IP notes

CCC

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Abstract

A brief note for solving integer programming in QUBO formalism.

MAP IP INTO QUBO

Goal: Solve for

$$\max\, c^T x \tag{1}$$

$$Ax \leq b \tag{2}$$

$$x \geq 0 \tag{3}$$

$$x \in \mathbf{Z}^n, \tag{4}$$

and A is an $m \times n$ matrix, b is an m-component vector.

1. Introduce slack variable:

$$max\ c^T x \tag{5}$$

$$Ax + s = b \tag{6}$$

$$s \geq 0 \tag{7}$$

$$x \geq 0 \tag{8}$$

$$x \in \mathbf{Z}^n. \tag{9}$$

Notice that s is an m-component vector.

2. Transform x and s vectors into binary vectors:

$$x_i = \sum_{j=0}^{u} 2^j q_j^{x_i} \tag{10}$$

$$s_i = \sum_{j=0}^{u} 2^j q_j^{s_i}, \tag{11}$$

where $i = 0, ..., (n-1)$. All $q \in \{0, 1\}$ and u is the upper bound to truncate the binary expansion.

3. Map to QUBO:

Define

$$\alpha = \begin{pmatrix} A & 2A & ... & 2^u A & \mathbf{1} & 2\mathbf{1} & 2^u\mathbf{1} \end{pmatrix} \tag{12}$$

$$C = \begin{pmatrix} c & 2c & ... & 2^u c \end{pmatrix} \tag{13}$$

$$q_x = \begin{pmatrix} q_0^{x_0} & ... & q_0^{x_{n-1}} & q_1^{x_0} & ... & q_1^{x_{n-1}} & ... & q_u^{x_0} & ... & q_u^{x_{n-1}} \end{pmatrix} \tag{14}$$

$$q_s = \begin{pmatrix} q_0^{s_0} & ... & q_0^{s_{m-1}} & q_1^{s_0} & ... & q_1^{s_{m-1}} & ... & q_u^{s_0} & ... & q_u^{s_{m-1}} \end{pmatrix} \tag{15}$$

$$q = \begin{pmatrix} q_x & q_s \end{pmatrix}. \tag{16}$$

Here $\mathbf{1}$ is $m \times m$ identity matrix. α is $m \times (n+m)(u+1)$ matrix.

Then the optimization problem is

$$\max \; C^T q_x - P(\alpha q - b)^T(\alpha q - b) \tag{17}$$

where P is the penalty.

This can be translated to

$$\max \; q^T Q q \tag{18}$$

$$Q = diag[C] - P(\alpha^T \alpha - 2diag[b\alpha]), \tag{19}$$

up to a constant shift.