ME 233 Spring 2010
Solution to Homework #2

1. (a) Since stabilizability and detectability both depend on stability of the system, we first check the stability of the system. We can use `MATLAB` to calculate the eigenvalues of A and the resulting eigenvalues are$\{1.2, 1.4142j, -1.4142j\}$. Thus, the system has no stable modes. Because of this, we can conclude that stabilizability and detectability and respectively equivalent to controllability and observability.

   To check the controllability of $[A, B]$, we type the command

```
>> rank([B A*B A^2*B])
```

   into `MATLAB`. Since this returns the value 3, we know that $[A, B]$ is controllable, which is equivalent to stabilizability in this case. To check the observability of $[A, B]$, we type command

```
>> rank([C; C*A; C*A^2])
```

   into `MATLAB`. Since this returns the value 1, we know that $[A, C]$ is not observable, which implies that $[A, C]$ is not detectable. We will see in following parts what effect this has on the optimal finite horizon LQR controller.

   i. Based on the following recursive relationship to calculate $P(k)$:

$$P(k-1) = A^T P(k) A + C^T C - A^T P(k) B \left[R + B^T P(k) B\right]^{-1} B^T P(k) A$$

   ii. Based on the results from Part i, calculate $J^o[x_0, m, S, N] = \frac{1}{2} x_0^T P(m) x_0$.

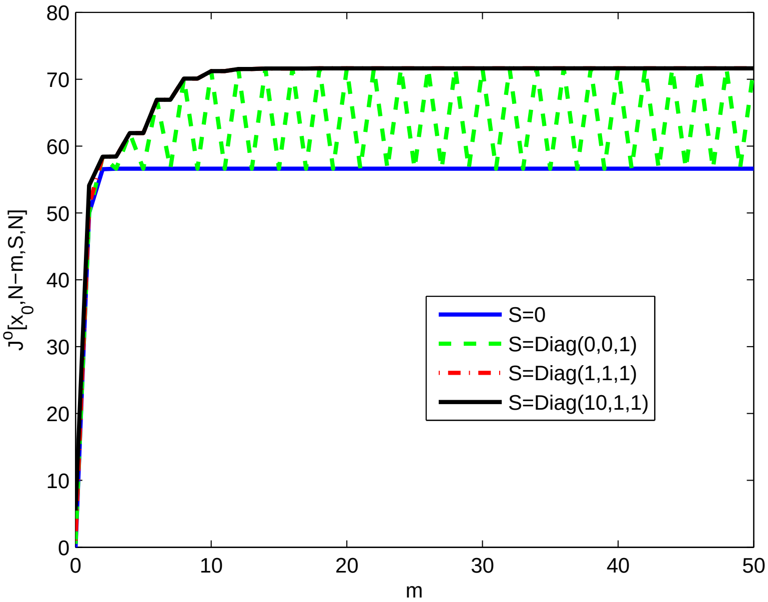

Figure 1: The optimal cost function for different S

iii. Based on the result from Part ii, plot $J^o[x_0, m, S, N]$ vs m as shown in Figure 1.

iv. To the solution of the `DARE`, we type the command

`>> dare(A,B,C'*C,R)`

into `MATLAB`. Then we get the following solution:

$$Pss = \begin{bmatrix} 113.2315 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 30 \end{bmatrix}.$$

For $S = 0$, we have:

$$P(0) = P(1) = \begin{bmatrix} 113.2315 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For $S =$Diag(0,0,1), we have:

$$P(0) = \begin{bmatrix} 113.2315 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 30 \end{bmatrix}, P(1) = \begin{bmatrix} 113.2315 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For $S =$Diag(1,1,1), we have:

$$P(0) = P(1) = \begin{bmatrix} 113.2315 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 30 \end{bmatrix}.$$

For $S =$Diag(10,1,1), we have:

$$P(0) = P(1) = \begin{bmatrix} 113.2315 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 30 \end{bmatrix}.$$

From parts i, ii, iii and iv, we can conclude that if the system is not detectable, then the convergence of the Riccati equation solution depends on S and even if the solution is converged, the converged solution is not unique, i.e. starting at different values of S can result in different steady state solutions to the Riccati equation. Sometime there is no converged solution with some specific S, for example, the S in (ii).

Besides, for $S = 0$, the steady state control $u(k) = -Kx(k)$, where $K = (B^T PB + R)^{-1}B^T PA$ with the converged P, does not result in a stable closed loop system. The reason for this closed loop instability is the poor choice of Q (i.e. C) and S in our LQR cost function. Since $[A, C]$ is not detectable, some unstable states do not manifest themselves in the transient cost, $y^2(k)+Ru^2(k)$. Because $S = 0$, these same states are not penalized in the final cost, $x^T(N)Sx(N)$. Thus, these states could be left unstable in the closed loop system without being penalized by our cost function.

For S chosen as (iii) and (iv), the Riccati equation solutions converge to the solution to DARE. It is easily shown that for the these two cases the steady state control law results in stable closed loop systems. The reason why we get closed loop stability in the two cases, unlike the previous case using (i), is the better choice of cost function. Although the unstable states are still not penalized in the transient cost, they *are* penalized in the final cost, $x^T(N)Sx(N) = ||x(N)||^2$. Thus, the system must be at least limitedly stable in order to guarantee a finite value of the LQR cost as $N \to \infty$.

(b) We can use the same methodologies in Part (a) to check the stablizability and detectability. Then, we know the system is controllable and observable, which means the system is stabilizable and detectable.

i. Based on the following recursive relationship to calculate $P(k)$:

$$P(k-1) = A^T P(k)A + C^T C - A^T P(k)B \left[R + B^T P(k)B\right]^{-1} B^T P(k)A$$

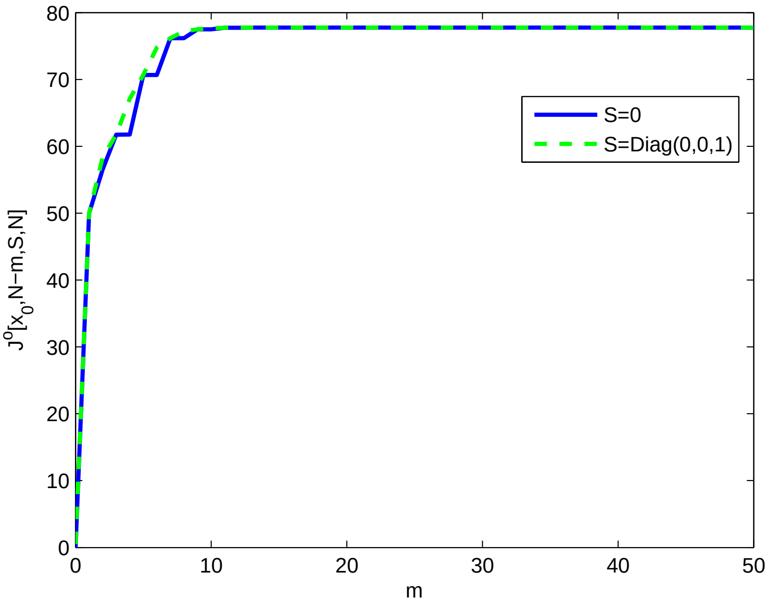

Figure 2: The optimal cost function for different S

ii. Based on the results from Part i, calculate $J^o[x_0, m, S, N] = \frac{1}{2}x_0^T P(m)x_0$.

iii. Based on the result from Part ii, plot $J^o[x_0, m, S, N]$ vs m as shown in Figure 2.

iv. To the solution of the `DARE`, we type the command

```
>> dare(A,B,C'*C,R)
```

into `MATLAB`. Then we get the following solution:

$$Pss = \begin{bmatrix} 113.2312 & 10.9845 & 1.0685 \\ 10.9845 & 41.1428 & 0.6637 \\ 1.0685 & 0.6637 & 40.1572 \end{bmatrix}.$$

For $S = 0$, we have:

$$P(0) = P(1) = \begin{bmatrix} 113.2312 & 10.9845 & 1.0685 \\ 10.9845 & 41.1428 & 0.6637 \\ 1.0685 & 0.6637 & 40.1572 \end{bmatrix}.$$

For $S =$Diag(0,0,1), we have:

$$P(0) = P(1) = \begin{bmatrix} 113.2312 & 10.9845 & 1.0685 \\ 10.9845 & 41.1428 & 0.6637 \\ 1.0685 & 0.6637 & 40.1572 \end{bmatrix}.$$

From the results, we can conclude that if the system is controllable and observable, the Riccati equation solution converges to a unique steady state solution which is equal to the solution to DARE regardless of the choice of S. Moreover, the steady state solution yields a control law that stabilizes the system.

(c) Since stabilizability and detectability both depend on stability of the system, we first check the stability of the system. For this system, the only stable eigenvalue/eigenvector pair is given by $\lambda_1 = 0.8$, $v_1 = [\ 1 \quad 0 \quad 0\ ]^T$. Using the same methodology as part (a), it is easily verified that $[A, B]$ is controllable, which in turn implies stabilizability. To check the observability of $[A, C]$, we type command

```
>> rank([C; C*A; C*A^2])
```

into MATLAB. Since this returns the value 2, we know that $[A, C]$ is not observable and, in particular, that the dimension of the unobservable subspace is $3 - 2 = 1$. However, since this system has a stable state, we must check whether or not the unobservable subspace is stable. Equivalently, we must check whether or not the unobservable subspace is a subspace of the stable subspace. Note that because

$$\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix},$$

a basis for the unobservable subspace is $\{v_1\}$. Thus, the unobservable subspace is equal to the stable subspace, which implies that $[A, C]$ is detectable.

i. Based on the following recursive relationship to calculate $P(k)$:

$$P(k-1) = A^T P(k)A + C^T C - A^T P(k)B\left[R + B^T P(k)B\right]^{-1} B^T P(k)A$$

ii. Based on the results from Part i, calculate $J^o[x_0, m, S, N] = \frac{1}{2} x_0^T P(m) x_0$.

iii. Based on the result from Part ii, plot $J^o[x_0, m, S, N]$ vs m as shown in Figure 3.

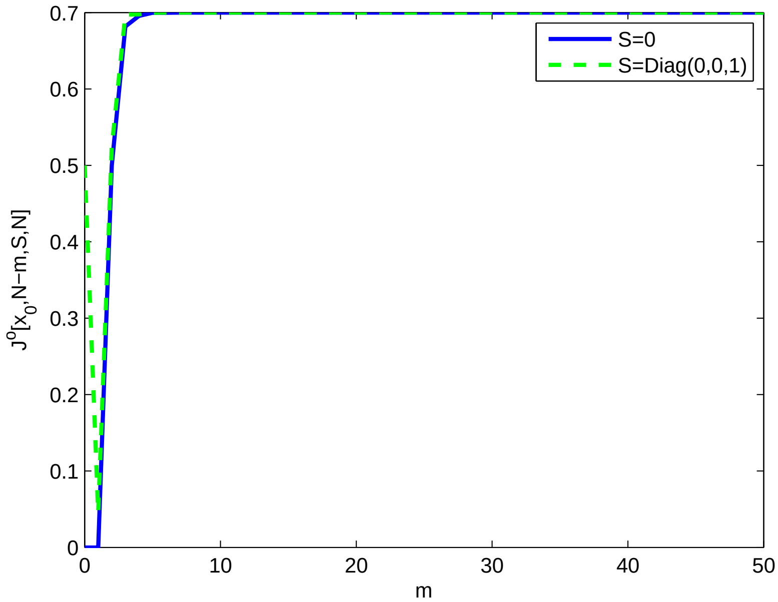

Figure 3: The optimal cost function for different S

iv. To the solution of the DARE, we type the command

```
>> dare(A,B,C'*C,R)
```

into MATLAB. Then we get the following solution:

$$Pss = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2.1429 & -1.2246 \\ 0 & -1.2246 & 1.3996 \end{bmatrix}.$$

For $S = 0$, we have:

$$P(0) = P(1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2.1429 & -1.2246 \\ 0 & -1.2246 & 1.3996 \end{bmatrix}.$$

For $S =$Diag(0,0,1), we have:

$$P(0) = P(1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2.1429 & -1.2246 \\ 0 & -1.2246 & 1.3996 \end{bmatrix}.$$

From the results, we can conclude that if the system is stabilizable and detectable, the Riccati equation solution converges to a unique steady state solution which is equal to the solution to DARE regardless of the choice of S. Moreover, the steady state solution yields a control law that stabilizes the system.

2. (a) By substituting $u(k) = -K_1 x(k) + u_1(k)$ into the cost function, we have:

$$\begin{aligned} J &= \frac{1}{2}\sum_{k=0}^{\infty}\left\{x^T(k)Qx(k) + 2x^T(k)Su(k) + u^T(k)Ru(k)\right\} \\ &= \frac{1}{2}\sum_{k=0}^{\infty}\left\{x^T(k)Qx(k) + 2x^T(k)S\left[-K_1x(k)+u_1(k)\right] + \left[-K_1x(k)+u_1(k)\right]^T R\left[-K_1x(k)+u_1(k)\right]\right\} \\ &= \frac{1}{2}\sum_{k=0}^{\infty}\left\{x^T(k)\left[Q - 2SK_1 + K_1^TRK_1\right]x(k) + 2x^T(k)\left[S - K_1^TR\right]u_1(k) + u_1(k)^TRu_1(k)\right\} \end{aligned}$$

In order to make the cross term be 0, let $S - K_1^TR = 0$ and then:

$$K_1 = R^{-1}S^T$$

The cost function changes to be:

$$\begin{aligned} J &= \frac{1}{2}\sum_{k=0}^{\infty}\left\{x^T(k)\left[Q - 2SR^{-1}S^T + SR^{-1}RR^{-1}S^T\right]x(k) + u_1(k)^TRu_1(k)\right\} \\ &= \frac{1}{2}\sum_{k=0}^{\infty}\left\{x^T(k)\left[Q - SR^{-1}S^T\right]x(k) + u_1(k)^TRu_1(k)\right\} \end{aligned}$$

Define $Q_1 = Q - SR^{-1}S^T \succeq 0$, then the cost function with the new input $u_1(k)$ is

$$J = \frac{1}{2}\sum_{k=0}^{\infty}\left\{x^T(k)Q_1x(k) + u_1(k)^TRu_1(k)\right\}$$

Obviously, by introducing the new input $u_1(k)$, the original problem can be transformed to a LQ problem.

3. (a) From the return difference equality, the closed-loop eigenvalues satisfy the following equation:

$$\begin{aligned} & 1 + \frac{1}{R}G(z)G(z^{-1}) = 0 \\ \Rightarrow\quad & 1 + \frac{1}{R}\left(\frac{z(z+2)}{(z-1)(z-2)(z+0.5)}\right)\left(\frac{z^{-1}(z^{-1}+2)}{(z^{-1}-1)(z^{-1}-2)(z^{-1}+0.5)}\right) = 0 \\ \Rightarrow\quad & 1 + \frac{1}{R}\frac{2z^2(z+2)(z+0.5)}{(z-1)(z-2)(z+0.5)(z-1)(z-0.5)(z+2)} = 0 \end{aligned}$$

The root locus is shown in Figure 4. It should be noted that zeros and poles coincide at $z = -0.5$ and $z = -2$. To keep the right number of eigenvalues, they should not be cancelled. Also note that since the closed-loop system is stable, the closed-loop eigenvalues are the 3 eigenvalues inside of the unit circle.

(b) When $R \to 0$, $\mathrm{eig}(A_c) \to \{0, 0, -0.5\}$. This makes intuitive sense because as R becomes small, control effort is lightly penalized. Thus, we should be able to apply greater control to the system and, thus, attain faster responses.

(c) When $R \to \infty$, $\mathrm{eig}(A_c) \to \{1, 0.5, -0.5\}$. This makes intuitive sense because as R becomes large, control effort is heavily penalized, which prevents the control system from applying much control to the system. Thus, we expect the system to have a slower responses.

(d) (See number 3a)

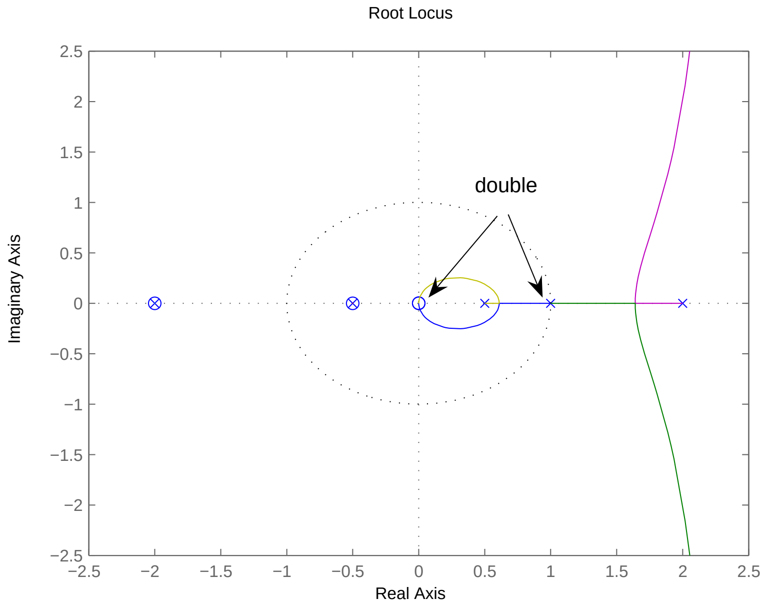

Figure 4: Locus of the closed-loop system eigenvalues

(e) Using the MATLAB-generated root locus plot from part (3d), you can get the plot shown in Figure 5. The gain at that point is 0.0312, so the value of R_o is $1/0.0312 = 32.05$.

(f) Expanding the numerator and denominator in $G(z)$, we get that

$$G(z) = \frac{z^2 + 2z}{z^3 - 2.5z^2 + 0.5z + 1}$$

Thus, our controllable canonical realization is given by

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -0.5 & 2.5 & 1 \\ \hline 0 & 2 & 1 & 0 \end{array}\right]$$

(g) To use the function `dare`, first define the matrices `A`, `B`, `C` as in part (3f) and `R0` as the value obtained in part (3e). Then, at the MATLAB command prompt, type

```
>> [P0,E0,K0] = dare(A,B,C'*C,R0)
```

This will assign the solution of the Discrete-time Algebraic Riccati Equation to `P0`, the closed-loop eigenvalues to `E0`, and the control gain to `K0`. This gives

$$P_o = \begin{bmatrix} 26.0945 & 23.6539 & -57.0703 \\ 23.6539 & 28.3387 & -43.9380 \\ -57.0703 & -43.9380 & 140.4052 \end{bmatrix}$$

$$E_o = \begin{bmatrix} 0.6156 & 0.6038 & -0.5000 \end{bmatrix}^T$$

$$K_o = \begin{bmatrix} -0.8141 & -0.7380 & 1.7806 \end{bmatrix}$$

Note that two of the closed-loop eigenvalues are very similar for R_o. Although we would ideally find these two closed-loop eigenvalues to be the same, this is the best we can do with the precision to which we know R_o. (It is easy to verify that if we increase or decrease $1/R_o$ by 0.0001, our closed-loop eigenvalues become complex or get farther apart, respectively.) In order to get these two closed-loop eigenvalues to be closer together, we would need to find R_o to greater precision.

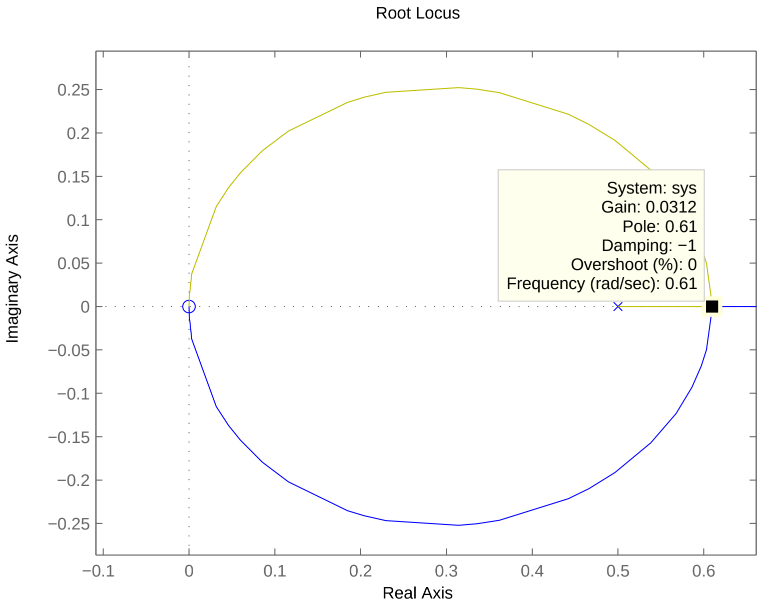

Figure 5: Locus of the closed-loop system eigenvalues with double root shown

(h) As discussed in class, our LQR controller can be graphically represented as shown in Figure 6. Since $G_o(z) = K_o(zI - A)^{-1}B$ is a SISO transfer function, we can use gain and phase margin as a measure of the robustness of the closed-loop stability.

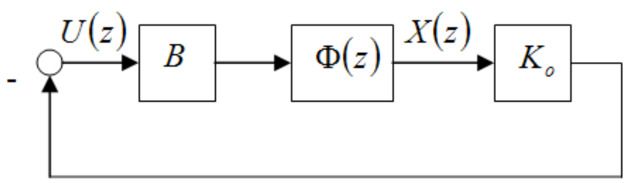

Figure 6: Representation of closed loop system

To do this, we create a state-space model of $G_o(z)$ by typing

```
>> sys_cl = ss(A,B,K0,0,-1)
```

At this point, we can either use `bode` or `nyquist` to find our gain and phase margins for G_o. For either case, the margins are shown by right-clicking on the diagram and selecting "All Stability Margins" from the "Characteristics" submenu. Figure 7 shows the results of using `nyquist` to find the margins of G_o. Reading the relevant numbers off of the diagram, we see that the phase margin is 25.5° and the gain margins are 4.91dB and -4.23dB.

Plotting a root locus of $1+\gamma G_o(z)$ gives an alternative method for finding the gain margins of our closed-loop system. This plot is shown in Figure 8. If we increase γ to 1.76 (4.91dB) or decrease γ to 0.615 (-4.22dB), the closed-loop system becomes marginally stable. Thus, our gain margins are verified.

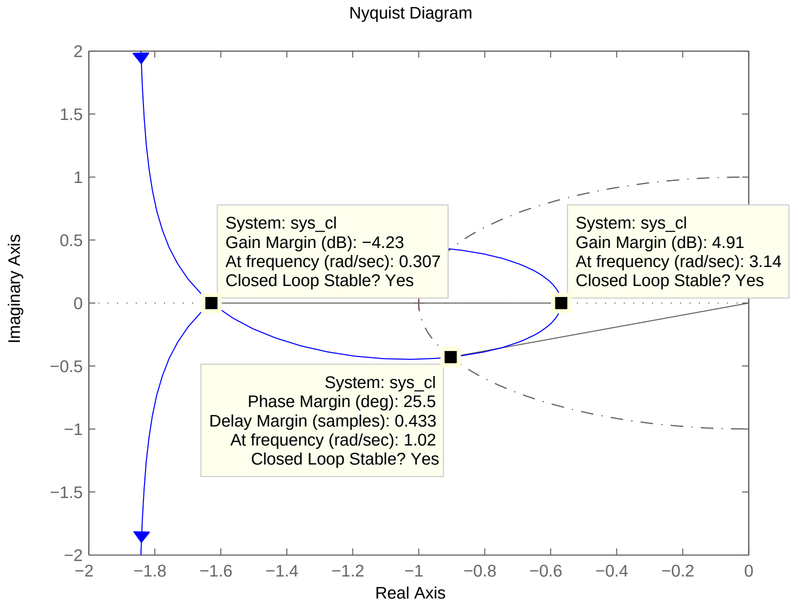

Figure 7: Nyquist plot of $G_o(z)$

(i) As mentioned in lecture, the guaranteed gain margins of the closed-loop system are given by

$$\gamma_1 = \frac{1}{1+\sqrt{R_o/(R_o+B^TP_oB)}} = 1.7595 = 4.91dB$$
$$\gamma_2 = \frac{1}{1-\sqrt{R_o/(R_o+B^TP_oB)}} = 0.6985 = -3.12dB$$

and the guaranteed phase margin of the closed-loop system is given by

$$PM = 2 * \sin^{-1}\left\{\frac{1}{2}\sqrt{\frac{R_o}{R_o+B^TP_oB}}\right\} = 24.9^\circ$$

Thus, although our upper gain margin (4.91dB) is not any better than the worst-case upper gain margin, our lower gain margin (-4.23dB) is a noticeable improvement over the worst-case lower gain margin. Our phase margin (25.5°) is also a slight improvement over the worst-case phase margin.

(j) Defining a coordinate change by $\overline{x}(k) = Tx(k)$ gives

$$\begin{aligned} \overline{x}(k+1) &= TAT^{-1}\overline{x}(k) + TBu(k) \\ y(k) &= CT^{-1}\overline{x}(k) \end{aligned}$$

Thus, we define

$$\begin{aligned} \overline{A} &= TAT^{-1} \\ \overline{B} &= TB \\ \overline{C} &= CT^{-1} \end{aligned}$$

to get a different state space realization of our system given by

$$\begin{aligned} \overline{x}(k+1) &= \overline{A}\overline{x}(k) + \overline{B}u(k) \\ y(k) &= \overline{C}\overline{x}(k) \end{aligned}$$

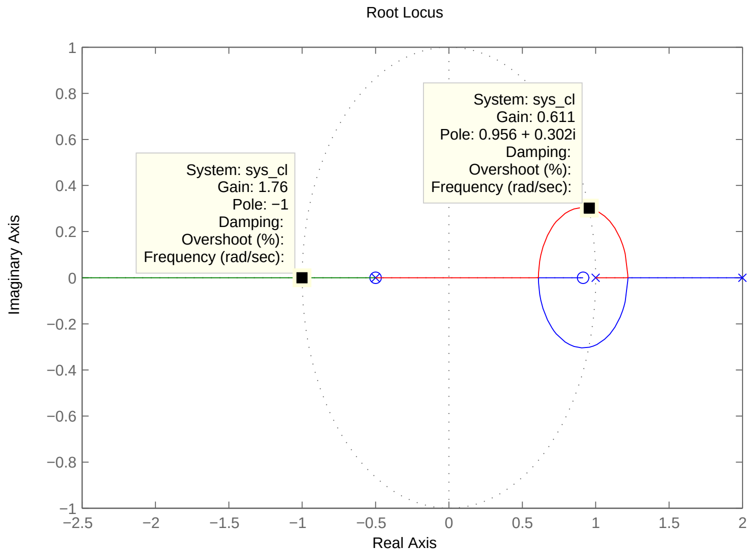

Figure 8: Root locus verification of gain margins

Now we assume that P is the positive definite solution of

$$0 = A^T PA - P + C^T C - A^T PB \left[R + B^T PB\right]^{-1} B^T PA$$

Substituting for A, B, and C in terms of $\overline{A}$, $\overline{B}$, and $\overline{C}$ gives

$$\begin{aligned} 0 &= \left(T^{-1}\overline{A}T\right)^T P \left(T^{-1}\overline{A}T\right) - P + T^T \overline{C}^T \overline{C} T \\ &\quad - \left(T^{-1}\overline{A}T\right)^T P \left(T^{-1}\overline{B}\right) \left[R + \left(T^{-1}\overline{B}\right)^T P \left(T^{-1}\overline{B}\right)\right]^{-1} \left(T^{-1}\overline{B}\right)^T P \left(T^{-1}\overline{A}T\right) \end{aligned}$$

Multiplying on the left by $(T^{-1})^T$ and on the right by T^{-1} gives

$$\begin{aligned} 0 &= \overline{A}^T \overline{PA} - \overline{P} + \overline{C}^T \overline{C} - \overline{A}^T \overline{PB} \left[R + \overline{B}^T \overline{PB}\right]^{-1} \overline{B}^T \overline{PA} \\ \overline{P} &= \left(T^{-1}\right)^T PT^{-1} \end{aligned}$$

Noticing that $\overline{P} > 0$, we see that $\overline{P}$ is the positive definite solution of the Riccati equation in the transformed coordinates. With this, it is trivial to see that $B^T PB = \overline{B}^T \overline{PB}$. This means that the quantity $\sqrt{R/(R + B^T PB)}$ is not a function of the realization. Thus, the worst-case gain and phase margins do not depend on the realization of the system being considered.