UNIVERSITY OF CALIFORNIA AT BERKELEY
Department of Mechanical Engineering
ME233 Advanced Control Systems II
Spring 2012

Homework #2

Assigned: Jan. 31 (Tu)
Due: Feb. 9 (Th)

1. Consider the following two-sided infinite sequence

$$h(k) = f(k) + f(-k) + c\delta(k)$$

where $c \in \mathcal{R}$ and $\delta(k)$ is the Kronecker delta. Throughout this problem, we will denote the two-sided Z-transforms of $f(k)$ and $h(k)$ respectively as $F(z)$ and $H(z)$.

   (a) Find $H(z)$ in terms of F and c.

   (b) Let

$$f(k) = \begin{cases} ba^k \ , & k \geq 1 \\ 0 \ , & k \leq 0 \end{cases}$$

   where $a, b \in \mathcal{R}$ and $|a| < 1$. Using the result of part (a), find $H(z)$. Express your answer in the form

$$H(z) = \frac{\alpha(z + z^{-1}) + \beta}{(z-a)(z^{-1}-a)} \ .$$

2. Consider the stochastic system

$$Y(k) - 0.5Y(k-1) = W(k) - 0.3W(k-1) \tag{1}$$

where $W(k)$ is a wide sense stationary (WSS) zero mean white random sequence with unit variance, i.e.

$$m_{_W} = 0 \qquad \Lambda_{_{WW}}(l) = E\{W(k+l)W(k)\} = \delta(l)$$

and $\delta(l)$ is the unit pulse function. In this problem, we will compare the theoretical value of the relevant covariances with empirical estimates of those quantities computed using MATLAB. [1]

   (a) Do a matlab simulation of the response of this system for one sample sequence $w(k)$:

      i. Generate the sample sequence $w(k)$ using `w = randn(N,1);` , where `N` is a large number (e.g. 5000).

[1]Since MATLAB requires initial conditions to perform time simulations, the output of the system given by Eq. (1) will not, strictly speaking be WSS. However, if the length of the sample sequence is taken to be sufficiently long, the relevant quantities will be approximately given by time averages.

ii. Generate the sample output sequence $y(k)$ using

```
[y,k] = lsim(sys1,w,k);
```

Notice that the vector `k` must be defined.

iii. Generate and plot the the estimates of the covariances and cross-covariances $\Lambda_{WW}(j)$, $\Lambda_{WY}(j)$, $\Lambda_{YW}(j)$, $\Lambda_{YY}(j)$, for $j = \{-10, -9, \cdots, 0, \cdots 10\}$ using the matlab command `xcov`, e.g.

```
cov_wy = xcov(w,y,10,'coeff');
```

Read the help on `xcov` to understand what the argument `'coeff'` does. [2]

(b) Determine the cross-covariance (cross-correlation) function

$$\Lambda_{YW}(l) = E\{Y(k+l)W(k)\}$$

and $\hat{\Lambda}_{YW}(z) = \sum_{l=-\infty}^{\infty} z^{-l}\Lambda_{YW}(l)$. Plot $\Lambda_{YW}(l)$ for $l = \{-10, -9, \cdots, 0, \cdots 10\}$ and compare your results with those empirically obtained using MATLAB. Notice that $\Lambda_{YW}(l)$ is a casual sequence, i.e. $\Lambda_{YW}(l) = 0$ for $l < 0$ and all the poles of $\hat{\Lambda}_{YW}(z)$ will be inside the unit circle.

(c) Determine the cross-covariance (cross-correlation) function

$$\Lambda_{WY}(l) = E\{W(k+l)Y(k)\}$$

and $\hat{\Lambda}_{WY}(z) = \sum_{l=-\infty}^{\infty} z^{-l}\Lambda_{WY}(l)$. Plot $\Lambda_{WY}(l)$ for $l = \{-10, -9, \cdots, 0, \cdots 10\}$ and compare your results with those empirically obtained using MATLAB. Notice that $\Lambda_{WY}(l)$ is an anti-casual sequence, i.e. $\Lambda_{WY}(l) = 0$ for $l > 0$ and all the poles of $\hat{\Lambda}_{WY}(z)$ will be outside the unit circle.

(d) Determine the auto-covariance (auto-correlation) function

$$\Lambda_{YY}(l) = E\{Y(k+l)Y(k)\}$$

and $\hat{\Lambda}_{YY}(z) = \sum_{l=-\infty}^{\infty} z^{-l}\Lambda_{YY}(l)$. Plot $\Lambda_{YY}(l)$ for $l = \{-10, -9, \cdots, 0, \cdots 10\}$ and compare your results with those empirically obtained using MATLAB. Notice that $\hat{\Lambda}_{YY}(z)$ will have poles both outside and inside the unit circle.

Hint: After finding $\hat{\Lambda}_{YY}(z)$, find its inverse Z-transform using the results of problem 1.

(e) Compute $\Lambda_{YW}(0)$ utilizing Eq. (1).

Hint: Multiply both sides of Eq. (1) by $W(k)$ and take expectations.

(f) Compute $\Lambda_{YW}(1)$ utilizing Eq. (1).

Hint: Multiply both sides of Eq. (1) by $W(k-1)$ and take expectations.

(g) Compute $\Lambda_{YY}(0)$ utilizing equation (1).

Hint: From Eq. (1) we have

$$Y(k) = 0.5Y(k-1) + W(k) - 0.3W(k-1)\ . \tag{2}$$

Square both sides of Eq. (2) and take expectations.

[2] The MATLAB function `xcov` is part of the signal processing toolbox. Those of you who do not have access to this toolbox can use a similar function that can be downloaded from the ME233 bSpace website.

3. Consider a second order discrete time system described by

$$\begin{bmatrix} X_1(k+1) \\ X_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.08 & -1 \\ 0.7 & 0.1 \end{bmatrix} \begin{bmatrix} X_1(k) \\ X_2(k) \end{bmatrix} + \begin{bmatrix} 0.34 \\ 0.3 \end{bmatrix} W(k)$$
$$Y(k) = \begin{bmatrix} 0 & 3 \end{bmatrix} \begin{bmatrix} X_1(k) \\ X_2(k) \end{bmatrix} + V(k)$$

where

- $E\{X(0)\} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $E\{X(0)X^T(0)\} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$
- $W(k)$ and $V(k)$ are white Gaussian sequences
- $m_w = E\{W(k)\} = 10, \quad E\{V(k)\} = 0$
- $E\left\{ \begin{bmatrix} W(k+j) - m_w \\ V(k+j) \end{bmatrix} \begin{bmatrix} (W(k) - m_w) & V(k) \end{bmatrix} \right\} = \begin{bmatrix} \Sigma_{ww} & 0 \\ 0 & \Sigma_{vv} \end{bmatrix} \delta(j)$ where $\Sigma_{ww} = 1$ and $\Sigma_{vv} = 0.5$.
- $E\left\{ \begin{bmatrix} W(k) - m_w \\ V(k) \end{bmatrix} X^T(0) \right\} = 0$

(a) Use MATLAB to plot $m_y(k) = E\{Y(k)\}$ for $k = 0, 1, \ldots$ until $m_y(k)$ reaches its steady state value $\bar{m}_y$.

(b) Using MATLAB, compute

$$\Lambda_{XX}(k, 0) = E\{(X(k) - m_{_X}(k))(X(k) - m_{_X}(k))^T\}$$

utilizing the covariance propagation Lyapunov equation and plot

$$\Lambda_{YY}(k, 0) = E\{(Y(k) - m_{_Y}(k))^2\}$$

for $k = 0, 1, \ldots$ until $\Lambda_{YY}(k, 0)$ reaches its steady state value $\bar{\Lambda}_{YY}(0)$.

(c) Using MATLAB, compute

$$\Lambda_{XX}(k, 5) = E\{(X(k+5) - m_x(k+5))(X(k) - m_x(k))^T\}$$

and plot

$$\Lambda_{YY}(k, 5) = E\{(Y(k+5) - m_y(k+5))(Y(k) - m_y(k))\}$$

for $k = 0, 1, \ldots,$ until it reaches its steady state value $\bar{\Lambda}_{YY}(5)$.

(d) Use the MATLAB function `dlyap` to compute $\bar{\Lambda}_{XX}(0)$. Then compute and plot the steady state covariances $\bar{\Lambda}_{YY}(j)$ for $j = \{-10, -9, \cdots, 0, 1, \cdots 10\}$.

(e) Let $G(z) = C(zI - A)^{-1}B$ be the transfer function from W to Y. Obtain an expression for the (steady state) output spectral density, $\Phi_{YY}(\omega)$ in terms of G, Σ_{ww} and Σ_{vv}. (Do not explicitly determine $G(z)$ or substitute in the numerical values for Σ_{ww} and Σ_{vv} in this problem.)

(f) Use MATLAB to plot the output spectral density $\Phi_{YY}(\omega)$ for $\omega \in [-\pi, \pi]$.

4. Let $X \sim N(10, 2)$, $V_1 \sim N(0, 1)$ and $V_2 \sim N(0, 2)$ be independent random variables. Assume that you are trying to make a measurement of X with two different instruments. Let $Y = X + V_1$ be the measurement of X using the first instrument and $Z = X + V_2$ be the measurement of X using the second instrument, where V_1 and V_2 are respectively the measurement noises of the first and second instruments.

   (a) Determine $m_{X|Y=9}$, i.e. the conditional expectation of X given that the first instrument yielded the measurement $Y = 9$.

   (b) Determine $m_{X|Z=11}$, i.e. the conditional expectation of X given that the second instrument yielded the measurement $Z = 11$.

   (c) Determine $m_{X|(Y=9,Z=11)}$, i.e. the conditional expectation of X given that the first and second instruments respectively yielded the measurements $Y = 9$ and $Z = 11$.

5. A random variable X is repeatedly measured, but the measurements are noisy. Assume that the measurement process can be described by

$$Y(k) = X + V(k)$$

where $X, V(0), V(1), V(2), \ldots$ are jointly Gaussian random variables with

$$\begin{aligned} E\{X\} &= 0 & E\{X^2\} &= X_0 \\ E\{V(k)\} &= 0 & E\{V(k+j)V(k)\} &= \Sigma_V \delta(j) \\ E\{XV(k)\} &= 0\,. \end{aligned}$$

Let $y(k)$ be the k-th measurement (i.e. outcome of $Y(k)$) and let $\bar{y}(k) = \{y(0), \ldots, y(k)\}$.

   (a) Obtain the least squares estimate of X given the $k+1$ measurements $y(0), \ldots, y(k)$ and the corresponding estimation error covariance, i.e. find $\hat{x}_{|\bar{y}(k)}$ and $\Lambda_{\tilde{X}_{|\bar{y}(k)}\tilde{X}_{|\bar{y}(k)}}$.
   Hint: You do not need to invert a $(k+1) \times (k+1)$ matrix to find these quantities. Instead express

$$\Lambda_{\bar{y}(k)\bar{y}(k)} = A + uv^T$$

   where A is a matrix that is easy to invert and u and v are vectors. In this case, the matrix inversion lemma says that

$$\Lambda^{-1}_{\bar{y}(k)\bar{y}(k)} = A^{-1} - \frac{1}{1 + v^T A^{-1} u} A^{-1} uv^T A^{-1}\,.$$

   (b) We now examine the case when $X_0 \to \infty$, i.e. when no prior information is available on X. Show the following:

$$\lim_{X_0 \to \infty} \left(\hat{x}_{|\bar{y}(k)} \right) = \frac{1}{k+1} \left[y(0) + y(1) + \cdots + y(k) \right]$$

$$\lim_{X_0 \to \infty} \left(\Lambda_{\tilde{X}_{|\bar{y}(k)}\tilde{X}_{|\bar{y}(k)}} \right) = \frac{\Sigma_V}{k+1}\,.$$