1. (10 points) Application of Dynamic Programming
Our goal is to solve the following problem:

$$\max_{U_0} J, \qquad s.t. \quad u(i) \geq 0, \ \sum_{i=0}^{N-1} u(i) = x_f \tag{1}$$

where $J := \prod_{i=0}^{N-1} u(i)$ and $U_k = [u(k),\, u(k+1),\, \cdots,\, u(N-1)]$.
Define

$$\begin{aligned} J_k\left(x(k)\right): &= \prod_{i=k}^{N-1} u(i) && (2)\\ J_k^o\left(x(k)\right): &= \max_{U_k} \prod_{i=k}^{N-1} u(i) && (3)\\ \Longrightarrow J_{N-1}^o\left(x(N-1)\right) &= u(N-1) = x_f - x(N-1) && (4) \end{aligned}$$

The central idea in dynamic programming is to express the optimal cost at time step k as a function of the optimal cost at time step $k+1$ so that a backward recursive scheme may be used. In other words,

$$\begin{aligned} J_k^o\left(x(k)\right): &= \max_{U_k} \prod_{i=k}^{N-1} u(i) && (5)\\ &= \max_{u(k),\, U_{k+1}} u(k) \prod_{i=k+1}^{N-1} u(i) && (6)\\ &= \max_{u(k)} u(k) \max_{U_{k+1}} \prod_{i=k+1}^{N-1} u(i) && (7)\\ &= \max_{u(k)} \left(u\left(k\right) J_{k+1}^o\left(x(k+1)\right)\right) && (8) \end{aligned}$$

You would need to convince yourself about some of the intermediate steps in the above set of equations. Consider:

$$\begin{aligned} J_{N-2}^o\left(x(N-2)\right) &= \max_{U_{N-2}} \left(u(N-2) J_{N-1}^o\left(x(N-1)\right)\right) && (9)\\ \Longrightarrow u^o(N-2) &= \arg\max_{u(N-2)} \Big(u(N-2) J_{N-1}^o\left(x(N-1)\right)\Big) && (10)\\ &= \arg\max_{u(N-2)} \Big(u(N-2)\Big(x_f - x(N-1)\Big)\Big) && (11)\\ &= \arg\max_{u(N-2)} \Big(u(N-2)\Big(x_f - x(N-2) - u(N-2)\Big)\Big) && (12)\\ &= \frac{x_f - x(N-2)}{2} && (13) \end{aligned}$$

Similarly,

$$\begin{aligned} u_o(N-3) &= \arg\max_{u(N-3)} \left(u(N-3) J_{N-1}^o\left(x(N-2)\right)\right) = \frac{x_f - x(N-3)}{3} && (14)\\ \vdots \quad &= \quad \vdots \\ u^o(0) &= \arg\max_{u(0)} \left(u(0) J_{N-1}^o\left(x(N-(N-1))\right)\right) = \frac{x_f - x(0)}{N} = \frac{x_f}{N} && (15) \end{aligned}$$

Given $u^o = \frac{x_f}{N}$, the above set of equations yield $u(i) = \frac{x_f}{N}$ for all i.
Note: when deriving the optimal control law for $k = N-3$, $N-4$, ..., the cost $J(N-k)$ is no longer a quadratic function of the control input. However, after some computation, you can find that there are at most two points that make $\partial J(N-k)/\partial u(N-k) = 0$. Easy evaluation at these points and the boundary points $(0 \le u(k) \le x_f)$ can tell you that $(x_f - x(N-k))/k$ is the one that gives you the maximum value.

2. (15 points) Optimal Tracking Problem
The LQ tracking problem is formulated as follow:

$$\min_{U_0} J := \frac{1}{2}\left[y_d(N) - y(N)\right]^T S \left[y_d(N) - y(N)\right] + \frac{1}{2}\sum_{k=0}^{N-1}\Big(\left[y_d(k) - y(k)\right]^T Q_y \left[y_d(k) - y(k)\right] + u(k)^T R u(k)\Big) \tag{16}$$

subject to $x(k+1) = Ax(k) + Bu(k); y(k) = Cx(k); x(0) = x_0$ with $y_d(k)$ specified for all k and $U_k := [u(k) \quad u(k+1) \quad \cdots \quad u(N-1)]$. Define the "cost to go":

$$J_k = \frac{1}{2}\left[y_d(N) - y(N)\right]^T S \left[y_d(N) - y(N)\right] + \frac{1}{2}\sum_{i=k}^{N-1}\Big(\left[y_d(i) - y(i)\right]^T Q_y \left[y_d(i) - y(i)\right] + u(i)^T R u(i)\Big) \tag{17}$$

Using Bellman's principle of optimality, we can obtain a recursive relation between $J_k^o(x(k))$(the optimal cost to go from $x(k)$ to $x(N)$), and $J_{k+1}^o(x(k+1))$ as:

$$J_k^o(x(k)) = \min_{u(k)}\{\frac{1}{2}\left[y_d(k) - y(k)\right]^T S \left[y_d(k) - y(k)\right] + \frac{1}{2}u(k)^T R u(k) + J_{k+1}^o(x(k+1))\} \tag{18}$$

We use the hint regarding the structure of J_k. Starting at $k = N$:

$$\begin{aligned} J_N^o(x(N)) &= \frac{1}{2}\{\left[y_d(N) - y(N)\right]^T S \left[y_d(N) - y(N)\right]\} \\ &= \frac{1}{2}x^T(N)C^T SCx(N) - y_d^T(N)SCx(N) + \frac{1}{2}y_d^T(N)Sy_d(N) \end{aligned} \tag{19}$$

Define $P(N) := C^T SC$, $b(N) := -y_d^T(N)SC$, $c(N) := \frac{1}{2}y_d^T(N)Sy_d(N)$. Then the optimal J_N is:

$$J_N^o(x(N)) = \frac{1}{2}x^T(N)P(N)x(N) + b(N)x(N) + c(N) \tag{20}$$

Now we use the recursive relation for $k = N-1$ to determine $u^o(N-1)$. The following general results are now useful: consider the quadratic function

$$f(u) = \frac{1}{2}u^T Ru + p^T u + q \tag{21}$$

The optimal (maximum when R is negative definite; minimum when R is positive definite) is achieved when

$$\frac{\partial f}{\partial u} = 0 \Rightarrow Ru^o + p = 0 \Rightarrow u^o = -R^{-1}p \tag{22}$$

and the optimal cost is

$$f^o = f(u^o) = -\frac{1}{2}p^T R^{-1}p + q \tag{23}$$

For the LQ problem

$$J_{N-1} = \min_{u(N-1)}\left\{J_N^o(x(N)) + \frac{1}{2}[y(N-1) - y_d(N-1)]^T Q_y [y(N-1) - y_d(N-1)] + u(N-1)^T R u(N-1)\right\}$$

Substituting in the system dynamic equation and after some algebra, we get

$$
\begin{aligned}
J_{N-1} = & \frac{1}{2}u^T(N-1)\left(B^T P(N) B + R\right)u(N-1) + u^T(N-1)B^T\left(P(N)Ax(N-1) + B^T b(N)\right) \\
& + \frac{1}{2}x^T(N-1)A^T P(N) A x(N-1) + b^T(N)Ax(N-1) + c(N) \\
& + \frac{1}{2}\left(Cx(N-1) - y_d(N-1)\right)^T Q_y \left(Cx(N-1) - y_d(N-1)\right)
\end{aligned}
$$

Regarding the above as a quadratic function of $u(N-1)$ and using the results of (21)-(23), we can get

$$
\begin{aligned}
u^o(N-1) &= -\left[R + B^T P(N) B\right]^{-1} B^T \left[P(N)Ax(N-1) + b^T(N)\right] \quad (24)\\
J^o_{N-1}\left(x(N-1)\right) &= \frac{1}{2}\Big\{ y_d^T(N-1)Q_y y_d(N-1) \\
&+ 2\left(-y_d^T(N-1)Q_y C + b(N)\left\{A - B(R + B^T P(N)B)^{-1}B^T P(N)A\right\}\right)x(N-1) \\
&+ x^T(N-1)\Big(C^T Q_y C + A^T P(N) A - A^T P(N) B\left[R + B^T P(N) B\right]^{-1} B^T P(N) A\Big)x(N-1) \\
&- b(N)B\left[R + B^T P B\right]^{-1} B^T b^T(N) + 2c(N)\Big\} \quad (25)
\end{aligned}
$$

Repeating this recursively for $k = N-2, \cdots, 1$ results in:

$$
\begin{aligned}
u^o(k) &= -\left[R + B^T P(k+1) B\right]^{-1} B^T \left[P(k+1)Ax(k) + b^T(k+1)\right] \quad (26)\\
J^o_k\left(x(k)\right) &= \frac{1}{2}x^T(k)P(k)x(k) + b(k)x(k) + c(k) \quad (27)
\end{aligned}
$$

where $P(k)$, $b(k)$, $c(k)$ satisfy:

$$
\begin{aligned}
P(k) &= C^T Q_y C + A^T P(k+1) A - A^T P(k+1) B\left[R + B^T P(k+1) B\right]^{-1} B^T P(k+1) A \quad (28)\\
b(k) &= -y_d^T(k) Q_y C + b(k+1)\{A - B(R + B^T P(k+1)B)^{-1}B^T P(k+1)A\} \quad (29)\\
c(k) &= c(k+1) + \frac{1}{2}y_d^T(k)Q_y y_d(k) - \frac{1}{2}b(k+1)B\left[R + B^T P(k+1)B\right]^{-1} B^T b(k+1) \quad (30)
\end{aligned}
$$

with the initial conditions for the backward recursion being:

$$
\begin{aligned}
P(N) &= C^T S C \quad (31)\\
b(N) &= -y_d^T(N) S C \quad (32)\\
c(N) &= \frac{1}{2}y_d^T(N) S y_d(N) \quad (33)
\end{aligned}
$$

Note: strictly speaking, the equation for $c(k)$ is not needed for computing the optimal control law. It is however beneficial to derive the full results for better understanding of the problem.
Understanding the solution: from the update equations, $b(N)$ and $c(N)$ depend on $y_d(N)$; $b(k)$ and $c(k)$ depend on $y_d(k)$, $b(k+1)$ and $c(k+1)$. Hence $b(0)$ and $c(0)$ depends on $y_d(0)$, $y(1)$, ..., $y_d(N)$, i.e., the full desired trajectory should be available to compute the initial control input. This makes intuitive sense, that to obtain the best strategy we would need to know a full "map" of the route we are planning to follow.

3. (10 points) Given that X_1, X_2 and X_3 are three independent random variables uniformly distributed over $[0, \quad 1]$, we need to obtain the probability distribution functions(pdf's) of:

$$
\begin{aligned}
Y: &= X_1 + X_2 \quad (34)\\
Z: &= X_1 + X_2 + X_3 \quad (35)
\end{aligned}
$$

First, let us consider the problem of computing the pdf of Y. The key to this problem is to note that, if given value $X_1 = x_1$, then Y becomes $Y = x_1 + X_2$, i.e. random variable $Y_{|X_1=x_1}$ looks exactly like random variable X_2 shifted by x_1.
Therefore the pdf of $Y_{|X_1=x_1}$ is easy to calculate.
If F and p stand for cumulative distribution function(cdf) and pdf respectively,

$$\begin{aligned} F_{Y_{|X_1=x_1}}(y) &= P(Y_{|X_1=x_1} < y) \\ &= P(X_2 + x_1 \le y) \\ &= P(X_2 \le y - x_1) \\ &= F_{X_2}(y - x_1) \\ \Longrightarrow p_{Y_{|X_1=x_1}} &= p_{X_2}(y - x_1) \end{aligned} \tag{36}$$

Convince yourself that:

$$p_{X_2}(y - x_1) = \begin{cases} 1 & \text{for} \quad -1 + y \le x_1 \le y \\ 0 & \text{otherwise} \end{cases} \tag{37}$$

Consider the joint distribution of the pair of random variables (Y, X_1). Then,

$$\begin{aligned} p_Y(y) &= \int_{-\infty}^{\infty} p_{Y,X_1}(y, x_1) dx_1 \\ &= \int_{-\infty}^{\infty} p_{Y_{X_1=x_1}}(y) p_{X_1}(x_1) dx_1 \\ &= \int_{-\infty}^{\infty} p_{X_2}(y - x_1) p_{X_1}(x_1) dx_1 \\ &= \begin{cases} \int_0^y dx_1 & \text{for} \quad 0 \le y \le 1 \\ \int_{y-1}^1 dx_1 & \text{for} \quad 1 \le y \le 1 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} y & \text{for} \quad 0 \le y \le 1 \\ 2 - y & \text{for} \quad 1 \le y \le 2 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{38}$$

Follow a similar procedure to get the pdf of Z. Using $p_Z(z) = \int_{-\infty}^{\infty} p_{X_3}(z - y) p_Y(y) dy$ and eq. (5), we will get

$$p_Z(z) = \begin{cases} \frac{1}{2} z^2 & \text{for} \quad 0 \le z \le 1 \\ 3z - z^2 - \frac{3}{2} & \text{for} \quad 1 \le z \le 2 \\ \frac{1}{2} z^2 - 3z + \frac{9}{2} & \text{for} \quad 2 \le z \le 3 \\ 0 & \text{otherwise} \end{cases} \tag{39}$$

Note that the shape of Y looks triangular and that of Z will be quadratic with a hump. As you compute the sum of a large number, if independent, identically distributed(IID) random variable, you can expect the shape of the pdf of the sum of random variables to look like a Gaussian.

4. (5 points) Positive Semi-definite Property of Covariance Matrix
We need to show that the covariance matrix is positive semi-definite, i.e. we need to show that $Z :=$

$E\left[(X-m_x)(X-m_x)^T\right]$ is p.s.d. Consider,

$$\begin{aligned}
\alpha^T Z\alpha &= \alpha^T\Big(\int_{-\infty}^{\infty}(x-m_x)(x-m_x)^T p_X(x)dx\Big)\alpha \\
&= \int_{-\infty}^{\infty}\alpha^T(x-m_x)(x-m_x)^T\alpha p_X(x)dx \\
&= \int_{-\infty}^{\infty} q^T q p_X(x)dx, \quad \text{where} \quad q := (x-m_x)^T\alpha \qquad (40)\\
&= \int_{-\infty}^{\infty} \parallel q \parallel_2^2 p_X(x)dx \\
&\geq 0 \qquad \text{since} \quad \parallel q \parallel_2 \geq 0 \quad (\text{for all } q), \quad p_x(x)\geq 0
\end{aligned}$$

That is, $\alpha^T Z\alpha \geq 0$, for all α and this means Z is p.s.d.
For the second part, we notice that for any deterministic α, $\alpha^T Z\alpha$ is the variance of $\alpha^T(x-m_x)$, since $E[\alpha^T(x-m_x)]=0$. So if the variance is positive for any nonzero deterministic α, Z is positive definite.

5. (15 points) Computing the autocorrelation function of the response of an LTI system to a wide-sense stationary(WSS) input
The response of an LTI system to a WSS random process is WSS at the steady state. In this problem, the discrete-time LTI system is described by $y(k)-0.8y(k-1)=e(k)+0.5e(k-1)$.
Also given: $e(k)$ is a zero mean white noise process.
The transfer function of the above LTI system is $G(z)=\frac{z+0.5}{z-0.8}$. Therefore the spectral density looks like:

$$\Phi_{yy}\big(e^{j\omega}\big) = G(z)G(z^{-1})|_{z=e^{j\omega}}\Phi_{ee}\big(e^{j\omega}\big) = \frac{1.25+0.5e^{-j\omega}+0.5e^{j\omega}}{1.64-0.8e^{-j\omega}-0.8e^{j\omega}}\times 1 = \frac{1.25+cos\omega}{1.64-1.6cos\omega} \qquad (41)$$

We have three different methods to compute the autocovariance of $y(k)$:
Method 1: Define $\Phi_{yy}(z)$ as the Z transform of $X_{yy}(k)$. $\Phi_{yy}(z)$ is related to $\Phi_{yy}(\omega)$ by $\Phi_{yy}(z)=\Phi_{yy}(\omega)|_{e^{j\omega}=z}$. Doing a partial fraction expansion of $\Phi_{yy}(z)$, we get

$$\begin{aligned}
\Phi_{yy}(z) &= -\frac{5}{8}\Big(1-\frac{91}{9}\frac{0.8z^{-1}}{1-0.8z^{-1}}+\frac{91}{9}\frac{\frac{1}{0.8}z^{-1}}{1-\frac{1}{0.8}z^{-1}}\Big)\\
X_{yy}(k) &= \mathcal{Z}^{-1}\big(\Phi_{yy}(z)\big) = -\frac{5}{8}\Big(\delta(k)-\frac{91}{9}0.8^k u(k-1)-\frac{91}{9}\Big(\frac{1}{0.8}\Big)^k u(-k)\Big),
\end{aligned}$$

where

$$u(k)=\begin{cases}1 & \text{for } k\geq 0\\ 0 & \text{for } k<0\end{cases}$$

Alternatively, you can just perform an inverse discrete-time Fourier transform on the spectral density function of y, which will give the same result.
Method 2: Suppose we have a system described by

$$\begin{aligned}
x(k+1) &= Ax(k)+Be(k)\\
y(k) &= Cx(k)+De(k) \quad \text{where} \quad E\left[e(k)\right], E[e(k)e^T(k+l)] = W\delta(l)=0
\end{aligned}$$

We want to determine $X_{yy}(l)$ at the steady state. As described in the notes, $X_{xx}(0)$ can be obtained by solving

$$X_{ss} = AX_{ss}A^T+BWB^T \qquad (42)$$

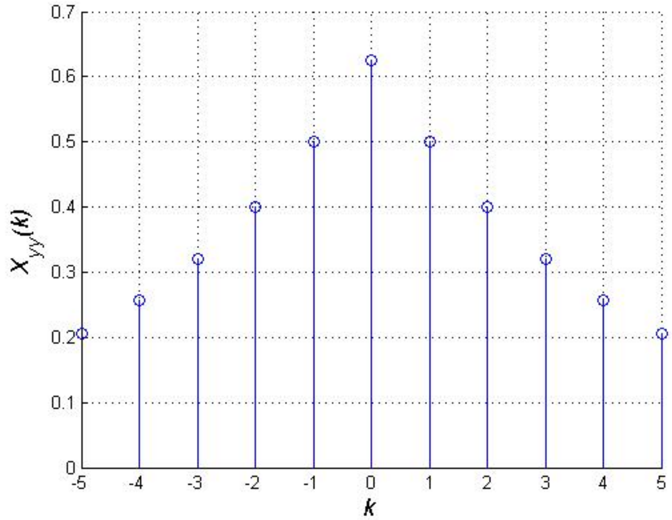

Figure 1: Autocorrelation function

To obtain $X_{yy}(l)$, $l \geq 0$,

$$\begin{aligned}
X_{yy}(l) =& E\left[y(k)y^T(k+l)\right] \\
=& E\left[\left(Cx(k)+De(k)\right)\left(Cx(k+l)+De(k+l)\right)^T\right] \\
=& E\left[Cx(k)x^T(k+l)C^T\right] + E\left[Cx(k)e^T(k+l)D^T\right] + E\left[De(k)x^T(k+l)C^T\right] + E\left[De(k)e^T(k+l)D^T\right] \\
=& CX_{xx}(l)C^T + 0 + DE\left[e(k)x^T(k+l)\right]C^T + 0
\end{aligned}$$

The second and fourth terms are zero based on the causality of the system and the whiteness of $e(k)$. The third term is computed as shown below.

$$\begin{aligned}
E\left[e(k)x^T(k+l)\right] =& E\left[e(k)\left(A^l x(k) + \sum_{j=0}^{l-1} A^{l-j-1}Be(k+j)\right)^T\right] \\
=& E\left[e(k)x^T(k)\right] + \sum_{j=0}^{l-1} E\left[e(k)\left(A^{l-j-1}Be(k+j)\right)^T\right] \\
=& 0 + E\left[e(k)\left(A^{l-1}Be(k)\right)^T\right] = WB^T\left(A^T\right)^{l-1}
\end{aligned}$$

Again, we have used causality and whiteness of $e(k)$ in the last step. Therefore $X_{yy}(l) = CX_{xx}(A^T)^l C^T + DWB^T\left(A^T\right)^{l-1}C^T$ if $l > 0$. We can use the fact that $X_{yy}(-l) = X_{yy}(l)^T$ to obtain the case for which $l < 0$. Finally for the case when $l = 0$, $X_{yy}(0) = CX_{xx}C^T + DWD^T$.

For this problem, A, B, C, D are $0.8, 1, 1.3, 1$ (Use any method to obtain the state space representation of the difference equation given). Solving for X_{ss} by hand or by using the MATLAB command *lyap*, we get $X_{ss} = \frac{25}{9}$. $X_{yy}(0) = 5.59$. Further,

$$\begin{aligned} X_{yy}(k) =& 1.3 \times \frac{25}{9} \times 0.8^k \times 1.3 + 0.8^{k-1} \times 1.3 \\ =& \frac{189}{36}(0.8)^k + (0.8)^{k-1} \text{when } k > 0 \end{aligned}$$

You can verify that this answer is consistent with the answer obtained from the first method.
Method 3: $X_{yy}(k)$ can also be computed using convolution.
The inverse Z transform of the transfer function $G(z) = \frac{z+0.5}{z-0.8} = 1 + \frac{1.3z^{-1}}{1-0.8z^{-1}}$ gives the impulse response

$$g(i) = \begin{cases} 1 & \text{for } i = 0 \\ \frac{1.3}{0.8}(0.8)^i & \text{for } i > 0 \\ 0 & \text{otherwise} \end{cases}$$

The cross-covariance of y and e is

$$\begin{aligned} X_{ye}(l) =& E[y(k)e(k+l)] = E\left[\left(\sum_{i=-\infty}^{\infty} g(i)e(k-i)\right) e(k+l)\right] \\ =& \sum_{i=-\infty}^{\infty} \{g(i)E[e(k-i)e(k+l)]\} = \sum_{i=-\infty}^{\infty} \{g(i)\delta_{-i,l}\} \\ =& g(-l) \end{aligned}$$

The covariance of y or the autocovariance of y at zero time difference is

$$\begin{aligned} X_{yy}(0) =& \sum_{i=-\infty}^{\infty} g(i)X_{ye}(-i) = \sum_{i=-\infty}^{\infty} g(i)g(i) \\ =& \sum_{i=0}^{\infty}[g(i)^2] = 1 + \sum_{i=1}^{\infty}[g(i)^2] \\ =& 1 + \sum_{i=1}^{\infty}[(\frac{1.3}{0.8})^2(0.8)^{2i}] \\ =& 1 + \frac{(\frac{1.3}{0.8})^2(0.8)^2}{1-(0.8)^2} \\ =& \frac{205}{36} \end{aligned}$$

For $l > 0$, the autocovariance of y is computed by

$$\begin{aligned} X_{yy}(l) =& \sum_{i=-\infty}^{\infty} g(i)X_{ye}(l-i) = \sum_{i=-\infty}^{\infty} [g(i)g(i-l)] \\ =& \sum_{i=l}^{\infty}[g(i)g(i-l)] = \frac{1.3}{0.8}(0.8)^l + \sum_{i=1}^{\infty}\left[(\frac{1.3}{0.8})^2(0.8)^{2i+l}\right] \\ =& \frac{1.3}{0.8}(0.8)^l + \frac{(\frac{1.3}{0.8})^2(0.8)^{2+l}}{1-(0.8)^2} \\ =& \frac{455}{72}(0.8)^l \end{aligned}$$

Similarly to method 2, $X_{yy}(l)$ for $l < 0$ is found by

$$X_{yy}(l) = X_{yy}(-l) = \tfrac{455}{72}(0.8)^{-l}$$

You can verify that this answer is consistent with the answers obtained from the first and the second methods.

6. (15 points) Given a system as:

$$\dot{x} = Ax + B_w w = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w \tag{43}$$

$$y = Cx = \begin{bmatrix} 1 & 0.5 \end{bmatrix} x \tag{44}$$

Since this is an LTI system, the steady state covariance of x satisfies the following Lyapunov Equation:

$$AX_{ss} + X_{ss}A^T = -B_w W B_w^T = -\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Longrightarrow X_{ss} = \begin{bmatrix} \frac{1}{12} & 0 \\ 0 & \frac{1}{6} \end{bmatrix} \tag{45}$$

Therefore, For $\tau \geq 0$,

$$X_{ss}(\tau) = X_{ss}e^{A^T\tau} = X_{ss}V \begin{bmatrix} e^{-\tau} & 0 \\ 0 & e^{-2\tau} \end{bmatrix} V^{-1}, \quad \text{with} \quad V = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{\sqrt{2}}{2} \\ \frac{1}{\sqrt{5}} & \frac{\sqrt{2}}{2} \end{bmatrix} \tag{46}$$

$$Y_{ss}(\tau) = CX_{ss}(\tau)C^T = CX_{ss}V \begin{bmatrix} e^{-\tau} & 0 \\ 0 & e^{-2\tau} \end{bmatrix} V^{-1}C' = \frac{1}{8}e^{-\tau} \tag{47}$$

Hence, $Y_{ss}(\tau) = \frac{1}{8}e^{-\tau}$ for $\tau \geq 0$, $Y_{ss}(\tau) = \frac{1}{8}e^{\tau}$ for $\tau < 0$. The variance is $Y_{ss}(0) = \frac{1}{8}$.
The spectral density is:

$$\Phi_{yy}(\omega) = \mathfrak{F}\{Y_{ss}(\tau)\} = \frac{1}{8}\left(\frac{1}{1+j\omega} + \frac{1}{1-j\omega}\right) = \frac{1}{4}\frac{1}{1+\omega^2} \tag{48}$$