ME 233 Advance Control II

Lecture 5
Random Vector Sequences

(ME233 Class Notes pp. PR6-PR10)

Outline

- Random vector sequences
  – Covariance, cross-covariance
- MIMO Linear Time Invariant Systems
- White noise driven state space systems
- Covariance propagation Lyapunov equation

Random Vector Sequences

A two-sided random vector sequence is a collection of random vectors

$$X = \{X(k)\}_{k=-\infty}^{\infty}$$

Where

$$X(k) = \begin{bmatrix} X_1(k) \\ \vdots \\ X_n(k) \end{bmatrix} \in \mathcal{R}^n$$

and $X_i(k)$ is defined over the same probability space $(\Omega,\, \mathcal{S},\, Pr)$

2nd order statistics

For a two-sided **Random Vector Sequence (RVS)**

$$\{X(k)\}_{k=-\infty}^{\infty}$$

Expected value or mean of $X(k)$,

$$\boxed{E\{X(k)\} = m_X(k) \in \mathcal{R}^n}$$

Auto-covariance

Define: $\tilde{X}(k) = X(k) - m_X(k)$

$$\boxed{\Lambda_{XX}(k,j) = E\left\{\tilde{X}(k+j)\tilde{X}^T(k)\right\}}$$

$$\Lambda_{XX}(k,j) = E\left\{\begin{bmatrix}\tilde{X}_1(k+j)\\ \vdots \\ \tilde{X}_n(k+j)\end{bmatrix}\begin{bmatrix}\tilde{X}_1(k) & \cdots & \tilde{X}_n(k)\end{bmatrix}\right\}$$

Cross-covariance

Define: $\tilde{X}(k) = X(k) - m_X(k)$

$\tilde{Y}(k) = Y(k) - m_Y(k)$

$$\boxed{\Lambda_{XY}(k,j) = E\left\{\tilde{X}(k+j)\tilde{Y}^T(k)\right\}}$$

$$\Lambda_{XY}(k,j) = E\left\{\begin{bmatrix}\tilde{X}_1(k+j)\\ \vdots \\ \tilde{X}_n(k+j)\end{bmatrix}\begin{bmatrix}\tilde{Y}_1(k) & \cdots & \tilde{Y}_n(k)\end{bmatrix}\right\}$$

Wide Sense Stationary (WSS)

A two-sided random vector sequence $\{X(k)\}_{k=-\infty}^{\infty}$

is **WSS** if:

1) $E\{X(k)\} = m_X$ (time invariant)

2) $\Lambda_{XX}(k,l) = \Lambda_{XX}(k+M,l)\}$

(only depends on l)

Notice that: SSS $\Rightarrow$ WSS

Auto-covariance function

For WSS RVS, the auto-covariance is only a function of the correlation index j

$$\Lambda_{XX}(j) = E\left\{\tilde{X}(k+j)\tilde{X}^T(k)\right\}$$

for ***any*** index k

$$\Lambda_{XX}(l) = \Lambda^T_{XX}(-l)$$

Trace of the auto-covariance function

- Since

$$\Lambda_{XX}(j) = E\left\{\tilde{X}(k+j)\tilde{X}^T(k)\right\}$$

$$\text{Trace}\left[\Lambda_{XX}(j)\right] = E\{\tilde{X}^T(k+j)\tilde{X}(k)\}$$

- Using Shwarz' inequality, it can be shown that

$$\text{Trace}\left[\Lambda_{XX}(0)\right] \geq \left|\text{Trace}\left[\Lambda_{XX}(j)\right]\right|$$

Cross-covariance function

$X(k)$ and $Y(k)$
are two **WSS** random vector sequences

$$\boxed{\Lambda_{XY}(j) = E\{\tilde{X}(k+j)\tilde{Y}^T(k)\}}$$

for ***any*** index k

Notice that:

$$\boxed{\Lambda_{XY}(l) = \Lambda_{YX}^T(-l)}$$

Power Spectral Density Function

$$\begin{aligned}\Phi_{XX}(\omega) &= \mathcal{F}\{\Lambda_{XX}(\cdot)\}\\ &= \sum_{l=-\infty}^{\infty}\Lambda_{XX}(l)e^{-j\omega l}\end{aligned}$$

$$\begin{aligned}\Lambda_{XX}(l) &= \mathcal{F}^{-1}\{\Phi_{XX}(\omega)\}\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega l}\Phi_{XX}(\omega)\,d\omega\end{aligned}$$

White noise vector sequence

A **WSS** random vector sequence $\{W(k)\}_{k=-\infty}^{\infty}$ **is white** if:

$$\boxed{\Lambda_{WW}(l) = \Sigma_{WW}\,\delta(l)}$$

where

$$\delta(l) = \begin{cases} 1 & l = 0 \\ 0 & l \neq 0 \end{cases}$$

$$\Sigma_{WW} = E\{\tilde{W}(k)\tilde{W}^T(k)\} \qquad \tilde{W}(k) = W(k) - m_W$$

$$\Sigma_{WW} = \Sigma_{WW}^T \succeq 0$$

White noise vector sequence

Given the white WSS random sequence $\{W(k)\}_{k=-\infty}^{\infty}$

with

$$\Lambda_{WW}(l) = \Sigma_{WW}\,\delta(l)$$

Its power spectral density (Fourier transform)

$$\Phi_{WW}(\omega) \;=\; \sum_{l=-\infty}^{\infty} \Lambda_{WW}(l) e^{-j\omega l}$$

is

$$\boxed{\Phi_{WW}(\omega) \;=\; \Sigma_{WW}}$$

MIMO Linear Time Invariant Systems

Let $\{G(k)\}_{k=-\infty}^{\infty}$ with $G(k) \in \mathcal{R}^{p\times m}$

be the pulse response of a causal LTI SISO system

Transfer function

$$G(z) = \mathcal{Z}\{G(k)\} = \sum_{k=-\infty}^{\infty} G(k)\, z^{-k}$$

MIMO Linear Time Invariant Systems

Let $U(k) \in \mathcal{R}^m$ be WSS

The forced response (zero initial state)

$$Y(k) = \sum_{i=-\infty}^{\infty} G(i)U(k-i)$$

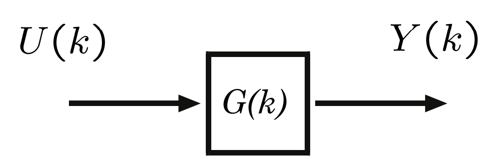
$$G(k) \in \mathcal{R}^{p\times m}$$

$Y(k) \in \mathcal{R}^p$ is also a WSS

MIMO Linear Time Invariant Systems

Let $U(k) \in \mathcal{R}^m$ be WSS

$$Y(k) = \sum_{i=-\infty}^{\infty} G(i)U(k-i)$$



MIMO Linear Time Invariant Systems

We will assume

$\{U(k)\}_{k=-\infty}^{\infty}$ is zero mean, I.e.

$$E\{U(k)\} = m_U = 0$$

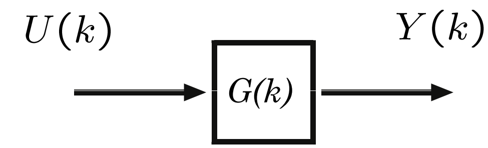
Thus, the forced response output is also zero mean

$$E\{Y(k)\} = m_Y = 0$$

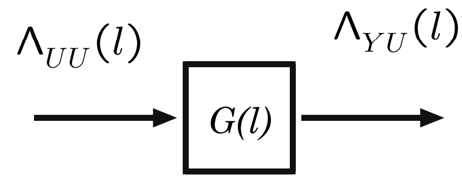
MIMO Linear Time Invariant Systems

Let $\{U(k)\}_{k=-\infty}^{\infty}$ be WSS

If

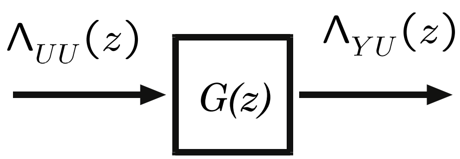


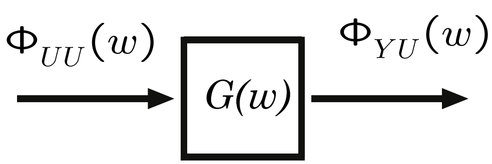
Then:



$$\Lambda_{YU}(l) = E\left\{\tilde{Y}(k+l)\tilde{U}^T(k)\right\}$$

MIMO Linear Time Invariant Systems





$\Phi_{UU}(w) = \Lambda_{UU}(z)|_{z=e^{j\omega}}$ $\quad$ $\Phi_{YU}(w) = \Lambda_{YU}(z)|_{z=e^{j\omega}}$

MIMO Linear Time Invariant Systems

Let $\{U(k)\}_{k=-\infty}^{\infty}$ be WSS and

$$Y(k) = \sum_{i=-\infty}^{\infty} G(i)U(k-i)$$

Then:

$$\Lambda_{YU}(l) = \sum_{i=-\infty}^{\infty} G(i)\,\Lambda_{UU}(l-i)$$

$$\Phi_{YU}(w) = G(w)\,\Phi_{UU}(w)$$

MIMO Linear Time Invariant Systems

$$\boxed{\Lambda_{YU}(l) = \sum_{i=-\infty}^{\infty} G(i)\,\Lambda_{UU}(l-i)}$$

Proof:

$$Y(k) = \sum_{i=-\infty}^{\infty} G(i)U(k-i) \qquad (m_U = 0)$$

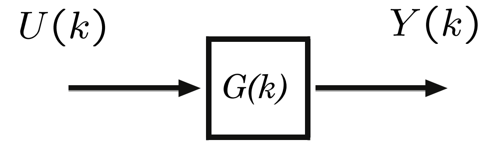
Then:

$$\begin{aligned}
\Lambda_{YU}(l) &= E\{Y(k+l)U^T(k)\} \\
&= E\left\{\left[\sum_{i=-\infty}^{\infty} G(i)\,U(k+l-i)\right] U^T(k)\right\} \\
&= \sum_{i=-\infty}^{\infty} G(i)\,E\left\{U(k+l-i)U^T(k)\right\} \\
&= \sum_{i=-\infty}^{\infty} G(i)\,\Lambda_{UU}(l-i)
\end{aligned}$$

MIMO Linear Time Invariant Systems

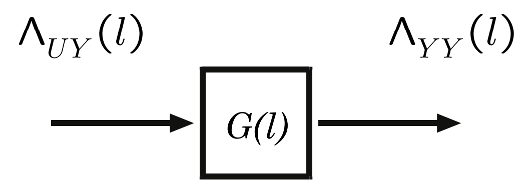
Let $\{U(k)\}_{k=-\infty}^{\infty}$ be WSS

If



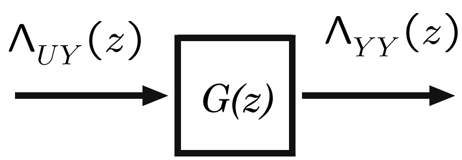
Then:

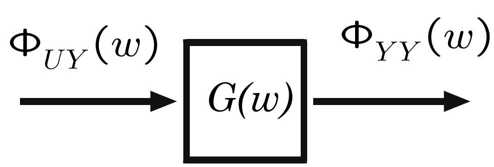
$E\left\{\tilde{U}(k+l)\tilde{Y}^T(k)\right\} =$



MIMO Linear Time Invariant Systems

Let $\{U(k)\}_{k=-\infty}^{\infty}$ be WSS





$$\Phi_{UY}(w) = \Lambda_{UY}(z)\big|_{z=e^{j\omega}} \qquad \Phi_{YY}(w) = \Lambda_{YY}(z)\big|_{z=e^{j\omega}}$$

MIMO Linear Time Invariant Systems

Let $\{U(k)\}_{k=-\infty}^{\infty}$ be a WSS VRS

If

$$Y(k) = \sum_{i=-\infty}^{\infty} G(i)U(k-i)$$

Then:

$$\boxed{\Lambda_{YY}(l) = \sum_{i=-\infty}^{\infty} G(i)\,\Lambda_{UY}(l-i)}$$

$$\boxed{\Phi_{YY}(w) = G(w)\,\Phi_{UY}(w)}$$

$$\Lambda_{UY}(l) = \Lambda_{YU}^T(-l)$$

MIMO Linear Time Invariant Systems

$$\boxed{\Lambda_{YY}(l) = \sum_{i=-\infty}^{\infty} G(i)\,\Lambda_{UY}(l-i)}$$

Proof:

$$Y(k) = \sum_{i=-\infty}^{\infty} G(i)U(k-i) \qquad (m_U = 0)$$

Then:

$$\begin{aligned}
\Lambda_{YY}(l) &= E\{Y(k+l)Y^T(k)\} \\
&= E\left\{\left[\sum_{i=-\infty}^{\infty} G(i)U(k+l-i)\right] Y^T(k)\right\} \\
&= \sum_{i=-\infty}^{\infty} G(i)\, E\left\{U(k+l-i)Y^T(k)\right\} \\
&= \sum_{i=-\infty}^{\infty} G(i)\,\Lambda_{UY}(l-i)
\end{aligned}$$

MIMO Linear Time Invariant Systems

$$\boxed{\Phi_{UY}(w) = \Phi_{YU}^T(-w)}$$

This is a consequence of the fact that

$$\Lambda_{UY}(l) = \Lambda_{YU}^T(-l)$$

$$\Phi_{UY}(w) = \Phi_{YU}^T(-w)$$

Proof:

$$\Phi_{UY}(\omega) = \sum_{l=-\infty}^{\infty} \Lambda_{UY}(l)e^{-j\omega l}$$

$$(\Lambda_{UY}(l) = \Lambda_{YU}^T(-l))$$

$$= \sum_{l=-\infty}^{\infty} \Lambda_{YU}^T(-l)e^{-j\omega l}$$

$$= \sum_{l=-\infty}^{\infty} \Lambda_{YU}^T(l)e^{j\omega l} \qquad = \Phi_{YU}^T(-\omega)$$

MIMO Linear Time Invariant Systems

If
$$Y(k) = \sum_{i=-\infty}^{\infty} G(i)U(k-i)$$

Then:

$$\boxed{\Phi_{YY}(\omega) = G(\omega)\,\Phi_{UU}(\omega)\,G^T(-\omega)}$$

$$G(\omega) = G(e^{j\omega}) \qquad G(-\omega) = G(e^{-j\omega})$$

$$\Phi_{YY}(\omega) = G(\omega)\,\Phi_{UU}(\omega)\,G^T(-\omega)$$

Proof: Use

$$\Phi_{YU}(w) = G(w)\,\Phi_{UU}(w)$$

$$\Phi_{UY}(w) = \Phi_{YU}^T(-w)$$

then

$$\Phi_{UY}(w) = \underbrace{\Phi_{UU}^T(-w)}_{\Phi_{UU}(w)}\,G^T(-w)$$

$$\Phi_{YY}(\omega) = G(\omega)\,\Phi_{UU}(\omega)\,G^T(-\omega)$$

Thus,

$$\Phi_{UY}(w) = \Phi_{UU}(w)\,G^T(-w)$$

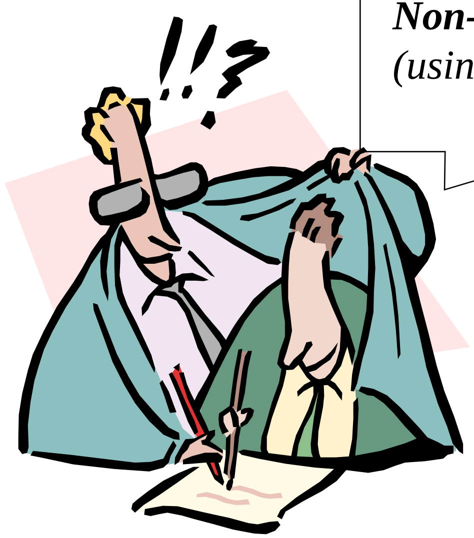
Since,

$$\Phi_{YY}(w) = G(w)\,\Phi_{UY}(w)$$

then

$$\Phi_{YY}(\omega) = G(\omega)\,\Phi_{UU}(\omega)\,G^T(-\omega)$$

Now we will do
Non-stationary systems!
(using state space models)



2nd order statistics of a random sequence

We now consider one-sided random sequence

$$\{X(k)\}_{k=0}^{\infty}$$

Expected value or mean of ***X(k)*,**

$$E\{X(k)\} = m_X(k)$$

Auto-covariance function:

$$\Lambda_{XX}(k,j) =$$

$$E\left\{[X(k+j) - m_X(k+j)]\,[X(k) - m_X(k)]^T\right\}$$

Subtracting the mean

- Define

$$\tilde{X}(k) = X(k) - m_X(k)$$

Auto-covariance

$$\Lambda_{XX}(k,j) = E\left\{\tilde{X}(k+j)\,\tilde{X}^T(k)\right\}$$

White noise driven state space systems

Consider a LTI system driven by white noise:

$$\begin{aligned} X(k+1) &= A\,X(k) + B\,W(k) \\ Y(k) &= C\,X(k) \end{aligned}$$

$X(k) \in \mathcal{R}^n$ $W(k) \in \mathcal{R}^p$

$Y(k) \in \mathcal{R}^m$

White noise driven state space systems

Consider a LTI system driven by white noise:

$$\begin{aligned} X(k+1) &= A\,X(k) + B\,W(k) \\ Y(k) &= C\,X(k) \end{aligned}$$

W(k) is white, but not stationary

$$m_W(k) = E\{W(k)\}$$

$$\Lambda_{WW}(k,l) = E\{\tilde{W}(k+l)\tilde{W}^T(k)\}$$

White noise driven state space systems

W(k) is white, but not stationary

$$\Lambda_{WW}(k,l) = \Sigma_{WW}(k)\,\delta(l)$$

$$\delta(l) = \begin{cases} 1 & l = 0 \\ 0 & l \neq 0 \end{cases}$$

$$\Sigma_{WW}(k) = E\{\tilde{W}(k)\tilde{W}^T(k)\} \in \mathcal{R}^{p\times p}$$

White noise driven state space systems

$$
\begin{aligned}
X(k+1) &= A\,X(k) + B\,W(k) \\
Y(k) &= C\,X(k)
\end{aligned}
$$

State Initial Conditions (IC):

$$
\begin{aligned}
m_X(0) &= E\{X(0)\} = m_X(0) \\
\Lambda_{XX}(0,0) &= E\left\{\tilde{X}(0)\tilde{X}^T(0)\right\}
\end{aligned}
$$

$$
E\left\{\tilde{X}(0)\tilde{W}^T(k)\right\} = 0
$$

Dynamics of the mean

$$
\begin{aligned}
X(k+1) &= A\,X(k) + B\,W(k) \\
Y(k) &= C\,X(k)
\end{aligned}
$$

Taking expectations on the equations above:

$$
\begin{aligned}
m_X(k+1) &= A\,m_X(k) + B\,m_W(k) \\
m_Y(k) &= C\,m_X(k)
\end{aligned}
$$

White noise driven state space systems

Subtracting the means we obtain,

$$
\begin{aligned}
\tilde{X}(k+1) &= A\,\tilde{X}(k) + B\,\tilde{W}(k) \\
\tilde{Y}(k) &= C\,\tilde{X}(k)
\end{aligned}
$$

Where now

$$
m_{\tilde{W}}(k) = 0
$$

$$
m_{\tilde{X}}(k) = 0
$$

Covariance propagation

$$
\tilde{X}(k+1) = A\,\tilde{X}(k) + B\,\tilde{W}(k)
$$

Notice that:

$$
\tilde{X}(k+1)\tilde{X}^T(k+1) = \left[A\,\tilde{X}(k) + B\,\tilde{W}(k)\right]\left[A\,\tilde{X}(k) + B\,\tilde{W}(k)\right]^T
$$

White noise driven covariance propagation

Taking expectations to:

$$
\begin{aligned}
\underbrace{\tilde{X}(k+1)\tilde{X}^T(k+1)}_{\Lambda_{XX}(k+1,0)} &= A\,\tilde{X}(k)\tilde{X}^T(k)\,A^T \\
&+ A\,\tilde{X}(k)\tilde{W}^T(k)\,B^T \\
&+ B\,\tilde{W}(k)\tilde{X}^T(k)\,A^T \\
&+ B\,\tilde{W}(k)\tilde{W}^T(k)\,B^T
\end{aligned}
$$

White noise driven covariance propagation

Taking expectations to:

$$
\begin{aligned}
\Lambda_{XX}(k+1,0) &= A\,\Lambda_{XX}(k,0)\,A^T \\
&+ A\,\Lambda_{XW}(k,0)\,B^T \\
&+ B\,\Lambda_{WX}(k,0)\,A^T \\
&+ B\,\Lambda_{WW}(k,0)\,B^T
\end{aligned}
$$

White noise driven covariance propagation

Notice that:

$$
\begin{aligned}
\Lambda_{XX}(k+1,0) &= A\,\Lambda_{XX}(k,0)\,A^T \\
&+ \cancelto{0}{A\,\Lambda_{XW}(k,0)\,B^T} \\
&+ \cancelto{0}{B\,\Lambda_{WX}(k,0)\,A^T} \\
&+ B\,\Lambda_{WW}(k,0)\,B^T
\end{aligned}
$$

(***W*** is white)

$$
\begin{aligned}
\Lambda_{XW}(k,0) &= \Lambda_{WX}^T(k,0) \\
&= E\left\{\tilde{X}(k)\tilde{W}^T(k)\right\} = 0
\end{aligned}
$$

Proof of $E\left\{\tilde{X}(k)\tilde{W}^T(k)\right\} = 0$

By induction:

1) For $k=0$: $E\left\{\tilde{X}(0)\tilde{W}^T(k)\right\} = 0$

2) Assume $E\left\{\tilde{X}(k-1)\tilde{W}^T(k)\right\} = 0$, then

$$
\tilde{X}(k) = A\,\tilde{X}(k-1) + B\,\tilde{W}(k-1)
$$

$$
\begin{aligned}
E\{\tilde{X}(k)\tilde{W}^T(k)\} &= \cancelto{0}{A\,E\{\tilde{X}(k-1)\tilde{W}^T(k)\}} \\
&+ \cancelto{0}{B\,E\{\tilde{W}(k-1)\tilde{W}^T(k)\}}
\end{aligned}
$$

White noise driven covariance propagation

We obtain the following Lyapunov equation:

$$\boxed{\Lambda_{XX}(k+1,0) \;=\; A\,\Lambda_{XX}(k,0)\,A^T + B\,\Sigma_{WW}(k)\,B^T}$$

$$\Lambda_{XX}(k,0) \;=\; E\left\{\tilde{X}(k)\tilde{X}^T(k)\right\}$$

$$\Lambda_{WW}(k,0) \;=\; E\left\{\tilde{W}(k)\tilde{W}^T(k)\right\} = \Sigma_{WW}(k)$$

White noise driven covariance propagation

From the output equation

$$\tilde{Y}(k) \;=\; C\,\tilde{X}(k)$$

we obtain

$$\Lambda_{YY}(k,0) \;=\; C\,\Lambda_{XX}(k,0)\,C^T$$

Covariance propagation

Lets now compute,

$$\Lambda_{XX}(k,l) \;=\; E\left\{\tilde{X}(k+l)\tilde{X}^T(k)\right\} \qquad l \geq 0$$

Using the solution of the LTI system,

$$\tilde{X}(k+l) \;=\; A^l\,\tilde{X}(k) + \sum_{j=k}^{k+l-1} A^{k+l-1-j} B\,\tilde{W}(j)$$

Covariance propagation

$$\tilde{X}(k+l) \;=\; A^l\,\tilde{X}(k) + \sum_{j=k}^{k+l-1} A^{k+l-1-j} B\,\tilde{W}(j)$$

$$\Lambda_{XX}(k,l) \;=\; E\left\{\tilde{X}(k+l)\tilde{X}^T(k)\right\}$$

$$= \; A^l\,E\{\tilde{X}(k)\tilde{X}^T(k)\}$$

$$+ \sum_{j=k}^{k+l-1} A^{k+l-1-j} B\,E\{\tilde{W}(j)\tilde{X}^T(k)\}$$

Covariance propagation

Since

$$\Lambda_{WX}(k,j) \;=\; E\left\{\tilde{W}(k+j)\tilde{X}^T(k)\right\}$$

$$=\; 0 \qquad\qquad j \geq 0$$

($\boldsymbol{W}$ is white)

$$\sum_{j=k}^{k+l-1} A^{k+l-1-j} B\, E\{\tilde{W}(j)\tilde{X}^T(k)\} = 0$$

Covariance propagation

$$\Lambda_{XX}(k,l) \;=\; E\left\{\tilde{X}(k+l)\tilde{X}^T(k)\right\}$$

Satisfies:

$$\boxed{\Lambda_{XX}(k,l) \;=\; A^l\,\Lambda_{XX}(k,0) \qquad l \geq 0}$$

$$\Lambda_{XX}(k,-l) \;=\; \Lambda_{XX}^T(k,l)$$

$$=\; \Lambda_{XX}(k,0)\,(A^l)^T$$

Stationary covariance equation

If $W(k)$ is WSS

and A is Schur (i.e. all eigenvalues inside unite circle):

and $X(k)$ and $Y(k)$ will converge to WSS zero mean VRS:

$$\lim_{k\to\infty} m_X(k) \;=\; \bar{m}_X \qquad\qquad \lim_{k\to\infty} m_Y(k) \;=\; C\bar{m}_X$$

$$\lim_{k\to\infty} \Lambda_{XX}(k,0) = \bar{\Lambda}_{XX}(0) \qquad \lim_{k\to\infty} \Lambda_{YY}(k,0) = \bar{\Lambda}_{YY}(0)$$

$$= C\bar{\Lambda}_{XX}(0)C^T$$

WSS Stationary covariance equation

For $W(k)$ WSS, and A Schur,

$$m_X(k+1) \;=\; A\,m_X(k) + B\,m_W$$

converges to

$$\bar{m}_X \;=\; [I-A]^{-1}\,B\,m_W$$

WSS Stationary covariance equation

For $W(k)$ WSS, and A Schur,

$$\bar{\Lambda}_{XX}(0) = \lim_{k\to\infty} E\{\tilde{X}(k)\tilde{X}^T(k)\}$$

Satisfies the Lyapunov equation:

$$\boxed{A\,\bar{\Lambda}_{XX}(0)\,A^T - \bar{\Lambda}_{XX}(0) = -B\,\Sigma_{WW}\,B^T}$$

WSS Stationary covariance equation

For $W(k)$ WSS, and A Schur,

$$\bar{\Lambda}_{XX}(l) = \lim_{k\to\infty} E\{\tilde{X}(k+l)\tilde{X}^T(k)\}$$

Satisfies

$$\boxed{\bar{\Lambda}_{XX}(l) = A^l\,\bar{\Lambda}_{XX}(0) \qquad l \geq 0}$$

Illustration – first order system

- Plant:

$$Y(k+1) \;=\; 0.5\,Y(k) + 1\,W(k)$$

- Input:

$$m_W(k) = 1 \qquad\qquad \Lambda_{WW}(k,l) = 0.2\,\delta(l)$$

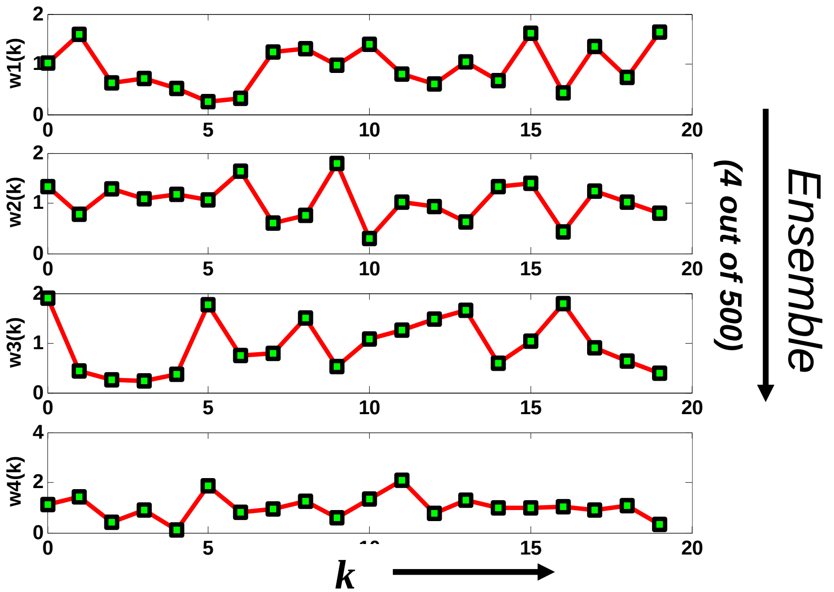
- State initial conditions:

$$m_Y(0) \;=\; 0 \qquad\qquad \Lambda_{YY}(0,0) \;=\; .1$$

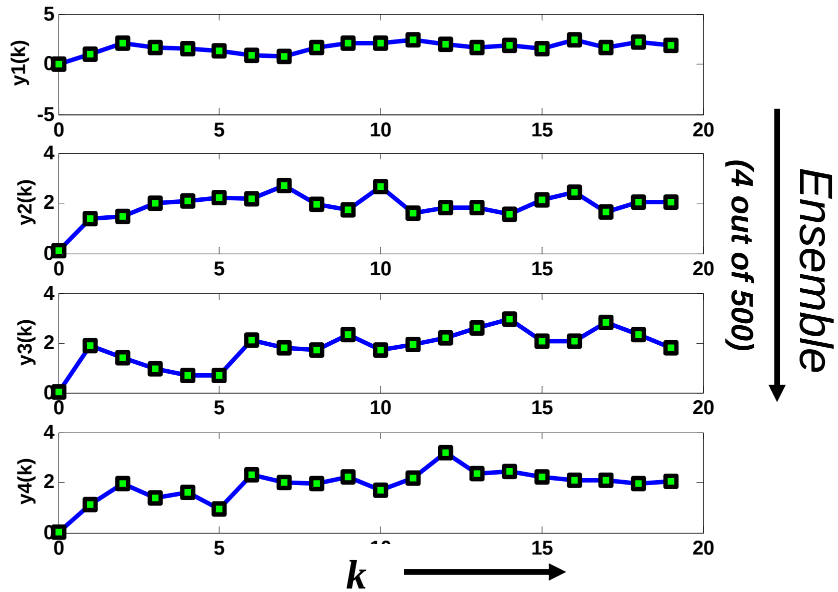
Matlab simulation: 500 sample sequences

```
lyy0 = 0.1
lww = 0.2
sys1=ss(.5,1,1,0,1)
N=20;
p=500;
w = sqrt(lww)*randn(N,p)+1;
y = zeros(N,p);
y0 = sqrt(lyy0)*randn(1,p);
k = (0:1:N-1)';

for j=1:p
   [y(:,j),k] = lsim(sys1,w(:,j),k,y0(1,j));
end

m_y=mean(y')
L_yy=diag(cov(y'));
```

W(k) $m_W(k) = 1$ $\Lambda_{WW}(k,l) = 0.2\,\delta(l)$



Y(k) $m_Y(0) \;=\; 0$ $\Lambda_{YY}(0,0) \;=\; .1$



Mean Transient Response

Actual:

$$m_Y(k+1) \;=\; 0.5\,m_Y(k) + 1$$

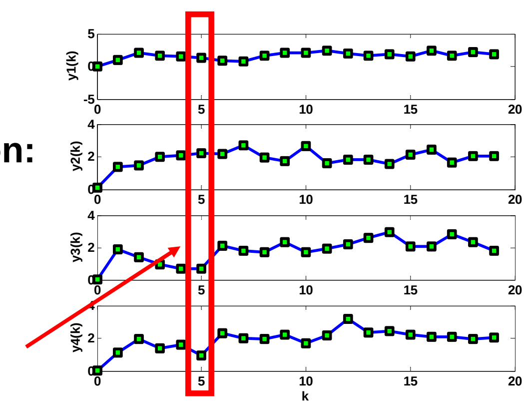
$$m_Y(0) \;=\; 0$$

Matlab calculation:

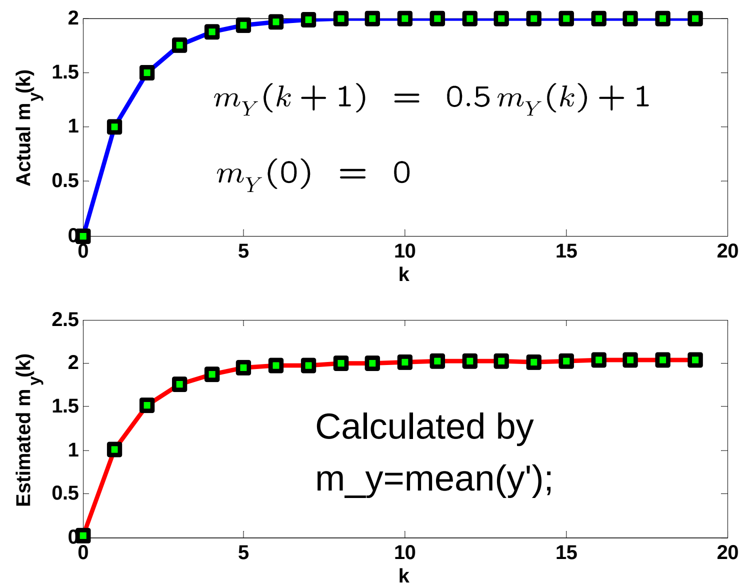
Ensemble mean

m_y=mean(y');

$\approx m_Y(5)$



Mean Transient Response



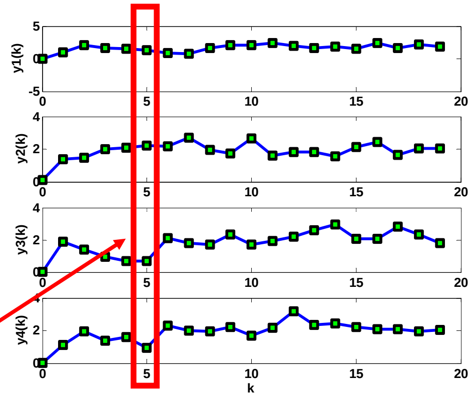
Covariance Transient Response

Actual:

$$\Lambda_{XX}(k+1,0) \;=\; 0.5^2\,\Lambda_{XX}(k,0)+0.2$$

$$\Lambda_{XX}(0,0) \;=\; .1$$

$$\Lambda_{WW}(k,l)=0.2\,\delta(l)$$



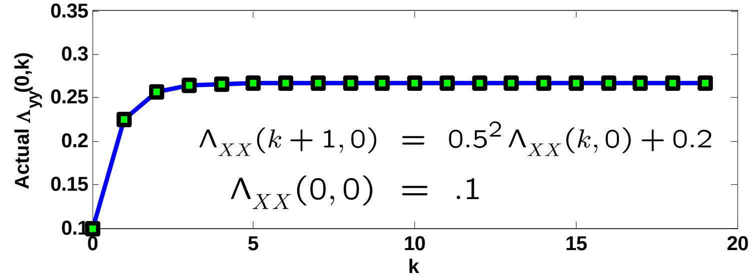
Matlab calculation:

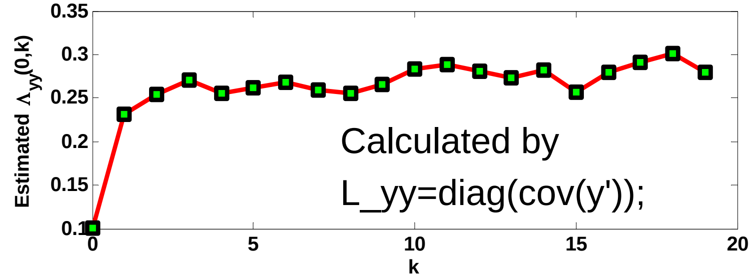
Ensemble covariance

L_yy=diag(cov(y'));

$$\approx \Lambda_{YY}(5,0)$$

Covariance Transient Response





Steady State Covariance