ME 233 Spring 2010
Solution to Homework #6

1. (a)

$$\begin{aligned}
\tilde{y}(k) &= y(k) - C\hat{x}(k) = y(k) - C\hat{x}^o(k) - CF(k)\tilde{y}^o(k) \\
&= \left[I - CF(k)\right]\tilde{y}^o(k)
\end{aligned}$$

Then, we have:

$$\begin{aligned}
\Lambda_{\tilde{y}\tilde{y}}(k,0) &= \left[I - CF(k)\right]\Lambda_{\tilde{y}^o\tilde{y}^o}(k,0)\left[I - CF(k)\right]^T \\
&= \left[I - CM(k)C^T\left[CM(k)C^T + V(k)\right]^{-1}\right]\left[CM(k)C^T + V(k)\right]\left[I - \left[CM(k)C^T + V(k)\right]^{-1} CM(k)C\right. \\
&= \left[CM(k)C^T + V(k) - CM(k)C^T\right]\left[CM(k)C^T + V(k)\right]^{-1}\left[CM(k)C^T + V(k) - CM(k)C^T\right]^T \\
&= V(k)\left[CM(k)C^T + V(k)\right]^{-1} V(k)
\end{aligned}$$

(b)

$$\hat{x}^o(k+1) = A\hat{x}(k) + Bu(k) = A\hat{x}^o(k)y(k) + AF(k)\tilde{y}^o(k) + Bu(k)$$

Obviously, $L(k) = AF(k)$.

(c)

$$\begin{aligned}
M(k+1) &= AZ(k)A^T + B_w W(k) B_w^T \\
&= A\left(M(k) - M(k)C^T\left[CM(k)C^T + V(k)\right]^{-1} CM(k)\right)A^T + B_w W(k) B_w^T \\
&= AM(k)A^T + B_w W(k) B_w^T - AM(k)C^T\left[CM(k)C^T + V(k)\right]^{-1} CM(k)A^T
\end{aligned}$$

2. (a) To find the estimation error covariances, use the equations

$$\begin{aligned}
M(k) &= AZ(k-1)A^T + BWB^T \\
Z(k) &= M(k) - M(k)C^T\left[CM(k)C^T + V\right]^{-1} CM(k) \\
\Lambda_{\tilde{y}^o\tilde{y}^o}(k,0) &= CM(k)C^T + V \\
\Lambda_{\tilde{y}\tilde{y}}(k,0) &= CZ(k)C^T - V + 2V\left[CM(k)C^T + V\right]^{-1} V
\end{aligned}$$

with initial condition

$$M(0) = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

to iteratively find the estimation error covariances. To quantify when the estimation error covariance matrices are approaching their steady state value, use matrix norms. For example, one could say that they are approaching their steady state values when

$$\max\left\{tr\left(M(k) - M(k-1)\right), tr\left(Z(k) - Z(k-1)\right)\right\} < 10^{-6}$$

Using these equations and this termination condition, the computed steady state estimation error covariances are

$$\begin{aligned} M_{ss} &= \begin{bmatrix} 0.1608 & 0.0764 \\ 0.0764 & 0.1586 \end{bmatrix} \\ Z_{ss} &= \begin{bmatrix} 0.1335 & 0.0198 \\ 0.0198 & 0.0411 \end{bmatrix} \\ (\Lambda_{\tilde{y}^o\tilde{y}^o})_{ss} &= 1.9275 \\ (\Lambda_{\tilde{y}\tilde{y}})_{ss} &= 0.1297 \end{aligned}$$

It is easy to verify that

$$\begin{aligned} (\Lambda_{\tilde{y}^o\tilde{y}^o})_{ss} - (\Lambda_{\tilde{y}\tilde{y}})_{ss} &> 0 \\ M_{ss} - Z_{ss} &> 0 \end{aligned}$$

Thus, our a-posteriori estimates always do "better" than the a-priori estimates in the sense that they have a smaller covariance.

(b) Figure 1 shows the output estimation error covariance as a function of time. As expected, the a-priori output estimate always has a much larger covariance than the a-posteriori output estimate.

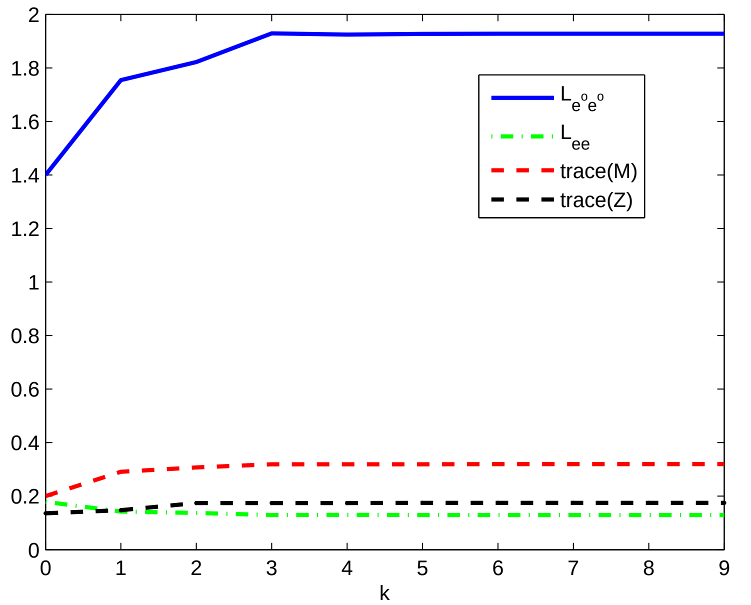

Figure 1: Output estimation error covariance vs. time

(c) There are two easy ways to find `M`, `Z`, `L`, `F`, and the vector of closed loop eigenvalues, `E`. The first method uses the function `kalman`. The code would look something like this:

```
>> sysd = ss(A,[B B],C,[0 0],-1);
>> [KF,L,M,F,Z] = kalman(sysd,W,V);
>> E = eig(A-L*C);
```

(The output `KF` is not needed.) Note that when we created our state space model of the system, we had to specify that it was a discrete time model by setting the sampling time to -1. If we did not do this, `kalman` would have assumed a continuous time system and return erroneous results.

The second method for finding these quantities makes use of the function `dare`. The code would look something like this:

```
>> [M,E,G] = dare(A',C',B*W*B',V);
>> Z = M - M*C' * (C*M*C'+V)^-1 * C*M;
>> L = G';
>> F = M*C' * (C*M*C'+V)^-1;
```

Using either method, the values of `M`, `Z`, `L`, `F`, and `E` are computed to be

$$
\begin{aligned}
M &= \begin{bmatrix} 0.1608 & 0.0764 \\ 0.0764 & 0.1586 \end{bmatrix} \\
Z &= \begin{bmatrix} 0.1335 & 0.0198 \\ 0.0198 & 0.0411 \end{bmatrix} \\
L &= \begin{bmatrix} -0.2564 \\ 0.1079 \end{bmatrix} \\
F &= \begin{bmatrix} 0.1189 \\ 0.2469 \end{bmatrix} \\
E &= \begin{bmatrix} -0.1519 + 0.3955j \\ -0.1519 - 0.3955j \end{bmatrix}
\end{aligned}
$$

Thus, the steady state output error covariances are given by

$$
\begin{aligned}
\Lambda_{\tilde{y}^o\tilde{y}^o} &= CMC^T + V = 1.9275 \\
\Lambda_{\tilde{y}\tilde{y}} &= CZC^T - V + 2V\left[CMC^T + V\right]^{-1} V = 0.1297
\end{aligned}
$$

Note that our values for M, Z, $\Lambda_{\tilde{y}^o\tilde{y}^o}$, and $\Lambda_{\tilde{y}\tilde{y}}$ agree with the values found in part (a).

(d) To simulate the system, first generate the random vectors and sequences. For instance

```
>> x0 = sqrt(0.1) * randn(2,1);
>> w = randn(1,K_STOP);
>> v = sqrt(V) * randn(1,K_STOP);
```

where `K_STOP` is the final value of k in the simulations. Note that you don't multiply the normally distributed vectors by their covariances, but by the square root of their covariances. From here, iteratively simulate the system using

$$
\begin{aligned}
F(k) &= M(k)C^T\left[CM(k)C^T + V\right]^{-1} \\
y(k) &= Cx(k) + v(k) \\
\hat{x}(k) &= \hat{x}^o(k) + F(k)\left[y(k) - C\hat{x}^o(k)\right] \\
x(k) &= Ax(k) + Bu(k) + Bw(k) \\
\hat{x}^o(k+1) &= A\hat{x}(k) + Bu(k) \\
M(k+1) &= AM(k)A^T + BB^T - AM(k)C^T\left[CM(k)C^T + V\right]^{-1} CM(k)A^T
\end{aligned}
$$

Assume we use measurements for `K_START` $\leq k \leq$ `K_STOP`. For good results, keep `K_START` large to minimize the effects of the transient response of the system and keep `K_STOP - K_START` large so that you will have a lot of data from which to compute the covariances.

(e) Our previously calculated covariance values are given by

$$
\begin{aligned}
M &= \begin{bmatrix} 0.1608 & 0.0764 \\ 0.0764 & 0.1586 \end{bmatrix} \\
Z &= \begin{bmatrix} 0.1335 & 0.0198 \\ 0.0198 & 0.0411 \end{bmatrix} \\
\Lambda_{\tilde{y}^o\tilde{y}^o} &= 1.9275 \\
\Lambda_{\tilde{y}\tilde{y}} &= 0.1297
\end{aligned}
$$

whereas the covariances computed by using `cov` on the simulation data are

$$
\begin{aligned}
\overline{M} &= \begin{bmatrix} 0.1588 & 0.0738 \\ 0.0738 & 0.1563 \end{bmatrix} \\
\overline{Z} &= \begin{bmatrix} 0.1336 & 0.0200 \\ 0.0200 & 0.0416 \end{bmatrix} \\
\overline{\Lambda}_{\tilde{y}^o\tilde{y}^o} &= 1.9308 \\
\overline{\Lambda}_{\tilde{y}\tilde{y}} &= 0.1299
\end{aligned}
$$

The covariances computed from the simulations closely match the actual values. To get better results, you would have to throw away more of the initial measurements and increase the simulation time.

(f) Repeating the process used in parts (a)-(f) for $V = 0.05$ and $V = 5$ gives the results shown in Table 1, Figure 2 and Figure 3. From Figure 2 and Figure 3, we see the steady state values are reached with different speed using different values of V. Notice that in all cases, the steady state Riccati equation solutions closely match the algebraic Riccati equation solution. Also, in all cases the values of the output estimation error covariances closely match those obtained using `cov` on the simulation data.

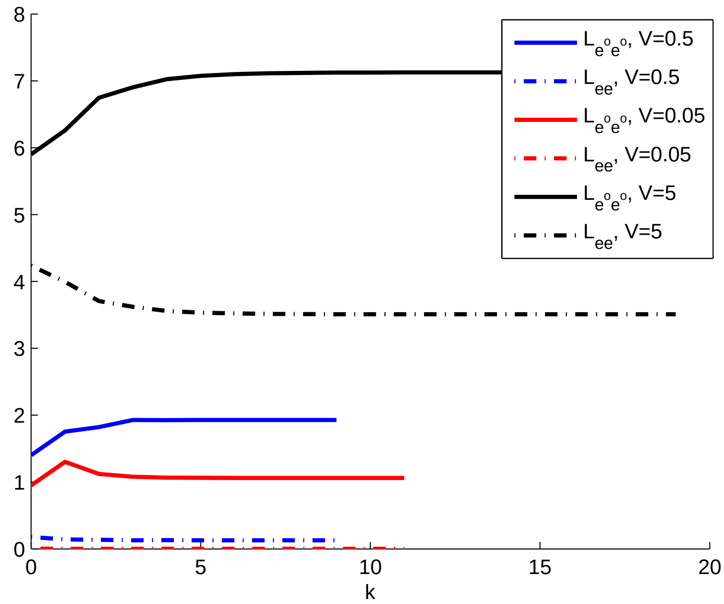

Figure 2: Output estimation error covariances vs. time with three values of V

Now, let's examine the effect of V on the relevant covariances. Wrom Figure 2 and Figure 3, it is easy to see that

$$
\begin{array}{rcrcr}
trace(M)\big|_{V=5} & > & trace(M)\big|_{V=0.5} & > & trace(M)\big|_{V=0.05} \\
trace(Z)\big|_{V=5} & > & trace(Z)\big|_{V=0.5} & > & trace(Z)\big|_{V=0.05} \\
\Lambda_{\tilde{y}^o\tilde{y}^o}\big|_{V=5} & > & \Lambda_{\tilde{y}^o\tilde{y}^o}\big|_{V=0.5} & > & \Lambda_{\tilde{y}^o\tilde{y}^o}\big|_{V=0.05} \\
\Lambda_{\tilde{y}\tilde{y}}\big|_{V=5} & > & \Lambda_{\tilde{y}\tilde{y}}\big|_{V=0.5} & > & \Lambda_{\tilde{y}\tilde{y}}\big|_{V=0.05}
\end{array}
$$

Thus, as we decrease V, we decrease the output covariance and all associated output estimation error covariances. Also note that as we increase V by a factor of 10, $\Lambda_{\tilde{y}^o\tilde{y}^o}$ increases by less than a factor of 10 whereas $\Lambda_{\tilde{y}\tilde{y}}$ increases by more than a factor of 10. This means that our a-posteriori estimation error covariances are much more sensitive to changes in V than the a-priori estimation error covariances. Also, we can see that it is important to have as little measurement noise as possible.

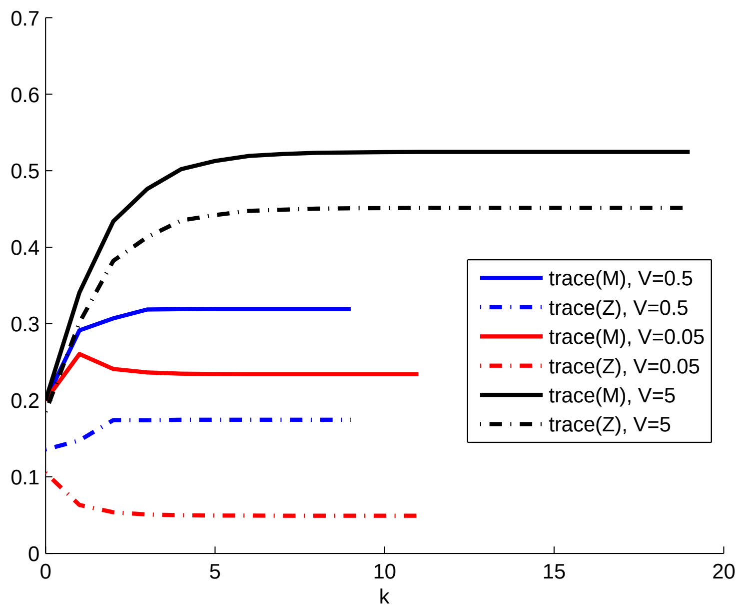

Figure 3: Trace vs. time with three values of V

Now let's examine the effect of V on the Kalman filter gains. It is trivial to see that

$$\begin{array}{ccccc} \|L\|_2\big|_{V=5} & < & \|L\|_2\big|_{V=0.5} & < & \|L\|_2\big|_{V=0.05} \\ \|F\|_2\big|_{V=5} & < & \|F\|_2\big|_{V=0.5} & < & \|F\|_2\big|_{V=0.05} \end{array}$$

Thus, as the measurement noise is increased, the Kalman filter gains have to be reduced so that noise is not amplified by the filter.

Now let's examine the effect of V on the closed loop eigenvalues. Recall that the transition matrix in discrete time is given by

$$A^k = \left[TDT^{-1}\right]^k = TD^kT^{-1}$$

Thus, if a system has eigenvalues that are smaller in magnitude, the system will respond more quickly. To quantify the speed of response of a system, we will use the eigenvalue of A with the largest magnitude. This is the equivalent to taking the infinity norm of our vector E. Thus we quantify the speed of our closed loop responses by

$$\begin{array}{rcl} \|E\|_\infty\big|_{V=5} & = & 0.6968 \\ \|E\|_\infty\big|_{V=0.5} & = & 0.4237 \\ \|E\|_\infty\big|_{V=0.05} & = & 0.5893 \end{array}$$

Interestingly enough, the speed of the response does not monotonically increase as V is decreased. Thus, even though we get small steady state estimation error covariances for $V = 0.05$, it takes longer to reach steady state than if $V = 0.5$.

3. In this problem, we will derive the Kalman filter for a system in which $w(k)$ and $v(k)$ are correlated. As the first step, we will transform the coordinates of $w(k)$ so that its transformed coordinates are uncorrelated with $v(k)$. For simplicity, we will choose our transformation to be

$$w'(k) := w(k) + Tv(k)$$

With this choice of transformation, we get

$$\begin{array}{rcl} \Lambda_{W'V}(0) & = & E\left\{\left[w(k) + Tv(k)\right]v^T(k)\right\} \\ & = & S + TV \end{array}$$

	$V=0.5$	$V=0.05$	$V=5$
M_{ss}	$\begin{bmatrix}0.1608 & 0.0764\\ 0.0764 & 0.1586\end{bmatrix}$	$\begin{bmatrix}0.1219 & 0.0958\\ 0.0958 & 0.1122\end{bmatrix}$	$\begin{bmatrix}0.2884 & 0.0464\\ 0.0464 & 0.2362\end{bmatrix}$
M	$\begin{bmatrix}0.1608 & 0.0764\\ 0.0764 & 0.1586\end{bmatrix}$	$\begin{bmatrix}0.1219 & 0.0958\\ 0.0958 & 0.1122\end{bmatrix}$	$\begin{bmatrix}0.2884 & 0.0464\\ 0.0464 & 0.2362\end{bmatrix}$
Z_{ss}	$\begin{bmatrix}0.1335 & 0.0198\\ 0.0198 & 0.0411\end{bmatrix}$	$\begin{bmatrix}0.0440 & 0.0045\\ 0.0045 & 0.0053\end{bmatrix}$	$\begin{bmatrix}0.2856 & 0.0325\\ 0.0325 & 0.1657\end{bmatrix}$
Z	$\begin{bmatrix}0.1335 & 0.0198\\ 0.0198 & 0.0411\end{bmatrix}$	$\begin{bmatrix}0.0440 & 0.0045\\ 0.0045 & 0.0053\end{bmatrix}$	$\begin{bmatrix}0.2856 & 0.0325\\ 0.0325 & 0.1657\end{bmatrix}$
$(\Lambda_{\tilde{y}^o\tilde{y}^o})_{ss}$	1.9275	1.0601	7.1256
$\overline{\Lambda}_{\tilde{y}^o\tilde{y}^o}$	1.9359	1.0571	7.0801
$\Lambda_{\tilde{y}^o\tilde{y}^o}$	1.9275	1.06	7.1256
$(\Lambda_{\tilde{y}\tilde{y}})_{ss}$	0.1297	0.0024	3.5085
$\overline{\Lambda}_{\tilde{y}\tilde{y}}$	0.1303	0.0024	3.4861
$\Lambda_{\tilde{y}\tilde{y}}$	0.1297	0.0024	3.5085
$\overline{M}$	$\begin{bmatrix}0.1588 & 0.0738\\ 0.0738 & 0.1563\end{bmatrix}$	$\begin{bmatrix}0.1221 & 0.0957\\ 0.0957 & 0.1115\end{bmatrix}$	$\begin{bmatrix}0.2784 & 0.0442\\ 0.0442 & 0.2282\end{bmatrix}$
$\overline{Z}$	$\begin{bmatrix}0.1336 & 0.0200\\ 0.0200 & 0.0416\end{bmatrix}$	$\begin{bmatrix}0.0439 & 0.0044\\ 0.0044 & 0.0052\end{bmatrix}$	$\begin{bmatrix}0.2762 & 0.0320\\ 0.0320 & 0.1601\end{bmatrix}$
L	$\begin{bmatrix}-0.2564\\ 0.1079\end{bmatrix}$	$\begin{bmatrix}-0.3393\\ 0.2216\end{bmatrix}$	$\begin{bmatrix}-0.1010\\ 0.0236\end{bmatrix}$
F	$\begin{bmatrix}0.1189\\ 0.2469\end{bmatrix}$	$\begin{bmatrix}0.2711\\ 0.3176\end{bmatrix}$	$\begin{bmatrix}0.0195\\ 0.0994\end{bmatrix}$
E	$\begin{bmatrix}-0.1519+0.3955j\\ -0.1519-0.3955j\end{bmatrix}$	$\begin{bmatrix}-0.0554\\ -0.5893\end{bmatrix}$	$\begin{bmatrix}-0.0254+0.6964i\\ -0.0254-0.6964i\end{bmatrix}$

Table 1: Summary of all relevant parameters for Kalman filters for three values of V

To make $w'(k)$ uncorrelated with $v(k)$, we will choose $T = -SV^{-1}$. The covariance of $w'(k)$ is given by

$$\begin{aligned}
\Lambda_{W'W'}(0) &= E\left\{[w(k)+Tv(k)]\,[w(k)+Tv(k)]^T\right\} \\
&= W + ST^T + TS^T + TVT^T \\
&= W - SV^{-1}S^T
\end{aligned}$$

Noting that

$$\begin{aligned}
w(k) &= w'(k) + SV^{-1}v(k) \\
&= w'(k) + SV^{-1}\big[y(k) - Cx(k)\big]
\end{aligned}$$

our governing equations become

$$\begin{aligned}
x(k+1) &= \big[A - SV^{-1}C\big]x(k) + \begin{bmatrix}B & SV^{-1}\end{bmatrix}\begin{bmatrix}u(k)\\ y(k)\end{bmatrix} + w'(k) \\
y(k) &= Cx(k) + v(k)
\end{aligned}$$

where $w'(k)$ and $v(k)$ are uncorrelated. Note that $\begin{bmatrix}u(k) & y(k)\end{bmatrix}^T$ is a deterministic quantity. Using the Kalman filter results derived in class, we get

$$\begin{aligned}
\hat{x}(k) &= \hat{x}^o(k) + M(k)C^T\left[CM(k)C^T + V\right]^{-1}\tilde{y}^o(k) \\
\hat{x}^o(k+1) &= \big[A - SV^{-1}C\big]\hat{x}(k) + \begin{bmatrix}B & SV^{-1}\end{bmatrix}\begin{bmatrix}u(k)\\ y(k)\end{bmatrix} \\
Z(k) &= M(k) - M(k)C^T\left[CM(k)C^T + V\right]^{-1}CM(k) \\
M(k+1) &= \big[A - SV^{-1}C\big]Z(k)\big[A - SV^{-1}C\big]^T + \big[W - SV^{-1}S^T\big]
\end{aligned}$$

To make the algebra easier, we will define

$$\begin{aligned}
L(k) &:= \left[A - SV^{-1}C\right]\left[M(k)C^T\right]\left[CM(k)C^T + V(k)\right]^{-1} + SV^{-1} \\
&= \left[AM(k)C^T - SV^{-1}CM(k)C^T\right]\left[CM(k)C^T + V(k)\right]^{-1} \\
&\quad + \left[SV^{-1}CM(k)C^T + SV^{-1}V\right]\left[CM(k)C^T + V(k)\right]^{-1} \\
&= \left[AM(k)C^T + S\right]\left[CM(k)C^T + V(k)\right]^{-1}
\end{aligned}$$

Simplifying the state estimation equations gives

$$\begin{aligned}
\hat{x}^o(k+1) &= \left[A - SV^{-1}C\right]\Big(\hat{x}^o(k) + M(k)C^T\left[CM(k)C^T + V(k)\right]^{-1}\tilde{y}^o(k)\Big) \\
&\quad + Bu(k) + SV^{-1}y(k) \\
&= \left[A - SV^{-1}C\right]\hat{x}^o(k) + \left[L(k) - SV^{-1}\right]\tilde{y}^o(k) + Bu(k) + SV^{-1}y(k) \\
&= A\hat{x}^o(k) + \left[L(k) - SV^{-1}\right]\tilde{y}^o(k) + Bu(k) + SV^{-1}\tilde{y}^o(k) \\
&= A\hat{x}^o(k) + Bu(k) + L(k)\tilde{y}^o(k) \\
&= A\hat{x}^o(k) + Bu(k) + L(k)\left[y(k) - C\hat{x}^o(k)\right]
\end{aligned}$$

Simplifying the state estimation covariance equations gives

$$\begin{aligned}
M(k+1) &= \left[A - SV^{-1}C\right]M(k)\left[A - SV^{-1}C\right]^T + \left[W - SV^{-1}S^T\right] \\
&\quad - \left[A - SV^{-1}C\right]M(k)C^T\left[CM(k)C^T + V\right]^{-1}CM(k)\left[A - SV^{-1}C\right] \\
&= \left[A - SV^{-1}C\right]M(k)\left[A - SV^{-1}C\right]^T + \left[W - SV^{-1}S^T\right] \\
&\quad - \left[L(k) - SV^{-1}\right]\Big(\left[L(k) - SV^{-1}\right]\left[CM(k)C^T + V\right]\Big)^T \\
&= AM(k)A^T - SV^{-1}CM(k)A^T - \left[A - SV^{-1}C\right]M(k)C^TV^{-1}S^T \\
&\quad + W - SV^{-1}S^T - L(k)\left[CM(k)C^T + V\right]L^T(k) \\
&\quad + SV^{-1}\left[CM(k)C^T + V\right]L^T(k) + \left[L(k) - SV^{-1}\right]\left[CM(k)C^T + V\right]V^{-1}S^T
\end{aligned}$$

Note that the third term in the last expression cancels with the last term in that expressions. Thus,

$$\begin{aligned}
M(k+1) &= AM(k)A^T - SV^{-1}CM(k)A^T + W - SV^{-1}S^T \\
&\quad - L(k)\left[CM(k)C^T + V\right]L^T(k) + SV^{-1}\left[CM(k)C^T + V\right]L^T(k) \\
&= AM(k)A^T + W - L(k)\left[CM(k)C^T + V\right]L^T(k) \\
&\quad - SV^{-1}\left[AM(k)C^T + S\right]^T + SV^{-1}\left[CM(k)C^T + V\right]L^T(k)
\end{aligned}$$

Note that the last two terms of the last expression cancel. Thus, in summary

$$\begin{aligned}
\hat{x}^o(k+1) &= A\hat{x}^o(k) + Bu(k) + L(k)\left[y(k) - C\hat{x}^o(k)\right] \\
L(k) &= \left[AM(k)C^T + S\right]\left[CM(k)C^T + V\right]^{-1} \\
M(k+1) &= AM(k)A^T + W - L(k)\left[CM(k)C^T + V\right]L^T(k)
\end{aligned}$$

4. We will solve this problem using two methods:

   (a) **Solution using Kalman filtering:** In order to use Kalman filtering results, we write the system in the standard state-space realization:

$$\begin{aligned}
x(k+1) &= Ax(k) + B_w w(k) \\
y(k) &= Cx(k) + v(k)
\end{aligned}$$

In this case, the parameters for the state space realization are: $A = 1, B_w = 0, C = 1, x(0) = x$. Then,

$$
\begin{aligned}
M(k) &= AM(k-1)A^T + B_w W B_w^T - AM(k-1)C^T \left[CM(k-1)C^T + V\right]^{-1} CM(k-1)A^T \\
&= M(k-1) - M(k-1)\left[M(k-1)+V\right]^{-1} M(k-1) \\
&= \frac{M(k-1)}{M(k-1)+V}
\end{aligned}
$$

Taking inverses we obtain:

$$
\begin{aligned}
M^{-1}(k) &= M^{-1}(k-1) + V^{-1} \\
&= M^{-1}(0) + kV^{-1} \text{ with } M(0) = X_0 \\
\Rightarrow M(k) &= \frac{VX_0}{kX_0 + V} \\
\Rightarrow F(k) &= \frac{M(k)}{M(k)+V} \\
&= \frac{X_0}{(k+1)X_0 + V}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\hat{x}(k) &= \hat{x}^o(k) + F(k)\left[y(k) - C\hat{x}^o(k)\right] \\
&= (1 - F(k))\,\hat{x}^o(k) + F(k)y(k) \\
&= (1 - F(k))\,\hat{x}(k-1) + F(k)y(k) \text{ with } \hat{x}^o(k) = A\hat{x}(k-1) \\
&= \frac{kX_0 + V}{(k+1)X_0 + V}\hat{x}(k-1) + \frac{X_0}{(k+1)X_0+V}y(k) \\
\Rightarrow ((k+1)X_0 + V)\,\hat{x}(k) &= (kX_0 + V)\hat{x}(k-1) + X_0 y(k)
\end{aligned}
$$

Let $t(k) = ((k+1)X_0 + V)\,\hat{x}(k)$ and notice that $t(k-1) = (kX_0 + V)\,\hat{x}(k-1)$. Then, we obtain

$$
t(k) = t(k-1) + X_0 y(k)
$$

The initial condition of $t(0)$ is calculated as follows

$$
\begin{aligned}
t(0) &= (X_0 + V)\hat{x}(0) = (X_0 + V)\left[\hat{x}^o(0) + F(0)\left[y(0) - C\hat{x}^o(0)\right]\right] \\
&= (X_0 + V)F(0)y(0) \text{ with } \hat{x}^o(0) = E\left[x(0)\right] = 0 \\
&= X_0 y(0)
\end{aligned}
$$

Thus,

$$
t(k) = X_0 \sum_{i=0}^{k} y(i)
$$

and

$$
\begin{aligned}
\hat{x}(k) &= \frac{1}{(k+1)X_0 + V} t(k) \\
&= \frac{X_0}{(k+1)X_0 + V} \sum_{i=0}^{k} y(i)
\end{aligned}
$$

Notice that, as $X_0 \to \infty$,

$$\hat{x}(k) = \frac{1}{(k+1)} \sum_{i=0}^{k} y(i)$$

The covariance of the a-posteriori state estimation error is given by

$$\begin{aligned} \Lambda_{\tilde{X}\tilde{X}}(k,0) &= Z(k) = M(k+1) \\ &= \frac{VX_0}{(k+1)X_0 + V} \end{aligned}$$

where we have used $M(k+1) = AZ(k)A^T + B_w W B_w^T = Z(k)$.

As $X_0 \to \infty$,

$$\Lambda_{\tilde{X}\tilde{X}}(k,0) = \frac{V}{(k+1)}$$

(b) **Solution using least squares estimation:**

First, we define

$$z := \begin{bmatrix} y(0) & \cdots & y(k) \end{bmatrix}^T = \begin{bmatrix} x + v(0) & \cdots & x + v(k) \end{bmatrix}^T$$

With this notation in mind, we are interested in finding $\hat{x}_{|z}$. Recall that

$$\begin{aligned} \hat{x}_{|z} &= E\{x\} + \Lambda_{xz}\Lambda_{zz}^{-1}\left(z - E\{z\}\right) \\ &= \Lambda_{xz}\Lambda_{zz}^{-1} z. \end{aligned}$$

Note that we used that x and z are zero mean. In order to find this quantity, we need to find expressions for Λ_{xz} and Λ_{zz}^{-1}. First, we will start by finding Λ_{xz}. Note that

$$\begin{aligned} E\left\{xy(j)\right\} &= E\left\{x^2\right\} + E\left\{xv(j)\right\} \\ &= X_0. \end{aligned}$$

Thus, if we define

$$w = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T \in \mathbb{R}^{k+1}$$

we can express

$$\Lambda_{xz} = X_0 w^T$$

Now we turn our attention to finding Λ_{zz}^{-1}. Note that

$$\begin{aligned} E\left\{y(k+j)y(k)\right\} &= E\left\{\left(x + v(k+j)\right)\left(x + v(k)\right)\right\} \\ &= E\left\{x^2\right\} + E\left\{xv(k)\right\} + E\left\{xv(k+j)\right\} + E\left\{v(k+j)v(k)\right\} \\ &= X_0 + V\delta(j). \end{aligned}$$

Thus, we can express

$$\begin{aligned} \Lambda_{zz} &= VI + X_0 ww^T \\ &= V\left(I + \frac{X_0}{V} ww^T\right). \end{aligned}$$

In order to invert this matrix, we must use the matrix inversion lemma, which states that

$$(I + AB)^{-1} = I - A\left(I + BA\right)^{-1} B.$$

Using this, we can say that

$$\begin{aligned}
\Lambda_{zz}^{-1} &= \frac{1}{V}\left(I + \frac{X_0}{V} w w^T\right)^{-1} \\
&= \frac{1}{V}\left[I - \frac{X_0}{V} w \left(1 + \frac{X_0}{V} w^T w\right)^{-1} w^T\right] \\
&= \frac{1}{V}\left[I - \frac{X_0}{V} \cdot \frac{V}{V + (k+1)X_0} w w^T\right] \\
&= \frac{1}{V}\left[I - \frac{X_0}{V + (k+1)X_0} w w^T\right].
\end{aligned}$$

Thus the estimate of x is given by

$$\begin{aligned}
\hat{x}(k) = \hat{x}_{|z} &= \frac{X_0}{V} w^T \left[I - \frac{X_0}{V + (k+1)X_0} w w^T\right] z \\
&= \frac{X_0}{V}\left[1 - \frac{X_0}{V + (k+1)X_0} w^T w\right] w^T z \\
&= \frac{X_0}{V + (k+1)X_0} w^T z \\
&= \frac{X_0}{V + (k+1)X_0} \sum_{i=0}^{k} y(i).
\end{aligned}$$

The covariance of the estimate is given by

$$\begin{aligned}
\Lambda_{\tilde{x}\tilde{x}}(k,0) &= \Lambda_{xx} - \Lambda_{xz}\Lambda_{zz}^{-1}\Lambda_{zx} \\
&= X_0 - \left(\frac{X_0}{V + (k+1)X_0} w^T\right)\left(X_0 w\right) \\
&= \frac{X_0 V}{V + (k+1)X_0}.
\end{aligned}$$

(c) Using the results of the previous part, it is trivial to see that

$$\begin{aligned}
\lim_{X_0 \to \infty} \hat{x}(k) &= \frac{1}{k+1} \sum_{i=0}^{k} y(k) \\
\lim_{X_0 \to \infty} \Lambda_{\tilde{x}\tilde{x}}(k,0) &= \frac{V}{k+1}.
\end{aligned}$$