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ME 233 – Homework 3

1.

$P(C_x)=P(C_y)=P(C_z)=1/3$

You choose door x so host can't open door x, therefore $P(H_x)=0$

Host opens a dud door so $P(H_y|C_y)=P(H_z|C_z)=0$

Only three doors for host to open so $P(H_x|C_y)+P(H_y|C_y)+P(H_z|C_y)=1$

and likewise $P(H_x|C_z)+P(H_y|C_z)+P(H_z|C_z)=1$

$P(H_x)=0$ so $P(H_x|C_y)=P(H_x|C_z)=0$

So we have $P(H_z|C_y)=P(H_y|C_z)=1$

By Bayes' rule, $P(H_z \cap C_y)=P(C_y)P(H_z|C_y)=1/3$

and $P(H_y \cap C_z)=P(C_z)P(H_y|C_z)=1/3$

$(H_z \cap C_y)$ and $(H_y \cap C_z)$ are mutually exclusive so

$P((H_z \cap C_y) \cup (H_y \cap C_z))=P(H_z \cap C_y)+P(H_y \cap C_z)=2/3$

2.(a)

$P_X(X=A)=1/2$, $P_X(X=B)=P_X(X=C)=1/4$

$P_{Y|X}(Y=D|X=A)=1/100$, $P_{Y|X}(Y=D|X=B)=1/100$, $P_{Y|X}(Y=D|X=C)=3/100$

$$P_{Y,X}(Y=D, X=A)=P_{Y|X}(Y=D|X=A)P_X(X=A)=\frac{1}{100}\cdot\frac{1}{2}$$

$$P_{Y,X}(Y=D, X=B)=P_{Y|X}(Y=D|X=B)P_X(X=B)=\frac{1}{100}\cdot\frac{1}{4}$$

$$P_{Y,X}(Y=D, X=C)=P_{Y|X}(Y=D|X=C)P_X(X=C)=\frac{3}{100}\cdot\frac{1}{4}$$

$$P_Y(Y=D)=P_{Y,X}(Y=D, X=A)+P_{Y,X}(Y=D, X=B)+P_{Y,X}(Y=D, X=C)=\frac{1}{200}+\frac{1}{400}+\frac{3}{400}$$

Bayes' rule: $P_{Y|X}(Y=D|X=A)P_X(X=A)=P_{X|Y}(X=A|Y=D)P_Y(Y=D)$

$$P_{X|Y}(X=A|Y=D)=\frac{P_{Y|X}(Y=D|X=A)P_X(X=A)}{P_Y(Y=D)}=\frac{1/200}{3/200}=\frac{1}{3}$$

2.(b)

$P_Y(Y\neq D)=1-P_Y(Y=D)=197/200$

$P_{Y|X}(Y\neq D|X=C)=1-P_{Y|X}(Y=D|X=C)=97/100$

Bayes' rule: $P_{Y|X}(Y\neq D|X=C)P_X(X=C)=P_{X|Y}(X=C|Y\neq D)P_Y(Y\neq D)$

$$P_{X|Y}(X=C|Y\neq D)=\frac{P_{Y|X}(Y\neq D|X=C)P_X(X=C)}{P_Y(Y\neq D)}=\frac{\frac{97}{100}\cdot\frac{1}{4}}{197/200}=\frac{97}{394}=0.2462$$

3.(a)

X_1, X_2, X_3 independent and uniform: $p_{X_1}(x)=p_{X_2}(x)=p_{X_3}(x)=\begin{cases}1 & 0<x<1\\ 0 & x<0 \text{ or } x>1\end{cases}$

Let $X=X_1$, $Y=X_2$, and $Z=X+Y$

$p_Z(z)=\int_{-\infty}^{\infty} p_X(x)p_Y(z-x)dx$

For $0<z<1$, $p_Z(z)=\int_0^z dx=z$

For $1<z<2$, $p_Z(z)=\int_{z-1}^{1} dx=2-z$

$$p_Z(z)=\begin{cases} z & 0<z<1 \\ 2-z & 1<z<2 \\ 0 & z<0 \text{ or } z>2 \end{cases}$$

3.(b)

Now let $X=X_1+X_2$, $Y=X_3$, and $Z=X+Y$

$p_Z(z)=\int_{-\infty}^{\infty} p_X(x)\,p_Y(z-x)\,dx$

For $0<z<1$, $p_Z(z)=\int_0^z x\,dx=\dfrac{z^2}{2}$

For $1<z<2$, $p_Z(z)=\int_{z-1}^{1} x\,dx+\int_1^z (2-x)\,dx=\dfrac{1}{2}-\dfrac{(z-1)^2}{2}+2z-\dfrac{z^2}{2}-2+\dfrac{1}{2}=-z^2+3z-\dfrac{3}{2}$

For $2<z<3$, $p_Z(z)=\int_{z-1}^{2} (2-x)\,dx=4-2-2(z-1)+\dfrac{(z-1)^2}{2}=\dfrac{z^2-6z+9}{2}=\dfrac{(z-3)^2}{2}$

$$p_Z(z)=\begin{cases} z^2/2 & 0<z<1 \\ -z^2+3z-3/2 & 1<z<2 \\ (z-3)^2/2 & 2<z<3 \\ 0 & z<0 \text{ or } z>3 \end{cases}$$

4.

$X\sim N(m_X,\sigma_X^2)$, $Y\sim N(m_Y,\sigma_Y^2)$ independent

$P_X(j\omega)=\mathcal{F}\{p_X(\cdot)\}=E\{e^{-j\omega X}\}=\exp\left(j\omega m_X-\dfrac{\sigma_X^2\omega^2}{2}\right)$

$Z=X+Y$, $P_Z(j\omega)=\mathcal{F}\{p_Z(\cdot)\}=E\{e^{-j\omega Z}\}=E\{e^{-j\omega(X+Y)}\}=E\{e^{-j\omega X}\}E\{e^{-j\omega Y}\}$ by independence of X and Y, so $P_Z(j\omega)=P_X(j\omega)P_Y(j\omega)=\exp\left(j\omega m_X-\dfrac{\sigma_X^2\omega^2}{2}\right)\exp\left(j\omega m_Y-\dfrac{\sigma_Y^2\omega^2}{2}\right)$

$P_Z(j\omega)=\exp\left(j\omega(m_X+m_Y)-\dfrac{(\sigma_X^2+\sigma_Y^2)\omega^2}{2}\right)=\exp\left(j\omega m_Z-\dfrac{\sigma_Z^2\omega^2}{2}\right)$

with $m_Z=m_X+m_Y$, and $\sigma_Z^2=\sigma_X^2+\sigma_Y^2$

$p_Z(\cdot)=\mathcal{F}^{-1}\{P_Z(j\omega)\}$, inverse Fourier transform of a Gaussian is a Gaussian

so $Z\sim N(m_Z,\sigma_Z^2)=N(m_X+m_Y,\sigma_X^2+\sigma_Y^2)$

5.(a)

$X\sim N(10,2)$, $V_1\sim N(0,1)$, $V_2\sim N(0,2)$ independent

$Y=X+V_1$, $Z=X+V_2$, so by result of problem 4, $m_Y=m_X+m_{V_1}=10$, $m_Z=m_X+m_{V_2}=10$

and $\Lambda_{YY}=\sigma_Y^2=\sigma_X^2+\sigma_{V_1}^2=3$, $\Lambda_{ZZ}=\sigma_Z^2=\sigma_X^2+\sigma_{V_2}^2=4$

$\Lambda_{XY}=E\{(X-m_X)(Y-m_Y)\}=E\{(X-m_X)(X+V_1-m_X-m_{V_1})\}$

$\Lambda_{XY}=E\{(X-m_X)(X-m_X)+(X-m_X)(V_1-m_{V_1})\}=E\{(X-m_X)(X-m_X)\}+E\{(X-m_X)(V_1-m_{V_1})\}$

$\Lambda_{XY}=\Lambda_{XX}+\Lambda_{XV_1}=\sigma_X^2=2$ since X and V_1 are independent

From lecture notes, $m_{X|y}=m_X+\Lambda_{XY}\Lambda_{YY}^{-1}(y-m_Y)$

$m_{X|Y=11}=10+2\cdot 3^{-1}(11-10)=32/3=10.667$

5.(b)

$$\Lambda_{XZ}=E\{(X-m_X)(Z-m_Z)\}=E\{(X-m_X)(X+V_2-m_X-m_{V_2})\}$$

$$\Lambda_{XZ}=E\{(X-m_X)(X-m_X)+(X-m_X)(V_2-m_{V_2})\}=E\{(X-m_X)(X-m_X)\}+E\{(X-m_X)(V_2-m_{V_2})\}$$

$\Lambda_{XZ}=\Lambda_{XX}+\Lambda_{XV_2}=\sigma_X^2=2$ since X and V_2 are independent

$$m_{X|z}=m_X+\Lambda_{XZ}\Lambda_{ZZ}^{-1}(z-m_Z)$$

$$m_{X|Z=9}=10+2\cdot 4^{-1}(9-10)=19/2=9.5$$

5.(c)

$$\Lambda_{YZ}=E\{(Y-m_Y)(Z-m_Z)\}=E\{(X+V_1-m_X-m_{V_1})(X+V_2-m_X-m_{V_2})\}$$

$$\Lambda_{YZ}=E\{(X-m_X)(X-m_X)+(X-m_X)(V_2-m_{V_2})+(V_1-m_{V_1})(X-m_X)+(V_1-m_{V_1})(V_2-m_{V_2})\}$$

$$=E\{(X-m_X)(X-m_X)\}+E\{(X-m_X)(V_2-m_{V_2})\}+E\{(V_1-m_{V_1})(X-m_X)\}+E\{(V_1-m_{V_1})(V_2-m_{V_2})\}$$

$\Lambda_{YZ}=\Lambda_{XX}+\Lambda_{XV_2}+\Lambda_{V_1X}+\Lambda_{V_1V_2}=\sigma_X^2=2$ since X, V_1, V_2 are all independent of one another

Let $W=\begin{bmatrix} Y \\ Z \end{bmatrix}$, then we have $m_W=\begin{bmatrix} m_Y \\ m_Z \end{bmatrix}=\begin{bmatrix} 10 \\ 10 \end{bmatrix}$ and $\Lambda_{WW}=\begin{bmatrix} \Lambda_{YY} & \Lambda_{YZ} \\ \Lambda_{ZY} & \Lambda_{ZZ} \end{bmatrix}=\begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}$, $\Lambda_{WW}^{-1}=\frac{1}{8}\begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix}$

$$\begin{bmatrix} \Lambda_{XX} & \Lambda_{XW} \\ \Lambda_{WX} & \Lambda_{WW} \end{bmatrix}=\begin{bmatrix} \Lambda_{XX} & \Lambda_{XY} & \Lambda_{XZ} \\ \Lambda_{YX} & \Lambda_{YY} & \Lambda_{YZ} \\ \Lambda_{ZX} & \Lambda_{ZY} & \Lambda_{ZZ} \end{bmatrix}=\begin{bmatrix} 2 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

$$m_{X|w}=m_X+\Lambda_{XW}\Lambda_{WW}^{-1}(w-m_W)$$

$$m_{X|(Y=y,Z=z)}=10+[2 \quad 2]\frac{1}{8}\begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} y-10 \\ z-10 \end{bmatrix}$$

$$m_{X|(Y=11,Z=9)}=10+\frac{1}{8}[4 \quad 2]\begin{bmatrix} 1 \\ -1 \end{bmatrix}=10+\frac{1}{4}=10.25$$