

1. 设 $f: I \rightarrow \mathbb{R}$ 为凸函数, $I = [-2, 2]$, 设 $f'(-2) = a$, $f'(2) = b$

(1) 求证 f 在 $(-2, 2)$ 上至多可数个点不可导.

Pf:

If $x_1 < x < x_2$, then $x = \lambda_1 x_1 + \lambda_2 x_2$ for $\lambda_1 = \frac{x_2 - x}{x_2 - x_1}$, $\lambda_2 = (1 - \lambda_1) = \frac{x - x_1}{x_2 - x_1}$, then

$$\begin{aligned} (\lambda_2 - \lambda_1) f(x) &\leq (\lambda_2 - x) f(x_1) + (x - \lambda_1) f(x_2) \\ \Rightarrow \frac{f(x) - f(x_1)}{x - x_1} &\leq \frac{f(x_2) - f(x)}{x_2 - x} \Rightarrow \frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_2) - f(x)}{x_2 - x} \end{aligned}$$

Hence we have $\frac{f(x+h) - f(x)}{h}$ is an increasing function of h when $x+h \in I$ and $h \neq 0$.

$\Rightarrow \forall x \in I$, $f'_l(x)$ and $f'_r(x)$ exist. If $-2 < x_1 < x_2 < 2$, then

$$a = f'_r(-2) \leq \frac{f(x_1) - f(-2)}{x_1 - (-2)} \leq f'_l(x_1) \leq f'_r(x_1) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq f'_l(x_2) \leq f'_r(x_2) \leq \frac{f(2) - f(x_2)}{2 - x_2} \leq f'_l(2) = b.$$

Now we define

$$X = \{x \in I: f \text{ is not differentiable at } x \text{ i.e. } f'_r(x_1) > f'_l(x_1)\}.$$

Suppose the X is uncountable, we define $X_0 = \{x \in X: f'_r(x_1) - f'_l(x_1) > 1\}$

and $X_n = \{x \in X: 2^{-n} < f'_r(x_1) - f'_l(x_1) \leq 2^{-n+1}\}$, $\forall n \geq 1$, then $\exists n_0 \in \mathbb{N}$ s.t.

X_{n_0} is countable. It implies that

$$\sum_{x \in X_{n_0}} f'_r(x_1) - f'_l(x_1) \geq \sum_{x \in X_{n_0}} 2^{-n_0} = +\infty.$$

Which contradicts to

$$\sum_{x \in X_{n_0}} f'_r(x_1) - f'_l(x_1) \leq f'_l(2) - f'_r(-2) = b - a < +\infty \quad \square$$

(2). 令 $f^*(x) = \sup_{y \in [-2, 2]} (xy - f(y))$, $\forall x \in [a, b]$. 求证: $f^*: [a, b] \rightarrow \mathbb{R}$ 为凸函数.

Pf:

$\forall x_1, x_2 \in [a, b]$, $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$, we have

$$\begin{aligned} f^*(\lambda_1 x_1 + \lambda_2 x_2) &= \sup_{y \in [-2, 2]} ((\lambda_1 x_1 + \lambda_2 x_2)y - f(y)) \\ &= \sup_{y \in [-2, 2]} (\lambda_1 (x_1 y - f(y)) + \lambda_2 (x_2 y - f(y))) \end{aligned}$$

$$\begin{aligned}
&= \sup_{y \in [-2, 2]} (\lambda_1 (\lambda_1 \cdot y - f(y)) + \lambda_2 (\lambda_2 \cdot y - f(y))) \\
&\leq \sup_{y \in [-2, 2]} \lambda_1 (\lambda_1 \cdot y - f(y)) + \sup_{y \in [-2, 2]} \lambda_2 (\lambda_2 \cdot y - f(y)) \\
&= \lambda_1 f^*(\lambda_1) + \lambda_2 f^*(\lambda_2)
\end{aligned}$$

(3). 求证: $(f^*)^* = f$.

Pf: From the definition we obtain at once

$$f^*(y) + f(x) \geq x \cdot y, \quad \forall x \in [-2, 2], y \in [a, b].$$

$$\Rightarrow f(x) \geq x \cdot y - f^*(y), \quad \forall x \in [-2, 2], y \in [a, b].$$

$$\Rightarrow f(x) \geq (f^*)^*(x) \quad \forall x \in [-2, 2].$$

$\forall x \in I$, $f'_l(x) \leq f'_r(x)$, we can choose $y \in [a, b]$ s.t. $f'_l(x) \leq y \leq f'_r(x)$, then

$$\frac{f(x) - f(z_1)}{x - z_1} \leq f'_l(x) \leq y \leq f'_r(x) \leq \frac{f(z_2) - f(x)}{z_2 - x}, \quad \forall z_2 > x > z_1.$$

$$\Rightarrow y \cdot z - f(z) \leq y \cdot x - f(x), \quad \forall z \in I$$

$$\Rightarrow f^*(y) \leq y \cdot x - f(x) \quad \text{for some } y \in [a, b]$$

$$\Rightarrow f(x) \leq y \cdot x - f^*(y) \leq (f^*)^*(x) \quad \square$$

(4). 若 f 在 $[-2, 2]$ 上可导, 求证: 对于 $y = f'(x) \in [a, b]$, 有 $f^*(y) = x \cdot f'(x) - f(x)$.

Pf:

$$f^*(f'(x)) = \sup_{y \in [-2, 2]} \{ f'(x) \cdot y - f(y) \}, \quad \text{let } g(y) = f'(x) \cdot y - f(y), \quad y \in [-2, 2]$$

Then $g'(y) = f'(x) - f'(y)$. Since $f'(y)$ is nondecreasing, we get

$$g'(y) \leq 0 \text{ on } [-2, x], \quad g'(y) \geq 0 \text{ on } [x, 2].$$

$$\Rightarrow \sup_{y \in [-2, 2]} g(y) = g(x) = f'(x) \cdot x - f(x) \quad \square$$

(5). 若 $f \in C^2[-2,2]$, $f^* \in C^2[a,b]$, 则 $\forall x \in (-2,2)$, 有 $f''(x) \cdot (f^*)''(f(x)) = 1$.

Pf:

$$(f^*)'(f(x)) = x$$

We only consider the case where $a < b$.

Claim 1: $f'(x)$ is strictly increasing on $[-2,2]$.

Pf:

Otherwise, $\exists x_0 < y_0$ s.t. $f'(x_0) < f'(y_0)$. WLOG, we may assume that $f'(y_0) < b$ and we replace y_0 by

$$y_0 = \sup \{x \in I : f'(x) = f'(x_0)\} < 2.$$

Thus $f'(x) \equiv f'(x_0)$ on $[x_0, y_0]$ and $f'(x) > f'(x_0)$ on $(y_0, 2]$.

$$\forall x \in (y_0, b), f'(x) > f'(x_0) \xrightarrow{\text{Lagrange}} \exists \{x_n\} \subset (y_0, b) \text{ s.t. } x_n \rightarrow y_0$$

$$\text{and } f''(x_n) > 0.$$

From Ex(4), $(f^*)(f'(x)) = x \cdot f'(x) - f(x)$, $\forall x \in [-2,2]$, then

$$(f^*)'(f(x)) \cdot f''(x) = x \cdot f''(x).$$

$$\Rightarrow (f^*)'(f(x_n)) \cdot f''(x_n) = x_n \cdot f''(x_n) \xrightarrow{f''(x_n) > 0} (f^*)'(f(x_n)) = x_n$$

Since $f, f^* \in C^2$, let $n \rightarrow \infty$, $(f^*)'(f(y_0)) = y_0$.

We have

$$\begin{aligned} & (f^*)''(f(y_0)) \\ &= \lim_{n \rightarrow \infty} \frac{(f^*)'(f(x_n)) - (f^*)'(f(y_0))}{f'(x_n) - f'(y_0)} \quad \left(\begin{array}{l} f'(x_n) \rightarrow f'(y_0) \\ \& f'(x_n) > f'(y_0) \end{array} \right) \\ &= \lim_{n \rightarrow \infty} \frac{x_n - y_0}{f'(x_n) - f'(y_0)} \\ &= \frac{1}{f''(y_0)}. \end{aligned}$$

On the other hand, $f'(x) \equiv f'(y_0)$, $\forall x \in [x_0, y_0] \Rightarrow f''(y_0) = 0$, which

contradicts $f''(y_0) = \frac{1}{(f^*)''(f(y_0))} \neq 0$. □

Claim 2: $f''(x) > 0$ on $[-2,2]$.

Pf:

Suppose not, $\exists x_0$ s.t. $f''(x_0) = 0$.

Since f is strictly increasing, then $\forall n \in \mathbb{N}$, $\exists x_n \in (x_0 - \frac{1}{n}, x_0 + \frac{1}{n})$ s.t. $f'(x_n) > 0$. Otherwise, $f'(x) \equiv 0$ on $(x_0 - \frac{1}{n}, x_0 + \frac{1}{n})$ for some n which implies that $f'(x) \equiv \text{constant}$ on $(x_0 - \frac{1}{n}, x_0 + \frac{1}{n})$.

Let's repeat what we did in the proof of Claim 1, we have

$$(f^*)''(f'(x_0)) = \frac{1}{f'(x_0)}$$

which contradicts to $f'(x_0) = 0$

□

Proof of (5):

From (4), $(f^*)(f'(x)) = x \cdot f'(x) - f(x)$, $\forall x \in [-2, 2]$, then

$$(f^*)'(f'(x)) \cdot f''(x) = x \cdot f''(x),$$

$$f''(x) > 0$$

\Rightarrow

$$(f^*)'(f'(x)) = x$$

\Rightarrow

$$(f^*)''(f'(x)) \cdot f''(x) = 1, \forall x \in [-2, 2].$$

□