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ME 233 Advance Control II

Continuous-time results 4

Linear Quadratic Gaussian
Loop Transfer Recovery

(ME233 Class Notes pp.LTR1-LTR9)

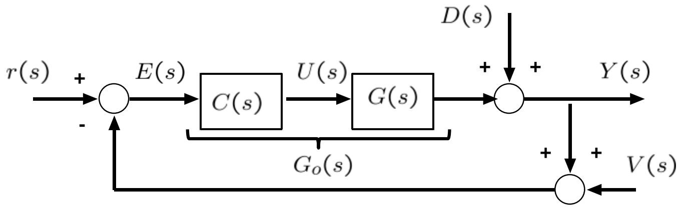
2

Outline

- Review of Feedback
- LQG stability margins
- LQG-LTR

3

Basic Feedback Transfer Functions (TF)



- $Y(s)$ is the controlled output
- $U(s)$ is the control input
- $E(s)$ is error signal fed to the controller
- $R(s)$ is the output reference
- $D(s)$ is the disturbance input
- $V(s)$ is the measurement noise

$E_T(s) = R(s) - Y(s)$ "true" error signal

$$E_T(s) = [I + G_o(s)]^{-1}\,[R(s) - D(s)] + [I + G_o(s)]^{-1} G_o(s)\, V(s)$$

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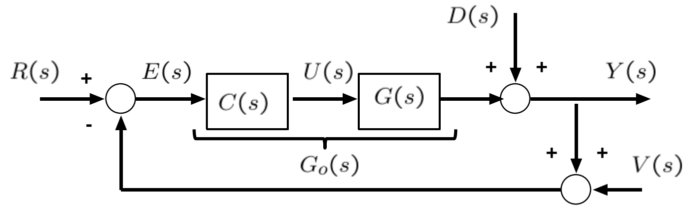
Basic Feedback Transfer Functions (TF)



$E_T(s) = R(s) - Y(s)$ "true" error signal

$$E_T(s) = \underbrace{[I + G_o(s)]^{-1}}_{\substack{S(s) \\ \text{sensitivity TF}}}\,[R(s) - D(s)] + \underbrace{[I + G_o(s)]^{-1} G_o(s)}_{\substack{T(s) \\ \text{complementary} \\ \text{sensitivity TF}}}\, V(s)$$

$T(s) + S(s) = I$

Basic Feedback Transfer Functions (TF)

$$E_T(s) = R(s) - Y(s) \qquad T(s) + S(s) = I$$

$$E_T(s) = \underbrace{[I + G_o(s)]^{-1}}_{S(s)} [R(s) - D(s)] + \underbrace{[I + G_o(s)]^{-1} G_o(s)}_{T(s)} V(s)$$

Frequency domain and singular values:

$$\sigma_{\max}[A(j\omega)] = (\lambda_{\max}[A^*(j\omega)A(j\omega)])^{\frac{1}{2}} \qquad \sigma_{\min}[A(j\omega)] = (\lambda_{\min}[A^*(j\omega)A(j\omega)])^{\frac{1}{2}}$$

$$Y(j\omega) = A(j\omega)U(j\omega) \quad \Rightarrow \quad \begin{cases} \|Y(j\omega)\| \geq \sigma_{\min}[A(j\omega)]\,\|U(j\omega)\| \\ \|Y(j\omega)\| \leq \sigma_{\max}[A(j\omega)]\,\|U(j\omega)\| \end{cases}$$

Basic Feedback Transfer Functions (TF)

$$E_T(s) = R(s) - Y(s) \qquad T(s) + S(s) = I$$

$$E_T(s) = \underbrace{[I + G_o(s)]^{-1}}_{S(s)} [R(s) - D(s)] + \underbrace{[I + G_o(s)]^{-1} G_o(s)}_{T(s)} V(s)$$

Frequency domain:

1) $\|R(j\omega)\|$ and $\|D(j\omega)\|$ are normally large at low frequencies

$\Rightarrow$ $\sigma_{\max}[S(j\omega)] < 1$ at low frequencies

2) $\|V(j\omega)\|$ and **plant model uncertainties** are normally large at high frequencies

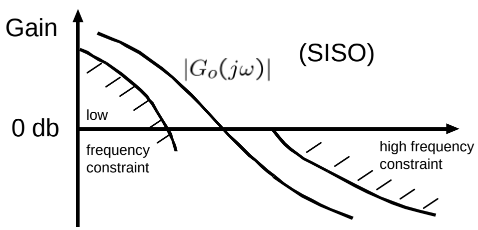
$\Rightarrow$ $\sigma_{\max}[T(j\omega)] < 1$ at high frequencies

Basic Feedback Transfer Functions (TF)

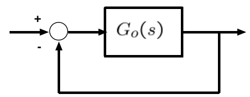
$$E_T(s) = \underbrace{[I + G_o(s)]^{-1}}_{S(s)} [R(s) - D(s)] + \underbrace{[I + G_o(s)]^{-1} G_o(s)}_{T(s)} V(s)$$

$\sigma_{\max}[S(j\omega)] < 1$ at low frequencies $\Rightarrow$ $\sigma_{\min}[G_o(j\omega)] >> 1$

$\sigma_{\max}[T(j\omega)] < 1$ at high frequencies $\Rightarrow$ $\sigma_{\max}[G_o(j\omega)] << 1$



Bode's integral theorem (SIS0)



$$S(s) = \frac{1}{1 + G_o(s)}$$

Let the open loop transfer function *Go(s)* have relative degree ≥ 2 and let *p1, p2 , … pm* be the unstable open loop poles (right have plane)

$$\int_0^\infty \ln(|S(j\omega)|dw = \pi \sum_{i=1}^{m} p_i$$

When *Go(s)* is stable,

$$\int_0^\infty \ln(|S(j\omega)|dw = 0$$

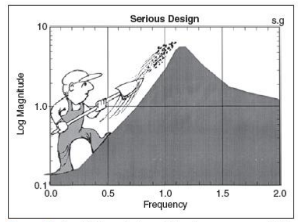
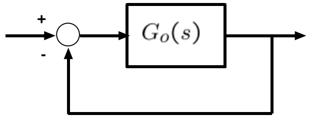


Figure 3. *Sensitivity reduction at low frequency unavoidably leads to sensitivity increase at higher frequencies.*

Multivariable Nyquist Stability Criterion



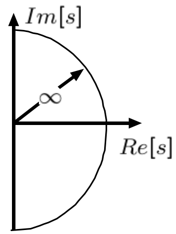
return difference

$$L(s) = [I + G_o(s)]$$

$$\det[L(s)] = \frac{A_c(s)}{A(s)}$$

roots of $A_c(s) = 0$ are the **closed loop** poles

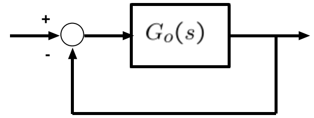
roots of $A(s) = 0$ are the **open loop** poles



Nyquist path ***D***

$N(0, \det[L(s)], D)$: number of counterclockwise encirclements around ***0*** by det[*L(s)*] when *s* is along the Nyquist path ***D***

Multivariable Nyquist Stability Criterion

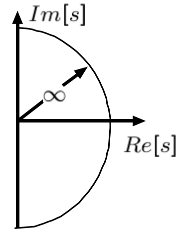


$$L(s) = [I + G_o(s)]$$

$$N(0, \det[L(s)], D) = P - Z$$

P = # of unstable **open** loop poles

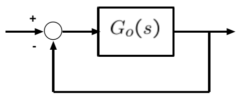
Z = # of unstable **closed** loop poles

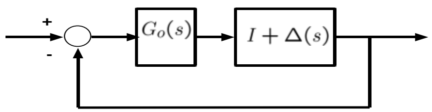


Nyquist path ***D***

$N(0, \det[L(s)], D)$: number of counterclockwise encirclements around ***0*** by det[*L(s)*] when *s* is along the Nyquist path ***D***

Robust Stability





Nominal closed loop system (asymptotically stable)

$$L(s) = [I + G_o(s)]$$

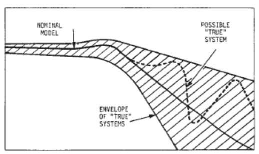
Actual system

$\Delta(s)$: output multiplicative uncertainty

Feedback system has robust stability iff

$$N(0, \det[L(s)], D) = N(0, \det[L'(s)], D)$$

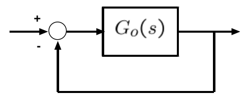
when *s* is along the Nyquist path ***D***

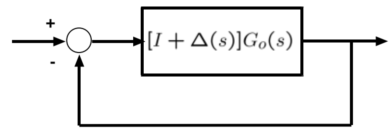


$$G_o'(s) = [I + \Delta(s)]G_o(s)$$

$$L'(s) = [I + G_o'(s)]$$

Robust Stability





$$G_o(s) = G(s)C(s)$$

$$L(s) = [I + G_o(s)]$$

$$G_o'(s) = [I + \Delta(s)]G_o(s)$$

$$L'(s) = [I + G_o'(s)]$$

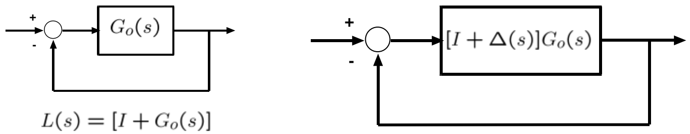
robust stability iff $N(0, \det[L(s)], D) = N(0, \det[L'(s)], D)$

⟺ $0 < \sigma_{\min}\,[I + [I + \epsilon\,\Delta(s)]G_o(s)]$ for all $\epsilon \in [0, 1]$

when *s* is along the Nyquist path ***D***

⟺ $0 < \sigma_{\min}\,[I + [I + \epsilon\Delta(j\omega)]G_o(j\omega)]$ for all $\begin{cases} \epsilon \in [0, 1] \\ \omega \in [0, \infty) \end{cases}$

Robust Stability



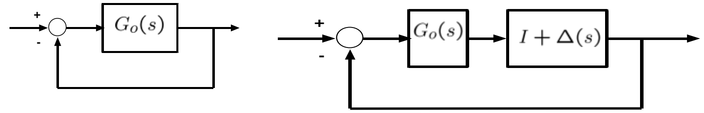
$L(s) = [I + G_o(s)]$

$$\left.\begin{array}{l} 0 < \sigma_{\min}\,[I + [I + \epsilon\Delta(j\omega)]G_o(j\omega)] \\ 0 < \sigma_{\min}\,[\underbrace{I + Go(j\omega)}_{L(j\omega)} + \epsilon\Delta(j\omega)G_o(j\omega)] \\ \Longleftrightarrow \\ 0 < \sigma_{\min}\left[I + \epsilon\Delta(j\omega)\underbrace{G_o(j\omega)[I + G_o(j\omega)]^{-1}}_{T(j\omega)}\right] \end{array}\right\} \text{for all } \begin{cases} \epsilon \in [0,1] \\ \omega \in [0,\infty) \end{cases}$$

$\Longleftrightarrow$

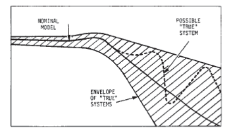
$$0 < 1 + \sigma_{\max}\,[\Delta(j\omega)T(j\omega)] \quad \Longleftrightarrow \quad \sigma_{\max}\,[T(j\omega)] < \frac{1}{\sigma_{\max}\,[\Delta(j\omega)]}$$

Robust Stability



$$\boxed{\sigma_{\max}\,[T(j\omega)] < \frac{1}{\sigma_{\max}\,[\Delta(j\omega)]}}$$

$$T(s) = G_o(s)[I + G_o(s)]^{-1}$$



at high frequencies when , $\sigma_{\max}\,[G_o(j\omega)] << 1$

$$\boxed{\sigma_{\max}\,[T(j\omega)] \approx \sigma_{\max}\,[G_o(j\omega)] < \frac{1}{\sigma_{\max}\,[\Delta(j\omega)]}}$$

Stationary LQR

Cost:

$$J_s = \frac{1}{2}E\{x^T(t)C_Q^T C_Q x(t) + u^T(t)Ru(t)\}$$

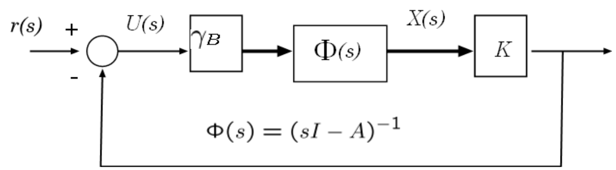
- **Optimal control:** $u^o(t) = -K\,x(t) + r$

Where the gain is obtained from the solution of the steady state LQR

$$K = R^{-1}B^T P$$

$$A^T P + P\,A + C_Q^T C_Q - P\,B\,R^{-1}\,B^T P = 0$$

LQR robustness properties



$$G_o(j\omega) = K\Phi(j\omega)B$$

$\gamma = 1$

Phase Margin ≥ *60º*

Close loop system is stable for:

$$\frac{1}{2} < \gamma < \infty$$

Stationary Kalman Filter

- **Kalman Filter Estimator:**

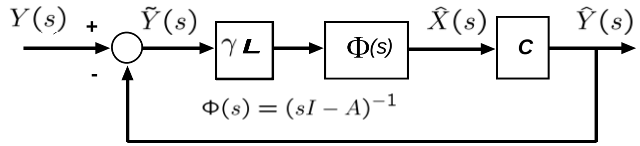
$$\frac{d}{dt}\hat{x}(t) = A\,\hat{x}(t) + B\,u(t) + L\,\tilde{y}(t)$$

$$\tilde{y}(t) = y(t) - C\,\hat{x}(t)$$

$$L = M\,C^T\,V^{-1}$$

$$AM + MA^T = -B_w W B_w^T + MC^T V^{-1} CM$$

KF dual robustness properties



$$G_o(j\omega) = C\Phi(j\omega)L$$

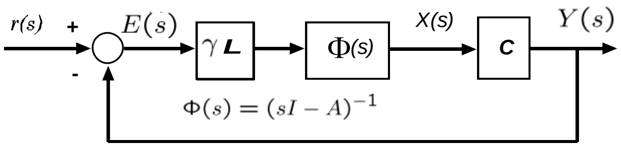
$\gamma = 1$

Phase Margin $\geq 60^o$

Close loop system is stable for:

$$\frac{1}{2} < \gamma < \infty$$

"Fictitious" KF robustness properties



$$G_o(j\omega) = C\Phi(j\omega)L$$

$\gamma = 1$

Phase Margin $\geq 60^o$

Close loop system is stable for:

$$\frac{1}{2} < \gamma < \infty$$

LQR example 1

Double integrator (example in pp ME232-143):

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$J = \frac{1}{2}\int_0^\infty \left\{ x^T\,{C_Q}^T\,C_Q\,x + R\,u^2 \right\} dt$$

with

$$C_Q = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad R > 0$$

Only position is penalized

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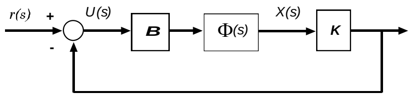
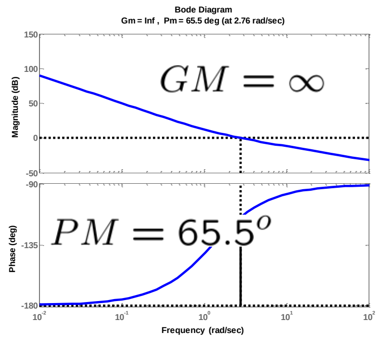
LQR example 1 margins



$$R = 0.1$$

$$G_o(s) = K\Phi(s)B$$

$$G_o(s) = \frac{2.5(s+1.26)}{s^2}$$



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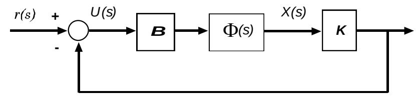
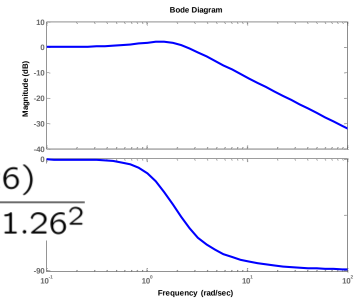
LQR example $T(s)$



$$R = 0.1$$

$$T(s) = \frac{G_o(s)}{1+G_o(s)}$$

$$T(s) = \frac{2.5(s+1.26)}{(s+1.26)^2 + 1.26^2}$$



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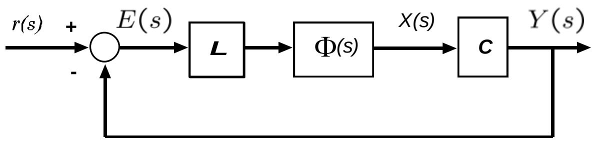
Fictitious KF Feedback Loop example 1



Controller design parameters B_w, W, V are chosen

$$W = 1 \qquad V = R = 0.1 \qquad B_w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

KF return difference equality = LQR return difference equality

$$G_w(s) = G_Q(s)$$

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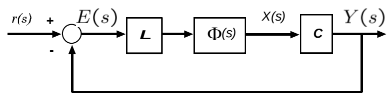
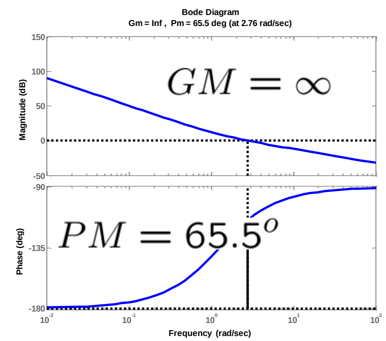
Fictitious KF example 1 margins



$$V = 0.1$$

$$G_o(s) = L\Phi(s)C$$

$$G_o(s) = \frac{2.5(s+1.26)}{s^2}$$



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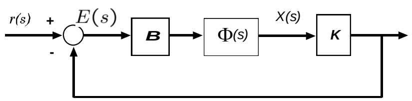
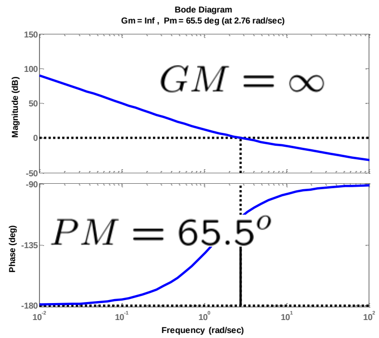
LQR example 1 margins



$$R = 0.1$$

$$G_o(s) = K\Phi(s)B$$

$$G_o(s) = \frac{2.5(s+1.26)}{s^2}$$



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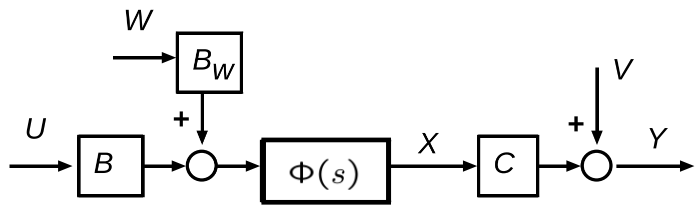
Stationary LQG



$$\dot{x}(t) = A\,x(t) + B\,u(t) + B_w\,w(t)$$

$$y(t) = C\,x(t) + v(t)$$

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Stationary LQG

Cost:

$$J_s = \frac{1}{2} E\{x^T(t) C_Q^T C_Q x(t) + u^T(t) R u(t)\}$$

- **Optimal control:** $\boxed{u^o(t) = -K\,\hat{x}(t)}$

Where the gain is obtained from the solution of the steady state LQR

$$K = R^{-1} B^T P$$

$$A^T P + P A + C_Q^T C_Q - P B R^{-1} B^T P = 0$$

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Stationary LQG

- **Kalman Filter Estimator:**

$$\frac{d}{dt}\hat{x}(t) = A\,\hat{x}(t) + B\,u(t) + L\,\tilde{y}(t)$$

$$\tilde{y}(t) = y(t) - C\,\hat{x}(t)$$

$$L = M\,C^T\,V^{-1}$$

$$AM + MA^T = -B_w W B_w^T + MC^T V^{-1} CM$$

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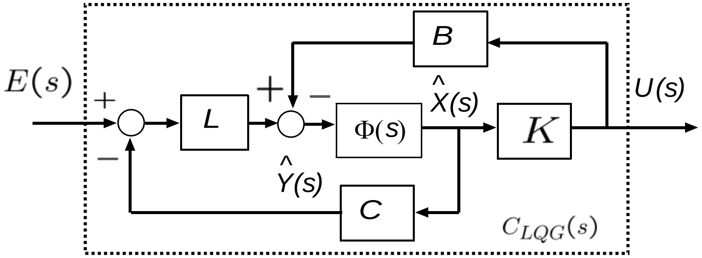
Stationary LQG Compensator



$$U(s) = C_{LQG}(s)\, E(s)$$

$$C_{LQG}(s) = K\ (sI - A + BK + LC)^{-1}\ L$$

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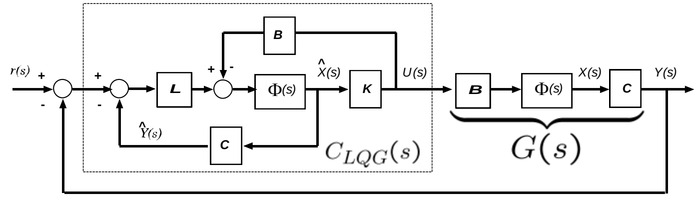
LQG Loop Transfer



$$C_{LQG}(s) = K\ (sI - A + BK + LC)^{-1}\ L$$

$$G(s) = C\ (sI - A)^{-1}\ B$$

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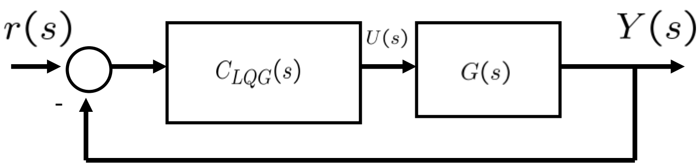
LQG Robustness Margins?



$$G_o(s) = G(s)\, C_{LQG}(s)$$

Unfortunately, there are no guaranteed robustness margins results for a general LQG controller

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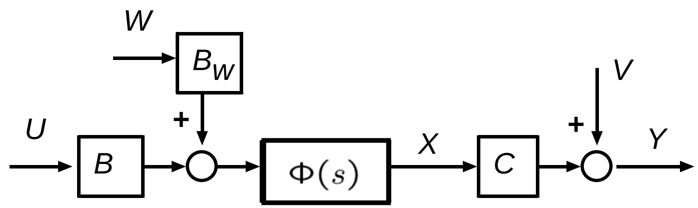
Example -1 Double integrator



$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B_w = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$W = 1 \qquad V = 0.1$$

LQG example 1

Double integrator (example in pp ME232-143):

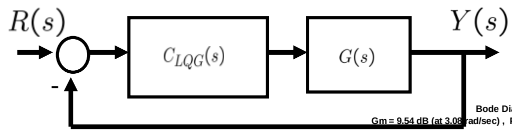
$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$J = \frac{1}{2}\int_0^\infty \left\{ x^T {C_Q}^T C_Q\, x + R\, u^2 \right\} dt$$

with

$$C_Q = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad R = 0.1$$

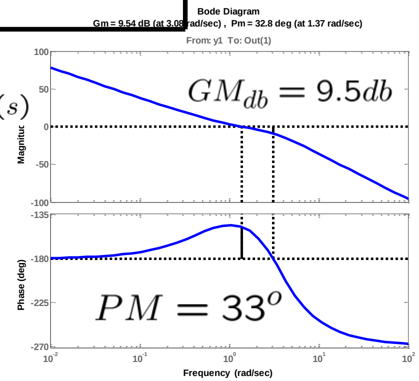
LQG example 1 margins



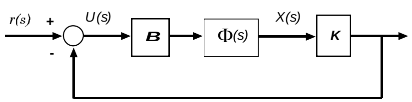
$$R = 0.1$$
$$V = 0.1$$

$$G_o(s) = G(s)\, C_{LQG}(s)$$

Margins could be much worst!



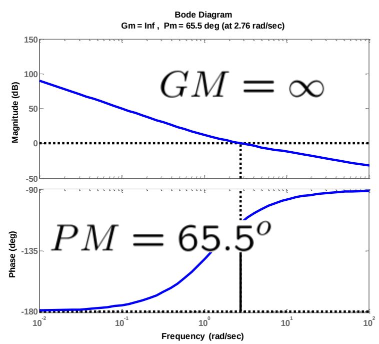
LQR example 1 margins



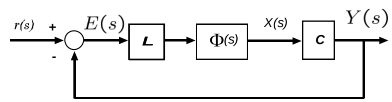
$$R = 0.1$$

$$G_o(s) = K\Phi(s)B$$

$$G_o(s) = \frac{2.5(s + 1.26)}{s^2}$$



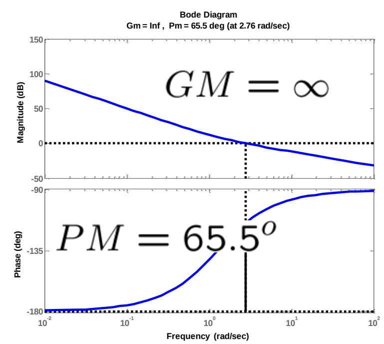
Fictitious KF Feedback Loop example 1



$$W = 1$$
$$V = 0.1$$

$$G_o(s) = L\Phi(s)C$$

$$G_o(s) = \frac{2.5(s + 1.26)}{s^2}$$



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LQG – Loop Transfer Recovery

LQG-LTR was developed by Prof. John Doyle (when he was a M.S. student at MIT).

- `Guaranteed margins for LQG regulators," J. Doyle, IEEE Trans. on Auto. Control (T-AC), August, 1978.
- ``Robustness with observers," J. Doyle and G. Stein, IEEE T-AC, August, 1979.

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John Doyle

Other important contributions in Robust Control

- `State-space solutions to standard H2 and H∞ optimal control problems," J. Doyle, K. Glover, P. Khargonekar, and B. Francis, IEEE T-AC, August, 1989 (Outstanding Paper Award Winner and Baker Prize Winner).
- ``Analysis of feedback systems with structured uncertainty (μ),"
  J. Doyle, IEE Proceedings, V129, Part D, No.6, November, 1982.

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LQG – Loop Transfer Recovery

LQG-LTR is a **robust control design methodology** that uses the LQG control structure

- LQG-LTR is not an optimal control design methodology.
- LQG-LTR is not even a stochastic control design methodology.
- A ***fictitious Kalman Filter*** is used as a robust control design methodology.
  - Output noise intensity and input noise vector (V & B_w) are used as design parameters – not true noise parameters.

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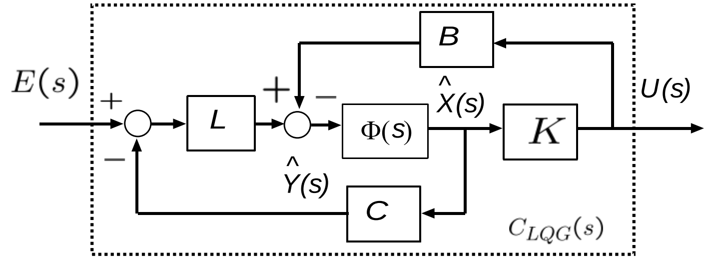
Stationary LQG Compensator

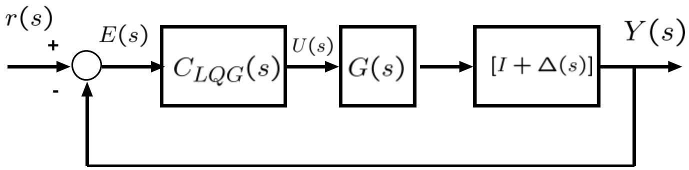


$$U(s) = C_{LQG}(s)\, E(s)$$

$$C_{LQG}(s) = K\ (sI - A + BK + LC)^{-1}\ L$$

LQG-LTR Method 1

- How to make an LQG compensator ***structure*** robust to unmodeled **output** multiplicative uncertainties



- $\Delta(s)$ is a multiplicative uncertainty which is stable and bounded, i.e.

$$\sigma_{\max}\left[\Delta(j\omega)\right] \leq m(j\omega) < \infty$$

LQG-LTR Theorem 1

Let $G_o(s) = G(s)\, C_{LQG}(s)$ where

$$C_{LQG}(s) = K\ (sI - A + BK + LC)^{-1}\ L$$

And let K be the state feedback gain that is obtained as follows

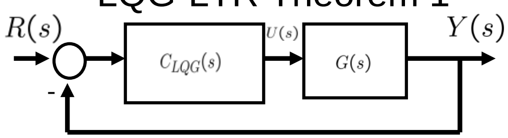
$$K = \frac{1}{\rho} N^{-1} B^T P_\rho \qquad N = N^T \succ 0 \qquad R = \rho N$$

$$A^T P_\rho + P_\rho A + C^T C - \frac{1}{\rho} P_\rho B\, N^{-1} B^T P_\rho = 0$$

$\rho > 0$

make LQR weight: $C_Q = C$

LQG-LTR Theorem 1



Under the assumptions in the previous page

- If $G(s) = C\Phi(s)B$ is square and has no unstable zeros, then point-wise in s

$$\lim_{\rho \to 0} G(s)\, C_{LQG}(s) = C\Phi(s)L$$

LQG-LTR Theorem 1

K is the state feedback solution of the following LQR

$$J = \frac{1}{2}\int_0^\infty \left\{ x^T C\, C\, x + \rho\, u^T N u \right\} dt \qquad N = N^T \succ 0$$

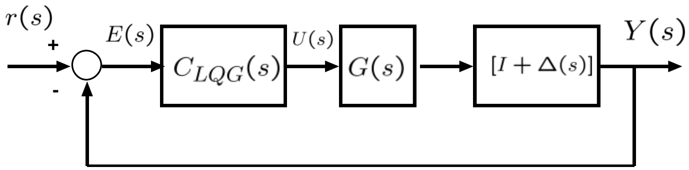
- C is the state output matrix in:

$$y(t) \;=\; C\, x(t) + v(t)$$

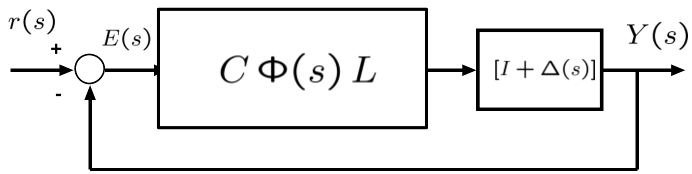
- $\rho > 0$ which is made very small, i.e.

$\rho \longrightarrow 0$ *"cheap" control LQR*

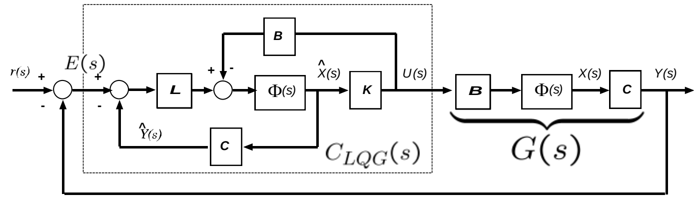
LQG-LTR Method 1



$\rho \longrightarrow 0$ *"cheap" control LQR* : $C_Q = C$



LQG-LTR-Method 1

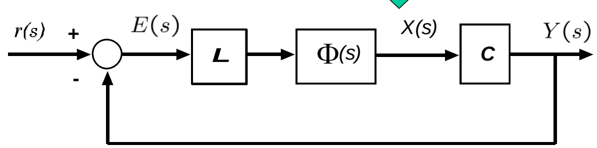


$K = \frac{1}{\rho} N^{-1} B^T P_\rho \qquad N = N^T \succ 0$

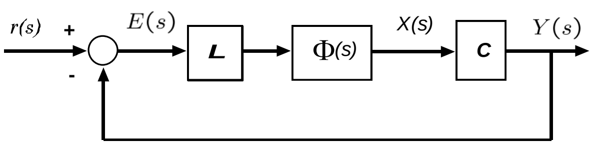
$A^T P_\rho + P_\rho A + C^T C - \frac{1}{\rho} P_\rho B N^{-1} B^T P_\rho = 0$

Make it approximate (point-wise in s)

$\rho \longrightarrow 0$



Fictitious KF is the target system



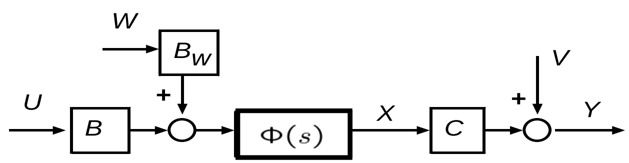
Since the LTR procedure achieves:

$$\lim_{\rho \to 0} G(s)\, C_{LQG}(s) = C\Phi(s)L$$

We need to determine the observer feedback L so that the target system has desirable properties

More on this later

Example -1 Double integrator



$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\boxed{G(s) = C\Phi(s)B = \frac{1}{s^2}}$$

no unstable zeros

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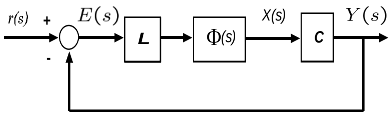
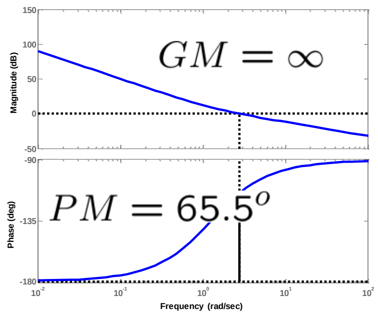
Design fictitious KF Target System



$$\boxed{G_o(s) = C\Phi(s)L}$$

$$G_o(j\omega)$$



Design Parameters:

$$B_w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\left.\begin{aligned} W &= 1 \\ V &= 0.1 \end{aligned}\right\}$$ *ratio adjusts gain crossover frequency*

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LTR procedure for computing K

1) For a small $\rho > 0$ compute:

$$K = \frac{1}{\rho} N^{-1} B^T P_\rho$$

where P_ρ is the solution of

$$A^T P_\rho + P_\rho A + C^T C - \frac{1}{\rho} P_\rho B N^{-1} B^T P_\rho = 0$$

2) Check if $G(s)\, C_{LQG}(s) \approx C\Phi(s)L$

otherwise, decrease ρ and repeat the process.

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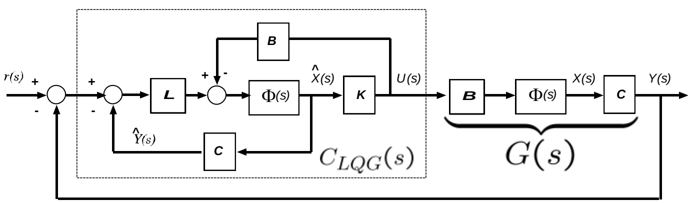
LQG-LTR-Method 1

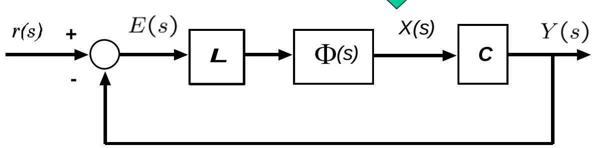


$K = \frac{1}{\rho} N^{-1} B^T P_\rho$ $\qquad N = N^T \succ 0$

$A^T P_\rho + P_\rho A + C^T C - \frac{1}{\rho} P_\rho B N^{-1} B^T P_\rho = 0$

Make it approximate (point-wise in s)

$\rho \rightarrow 0$



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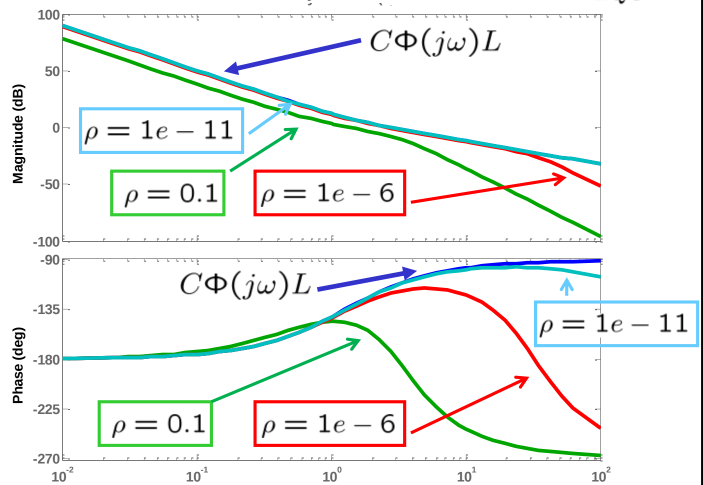
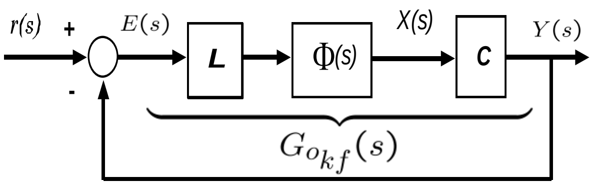
LQG-LTR KF example 1 $G_o(s) = G(s)\, C_{LQG}(s)$



Fictitious KF design parameters

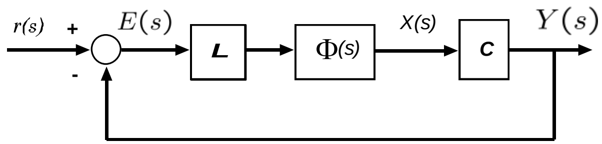
$$\lim_{\rho \to 0} G(s)\, C_{LQG}(s) = C\Phi(s)L$$

$G_{o_{kf}}(s)$



Select ***B***$_w$, ***W***, and ***V*** as design parameters to shape the open loop transfer function $G_{o_{kf}}(s) = C\Phi(s)L$

Fictitious KF Feedback Loop



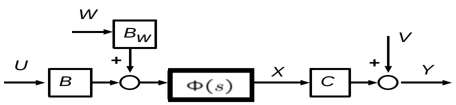
$$G_{o_{kf}}(s) = C\Phi(s)L$$

Sensitivity and Complementary sensitivity Transfer Functions:

$$S(s) = \left[I + G_{o_{kf}}(s)\right]^{-1} \qquad r(s) \to U(s)$$

$$T(s) = G_{o_{kf}}(s)\left[I + G_{o_{kf}}(s)\right]^{-1} \qquad r(s) \to Y(s)$$

Simplify fictitious noise covariance description



$E\{w(t)w(t)^T\} = I\delta(t)$ ⇒ $W = I$

$E\{v(t)v(t)^T\} = \mu^2 I\delta(t)$ ⇒ $V = \mu^2 I$

only the ratio W/V is important

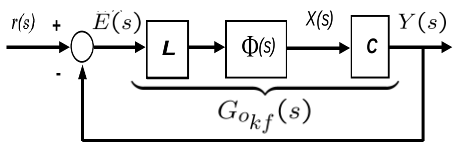
μ: measurement noise standard deviation

KF gain L is calculated by:

$$L = \frac{1}{\mu^2} M\, C^T$$

$$AM + MA^T = -B_w B_w^T + \frac{1}{\mu^2} M C^T C M$$

Simplify fictitious noise covariance description



$$G_{o_{kf}}(s) = C\Phi(s)L$$

$$G_w(s) = C\Phi(s)B_w$$

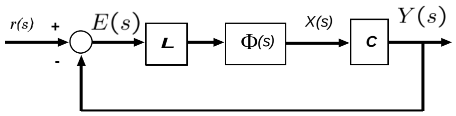
KF gain L is calculated by: $L = \frac{1}{\mu^2} M\, C^T$

$$AM + MA^T = -B_w B_w^T + \frac{1}{\mu^2} M C^T C M$$

Return difference equality:

$$(1 + G_{o_{kf}}(s))(1 + G_{o_{kf}}(-s))^T = I + \frac{1}{\mu^2} G_w(s)\, G_w(-s)^T$$

Fictitious KF Feedback Loop Design



$$G_{o_{kf}}(s) = C\Phi(s)L$$

$$G_w(s) = C\Phi(s)B_w$$

affects zeros of $G_w(s)$

Design parameters:

- Fictitious input noise input vector: B_w
- Fictitious output noise standard deviation: μ

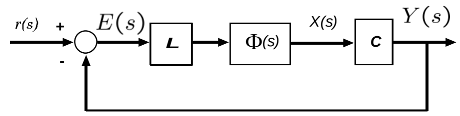
  (affects bandwidth of close loop system)

Design equation: (return difference equation)

$$\sigma_i[1 + G_{o_{kf}}(j\omega)] = \sqrt{1 + \left(\frac{\sigma_i[G_w(j\omega)]}{\mu}\right)^2}$$

i^{th} singular value

Fictitious KF Feedback Loop Design



$$G_{o_{kf}}(s) = C\Phi(s)L$$

$$G_w(s) = C\Phi(s)B_w$$

$$\sigma_i[1 + G_{o_{kf}}(j\omega)] = \sqrt{1 + \left(\frac{\sigma_i[G_w(j\omega)]}{\mu}\right)^2}$$

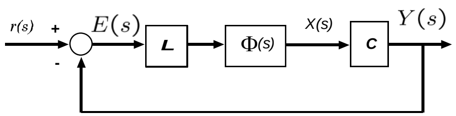
1. Designer-specified shapes: When $\boxed{\frac{\sigma_{\min}[G_w(j\omega)]}{\mu} >> 1}$

   (generally at low frequency)

$$\sigma_i[G_{o_{kf}}(j\omega)] \approx \frac{\sigma_i[G_w(j\omega)]}{\mu} \Rightarrow \begin{cases} \sigma_i[T(j\omega)] \approx 1 \\ \sigma_i[S(j\omega)] \approx \dfrac{1}{\sigma_i[G_{o_{kf}}(j\omega)]} \end{cases}$$

use $\mathbf{B_w}$ to place zeros of $\mathbf{G_w(j\omega)}$

Fictitious KF Feedback Loop Design



$$G_{o_{kf}}(s) = C\Phi(s)L$$

$$G_w(s) = C\Phi(s)B_w$$

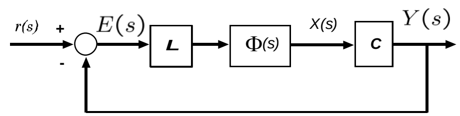
$$\sigma_i[1 + G_{o_{kf}}(j\omega)] = \sqrt{1 + \left(\frac{\sigma_i[G_w(j\omega)]}{\mu}\right)^2}$$

2. High frequency attenuation: As $\boxed{\omega \longrightarrow \infty}$

$$\sigma_i[G_{o_{kf}}(j\omega)] \approx \frac{\sigma_i[CL]}{\omega} \Rightarrow \begin{cases} \sigma_i[T(j\omega)] \approx \sigma_i[G_{o_{kf}}(j\omega)] \\ \sigma_i[S(j\omega)] \approx 1 \end{cases}$$

(gain Bode plot has -20 db/dec slope)

Fictitious KF Feedback Loop Design



$$G_{o_{kf}}(s) = C\Phi(s)L$$

$$G_w(s) = C\Phi(s)B_w$$

$$\sigma_i[1 + G_{o_{kf}}(j\omega)] = \sqrt{1 + \left(\frac{\sigma_i[G_w(j\omega)]}{\mu}\right)^2}$$

3. Well-behaved crossover frequency :

Sensitivitiy and complementary sensitivity TFs never become too large
(even in the vicinity of the gain crossover frequency)

$$\sigma_i[S(j\omega)] \le 1 \qquad \sigma_i[T(j\omega)] \le \underset{\approx 6\text{ db}}{2}$$

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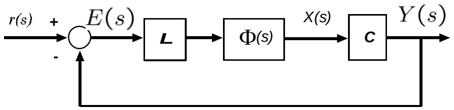
Fictitious KF Target Design



$$W = I$$
$$V = \mu^2 I$$

$$L = \frac{1}{\mu^2} M\, C^T \qquad AM + MA^T = -B_w B_w^T + \frac{1}{\mu^2} M C^T C M$$

Goal: "Shape" the fictitious KF open loop transfer function

$$\boxed{G_{o_{kf}}(s) = C\Phi(s)L}$$

$$\boxed{G_w(s) = C\Phi(s)B_w}$$

- Design parameters:

B_w *places zeros of* $G_w(s)$

μ *adjusts gain crossover frequency of* $G_{o_{kf}}(s)$

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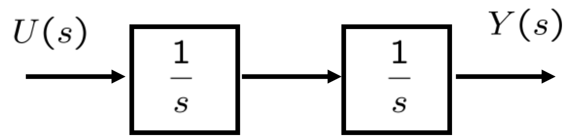
Example 1 – double integrator



$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x$$

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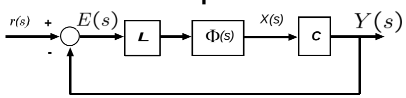
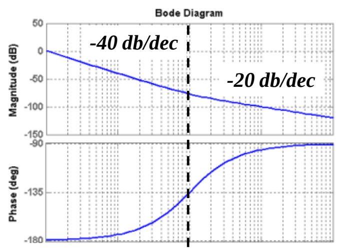
Example 1: selection of B_w



$$G_w(s) = C\Phi(s)B_w \qquad B_w = \begin{bmatrix} B_{w1} \\ B_{w2} \end{bmatrix} = \begin{bmatrix} B_{w1} \\ 1 \end{bmatrix}$$



$1/B_{w1}$

$$G_w(s) = \frac{B_{w1}s + 1}{s^2}$$

sets the location of the zero

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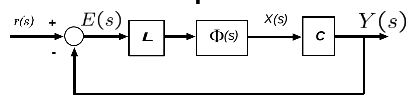
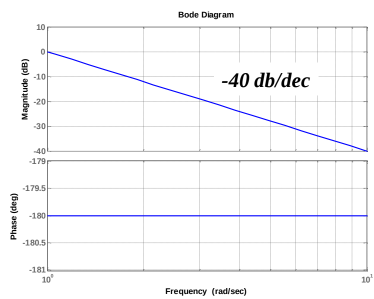
Example 1: selection of B_w



$$G_w(s) = C\Phi(s)B_w \qquad G_w(s) = \frac{B_{w1}s + 1}{s^2}$$



In this example we will set

B_{w1} = *0*

$$G_w(s) = \frac{1}{s^2}$$

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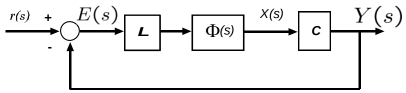
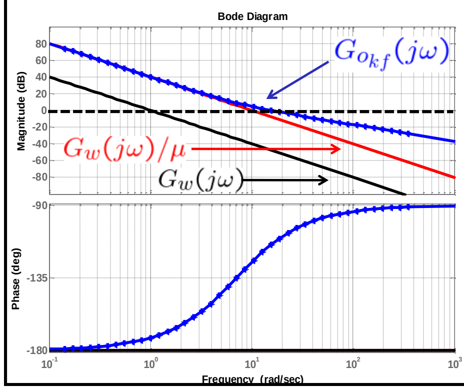
Example 1: selection of μ



$$G_w(s) = \frac{1}{s^2}$$



μ: adjusts gain crossover frequency of

$$G_{o_{kf}}(s) = C\Phi(s)L$$

In this example we will set

$$\mu = 0.01$$

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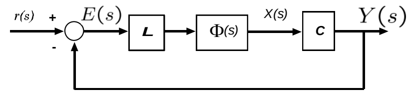
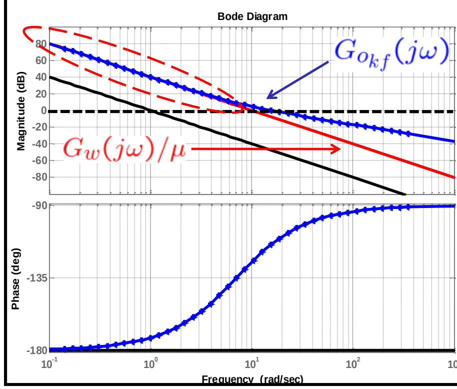
Example 1: selection of μ



$$G_w(s) = \frac{1}{s^2}$$

$$\mu = 0.01$$



1. <u>Designer-specified shapes</u>: (low frequencies)

$$\frac{\sigma_{\min}[G_w(j\omega)]}{\mu} >> 1$$

$$\sigma_i[G_{o_{kf}}(j\omega)] \approx \frac{\sigma_i[G_w(j\omega)]}{\mu}$$

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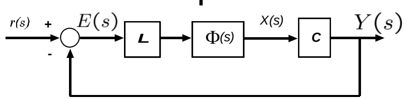
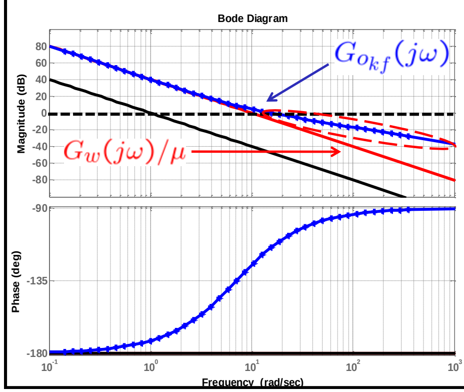
Example 1: selection of μ



$$G_w(s) = \frac{1}{s^2}$$

$$\mu = 0.01$$



2. <u>High frequency attenuation</u>:

$$\omega \to \infty$$

$$\sigma_i[G_{o_{kf}}(j\omega)] \approx \frac{\sigma_i[CL]}{\omega}$$

(gain Bode plot has -20 db/dec slope)

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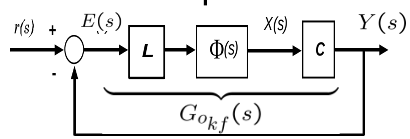
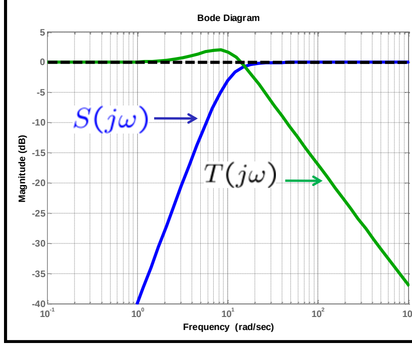
Example 1: selection of μ



$$G_w(s) = \frac{1}{s^2}$$

$$\mu = 0.01$$



3. <u>Well-behaved crossover frequency</u>:

$$\sigma_i[S(j\omega)] \leq 1$$

$$\sigma_i[T(j\omega)] \leq 2$$

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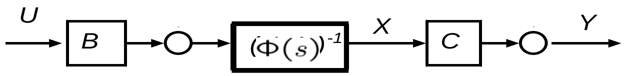
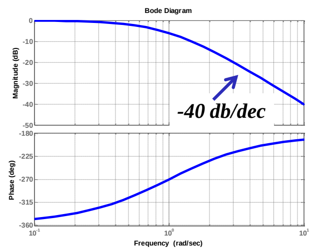
Example-2: Unstable Plant



$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$G(s) = C\Phi(s)B = \frac{1}{(s-1)^2}$$

no unstable zeros



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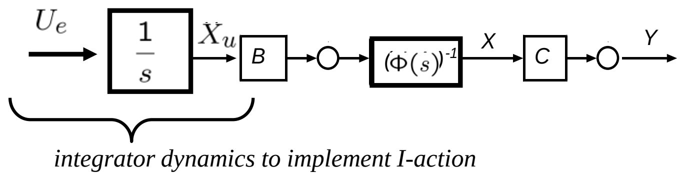
Example-2: I-action

- Introduce I-action to achieve 0 steady-state error to constant reference input
- Define I-action extended system



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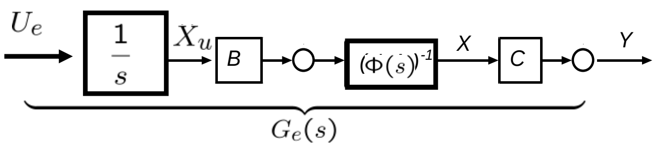
Example-2: I-action

- Define I-action extended system



$$A_e = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \qquad B_e = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C_e = \begin{bmatrix} C & 0 \end{bmatrix}$$

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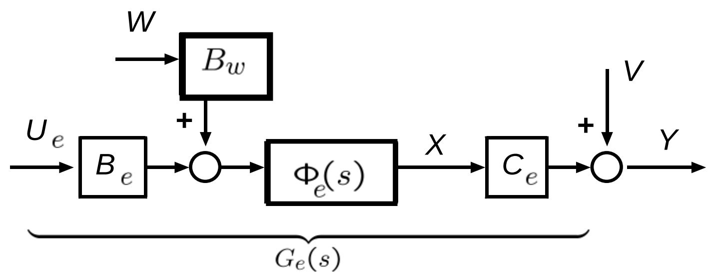
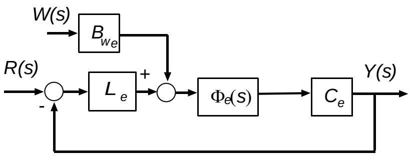
Example-2: I-action

- I-action extended system



$$A_e = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad B_e = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad C_e = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Example-2: selection of B_w



$$G_w(s) = C_e \Phi_e(s) B_w$$

Design parameter:

$$B_w = \begin{bmatrix} B_{w1} \\ B_{w2} \\ B_{w3} \end{bmatrix}$$

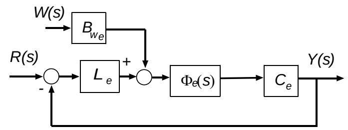
We can "place" two zeros of $G_w(s)$

$$G_w(s) = \frac{B_{w1}s + (B_{w2} - B_{w1})s + B_{w3}}{s(s-1)^2}$$

remember that, at low frequencies,

$$\frac{\sigma_{\min}[G_w(j\omega)]}{\mu} >> 1 \quad \Rightarrow \quad \sigma_i[G_{o_{kf}}(j\omega)] \approx \frac{\sigma_i[G_w(j\omega)]}{\mu}$$

Example-2: selection of B_w



$$G_w(s) = C_e \Phi_e(s) B_w$$

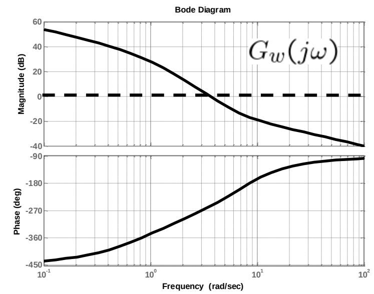
$$G_w(s) = \frac{(s+5)^2 + 5^2}{s(s-1)^2}$$

Example:

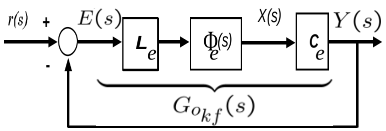
Place two zeros of $G_w(s)$ at

$$z_{1,2} = -5 \pm 5j$$

$$B_w = \begin{bmatrix} 1 \\ 11 \\ 50 \end{bmatrix}$$



Example-2: selection of μ



$$G_w(s) = \frac{(s+5)^2 + 5^2}{s(s-1)^2}$$

$$W = I \qquad V = \mu^2 I$$

Example:

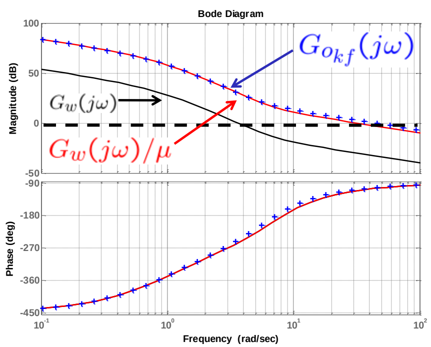
$$\mu^2 = 0.01$$

1. Designer-specified shapes: (low frequencies)

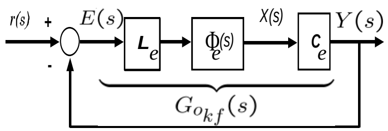
$$|G_{o_{kf}}(j\omega)| \approx \frac{|G_w(j\omega)|}{\mu}$$

for

$$\frac{|G_w(j\omega)|}{\mu} >> 1$$



Example-2: selection of μ



$$G_w(s) = \frac{(s+5)^2 + 5^2}{s(s-1)^2}$$

$$W = I \qquad V = \mu^2 I$$

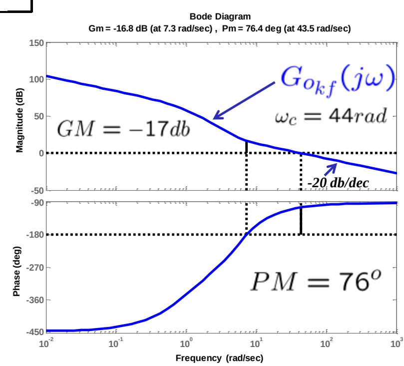
Example: $\mu^2 = 0.01$

$1/\mu \approx 32$

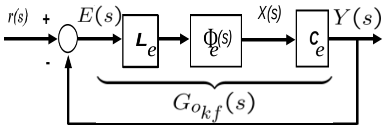
$$G_{o_{kf}}(s) \approx \frac{44[(s+4)^2 + 4.4^2]}{s(s-1)^2}$$

2. High frequency attenuation:

$$\omega \to \infty \quad |G_{o_{kf}}(j\omega)| \approx \frac{CL}{\omega}$$



Example-2: Fictitious KF Design

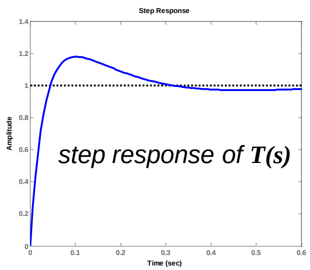


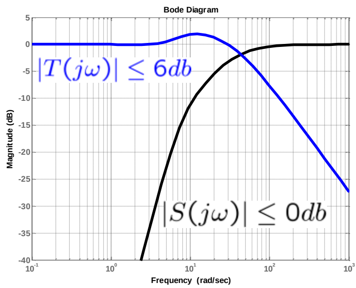
$$G_w(s) = \frac{(s+5)^2+5^2}{s(s-1)^2}$$

$$G_{o_{kf}}(s) \approx \frac{44[(s+4)^2+4.4^2]}{s(s-1)^2}$$

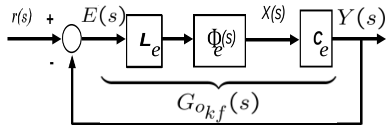
Example: $\mu^2 = 0.01$

3. Well-behaved crossover frequency:





Example-2: selection of B_w



$$G_w(s) = C_e \Phi_e(s) B_w$$

$$W = I \quad V = \mu^2 I$$

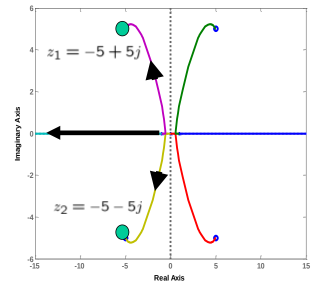
Close loop poles: As $\mu \to 0$

1. *2* close loop poles converge to the zeros of $G_w(s)$

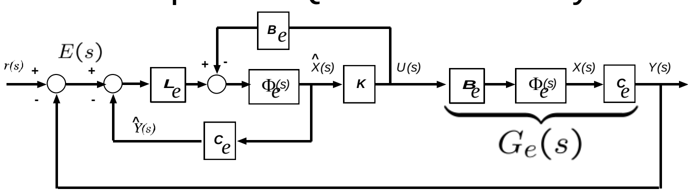
$$G_w(s) = \frac{(s+5)^2+5^2}{s(s-1)^2}$$

2. The reminder pole goes to $-\infty$

Symmetric root locus:



Example-2: LQG-LTR recovery



Use on extended system (including integrator dynamics)

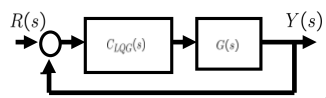
$$K = \frac{1}{\rho} N^{-1} B_e^T P_\rho$$

Keep decreasing ρ *until*

$$G_e(s)\, C_{LQG}(s) \approx C\Phi(s)L$$

$$A_e^T P_\rho + P_\rho A_e + C_e^T C_e - \frac{1}{\rho} P_\rho B_e N^{-1} B_e^T P_\rho = 0$$

Example-2: LQG-LTR



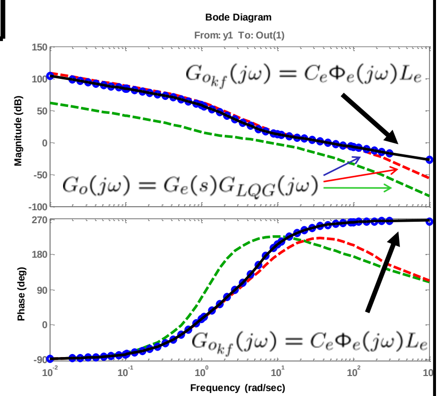
Keep decreasing ρ *until*

$$G_e(s)\, C_{LQG}(s) \approx C_e \Phi_e(s) L_e$$

$\rho = 10^{-5}$

$\rho = 10^{-11}$

$\rho = 10^{-17}$



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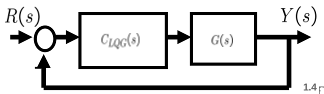
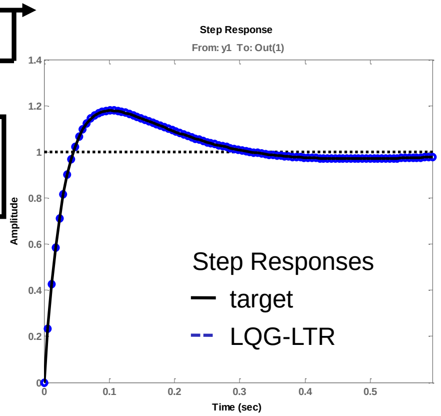
Example-2: LQG-LTR



Keep decreasing ρ until

$G_e(s)\, C_{LQG}(s) \approx C_e \Phi_e(s) L_e$

$\rho = 10^{-17}$



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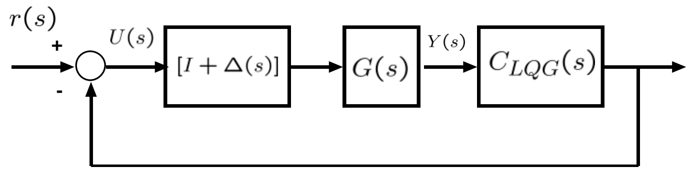
LQG-LTR Method 2

- How to make an LQG compensator ***structure*** robust to unmodeled **input** multiplicative uncertainties



- $\Delta(s)$ is a multiplicative uncertainty which is stable and bounded, i.e.

$$\sigma_{\max}\,[\Delta(j\omega)] \leq m(j\omega) < \infty$$

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LQG-LTR Theorem 2

Let $G_o(s) = C_{LQG}(s)\, G(s)$ where

$$C_{LQG}(s) = K\ (sI - A + BK + LC)^{-1}\ L$$

And let L be the Kalman Filter feedback gain that is obtained as follows

$$L = \frac{1}{\rho} M_\rho C^T N^{-1} \qquad N = N^T \succ 0$$

$$AM_\rho + M_\rho A^T + BB^T - \frac{1}{\rho} M_\rho C^T N^{-1} C M_\rho = 0$$

$$\rho > 0$$

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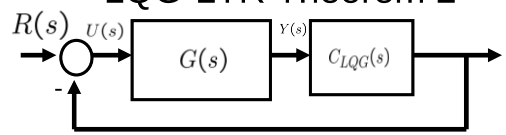
LQG-LTR Theorem 2



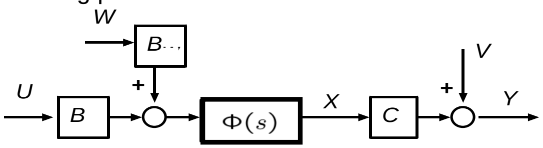
Under the assumptions in the previous page

- If $G(s) = C\Phi(s)B$ is square and has no unstable zeros, then point-wise in s

$$\lim_{\rho \to 0} C_{LQG}(s)\, G(s) = K\Phi(s)B$$

LQG-LTR Theorem 2

L is the Kalman Filter gain solution of the following filtering problem



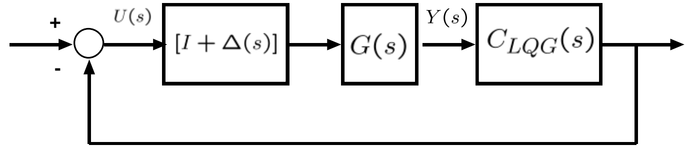
$B_w = B$ $\qquad$ $E\{w(k)w(k)^T\} = W = I$

$E\{v(k)v(k)^T\} = V = \rho N \succ 0$

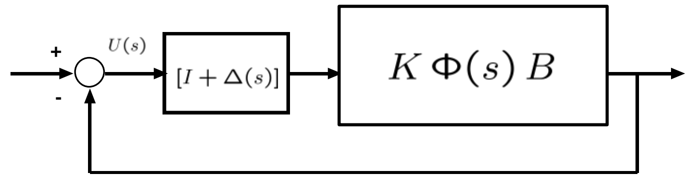
- $\rho > 0$ which is made very small, i.e.

$\rho \longrightarrow 0$ *"noiseless" output measurement*

LQG-LTR Method 2



$\rho \longrightarrow 0$ *"noiseless" output measurement* $\qquad B_w = B$



More on LQG-LTR

- **LTR Theorem Proof:** Read ME233 Class Notes, pages LTR-3 to LTR- 5
  (also back of these notes)
- Fictitious Kalman Filter Design Techniques: Read ME233 Class Notes, pages LTR-6 to LTR- 9
- Stein and Athans "The LQG/LTR Procedure for Multivariable Feedback Control Design," *IEEE TAC*. Vol. AC-32. NO. 2, Feb 1987

Outline

- Continuous time LQR stability margins
- Continuous time Kalman Filter stability margins
- Fictitious Kalman Filter
- LQG stability margins
- LQG-LTR

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LQG-LTR Theorem 1

Assume that:

- $G_o(s) = G(s)\, C_{LQG}(s)$ where
  - $C_{LQG}(s) = K\ (sI - A + BK + LC)^{-1}\ L$
  - The feedback gain K is satisfies

$$K = \frac{1}{\rho} N^{-1} B^T P_\rho \qquad N = N^T \succ 0$$
$$A^T P_\rho + P_\rho A + C^T C - \frac{1}{\rho} P_\rho B N^{-1} B^T P_\rho = 0$$

- If $G(s) = C\Phi(s)B$ is square and has no unstable zeros, then point-wise in s

$$\lim_{\rho \to 0} G(s)\, C_{LQG}(s) = C\Phi(s)L$$

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Notation

- For convenience, we define:

$$\Phi(s) = (sI - A)^{-1}$$

$$\Phi_{LC}(s) = (sI - A + LC)^{-1}$$

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Linear Algebra Result

- We often use results like:

$$K\ [I + \Phi(s)BK]^{-1} = [I + K\Phi(s)B]^{-1}\ K$$

- which can be easily verified by multiplying left and right by the appropriate matrices:

$$[I + K\Phi(s)B]\ K\ =\ K\ [I + \Phi(s)BK]$$

$$K + K\Phi(s)BK\ = K + K\Phi(s)BK$$

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LQG-LTR – Theorem 1 Proof

Proof: The result is obtained in 4 steps:

Step 1: Alternate expression for the LQG compensator $C_{LQG}(s)$

$$C_{LQG}(s) = [I + K\Phi_{LC}(s)B]^{-1}\ K\Phi_{LC}(s)\, L$$

where

$$\Phi_{LC}(s) = (sI - A + LC)^{-1}$$

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LQG-LTR – Theorem 1 Proof

Step 2: Let $K(\rho)$ be given by

$$K(\rho) = \frac{1}{\rho} N^{-1} B^T P_\rho$$

where P_ρ is the solution of

$$A^T P_\rho + P_\rho A + C^T C - \frac{1}{\rho} P_\rho B N^{-1} B^T P_\rho = 0$$

(LTR procedure for computing $K(\rho)$)

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LQG-LTR – Theorem 1 Proof

If $G(s) = C\Phi(s)B$ has no unstable zeros

Then as $\rho \to 0$

$$K(\rho) \to \frac{1}{\sqrt{\rho}} N^{-1/2} T C$$

where T is unitary, i.e.

$$T^T T = I$$

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Lemma: maximally achievable accuracy of LQR

To proof step 2 we use the following lemma from:

Kwakernaak, H. and Sivan, R. "The maximally achievable accuracy of linear optimal regulators and linear optimal filters." *IEEE Transactions on Automatic Control*, vol.AC-17, no.1, Feb. 1972, pp. 79-86. USA.

Let P_ρ be the solution of the following algebraic Riccati equation

$$A^T P_\rho + P_\rho A + C^T C - \frac{1}{\rho} P_\rho B N^{-1} B^T P_\rho = 0$$

where $N = N^T \succ 0$ and $G(s) = C\Phi(s)B$ is square.

Then

$G(s) = C\Phi(s)B$ has no unstable zeros <u>if and only if</u> $\lim_{\rho \to 0} P_\rho = 0$

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Sketch of proof of step 2

Rewriting the Riccati equation

$$A^T P_\rho + P_\rho A + C^T C - \rho K^T(\rho) N K(\rho) = 0$$

and utilizing $P_\rho \to 0$

results in $\rho K^T(\rho) N K(\rho) \to C^T C$

Thus, $K(\rho) \to \frac{1}{\sqrt{\rho}} N^{-1/2} T C \qquad T^T T = I$

LQG-LTR - Proof

Step 3: If $G(s) = C\Phi(s)B$ is square and has no unstable zeros, then as $\rho \to 0$

$$C_{LQG}(s) \to [C\Phi_{LC}(s)B]^{-1}\, C\Phi_{LC}(s)L$$

where

$$\Phi_{LC}(s) = (sI - A + LC)^{-1}$$

Proof of Step 3

$$C_{LQG}(s) = [I + K\Phi_{LC}(s)B]^{-1}\, K\Phi_{LC}(s)\, L$$

substitute:

$$K(\rho) \to \frac{1}{\sqrt{\rho}} N^{-1/2}\, T\, C$$

$$C_{LQG}(s) \to \left[\xrightarrow{0}\sqrt{\rho}T^T N^{1/2} + C\Phi_{LC}(s)B\right]^{-1} C\Phi_{LC}(s)\, L$$

$$C_{LQG}(s) \to [C\Phi_{LC}(s)B]^{-1}\, C\Phi_{LC}(s)\, L$$

LQG-LTR – Theorem 1 Proof

Step 4: If $G(s) = C\Phi(s)B$ is square and has no unstable zeros, then as $\rho \to 0$

$$C_{LQG}(s) \to [C\Phi(s)B]^{-1}\, [C\Phi(s)L]$$

where

$$\Phi(s) = (sI - A)^{-1}$$

LQG-LTR Theorem 2

Let:

- $G_o(s) = C_{LQG}(s)\, G(s)$ where
  - $C_{LQG}(s) = K\,(sI - A + BK + LC)^{-1}\, L$
  - The feedback gain L is satisfies

$$L = \frac{1}{\rho} M_\rho C^T N^{-1} \qquad N = N^T \succ 0$$

$$AM_\rho + M_\rho A^T + BB^T - \frac{1}{\rho} M_\rho C^T N^{-1} C M_\rho = 0$$

- If $G(s) = C\Phi(s)B$ is square and has no unstable zeros, then point-wise in s

$$\lim_{\rho \to 0} C_{LQG}(s)\, G(s) = K\Phi(s)B$$

Proof LQG-LTR Theorem 2

- Start with LQG-LTR Theorem 1
- Apply LQG – KF duality