Supplementary Material for Energy-Bounded Caging: Formal Definition and 2D Energy Lower Bound Algorithm Based on Weighted Alpha Shapes

Jeffrey Mahler[1], Florian T. Pokorny[1], Zoe McCarthy[1], A. Frank van der Stappen[3], Ken Goldberg[2]

This document contains select proofs for Lemmas and Theorems in the paper "Energy-Bounded Caging: Formal Definition and 2D Energy Lower Bound Algorithm Based on Weighted Alpha Shapes," currently under review for the 2016 IEEE Robotics and Automation Letters.

I. Translational Penetration Depth

Recall that 2D generalized penetration depth (GPD) $p : SE(2) \to \mathbb{R}$ between an object $\mathcal{O}(\mathbf{q}_i)$ in pose $\mathbf{q}_i = (x_i, y_i, \theta_i) \in SE(2)$ and obstacle $\mathcal{G}$ is defined as [3]:

$$p(\mathbf{q}_i) = \min_{\mathbf{q}_j \in SE(2)} \left\{ d(\mathbf{q}_i, \mathbf{q}_j) \middle| int\left(\mathcal{O}(\mathbf{q}_j)\right) \cap \mathcal{G} = \varnothing \right\}.$$

where $d : SE(2) \times SE(2) \to \mathbb{R}$ is a distance metric between poses. Following Zhang et. al [2], we use $d(\mathbf{q}_i, \mathbf{q}_j) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} + \min\limits_{m \in \mathbb{Z}} \rho|\theta_i - (\theta_j + 2\pi m)|$ which has the following important property:

Lemma 4.1: Let $\mathcal{Z}$ be the collision space between the object and obstacles. Let $r_i = r(\mathbf{q}_i) : SE(2) \to \mathbb{R}$ be an approximate solution to the above equation such that $r_i \leqslant p(\mathbf{q}_i)$ for all $\mathbf{q}_i \in \mathcal{C}$ and let $\mathbb{B}_r(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^3 : \|\mathbf{x} - \mathbf{y}\| \leqslant r\}$ be a standard Euclidean ball of radius r centered at $\mathbf{x} \in \mathbb{R}^3$. For any embedded pose $\hat{\mathbf{q}}_i$, if $\pi(\hat{\mathbf{q}}_i) \in \mathcal{Z}$, then any embedded pose $\hat{\mathbf{q}}_j \in \mathbb{B}_{r_i}(\hat{\mathbf{q}}_i)$ also satisfies $\pi(\hat{\mathbf{q}}_j) \in \mathcal{Z}$.

Proof: See Fig. 1 for an illustration, and Zhang et al. [2] for an alternative proof. Let $\mathbf{v}_i \in \mathcal{O}$ be the point of maximum interpenetration between $\mathcal{O}$ and $\mathcal{G}$ when $\mathcal{O}$ is in pose $\mathbf{q}_i$. Also, let $\hat{\mathbf{q}}_j \in \mathbb{B}_{r_i}(\hat{\mathbf{q}}_i)$ for $\pi(\hat{\mathbf{q}}_i) = \mathbf{q}_i$. Then define $\mathbf{v}_j$ to be the point of $\mathcal{O}$ in pose $\mathbf{q}_j$ that corresponds to $\mathbf{v}_i$. We prove the result by showing that $\mathbf{v}_j$ must still penetrate $\mathcal{G}$.

Recall that $\mathbf{z}$ denotes the center of rotation of $\mathcal{O}$ and $\rho = \max\limits_{\mathbf{v} \in \mathcal{O}} \|\mathbf{v} - \mathbf{z}\|_2$ denotes the maximum moment arm of $\mathcal{O}$. Then let $\Delta_t(\hat{\mathbf{q}}_i, \hat{\mathbf{q}}_j) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$ and $\Delta_\theta(\hat{\mathbf{q}}_i, \hat{\mathbf{q}}_j) = |z_i - z_j|$ be the translational and rotation distance between $\hat{\mathbf{q}}_i$ and $\hat{\mathbf{q}}_j$, respectively. We have

$$\begin{aligned} d^2(\hat{\mathbf{q}}_i, \hat{\mathbf{q}}_j) &= \Delta_t^2(\hat{\mathbf{q}}_i, \hat{\mathbf{q}}_j) + \rho^2 \Delta_\theta^2(\hat{\mathbf{q}}_i, \hat{\mathbf{q}}_j) < r_i^2 \\ \Rightarrow \rho^2 \Delta_\theta^2(\hat{\mathbf{q}}_i, \hat{\mathbf{q}}_j) &< r_i^2 - \Delta_t^2(\mathbf{q}_i, \mathbf{q}_j). \end{aligned}$$

As $\mathcal{O}$ is rotated by angle $\Delta_\theta(\hat{\mathbf{q}}_i, \hat{\mathbf{q}}_j)$, $\mathbf{v}_i$ translates moves along the arc of length $\Delta_\theta(\mathbf{q}_i, \mathbf{q}_j)\|\mathbf{v}_i - \mathbf{z}\|_2 \leqslant \rho\Delta_\theta(\mathbf{q}_i, \mathbf{q}_j)$.

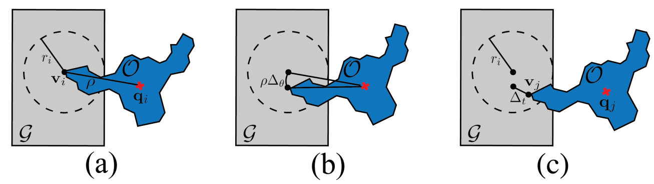

Fig. 1: Illustration of the proof that any pose $\mathbf{q}_j$ in the ball of radius r_i around a pose $\mathbf{q}_i$ in collision is also in collision when r_i lower bounds the translational penetration depth in 2D. (a) The point of maximum penetration $\mathbf{v}_i$ must move a distance of at least r_i for the object $\mathcal{O}$ to be out of collision with the obstacle $\mathcal{G}$. (b) Rotating $\mathcal{O}$ by $\Delta_\theta(\mathbf{q}_i, \mathbf{q}_j)$ moves $\mathbf{v}_i$ along the arc of length less than $\rho\Delta_\theta$. (c) Subsequently translating $\mathcal{O}$ by $\Delta_t(\mathbf{q}_i, \mathbf{q}_j)$ moves the original point of maximum penetration to point $\mathbf{v}_j$, which is guaranteed to still be in the ball or radius r_i because $\Delta_t \leqslant r_i - \rho\Delta_\theta$.

Furthermore, $\mathbf{v}_i$ is translated by exactly $\Delta_t(\hat{\mathbf{q}}_i, \hat{\mathbf{q}}_j)$ as $\mathcal{O}$ is translated between pose $\mathbf{q}_i$ and $\mathbf{q}_j$. Thus $\|\mathbf{v}_i - \mathbf{v}_j\|_2^2 \leqslant \Delta_t^2(\mathbf{q}_i, \mathbf{q}_j) + \rho^2 \Delta_\theta^2(\mathbf{q}_i, \mathbf{q}_j) < r_i^2 \leqslant p^2(\mathbf{q}_i)$, which implies that $\mathbf{v}_j$ penetrates $\mathcal{G}$ by the defintion of penetration depth. ■

II. Verifying Path Non-existence

Recall that $\mathcal{X}$ is a set of sampled object poses embedded in $\mathbb{R}^3$ and $\mathcal{R}$ is the corresponding estimate of the translational penetration depth for each sample. We use Algorithm 1, a modified version of the algorithm by McCarthy et al. [1], to verify that a polygonal object $\mathcal{O}$ has no escape path that avoids a forbidden subcomplex $\mathcal{V}_u$.

Input: Lifted initial pose $\hat{\mathbf{q}}_0$, weighted Delaunay triangulation $D(\mathcal{X}, \mathcal{R})$, u-Energy Forbidden Subcomplex $\mathcal{V}_u$
Result: True if $\mathcal{V}_u$ cages $\mathcal{O}$ in pose $\pi(\mathbf{x})$, False otherwise
`// Init free subcomplex and boundary`
$\mathcal{U} = \{\sigma_i \mid \sigma_i \in D(\mathcal{X}, \mathcal{R}) \setminus \mathcal{V}_u, |\sigma_i| = 3, \}$;
$\mathcal{W} = \{\sigma_j \mid \sigma_j \in \partial D(\mathcal{X}, \mathcal{R}) \setminus \mathcal{V}_u, |\sigma_j| = 2\}$;
`// Compute connected components`
$\mathcal{Q}$ =DisjointSetStructure($\mathcal{U} \cup \mathcal{W}$);
for $\sigma_i \in \mathcal{U} \cup \mathcal{W}$ **do**
    for $\sigma_j \in$ *Neighbors*($\sigma_i, D(\mathcal{X}, \mathcal{R}) \setminus \mathcal{V}_u$) **do**
        if $\sigma_i \cap \sigma_j \notin \mathcal{V}_u$ **then**
            $\mathcal{Q}$.UnionSets(σ_i, σ_j);
    end
end
`// Check connectivity`
σ_0 = Locate($\hat{\mathbf{q}}_0$, $D(X, R)$);
for $\sigma_i \in \mathcal{W}$ **do**
    if $\mathcal{Q}$.*SameSet*(σ_0, σ_i) **then**
        return False;
end
return True;

Algorithm 1: Verifying u-Energy-Bounded Cages

[1]Department of Electrical Engineering and Computer Sciences; {jmahler, ftpokorny, zmccarthy}@berkeley.edu
[2]Department of Industrial Engineering and Operations Research; goldberg@berkeley.edu
[1−2] University of California, Berkeley, USA
[3]Department of Information and Computing Sciences, Utrecht University, The Netherlands; A.F.vanderStappen@uu.nl

Theorem 4.3: Let $D(\mathcal{X}, \mathcal{R})$ denote the weighted Delaunay triangulation and $A(\mathcal{X}, \mathcal{R})$ denote the weighted α-shape at $\alpha = 0$ for this point and weight set. If $\mathcal{V}_u$ is any subcomplex of $D(\mathcal{X}, \mathcal{R})$ in $\mathbb{R}^3$ such that $\hat{\mathbf{q}}_0 \in \operatorname{Conv}(\mathcal{X}) - \mathcal{V}_u$ and Algorithm 1 returns True, then there exists no continuous path from $\hat{\mathbf{q}}_0$ to $\partial \operatorname{Conv}(\mathcal{X})$ in $D(\mathcal{X}, \mathcal{R}) - \mathcal{V}_u$.

Proof: A similar proof was given by McCarthy et al. [1], who showed that a modification of Algorithm 1 is correct for disproving the existence of paths between a start configuration $\mathbf{q}_0$ and a single goal configuration $\mathbf{q}_f$. Suppose instead that Algorithm 1 returns True but there exists a continuous path γ such that $\gamma(0) = \hat{\mathbf{q}}_0$ and $\gamma(1) \in \partial D(\mathcal{X}, \mathcal{R})$. Since $D(\mathcal{X}, \mathcal{R})$ spans the convex hull of $\mathcal{X}$ and γ is continuous, γ must pass through a discrete sequence of N_s tetrahedra $\mathcal{S} = \{\sigma_1, ..., \sigma_{N_s}\}$ and triangles $\mathcal{T} = \{\sigma_1 \cap \sigma_2, ..., \sigma_{N_s - 1} \cap \sigma_{N_s}, \}$, which need not be unique. Let $\mathcal{P} \subset \mathcal{S}$ be the subset of tetrahedra in the forbidden set $\mathcal{V}_\tau$ and let $\mathcal{Q} \subset \mathcal{T}$ be the subset of separating triangles in $\mathcal{V}_\tau$. Since Algorithm 1 returned True, we cannot have $\mathcal{P} = \varnothing$ and $\mathcal{Q} = \varnothing$. However, by the continuity of γ, for all $\sigma \in \mathcal{S}$ or $\sigma \in \mathcal{T}$ there exists $t_0 \in [0, 1]$ such that $\gamma(t_0) \in \sigma$. Therefore γ does not belong strictly to $D(\mathcal{X}, \mathcal{R}) - \mathcal{V}_\tau$, contradicting our assumption of the existence of an escape path. ■

References

[1] Z. McCarthy, T. Bretl, and S. Hutchinson, "Proving path non-existence using sampling and alpha shapes," in *Proc. IEEE Int. Conf. Robotics and Automation (ICRA)*. IEEE, 2012, pp. 2563–2569.

[2] L. Zhang, Y. J. Kim, and D. Manocha, "Efficient cell labelling and path non-existence computation using c-obstacle query," *Int. J. Robotics Research (IJRR)*, vol. 27, no. 11-12, pp. 1246–1257, 2008.

[3] L. Zhang, Y. J. Kim, G. Varadhan, and D. Manocha, "Generalized penetration depth computation," *Computer-Aided Design*, vol. 39, no. 8, pp. 625–638, 2007.