UNIVERSITY OF CALIFORNIA AT BERKELEY
Department of Mechanical Engineering
ME233 Advanced Control Systems II
Spring 2012

Homework #5

Assigned: Mar. 6 (Tu)
Due: Mar. 15 (Th)

1. Consider the state-space realization $G(z) = C(zI - A)^{-1}B + D$, where $D^T D$ is invertible. Assume that the dimensions of the matrices are given by $A \in \mathcal{R}^{n_x \times n_x}$, $B \in \mathcal{R}^{n_x \times n_u}$, $C \in \mathcal{R}^{n_y \times n_x}$, and $D \in \mathcal{R}^{n_y \times n_u}$. In this problem, we will establish the relationship between the transmission zeros of this realization and the unobservable modes of $(\hat{C}, \hat{A})$, where

$$\hat{A} = A - B(D^T D)^{-1} D^T C$$
$$\hat{C} = C - D(D^T D)^{-1} D^T C \ .$$

   (a) Show that, for any matrix M, the columns of the matrix X are linearly independent if and only if the columns of the matrix

$$Z := \begin{bmatrix} I \\ M \end{bmatrix} X$$

   are linearly independent

   Hint: A good way to start is by showing that the null space of X is equal to the null space of Z.

   (b) Using the result from part (a), prove that the following conditions are equivalent:

   - $\begin{bmatrix} \hat{A} - \lambda I \\ \hat{C} \end{bmatrix} X = 0$ and the columns of X are linearly independent
   - $\exists Y$ such that $\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = 0$ and the columns of $\begin{bmatrix} X \\ Y \end{bmatrix}$ are linearly independent

   (c) Using the result from part (b), prove that

$$\text{nullity} \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} = \text{nullity} \begin{bmatrix} \hat{A} - \lambda I \\ \hat{C} \end{bmatrix}, \qquad \forall \lambda \in \mathcal{C} \ .$$

   (d) Using the result from part (c), find the relationship between the transmission zeros of the state-space realization $G(z) = C(zI - A)^{-1}B + D$ and the unobservable modes of $(\hat{C}, \hat{A})$.

   Hint: First convert the condition in part (b) into a condition relating the rank of the two matrices. Then show that

$$\text{rank} \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} < \text{normalrank} \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} \quad \Leftrightarrow \quad \text{rank} \begin{bmatrix} \hat{A} - \lambda I \\ \hat{C} \end{bmatrix} < n_x \ .$$

2. When designing an infinite-horizon LQR for the discrete-time system

$$x(k+1) = Ax(k) + Bu(k)$$

under the cost function

$$J = \sum_{k=0}^{\infty} p^T(k)p(k)$$

where $p(k) = Cx(k) + Du(k)$ and $D^T D$ is invertible, we can guarantee that the optimal solution exists provided that (A, B) is stabilizable and a condition involving the transmission zeros of the state space realization $C(zI - A)^{-1}B + D$ holds. Using the result of problem 1, show that when $C^T D = 0$, the transmission zeros of the state space realization $C(zI - A)^{-1}B + D$ are the unobservable modes of (C, A).

3. In this problem we will verify some results concerning the convergence of the LQR's discrete Riccati equation (DRE) to a steady state solution and the existence, uniqueness and closed loop stability of the discrete algebraic Riccati equation (DARE) solution.

   Consider the design of an optimal LQR for the LTI discrete-time system

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\,x(k) \end{aligned} \tag{1}$$

   where $u(k) = -K(k+1)\,x(k)$ is the optimal control input that minimizes the following cost criteria

$$J[x_o, m, Q_f, N] = x^T(N)\,Q_f\,x(N) + \sum_{k=m}^{N-1} \left\{ y^2(k) + R\,u^2(k) \right\} \quad \text{s.t.} \quad x(m) = x_o\,,$$

   for $m = 0$, $Q_f = Q_f^T \succeq 0$ and $R = R^T \succ 0$, and any arbitrary initial condition $x_o \in \mathcal{R}^n$. Define the optimal value value function

$$J^o[x_o, m, Q_f, N] = \min_{U_{[m,N-1]}} J[x_o, m, Q_f, N]$$

   where $U_{[m,N-1]} = \{u(m), \cdots, u(N-1)\}$ is the set of all possible control actions from $k = m$ to $k = N-1$.

   (a) Let

$$A = \begin{bmatrix} 1.2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 10 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

   and verify that (A, B) is stabilizable but (C, A) is not detectable. Let $P(N) = Q_f$. For each of the four cases

   i. $Q_f = 0$

ii. $Q_f = \text{diag}(0,\ 0,\ 1)$,
iii. $Q_f = \text{diag}(1,\ 1,\ 1)$,
iv. $Q_f = \text{diag}(10,\ 1,\ 1)$,

do the following:

- For $x_o = [1\ \ 0\ \ 1]^T$, plot $J^o[x_o, N-m, Q_f, N]$ vs m. (Note that this will require computing $P(k)$, the solution of the Riccati difference equation, backwards from $P(50) = Q_f$.)
- Compute the solution of the DARE using the MATLAB command `dare` and compare it with values of $P(0)$ and $P(1)$.

Discuss your results.

(b) Let

$$A = \begin{bmatrix} 0.8 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \qquad R = 0.1\ .$$

Repeat part (a) for the two cases

i. $Q_f = 0$
ii. $Q_f = \text{diag}(0,\ 0,\ 1)$.

Discuss your results.

4. Consider the design of an optimal LQR for the SISO controllable and observable LTI discrete-time system described by Eq. (1),

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\,x(k) \end{aligned}$$

where $u(k) = -K\,x(k)$ is the optimal control input that minimizes the following cost criteria

$$J = \sum_{k=0}^{\infty} \left\{ y^2(k) + u^T(k) R u(k) \right\}\ ,$$

$R \in (0, \infty)$ is the input weight and

$$G(z) = C(zI - A)^{-1}B = \frac{z(z+2)}{(z-1)(z+0.5)(z-2)} \tag{2}$$

(a) Draw (by hand) the locus of the eigenvalues of $A_c = A - BK$ and their respective reciprocals for $R \in (0, \infty)$.

(b) What are the eigenvalues of $A_c = A - BK$ for $R \to 0$?

(c) What are the eigenvalues of $A_c = A - BK$ for $R \to \infty$?

(d) Use the MATLAB function `rlocus` to verify your answers to parts (a)-(c).

(e) Use the MATLAB function `rlocfind` (or `rlocus`) to determine the unique value of the input weight R_o for which all closed-loop eigenvalues are real and two eigenvalues are equal (i.e. double roots) and nonzero.

(f) Find the controllable canonical realization for the transfer function $G(z)$ in Eq. (2).

(g) Using the canonical realization obtained in (f), compute the following quantities (using MATLAB) for the LQR problem defined above with $R = R_o$ (the value determined in part (e)): the solution of the algebraic Riccati equation P_o, the optimal gain K_o and the location of the closed-loop eigenvalues.

5. (Former midterm problem) Consider the following stationary stochastic system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.8 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(k) \tag{3}$$
$$y(k) = x_1(k) + v(k)$$

where $u(k)$ is a deterministic (known) input, $y(k)$ is the measured output, $w(k)$ and $v(k)$ are both white, zero mean, Gaussian and stationary random noises, and

$$E\left\{ \begin{bmatrix} w(k+j) \\ v(k+j) \end{bmatrix} \begin{bmatrix} w(k) & v(k) \end{bmatrix} \right\} = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix} \delta(j)$$

(a) Determine if there exists a unique steady state solution to the Riccati difference equation governing the Kalman filter a-priori state estimation error covariance, $M(k)$.

(b) Sketch the root locus of the stationary Kalman filter closed-loop poles and their reciprocals for $\frac{W}{V} \in (0 \in \infty)$.

(c) The system in Eq. (3) can be described by the following ARMAX model:

$$(1 - 0.8z^{-1})Y(z) = z^{-2}U(z) + (1 - 0.5z^{-1})\tilde{Y}^o(z)$$

where

- $Y(z) = \mathcal{Z}\{y(k)\}$, $U(z) = \mathcal{Z}\{u(k)\}$, $\tilde{Y}^o(z) = \mathcal{Z}\{\tilde{y}^o(k)\}$.
- $\tilde{y}^o(k) = y(k) - \hat{y}^o(k)$ is the steady state Kalman filter residual (i.e. a-priori output estimation error) and

$$E\{(\tilde{y}^o(k))^2\} = 1\,.$$

Determine the values of noise variances W and V.

6. Consider a discrete-time process described by

$$x(k+1) = Ax(k) + Bu(k) + B_w w(k)$$

where the stationary, Gaussian white noise $w \in \mathcal{R}^s$ satisfies

$$E\{w(k)\} = 0, \qquad E\{w(k+j)w^T(k)\} = W\delta(j) \ .$$

The signals x and u respectively have dimension n and m.

There are two sensors configurations we will be considering.

Sensor Configuration A: In this configuration, the output equation is

$$y(k) = Cx(k) + v_A(k)$$

where $y \in \mathcal{R}^r$ is output vector and $v_A \in \mathcal{R}^r$ is a stationary Gaussian measurement noise. The measurement noise is independent from the initial state and input noise, and the following quantities are given

$$E\{v_A(k)\} = 0, \qquad E\{v_A(k+j)v_A^T(k)\} = V_A\delta(j) \ .$$

Sensor Configuration B: For the purpose of preparing for any sensor failures, this configuration uses two identical sets of sensors and measures the output twice. With this sensor configuration the measurement vector is $2r$ dimensional, and it is given by

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} C \\ C \end{bmatrix} x(k) + \begin{bmatrix} v_{B1}(k) \\ v_{B2}(k) \end{bmatrix}$$

where the measurement noises are independent from the initial state and the input noise, and the following quantities are given for $i = 1, 2$:

$$E\{v_{Bi}(k)\} = 0, \quad E\{v_{Bi}(k+j)v_{Bi}^T(k)\} = V_B\delta(j), \quad E\{v_{B1}(k+j)v_{B2}^T(k)\} = 0$$

While the duplication of the output measurement increases the hardware cost in Sensor Configuration B, it is also true that the specification for each sensor may be relaxed if the same output is measured twice and the two measurements are used in the Kalman filter.

(a) List a set of conditions that guarantee that the stationary Kalman filter exists for Sensor Configuration A. Show that these conditions also guarantee that the stationary Kalman filter exists for Sensor Configuration B. (Do not assume that A is Schur.)

(b) Let the assumptions you listed in part (a) hold and assume that you design and use a Kalman filter for each sensor configuration. Determine a relationship between the sensor noise covariances V_A and V_B so that $M_A = M_B$, where M_A is the steady state a-priori state estimation error covariance for Sensor Configuration A, and M_B is the steady state a-priori state estimation error covariance for Sensor Configuration B.

Hint: The following relationships for stationary Kalman filters are helpful in solving this problem:

- $(I + MC^T V^{-1} C)^{-1} = I - MC^T (CMC^T + V)^{-1} C$
- $A - LC = A\left[I - MC^T (CMC^T + V)^{-1} C\right]$
- $M = (A - LC)MA^T + B_w W B_w^T$

The first relationship comes from the matrix inversion lemma, the second relationship expresses $A - LC$ as a function of M, and the third relationship is a restatement of the discrete algebraic Riccati equation.