10.

Pf: If $u_n \le v_n$, then

$\qquad\qquad v_n \ge v_{n+1} = \dfrac{u_n + v_n}{2} \ge \sqrt{u_n v_n} = u_{n+1} \ge u_n.$ $\leftarrow$ fundamental inequality.

Since $u_1 = \sqrt{u_0 v_0} \le \dfrac{u_0 + v_0}{2} = v_0$, hence we have

$$v_1 \ge v_2 \ge \cdots \ge v_n \ge u_n \ge \cdots \ge u_2 \ge u_1, \quad \forall n \in \mathbb{N}_+$$

$\Rightarrow$ $\{u_n\}$ is nondecreasing and $u_n \le v_0, \forall n.$ $\Rightarrow$ $\lim\limits_{n \to \infty} u_n$ exists.

$\{v_n\}$ is increasing and $v_n \ge u_0, \forall n.$ $\lim\limits_{n \to \infty} v_n$ exists.

$\Rightarrow$ $\lim\limits_{n \to \infty} u_n = \lim\limits_{n \to \infty} \dfrac{u_n + v_n}{2} = \dfrac{\lim\limits_{n \to \infty} u_n + \lim\limits_{n \to \infty} v_n}{2}$ $\Rightarrow$ $\lim\limits_{n \to \infty} u_n = \lim\limits_{n \to \infty} v_n.$

12.

Pf: $u_n = \left(\dfrac{1}{n} - \dfrac{1}{n+1}\right) \dfrac{1}{n+2} = \dfrac{1}{2}\left(\dfrac{1}{n} - \dfrac{1}{n+2}\right) - \left(\dfrac{1}{n+1} - \dfrac{1}{n+2}\right) = \dfrac{\frac{1}{2}}{n} - \dfrac{1}{n+1} + \dfrac{\frac{1}{2}}{n+2}$ i.e. $a = \dfrac{1}{2}, b = -1, c = \dfrac{1}{2}.$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \dfrac{1}{2}\left(\dfrac{1}{n} - \dfrac{1}{n+1}\right) - \dfrac{1}{2}\left(\dfrac{1}{n+1} - \dfrac{1}{n+2}\right)$

$v_n = \sum\limits_{k=1}^{n} u_k = \dfrac{1}{2}\left(\sum\limits_{k=1}^{n}\left(\dfrac{1}{k} - \dfrac{1}{k+1}\right) - \sum\limits_{k=1}^{n}\left(\dfrac{1}{k+1} - \dfrac{1}{k+2}\right)\right)$

$\qquad = \dfrac{1}{2}\left(1 - \dfrac{1}{n+1} - \dfrac{1}{2} + \dfrac{1}{n+2}\right) = \dfrac{1}{4} + \dfrac{1}{2}\left(-\dfrac{1}{n+1} + \dfrac{1}{n+2}\right)$

$\Rightarrow$ $\lim\limits_{n \to \infty} v_n = \dfrac{1}{4}$

14.

Pf: Let $\{a_k\}_{k=1}^{\infty}$ be a sequence which has no upper bound.

$\forall n \in \mathbb{N}, \exists k_n \in \mathbb{N}$ s.t. $a_{k_n} \ge n$, otherwise $a_k \le n, \forall k \in \mathbb{N}$, which is a contradiction

$\Rightarrow \{a_{k_n}\}_{n=1}^{\infty}$ is a subsequence which diverges to $+\infty.$

16

Pf: (1) $\quad v_{n+1} - v_n$

$\quad = \dfrac{1}{(n+1)!} + \dfrac{1}{(n+1)(n+1)!} - \dfrac{1}{n \times n!}$

$\quad = \dfrac{n(n+1) + n - (n+1)^2}{n(n+1) \cdot (n+1)!} = -\dfrac{1}{n(n+1) \cdot (n+1)!} \le 0$

$$\Rightarrow \quad u_4 \leq \cdots \leq u_{n-1} \leq u_n \leq v_n \leq v_{n-1} \leq \cdots \leq v_1, \ \forall n$$

$$\Rightarrow \quad \lim_{n \to \infty} u_n, \ \lim_{n \to \infty} v_n \text{ exist and } \lim_{n \to \infty} u_n - \lim_{n \to \infty} v_n = \lim_{n \to \infty} \frac{1}{n \cdot n!} = 0$$

(2) Let $L = \lim_{n \to \infty} u_n = \lim_{n \to \infty} v_n$. Suppose that $L = \frac{p}{q} \in \mathbb{Q}$ s.t p, q are coprime integers.

$$\Rightarrow \quad u_{q-1} + \frac{1}{q!} < \frac{p}{q} < u_{q+1} + \frac{1}{q!} + \frac{1}{q \cdot q!}$$

$$\Rightarrow \quad \frac{q \cdot q! \cdot u_{q-1} + q}{q \cdot q!} < \frac{p \cdot q!}{q \cdot q!} < \frac{q \cdot q! \cdot u_{q-1} + (q+1)}{q \cdot q!} \Rightarrow p \cdot q! \in (q \cdot q! \cdot u_{q-1} + q, \ q \cdot q! \cdot u_{q-1} + q + 1)$$

Since $q \cdot q! \cdot u_{q-1}$ is a positive integer by the def of u_{q-1},

$(q \cdot q! u_{q-1} + q, \ q \cdot q! \cdot u_{q-1} + q - 1)$ contains no integer ! $\qquad \Box$

(3) $$\sum_{k=0}^{n} \frac{ak + b}{k!} = a \sum_{k=1}^{n} \frac{1}{(k-1)!} + b \sum_{k=0}^{n} \frac{1}{k!}$$

$$= a \sum_{k=0}^{n-1} \frac{1}{k!} + b \sum_{k=0}^{n} \frac{1}{k!} = a u_{n-1} + b u_n \to (a+b) L \text{ as } n \to \infty$$

18.

Pf: Since $\left| \sin \frac{1}{n^2} \right| < \frac{1}{n^2}$, we have

$$|u_n| = \left(5 \left| \sin \frac{1}{n^2} \right| + \frac{1}{5} |\cos n| \right)^n$$

$$< \left(\frac{5}{n^2} + \frac{1}{5} \right)^n$$

$$\underset{n > 5}{<} \left(\frac{2}{5} \right)^n$$

$$\Rightarrow \lim_{n \to \infty} u_n = 0$$

20.

Pf: $\forall \varepsilon > 0, \ \exists p$ s.t. $\alpha_p < \frac{\varepsilon}{2}$, take $N = \left[\frac{2p}{\varepsilon} - 1 \right] + 1$ i.e. $\frac{p}{N+1} \leq \frac{\varepsilon}{2}$, then

$$|u_n| \leq \alpha_p + \frac{p}{n+1}$$

$$< \frac{\varepsilon}{2} + \frac{p}{N+1} \leq \varepsilon, \ \forall n \geq N \qquad \Box$$

22.

Pf: Let $L = \lim\limits_{n\to\infty} u_n$.

$$u_n - \frac{n^2+n}{2n^2}\cdot L = \frac{(u_1-L)+2(u_1-L)+\cdots n(u_n-L)}{n^2}$$

$\forall \varepsilon > 0$, $\exists N_1 \in \mathbb{N}$ s.t. $\forall n \geq N_1$ $|u_n - L| \leq \frac{\varepsilon}{2}$, then

$$\left| \frac{N_1(u_{N_1}-L)+\cdots+n(u_n-L)}{n^2} \right|$$

$$\leq \frac{N_1+\cdots+n}{n^2}\cdot\frac{\varepsilon}{2} \leq \frac{\frac{n^2+n}{2}}{n^2}\cdot\frac{\varepsilon}{2} = \left(\frac{1}{2}+\frac{1}{2n}\right)\cdot\frac{\varepsilon}{2} \leq \frac{\varepsilon}{2}, \quad \forall n \geq N_1$$

Since $(u_1-L)+2(u_2-L)+\cdots+(N_1-1)(u_{N_1-1}-L)$ is finite, $\exists N_2$ s.t. $\forall n \geq N_2$

$$\frac{|(u_1-L)+2(u_2-L)+\cdots+(N_1-1)(u_{N_1-1}-L)|}{n^2} \leq \frac{\varepsilon}{2}$$

$\Rightarrow \forall n \geq \max\{N_1, N_2\}$, $\left|u_n - \frac{n^2+n}{2n^2}\cdot L\right| \leq \varepsilon$

$\Rightarrow \lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{n^2+n}{2n^2}\cdot L = \frac{L}{2}$ □

24.

Pf: (1). Since u_n diverges to $+\infty$, $\exists n_1 > n_0$ s.t. $u_{n_1} > x$, then

$$u_{n_0} \leq x < u_{n_1}.$$

$\Rightarrow \exists n_0 \leq n \leq n_1 - 1$ s.t. $u_n \leq x \leq u_{n+1}$

$\Rightarrow |x - u_n| \leq \varepsilon$

(2). $\forall x \in \mathbb{R}$, $\exists m \in \mathbb{N}$ s.t. $v_m + x \geq u_{n_0}$.

By $\boxed{x}$ (1), $\exists p \in \mathbb{N}$ s.t. $|u_p - (v_m + x)| \leq \varepsilon$ □

26.

Pf: (1) $u_n = \sqrt{n+2} - \sqrt{n+1} = \frac{1}{\sqrt{n+2}+\sqrt{n+1}} \sim 0$

(2) $v_n = e^{\frac{1}{n}} - e^{\frac{1}{n+1}} \sim e^0 - e^0 = 1$

(3) $w_n = \sqrt{1+\frac{1}{\ln(n+1)}} - 1 = \frac{\frac{1}{\ln(n+1)}}{} \sim 0$

$$)+ \int\int t \frac{1}{\ln(n+1)}$$

28.

Pf: (1) $\sqrt{k+1} - \sqrt{k} = \frac{k+1-k}{\sqrt{k+1}+\sqrt{k}} = \frac{1}{\sqrt{k+1}+\sqrt{k}}$

$\Rightarrow \frac{1}{2\sqrt{k+1}} \leq \sqrt{k+1} - \sqrt{k} \leq \frac{1}{2\sqrt{k}}$

(2).

By (1), $\sqrt{k+1} - \sqrt{k} \leq \frac{1}{2\sqrt{k}} \leq \sqrt{k} - \sqrt{k-1}$.

$\Rightarrow 2\sqrt{n+1} - 2 \leq u_n \leq 2\sqrt{n}$

$\Rightarrow \sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}} \leq \frac{u_n}{2\sqrt{n}} \leq 1$

$\Rightarrow \lim_{n\to\infty} \frac{u_n}{2\sqrt{n}} = 1$

$\Rightarrow u_n \sim 2\sqrt{n}$ $\square$

30.

Pf: $1 \leq \frac{\sum_{k=1}^{n} k!}{n!} = 1 + \frac{1}{n} + \frac{\sum_{k=1}^{n-2} k!}{n!}$

$\leq 1 + \frac{1}{n} + \frac{\sum_{k=1}^{n-2} (n-2)!}{n!}$

$= 1 + \frac{1}{n} + \frac{n-2}{n(n-1)}$

$\leq 1 + \frac{1}{n} + \frac{1}{n} = 1 + \frac{2}{n}$

$\Rightarrow \lim_{n\to\infty} \frac{\sum_{k=1}^{n} k!}{n!} = 1 \quad \Rightarrow \sum_{k=1}^{n} k! \sim n!$