ME 233 Spring 2016
Solution to Homework #2

1. Upload later

2. (a) Figure 1 shows the MATLAB estimates of the auto-covariances and cross-covariances of W and Y. As we would expect, $\Lambda_{WW}(j)$ is approximately a unit pulse and $\Lambda_{YY}(j)$ is approximately symmetric. Also, $\Lambda_{YW}(-j) \approx \Lambda_{WY}(j)$ is approximately 0 for positive j, as causality dictates.

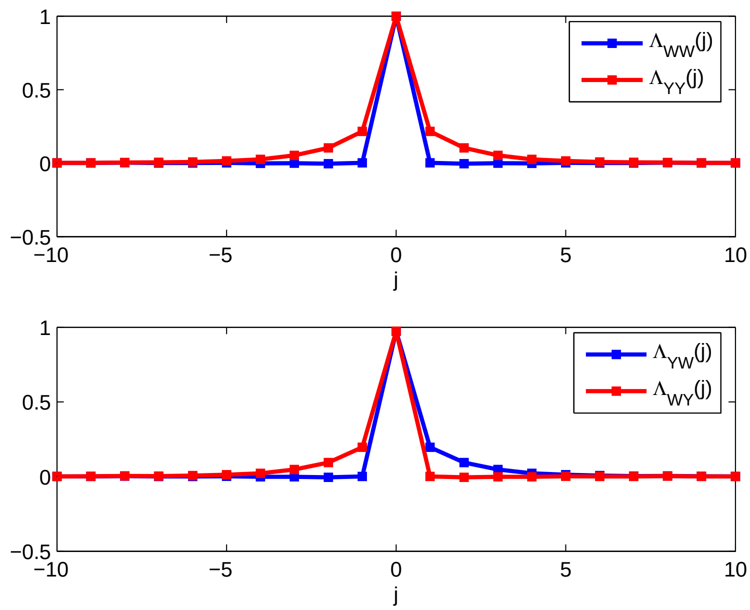

Figure 1: MATLAB estimates of auto-covariances and cross-covariances

(b) To find $\Lambda_{YW}(l)$, it is easiest to first find $\hat{\Lambda}_{YW}(z)$. Thus, we first note that

$$\begin{aligned}
\hat{\Lambda}_{YW}(z) &= G(z)\hat{\Lambda}_{WW}(z) \\
G(z) &= \frac{z-0.3}{z-0.5} \\
\hat{\Lambda}_{WW}(z) &= \mathcal{Z}\left\{\delta(l)\right\} = 1 \\
\Rightarrow \hat{\Lambda}_{YW}(z) &= \frac{z-0.3}{z-0.5}.
\end{aligned}$$

Now, with the aid of inverse Z-transform tables, we get that

$$\begin{aligned}
\Lambda_{YW}(l) &= \mathcal{Z}^{-1}\left\{\frac{0.4z}{z-0.5} + 0.6\right\} \\
&= \begin{cases} 0.4(0.5)^l + 0.6\delta(l) & l \geq 0 \\ 0 & l < 0 \end{cases}.
\end{aligned}$$

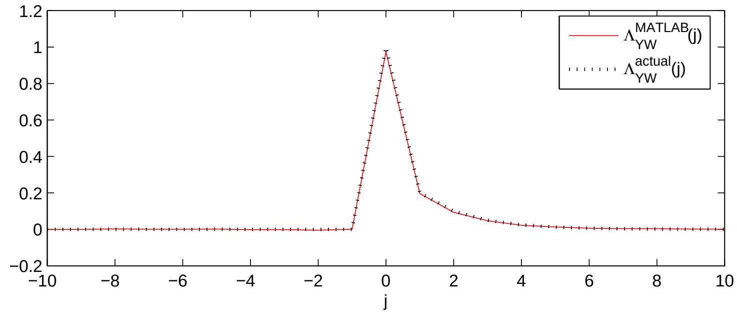

Figure 2: Comparison of MATLAB-determined cross-covariance to actual values

Figure 2 shows that the values of $\Lambda_{YW}(l)$ determined through MATLAB simulation match up well with the values determined above.

(c) Now that we have $\Lambda_{YW}(l)$, finding $\Lambda_{WY}(l)$ is a trivial matter. Using the property that $\Lambda_{YW}(l) = \Lambda_{WY}(-l)$, we see that

$$\Lambda_{WY}(l) = \begin{cases} 0.4(0.5)^{-l} + 0.6\delta(l) & l \leq 0 \\ 0 & l > 0 \end{cases}.$$

Figure 3 shows that the values of $\Lambda_{WY}(l)$ determined through MATLAB simulation match up well with the values determined above.

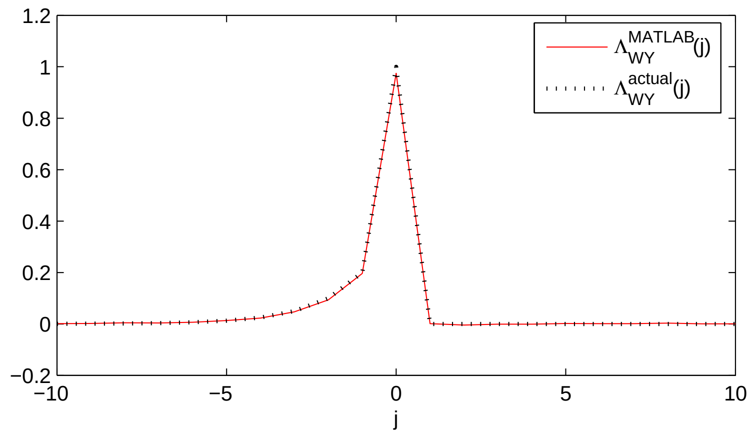

Figure 3: Comparison of MATLAB-determined cross-covariance to actual values

To find $\hat{\Lambda}_{WY}(z)$, it is easiest to recognize that the following general property applies to any random variables X and U:

$$\begin{aligned} \hat{\Lambda}_{XU}(z) &= \sum_{l=-\infty}^{\infty} z^{-l}\Lambda_{XU}(l) \\ &= \sum_{l=-\infty}^{\infty} (z^{-1})^{l}\Lambda_{UX}(-l) \\ &= \sum_{l=-\infty}^{\infty} (z^{-1})^{-l}\Lambda_{UX}(l) \\ &= \hat{\Lambda}_{UX}(z^{-1}). \end{aligned}$$

Applying this property to our system here gives

$$\hat{\Lambda}_{WY}(z) = \hat{\Lambda}_{YW}(z^{-1}) = \frac{z^{-1} - 0.3}{z^{-1} - 0.5} = \frac{0.3z - 1}{0.5z - 1}.$$

(d) We have the following:

$$\begin{aligned}\hat{\Lambda}_{YY}(z) &= \left(\frac{z - 0.3}{z - 0.5}\right)\left(\frac{z^{-1} - 0.3}{z^{-1} - 0.5}\right) \\ &= \frac{-0.3(z + z^{-1}) + 1.09}{(z - 0.5)(z^{-1} - 0.5)}.\end{aligned}$$

Using the results obtain in problem 1, we obtain:

$$\begin{aligned}a &= 0.5 \\ \alpha &= -0.3 \\ \beta &= 1.09.\end{aligned}$$

Then we can deduce b and c:

$$\begin{aligned}b &= 0.4533 \\ c &= 1.0533.\end{aligned}$$

So we obtain:

$$\hat{\Lambda}_{YY}(l) = f(l) + f(-l) + c\delta(l)$$

Where $c = 1.0533$ and where $f(l)$ is defined as:

$$f(l) = \begin{cases} 0.4533(0.5)^l, & l \geq 1 \\ 0, & l \leq 0 \end{cases}.$$

Figure 4 shows that the values of $\Lambda_{YY}(l)$ determined through MATLAB simulation match up well with the values determined above. (Note that the auto-covariance was normalized in this figure, i.e. $\Lambda_{YY}(l)$ was scaled so that its maximum value was 1.)

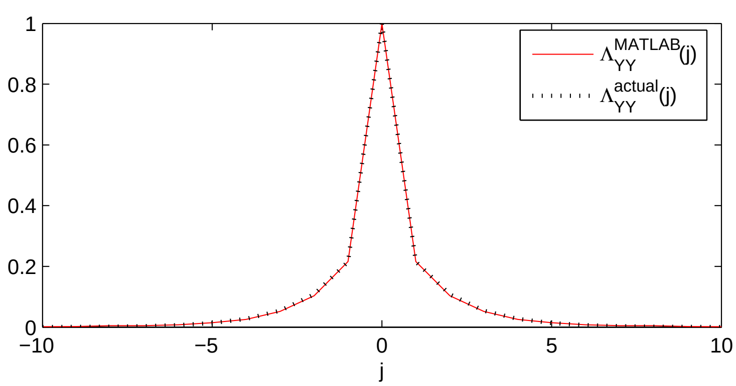

Figure 4: Comparison of MATLAB-determined auto-covariance to actual values

(e) Here, we want to compute covariances using the orginal series equation and compare our results to those obtained using transforms. To start, note that

$$\begin{aligned}\Lambda_{YW}(0) &= E\{Y(k)W(k)\} \\ &= E\Big\{\big[0.5Y(k-1) + W(k) - 0.3W(k-1)\big]W(k)\Big\} \\ &= E\left\{W^2(k)\right\} + 0.5E\{Y(k-1)W(k)\} - 0.3E\{W(k-1)W(k)\}.\end{aligned}$$

Since the system is causal we know that the system's output should not depend on future inputs. Thus, the system's output should be independent of future inputs. Also, since W is white, its value should be independent of its value at any other timestep. Using these two facts gives

$$\begin{aligned}\Lambda_{YW}(0) &= E\left\{W^2(k)\right\} + E\left\{W(k)\right\}\left[0.5E\left\{Y(k-1)\right\} - 0.3E\left\{W(k-1)\right\}\right] \\ &= E\left\{W^2(k)\right\} = 1\end{aligned}$$

where we have used the fact that W is zero-mean. Note that this result agrees with the result found in part (b).

(f) Using the wide-sense stationarity of the signals and the results from the previous part,

$$\begin{aligned}\lambda_{YW}(1) &= E\left\{Y(k+1)W(k)\right\} \\ &= E\left\{Y(k)W(k-1)\right\} \\ &= -0.3E\left\{W^2(k-1)\right\} + 0.5E\left\{Y(k-1)W(k-1)\right\} + E\left\{W(k)W(k-1)\right\} \\ &= -0.3E\left\{W^2(k-1)\right\} + 0.5E\left\{Y(k-1)W(k-1)\right\} \\ &= -0.3E\left\{W^2(k)\right\} + 0.5E\left\{Y(k)W(k)\right\} \\ &= -0.3 + 0.5\Lambda_{YW}(0) = 0.2.\end{aligned}$$

Note that this result agrees with the result found in part (b).

(g) To solve this problem, we will first note that

$$Y^2(k) = \left[0.5Y(k-1) + W(k) - 0.3W(k-1)\right]^2.$$

Taking the expected value of both sides gives

$$\begin{aligned}\Lambda_{YY}(0) &= 0.25E\left\{Y^2(K-1)\right\} + E\left\{W^2(k)\right\} + 0.09E\left\{W^2(k-1)\right\} \\ &\quad + E\left\{Y(k-1)W(k)\right\} - 0.3E\left\{Y(k-1)W(k-1)\right\} - 0.6E\left\{W(k)W(k-1)\right\} \\ &= 0.25\Lambda_{YY}(0) + 1 + 0.09 + 0 - 0.3\Lambda_{YW}(0) + 0 \\ &= \frac{0.79}{0.75} = 1.0533.\end{aligned}$$

Note that this result agrees with the result found in part (e).

3. (a) To begin, we find the conditional expectation of X given y:

$$m_{X|y} = m_X + \Lambda_{XY}\Lambda_{YY}^{-1}(y - m_Y)$$

Since X and V_1 are two independent normal distributed random variables, with the results from Problem 5 in HW#1 we see that

$$\begin{aligned}\Lambda_{YY} &= \Lambda_{XX} + \Lambda_{V_1V_1} \\ m_Y &= m_X\end{aligned}$$

Noting that $X - m_X$ is independent of V_1, we calculate the cross-covariance of X and Y as

$$\begin{aligned}\Lambda_{XY} &= E\left[(X - m_X)(Y - m_Y)\right] \\ &= E\left[(X - m_X)(X + V_1 - m_X)\right] \\ &= E\left[(X - m_X)^2\right] + E\left[(X - m_X)V_1\right] \\ &= E\left[(X - m_X)^2\right] + E\left[X - m_X\right]E\left[V_1\right] \\ &= E\left[(X - m_X)^2\right] \\ &= \Lambda_{XX}\end{aligned}$$

Substituting the relevant values gives

$$m_{X|Y=9} = 10 + \frac{2(9-10)}{2+1} = 9\frac{1}{3}$$

(b) Using the same methodology as before, we see that

$$\begin{aligned} m_{X|z} &= m_X + \Lambda_{XZ}\Lambda_{ZZ}^{-1}(z - m_Z) \\ \Lambda_{ZZ} &= \Lambda_{XX} + \Lambda_{V_2V_2} \\ m_Z &= m_X \\ \Lambda_{XZ} &= \Lambda_{XX} \end{aligned}$$

Thus,

$$m_{X|Z=11} = 10 + \frac{2(11-10)}{2+2} = 10\frac{1}{2}$$

(c) First, we define the random vector W as

$$W = \begin{bmatrix} Y \\ Z \end{bmatrix}$$

The mean and covariance of this vector are given by

$$\begin{aligned} m_W &= \begin{bmatrix} m_Y \\ m_Z \end{bmatrix} \\ \Lambda_{WW} &= \begin{bmatrix} \Lambda_{YY} & \Lambda_{YZ} \\ \Lambda_{ZY} & \Lambda_{ZZ} \end{bmatrix} \end{aligned}$$

As before,

$$\begin{aligned} \Lambda_{YY} &= \Lambda_{XX} + \Lambda_{V_1V_1} \\ \Lambda_{ZZ} &= \Lambda_{XX} + \Lambda_{V_2V_2} \end{aligned}$$

The cross-covariance between Y and Z can be calculated as

$$\begin{aligned} \Lambda_{ZY} = \Lambda_{YZ} &= E\left[(X - m_X + V_1)(X - m_X + V_2)\right] \\ &= E\left[(X - m_X)^2\right] + E\left[(X - m_X)(V_1 + V_2)\right] + E\left[V_1V_2\right] \\ &= E\left[(X - m_X)^2\right] \\ &= \Lambda_{XX} \end{aligned}$$

The cross-covariance between X and W can be expressed as

$$\Lambda_{XW} = \begin{bmatrix} \Lambda_{XY} & \Lambda_{XZ} \end{bmatrix} = \begin{bmatrix} \Lambda_{XX} & \Lambda_{XX} \end{bmatrix}$$

Thus,

$$\begin{aligned} m_{X|Y=9,Z=11} &= m_{X|W=[9\ 11]^T} \\ &= m_X + \Lambda_{XW}\Lambda_{WW}^{-1}(w - m_W) \\ &= 10 + \begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}^{-1} \left(\begin{bmatrix} 9 \\ 11 \end{bmatrix} - \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right) \\ &= 9\frac{3}{4} \end{aligned}$$

Note that the Y measurement has a greater impact on the conditional mean for X than the Z measurement. This means that our estimation is making use of the fact that Y is a more "reliable" measurement than Z, i.e. $\Lambda_{YY} < \Lambda_{ZZ}$.

4. (a) First, we define

$$Z := \begin{bmatrix} Y(0) & Y(1) & \cdots & Y(k) \end{bmatrix}^T.$$

And Z takes the outcome of $\bar{y}(k) = \begin{bmatrix} y(0) & \cdots & y(k) \end{bmatrix}^T$.
With this notation in mind, we are interested in finding $\hat{x}_{|z}$. Recall that

$$\begin{aligned}
\hat{x}_{|\bar{y}(k)} &= E\{X\} + \Lambda_{XZ}\Lambda_{ZZ}^{-1}\left(\bar{y}(k) - E\{Z\}\right) \\
&= \Lambda_{XZ}\Lambda_{ZZ}^{-1}\bar{y}(k).
\end{aligned}$$

Note that we used that X and Z are zero mean. In order to find this quantity, we need to find expressions for Λ_{XZ} and Λ_{ZZ}^{-1}. First, we will start by finding Λ_{XZ}. Note that

$$\begin{aligned}
E\left\{XY(j)\right\} &= E\left\{X^2\right\} + E\left\{XV(j)\right\} \\
&= X_0.
\end{aligned}$$

Thus, if we define

$$w = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T \in \mathbb{R}^{k+1}$$

we can express

$$\Lambda_{XZ} = X_0 w^T$$

Now we turn our attention to finding Λ_{ZZ}^{-1}. Note that

$$\begin{aligned}
E\left\{Y(k+j)Y(k)\right\} &= E\left\{\left(X + V(k+j)\right)\left(X + V(k)\right)\right\} \\
&= E\left\{X^2\right\} + E\left\{XV(k)\right\} + E\left\{XV(k+j)\right\} + E\left\{V(k+j)V(k)\right\} \\
&= X_0 + \Sigma_V \delta(j).
\end{aligned}$$

Thus, we can express

$$\begin{aligned}
\Lambda_{ZZ} &= \Sigma_V I + X_0 w w^T \\
&= \Sigma_V \left(I + \frac{X_0}{\Sigma_V} w w^T\right).
\end{aligned}$$

In order to invert this matrix, we must use the matrix inversion lemma, which states that

$$(I + AB)^{-1} = I - A\left(I + BA\right)^{-1} B.$$

Using this, we can say that

$$\begin{aligned}
\Lambda_{ZZ}^{-1} &= \frac{1}{\Sigma_V}\left(I + \frac{X_0}{\Sigma_V} w w^T\right)^{-1} \\
&= \frac{1}{\Sigma_V}\left[I - \frac{X_0}{\Sigma_V} w \left(1 + \frac{X_0}{\Sigma_V} w^T w\right)^{-1} w^T\right] \\
&= \frac{1}{\Sigma_V}\left[I - \frac{X_0}{\Sigma_V} \cdot \frac{\Sigma_V}{\Sigma_V + (k+1)X_0} w w^T\right] \\
&= \frac{1}{\Sigma_V}\left[I - \frac{X_0}{\Sigma_V + (k+1)X_0} w w^T\right].
\end{aligned}$$

Thus the estimate of X is given by

$$\begin{aligned}
\hat{x}(k) = \hat{x}_{|\bar{y}(k)} &= \frac{X_0}{\Sigma_V} w^T \left[I - \frac{X_0}{\Sigma_V + (k+1)X_0} w w^T\right] \bar{y}(k) \\
&= \frac{X_0}{\Sigma_V}\left[1 - \frac{X_0}{\Sigma_V + (k+1)X_0} w^T w\right] w^T \bar{y}(k) \\
&= \frac{X_0}{\Sigma_V + (k+1)X_0} w^T \bar{y}(k) \\
&= \frac{X_0}{\Sigma_V + (k+1)X_0} \sum_{i=0}^{k} y(i).
\end{aligned}$$

The covariance of the estimate is given by

$$
\begin{aligned}
\Lambda_{\tilde{X}\tilde{X}}(k,0) &= \Lambda_{XX} - \Lambda_{XZ}\Lambda_{ZZ}^{-1}\Lambda_{ZX} \\
&= X_0 - \left(\frac{X_0}{\Sigma_V + (k+1)X_0} w^T\right)\left(X_0 w\right) \\
&= \frac{X_0 \Sigma_V}{\Sigma_V + (k+1)X_0}.
\end{aligned}
$$

(b) Using the results of the previous part, it is trivial to see that

$$
\begin{aligned}
\lim_{X_0 \to \infty} \hat{x}(k) &= \frac{1}{k+1} \sum_{i=0}^{k} y(k) \\
\lim_{X_0 \to \infty} \Lambda_{\tilde{X}\tilde{X}}(k,0) &= \frac{\Sigma_V}{k+1}.
\end{aligned}
$$