The full vector (complex) expression for cavity accelerating voltage $\vec{V}$, combining shunt impedance, detuning (primarily microphonics), and beam loading, is

$$\left(1 - j\frac{\omega_d}{\omega_f}\right)\vec{V} + \frac{1}{\omega_f}\frac{d\vec{V}}{dt} = 2\vec{K}_1\sqrt{R_1} - R_c\vec{I}$$

where $\vec{K}_1$ is the incident wave amplitude in $\sqrt{\text{Watts}}$, $R_1 = Q_1(R/Q)$ is the coupling impedance of port 1, $\vec{I}$ is the beam current, $R_c = Q_L(R/Q)$ is the coupling impedance to the beam, and $\omega_f = \omega_0/2Q_L$ is the mode's bandwidth, and ω_0 is the nominal resonant frequency of the mode. $\omega_d = 2\pi\Delta f$ is the (time varying) detune frequency, *i.e.*, the difference between actual eigenmode frequency and the accelerator's time base; that term will be discussed more later. The overall Q_L is given as $1/Q_L = 1/Q_0 + 1/Q_1 + 1/Q_2$, where $1/Q_0$ represents losses to the cavity walls, $1/Q_1$ represents coupling to the input coupler, and $1/Q_2$ represents coupling to the field probe. (R/Q) is the shunt impedance of the mode in Ohms, a pure geometry term computable for each particular eigenmode using E&M codes like Superfish. Physically, shunt impedance relates a mode's stored energy U to the accelerating voltage it produces, according to

$$U = \frac{V^2}{(R/Q)\omega_0} \quad .$$

The only assumptions in the above formulation are that the cavity losses are purely resistive, and thus expressible with a fixed Q_0, and that no power is launched into the cavity from the field probe. If other ports have incoming power, there would be additional terms of the same form as $2\vec{K}_1\sqrt{R_1}$.

The output wave $\vec{E}_2$ from the field probe is

$$\vec{E}_2 = \vec{V}/\sqrt{Q_2(R/Q)} \quad .$$

The discussion so far applies independently to every cavity eigenmode. Each such mode has its own value of $\vec{V}$, ω_d, (R/Q), Q_i, and therefore ω_f and R_C. The fields from all the eigenmodes superimpose. If one assigns the subscript μ to a particular such mode, the expression for emitted (a.k.a. reflected) wave travelling outward from the fundamental port includes a prompt reflection term, yielding

$$\vec{E}_1 = \sum_\mu \vec{V}_\mu/\sqrt{Q_{\mu 1}(R/Q)_\mu} - \vec{K}_1 \quad .$$

It's possible to rewrite equation (1) in the frame of the eigenmode itself. Define $\vec{S}$ such that $\vec{V} = \vec{S}e^{j\theta}$, where $d\theta/dt = \omega_d$, then

$$\left(1 - j\frac{\omega_d}{\omega_f}\right)\vec{S}e^{j\theta} + \frac{1}{\omega_f}\left(\frac{d\vec{S}}{dt}e^{j\theta} + \vec{S}\cdot j\omega_d e^{j\theta}\right) = 2\vec{K}_1\sqrt{R_c} - R_c\vec{I}$$

$$\frac{d\vec{S}}{dt} = -\omega_f \vec{S} + \omega_f e^{-j\theta} \left(2\vec{K}_1 \sqrt{R_c} - R_c \vec{I} \right)$$

This state-variable equation is a pure low-pass filter, an advantage especially in the FPGA implementation.

These electromagnetic fields interact mechanically. Each mode's fields generate a force proportional to $V_\mu^2 = |\vec{V}_\mu|^2$, and mechanical displacements influence each mode's instantaneous detune frequency. Construct the previous section's ω_d as a baseline ω_{d0} from the electrical mode solution (*e.g.*, $-2\pi(800\,\mathrm{kHz})$ for the TTF cavity's $8\pi/9$ mode), plus a perturbation ω_μ contributed from the mechanical mode deflections. Consider the electrical mode index μ to include not only electrical eigenmodes of one cavity, but modes of all cavities in the mechanical assembly (*e.g.*, cryomodule). Also include the dependence on piezoelectric actuator voltages V_κ. Then if the assembly's mechanical eigenmodes are indexed by ν, mechanical forces F_ν and displacements x_ν of those eigenmodes are related to the electrical system by

$$F_\nu = \sum_\mu A_{\nu\mu} V_\mu^2 + \sum_\kappa B_{\nu\kappa} V_\kappa$$

$$\omega_\mu = \sum_\nu C_{\mu\nu} x_\nu \quad ,$$

where A, B, and C are constant matrices. These expressions are understood to apply at every time instant; the quantities V, F, x, and ω all vary with time.

The differential equation governing the dynamics of each mechanical eigenmode is that of a textbook second order low-pass filter. In Laplace form,

$$k_\nu x_\nu = \frac{F_\nu}{1 + \dfrac{1}{Q_\nu} \dfrac{s}{\omega_\nu} + \left(\dfrac{s}{\omega_\nu} \right)^2} \quad ,$$

where k_ν is the spring constant. For computational purposes, we want it expressed in terms of the state-space formulation

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + c \cdot \begin{pmatrix} 0 \\ F \end{pmatrix} \quad ,$$

where a scaled velocity coordinate y has been introduced. Convert the latter equation to Laplace form and solve to get

$$\begin{pmatrix} x \\ y \end{pmatrix} = -c \begin{pmatrix} a-s & -b \\ b & a-s \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0 \\ F \end{pmatrix} \quad .$$

Analytically invert that 2×2 matrix, and multiply out to get

$$x = \frac{-bcF}{(a-s)^2 + b^2} \quad .$$

Equate coefficients with the earlier low-pass filter form, in the case $Q > \frac{1}{2}$, to get

$$a \pm jb = \omega \left(\frac{-1}{2Q} \pm j\sqrt{1 - \frac{1}{4Q^2}} \right)$$

$$c = -\frac{1}{k} \cdot \frac{a^2 + b^2}{b} = -\frac{\omega^2}{kb} \quad .$$

All the symbols above, including the mechanical resonance frequency ω, apply to a single mechanical eigenmode, and thus have an implied ν subscript.

A deeper understanding of the forces and responses of a single electrical eigenmode μ of the cavity comes from Slater's perturbation theory. For an eigenmode solution $\vec{H}(\vec{r})\sin(\omega_\mu t)$, $\vec{E}(\vec{r})\cos(\omega_\mu t)$ to Maxwell's equations in a closed conducting cavity (volume Φ), the stored energy U is given by

$$U = \int_\Phi \left[\frac{\mu_0}{4} H^2(\vec{r}) + \frac{\varepsilon_0}{4} E^2(\vec{r}) \right] d\Phi \quad .$$

Suppose a mechanical eigenmode ν involves small deflections $x \cdot \vec{\xi}(\vec{r})$, where x gives the amount of deflection, and the dimensionless quantity $\xi(\vec{r})$ represents the mode shape. Both the force on the mode and the response to a deflection x are given in terms of the Slater integral

$$F = \int_S \left[\frac{\mu_0}{4} H^2(\vec{r}) - \frac{\varepsilon_0}{4} E^2(\vec{r}) \right] \vec{n}(\vec{r}) \cdot \vec{\xi}(\vec{r}) dS \quad ,$$

where $\vec{n}(\vec{r})$ is the normal vector to the cavity surface S, and F directly gives the force. Note in particular the subtraction of E and H terms, contrasted with the addition in the energy integral. Also notice the dot product of the deflection shape with the surface normal. Then the resonance frequency shift of the electical eigenmode is given by

$$\Delta\omega_\mu = -x\omega_\mu \frac{F}{U}$$

and the force by

$$F = \frac{F}{U} \frac{1}{(R/Q)\omega_\mu} V^2 \quad ,$$

where F/U is a property of the electrical eigenmode, independent of amplitude, with units of m^{-1}. Thus $A_{\nu\mu} = (F/U)/((R/Q)\omega_\mu)$, and $C_{\mu\nu} = -\omega_\mu F/U$.

Slater's analysis above lets us express the static Lorentz response of an electrical mode to its own stored energy as

$$\frac{\Delta\omega_\mu}{V^2} = \frac{C_{\mu\nu} A_{\nu\mu}}{k_\nu} = -\left(\frac{F}{U}\right)^2 \frac{1}{k_\nu (R/Q)}$$

correctly showing that this constant is always negative: the mode's static resonant frequency gets lower as it is filled. Summing over all mechanical modes ν gives the total DC response, often quoted in units of $\mathrm{Hz}/(\mathrm{MV/m})^2$.

Using electrical measurements alone, it's not possible to constrain the scaling of x_ν. It is therefore helpful to rescale x_ν and F_ν each by a factor of $\sqrt{k_\nu}$, and eliminate k_ν from the equations. Instead of conventional units (m and N) for x and F, they now both have units of $\sqrt{\mathrm{Joules}}$, so that $x \cdot F$ still represents energy. In this rescaled no-k case,

$$A_{\nu\mu} = \frac{1}{\omega_0}\sqrt{-\frac{1}{(R/Q)}\frac{\Delta\omega}{V^2}}$$

$$C_{\mu\nu} = -\omega_0\sqrt{-(R/Q)\frac{\Delta\omega}{V^2}} \quad .$$

It is perhaps an unexpected result that the cross-coupling between cavity modes (*e.g.*, excite the π mode, measure $\Delta\omega$ for the $8\pi/9$ mode) is quantitatively predicted from measurements of each mode individually, with the exception of the choice of sign of the above radicals. All that is required is confidence that mechanical modes are correctly identified and non-degenerate.

See also *Ponderomotive Instabilities and Microphonics – A Tutorial*, J. R. Delayen