

## 1 Risk Minimization with Doubt

Suppose we have a classification problem with classes labeled  $1, \dots, c$  and an additional “doubt” category labeled  $c + 1$ . Let  $f : \mathbb{R}^d \rightarrow \{1, \dots, c + 1\}$  be a decision rule. Define the loss function

$$L(f(\mathbf{x}), y) = \begin{cases} 0 & \text{if } f(\mathbf{x}) = y \quad f(\mathbf{x}) \in \{1, \dots, c\}, \\ \lambda_c & \text{if } f(\mathbf{x}) \neq y \quad f(\mathbf{x}) \in \{1, \dots, c\}, \\ \lambda_d & \text{if } f(\mathbf{x}) = c + 1 \end{cases} \quad (1)$$

where  $\lambda_c \geq 0$  is the loss incurred for making a misclassification and  $\lambda_d \geq 0$  is the loss incurred for choosing doubt. In words this means the following:

- When you are correct, you should incur no loss.
- When you are incorrect, you should incur some penalty  $\lambda_c$  for making the wrong choice.
- When you are unsure about what to choose, you might want to select a category corresponding to “doubt” and you should incur a penalty  $\lambda_d$ .

The risk of classifying a new data point  $\mathbf{x}$  as class  $f(\mathbf{x}) \in \{1, 2, \dots, c + 1\}$  is

$$R(f(\mathbf{x})|\mathbf{x}) = \sum_{i=1}^c L(f(\mathbf{x}), i) P(Y = i|\mathbf{x}).$$

(a) Show that the following policy  $f_{opt}(x)$  obtains the minimum risk:

- **(R1)** Find class  $i$  such that  $P(Y = i|\mathbf{x}) \geq P(Y = j|\mathbf{x})$  for all  $j$ , meaning you pick the class with the highest probability given  $\mathbf{x}$ .
- **(R2)** Choose class  $i$  if  $P(Y = i|\mathbf{x}) \geq 1 - \frac{\lambda_d}{\lambda_c}$
- **(R3)** Choose doubt otherwise.

- (b) How would you modify your optimum decision rule if  $\lambda_d = 0$ ? What happens if  $\lambda_d > \lambda_c$ ? Explain why this is or is not consistent with what one would expect intuitively.

## 2 The Classical Bias-Variance Tradeoff

Consider a random variable  $X$ , which has unknown mean  $\mu$  and unknown variance  $\sigma^2$ . Given  $n$  iid realizations of training samples  $X_1 = x_1, X_2 = x_2, \dots, X_n = x_n$  from the random variable, we wish to estimate the mean of  $X$ . We will call our estimate of  $X$  the random variable  $\hat{X}$ , which has mean  $\hat{\mu}$ . There are a few ways we can estimate  $\mu$  given the realizations of the  $n$  samples:

1. Average the  $n$  samples:  $\frac{x_1 + x_2 + \dots + x_n}{n}$ .
2. Average the  $n$  samples and one sample of 0:  $\frac{x_1 + x_2 + \dots + x_n}{n+1}$ .
3. Average the  $n$  samples and  $n_0$  samples of 0:  $\frac{x_1 + x_2 + \dots + x_n}{n+n_0}$ .
4. Ignore the samples: just return 0.

In the parts of this question, we will measure the *bias* and *variance* of each of our estimators. The *bias* is defined as

$$E[\hat{X} - \mu]$$

and the *variance* is defined as

$$\text{Var}[\hat{X}].$$

(a) What is the bias of each of the four estimators above?

(b) What is the variance of each of the four estimators above?

- (c) Suppose we have constructed an estimator  $\hat{X}$  from some samples of  $X$ . We now want to know how well  $\hat{X}$  estimates a new independent sample of  $X$ . Denote this new sample by  $X'$ . Derive a general expression for  $E[(\hat{X} - X')^2]$  in terms of  $\sigma^2$  and the bias and variance of the estimator  $\hat{X}$ . Similarly, derive an expression for  $E[(\hat{X} - \mu)^2]$ . Compare the two expressions and comment on the differences between them.
- (d) It is a common mistake to assume that an unbiased estimator is always “best.” Let’s explore this a bit further. Compute  $E[(\hat{X} - \mu)^2]$  for each of the estimators above.
- (e) Demonstrate that the four estimators are each just special cases of the third estimator, but with different instantiations of the hyperparameter  $n_0$ .
- (f) What happens to bias as  $n_0$  increases? What happens to variance as  $n_0$  increases?