A_c must be Schur. This in turn implies that the pair $[A,B]$ must be stabilizable.

4 Proof of Eq. (14)

Consider the optimal cost

$$J^o[x_m, m, 0, N] \;=\; \min_{U_{(m,N-1)}} \left\{ \frac{1}{2} \sum_{k=m}^{N-1} \Big(x^T(k)\, C^T C\; x(k) + u^T(k) R u(k) \Big) \right\} \tag{26}$$

where

$$U_{(m,N)} = \{u(m),\, u(m+1),\, \cdots,\, u(N)\}$$

is the set of all available control sequences $u(k)$'s for $m \le k \le N$.

By the principle of optimality and noticing that $S = 0$ and $C^T C \succeq 0$ and $R \succ 0$ are constant,

$$\begin{aligned} 2\, J^o[x_m, m, 0, N+1] &= \min_{U_{(m,N)}} \sum_{k=m}^{N} \Big(x^T(k)\, C^T C\; x(k) + u^T(k) R u(k) \Big) \\ &= \min_{u(N)} \left\{ \min_{U_{(m,N-1)}} \left[x^T(N)\, C^T C\, x(N) + u^T(N)\, R\, u(N) + \sum_{k=m}^{N-1} \Big(x^T(k)\, C^T C\; x(k) + u^T(k) R u(k) \Big) \right] \right\} \end{aligned}$$

Notice that, for any x_m and any control sequence in the set $U_{(m,N-1)}$,

$$\begin{aligned} x^T(N)\, C^T C\, x(N) + u^T(N)\, R\, u(N) \;&+\; \sum_{k=m}^{N-1} \Big(x^T(k)\, C^T C\; x(k) + u^T(k) R u(k) \Big) \\ &\ge\; \sum_{k=m}^{N-1} \Big(x^T(k)\, C^T C\; x(k) + u^T(k) R u(k) \Big)\;. \end{aligned}$$

Therefore,

$$\begin{aligned} \min_{U_{(m,N-1)}} [\, x^T(N)\, C^T C\, x(N) \;&+\; u^T(N)\, R\, u(N) + \sum_{k=m}^{N-1} \Big(x^T(k)\, C^T C\; x(k) + u^T(k) R u(k) \Big)] \\ &\ge\; \min_{U_{(m,N-1)}} \sum_{k=m}^{N-1} \Big(x^T(k)\, C^T C\; x(k) + u^T(k) R u(k) \Big) = 2\, J^o[x_m, m, 0, N] \end{aligned}$$

and

$$J^o[x_m, m, 0, N+1] \ge J^o[x_m, m, 0, N]$$

5 Proof of (28)

Consider the n-th order *asymptotically stable* discrete time LTI system

$$x(k+1) = A\, x(k) \qquad x(0) = x_o\,. \tag{27}$$

Then $x(k)$ converges to 0 geometrically and

$$\sum_{k=0}^{\infty} \|x(k)\|^2 < \infty$$

Sketch of Proof:

Since A is Schur (all of it eigenvalues are inside the unit circle), for any matrix $Q \succ 0$ there exists a matrix $P \succ 0$, which is the solution of the Lyapunov equation

$$A^T P A - P = -Q$$

Moreover, the positive definite Lyapunov function $V(k) = x^T(k)Px(k)$ satisfies

$$V(k+1) - V(k) = -x^T(k)Qx(k) \tag{28}$$

Let $\lambda_{\max}(P) \geq \lambda_{\min}(P) > 0$ be respectively the maximum and minimum eigenvalues of P and $\lambda_{\max}(Q) \geq \lambda_{\min}(Q) > 0$ be respectively the maximum and minimum eigenvalues of Q. Then, form (28), we obtain

$$V(k+1) - V(k) \leq -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} V(k)$$

since

$$\begin{aligned} \lambda_{\min}(Q)\|x(k)\|^2 \leq x^T(k)Qx(k) \leq \lambda_{\max}(Q)\|x(k)\|^2 \\ \lambda_{\min}(P)\|x(k)\|^2 \leq x^T(k)Px(k) \leq \lambda_{\max}(P)\|x(k)\|^2 \,. \end{aligned}$$

Moreover,

$$V(k+1) \leq (1 - \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}) V(k) = -\alpha V(k)$$

where $\alpha = 1 - \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}$ and $0 \leq \alpha < 1$, since $V(k) \geq 0$ and $V(k) = 0 \Leftrightarrow \|x(k)\| = 0$.

Therefore, $V(k)$ converges to zero geometrically and

$$\lim_{k \to \infty} V(k) = 0$$

From Eq. 28 we also obtain

$$\sum_{k=0}^{\infty} x^T(k)Qx(k) = \sum_{k=0}^{\infty} \{V(k) - V(k+1)\} = V(0) - \lim_{k \to \infty} V(k)$$

Therefore,

$$\sum_{k=0}^{\infty} \|x(k)\|^2 = \frac{V(0)}{\lambda_{\min}(Q)} < \infty \,.$$

References

Anderson, B. and Moore, J. (1977). *Linear Optimal Control*. Prentice Hall.

Lewis, F. L. (1986). *Optimal Control*. John Wiley.