

1 Back to Basics: Linear Algebra

Let $X \in \mathbb{R}^{m \times n}$. We do not assume that X has full rank.

- (a) Give the definition of the rowspace, columnspace, and nullspace of X .
- (b) Check the following facts:
 - (a) The rowspace of X is the columnspace of X^\top , and vice versa.
 - (b) The nullspace of X and the rowspace of X are orthogonal complements.
 - (c) The nullspace of $X^\top X$ is the same as the nullspace of X . *Hint: if v is in the nullspace of $X^\top X$, then $v^\top X^\top X v = 0$.*
 - (d) The columnspace and rowspace of $X^\top X$ are the same, and are equal to the rowspace of X . *Hint: Use the relationship between nullspace and rowspace.*

2 Probability Review

There are n archers all shooting at the same target (bulls-eye) of radius 1. Let the score for a particular archer be defined to be the distance away from the center (the lower the score, the better, and 0 is the optimal score). Each archer's score is independent of the others, and is distributed uniformly between 0 and 1. What is the expected value of the worst (highest) score?

- (a) Define a random variable Z that corresponds with the worst (highest) score.
- (b) Derive the Cumulative Distribution Function (CDF) of Z .
- (c) Derive the Probability Density Function (PDF) of Z .
- (d) Calculate the expected value of Z .

3 Vector Calculus

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Below, $\mathbf{x} \in \mathbb{R}^d$ means that \mathbf{x} is a $d \times 1$ column vector with real-valued entries. Likewise, $\mathbf{A} \in \mathbb{R}^{d \times d}$ means that \mathbf{A} is a $d \times d$ matrix with real-valued entries. In this course, we will by convention consider vectors to be column vectors.

Consider $\mathbf{x}, \mathbf{w} \in \mathbb{R}^d$ and $\mathbf{A} \in \mathbb{R}^{d \times d}$. In the following questions, $\frac{\partial}{\partial \mathbf{x}}$ denotes the derivative with respect to \mathbf{x} , while $\nabla_{\mathbf{x}}$ denotes the gradient with respect to \mathbf{x} .

Derive the following derivatives.

- (a) $\frac{\partial \mathbf{w}^\top \mathbf{x}}{\partial \mathbf{x}}$ and $\nabla_{\mathbf{x}}(\mathbf{w}^\top \mathbf{x})$
- (b) $\frac{\partial (\mathbf{w}^\top \mathbf{A} \mathbf{x})}{\partial \mathbf{x}}$ and $\nabla_{\mathbf{x}}(\mathbf{w}^\top \mathbf{A} \mathbf{x})$
- (c) $\frac{\partial (\mathbf{w}^\top \mathbf{A} \mathbf{x})}{\partial \mathbf{w}}$ and $\nabla_{\mathbf{w}}(\mathbf{w}^\top \mathbf{A} \mathbf{x})$
- (d) $\frac{\partial (\mathbf{w}^\top \mathbf{A} \mathbf{x})}{\partial \mathbf{A}}$ and $\nabla_{\mathbf{A}}(\mathbf{w}^\top \mathbf{A} \mathbf{x})$
- (e) $\frac{\partial (\mathbf{x}^\top \mathbf{A} \mathbf{x})}{\partial \mathbf{x}}$ and $\nabla_{\mathbf{x}}(\mathbf{x}^\top \mathbf{A} \mathbf{x})$
- (f) $\nabla_{\mathbf{x}}^2(\mathbf{x}^\top \mathbf{A} \mathbf{x})$

¹Good resources for matrix calculus are:

- The Matrix Cookbook: <https://www.math.uwaterloo.ca/~hwolkowi/matrixcookbook.pdf>
- Wikipedia: https://en.wikipedia.org/wiki/Matrix_calculus
- Khan Academy:
<https://www.khanacademy.org/math/multivariable-calculus/multivariable-derivatives>
- YouTube: <https://www.youtube.com/playlist?list=PLSQL0a2vh4HC5feHa6Rc5c0wbRTx56nF7>.