

# ME 233 Advanced Control II

## Lecture 4 Random Vector Sequences

(ME233 Class Notes pp. PR6-PR10)

# Outline

- Random vector sequences
  - Mean, auto-covariance, cross-covariance
- MIMO Linear Time Invariant Systems
- State space systems driven by white noise
- Lyapunov equation for covariance propagation

# Random vector sequences

A two-sided random vector sequence is a collection of random vectors

$$X = \{\dots X(-1), X(0), X(1),$$

$$X(2), X(3), \dots\}$$

each  $X(k) \in \mathcal{R}^n$  is itself a random vector

defined over the same probability space  $(\Omega, \mathcal{S}, P)$

# Random vector sequences

We either will use

$$\{X(k)\}_{k=-\infty}^{\infty}$$

or

$$X(k)$$

*Shorthand  
(sloppy) notation*

to denote the two-sided random vector sequence.

Each element  $X(k)$  of the sequence is a random vector:

$$X(k) : \Omega \rightarrow \mathcal{R}^n$$

# Random vector sequences

A sample sequence

corresponds to the value of

$$\{\dots, X(-1), X(0), X(1), X(2), \dots\}$$

obtained after performing an experiment

# 2<sup>nd</sup> order statistics

For a two-sided **Random Vector Sequence (RVS)**

$$\{X(k)\}_{k=-\infty}^{\infty}$$

**Expected value or mean of  $X(k)$ ,**

$$E \{X(k)\} = m_X(k) \in \mathcal{R}^n$$

# Auto-covariance

Define:  $\tilde{X}(k) = X(k) - m_X(k)$

$$\Lambda_{XX}(k, \underline{j}) = E \left\{ \tilde{X}(k + \underline{j}) \tilde{X}^T(k) \right\}$$

$$\Lambda_{XX}(k, \underline{j}) = E \left\{ \begin{bmatrix} \tilde{X}_1(k + \underline{j}) \\ \vdots \\ \tilde{X}_n(k + \underline{j}) \end{bmatrix} \begin{bmatrix} \tilde{X}_1(k) & \cdots & \tilde{X}_n(k) \end{bmatrix} \right\}$$

# Cross-covariance

Define:  $\tilde{X}(k) = X(k) - m_X(k)$

$$\tilde{Y}(k) = Y(k) - m_Y(k)$$

$$\Lambda_{XY}(k, \underline{j}) = E \left\{ \tilde{X}(k + \underline{j}) \tilde{Y}^T(k) \right\}$$

$$\Lambda_{XY}(k, \underline{j}) = E \left\{ \begin{bmatrix} \tilde{X}_1(k + \underline{j}) \\ \vdots \\ \tilde{X}_n(k + \underline{j}) \end{bmatrix} \begin{bmatrix} \tilde{Y}_1(k) & \cdots & \tilde{Y}_n(k) \end{bmatrix} \right\}$$

# Wide Sense Stationary (WSS)

A two-sided random vector sequence  $\{X(k)\}_{k=-\infty}^{\infty}$

is **WSS** if:

$$1) \quad E\{X(k)\} = m_X \quad (\text{time invariant})$$

$$2) \quad \Lambda_{XX}(\underline{k}, l) = \Lambda_{XX}(\underline{k+M}, l)$$

(only depends on  $\lambda$ )

# Auto-covariance function

For WSS RVS, the auto-covariance is only a function of the correlation index  $j$

$$\Lambda_{XX}(j) = E \left\{ \tilde{X}(k + j) \tilde{X}^T(k) \right\}$$

for **any** index  $k$

$$\Lambda_{XX}(l) = \Lambda_{XX}^T(-l)$$

# Auto-covariance function Z-transform

$$\Lambda_{XX}(l) = \Lambda_{XX}^T(-l)$$



$$\hat{\Lambda}_{XX}(z) = \hat{\Lambda}_{XX}^T(z^{-1})$$

# Auto-covariance function

$$\Lambda_{XX}(l) = \Lambda_{XX}^T(-l)$$

*Proof:*

$$\begin{aligned}\Lambda_{XX}^T(-l) &= E\{(X(k-l) - m_X)(X(k) - m_X)^T\}^T \\ &= E\{(X(k) - m_X)(X(k-l) - m_X)^T\}\end{aligned}$$

Define  $\bar{k} := k - l$

$$\begin{aligned}\Lambda_{XX}^T(-l) &= E\{(X(\bar{k} + l) - m_X)(X(\bar{k}) - m_X)^T\} \\ &= \Lambda_{XX}(l)\end{aligned}$$



# Auto-covariance function Z-transform

$$\hat{\Lambda}_{XX}(z) = \hat{\Lambda}_{XX}^T(z^{-1})$$

**Proof:**

$$\hat{\Lambda}_{XX}^T(z^{-1}) = \left[ \sum_{l=-\infty}^{\infty} \Lambda_{XX}(l) z^{+l} \right]^T = \sum_{l=-\infty}^{\infty} \Lambda_{XX}^T(l) z^{+l}$$

Define  $n := -l$

$$\hat{\Lambda}_{XX}^T(z^{-1}) = \sum_{n=-\infty}^{\infty} \Lambda_{XX}^T(-n) z^{-n} = \sum_{n=-\infty}^{\infty} \Lambda_{XX}(n) z^{-n}$$

$$= \hat{\Lambda}_{XX}(z)$$



# Cross-covariance function

$X(k)$  and  $Y(k)$

are two **WSS** random vector sequences

$$\Lambda_{XY}(j) = E\{\tilde{X}(k+j)\tilde{Y}^T(k)\}$$

for **any** index  $k$

Notice that:

$$\Lambda_{XY}(l) = \Lambda_{YX}^T(-l)$$

# Auto-covariance function Z-transform

$$\Lambda_{XY}(l) = \Lambda_{YX}^T(-l)$$



$$\hat{\Lambda}_{XY}(z) = \hat{\Lambda}_{YX}^T(z^{-1})$$

# Cross-covariance function

$$\Lambda_{XY}(l) = \Lambda_{YX}^T(-l)$$

*Proof:*

$$\begin{aligned}\Lambda_{YX}^T(-l) &= E\{(Y(k-l) - m_Y)(X(k) - m_X)^T\}^T \\ &= E\{(X(k) - m_X)(Y(k-l) - m_Y)^T\}\end{aligned}$$

Define  $\bar{k} := k - l$

$$\begin{aligned}\Lambda_{YX}^T(-l) &= E\{(X(\bar{k} + l) - m_X)(Y(\bar{k}) - m_Y)^T\} \\ &= \Lambda_{XY}(l)\end{aligned}$$



# Auto-covariance function Z-transform

$$\hat{\Lambda}_{XY}(z) = \hat{\Lambda}_{YX}^T(z^{-1})$$

**Proof:**

$$\hat{\Lambda}_{YX}^T(z^{-1}) = \left[ \sum_{l=-\infty}^{\infty} \Lambda_{YX}(l) z^{+l} \right]^T = \sum_{l=-\infty}^{\infty} \Lambda_{YX}^T(l) z^{+l}$$

Define  $n := -l$

$$\hat{\Lambda}_{YX}^T(z^{-1}) = \sum_{n=-\infty}^{\infty} \Lambda_{YX}^T(-n) z^{-n} = \sum_{n=-\infty}^{\infty} \Lambda_{XY}(n) z^{-n}$$

$$= \hat{\Lambda}_{XY}(z)$$



# Ergodicity

A **Wide Sense Stationary** random sequence

is **ergodic**

if its ensemble average = time average (constant)

$$E \{X(k)\} = m_X$$

$$= \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{k=-N}^N x(k)$$

with probability 1  
(almost surely)

*sample sequence*

# Ergodicity

For any **WSS ergodic** random sequence

we can approximate the covariance as a “time average”

$$\begin{aligned}\Lambda_{XX}(j) &= E\{\tilde{X}(k+j)\tilde{X}^T(k)\} \\ &= \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{k=-N}^N \tilde{x}(k+j)\tilde{x}^T(k)\end{aligned}$$

with probability 1  
(almost surely)

$$\tilde{x}(k) = x(k) - m_X$$

*sample sequence*

# Power Spectral Density Function

Fourier transform of the auto-covariance function:

$$\Phi_{XX}(\omega) = \mathcal{F}\{\Lambda_{XX}(\cdot)\}$$

$$= \sum_{l=-\infty}^{\infty} \Lambda_{XX}(l) e^{-j\omega l}$$


*Complex-valued matrix*

*l: correlation index*

**Note:**

The power spectral density function is periodic, with period  $T = 2\pi$

$$e^{-j\omega l} = \cos(\omega l) - j \sin(\omega l)$$

# Power Spectral Density Function

Using the inverse Fourier transform we obtain:

$$\Lambda_{XX}(l) = \mathcal{F}^{-1}\{\Phi_{XX}(\omega)\}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega l} \Phi_{XX}(\omega) d\omega$$

# Power Spectral Density Function

Properties of the power spectral density function:

1.  $\Phi_{XX}(\omega) = \Phi_{XX}^T(-\omega)$
2.  $\Phi_{XX}(\omega) = \Phi_{XX}^*(\omega) \quad \omega \in [-\pi, \pi]$
3.  $\Phi_{XX}(\omega) \succeq 0 \quad \omega \in [-\pi, \pi]$
4.  $\Lambda_{XX}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_{XX}(\omega) d\omega$

# Power Spectral Density Function

$$1. \quad \Phi_{XX}(\omega) = \Phi_{XX}^T(-\omega) \quad \omega \in [-\pi, \pi]$$

*Proof:*

$$\Phi_{XX}^T(-\omega) = \left[ \sum_{l=-\infty}^{\infty} \Lambda_{XX}(l) e^{j\omega l} \right]^T = \sum_{l=-\infty}^{\infty} \Lambda_{XX}^T(l) e^{j\omega l}$$

Define  $n := -l$

$$\begin{aligned} \Phi_{XX}^T(-\omega) &= \sum_{n=-\infty}^{\infty} \Lambda_{XX}^T(-n) e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \Lambda_{XX}(n) e^{-j\omega n} \\ &= \Phi_{XX}(\omega) \end{aligned}$$



# Power Spectral Density Function

$$2. \quad \Phi_{XX}(\omega) = \Phi_{XX}^*(\omega) \quad \omega \in [-\pi, \pi]$$

*Proof:*

$$\Phi_{XX}^*(\omega) = \left[ \sum_{l=-\infty}^{\infty} \Lambda_{XX}(l) e^{-j\omega l} \right]^* = \sum_{l=-\infty}^{\infty} \Lambda_{XX}^T(l) e^{j\omega l}$$

Define  $n := -l$

$$\begin{aligned} \Phi_{XX}^*(\omega) &= \sum_{n=-\infty}^{\infty} \Lambda_{XX}^T(-n) e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \Lambda_{XX}(n) e^{-j\omega n} \\ &= \Phi_{XX}(\omega) \end{aligned}$$



# Power Spectral Density Function

Properties of the power spectral density function:  
**(scalar case)**

1.  $\Phi_{XX}(\omega) = \Phi_{XX}(-\omega)$
  
2.  $\Phi_{XX}(\omega)$  is real  $\omega \in [-\pi, \pi]$
  
3.  $\Phi_{XX}(\omega) \geq 0$   $\omega \in [-\pi, \pi]$
  
4.  $\Lambda_{XX}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_{XX}(\omega) d\omega$

# White noise vector sequence

A **WSS** random vector sequence  $\{W(k)\}_{k=-\infty}^{\infty}$  is **white** if:

$$\Lambda_{WW}(l) = \Sigma_{WW} \delta(l)$$

where

$$\delta(l) = \begin{cases} 1 & l = 0 \\ 0 & l \neq 0 \end{cases}$$

$$\Sigma_{WW} = E\{\tilde{W}(k)\tilde{W}^T(k)\} \quad \tilde{W}(k) = W(k) - m_W$$

$$\Sigma_{WW} = \Sigma_{WW}^T \succeq 0$$

# White noise vector sequence

Given the white WSS random sequence  $\{W(k)\}_{k=-\infty}^{\infty}$

with

$$\Lambda_{WW}(l) = \Sigma_{WW} \delta(l)$$

Its power spectral density (Fourier transform)

$$\Phi_{WW}(\omega) = \sum_{l=-\infty}^{\infty} \Lambda_{WW}(l) e^{-j\omega l}$$

is

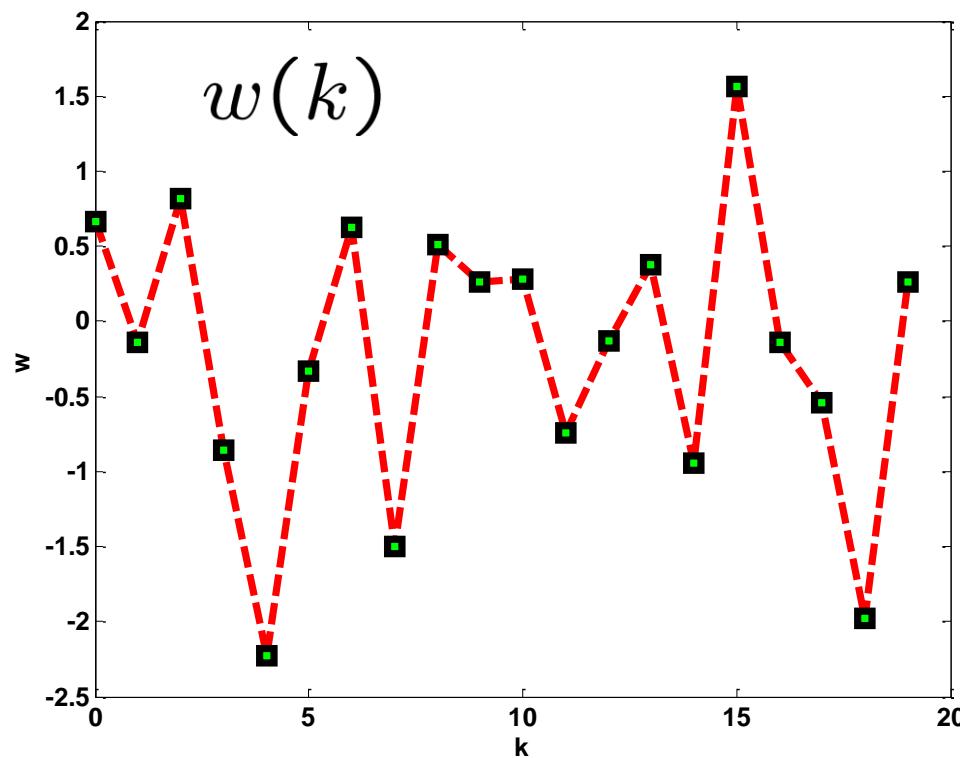
$$\Phi_{WW}(\omega) = \Sigma_{WW}$$

# White noise illustration (scalar case)

- zero-mean white noise  $W(k)$

Matlab commands:

**w = randn(N,1);**



first 20 samples

# MIMO Linear Time Invariant Systems

Let  $\{g(k)\}_{k=-\infty}^{\infty}$  with  $g(k) \in \mathcal{R}^{p \times m}$

be the pulse response of an **asymptotically stable**  
MIMO LTI system

Transfer function

$$G(z) = \mathcal{Z}\{g(k)\} = \sum_{k=-\infty}^{\infty} g(k) z^{-k}$$

# MIMO Linear Time Invariant Systems

Let  $U(k) \in \mathcal{R}^m$  be WSS

The forced response (zero initial state)

$$Y(k) = \sum_{i=-\infty}^{\infty} g(i)U(k-i)$$

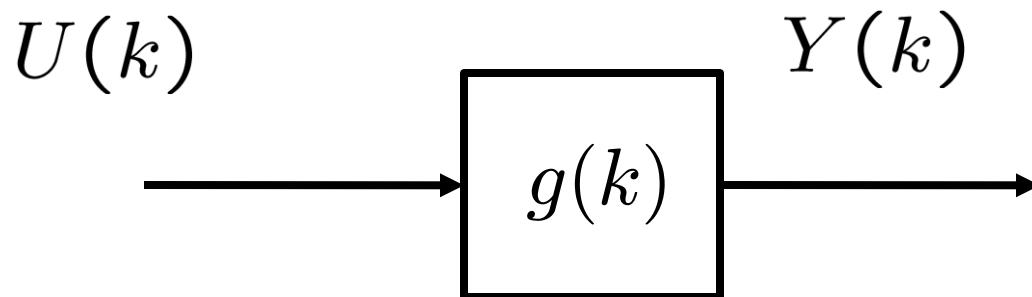
$$g(k) \in \mathcal{R}^{p \times m}$$

$Y(k) \in \mathcal{R}^p$  is also WSS

# MIMO Linear Time Invariant Systems

Let  $U(k) \in \mathcal{R}^m$  be WSS

$$Y(k) = \sum_{i=-\infty}^{\infty} g(i)U(k-i)$$



# MIMO Linear Time Invariant Systems

We will assume

$\{U(k)\}_{k=-\infty}^{\infty}$  is zero mean, i.e.

$$E \{U(k)\} = m_U = 0$$

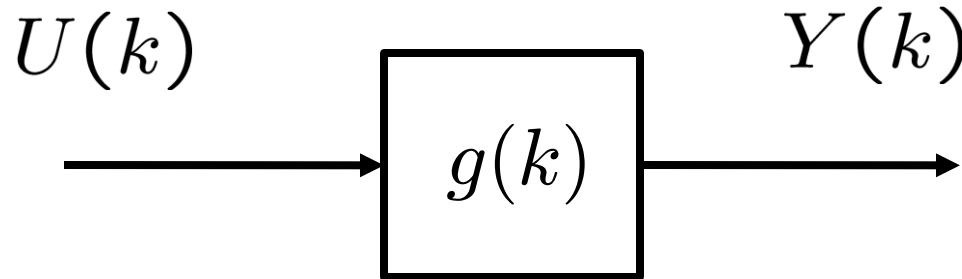
Thus, the forced response output is also zero mean

$$E \{Y(k)\} = m_Y = 0$$

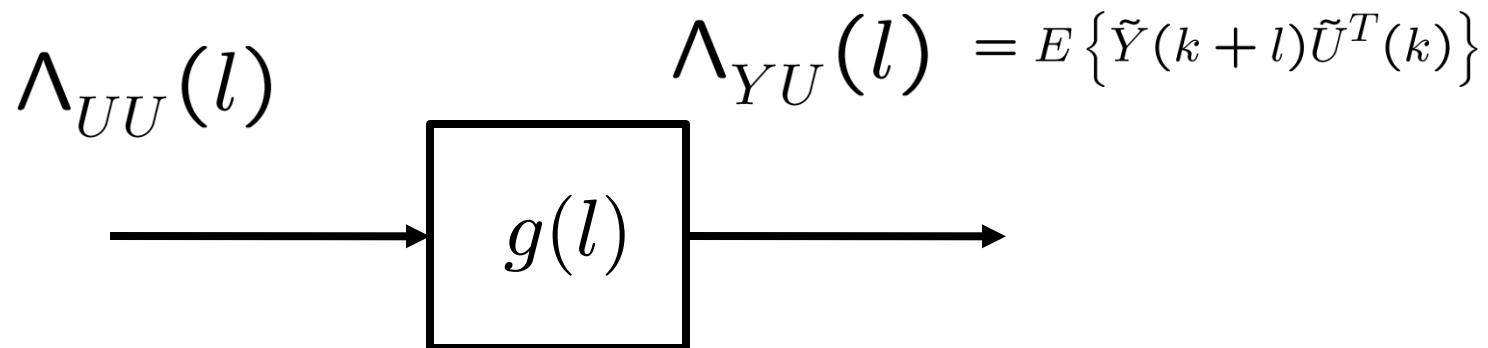
# MIMO Linear Time Invariant Systems

Let  $\{U(k)\}_{k=-\infty}^{\infty}$  be WSS

If



Then:



# MIMO Linear Time Invariant Systems

$$\Lambda_{YU}(l) = \sum_{i=-\infty}^{\infty} g(i) \Lambda_{UU}(l-i)$$

**Proof:**  $Y(k) = \sum_{i=-\infty}^{\infty} g(i)U(k-i) \quad (m_U = 0)$

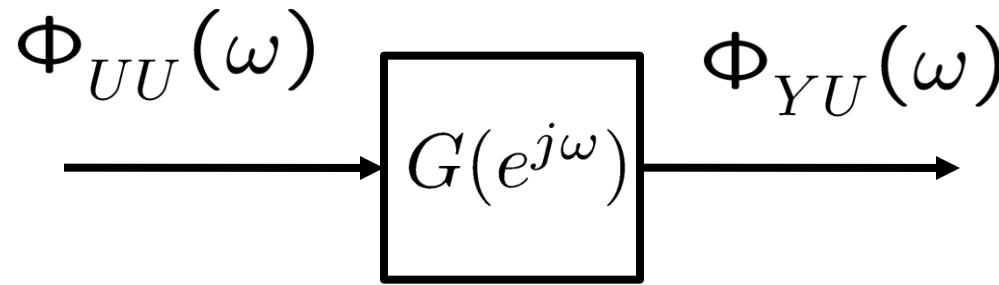
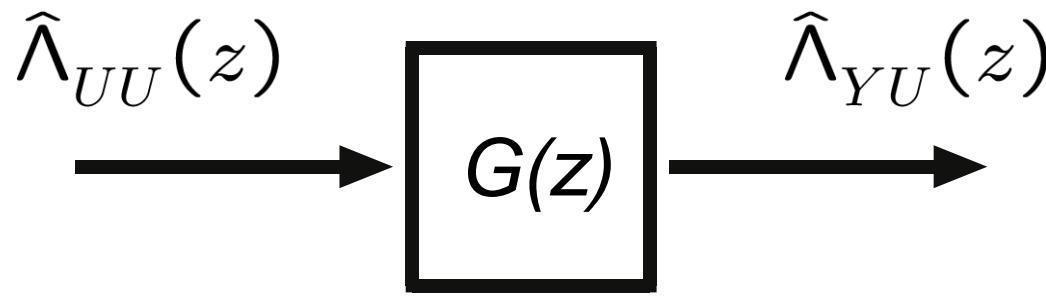
Then:

$$\begin{aligned}
 \Lambda_{YU}(l) &= E\{Y(k+l)U^T(k)\} \\
 &= E\left\{\left[\sum_{i=-\infty}^{\infty} g(i) U(k+l-i)\right] U^T(k)\right\} \\
 &= \sum_{i=-\infty}^{\infty} g(i) E\{U(k+l-i)U^T(k)\} \\
 &= \sum_{i=-\infty}^{\infty} g(i) \Lambda_{UU}(l-i)
 \end{aligned}$$



# MIMO Linear Time Invariant Systems

Let  $\{U(k)\}_{k=-\infty}^{\infty}$  be WSS



$$\hat{\Lambda}_{UU}(z) = \mathcal{Z}\{\Lambda_{UU}(l)\}$$

$$\hat{\Lambda}_{YU}(z) = \mathcal{Z}\{\Lambda_{YU}(l)\}$$

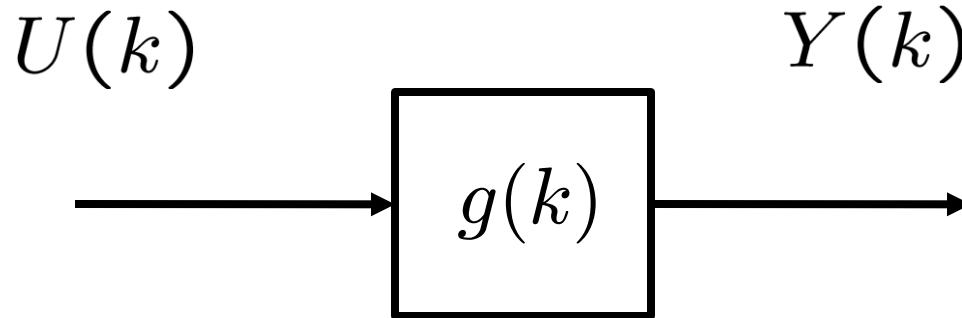
$$\Phi_{UU}(\omega) = \hat{\Lambda}_{UU}(z) \Big|_{z=e^{j\omega}}$$

$$\Phi_{YU}(\omega) = \hat{\Lambda}_{YU}(z) \Big|_{z=e^{j\omega}}$$

# MIMO Linear Time Invariant Systems

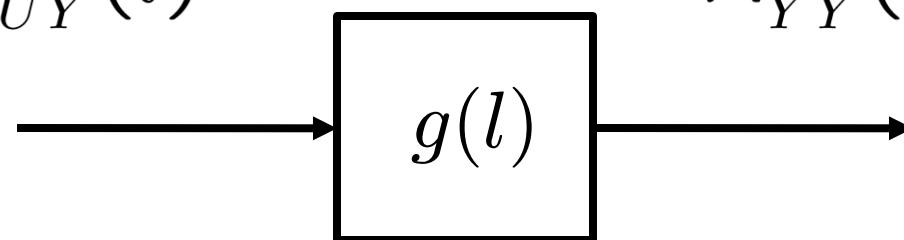
Let  $\{U(k)\}_{k=-\infty}^{\infty}$  be WSS

If



Then:

$$E \left\{ \tilde{U}(k+l) \tilde{Y}^T(k) \right\} = \Lambda_{UY}(l) \quad \Lambda_{YY}(l)$$



# MIMO Linear Time Invariant Systems

$$\Lambda_{YY}(l) = \sum_{i=-\infty}^{\infty} g(i) \Lambda_{UY}(l-i)$$

**Proof:**

$$Y(k) = \sum_{i=-\infty}^{\infty} g(i) U(k-i) \quad (m_U = 0)$$

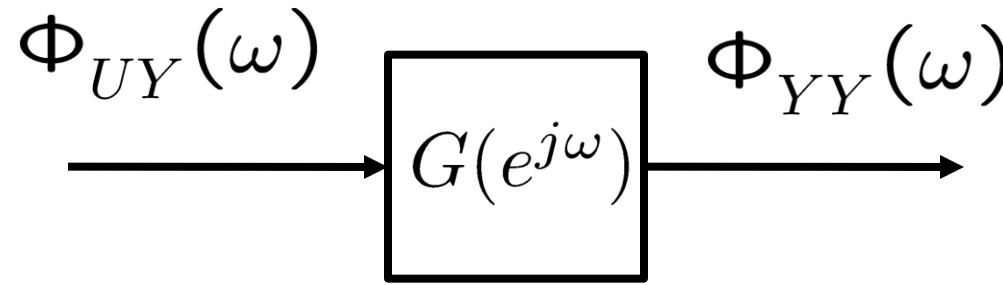
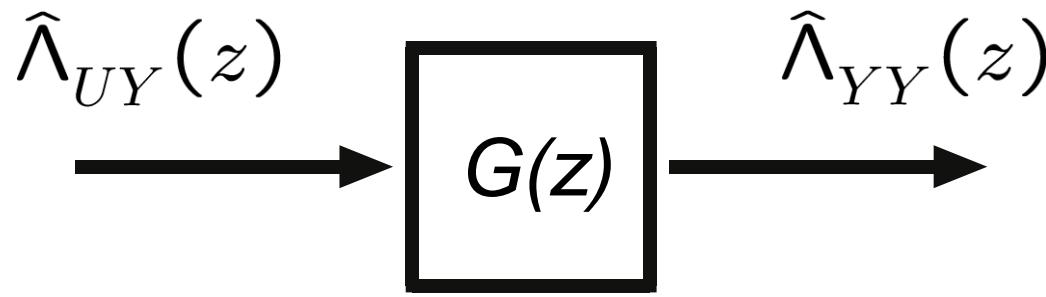
Then:

$$\begin{aligned}
 \Lambda_{YY}(l) &= E\{Y(k+l)Y^T(k)\} \\
 &= E\left\{\left[\sum_{i=-\infty}^{\infty} g(i) U(k+l-i)\right] Y^T(k)\right\} \\
 &= \sum_{i=-\infty}^{\infty} g(i) E\{U(k+l-i)Y^T(k)\} \\
 &= \sum_{i=-\infty}^{\infty} g(i) \Lambda_{UY}(l-i)
 \end{aligned}$$



# MIMO Linear Time Invariant Systems

Let  $\{U(k)\}_{k=-\infty}^{\infty}$  be WSS



$$\hat{\Lambda}_{UY}(z) = \mathcal{Z}\{\Lambda_{UY}(l)\}$$

$$\Phi_{UY}(\omega) = \hat{\Lambda}_{UY}(z) \Big|_{z=e^{j\omega}}$$

$$\hat{\Lambda}_{YY}(z) = \mathcal{Z}\{\Lambda_{YY}(l)\}$$

$$\Phi_{YY}(\omega) = \hat{\Lambda}_{YY}(z) \Big|_{z=e^{j\omega}}$$

# MIMO Linear Time Invariant Systems

$$\Phi_{UY}(\omega) = \Phi_{YU}^T(-\omega)$$

This is a consequence of the fact that

$$\Lambda_{UY}(l) = \Lambda_{YU}^T(-l)$$

# MIMO Linear Time Invariant Systems

$$\Phi_{UY}(\omega) = \Phi_{YU}^T(-\omega)$$

**Proof:**

$$\Phi_{YU}^T(-\omega) = \left[ \sum_{l=-\infty}^{\infty} \Lambda_{YU}(l) e^{j\omega l} \right]^T = \sum_{l=-\infty}^{\infty} \Lambda_{YU}^T(l) e^{j\omega l}$$

Define  $n := -l$

$$\begin{aligned} \Phi_{YU}^T(-\omega) &= \sum_{n=-\infty}^{\infty} \Lambda_{YU}^T(-n) e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} \Lambda_{UY}(n) e^{-j\omega n} = \Phi_{UY}(\omega) \end{aligned}$$



# MIMO Linear Time Invariant Systems

If

$$Y(k) = \sum_{i=-\infty}^{\infty} g(i)U(k-i)$$

Then:

$$\hat{\Lambda}_{YY}(z) = G(z) \hat{\Lambda}_{UU}(z) G^T(z^{-1})$$

# MIMO Linear Time Invariant Systems

$$\widehat{\Lambda}_{YY}(z) = G(z) \widehat{\Lambda}_{UU}(z) G^T(z^{-1})$$

**Proof:**

$$\begin{aligned}\widehat{\Lambda}_{YY}(z) &= G(z) \widehat{\Lambda}_{UY}(z) \\&= G(z) [\widehat{\Lambda}_{YU}(z^{-1})]^T \\&= G(z) [G(z^{-1}) \widehat{\Lambda}_{UU}(z^{-1})]^T \\&= G(z) \widehat{\Lambda}_{UU}^T(z^{-1}) G^T(z^{-1}) \\&= G(z) \widehat{\Lambda}_{UU}(z) G^T(z^{-1})\end{aligned}$$

# MIMO Linear Time Invariant Systems

If

$$Y(k) = \sum_{i=-\infty}^{\infty} g(i)U(k-i)$$

Then:

$$\Phi_{YY}(\omega) = G(e^{j\omega}) \Phi_{UU}(\omega) G^*(e^{j\omega})$$

# MIMO Linear Time Invariant Systems

$$\Phi_{YY}(\omega) = G(e^{j\omega}) \Phi_{UU}(\omega) G^*(e^{j\omega})$$

**Proof:**

$$\widehat{\Lambda}_{YY}(z) = G(z) \widehat{\Lambda}_{UU}(z) G^T(z^{-1})$$

Let  $z = e^{j\omega}$

$$\widehat{\Lambda}_{YY}(e^{j\omega}) = G(e^{j\omega}) \widehat{\Lambda}_{UU}(e^{j\omega}) G^T(e^{-j\omega})$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \Phi_{YY}(\omega) & \Phi_{UU}(\omega) & G^*(e^{j\omega}) \end{array}$$

# Next Topic

- Stable causal LTI systems driven by  
uncorrelated random vector sequences
  - *Similar to “white”*
  - *Definition in 2 slides*
- State-space
- No WSS assumption

# 2<sup>nd</sup> order statistics of a random sequence

We now consider one-sided random sequence

$$\{X(k)\}_{\underline{k=0}}^{\infty}$$

**Expected value or mean of  $X(k)$ ,**

$$E \{X(k)\} = m_X(k)$$

**Auto-covariance function:**

$$\Lambda_{XX}(k, j) =$$

$$E \left\{ [X(\underline{k+j}) - m_X(\underline{k+j})] [X(\underline{k}) - m_X(\underline{k})]^T \right\}$$

# Uncorrelated random vector sequence

A random vector sequence  $\{W(k)\}_{k=-\infty}^{\infty}$  is **uncorrelated** if:

$$\Lambda_{WW}(k, l) = \Sigma_{WW}(k) \delta(l)$$

where

$$\delta(l) = \begin{cases} 1 & l = 0 \\ 0 & l \neq 0 \end{cases}$$

$$\Sigma_{WW}(k) = E\{\tilde{W}(k)\tilde{W}^T(k)\} \quad \tilde{W}(k) = W(k) - m_W(k)$$

$$\Sigma_{WW}(k) = \Sigma_{WW}^T(k) \succeq 0$$

# Subtracting the mean

- Define

$$\tilde{X}(k) = X(k) - m_X(k)$$

Auto-covariance

$$\Lambda_{XX}(k, j) = E \left\{ \tilde{X}(k + j) \tilde{X}^T(k) \right\}$$

# State space systems

Consider a LTI system driven by an uncorrelated RVS:

$$X(k+1) = A X(k) + B W(k)$$

$$Y(k) = C X(k)$$

$$X(k) \in \mathcal{R}^n$$

$$W(k) \in \mathcal{R}^p$$

$$Y(k) \in \mathcal{R}^m$$

# State space systems

$W(k)$  is an uncorrelated RVS

$$m_W(k) = E\{W(k)\}$$

$$\Lambda_{WW}(k, l) = \Sigma_{WW}(k) \delta(l)$$

$$\delta(l) = \begin{cases} 1 & l = 0 \\ 0 & l \neq 0 \end{cases}$$

$$\Sigma_{WW}(k) = E\{\tilde{W}(k)\tilde{W}^T(k)\} \in \mathcal{R}^{p \times p}$$

# State space systems

$$X(k+1) = A X(k) + B W(k)$$

$$Y(k) = C X(k)$$

State Initial Conditions (IC):

$$m_X(0) = E\{X(0)\}$$

$$\Lambda_{XX}(0,0) = E\{\tilde{X}(0)\tilde{X}^T(0)\}$$

$$E\{\tilde{X}(0)\tilde{W}^T(k)\} = 0, \quad \forall k \geq 0$$

# Dynamics of the mean

$$X(k+1) = A X(k) + B W(k)$$

$$Y(k) = C X(k)$$

Taking expectations on the equations above:

$$m_X(k+1) = A m_X(k) + B m_W(k)$$

$$m_Y(k) = C m_X(k)$$

# State space systems

Subtracting the means we obtain,

$$\tilde{X}(k+1) = A \tilde{X}(k) + B \tilde{W}(k)$$

$$\tilde{Y}(k) = C \tilde{X}(k)$$

Where now

$$m_{\tilde{W}}(k) = 0$$

$$m_{\tilde{X}}(k) = 0$$

# Causality in cross-covariance

$$E\{\tilde{W}(k+j)\tilde{X}^T(k)\} = 0 \quad \forall j \geq 0, k \geq 0$$

**Proof:** (by induction on  $k$ )

1. Base case,  $k=0$ : trivial by assumptions on system
2. Case  $k>0$ :

$$\begin{aligned}
 E\{\tilde{W}(k+j)\tilde{X}^T(k)\} &= E\{\tilde{W}(k+j)[A\tilde{X}(k-1) + B\tilde{W}(k-1)]^T\} \\
 &= E\{\tilde{W}(k+j)\tilde{X}^T(k-1)\}A^T \\
 &\quad + \cancel{E\{\tilde{W}(k+j)\tilde{W}^T(k-1)\}B^T} \xrightarrow{\text{Red}} \mathbf{0} \\
 &= 0 \quad (\text{by induction hypothesis})
 \end{aligned}$$

# Covariance propagation

$$\tilde{X}(k+1) = A \tilde{X}(k) + B \tilde{W}(k)$$

Notice that:

$$\tilde{X}(k+1) \tilde{X}^T(k+1) =$$

$$\left[ A \tilde{X}(k) + B \tilde{W}(k) \right] \left[ A \tilde{X}(k) + B \tilde{W}(k) \right]^T$$

# Covariance propagation

Taking expectations to:

$$\begin{aligned}
 \tilde{X}(k+1)\tilde{X}^T(k+1) &= A\tilde{X}(k)\tilde{X}^T(k)A^T \\
 \underbrace{\Lambda_{XX}(k+1, 0)} &\quad + A\tilde{X}(k)\tilde{W}^T(k)B^T \\
 &\quad + B\tilde{W}(k)\tilde{X}^T(k)A^T \\
 &\quad + B\tilde{W}(k)\tilde{W}^T(k)B^T
 \end{aligned}$$

# Covariance propagation

Notice that:

$$\begin{aligned}
 \Lambda_{XX}(k+1, 0) &= A \Lambda_{XX}(k, 0) A^T \\
 &\quad + A \Lambda_{XW}(k, 0) B^T \xrightarrow{\text{0}} \\
 &\quad + B \Lambda_{WX}(k, 0) A^T \xrightarrow{\text{0}} \\
 &\quad + B \Lambda_{WW}(k, 0) B^T
 \end{aligned}$$

( $W(k)$  is an uncorrelated RVS)

$$\begin{aligned}
 \Lambda_{XW}(k, 0) &= \Lambda_{WX}^T(k, 0) \\
 &= E \left\{ \tilde{X}(k) \tilde{W}^T(k) \right\} = 0
 \end{aligned}$$

# Covariance propagation

We obtain the following Lyapunov equation:

$$\Lambda_{XX}(k+1, 0) = A \Lambda_{XX}(k, 0) A^T + B \Sigma_{WW}(k) B^T$$

$$\Lambda_{XX}(k, 0) = E \left\{ \tilde{X}(k) \tilde{X}^T(k) \right\}$$

$$\Lambda_{WW}(k, 0) = E \left\{ \tilde{W}(k) \tilde{W}^T(k) \right\} = \Sigma_{WW}(k)$$

# Covariance propagation

From the output equation

$$\tilde{Y}(k) = C \tilde{X}(k)$$

we obtain

$$\Lambda_{YY}(k, 0) = C \Lambda_{XX}(k, 0) C^T$$

# Covariance propagation

Lets now compute,

$$\Lambda_{XX}(k, l) = E \left\{ \tilde{X}(k + l) \tilde{X}^T(k) \right\} \quad l \geq 0$$

Using the solution of the LTI system,

$$\tilde{X}(k + l) = A^l \tilde{X}(k) + \sum_{j=k}^{k+l-1} A^{k+l-1-j} B \tilde{W}(j)$$

# Covariance propagation

$$\tilde{X}(k+l) = A^l \tilde{X}(k) + \sum_{j=k}^{k+l-1} A^{k+l-1-j} B \tilde{W}(j)$$

$$\Lambda_{XX}(k, l) = E \left\{ \tilde{X}(k+l) \tilde{X}^T(k) \right\}$$

$$= A^l E \{ \tilde{X}(k) \tilde{X}^T(k) \}$$

$$+ \sum_{j=k}^{k+l-1} A^{k+l-1-j} B E \{ \tilde{W}(j) \tilde{X}^T(k) \} \xrightarrow{\text{0}}$$

$$= A^l \Lambda_{XX}(k, 0)$$

# Covariance propagation

Lets now compute

$$\begin{aligned}\Lambda_{XX}(k, -l) &= E \left\{ \tilde{X}(k-l) \tilde{X}^T(k) \right\} \quad l \geq 0 \\ &= E \{ \tilde{X}(k) \tilde{X}^T(k-l) \}^T\end{aligned}$$

define  $\hat{k} := k - l$

$$\begin{aligned}\Lambda_{XX}(k, -l) &= E \{ \tilde{X}(\hat{k} + l) \tilde{X}^T(\hat{k}) \}^T \\ &= \Lambda_{XX}^T(\hat{k}, l) = [A^l \Lambda_{XX}(k - l, 0)]^T \\ &= \Lambda_{XX}(k - l, 0) (A^l)^T\end{aligned}$$

# Covariance propagation

$$\Lambda_{XX}(k, l) = E \left\{ \tilde{X}(k + l) \tilde{X}^T(k) \right\}$$

Satisfies:

$$\Lambda_{XX}(k, l) = A^l \Lambda_{XX}(k, 0) \quad l \geq 0$$

$$\Lambda_{XX}(k, -l) = \Lambda_{XX}(k - l, 0) (A^l)^T \quad l \geq 0$$

# Stationary covariance equation

If  $W(k)$  is WSS

and  $A$  is Schur (i.e. all eigenvalues inside unit circle):

and  $X(k)$  and  $Y(k)$  will converge to WSS RVS:

$$\lim_{k \rightarrow \infty} m_X(k) = \bar{m}_X$$

$$\lim_{k \rightarrow \infty} m_Y(k) = C\bar{m}_X$$

$$\lim_{k \rightarrow \infty} \Lambda_{XX}(k, 0) = \bar{\Lambda}_{XX}(0)$$

$$\lim_{k \rightarrow \infty} \Lambda_{YY}(k, 0) = \bar{\Lambda}_{YY}(0)$$

$$= C\bar{\Lambda}_{XX}(0)C^T$$

## WSS Stationary covariance equation

For  $W(k)$  WSS, and  $A$  Schur,

$$m_X(k+1) = A m_X(k) + B m_W$$

converges to

$$\bar{m}_X = [I - A]^{-1} B m_W$$

# WSS Stationary covariance equation

For  $W(k)$  WSS, and  $A$  Schur,

$$\bar{\Lambda}_{XX}(0) = \lim_{k \rightarrow \infty} E\{\tilde{X}(k)\tilde{X}^T(k)\}$$

Satisfies the Lyapunov equation:

$$A \bar{\Lambda}_{XX}(0) A^T - \bar{\Lambda}_{XX}(0) = -B \Sigma_{WW} B^T$$

# WSS Stationary covariance equation

For  $W(k)$  WSS, and  $A$  Schur,

$$\bar{\Lambda}_{XX}(l) = \lim_{k \rightarrow \infty} E\{\tilde{X}(k+l)\tilde{X}^T(k)\}$$

Satisfies

$$\bar{\Lambda}_{XX}(l) = A^l \bar{\Lambda}_{XX}(0) \quad l \geq 0$$

$$\bar{\Lambda}_{XX}(-l) = \bar{\Lambda}_{XX}(0)(A^l)^T$$

# Illustration – first order system

- Plant:

$$Y(k+1) = 0.5 Y(k) + 1 W(k)$$

- Input:

$$m_W(k) = 1 \quad \Lambda_{WW}(k, l) = 0.2 \delta(l)$$

- State initial conditions:

$$m_Y(0) = 0 \quad \Lambda_{YY}(0, 0) = .1$$

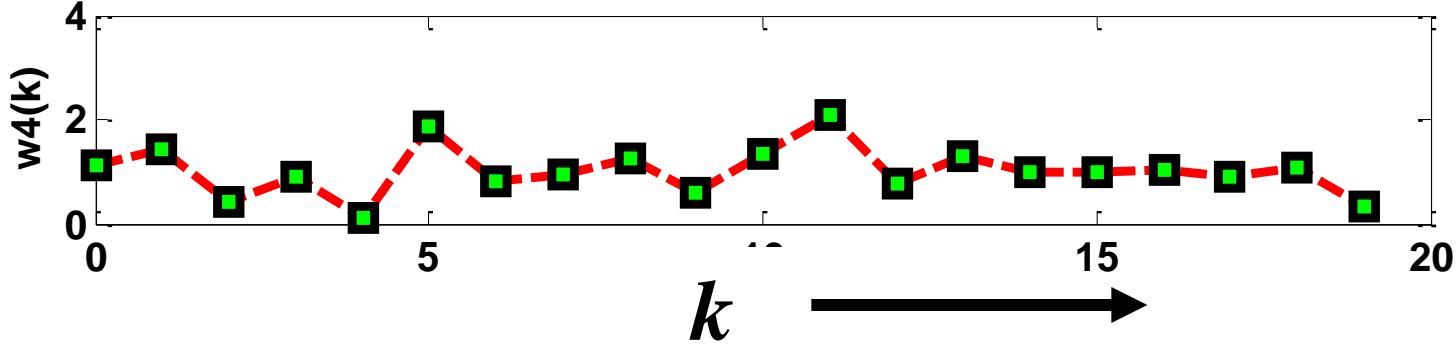
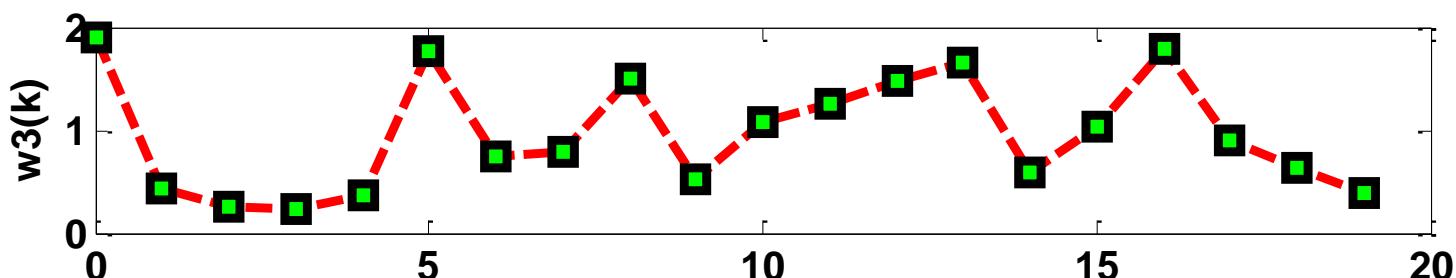
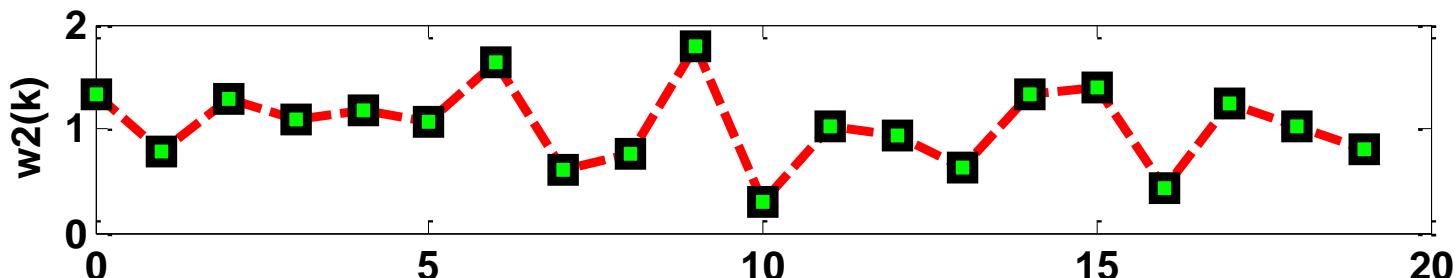
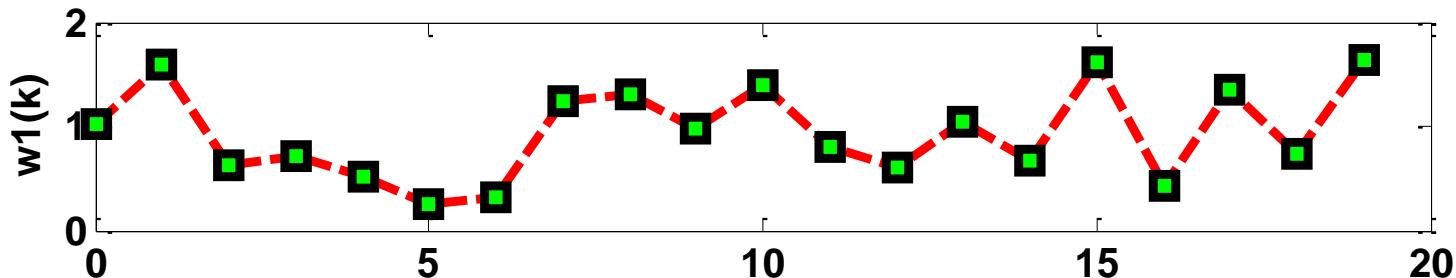
# Matlab simulation: 500 sample sequences

```
lyy0 = 0.1
lww = 0.2
sys1=ss(.5,1,1,0,1)
N=20;
p=500;
w = sqrt(lww)*randn(N,p)+1;
y = zeros(N,p);
y0 = sqrt(lyy0)*randn(1,p);
k = (0:1:N-1)';

for j=1:p
    [y(:,j),k] = lsim(sys1,w(:,j),k,y0(1,j));
end

m_y=mean(y')
L_yy=diag(cov(y'));
```

$$W(k) \quad m_W(k) = 1 \quad \Lambda_{WW}(k, l) = 0.2 \delta(l)$$

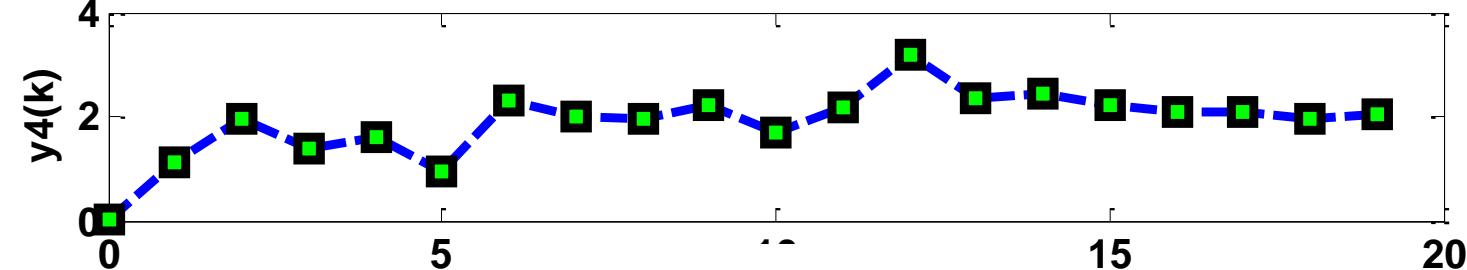
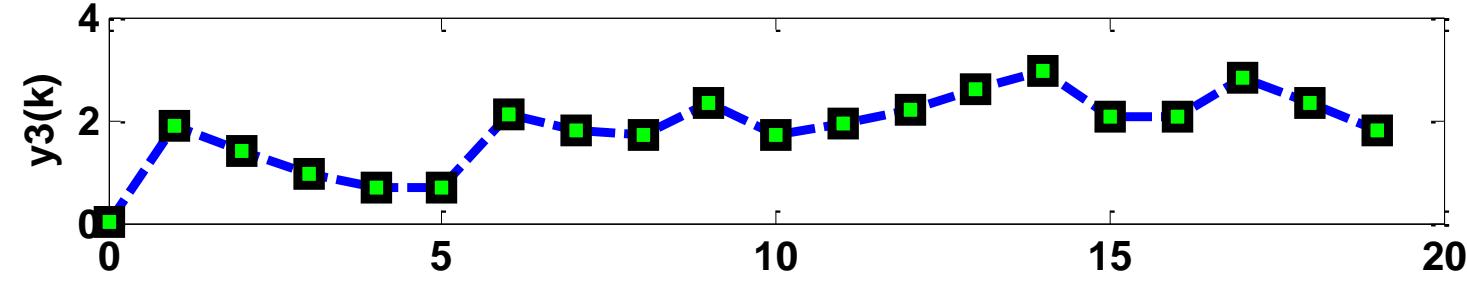
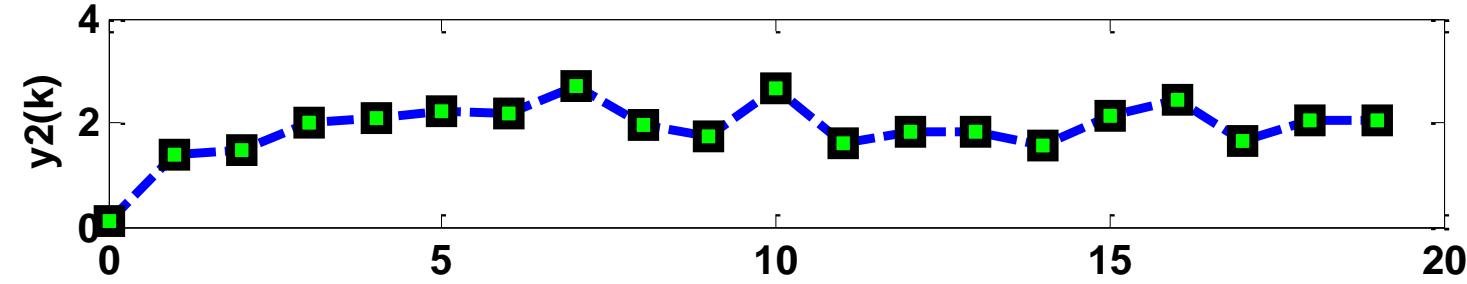
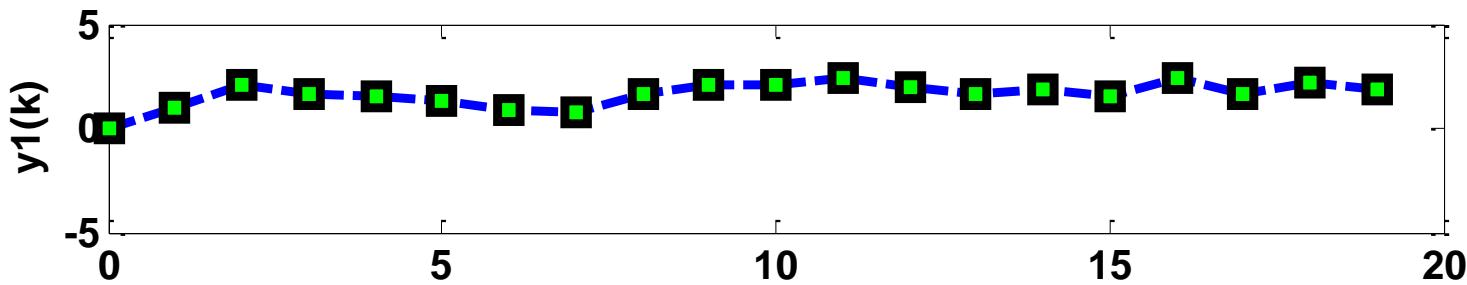


*4 out of 500*

*Ensemble*

$k$

$$Y(k) \quad m_Y(0) = 0 \quad \Lambda_{YY}(0,0) = .1$$

 $k$ 

4 out of 500

Ensemble

# Mean Transient Response

**Actual:**

$$m_Y(k+1) = 0.5 m_Y(k) + 1$$

$$m_Y(0) = 0$$

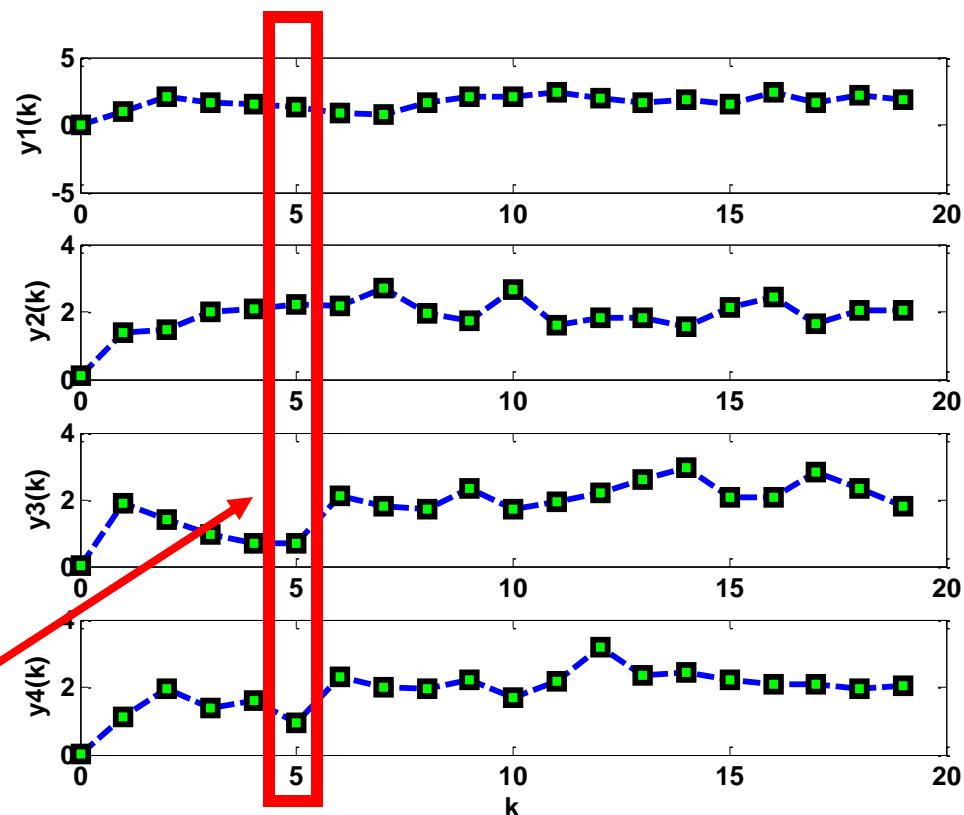
**Matlab calculation:**

Ensemble mean

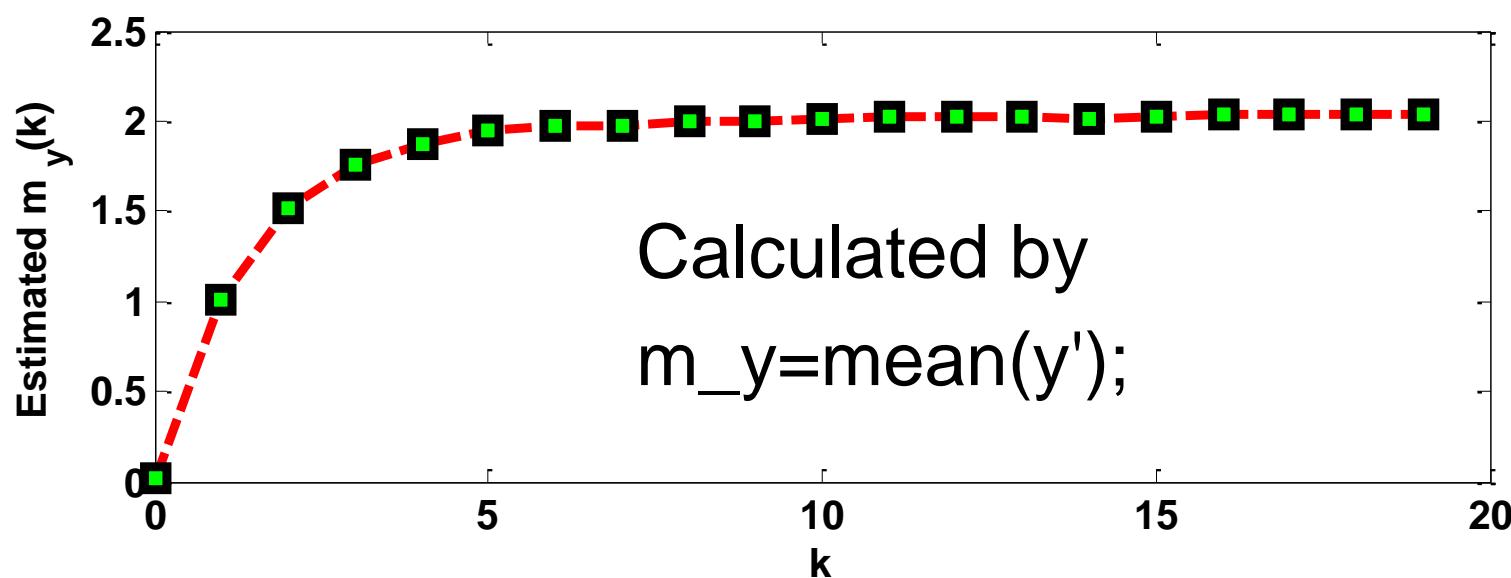
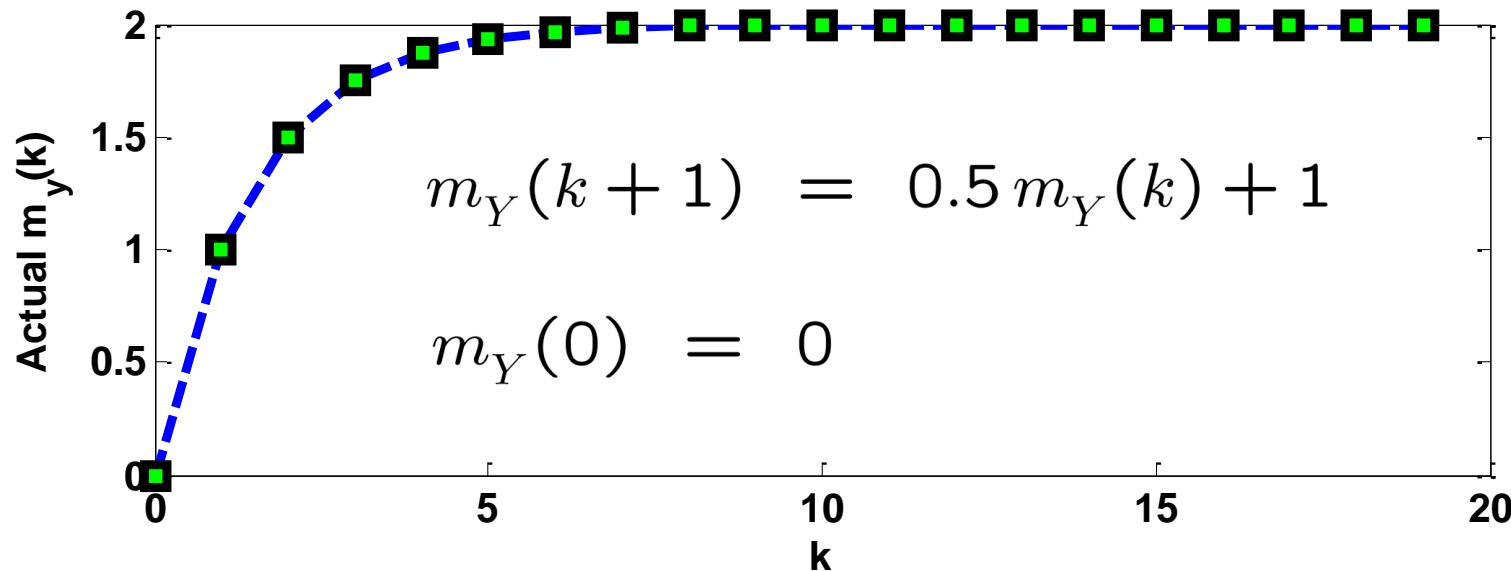
`m_y=mean(y');`

*Not using ergodicity  
because not WSS!!*

$$\approx m_Y(5)$$



# Mean Transient Response



# Covariance Transient Response

**Actual:**

$$\Lambda_{XX}(k+1, 0) = 0.5^2 \Lambda_{XX}(k, 0) + 0.2$$

$$\Lambda_{XX}(0, 0) = .1$$

$$\Lambda_{WW}(k, l) = 0.2 \delta(l)$$

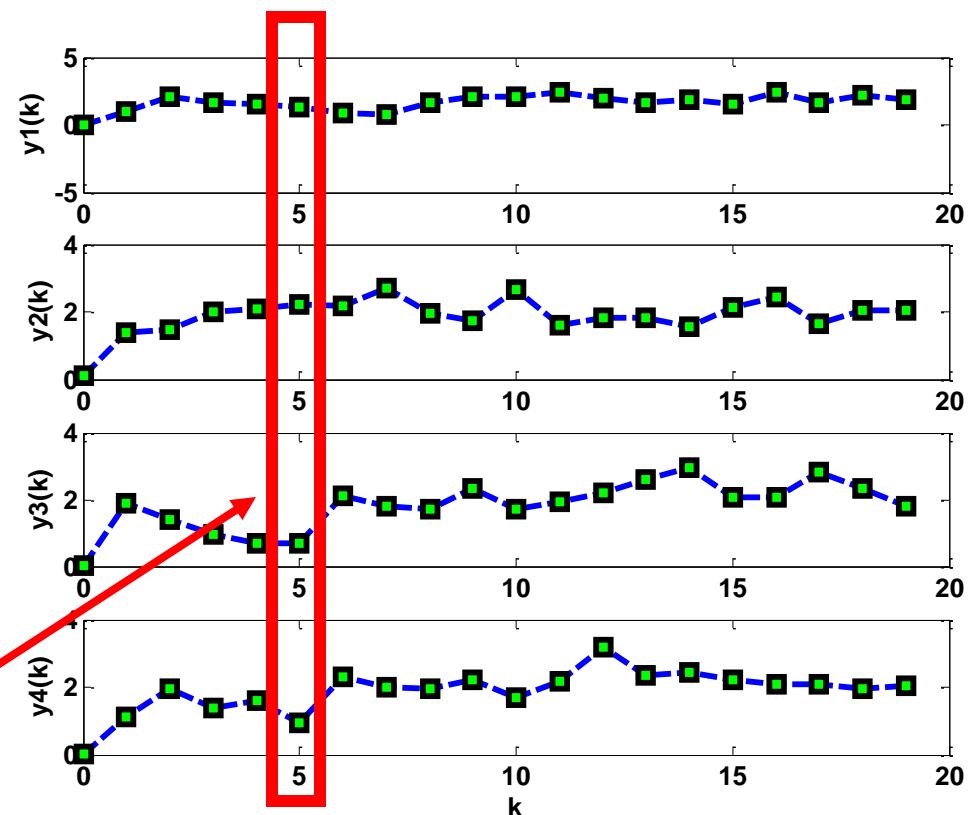
**Matlab calculation:**

Ensemble covariance

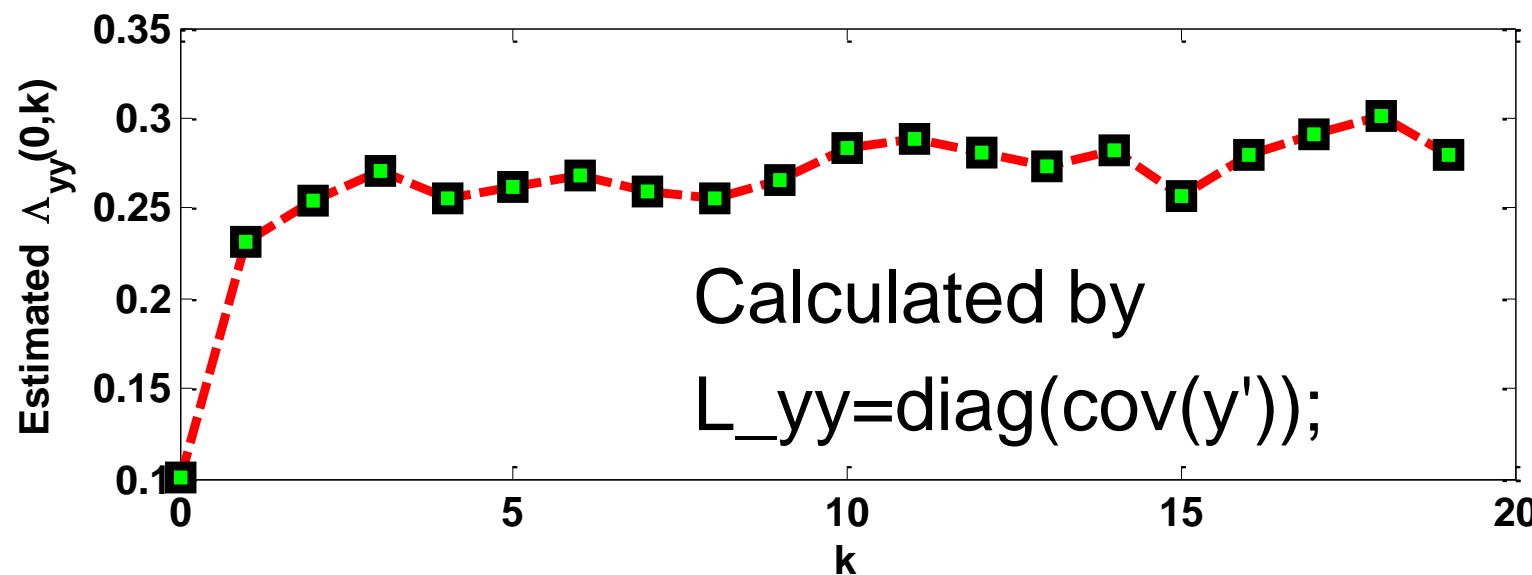
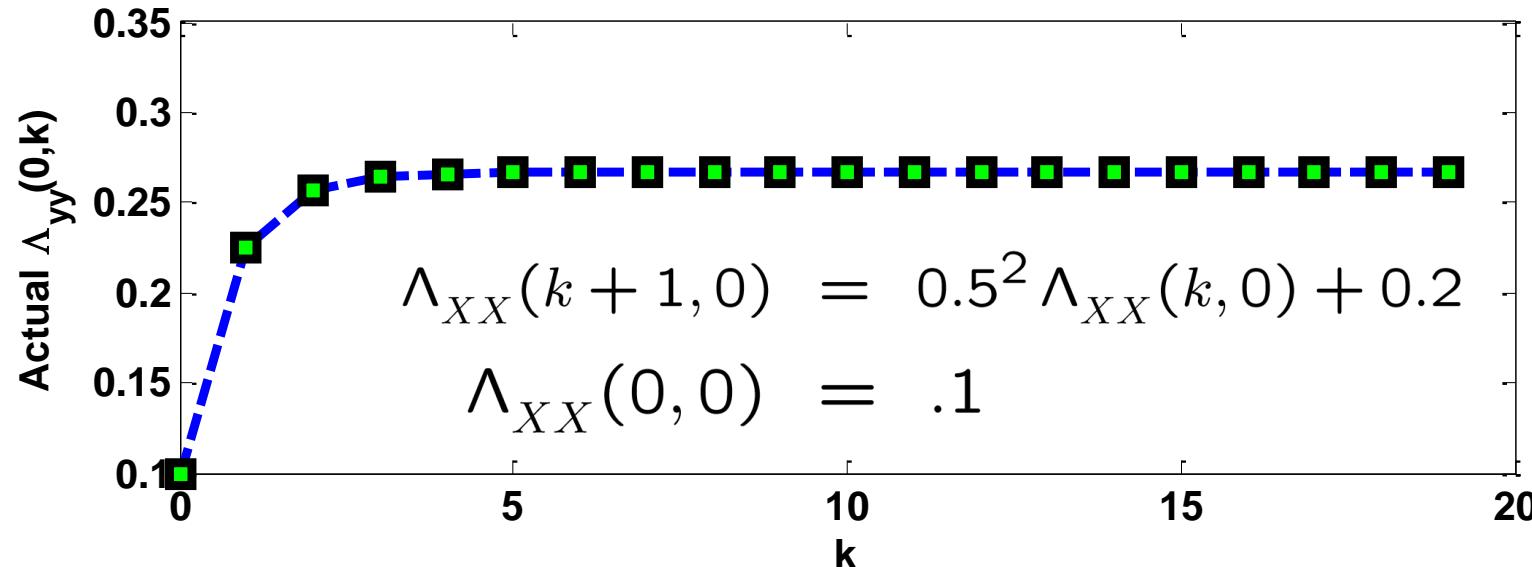
`L_yy=diag(cov(y'));`

*Not using ergodicity  
because not WSS!!*

$$\approx \Lambda_{YY}(5, 0)$$



# Covariance Transient Response



# Steady State Covariance

