UNIVERSITY OF CALIFORNIA, BERKELEY
Department of Mechanical Engineering
ME233 Advanced Control Systems II, Spring 2010

Homework #6

Assigned:	Tu., March 9
Due:	Th., March 18

The first ME233 Midterm will be on Thursday, March 11th. The exam will be closed book and notes, but you are allowed to bring two 8.5×11 pages of handwritten notes. You won't need a calculator.The material covered will include up to and including Homework 5 and the least squares estimation lecture in 3/2.

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1) Consider the Kalman filter for the system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + B_w w(k) \qquad E\{x(0)\} = x_o \quad E\{(x(0)-x_o)(x(0)-x_o)^T\} = X_o \\ y(k) &= Cx(k) + v(k) \end{aligned}$$

under the standard assumptions:

- $u(k)$ is the known control input
- $w(k)$ and $v(k)$ are white, zero mean and Gaussian noises with respective covariances $E\{w(k)w(k+l)^T\} = W(k)\delta(l)$ and $E\{v(k)v(k+l)^T\} = V(k)\delta(l)$, that are uncorrelated with each other and with the initial state.

The Kalman filter for this system was derived in class to be

$$\tilde{y}^o(k) = y(k) - C\,\hat{x}^o(k) \tag{1}$$

$$\hat{x}(k) = \hat{x}^o(k) + F(k)\,\tilde{y}^o(k) \qquad F(k) = M(k)C^T\left[C\,M(k)C^T + V(k)\right]^{-1} \tag{2}$$

$$Z(k) = M(k) - M(k)C^T\left[CM(k)C^T + V(k)\right]^{-1}CM(k) \tag{3}$$

$$\hat{x}^o(k+1) = A\,\hat{x}(k) + B\,u(k) \tag{4}$$

$$M(k+1) = A\,Z(k)A^T + B_w W(k) B_w^T \tag{5}$$

with initial conditions:$\hat{x}^o(0) = x_o$, and $M(0) = X_o$.

1. The a-priori output estimation error $\tilde{y}^o(k) = y(k) - C\,\hat{x}^o(k)$ is known as the *residual* or *innovation sequence*. It was shown in class that the covariance of the innovation sequence is given by

$$\Lambda_{\tilde{y}^o\tilde{y}^o}(k,0) = E\{\tilde{y}^o(k)(\tilde{y}^o(k))^T\} = CM(k)C^T + V(k).$$

In fact, it can be shown that

$$E\{\tilde{y}^o(k)(\tilde{y}^o(k-j))^T\} = 0 \quad \text{for} \quad j = 1, \cdots, k\,,$$

which explains why it is called the *innovation sequence*.

The a-posteriori output estimation error, is given by $\tilde{y}(k) = y(k) - C\,\hat{x}(k)$. This sequence is however not used in the filter. Prove that

$$\Lambda_{\tilde{y}\tilde{y}}(k,0) = E\{\tilde{y}(k)(\tilde{y}(k))^T\} = V(k)[CM(k)C + V(k)]^{-1}V(k).$$

2. Show that (2) and (4) can be combined into the following a-priori state estimate update law

$$\hat{x}^o(k+1) \;=\; A\,\hat{x}^o(k) + B\,u(k) + L(k)\,\tilde{y}^o(k)$$

with initial condition $\hat{x}^o(0) = x_o$ and obtain and expression for $L(k)$.

3. Show that Eqs. (3) and (5) can be combined into the following single Riccati equation

$$M(k+1) \;=\; AM(k)A^T + B_w W(k) B_w^T - AM(k)C^T\left[CM(k)C^T + V(k)\right]^{-1} CM(k)A^T \quad (6)$$

with initial condition $M(0) = X_o$.

2) Consider again the stochastic system in Problem No. 3), Homework No 5.[1]

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.08 & -1 \\ 0.7 & 0.1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.34 \\ 0.3 \end{bmatrix} (u(k) + w(k))$$

$$y(k) \;=\; \begin{bmatrix} 0 & 3 \end{bmatrix} x(k) + v(k)$$

where $u(k)$ is a deterministic input, to be defined subsequently and

$$x(0) \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}\right), \qquad \begin{bmatrix} w(k) \\ v(k) \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & V \end{bmatrix}\right)$$

$$E\{x(0) \begin{bmatrix} w(k) & v(k) \end{bmatrix}\} = 0$$

In this problem we will compute the time varying gains and run the steady-state Kalman filter for this system.

(a) Starting from the initial conditions and setting the noise intensity $V = 0.5$, recursively compute the following time varying matrices and scalars, utilizing Eqs. (6) and (3), until their respective steady state values are reached:

- The traces of the a-priori and a-posteriori state estimation error covariances, respectively trace$\{M(k)\}$ and trace$\{Z(k)\}$ [2].
- The a-priori and a-posteriorly output estimation error covariances, respectively [3]

$$\Lambda_{\tilde{y}^o\tilde{y}^o}(k,0) \;=\; E\{|\tilde{y}^o(k)|^2\} = CM(k)C^T + V, \quad \tilde{y}^o(k) = y(k) - \hat{y}^o(k)$$

$$\Lambda_{\tilde{y}\tilde{y}}(k,0) \;=\; E\{|\tilde{y}(k)|^2\} = V[CM(k)C + V]^{-1}V, \quad \tilde{y}(k) = y(k) - \hat{y}(k)$$

(b) Plot the response of the sequences $\Lambda_{\tilde{y}^o\tilde{y}^o}(k,0)$ and $\Lambda_{\tilde{y}\tilde{y}}(k,0)$ and the sequences trace$\{M(k)\}$ and trace$\{Z(k)\}$.

(c) Compute the steady state covariances $\bar{M}$, $\bar{Z}$, $\bar{\Lambda}_{\tilde{y}^o\tilde{y}^o}$ and $\bar{\Lambda}_{\tilde{y}\tilde{y}}$, and Kalman filter gains $\bar{L}$ and $\bar{F}$ by solving the algebraic Riccati equation

$$A\bar{M}A^T - \bar{M} + BWB^T - A\bar{M}C^T\left[C\bar{M}C^T + V\right]^{-1} C\bar{M}A^T = 0$$

using either the Matlab commands `dare` or `kalman`. Read the Matlab help entries on these functions to understand what they do. Also compute the close loop eigenvalues $A_c = A - \bar{L}C$.

[1]Notice that we have changed the notation to better conform with the notation in the Kalman filter notes.
[2]Remember that $E\{\|\tilde{x}(k)\|^2\} = \text{trace}\{Z(k)\}$ and $E\{\|\tilde{x}^o(k)\|^2\} = \text{trace}\{M(k)\}$.
[3]The a-priori output estimation error $\tilde{y}^o(k) = y(k) - \hat{y}^o(k)$ is often called the *innovation sequence*.

(d) Perform a simulation by running the plant and the Kalman filter for a sufficiently long period of time, so that expected values and covariances can be approximated by time averages. Use $u(k) = 10$ as the deterministic input and generate a set of sample outcomes $x(0)$, $w(k)$ and $v(k)$.

(e) From the resulting simulation data, calculate, using the function `cov`, the approximate steady state values of $\bar{M}$, $\bar{Z}$, $\bar{\Lambda}_{\tilde{y}^o\tilde{y}^o}$ and $\bar{\Lambda}_{\tilde{y}\tilde{y}}$, and compare them with their actual values, which were previously calculated.

(f) Repeat steps (a)-(e) but change the measurement noise intensity as follows i) $V = 0.05$, ii) $V = 5$. Discuss your results for all three cases.

3) Consider the discrete time system given by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + w(k) \qquad (7) \\ y(k) &= Cx(k) + v(k) \qquad (8) \end{aligned}$$

where $E\{x(0)\} = x_o$, $E\left\{(x(0) - x_o)(x(0) - x_o)^T\right\} = X_o$ and

$$E\left\{\begin{bmatrix} w(k) \\ v(k) \end{bmatrix}\begin{bmatrix} w(j)^T & v(j)^T \end{bmatrix}\right\} = \begin{bmatrix} W & S \\ S^T & V \end{bmatrix}\delta(k-j)$$

and $W \in \mathcal{R}^{n\times n}$ is a positive semi-definite matrix and $V \in \mathcal{R}^{m\times m}$ is positive definite matrix. The a-prior Kalman filter for this system can be written as

$$\begin{aligned} \hat{x}^o(k+1) &= A\hat{x}^o(k) + Bu(k) + L(k)[y(k) - C\hat{x}^o(k)] \qquad (9) \\ L(k) &= [AM(k)C^T + S][CM(k)C^T + V]^{-1} \qquad (10) \\ M(k+1) &= AM(k)A^T + W - L(k)[CM(k)C^T + V]L^T(k) \qquad (11) \end{aligned}$$

with initial conditions $\hat{x}^o(0) = x_o$ and $M(0) = X_o$.

Derive Eqs. (9) - (11) using previously derived results in Kalman filtering and noticing that Eqs. (7) - (8) can be written as

$$x(k+1) = A'x(k) + Bu(k) + w'(k) + SV^{-1}y(k)\,,$$

where $A' = A - SV^{-1}C$ and

$$E\left\{\begin{bmatrix} w'(k) \\ v(k) \end{bmatrix}\begin{bmatrix} w'(j)^T & v(j)^T \end{bmatrix}\right\} = \begin{bmatrix} W' & 0 \\ 0 & V \end{bmatrix}\delta(k-j), \qquad W' = W - SV^{-1}S^T$$

4) A random variable x is repeatedly measured, but the measurements process are noisy. Assume that the system can be described by

$$y(k) = x + v(k)$$

where $y(k)$ is the k-th measurement, $v(k)$ is the measurement noise, and x and $v(k)$ are Gaussian distributed with $E\{x\} = 0$, $E\{x^2\} = X_0$, $E\{v(k)\} = 0$, $E\{v(k)v(k+j)\} = V\,\delta(j)$ and $E\{x(0)v(k)\} = 0$.

(a) Obtain the least square estimate

$$\hat{x}(k) = E\{x|y(0)\cdots y(k)\}$$

and the estimation error covariance

$$E\{(x - \hat{x}(k))^2\}$$

(b) Show that in the limit when $X_0 \to \infty$, i.e. no prior information is available on x,

$$\lim_{X_0 \to \infty} \hat{x}(k) = \frac{1}{k+1}\left[y(0) + y(1) + \cdots + y(k)\right]$$

and

$$\lim_{X_0 \to \infty} E\{(x - \hat{x}(k))^2\} = \frac{V}{k+1}$$