ME 233 Advance Control II

Continuous time results 2

Kalman filters

(ME233 Class Notes pp.KF7-KF10)

Outline

- Continuous time Kalman Filter
- LQ-KF duality
- KF return difference equality
  - symmetric root locus
- ARMAX models

Stochastic state model

Consider the following nth order LTI system with stochastic input and measurement noise:

$$\begin{aligned} \frac{d}{dt}x(t) &= A\,x(t) + B\,u(t) + B_w\,w(t) \\ y(t) &= C\,x(t) + v(t) \end{aligned}$$

Where:

- $u(t)$ **deterministic (known) input**
- $w(t)$ Gaussian, white noise, zero mean, input noise
- $v(t)$ Gaussian, white noise, zero mean, meas. noise
- $x(0)$ Gaussian

Assumptions

- Initial conditions:

$$E\{x(0)\} = x_o \qquad E\{\tilde{x}^o(0)\tilde{x}^{oT}(0)\} = X_o$$

- Noise properties (in addition to Gaussian),:

$$E\{w(t+\tau)w^T(t)\} = W(t)\,\delta(\tau)$$

$$E\{v(t+\tau)v^T(t)\} = V(t)\,\delta(\tau)$$

$$E\{w(t+\tau)v^T(t)\} = 0$$

$$E\{\tilde{x}^o(0)w^T(t)\} = 0 \qquad E\{\tilde{x}^o(0)v^T(t)\} = 0$$

Conditional estimation

- Conditional state estimate

$$Y_t \;=\; \{y(\tau)\} \qquad \tau \in [0, t]$$

$$\hat{x}(t) = E\{x(t) | Y_t\}$$

- Conditional state estimation error covariance

$$M(t) = E\{\tilde{x}(t)\tilde{x}^T(t)\}$$

$$\tilde{x}(t) = x(t) - \hat{x}(t)$$

CT Kalman Filter

Kalman filter:

$$\begin{aligned} \frac{d}{dt}\hat{x}(t) &= A\,\hat{x}(t) + B\,u(t) + L(t)\,\tilde{y}(t) \\ \tilde{y}(t) &= y(t) - C\,\hat{x}(t) \qquad \hat{x}^o(0) = x_o \end{aligned}$$

Where:

$$L(t) = M\,C^T\,V^{-1}$$

$$\begin{aligned} \frac{d}{dt}M(t) &= AM + MA^T + B_w W B_w^T - \; MC^T V^{-1} CM \\ M(0) &= X_o \end{aligned}$$

Steady State KF

Theorem:

1) If the pair (C, A) is observable (or detectable):

The solution of the Riccati differential equation

$$\frac{d}{dt}M(t) \;=\; AM + MA^T + B_w W B_w^T - \; MC^T V^{-1} CM$$

$$M(0) = 0$$

Converges to a stationary solution, which satisfies the Algebraic Riccati Equation (ARE):

$$AM + MA^T = -B_w W B_w^T + MC^T V^{-1} CM$$

Steady State KF

Theorem:

2) If in addition to 1) the pair (A,B'_w) is controllable (stabilizable), where

$$B'_w\, B'^T_w \;=\; B_w W B_w^T$$

The solution of the Algebraic Riccati Equation (ARE):

$$AM + MA^T = -B_w W B_w^T + MC^T V^{-1} CM$$

is unique, positive definite (semi-definite) , and the close loop observer matrix

$$A_c = A - LC$$

is **Hurwitz**.

$$L = M\,C^T\,V^{-1}$$

Steady State Kalman Filter

Theorem:

3) Under stationary noise and the conditions in 1) and 2),

The observer residual

$$\tilde{y}(t) \;=\; y(t) - C\,\hat{x}(t)$$

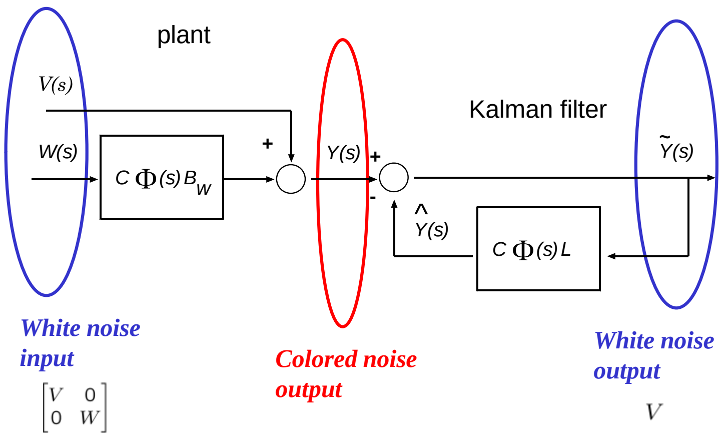
of the KF:

$$\frac{d}{dt}\hat{x}(t) \;=\; A\,\hat{x}(t) + B\,u(t) + L\,\tilde{y}(t)$$

becomes white

$$\boxed{E\left\{\tilde{y}(t+\tau)\tilde{y}^T(t)\right\} = V\,\delta(\tau)}$$

KF as a innovations (whitening) filter



LQR duality

Cost:

$$J = x^T(t_f)\,Q_f\,x(t_f) + \int_0^{t_f}\left\{x^T\,C_Q^T C_Q\,x + u^T\,R\,u\right\}\,dt$$

$$u(t) \;=\; -K(t)\,x(t)$$

Where:

$$K(t) \;=\; R^{-1}\,B^T\,P$$

$$-\frac{d}{dt}P(t) \;=\; A^T\,P + P\,A + C_Q^T C_Q - P\,B\,R^{-1}\,B^T P$$

$$P(t_f) \;=\; S$$

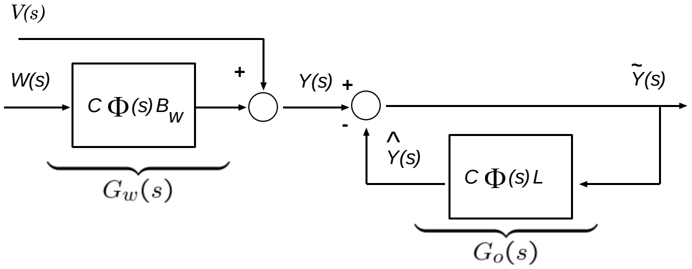
Kalman Filter & LQR Duality

Comparing ARE's and feedback gains, we obtain the following duality

$$\xrightarrow{\textit{duality}}$$

LQR	**KF**
P	M
A	A^T
B	C^T
R	V
$C_Q{}^T$	$B'_w = B_w W^{1/2}$
K	L^T
$(A-BK)$	$(A-LC)^T$

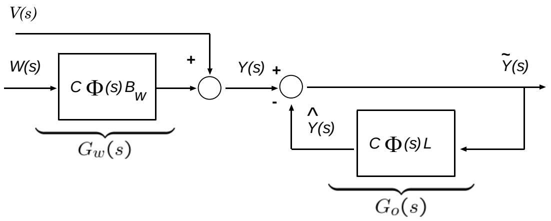
KF return difference equality



$$[I+G_o(s)]\ \underbrace{V}_{\Lambda_{\tilde y\tilde y}(s)}\ [I+G_o(-s)]^T = V + G_w(s)\,W G_w^T(-s)$$

$$\Lambda_{\tilde y\tilde y}(\tau) = E\left\{\tilde y(t+\tau)\tilde y^T(t)\right\}$$

KF root locus for SISO Systems



$$[1+G_o(s)] = [1+C\Phi(s)L]$$

$$= \frac{\det(sI-A+LC)}{\det(sI-A)}$$

$$= \frac{C(s)}{A(s)}$$

⟸ ***roots are Kalman filter poles***

⟸ ***roots are plant poles***

KF symmetric root locus for SISO Systems

$$\frac{C(-s)C(s)}{A(-s)A(s)} = \left[1+\rho\frac{B_w(-s)B_w(s)}{A(-s)A(s)}\right]$$

$$\frac{C(s)}{A(s)} = \frac{\det(sI-A+LC)}{\det(sI-A)}$$

⟸ ***roots are Kalman filter poles***

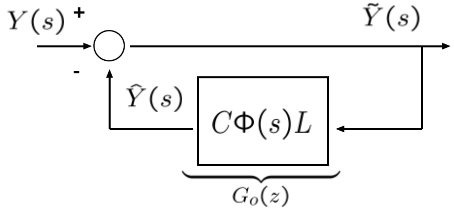
⟸ ***roots are plant poles***

$$\frac{B_w(s)}{A(s)} = G_w(s) = C\Phi(s)B_w$$

$$\rho = \frac{W}{V} \geq 0$$

KF Loop gain and phase margins (SISO)

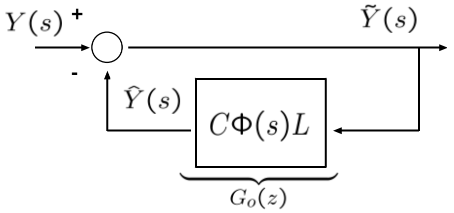
Consider the closed loop sensitive transfer function



$$S(s) = \frac{\tilde Y^o(s)}{Y(s)} = \frac{1}{1+G_o(s)}$$

$$|1+G_o(j\omega)|^2 = 1 + \frac{W}{V}|G_w(j\omega)|^2$$

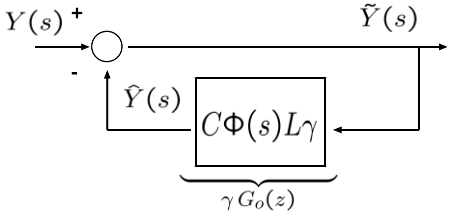
KF Loop phase margins (SISO)



Since, $|(1+G_o(e^{j\omega}))| \geq 1$

The phase margin of $G_o(e^{j\omega})$ is greater than or equal to 60 degrees.

KF Loop gain margins (SISO)



Estimator was designed for $\gamma = 1$

Estimator is guaranteed to remain asymptotically stable for

$$\frac{1}{2} < \gamma < \infty$$

SISO ARMAX stochastic models

SISO ARMAX model:

$$A(s)\,Y(s) = B(s)\,U(s) + C(s)\,\tilde{Y}(s)$$

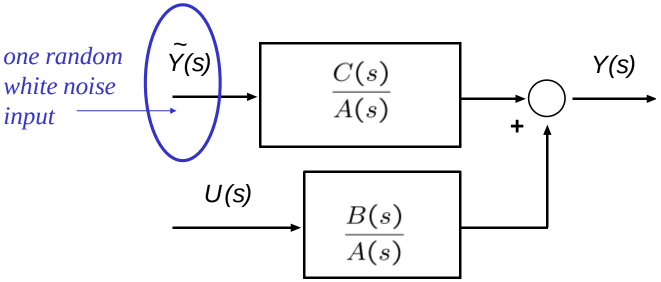
$$C(s) \;=\; \det\{(sI - A + LC)\} = 0 \qquad \textbf{(Hurwitz)}$$

$$A(s) \;=\; \det\{(sI - A)\} = 0$$

$\tilde{y}(t)$ Kalman filter innovations (residual)

SISO ARMAX stochastic models

$$Y(s) = \frac{B(s)}{A(s)}\,U(s) + \frac{C(s)}{A(s)}\,\tilde{Y}(s)$$



Outline

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Additional material:

- Derivation of continuous time Kalman Filter

Derivation of the CT Kalman Filter

1. Approximate the CT state estimation problem by a DT state estimation problem .
2. Obtain the DT Kalman filter for the DT state estimation problem.
3. Obtain the CT Kalman filter from the DT Kalman filter by taking the limit as the sampling time approaches to zero.

CT Kalman Filter

Consider the following nth order LTI system with stochastic input and measurement noise:

$$\begin{aligned} \frac{d}{dt}x(t) &= A\,x(t) + B\,u(t) + B_w\,w(t) \\ y(t) &= C\,x(t) + v(t) \end{aligned}$$

Where:

- $u(t)$ **deterministic input**
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Derivation of the CT Kalman Filter

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Derivation of the CT Kalman Filter

1. Approximate the CT state estimation problem by a DT state estimation problem :

- **State and output equations:**

$$x(k+1) \;\approx\; \underbrace{[I + \Delta t\, A]}_{A_d} x(k) + \underbrace{B\,\Delta t}_{B_d}\, u(k) + \underbrace{B_w\,\Delta t}_{B_{dw}}\, w(k)$$

$$y(k) \;\approx\; Cx(k) + v(k)$$

$$w(k) \approx \frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t} w(t)dt \qquad v(k) \approx \frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t} v(t)dt$$

Derivation of the CT Kalman Filter

- **Covariances** (from pages 48-52 in random process lecture 8):

$$\Lambda_{ww}(k,l) = W_d(k)\,\delta(l)$$

$$W_d(k) \;=\; \frac{1}{\Delta t}\bar{W}(k)$$

$$\bar{W}(k) \;=\; \frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t} W(t)dt$$

Notice that:

$$\lim_{\Delta t \to 0} = \bar{W}(k) = W(t)$$

Derivation of the CT Kalman Filter

- **Covariances:**

$$\Lambda_{vv}(k,l) = V_d(k)\,\delta(l)$$

$$V_d(k) \;=\; \frac{1}{\Delta t}\bar{V}(k)$$

$$\bar{V}(k) \;=\; \frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t} V(t)dt$$

Notice that:

$$\lim_{\Delta t \to 0} = \bar{V}(k) = V(t)$$

Derivation of the CT Kalman Filter

2. Obtain the DT Kalman filter for the DT state estimation problem.

$$\hat{x}^o(k+1) \;=\; A_d\,\hat{x}^o(k) + B_d\,u(k) + L_d(k)\,\tilde{y}^o(k)$$

$$\tilde{y}^o(k) \;=\; y(k) - C\,\hat{x}^o(k)$$

$$L_d(k) \;=\; A_d\,M(k)C^T\left[C\,M(k)C^T + V_d(k)\right]^{-1}$$

$$M(k+1) \;=\; A_d M(k) A_d^T + B_{dw} W_d(k) B_{dw}^T - AM(k)C^T\left[CM(k)C^T + V_d(k)\right]^{-1} CM(k)A_d^T$$

Derivation of the CT Kalman Filter

3) Obtain the CT Kalman filter from the DT Kalman filter.

- **State Equation:**

$$\hat{x}^o(k+1) = A_d\,\hat{x}^o(k) + B_d\,u(k) + L_d(k)\,\tilde{y}^o(k)$$

$$\hat{x}^o(k+1) = \underbrace{[I+\Delta t\,A]}_{A_d}\hat{x}^o(k) + \underbrace{B\,\Delta t}_{B_d}\,u(k) + L_d(k)\,\tilde{y}^o(k)$$

$$\begin{aligned}\frac{\hat{x}^o(k+1)-\hat{x}^o(k)}{\Delta t} &= A\,\hat{x}^o(k) + B\,u(k)\\ &+ \frac{1}{\Delta t}L_d(k)\,\tilde{y}^o(k)\end{aligned}$$

Derivation of the CT Kalman Filter

Taking limit as $\Delta t \to 0$

$$\begin{aligned}\frac{d}{dt}\hat{x}(t) &= A\,\hat{x}(t) + B\,u(t)\\ &+ \lim_{\Delta t\to 0}\{\frac{1}{\Delta t}L_d(k)\}\,\tilde{y}(t)\end{aligned}$$

$$\tilde{y}(t) = y(t) - C\,\hat{x}(t)$$

$$\lim_{\Delta t\to 0}\{\frac{1}{\Delta t}L_d(k)\} = L(t) = M(t)C^TV(t)^{-1}$$

Derivation of the CT Kalman Filter

Kalman filter gain

$$L_d(k) = A_d\,M(k)C^T\left[C\,M(k)C^T + V_d(k)\right]^{-1}$$

$$L_d(k) = (1+\Delta tA)\,M(k)C^T\left[C\,M(k)C^T + \frac{1}{\Delta t}\bar{V}(k)\right]^{-1}$$

$$L_d(k) = \Delta t(1+\Delta tA)\,M(k)C^T\left[\Delta tC\,M(k)C^T + \bar{V}(k)\right]^{-1}$$

$$\begin{aligned}\lim_{\Delta t\to 0}\{\frac{1}{\Delta t}L_d(k)\} &= L(t)\\ &= M(t)C^TV(t)^{-1}\end{aligned}$$

Derivation of the CT Kalman Filter

Riccati equation

$$\begin{aligned}M(k+1) &= A_dM(k)A_d^T + B_{dw}W_d(k)B_{dw}^T\\ &- AM(k)C^T\left[CM(k)C^T + V_d(k)\right]^{-1}CM(k)A_d^T\end{aligned}$$

Subtracting $M(k)$ from both sides and dividing by Δt

$$\begin{aligned}\frac{M(k+1)-M(k)}{\Delta t} &= AM(k) + M(k)A^T + \Delta tAM(k)A^T\\ &+ B_w\bar{W}(k)B_w^T - M(k)C^T\left[\Delta tCM(k)C^T + \bar{V}(k)\right]^{-1}CM(k)\\ &-\Delta t\,AM(k)C^T\left[\Delta tCM(k)C^T + \bar{V}(k)\right]^{-1}CM(k)A_d^T\end{aligned}$$

Derivation of the CT Kalman Filter

Taking $\Delta t \rightarrow 0$

$$\frac{M(k+1)-M(k)}{\Delta t} = AM(k) + M(k)A^T + \Delta t AM(k)A^T$$
$$+ B_w \bar{W}(k) B_w^T - M(k)C^T \left[\Delta t C M(k) C^T + \bar{V}(k)\right]^{-1} CM(k)$$
$$-\Delta t\, AM(k)C^T \left[\Delta t C M(k) C^T + \bar{V}(k)\right]^{-1} CM(k)A_d^T$$

we obtain

$$\begin{aligned}\frac{d}{dt}M(t) &= AM(t) + M(t)A^T + B_w W(t) B_w^T \\ &- M(t)C^T V^{-1}(t) CM(t)\end{aligned}$$