ME 233 Spring 2010
Solution to Homework #7

1. (a) The poles of this system, which are the eigenvalues of the A matrix of the realization, are at $z = 0, 0.8$. This means that the system is stable, which in turn implies that it must be stabilizable and detectable. Thus, there exists a unique steady state solution to the Riccati equation for the Kalman filter a priori state estimation error covariance, M.

   (b) Notice that the pair $[A, B_w W]$ is stabilizable but not controllable, since the disturbance $w(k)$ does not contaminate the state element $x_2(k)$. As a consequence the transfer function $G_w(z)$ has a pole-zero cancelation at the origin.

$$\begin{aligned} G_w(z) &= C\left[zI - A\right]^{-1} B_w \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z-0.8 & -1 \\ 0 & z \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{z}{z(z-0.8)} \\ G_w(z^{-1})G_w(z) &= -1.25\frac{z^2}{z(z-0.8)(z-1.25)} \end{aligned}$$

From the return difference equality, we obtain

$$\begin{aligned} \frac{(z-a_c)(z^{-1}-a_c)}{(z-0.8)(z^{-1}-0.8)} &= \gamma\left[1 + \frac{W}{V}(G_w(z^{-1})G_w(z)\right] \\ \frac{(z-a_c)(z-\frac{1}{a_c})}{(z-0.8)(z-1.25)} &= \gamma\left[1 - 1.25\frac{W}{V}\,\frac{z^2}{z(z-0.8)(z-1.25)}\right] \end{aligned} \tag{1}$$

Because $G_w(z^{-1})G_w(z)$ has a negative gain, we use positive feedback rules for the root locus plot, as shown in Figure 1.

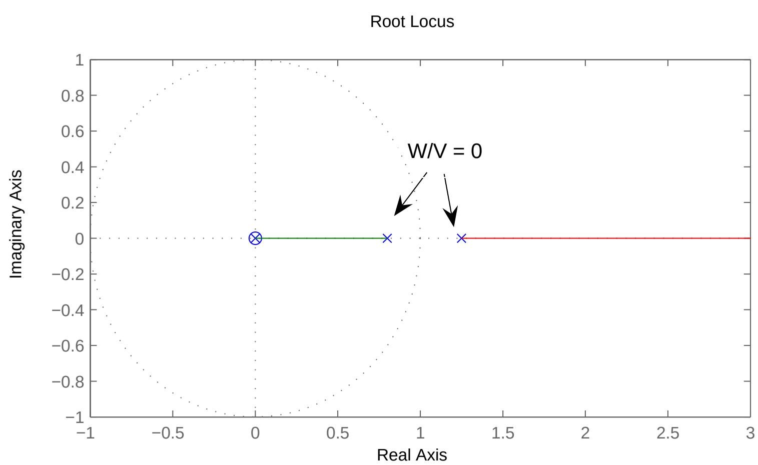

Figure 1: Locus of the closed-loop system eigenvalues

(c) **First method** Notice that $x_2(k+1) = u(k)$, which is deterministic. Therefore, the plant and a-priori Kalman filter dynamics for this system can respectively be written as follows:

$$\begin{aligned} x_2(k+1) &= u(k) \\ x_1(k+1) &= a\,x_1(k) + u(k-1) + w(k) \\ y(k) &= x_1(k) \end{aligned}$$

Hence, the a-priori Kalman filter can be written as

$$\begin{aligned} \hat{x}_2^o(k+1) &= u(k) \\ \hat{x}_1^o(k+1) &= a\,x_1^o(k) + u(k-1) + L\,\tilde{y}^o(k) \\ y(k) &= x_1^o(k) + \tilde{y}^o(k) \end{aligned} \tag{2}$$

where $a = 0.8$ and $L = \frac{a\,m}{m+V} \in \mathcal{R}$, $m = E\{(\tilde{x}_1^o(k))^2\}$ is the covariance of the a-priori state $x_1(k)$ estimation error and $\tilde{y}^o(k)$ is the Kalman filter residual.
From (2) we obtain the ARMAX equation

$$\begin{aligned} (1 - az^{-1})Y(z) &= z^{-2}U(z) + (1 - (a-L)z^{-1})\tilde{Y}^o(z) \\ z(z-a)Y(z) &= U(z) + z(z - (a-L))\tilde{Y}^o(z) \\ z(z-0.8)Y(z) &= U(z) + z(z-0.5)\tilde{Y}^o(z) \end{aligned}$$

Thus, since $a = 0.8$ and $a - L = 0.5$, we obtain $L = 0.3$. Moreover, from $L = \frac{a\,m}{m+V}$ and since $m + V = 1$, we obtain $m = \frac{L}{a} = 0.375$ and $V = 1 - 0.375 = 0.625$.

To obtain W we utilize the root locus gain rule. Canceling the pole and zero at the origin in the right hand side of equation (1), we use the root locus gain rule to determine the value of $\frac{W}{V}$ so that $a_c = 0.5$:

$$1.25\frac{W}{V}\,\frac{0.5}{|0.5-0.8|\,|0.5-1.25|} = 1 \quad \Rightarrow \quad \frac{W}{V} = \frac{|0.5-0.8|\,|0.5-1.25|}{0.5 \times 1.25} = 0.36$$

Therefore, W is given by

$$W = 0.36\,V = 0.225.$$

Second method Notice that $x_2(k+1) = u(k)$, which is deterministic. Therefore, the stationary a-priori state estimation error covariance M must satisfy:

$$M = E\left\{ \begin{bmatrix} \tilde{x}_1^o(k) \\ \tilde{x}_2^o(k) \end{bmatrix} \begin{bmatrix} \tilde{x}_1^o(k) \\ \tilde{x}_2^o(k) \end{bmatrix}^T \right\} = \begin{bmatrix} m & 0 \\ 0 & 0 \end{bmatrix}$$

Moreover, since

$$E\left\{ (\tilde{y}^o(k))^2 \right\} = CMC^T + V = m + V = 1$$

The a-priori Kalman filter gain matrix is given by

$$\begin{aligned} L &= AMC^T(CMC^T + V)^{-1} = AMC^T \\ &= \begin{bmatrix} 0.8\,m \\ 0 \end{bmatrix} \end{aligned}$$

Therefore, we have the Kalman filter closed-loop a-priori characteristic equation

$$\begin{aligned}
C(z) &= \det(zI - A + LC) \\
&= \det\left(zI - \begin{bmatrix} 0.8 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0.8m & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&= \det\begin{bmatrix} z - 0.8(1-m) & -1 \\ 0 & z \end{bmatrix} \\
&= z^2 - 0.5z = z(z-0.5) \\
\Rightarrow 0.8(1-m) &= 0.5 \\
\Rightarrow m &= 0.375 \\
\Rightarrow V &= 1 - m = 0.625
\end{aligned}$$

To obtain W we use the Kalman filter algebraic Riccati equation:

$$\begin{aligned}
0 &= AMA^T - M - AMC^T\left[CMC^T + V\right]^{-1} CMA^T + B_w W B_w^T \\
\Rightarrow W &= -0.64m + m + 0.64m^2 = 0.225
\end{aligned}$$

2. (a) The easiest way to find the steady state Kalman filter gain and error estimation error covariance is to use the MATLAB function `kalman`:

```
>> sys = ss(A,[B Bw],C,[0 0]);
>> [Kest,L,M] = kalman(sys,W,V);
```

To find M and L by hand, you would need to first solve the algebraic Riccati equation for M:

$$0 = AM + MA^T + B_w W B_w^T - MC^T V^{-1} CM$$

Then L is given by

$$L = MC^T V^{-1}$$

Either method gives

$$\begin{aligned}
L &= \begin{bmatrix} 0.8809 \\ 0.3880 \end{bmatrix} \\
M &= \begin{bmatrix} 0.4404 & 0.1940 \\ 0.1940 & 1.1488 \end{bmatrix}
\end{aligned}$$

(b) We apply the root locus methodology to $G_w(s)G_w(-s)$ where

$$G_w(s) = C\left[sI - A\right]^{-1} B_w$$

Thus, we apply the root locus methods to

$$G_w(s)G_w(-s) = \left(\frac{1}{s^2 + 0.5s + 2}\right)\left(\frac{1}{s^2 - 0.5s + 2}\right)$$

Figure 2 shows the symmetric root locus plot for this system obtained using negative feedback rules. Note that in this particular case, the error dynamics monotonically increase in speed as V becomes small relative to W. This behavior is not true for a general system.

(c) i. Taking the Laplace transform of the innovation driven model of the plant output given by

$$\begin{aligned}
\frac{d\hat{x}(t)}{dt} &= A\hat{x}(t) + Bu(t) + L\epsilon(t) \\
y(t) &= C\hat{x}(t) + \epsilon(t)
\end{aligned}$$

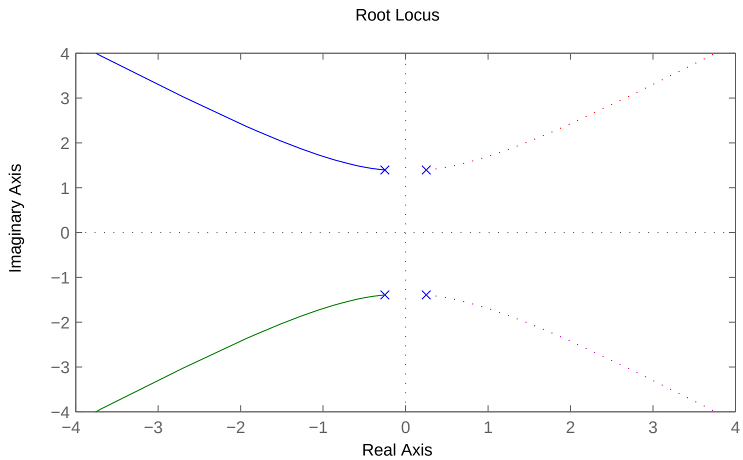

Figure 2: Kalman filter symmetric root locus plot

we have

$$\begin{aligned}
s\hat{X}(s) &= A\hat{X}(s) + BU(s) + LE(s) \\
\Rightarrow \hat{X}(s) &= [sI - A]^{-1} BU(s) + [sI - A]^{-1} LE(s) \\
\Rightarrow Y(s) &= C\hat{X}(s) + E(s) \\
&= C[sI - A]^{-1} BU(s) + C[sI - A]^{-1} LE(s) + E(s) \\
&= C[sI - A]^{-1} BU(s) + \left(I + C[sI - A]^{-1} L\right) E(s)
\end{aligned}$$

If we define

$$\begin{aligned}
G_u(s) &= C[sI - A]^{-1} B \\
G_\epsilon(s) &= I + C[sI - A]^{-1} L
\end{aligned}$$

we get that

$$Y(s) = G_u(s)U(s) + G_\epsilon(s)E(s)$$

First, let's examine G_ϵ. Since it is a scalar quantity, it is not affected by taking a determinant. Thus

$$\begin{aligned}
G_\epsilon(s) &= \det\left(I + C[sI - A]^{-1} L\right) \\
&= \det\left(I + [sI - A]^{-1} LC\right) \\
&= \det(sI - A)^{-1} \det(sI - A + LC) \\
&= \frac{\det(sI - [A - LC])}{\det(sI - A)}
\end{aligned}$$

Note that the two determinants here are monic polynomials. Thus

$$\begin{aligned}
C(s) &= \det(sI - [A - LC]) \\
A(s) &= \det(sI - A)
\end{aligned}$$

(For the remainder of the problem, keep in mind that "$A(s)$," "$B(s)$," and "$C(s)$" refer to polynomials whereas "A," "B," and "C" refer to matrices.) Now we will show that the denominator of $G_u(s)$ is also given by $A(s)$. First note that, using similar arguments as above,

$$\begin{aligned}
G_u(s) + 1 &= \det\left(I + C[sI - A]^{-1} B\right) \\
&= \frac{\det(sI - [A - BC])}{\det(sI - A)}
\end{aligned}$$

Note that the denominator of this expression is the same as the denominator of $G_\epsilon(s)$. Since the denominator of $G_u(s)+1$ is the same as the denominator of $G_u(s)$, the denominator of $G_u(s)$ is also given by $A(s)$.

ii. Using our results from part (i), we see that

$$\begin{aligned}\frac{B(s)}{A(s)} &= G_u(s) \\ &= C\left[sI - A\right]^{-1} B \\ &= \frac{1}{z^2 + 0.5z + 2}\end{aligned}$$

Thus, $A(s)$ has order 2 whereas $B(s)$ has order 0.

iii. Using the results of part (i),

$$\begin{aligned}C(s) &= \det\left[sI - (A - LC)\right] \\ &= \det\begin{bmatrix} s + 0.8809 & -1.0000 \\ 2.3880 & s + 0.5000\end{bmatrix} \\ &= s^2 + 1.3809s + 2.8284\end{aligned}$$

3. (a) Assuming stabilizability of $[A, B_w]$ and detectability of $[A, C]$, the AREs for the steady state a priori estimation error covariances for Configurations A (single set of sensors) and B (two sets of sensors) are respectively

$$AM_A + M_A A^T + B_w W B_w^T - M_A C^T V_A^{-1} C M_A = 0$$
$$AM_B + M_B A^T + B_w W B_w^T - M_B \begin{bmatrix} C^T & C^T \end{bmatrix} \begin{bmatrix} V_B^{-1} & 0 \\ 0 & V_B^{-1} \end{bmatrix} \begin{bmatrix} C \\ C \end{bmatrix} M_B = 0.$$

Setting $M_A = M_B$ and subtracting these two equations gives

$$\begin{aligned}0 &= M_A \left(\begin{bmatrix} C^T & C^T \end{bmatrix} \begin{bmatrix} V_B^{-1} & 0 \\ 0 & V_B^{-1} \end{bmatrix} \begin{bmatrix} C \\ C \end{bmatrix} - C^T V_A^{-1} C \right) M_A \\ &= M_A C^T \left(2V_B^{-1} - V_A^{-1} \right) C M_A.\end{aligned}$$

Thus, we can see that choosing $V_B = 2V_A$ guarantees that $M_A = M_B$.

(b) The optimal filtering gains for Configuration B (when everything is working properly) is given by

$$L_B = M_B \begin{bmatrix} C^T & C^T \end{bmatrix} \begin{bmatrix} V_B^{-1} & 0 \\ 0 & V_B^{-1} \end{bmatrix}.$$

In the case that we replace the measurements $y_2(t)$ by $y_1(t)$, the state estimate update equation is given by

$$\begin{aligned}\frac{d}{dt}\hat{x}(t) &= A\hat{x}(t) + Bu(t) + L_B \left(\begin{bmatrix} y_1(t) \\ y_1(t) \end{bmatrix} - \begin{bmatrix} C \\ C \end{bmatrix} \hat{x}(t) \right) \\ &= A\hat{x}(t) + Bu(t) + L_B \left(\begin{bmatrix} C \\ C \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} v_{B1}(t) \\ v_{B1}(t) \end{bmatrix} \right).\end{aligned}$$

If we subtract this equation from the state update equation of the plant, we get the estimation error dynamics:

$$\begin{aligned}\frac{d}{dt}\tilde{x}(t) &= \left(A - L_B \begin{bmatrix} C \\ C \end{bmatrix} \right) \tilde{x}(t) + B_w w(t) - L_B \begin{bmatrix} v_{B1}(t) \\ v_{B1}(t) \end{bmatrix} \\ &= \left(A - 2M_B C^T V_B^{-1} C \right) \tilde{x}(t) + B_w w(t) - 2M_B C^T V_B^{-1} v_{B1}(t) \\ &= \left(A - 2M_B C^T V_B^{-1} C \right) \tilde{x}(t) + \begin{bmatrix} B_w & -2M_B C^T V_B^{-1} \end{bmatrix} \begin{bmatrix} w(t) \\ v_{B1}(t) \end{bmatrix}.\end{aligned}$$

Note that since this is a system driven by white noise, we can find the covariance of the state (i.e. the state estimation error) with the Lyapunov equation

$$\begin{aligned}
0 &= \left(A - 2M_B C^T V_B^{-1} C\right) M + M \left(A - 2M_B C^T V_B^{-1} C\right)^T \\
&\quad + \begin{bmatrix} B_w & -2M_B C^T V_B^{-1} \end{bmatrix} \begin{bmatrix} W & 0 \\ 0 & V_B \end{bmatrix} \begin{bmatrix} B_w^T \\ -2V_B^{-1} C M_B \end{bmatrix} \\
&= \left(A - 2M_B C^T V_B^{-1} C\right) M + M \left(A - 2M_B C^T V_B^{-1} C\right)^T + B_w W B_w^T + 4M_B C^T V_B^{-1} C M_B \\
&= AM + MA^T + B_w W B_w^T - 2M_B C^T V_B^{-1} C M - 2M C^T V_B^{-1} C M_B + 4M_B C^T V_B^{-1} C M_B
\end{aligned}$$

Notice that this covariance is not the same as for either of the cases in (a). Now we would like to analyze how we would have to change the covariance of $v_{B1}(t)$ in order to recover the optimal covariance, M_B. Denoting the new covariance of $v_{B1}(t)$ as V_C, the Lyapunov equation for the estimation error covariance (which we are setting to M_B) is

$$\begin{aligned}
0 &= \left(A - 2M_B C^T V_B^{-1} C\right) M_B + M_B \left(A - 2M_B C^T V_B^{-1} C\right)^T \\
&\quad + \begin{bmatrix} B_w & -2M_B C^T V_B^{-1} \end{bmatrix} \begin{bmatrix} W & 0 \\ 0 & V_C \end{bmatrix} \begin{bmatrix} B_w^T \\ -2V_B^{-1} C M_B \end{bmatrix} \\
&= AM_B + M_B A^T + B_w W B_w^T - 4M_B C^T V_B^{-1} C M_B + 4M_B C^T V_B^{-1} V_C V_B^{-1} C M_B.
\end{aligned}$$

Subtracting the ARE for M_B, we get that

$$\begin{aligned}
0 &= -2M_B C^T V_B^{-1} C M_B + 4M_B C^T V_B^{-1} V_C V_B^{-1} C M_B \\
&= 2M_B C^T V_B^{-1} \left(-V_B + 2V_C\right) V_B^{-1} C M_B.
\end{aligned}$$

Thus, we could recover the earlier covariance if V_B were cut in half.

4. Before calculating the optimal $u(k)$, we define:

$$J_k[x(k)] = E_{\mathcal{W}_k} \left\{ x(N) + \sum_{j=k}^{N-1} \left(1 - u(k)\right) x(k) \right\}$$

$$\begin{aligned}
J_k^o[x(k)] &= \max_{U_k} E_{\mathcal{W}_k} \left\{ x(N) + \sum_{j=k}^{N-1} \left(1 - u(k)\right) x(k) \right\} \\
U_k &= \{u(k), u(k+1), \cdots, u(N-1)\}
\end{aligned}$$

With recursive formula derived in the lecture notes, we have:

$$J_k^o[x(k)] = \max_{u(k)} \left\{ \left(1 - u(k)\right) x(k) + E_{w(k)} \left(J_{k+1}^o[x(k+1)]\right) \right\}$$

Obviously, $J_N^o[x(N)] = x(N)$.

When $k = N - 1$,

$$\begin{aligned}
J_{N-1}^o[x(N-1)] &= \max_{u(N-1)} \left\{ \left(1 - u(N-1)\right) x(N-1) + E_{w(N-1)} \left(J_N^o[x(N)]\right) \right\} \\
&= \max_{u(N-1)} \left\{ \left(1 - u(N-1)\right) x(N-1) + E_{w(N-1)} \left(x(N)\right) \right\} \\
&= \max_{u(N-1)} \left\{ \left(1 - u(N-1)\right) x(N-1) + E_{w(N-1)} \left(x(N-1) + w(N-1)u(N-1)x(N-1)\right) \right\} \\
&= \max_{u(N-1)} \left\{ \left(1 - u(N-1)\right) x(N-1) + \left(1 + \bar{w} u(N-1)\right) x(N-1) \right\} \\
&= x(N-1) \max_{u(N-1) \in [0,1]} \left\{ 2 + \left(\bar{w} - 1\right) u(N-1) \right\}
\end{aligned}$$

(a) If $\bar{w} > 1$, we have

$$\begin{aligned} J^o_{N-1}[x(N-1)] &= x(N-1) \max_{u(N-1)\in[0,1]} \{2 + (\bar{w}-1)\,u(N-1)\} \\ &= (\bar{w}+1)\,x(N-1) \text{ with } u^o(N-1) = 1 \end{aligned}$$

Now we assume that $J^o_k[x(k)]$ can be express as

$$J^o_k[x(k)] = (\bar{w}+1)^{N-k}\,x(k)$$

Then we will prove the above assumption by induction.

$$\begin{aligned} J^o_k[x(k)] &= \max_{u(k)} \left\{(1-u(k))\,x(k) + E_{w(k)}\left(J^o_{k+1}[x(k+1)]\right)\right\} \\ &= \max_{u(k)} \left\{(1-u(k))\,x(k) + E_{w(k)}\left((\bar{w}+1)^{N-k-1}\,x(k+1)\right)\right\} \\ &= \max_{u(k)} \left\{(1-u(k))\,x(k) + E_{w(k)}\left((\bar{w}+1)^{N-k-1}\,(1+w(k)u(k))\,x(k)\right)\right\} \\ &= x(k) \max_{u(k)} \left\{1 - u(k) + (\bar{w}+1)^{N-k-1}\,(1+\bar{w}u(k))\right\} \\ &= x(k) \max_{u(k)\in[0,1]} \left\{1 + (\bar{w}+1)^{N-k-1} + \left((\bar{w}+1)^{N-k-1}\,\bar{w} - 1\right) u(k)\right\} \end{aligned}$$

With $\bar{w} > 1$, $(\bar{w}+1)^{N-k-1}\,\bar{w} > 1$. Therefore, $u^o(k) = 1$ and

$$\begin{aligned} J^o_k[x(k)] &= x(k) \left\{(\bar{w}+1)^{N-k-1} + (\bar{w}+1)^{N-k-1}\,\bar{w}\right\} \\ &= (\bar{w}+1)^{N-k}\,x(k) \end{aligned}$$

By induction, we can conclude that $u^o(k) = 1$ and $J^o_k[x(k)] = (\bar{w}+1)^{N-k}\,x(k)$ for all $k = 0, 1, ..., N-1$.

(b) By using the similar arguments in Part (a), we have the following results.
When $k = N-1$,

$$J^o_{N-1}[x(N-1)] = x(N-1) \max_{u(N-1)\in[0,1]} \{2 + (\bar{w}-1)\,u(N-1)\}$$

If $0 < \bar{w} < 1/N$, we have

$$\begin{aligned} J^o_{N-1}[x(N-1)] &= x(N-1) \max_{u(N-1)\in[0,1]} \{2 + (\bar{w}-1)\,u(N-1)\} \\ &= 2x(N-1) \text{ with } u^o(N-1) = 0 \end{aligned}$$

Now we assume that $J^o_k[x(k)]$ can be express as

$$J^o_k[x(k)] = (N-k+1)x(k)$$

Then we will prove the above assumption by induction.

$$\begin{aligned} J^o_k[x(k)] &= \max_{u(k)} \left\{(1-u(k))\,x(k) + E_{w(k)}\left(J^o_{k+1}[x(k+1)]\right)\right\} \\ &= \max_{u(k)} \left\{(1-u(k))\,x(k) + E_{w(k)}\left((N-k)x(k+1)\right)\right\} \\ &= \max_{u(k)} \left\{(1-u(k))\,x(k) + E_{w(k)}\left((N-k)\,(1+w(k)u(k))\,x(k)\right)\right\} \\ &= x(k) \max_{u(k)} \{1 - u(k) + (N-k)\,(1+\bar{w}u(k))\} \\ &= x(k) \max_{u(k)\in[0,1]} \{1 + N - k + ((N-k)\bar{w} - 1)\,u(k)\} \end{aligned}$$

With $\bar{w} < 1/N$, $(N-k)\bar{w} < 1$. Therefore, $u^o(k) = 0$ and

$$J_k^o[x(k)] = (N-k+1)x(k)$$

By induction, we can conclude that $u^o(k) = 0$ and $J_k^o[x(k)] = (N-k+1)x(k)$ for all $k = 0, 1, ..., N-1$.

(c) For $1/N \le \bar{w} \le 1$, suppose we can find $\bar{k}$ such that

$$\frac{1}{\bar{k}+1} < \bar{w} \le \frac{1}{\bar{k}}$$

When $k \ge N - \bar{k}$, $(N-k)\bar{w} < 1$. Therefore, we still have the same results as in Part (b) for $k = N - \bar{k}, ..., N-1$:

$$\begin{aligned} J_k^o[x(k)] &= (N-k+1)x(k) \\ u^o(k) &= 0 \end{aligned}$$

When $k = N - \bar{k} - 1$, from Part (b) we know

$$J_k^o[x(k)] = x(k) \max_{u(k)\in[0,1]} \{1 + N - k + ((N-k)\bar{w} - 1)\, u(k)\}$$

By considering that $(N-k)\bar{w} = (\bar{k}+1)\bar{w} > 1$, we have $u^o(k) = 1$ and $J_k^o[x(k)] = (N-k)(\bar{w}+1)$. When $k < N - \bar{k} - 1$, we assume that $J_k^o[x(k)]$ can be express as

$$J_k^o[x(k)] = (\bar{k}+1)(\bar{w}+1)^{N-\bar{k}-k} x(k)$$

Again, we will prove the above assumption by induction.

$$\begin{aligned} J_k^o[x(k)] &= \max_{u(k)} \left\{(1-u(k))\, x(k) + E_{w(k)} \left(J_{k+1}^o[x(k+1)]\right)\right\} \\ &= \max_{u(k)} \left\{(1-u(k))\, x(k) + E_{w(k)} \left((\bar{k}+1)\,(\bar{w}+1)^{N-k-1-\bar{k}}\, x(k+1)\right)\right\} \\ &= \max_{u(k)} \left\{(1-u(k))\, x(k) + E_{w(k)} \left((\bar{k}+1)\,(\bar{w}+1)^{N-k-1-\bar{k}}\, (1 + w(k)u(k))\, x(k)\right)\right\} \\ &= x(k) \max_{u(k)} \left\{1 - u(k) + (\bar{k}+1)\,(\bar{w}+1)^{N-k-1-\bar{k}}\, (1 + \bar{w}u(k))\right\} \\ &= x(k) \max_{u(k)\in[0,1]} \left\{1 + (\bar{k}+1)\,(\bar{w}+1)^{N-k-1-\bar{k}} + \left((\bar{k}+1)\,(\bar{w}+1)^{N-k-1-\bar{k}}\, \bar{w} - 1\right) u(k)\right\} \end{aligned}$$

Since $\bar{w} > \frac{1}{\bar{k}+1}$ and $\bar{w}+1 > 1$, $(\bar{k}+1)\,(\bar{w}+1)^{N-k-1-\bar{k}}\, \bar{w} - 1 > 0$. Therefore, we have the optimal value $u^o(k) = 1$ and

$$\begin{aligned} J_k^o[x(k)] &= (\bar{k}+1)(\bar{w}+1)^{N-k-1-\bar{k}}(\bar{w}+1)x(k) \\ &= (\bar{k}+1)(\bar{w}+1)^{N-\bar{k}-k}x(k) \end{aligned}$$

In conclusion, we get the following optimal polices:

$$u^o(k) = \begin{cases} 1 & \text{for } k = 0, 1, \cdots, N - \bar{k} - 1 \\ 0 & \text{for } k = N - \bar{k}, \cdots, N-1 \end{cases}$$