ME 233 Spring 2010
Solution to Homework #5

1. (a) To begin, we first note that

$$\Phi_{WW}(\omega) = \Lambda_{WW}(e^{j\omega}) = 1.$$

Thus,

$$\begin{aligned}
\Phi_{YY}(\omega) &= G(e^{j\omega})G(e^{-j\omega})\Phi_{WW}(\omega) \\
&= \left(\frac{e^{j\omega} - 0.3}{e^{j\omega} - 0.5}\right)\left(\frac{e^{-j\omega} - 0.3}{e^{-j\omega} - 0.5}\right) \\
&= \frac{1.09 - 0.3(e^{j\omega} + e^{-j\omega})}{1.25 - 0.5(e^{j\omega} + e^{-j\omega})} \\
&= \frac{1.09 - 0.6\cos(\omega)}{1.25 - \cos(\omega)}.
\end{aligned}$$

Figure 1 shows a plot of $\Phi_{YY}(\omega)$. Note that the plot is symmetric about $\omega = 0$. This can be seen by examining the equation for the spectral density:

$$\begin{aligned}
\Phi_{YY}(-\omega) &= \Lambda_{YY}(z)|_{z=e^{-j\omega}} \\
&= \Lambda_{YY}(z^{-1})|_{z=e^{j\omega}} \\
&= \Lambda_{YY}(z)|_{z=e^{j\omega}} \\
&= \Phi_{YY}(\omega).
\end{aligned}$$

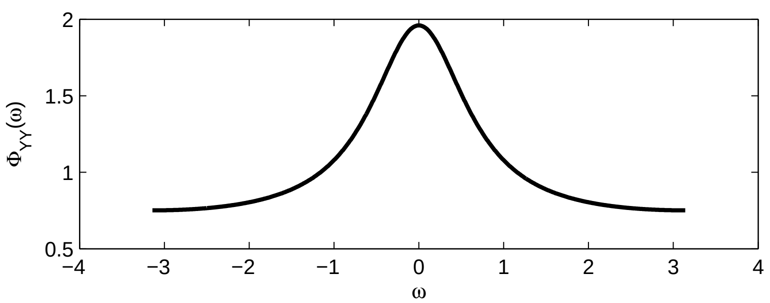

Figure 1: Spectral density of Y

(b) Noting that $\Lambda_{YY}(l) = \Lambda_{YY}(-l)$, we know that

$$\begin{aligned}
\Lambda_{YY}(z) &= \sum_{l=-\infty}^{\infty} z^{-l}\Lambda_{YY}(l) \\
&= \sum_{l=-\infty}^{0} z^{-l}\Lambda_{YY}(l) + \sum_{l=0}^{\infty} z^{-l}\Lambda_{YY}(l) - \Lambda_{YY}(0) \\
&= \sum_{l=0}^{\infty}(z^{-1})^{-l}\Lambda_{YY}(l) + \sum_{l=0}^{\infty} z^{-l}\Lambda_{YY}(l) - \Lambda_{YY}(0).
\end{aligned}$$

If we denote

$$F(z) = \sum_{l=0}^{\infty} z^{-l} \Lambda_{YY}(l) - \frac{1}{2}\Lambda_{YY}(0)$$

we see that

$$\begin{aligned}
\Lambda_{YY}(z) &= F(z) + F(z^{-1}) \\
\mathcal{Z}^{-1}\{F(z)\} &= \Lambda_{YY}(l) - \frac{1}{2}\Lambda_{YY}(0)\delta(l), \qquad l \geq 0 \\
&= \begin{cases} \frac{1}{2}\Lambda_{YY}(l), & l = 0 \\ \Lambda_{YY}(l), & l > 0 \end{cases}.
\end{aligned}$$

We know that $\Lambda_{YY}(l)$ should get smaller as l approaches infinity. Therefore, $F(z)$ should have stable poles and $F(z^{-1})$ should have unstable poles. This tells us how we should group our terms in $\Lambda_{YY}(l)$. Using this argument as our basis, we now express

$$\begin{aligned}
\Lambda_{YY}(z) &= \left(\frac{z - 0.3}{z - 0.5}\right)\left(\frac{z^{-1} - 0.3}{z^{-1} - 0.5}\right) \\
&= \frac{Az}{z - 0.5} + \frac{Az^{-1}}{z^{-1} - 0.5} + B \\
&= \frac{Az}{z - 0.5} - \frac{A}{0.5z - 1} + B.
\end{aligned}$$

Using standard partial fraction expansion techniques, we find that

$$\begin{aligned}
A &= 0.4533 \\
B &= 0.1467
\end{aligned}$$

Thus, if we define

$$F(z) = \frac{Az}{z - 0.5} + \frac{B}{2}$$

our equation for $\Lambda_{YY}(z)$ takes the form shown above. Thus,

$$\begin{aligned}
\mathcal{Z}^{-1}\{F(z)\} &= A(0.5)^l + \frac{B}{2}\delta(l) = \begin{cases} \frac{1}{2}\Lambda_{YY}(l), & l = 0 \\ \Lambda_{YY}(l), & l > 0 \end{cases} \\
\Rightarrow \Lambda_{YY}(l) &= \begin{cases} A(0.5)^l + (B + A)\,\delta(l) & l \geq 0 \\ A(0.5)^{-l} & l < 0 \end{cases} \\
&= \begin{cases} 0.4533(0.5)^l + (0.6)\,\delta(l) & l \geq 0 \\ 0.4533(0.5)^{-l} & l < 0 \end{cases}.
\end{aligned}$$

If we define

$$\begin{aligned}
\Lambda^C_{YY}(l) &= \begin{cases} 0.4533(0.5)^l + (0.6)\,\delta(l) & l \geq 0 \\ 0 & l < 0 \end{cases} \\
\Lambda^A_{YY}(l) &= \begin{cases} 0.4533(0.5)^{-l} & l < 0 \\ 0 & l \geq 0 \end{cases}
\end{aligned}$$

then

$$\Lambda_{YY}(l) = \Lambda^C_{YY}(l) + \Lambda^A_{YY}(l)$$

where $\Lambda^C_{YY}(l)$ is a causal sequences and $\Lambda^A_{YY}(l)$ is a strictly anticausal sequence.
Figure 2 shows that the values of $\Lambda_{YY}(l)$ determined through MATLAB simulation match up well with the values determined above. (Note that the auto-covariance was normalized in this figure, i.e. $\Lambda_{YY}(l)$ was scaled so that its maximum value was 1.)

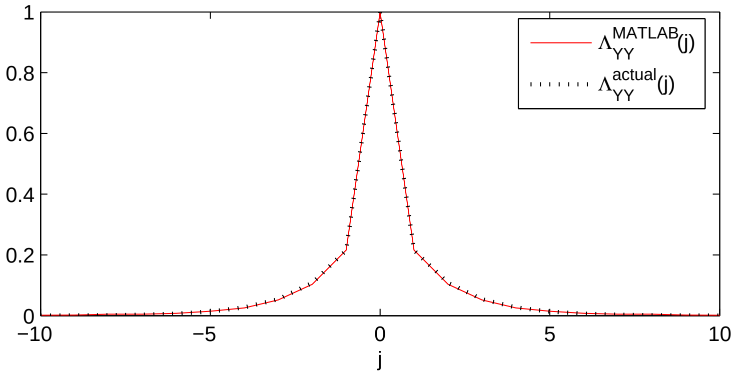

Figure 2: Comparison of MATLAB-determined auto-covariance to actual values

(c) Here, we want to compute covariances using the orginal series equation and compare our results to those obtained using transforms. To start, note that

$$\begin{aligned}\Lambda_{YW}(0) &= E\{Y(k)W(k)\}\\ &= E\Big\{\big[0.5Y(k-1)+W(k)-0.3W(k-1)\big]W(k)\Big\}\\ &= E\left\{W^2(k)\right\}+0.5E\{Y(k-1)W(k)\}-0.3E\{W(k-1)W(k)\}.\end{aligned}$$

Since the system is causal we know that the system's output should not depend on future inputs. Thus, the system's output should be independent of future inputs. Also, since W is white, its value should be independent of its value at any other timestep. Using these two facts gives

$$\begin{aligned}\Lambda_{YW}(0) &= E\left\{W^2(k)\right\}+E\{W(k)\}\left[0.5E\{Y(k-1)\}-0.3E\{W(k-1)\}\right]\\ &= E\left\{W^2(k)\right\}=1\end{aligned}$$

where we have used the fact that W is zero-mean. Note that this result agrees with the result found in part (b).

(d) Using the wide-sense stationarity of the signals and the results from the previous part,

$$\begin{aligned}\lambda_{YW}(1) &= E\{Y(k+1)W(k)\}\\ &= E\{Y(k)W(k-1)\}\\ &= -0.3E\left\{W^2(k-1)\right\}+0.5E\{Y(k-1)W(k-1)\}+E\{W(k)W(k-1)\}\\ &= -0.3E\left\{W^2(k-1)\right\}+0.5E\{Y(k-1)W(k-1)\}\\ &= -0.3E\left\{W^2(k)\right\}+0.5E\{Y(k)W(k)\}\\ &= -0.3+0.5\Lambda_{YW}(0)=0.2.\end{aligned}$$

Note that this result agrees with the result found in part (b).

(e) To solve this problem, we will first note that

$$Y^2(k) = [0.5Y(k-1)+W(k)-0.3W(k-1)]^2.$$

Taking the expected value of both sides gives

$$\begin{aligned}\Lambda_{YY}(0) &= 0.25E\left\{Y^2(K-1)\right\}+E\left\{W^2(k)\right\}+0.09E\left\{W^2(k-1)\right\}\\ &\quad +E\{Y(k-1)W(k)\}-0.3E\{Y(k-1)W(k-1)\}-0.6E\{W(k)W(k-1)\}\\ &= 0.25\Lambda_{YY}(0)+1+0.09+0-0.3\Lambda_{YW}(0)+0\\ &= \frac{0.79}{0.75}=1.0533.\end{aligned}$$

Note that this result agrees with the result found in part (e).

2. We will show two methods for proving the desired result.

   First Method: To begin, we note that

$$\begin{aligned} E\left[\left(\tilde{X}(k+j)-\tilde{X}(k)\right)\left(\tilde{X}(k+j)-\tilde{X}(k)\right)^T\right] &\succeq 0 \\ E\left[\left(\tilde{X}(k+j)+\tilde{X}(k)\right)\left(\tilde{X}(k+j)+\tilde{X}(k)\right)^T\right] &\succeq 0. \end{aligned}$$

   Evaluating these expectations and using the property that the trace of a matrix is the sum of its eigenvalues gives

$$\begin{aligned} 2\Lambda_{XX}(0)-\left(\Lambda_{XX}(j)+\Lambda_{XX}^T(j)\right) &\succeq 0 \\ 2\Lambda_{XX}(0)+\left(\Lambda_{XX}(j)+\Lambda_{XX}^T(j)\right) &\succeq 0 \\ \Rightarrow \text{Trace}\left[2\Lambda_{XX}(0)\pm\left(\Lambda_{XX}(j)+\Lambda_{XX}^T(j)\right)\right] &\geq 0. \end{aligned}$$

   Since finding the trace of a matrix is a linear operator, we can express this as

$$\text{Trace}\left[\Lambda_{XX}(0)\right] \geq \pm\frac{1}{2}\Big(\text{Trace}\left[\Lambda_{XX}(j)\right]+\text{Trace}\left[\Lambda_{XX}^T(j)\right]\Big).$$

   Using the property that $\text{Trace}(A)=\text{Trace}(A^T)$, we can now say that

$$\begin{aligned} \text{Trace}\left[\Lambda_{XX}(0)\right] &\geq \pm\,\text{Trace}\left[\Lambda_{XX}(j)\right] \\ \Rightarrow \text{Trace}\left[\Lambda_{XX}(0)\right] &\geq \left|\text{Trace}\left[\Lambda_{XX}(j)\right]\right|. \end{aligned}$$

   Second Method: In this method, we will prove the result using the elements in the random sequence $X(k)$. First, we partition $X(k)$ as

$$X(k)=\begin{bmatrix}X_1(k) & \cdots X_n(k)\end{bmatrix}^T$$

   where $X_i(k)$ is a scalar random sequence. Now note that

$$\begin{aligned} \text{Trace}\left[\Lambda_{XX}(0)\right]\pm\text{Trace}\left[\Lambda_{XX}(j)\right] &= \text{Trace}\left[E\left\{X(k)X^T(k)\right\}\right]\pm\text{Trace}\left[E\left\{X(k+j)X^T(k)\right\}\right] \\ &= \sum_{i=1}^n E\left\{X_i^2(k)\right\}\pm\sum_{i=1}^n E\left\{X_i(k+j)X_i(k)\right\} \\ &= \sum_{i=1}^n\left(\Lambda_{X_iX_i}(0)\pm\Lambda_{X_iX_i}(j)\right). \end{aligned}$$

   Since $X_i(k)$ is a scalar random sequence, we know that

$$\begin{gathered} \Lambda_{X_iX_i}(0)\geq\left|\Lambda_{X_iX_i}(j)\right| \\ \Rightarrow \Lambda_{X_iX_i}(0)\pm\Lambda_{X_iX_i}(j)\geq 0, \quad \forall i \\ \Rightarrow \sum_{i=1}^n\left(\Lambda_{X_iX_i}(0)\pm\Lambda_{X_iX_i}(j)\right)\geq 0 \\ \Rightarrow \text{Trace}\left[\Lambda_{XX}(0)\right]\pm\text{Trace}\left[\Lambda_{XX}(j)\right]\geq 0 \\ \Rightarrow \text{Trace}\left[\Lambda_{XX}(0)\right]\geq\left|\text{Trace}\left[\Lambda_{XX}(j)\right]\right| \end{gathered}$$

   which concludes the proof.

3. (a) First, we express our system as

$$\begin{aligned} X(k+1) &= AX(k)+BW(k) \\ Y(k) &= CX(k)+V(k). \end{aligned}$$

Taking expectation of our system equations gives

$$\begin{aligned} m_x(k+1) &= Am_x(k) + Bm_w(k) \\ m_y(k) &= Cm_x(k). \end{aligned}$$

Thus, finding $m_y(k)$ is equivalent to finding a step response of this system with magnitude 10. Figure 3 shows a plot of $m_y(k)$ versus the time step. Note that because $W(k)$ is not a zero-mean sequence, $Y(k)$ does not settle out to 0; the steady state value of m_y is given by

$$\overline{m}_y = 10.084.$$

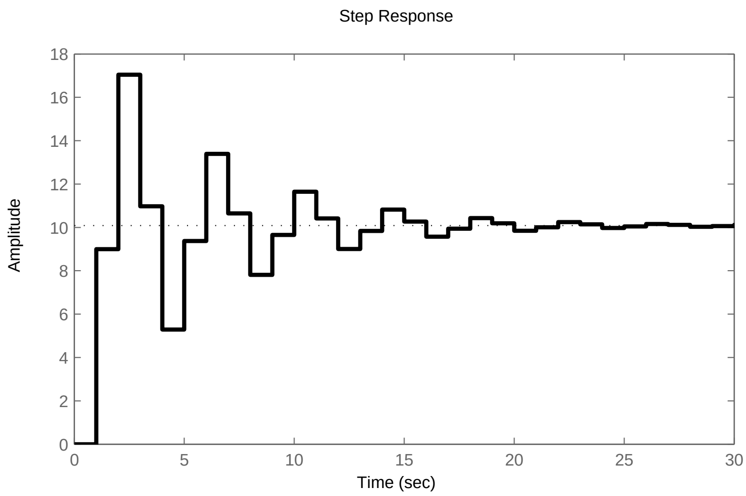

Figure 3: Evolution of $m_y(k)$ with time

(b) As discussed in lecture, the covariance of X propagates in the following way:

$$\Lambda_{XX}(k+1,0) = A\Lambda_{XX}(k,0)A^T + B\Sigma_{WW}(k)B^T.$$

Since we know the initial condition $\Lambda_{XX}(0,0)$, we can find $\Lambda_{XX}(k,0)$ iteratively using this Lyapunov equation. To find the covariance of Y, note that

$$\begin{aligned} \Lambda_{YY}(k,0) &= E\left\{\tilde{Y}^2(k)\right\} \\ &= E\left\{\left[C\tilde{X}(k) + V(k)\right]\left[C\tilde{X}(k) + V(k)\right]^T\right\} \\ &= C\Lambda_{XX}(k,0)C^T + \Sigma_{vv} \end{aligned}$$

where we made use of the fact that $X(k)$ and $V(k)$ are uncorrelated. Thus, we can use our simulation results for $\Lambda_{XX}(k,0)$ to find $\Lambda_{YY}(k,0)$. Figure 4 shows the results of simulating the evolution of $\Lambda_{XX}(k,0)$ and then using it to find $\Lambda_{YY}(k,0)$. This set of simulations terminated when

$$\left\|\Lambda_{XX}(k,0) - \Lambda_{XX}(k-1,0)\right\|_{i2} \leq 10^{-5}.$$

Note that we could have used any matrix norm in this termination condition (Frobenius norm, i1 norm, i2 norm, i∞ norm, etc). The steady state covariance of y was found to be

$$\Lambda_{YY}(0) = 3.27.$$

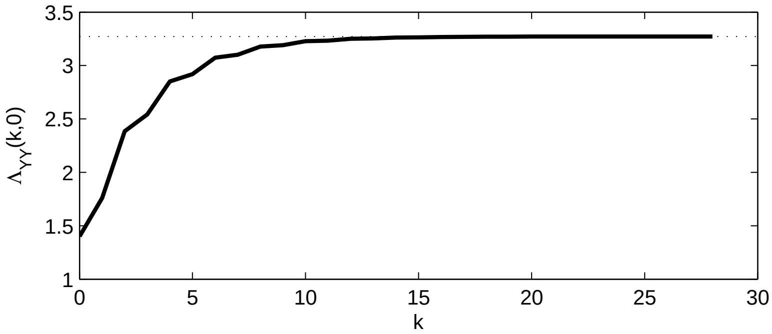

Figure 4: Evolution of $\Lambda_{YY}(k, 0)$ with time

(c) To find $\Lambda_{XX}(5)$, recall that

$$\begin{aligned}\Lambda_{XX}(k,l) &= A^l \Lambda_{XX}(k,0) \\ \Rightarrow \Lambda_{XX}(k,5) &= A^5 \Lambda_{XX}(k,0).\end{aligned}$$

To find $\Lambda_{YY}(5)$, note that

$$\begin{aligned}\Lambda_{YY}(k,5) &= E\left\{\left[C\tilde{X}(k+5) + V(k+5)\right]\left[C\tilde{X}(k) + V(k)\right]^T\right\} \\ &= C\Lambda_{XX}(k,5)C^T\end{aligned}$$

where we have used that the measurement noise is white and uncorrelated with the state. Figure 5 shows the simulation results. At steady state,

$$\Lambda_{YY}(5) = 0.27.$$

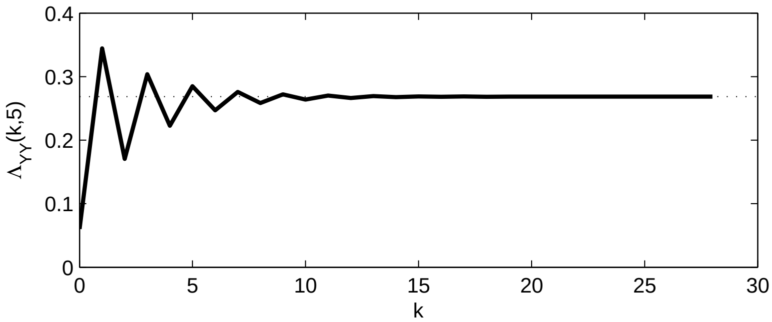

Figure 5: Evolution of $\Lambda_{YY}(k, 5)$ with time

(d) At steady state,

$$A\Lambda_{XX}(0)A^T - \Lambda_{XX}(0) = -B\Sigma_{ww}B^T.$$

A call to `dlyap(A,B*Sigma_ww*B')` gives

$$\Lambda_{XX}(0) = \begin{bmatrix} 0.4308 & 0.0276 \\ 0.0276 & 0.3080 \end{bmatrix}.$$

At steady state, the stationary covariances of x and y are given by

$$\begin{aligned}
\Lambda_{XX}(l) &= \begin{cases} \Lambda_{XX}(0)\left(A^{-l}\right)^T & l < 0 \\ \Lambda_{XX}(0) & l = 0 \\ A^l \Lambda_{XX}(0) & l > 0 \end{cases} \\
\Lambda_{YY}(l) &= E\left\{\left[C\tilde{X}(k+l) + V(k+l)\right]\left[C\tilde{X}(k) + V(k)\right]^T\right\} \\
&= C\Lambda_{XX}(l)C^T + \Sigma_{vv}\delta(l).
\end{aligned}$$

Figure 6 shows the computed stationary covariance of Y. As expected, the plot is symmetric and the largest value occurs at $j = 0$. Note that the values of $\Lambda_{YY}(0)$ and $\Lambda_{YY}(5)$ are the same as the steady state covariances found in the two previous parts.

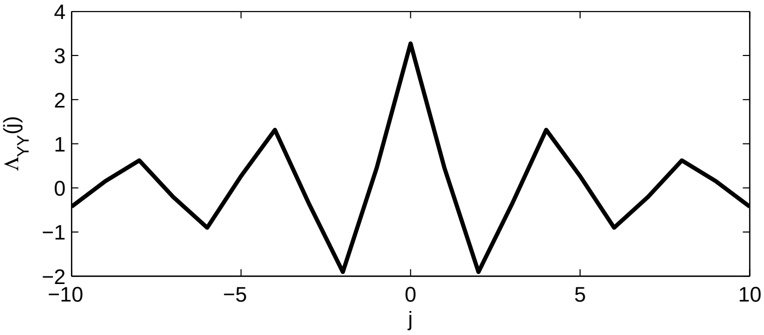

Figure 6: Stationary covariance of Y

(e) First, we define

$$\begin{aligned}
\overline{W}(k) &= \begin{bmatrix} W(k) \\ V(k) \end{bmatrix} \\
\overline{G}(z) &= \begin{bmatrix} G(z) & 1 \end{bmatrix}
\end{aligned}$$

so that our governing equations in the Z domain become

$$Y(z) = \overline{G}(z)\overline{W}(z).$$

Thus, the output spectral density is given by

$$\begin{aligned}
\Phi_{YY}(\omega) &= \overline{G}(\omega)\Phi_{\overline{WW}}(\omega)\overline{G}^T(-\omega) \\
&= \begin{bmatrix} G(\omega) & 1 \end{bmatrix} \begin{bmatrix} \Sigma_{ww} & 0 \\ 0 & \Sigma_{vv} \end{bmatrix} \begin{bmatrix} G(-\omega) \\ 1 \end{bmatrix} \\
&= |G(\omega)|^2 \Sigma_{ww} + \Sigma_{vv}.
\end{aligned}$$

(f) Figure 7 shows the spectral density of Y. As expected this graph is symmetric. Notice, however, that Φ_{YY} is not a maximum when $\omega = 0$. Unlike auto-covariances, spectral densities do not have to be a maximum when the argument is zero.

To see where these peaks come from, we will find the equivalent damping and natural frequency of this system. Recall from classical controls that for a continuous time second-order underdamped system with damping ζ and natural frequency ω_n, the poles are given by

$$\begin{aligned}
q_1 &= \sigma + j\omega_d \\
q_2 &= \sigma - j\omega_d
\end{aligned}$$

where

$$\begin{aligned}
\sigma &= -\zeta\omega_n \\
\omega_d &= \omega_n\sqrt{1 - \zeta^2} \\
&= \sqrt{\omega_n^2 - \sigma^2}.
\end{aligned}$$

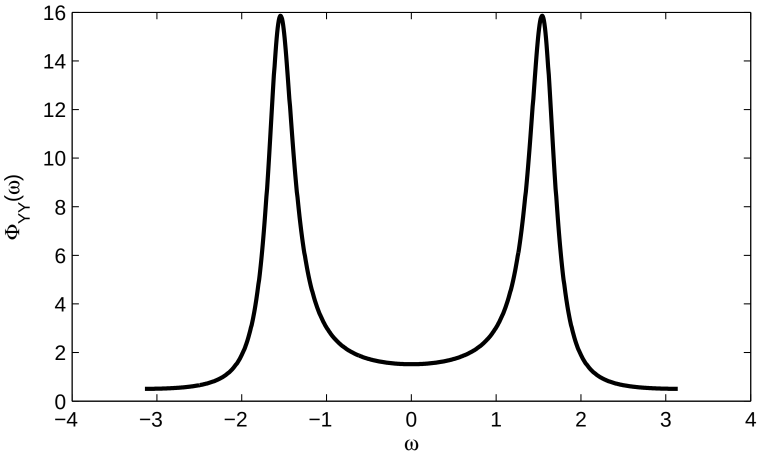

Figure 7: Spectral density of Y

Now recall from ME232 that if we have poles $\lambda_1, \ldots, \lambda_n$ in continuous time, the poles of the discrete time system obtained using a zero-order hold are given by $e^{\lambda_1 T}, \ldots, e^{\lambda_n T}$, where T is the sampling time. (Refer to page `ME232-36` of the ME232 class notes.) Thus, letting $T = 1$, our discrete time poles are given by

$$\begin{aligned} p_1 &= e^{q_1} = e^{\sigma} e^{j\omega_d} = e^{\sigma}\big[cos(\omega_d) + j sin(\omega_d)\big] \\ p_2 &= e^{q_2} = e^{\sigma} e^{-j\omega_d} = e^{\sigma}\big[cos(\omega_d) - j sin(\omega_d)\big]. \end{aligned}$$

Thus, we can solve for

$$\begin{aligned} \sigma &= \ln\left\{\, |p_1| \,\right\} \\ \omega_d &= \tan^{-1}\left\{ \left| \frac{\mathrm{Im}(p_1)}{\mathrm{Re}(p_1)} \right| \right\} \\ \omega_n &= \sqrt{\sigma^2 + \omega_d^2} \\ \zeta &= \frac{-\sigma}{\omega_n}. \end{aligned}$$

In our system here, these values are

$$\begin{aligned} \sigma &= -0.1841 \\ \omega_d &= 1.5588 \\ \omega_n &= 1.5696 \\ \zeta &= 0.1173. \end{aligned}$$

Because we have a small value of ζ, our system is lightly damped, resulting in a relatively high peak gain at ω_d. Thus, we can expect a lot of the system output to be at the frequency ω_d. This explains our peaks in our output spectral density. This analysis can be verified by looking at the Bode plot for $G(z)$, shown in Figure 8. (Note that you would have to square the peak gain in the Bode plot and then add Σ_{vv} to get the peak in the output spectral density.)

Also note that $\Phi_{YY}(\omega)$ is large when $\omega \sim \pi/2$, i.e. half of our sampling frequency. This corresponds to the relative extrema of Figure 6, which occur at even correlation indices.

4. First, recall that

$$\begin{aligned} \Phi_{YY}(\omega) &= |G(j\omega)|^2 \Phi_{WW}(\omega) \\ &= |G(j\omega)|^2 \\ &= \big(G(s)G(-s)\big)\Big|_{s=j\omega} \end{aligned}$$

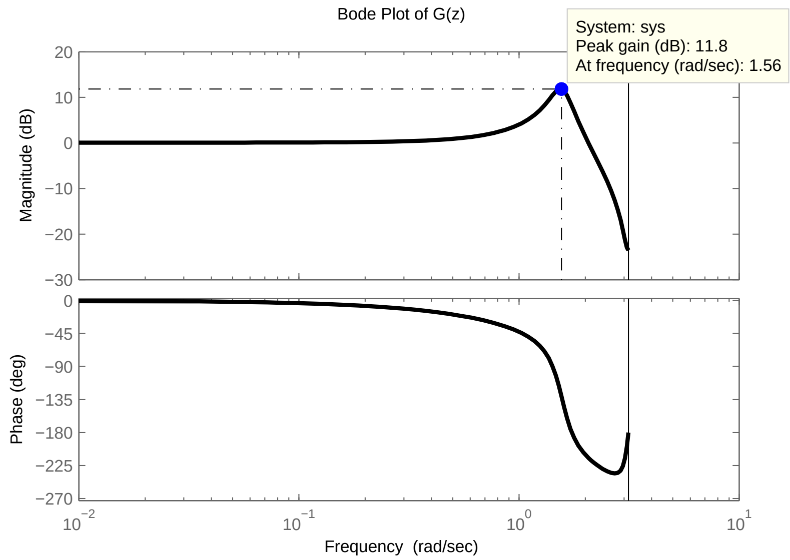

Figure 8: Bode plot of $G(z)$

where we used that $w(t)$ is zero-mean, unit-variance, white noise. By looking at the order of the numerator and denominator in $\Phi_{YY}(\omega)$, we see that we can express

$$G(s) = K\frac{(s-z_1)}{(s-p_1)(s-p_2)}.$$

Plugging this into the equation for $\Phi_{YY}(\omega)$ gives

$$\begin{aligned}\frac{0.25\omega^2+1}{\omega^4+5\omega^2+1} &= K^2\frac{z_1^2-s^2}{(p_1^2-s^2)(p_2^2-s^2)}\Big|_{s=j\omega}\\ &= \frac{K^2(\omega^2+z_1^2)}{(\omega^2+p_1^2)(\omega^2+p_2^2)}.\end{aligned}$$

Since $G(s)$ is stable and minimum phase, we know that all poles and zeros have negative real parts. Thus, by comparing coefficients, we see that

$$\begin{aligned}K &= \pm 0.5\\ z_1 &= -2\\ p_1 &= -\sqrt{\frac{5\pm\sqrt{21}}{2}} = -\frac{\sqrt{7}\pm\sqrt{3}}{2}\\ p_2 &= p_1^{-1}.\end{aligned}$$

Hence, our solution is given by

$$G(s) = \frac{0.5(s+2)}{(s+\frac{\sqrt{7}+\sqrt{3}}{2})(s+\frac{\sqrt{7}-\sqrt{3}}{2})}.$$

5. (a) The linear estimator is of the form:

$$\frac{d\hat{x}}{dt} = a\hat{x} + L[y-\hat{x}].$$

Defining the estimation error as

$$\tilde{X} = X - \hat{X}$$

the dynamic equation for the estimation error becomes

$$\begin{aligned}\frac{d\tilde{X}}{dt} &= \frac{dX}{dt} - \frac{d\hat{X}}{dt} \\ &= \big(aX + b_w W\big) - \big(a\hat{x} + L[(X+V) - \hat{X}]\big) \\ &= (a-L)\tilde{X} + \begin{bmatrix} b_w & -L \end{bmatrix} \begin{bmatrix} W \\ V \end{bmatrix}.\end{aligned}$$

(b) If the error dynamics are stable, i.e $(a-L)$ is negative, the steady state estimation error variance, $\overline{\Lambda}_{\tilde{X}\tilde{X}}$, satisfies the following Lyapunov equation:

$$(a-L)\overline{\Lambda}_{\tilde{X}\tilde{X}}(0) + \overline{\Lambda}_{\tilde{X}\tilde{X}}(0)(a-L) = -\begin{bmatrix} b_w & -L \end{bmatrix} \begin{bmatrix} \sigma_W^2 & 0 \\ 0 & \sigma_V^2 \end{bmatrix} \begin{bmatrix} b_W \\ -L \end{bmatrix}.$$

Thus, our steady state estimation error variance is given by

$$\overline{\Lambda}_{\tilde{X}\tilde{X}}(0) = \frac{-(b_w^2\sigma_W^2 + L^2\sigma_V^2)}{2(a-L)}.$$

(c) When $\overline{\Lambda}_{\tilde{X}\tilde{X}}(0)$ is minimized with respect to L,

$$\begin{aligned} &\frac{d\overline{\Lambda}_{\tilde{X}\tilde{X}}(0)}{dL} = 0 \\ \Rightarrow\ & (a-L)2L\sigma_V^2 + (b_w^2\sigma_W^2 + L^2\sigma_V^2) = 0 \\ \Rightarrow\ & \sigma_V^2 L^2 - 2a\sigma_V^2 L - b_w^2\sigma_W^2 = 0 \\ \Rightarrow\ & L = a \pm \sqrt{a^2 + \frac{b_w^2\sigma_W^2}{\sigma_V^2}}. \end{aligned}$$

To make the estimation error dynamics stable, the positive sign should be chosen. Thus

$$\begin{aligned} L &= a + \sqrt{a^2 + \frac{b_w^2\sigma_W^2}{\sigma_V^2}} \\ \overline{\Lambda}_{\tilde{X}\tilde{X}}(0) &= \sqrt{a^2\sigma_V^4 + b_w^2\sigma_W^2\sigma_V^2} + a\sigma_V^2. \end{aligned}$$

6. (a) Defining

$$\begin{aligned} a &= \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}^T \\ Z(k) &= \begin{bmatrix} Y(k-1) & Y(k-2) & \cdots & Y(k-n) \end{bmatrix}^T \end{aligned}$$

we can rewrite $\hat{y}(k)|_{k-1,\ldots,k-n}$ as

$$\hat{y}(k)|_{k-1,\ldots,k-n} = a^T z(k).$$

Thus, the cost function to be minimized is given by

$$\begin{aligned} \sigma_{\tilde{Y}} &= E\left\{\left[Y(k) - a^T Z(k)\right]^2\right\} \\ &= E\left\{Y^2(k)\right\} - 2a^T E\left\{Z(k)Y(k)\right\} + a^T E\left\{Z(k)Z^T(k)\right\} a \\ &= \sigma_0 - 2a^T\Lambda_{ZY}(0) + a^T\Lambda_{ZZ}(0)a. \end{aligned}$$

Note that we have used that $Y(k)$ and $Z(k)$ have zero mean. Minimizing with respect to a gives

$$\begin{aligned} &\frac{\partial}{\partial a}\Big(\sigma_0 - 2a^T\Lambda_{ZY}(0) + a^T\Lambda_{ZZ}(0)a\Big) = 0 \\ &\Rightarrow -2\Lambda_{ZY}(0) + 2\Lambda_{ZZ}(0)a = 0 \\ &\Rightarrow \Lambda_{ZZ}(0)a = \Lambda_{ZY}(0). \end{aligned}$$

To finish, note that this matrix equation is the same as

$$\begin{bmatrix} \sigma_0 & \sigma_1 & \cdots & \sigma_{n-1} \\ \sigma_1 & \sigma_0 & \cdots & \sigma_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n-1} & \sigma_{n-2} & \cdots & \sigma_0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_n \end{bmatrix}.$$

(b) Plugging the result from the previous part into the expression for $\sigma_{\tilde{Y}}$ gives

$$\begin{aligned} \sigma_{\tilde{Y}} &= \sigma_0 - 2a^T \Lambda_{ZY}(0) + a^T \Lambda_{ZZ}(0) a \\ &= \sigma_0 - 2a^T \Lambda_{ZY}(0) + a^T \Lambda_{ZY}(0) \\ &= \sigma_0 - a^T \Lambda_{ZY}(0) \\ &= \sigma_0 - \sum_{i=1}^{n} a_i \sigma_i. \end{aligned}$$

(c) In parts (a) and (b), we saw that a and $\sigma_{\tilde{Y}}$ were only functions of the auto-covariance of Y. Thus, to solve this problem, we only need to find the auto-covariance of Y. To do this, first find any state space realization of the system. For example, a controllable canonical realization could be used:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -0.32 & -1.2 & 1 \\ \hline 0.2 & 1 & 0 \end{array}\right].$$

Recall that if $X(k)$ is the state, the steady state auto-covariance satisfies

$$\begin{aligned} \Lambda_{XX}(0) &= A\Lambda_{XX}(0)A^T + B\Sigma_{WW}B^T \\ \Lambda_{XX}(l) &= A^l \Lambda_{XX}(0), \qquad l \geq 0 \\ \sigma_l &= E\left\{CX(k+l)X^T(k)C^T\right\} \\ &= C\Lambda_{XX}(l)C^T \end{aligned}$$

where Σ_{WW} is the covariance of W. Although Λ_{XX} will depend on the realization, σ_l will not. (This can be easily verified by letting $\overline{X}(k) = T^{-1}X(k)$, showing that $\Lambda_{\overline{XX}}(0) = T\Lambda_{XX}(0)T^T$ and then showing that σ_l is the same for both realizations.) To solve the Lyapunov equation for $\Lambda_{XX}(0)$, use `dlyap(A,B*B')`. Applying these results gives

$$\begin{aligned} \sigma_0 &= 4.3417 \\ \sigma_1 &= -3.7955 \\ \sigma_2 &= 3.1653 \end{aligned}$$

Thus, we can now solve for

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sigma_0 & \sigma_1 \\ \sigma_1 & \sigma_0 \end{bmatrix}^{-1} \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} -1.0046 \\ -0.1492 \end{bmatrix}$$

With this, we see that

$$\begin{aligned} \sigma_{\tilde{Y}} &= \sigma_0 - \sum_{i=1}^{2} a_i \sigma_i = 1.0009 \\ \tilde{y}(k) &= y(k) - \hat{y}(k) = y(k) - a_1 y(k-1) - a_2 y(k-2) \\ &= \left(1 - a_1 q^{-1} - a_2 q^{-2}\right) y(k) \\ \Rightarrow H(q^{-1}) &= 1 + 1.0046q^{-1} + 0.1492q^{-2} \end{aligned}$$

(d) Use the `MATLAB` function `randn` and `lsim` to generate $w(k)$ and $y(k)$ respectively. In the simulation, with $N = 500000$ and $M = 100000$, the calculated results are $\sigma_0 = 4.3651$ and $\sigma_\epsilon = 1.0041$.