ME 233 Advanced Control II

Lecture 21

Stability Analysis of a discrete-time Series-Parallel Least Squares Parameter Identification Algorithm

ARMA Model

Consider the following system

$$A(q^{-1})y(k) = q^{-d}\, B(q^{-1})u(k)$$

Where

- $u(k)$ known ***bounded*** input
- $y(k)$ measured output

ARMA Model

$$A(q^{-1})y(k) = q^{-d}\, B(q^{-1})u(k)$$

$$A(q^{-1}) = 1 + a_1 q^{-1} + \cdots + a_n q^{-n}$$ **(anti-Schur)**

$$B(q^{-1}) = b_o + b_1 q^{-1} + \cdots + b_m q^{-m}$$

- Orders n and m are ***known***
- a's and b's are ***unknown*** but ***constant*** coefficients

ARMA Model

ARMA model can be written as:

$$\begin{aligned} y(k+1) &= -\sum_{i=1}^{n} a_i\, y(k-i+1) + \sum_{i=0}^{m} b_i\, u(k-i-d+1) \\ &= \theta^T \phi(k) \end{aligned}$$

Where:

$$\theta = \left[\begin{array}{ccccc} a_1 & \cdots & a_n & b_o \cdots & b_m \end{array}\right]^T \in \mathcal{R}^{n+m+1}$$

$$\phi(k) = \left[-y(k) \;\; \cdots \;\; -y(k-n+1) \;\; u(k-d+1) \;\; \cdots \;\; u(k-d-m+1)\right]^T$$

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Series-parallel estimation model

A-posteriori series-parallel estimation model

$$\hat{y}(k+1) = -\sum_{i=1}^{n} \hat{a}_i(k+1)\, y(k-i+1) + \sum_{i=0}^{m} \hat{b}_i(k+1)\, u(k-i-d+1)$$

Where

- $\hat{y}(k)$ a-posteriori estimate of $y(k)$
- $\hat{a}_i(k)$ estimate of a_i at sampling time k
- $\hat{b}_i(k)$ estimate of b_i at sampling time k

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Series-parallel estimation model

A-posteriori series-parallel estimation model

$$\hat{y}(k+1) = \hat{\theta}^T(k+1)\, \phi(k)$$

Where

- $\hat{y}(k)$ a-posteriori estimate of $y(k)$

$$\hat{\theta}(k) = \begin{bmatrix} \hat{a}_1(k) & \cdots & \hat{a}_n(k) & \hat{b}_o(k) \cdots & \hat{b}_m(k) \end{bmatrix}^T$$

$$\phi(k) = \begin{bmatrix} -y(k) & \cdots & -y(k-n+1) & u(k-d+1) & \cdots & u(k-d-m+1) \end{bmatrix}^T$$

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Series-parallel estimation model

A-priori series-parallel estimation model

$$\hat{y}^o(k+1) = \hat{\theta}^T(k)\, \phi(k)$$

Where

- $\hat{y}^o(k)$ a-priori estimate of $y(k)$

$$\hat{\theta}(k) = \begin{bmatrix} \hat{a}_1(k) & \cdots & \hat{a}_n(k) & \hat{b}_o(k) \cdots & \hat{b}_m(k) \end{bmatrix}^T$$

$$\phi(k) = \begin{bmatrix} -y(k) & \cdots & -y(k-n+1) & u(k-d+1) & \cdots & u(k-d-m+1) \end{bmatrix}^T$$

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Additional Notation

- Unknown parameter vector:

$$\theta = \begin{bmatrix} a_1 & \cdots & a_n & b_o \cdots & b_m \end{bmatrix}^T \in \mathcal{R}^{n+m+1}$$

- Parameter vector estimate:

$$\hat{\theta}(k) = \begin{bmatrix} \hat{a}_1(k) & \cdots & \hat{a}_n(k) & \hat{b}_o(k) \cdots & \hat{b}_m(k) \end{bmatrix}^T$$

- **Parameter error estimate**:

$$\boxed{\tilde{\theta}(k) = \theta - \hat{\theta}(k)}$$

- Regressor vector:

$$\phi(k) = \begin{bmatrix} -y(k) & \cdots & -y(k-n+1) & u(k-d+1) & \cdots & u(k-d-m+1) \end{bmatrix}^T$$

Additional Notation

- ***A-posteriori*** output estimation error:

$$\begin{aligned} e(k) &= y(k) - \hat{y}(k) \\ &= \tilde{\theta}^T(k)\phi(k-1) \end{aligned}$$

- ***A-priori*** output estimation error:

$$\begin{aligned} e^o(\underline{k}) &= y(k) - \hat{y}^o(k) \\ &= \tilde{\theta}^T\underline{(k-1)}\phi(k-1) \end{aligned}$$

Parameter Adaptation Algorithm (PAA)

A-posteriori version

- **Parameter estimate update**

$$\hat{\theta}(k+1) = \hat{\theta}(k) + \frac{1}{\lambda_1(k)} F(k)\phi(k)e(k+1)$$

- **Gain update**

$$F^{-1}(k+1) = \lambda_1(k)\, F^{-1}(k) + \lambda_2(k)\, \phi(k)\phi^T(k)$$

- **We make the restriction**

$$0 < \lambda_1(k) \le 1 \qquad 0 \le \lambda_2(k) < 2$$

PAA Special Cases

$$\hat{\theta}(k+1) = \hat{\theta}(k) + \frac{1}{\lambda_1(k)} F(k)\phi(k)e(k+1)$$

$$F^{-1}(k+1) = \lambda_1(k)\, F^{-1}(k) + \lambda_2(k)\, \phi(k)\phi^T(k)$$

- **Least squares**

$$\lambda_1(k) = 1 \qquad \lambda_2(k) = 1$$

- **Least squares with forgetting factor**

$$0 < \lambda_1(k) < 1 \qquad \lambda_2(k) = 1$$

- **Constant gain**

$$\lambda_1(k) = 1 \qquad \lambda_2(k) = 0$$

Example

- Plant:

$$y(k) = \frac{q^{-1}\, 0.1(1 + 0.5q^{-1})}{(1 + 0.9q^{-1})(1 + 0.8q^{-1})} u(k)$$

$$y(k+1) = \theta^T \phi(k)$$

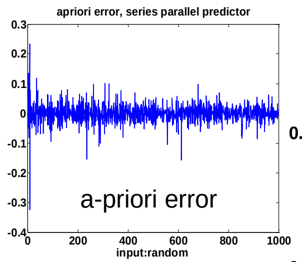
$$\theta = \begin{bmatrix} 1.7 \\ 0.72 \\ 0.1 \\ 0.05 \end{bmatrix} \qquad \phi(k) = \begin{bmatrix} -y(k) \\ -y(k-1) \\ u(k) \\ u(k-1) \end{bmatrix}$$

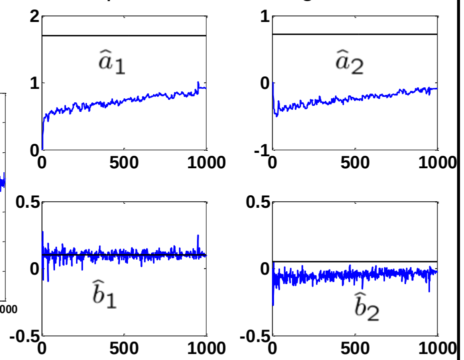
Example: Constant gain

u(k) : zero mean uniform white noise between [-1,1]

$F = 100 * I_4$

parameter convergence





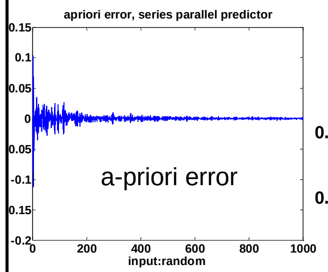
Example: Least Squares

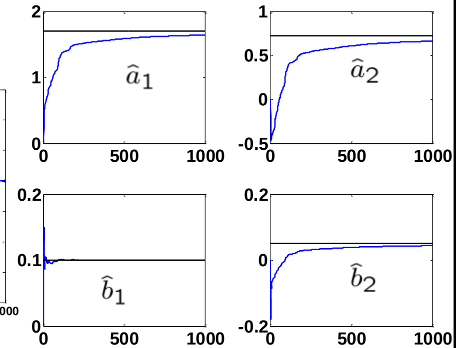
u(k) : zero mean uniform white noise between [-1,1]

$F(0) = 100 * I_4$

parameter convergence

$\lambda_1 = 1 \quad \lambda_2 = 1$





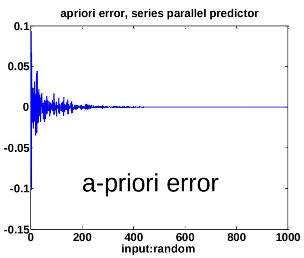
Example: Least Squares & forgetting factor

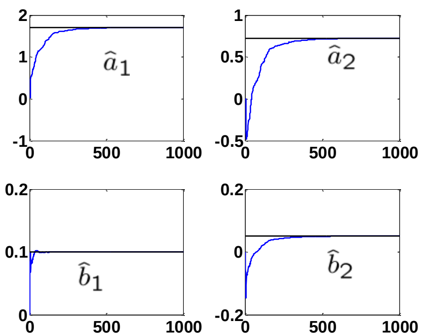
u(k) : zero mean uniform white noise between [-1,1]

$F(0) = 100 * I_4$

parameter convergence

$\lambda_1 = 0.99 \quad \lambda_2 = 1$





Theorem

Under the following conditions:

1. The input *u(k)* is bounded, i.e. $|u(k)| < \infty$
2. $A(q^{-1})$ is anti-Schur
3. Maximum singular value of *F(k)* is uniformly bounded

$$\lambda_{\max}\{F(k)\} < K_{max} < \infty \,.$$

$$\boxed{\lim_{k\to\infty} e(k) = 0 \quad \text{and} \quad \lim_{k\to\infty} e^o(k) = 0}$$

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Parameter Adaptation Algorithm (PAA)

Since the unknown parameters are constant and

$$\tilde{\theta}(k) = \theta - \hat{\theta}(k)$$

the PAA

$$\hat{\theta}(k+1) = \hat{\theta}(k) + \frac{1}{\lambda_1(k)} F(k)\phi(k)e(k+1)$$

implies that

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) - \frac{1}{\lambda_1(k)} F(k)\phi(k)e(k+1)$$

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A-posteriori dynamics

- Error dynamics

$$e(k+1) \;=\; \underbrace{\tilde{\theta}^T(k+1)\phi(k)}_{m(k+1) = -w(k+1)}$$

- PAA

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) - \frac{1}{\lambda_1(k)} F(k)\phi(k)e(k+1)$$

$$F^{-1}(k+1) = \lambda_1(k)\, F^{-1}(k) + \lambda_2(k)\, \phi(k)\phi^T(k)$$

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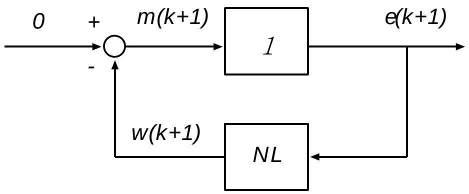
Equivalent Feedback Loop



$$m(k+1) \;=\; \tilde{\theta}^T(k+1)\phi(k) \;= e(k+1)$$

$$w(k+1) \;=\; -m(k+1)$$

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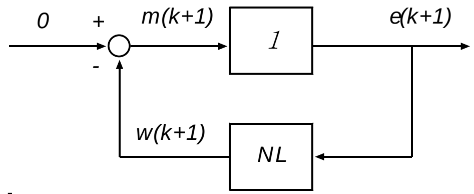
Equivalent Feedback Loop



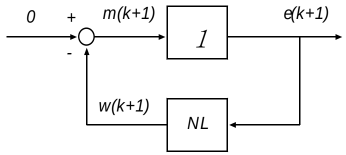
NL:

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) - \frac{1}{\lambda_1(k)} F(k)\phi(k)e(k+1)$$

$$w(k+1) \;=\; -\phi(k)^T\tilde{\theta}(k+1)$$

$$F^{-1}(k+1) = \lambda_1(k)\, F^{-1}(k) + \lambda_2(k)\, \phi(k)\phi^T(k)$$

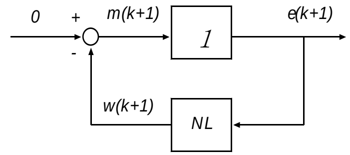
Stability analysis using Hyperstability



1. Verify that the LTI dynamics are SPR

2 Verify that the PAA dynamics are P-class

Good News: LTI "***very***" SPR



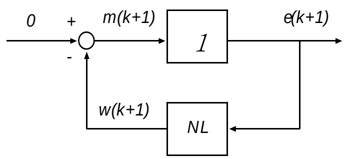
1. Verify that the LTI dynamics are SPR

$$e(k+1) = m(k+1)$$

$$\boxed{G(z) = 1}$$

Always SPR

Bad News: NL is ***not*** P-class



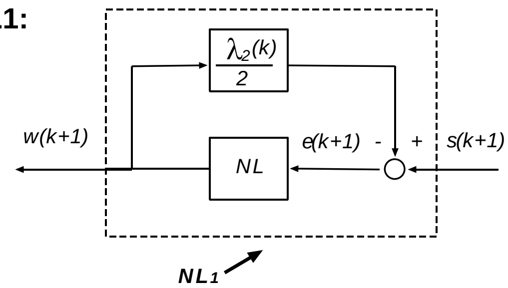
- Unfortunately the NL block is **not** P-class

NL:
$$\begin{cases} \tilde{\theta}(k+1) = \tilde{\theta}(k) - \dfrac{1}{\lambda_1(k)} F(k)\phi(k)e(k+1) \\ w(k+1) = -\phi(k)^T\tilde{\theta}(k+1) \\ F^{-1}(k+1) = \lambda_1(k)\, F^{-1}(k) + \lambda_2(k)\, \phi(k)\phi^T(k) \end{cases}$$

Solution: Modify the NL block

- **Add a feedback term to NL to make it P-class**

NL1:



$$\sum_{j=0}^{k} w(j)s(j) \geq -\gamma_o^2$$

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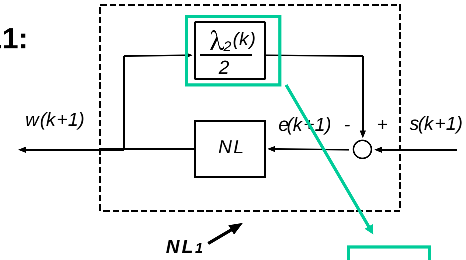
Modifying the NL block

- **Add a feedback term to NL to make it P-class**

NL1:



$$F^{-1}(k+1) = \lambda_1(k)\, F^{-1}(k) + \boxed{\lambda_2(k)}\, \phi(k)\phi^T(k)$$

Proof: See Additional Material at end of this lecture (the class notes on bSpace are incorrect)

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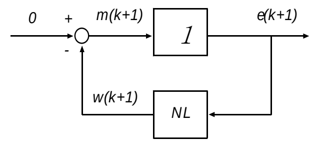
What happens to the feedback structure?

- **Add and subtract the same blocks:**



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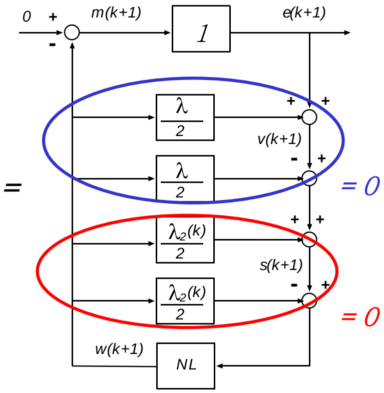

$$\lambda = \sup_k \lambda_2(k)$$

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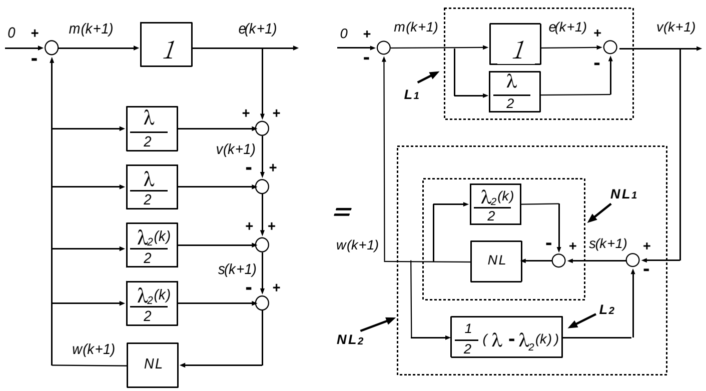
What happens to the feedback structure?

- **Rearranging blocks,**



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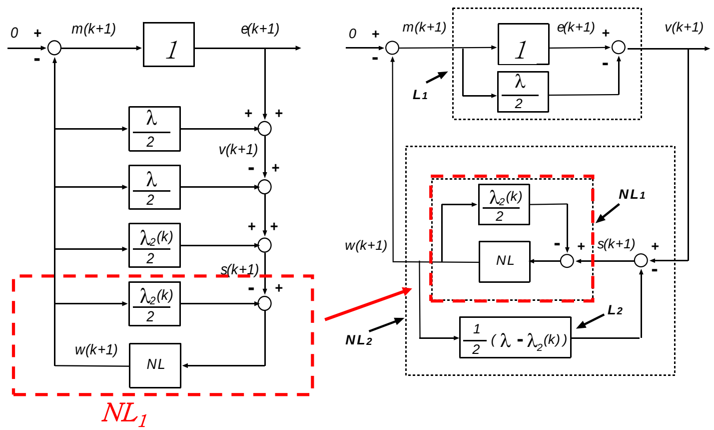
What happens to the feedback structure?

- **Rearranging blocks,**



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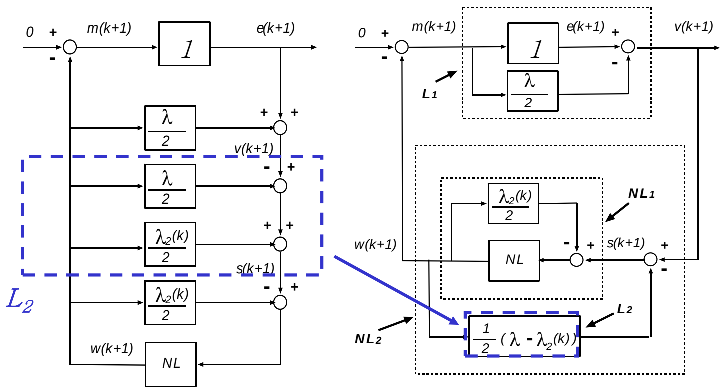
What happens to the feedback structure?

- **Rearranging blocks,**



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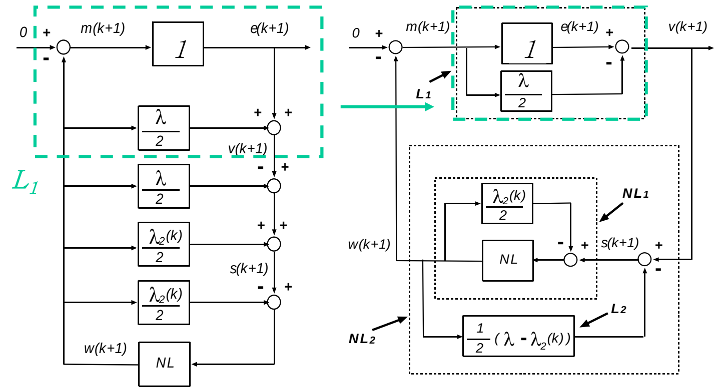
What happens to the feedback structure?

- **Rearranging blocks,**



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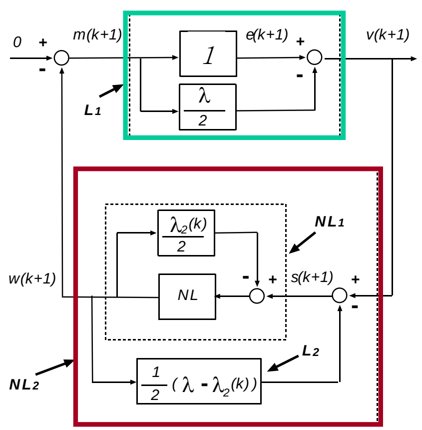
Can we now use Hyperstability Theory?



For Asymptotic Hyperstability:

1. L_1 **must be SPR**
2. NL_2 **must be P-class**

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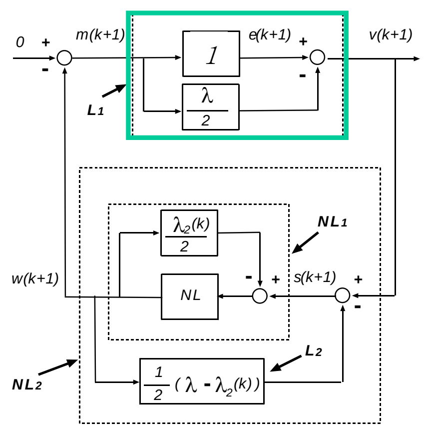
Linear Block L_1



Since:

$$L_1: \quad 1 - \frac{\lambda}{2}$$

$$\lambda = \sup_k \lambda_2(k)$$

L_1 is SPR **iff**

$$\sup_k \lambda_2(k) < 2$$

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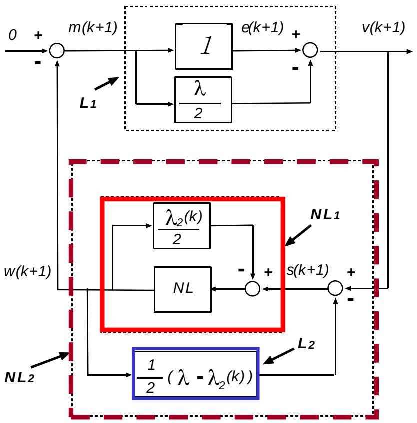
Nonlinear Block NL_2



$$\lambda = \sup_k \lambda_2(k)$$

NL_2 :

Feedback combination of two blocks:

1. NL_1 : **P-class**
2. L_2 : **P-class**

Therefore NL_2 is P-class

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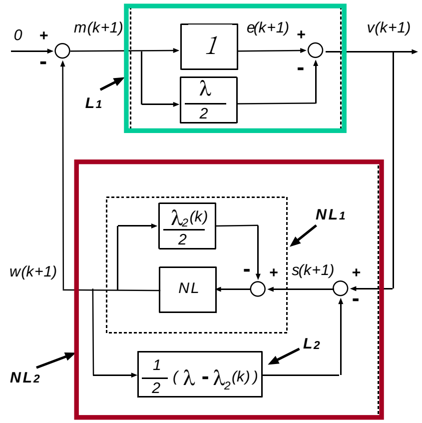
Hyperstability Theorem



If

$$\epsilon < \lambda_1(k) \le 1$$
$$0 \le \lambda_2(k) < 2 - \epsilon$$
for some $\epsilon > 0$

Then

1. L_1 **is SPR**
2. NL_2 **is P-class**

Therefore:

The interconnection is asymptotically hyperstable

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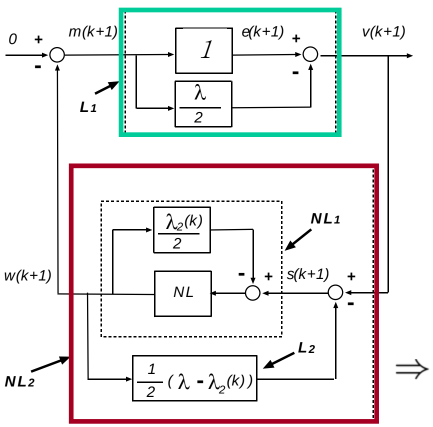
Hyperstability Theorem



If

$$\epsilon < \lambda_1(k) \le 1$$
$$0 \le \lambda_2(k) < 2 - \epsilon$$
for some $\epsilon > 0$

Then

$$v(k) \longrightarrow 0$$
$$m(k) \longrightarrow 0$$

$$\Rightarrow \boxed{\begin{array}{l} e(k) \longrightarrow 0 \\ \hat{y}(k) \longrightarrow y(k) \end{array}}$$

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A-posteriori error convergence

We have concluded that the a-posteriori output error converges to zero:

$$\lim_{k\to\infty} e(k) = 0$$

where

$$\begin{aligned} e(k) &= y(k) - \hat{y}(k) \\ &= \tilde{\theta}(k)^T \phi(k) \end{aligned}$$

What about the a-priori output error?

A-posteriori error convergence

- Notice that

$$e(k+1) = \frac{\lambda_1(k)}{\lambda_1(k) + \phi^T(k)F(k)\phi(k)} e^o(k+1)$$

- Therefore, $e(k) \longrightarrow 0$ does not necessarily imply that $e^o(k) \longrightarrow 0$
- To prove $e^o(k) \longrightarrow 0$, we need to first show

$$\|\phi(k)\| < \infty \qquad \forall\, k \geq 0$$

Bondedness of the regressor vector

Remember that:

$$\phi(k) = \begin{bmatrix} -y(k) & \cdots & -y(k-n+1) & u(k-d+1) & \cdots & u(k-d-m+1) \end{bmatrix}^T$$

Therefore,

$$\|\phi(k)\|^2 = \sum_{i=1}^{n} y^2(k-i+1) + \sum_{j=0}^{m} u^2(k-j-d+1)$$

By assumption,

$$|u(k)| < \infty \qquad \forall\, k \geq 0$$

Bondedness of the regressor vector

Since

$$\|\phi(k)\|^2 = \sum_{i=1}^{n} y^2(k-i+1) + \sum_{j=0}^{m} u^2(k-j-d+1)$$

and,

$$|u(k)| < \infty \qquad \forall\, k \geq 0$$

we only need to show that

$$|y(k)| < \infty \qquad \forall\, k \geq 0$$

Bondedness of the regressor vector

Remember that:

$$y(k) = \frac{q^{-\mathrm{d}}B(q^{-1})}{A(q^{-1})} u(k)$$

and $A(q^{-1})$ is anti-Schur.

Therefore LTI system $G(q^{-1}) = \dfrac{q^{-\mathrm{d}}B(q^{-1})}{A(q^{-1})}$ is BIBO

Thus,

$$|u(k)| < \infty \Rightarrow |y(k)| < \infty$$

$\blacksquare$

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Additional Material
(you are not responsible for this)

- Proof that NL_1 is P-class

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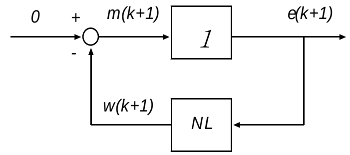
Equivalent feedback loop (review)



- Recall that the *NL* block is **not** P-class

NL:

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) - \frac{1}{\lambda_1(k)} F(k)\phi(k)e(k+1)$$

$$w(k+1) \;=\; -\phi(k)^T\tilde{\theta}(k+1)$$

$$F^{-1}(k+1) = \lambda_1(k)\, F^{-1}(k) + \lambda_2(k)\, \phi(k)\phi^T(k)$$

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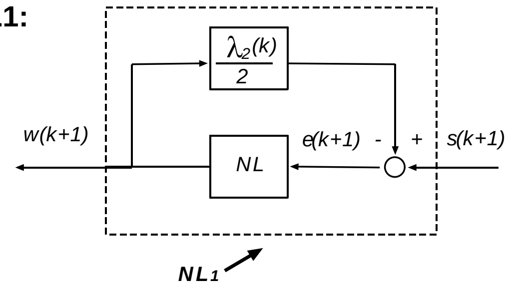
Solution: Modify the *NL* block (review)

- **Add a feedback term to *NL* to make it P-class**

NL1:



We want to show: $\displaystyle\sum_{j=0}^{k} w(j)s(j) \geq -\gamma_o^2$

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Simplified Notation

$\hat{\theta}_k = \hat{\theta}(k)$ $\quad$ $\tilde{\theta}_k = \tilde{\theta}(k)$

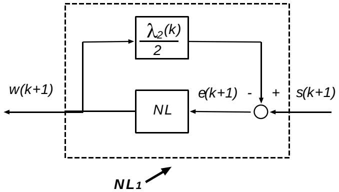
$\phi_k = \phi(k)$ $\quad$ $e_k = e(k)$

$w_k = w(k)$ $\quad$ $s_k = s(k)$

$\lambda_{1,k} = \lambda_1(k)$ $\quad$ $\lambda_{2,k} = \lambda_2(k)$

$F_k = F(k)$

Proof that NL_1 is P-class



- From the summing junction, we have

$$e_{k+1} = s_{k+1} - \frac{\lambda_{2,k}}{2} w_{k+1}$$

$$\Downarrow$$

$$s_{k+1} = e_{k+1} + \frac{\lambda_{2,k}}{2} w_{k+1}$$

Proof that NL_1 is P-class

$$\textbf{NL:} \begin{cases} \tilde{\theta}_{k+1} = \tilde{\theta}_k - \lambda_{1,k}^{-1} F_k \phi_k e_{k+1} \\ w_{k+1} = -\phi_k^T \tilde{\theta}_{k+1} \\ F_{k+1}^{-1} = \lambda_{1,k} F_k^{-1} + \lambda_{2,k} \phi_k \phi_k^T \end{cases}$$

- Multiply the input of NL_1 by its output

$$2w_{k+1}s_{k+1} = \underbrace{w_{k+1}}_{-\tilde{\theta}_{k+1}^T \phi_k} \left[2e_{k+1} + \lambda_{2,k} \underbrace{w_{k+1}}_{-\phi_k^T \tilde{\theta}_{k+1}}\right]$$

$$2w_{k+1}s_{k+1} = \tilde{\theta}_{k+1}^T \underbrace{\left[\lambda_{2,k}\phi_k\phi_k^T\right]}_{F_{k+1}^{-1} - \lambda_{1,k}F_k^{-1}} \tilde{\theta}_{k+1} - 2\tilde{\theta}_{k+1}^T \underbrace{\phi_k e_{k+1}}_{\lambda_{1,k}F_k^{-1}(\tilde{\theta}_k - \tilde{\theta}_{k+1})}$$

Proof that NL_1 is P-class

- From the previous slide

$$2w_{k+1}s_{k+1} = \tilde{\theta}_{k+1}^T \left[F_{k+1}^{-1} - \lambda_{1,k}F_k^{-1}\right] \tilde{\theta}_{k+1} + 2\,\lambda_{1,k}\,\tilde{\theta}_{k+1}^T F_k^{-1}(\tilde{\theta}_{k+1} - \tilde{\theta}_k)$$

$$= \tilde{\theta}_{k+1}^T F_{k+1}^{-1} \tilde{\theta}_{k+1} + \lambda_{1,k}\left[-\tilde{\theta}_{k+1}^T F_k^{-1} \tilde{\theta}_{k+1} + 2\tilde{\theta}_{k+1}^T F_k^{-1}(\tilde{\theta}_{k+1} - \tilde{\theta}_k)\right]$$

Define $\triangle = \tilde{\theta}_k - \tilde{\theta}_{k+1}$

Proof that NL_1 is P-class

- From the previous slide

$\triangle = \tilde{\theta}_k - \tilde{\theta}_{k+1}$

$$2w_{k+1}s_{k+1} = \tilde{\theta}_{k+1}^T F_{k+1}^{-1} \tilde{\theta}_{k+1} - \lambda_{1,k}\left[\tilde{\theta}_{k+1}^T F_k^{-1} \tilde{\theta}_{k+1} + 2\tilde{\theta}_{k+1}^T F_k^{-1} \triangle\right]$$

$$= \tilde{\theta}_{k+1}^T F_{k+1}^{-1} \tilde{\theta}_{k+1} - \lambda_{1,k}\left[\underbrace{\left(\tilde{\theta}_{k+1} + \triangle\right)^T}_{\tilde{\theta}_k^T} F_k^{-1} \underbrace{\left(\tilde{\theta}_{k+1} + \triangle\right)}_{\tilde{\theta}_k} - \triangle^T F_k^{-1} \triangle\right]$$

Proof that NL_1 is P-class

$$\Delta = \tilde{\theta}_k - \tilde{\theta}_{k+1}$$

- From the previous slide

$$2w_{k+1}s_{k+1} = \tilde{\theta}_{k+1}^T F_{k+1}^{-1}\tilde{\theta}_{k+1} \underbrace{- \lambda_{1,k}\tilde{\theta}_k^T F_k^{-1}\tilde{\theta}_k}_{\geq -\tilde{\theta}_k^T F_k^{-1}\tilde{\theta}_k} + \underbrace{\lambda_{1,k}\Delta^T F_k^{-1}\Delta}_{\geq 0}$$

because $\lambda_{1,k} \leq 1$

because $F_k^{-1} \succ 0$, $\lambda_{1,k} > 0$

- Therefore

$$2w_{k+1}s_{k+1} \geq \tilde{\theta}_{k+1}^T F_{k+1}^{-1}\tilde{\theta}_{k+1} - \tilde{\theta}_k^T F_k^{-1}\tilde{\theta}_k$$

Proof that NL_1 is P-class

$$w_{k+1}s_{k+1} \geq \frac{1}{2}\left[\tilde{\theta}_{k+1}^T F_{k+1}^{-1}\tilde{\theta}_{k+1} - \tilde{\theta}_k^T F_k^{-1}\tilde{\theta}_k\right]$$

- Now check the Popov inequality

$$\sum_{i=0}^{k} w_i s_i \geq \frac{1}{2}\sum_{i=0}^{k}\left[\tilde{\theta}_i^T F_i^{-1}\tilde{\theta}_i - \tilde{\theta}_{i-1}^T F_{i-1}^{-1}\tilde{\theta}_{i-1}\right]$$

$$= \frac{1}{2}\left[\tilde{\theta}_k^T F_k^{-1}\tilde{\theta}_k - \tilde{\theta}_{-1}^T F_{-1}^{-1}\tilde{\theta}_{-1}\right]$$

$$\geq \underbrace{-\frac{1}{2}\tilde{\theta}_{-1}^T F_{-1}^{-1}\tilde{\theta}_{-1}}_{\gamma_0^2}$$

■