ME 233 Advance Control II

Lecture 9
Review of some topics for infinite-horizon control and estimation

(Not in the ME233 Class Notes)

Outline

- Controllability
- Observability
- Stabilizability
- Detectability
- Transmission Zeros

Controllability

Let $A \in \mathcal{R}^{n \times n}, \ B \in \mathcal{R}^{n \times n_u}$

Theorem: The following are equivalent:

1. *(A,B)* is controllable
2. $\operatorname{rank}[B \ AB \ \cdots \ A^{n-1}B] = n$
3. There does not exist T such that

$$T^{-1}AT = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} \qquad T^{-1}B = \begin{bmatrix} \bar{B} \\ 0 \end{bmatrix}$$

4. $\operatorname{rank}[A - \lambda I \ \ B] = n, \quad \forall \lambda \in \mathcal{C}$
5. The eigenvalues of $A+BK$ can be arbitrarily assigned via choice of K

Observability

Let $A \in \mathcal{R}^{n \times n}, \ C \in \mathcal{R}^{n_y \times n}$

Theorem: The following are equivalent:

1. *(C,A)* is observable
2. $$\operatorname{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$
3. There does not exist T such that

$$T^{-1}AT = \begin{bmatrix} \bar{A}_{11} & 0 \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}$$

$$CT = \begin{bmatrix} \bar{C} & 0 \end{bmatrix}$$

Observability

Let $A \in \mathcal{R}^{n\times n},\ C \in \mathcal{R}^{n_y\times n}$

Theorem (cont'd):

4. $\operatorname{rank}\begin{bmatrix} A-\lambda I \\ C \end{bmatrix} = n, \quad \forall\lambda \in \mathcal{C}$

5. The eigenvalues of $A+LC$ can be arbitrarily assigned via choice of L

Stabilizability (discrete-time)

Let $A \in \mathcal{R}^{n\times n},\ B \in \mathcal{R}^{n\times n_u}$

Theorem: The following are equivalent:

1. (A,B) is stabilizable
2. There does not exist T such that

   $$T^{-1}AT = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} \qquad T^{-1}B = \begin{bmatrix} \bar{B} \\ 0 \end{bmatrix}$$

   where $\bar{A}_{22}$ is not Schur
3. $\operatorname{rank}[A-\lambda I\ \ B] = n$ whenever $|\lambda| \geq 1$
4. There exists K such that $A+BK$ is Schur

Detectability (discrete-time)

Let $A \in \mathcal{R}^{n\times n},\ C \in \mathcal{R}^{n_y\times n}$

Theorem: The following are equivalent:

1. (C,A) is detectable
2. There does not exist T such that

   $$T^{-1}AT = \begin{bmatrix} \bar{A}_{11} & 0 \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \qquad CT = \begin{bmatrix} \bar{C} & 0 \end{bmatrix}$$

   where $\bar{A}_{22}$ is not Schur
3. $\operatorname{rank}\begin{bmatrix} A-\lambda I \\ C \end{bmatrix} = n$ whenever $|\lambda| \geq 1$
4. There exists L such that $A+LC$ is Schur

Normalrank of a MIMO transfer function

- Let $Q(z)$ be a matrix transfer function
- Define

$$\operatorname{normalrank}(Q(z)) := \max_{z\in\mathcal{C}} \Big(\operatorname{rank}(Q(z))\Big)$$

- Example:

$$\operatorname{rank}\begin{bmatrix} z & 1 \\ z^2 & 1 \\ z & 1 \end{bmatrix}\Bigg|_{z=2} = \operatorname{rank}\begin{bmatrix} 2 & 1 \\ 4 & 1 \\ 2 & 1 \end{bmatrix} = 2 \implies \operatorname{normalrank}\begin{bmatrix} z & 1 \\ z^2 & 1 \\ z & 1 \end{bmatrix} = 2$$

even though $\operatorname{rank}\begin{bmatrix} z & 1 \\ z^2 & 1 \\ z & 1 \end{bmatrix}\Bigg|_{z=1} = \operatorname{rank}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} = 1$

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Transmission zeros

- Let $G(z)$ be a transfer function with the state-space realization

$$G(z) = C(zI - A)^{-1}B + D$$

- $z_0 \in \mathcal{C}$ is called a transmission zero of this realization if

$$\text{rank}\begin{bmatrix} A - z_0 I & B \\ C & D \end{bmatrix} < \text{normalrank}\begin{bmatrix} A - zI & B \\ C & D \end{bmatrix}$$

- MATLAB command: `zero(sys)`

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SISO transmission zeros

- Suppose $G(z)=C(zI-A)^{-1}B+D$ is a SISO transfer function that is not identically zero
- Let $G(z) = \frac{b(z)}{a(z)}$ where $a(z)$ and $b(z)$ are polynomials
- Assume without loss of generality that $a(z) = det(zI - A)$

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SISO transmission zeros

- Note that $\det\begin{bmatrix} A - zI & B \\ C & D \end{bmatrix}$ is a polynomial in z

  $\Rightarrow$ continuous function of z

- Choose z_0 such that $\det(A - z_0 I) \neq 0$

$$\begin{bmatrix} A - z_0 I & B \\ C & D \end{bmatrix}\begin{bmatrix} I & -(A - z_0 I)^{-1}B \\ 0 & I \end{bmatrix} = \begin{bmatrix} A - z_0 I & 0 \\ C & G(z_0) \end{bmatrix}$$

$$\Rightarrow \det\begin{bmatrix} A - z_0 I & B \\ C & D \end{bmatrix}\det\begin{bmatrix} I & -(A - z_0 I)^{-1}B \\ 0 & I \end{bmatrix} = \det\begin{bmatrix} A - z_0 I & 0 \\ C & G(z_0) \end{bmatrix}$$

$$\Rightarrow \det\begin{bmatrix} A - z_0 I & B \\ C & D \end{bmatrix} = \det(A - z_0 I)G(z_0) = (-1)^n a(z_0)\frac{b(z_0)}{a(z_0)} = (-1)^n b(z_0)$$

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SISO transmission zeros

- Whenever $\det(z_0 I - A) \neq 0$,

  Numerator of $G(z)$

$$\det\begin{bmatrix} A - z_0 I & B \\ C & D \end{bmatrix} = (-1)^n b(z_0)$$

- By continuity of the left-hand side

$$\det\begin{bmatrix} A - zI & B \\ C & D \end{bmatrix} = (-1)^n b(z) \qquad \forall z \in \mathcal{C}$$

- Since $b(z)$ is not identically 0, $\text{rank}\begin{bmatrix} A - zI & B \\ C & D \end{bmatrix}$ drops if and only if $b(z)=0$

$\boxed{\text{$z$ is a transmission zero if and only if it is a zero of the transfer function $G(z)$}}$

Transmission zeros

Theorem:

Let $D^T D$ be invertible and define

$$\hat{A} := A - B(D^T D)^{-1} D^T C$$
$$\hat{C} := C - D(D^T D)^{-1} D^T C$$

Then λ is a transmission zero of the state-space realization $G(z) = C(zI - A)^{-1}B + D$ if and only if

$$\operatorname{rank} \begin{bmatrix} \hat{A} - \lambda I \\ \hat{C} \end{bmatrix} < n$$

Unobservable mode of $(\hat{C}, \hat{A})$ at $z = \lambda$