Non-linear least squares method for Newton Gravity law using Gauss-Newton method

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1 Newton's Gravity law (ODE)

Newton's Second law

$$\vec{F} = m\vec{a} \Rightarrow \frac{d^2x}{dt^2} = \frac{\vec{F}}{m}$$

Newton's Gravity law

$$\vec{F} = \frac{GMm}{r^2}\frac{\vec{r}}{r} = \frac{GMm}{r^3}\vec{r} \Rightarrow \frac{d^2x}{dt^2} = \frac{GM}{r^3}\vec{r},$$

$$\text{where } x = \overrightarrow{r(t)} = (x(t), y(t), z(t))$$

$$\frac{d^2\vec{r}}{dt^2} = \frac{GM}{(x^2+y^2+z^2)^{\frac{3}{2}}}(x\ y\ z)$$

Split vector into scalar equations

$$\begin{cases} \frac{d^2x}{dt^2} = \frac{GM}{(x^2+y^2+z^2)^{\frac{3}{2}}}x \\ \frac{d^2y}{dt^2} = \frac{GM}{(x^2+y^2+z^2)^{\frac{3}{2}}}y \\ \frac{d^2z}{dt^2} = \frac{GM}{(x^2+y^2+z^2)^{\frac{3}{2}}}z \end{cases}$$

Make system first order ODE

$$\begin{cases} \frac{dv_x}{dt} = \frac{GM}{(x^2+y^2+z^2)^{\frac{3}{2}}}x \\ \frac{dv_y}{dt} = \frac{GM}{(x^2+y^2+z^2)^{\frac{3}{2}}}y \\ \frac{dv_z}{dt} = \frac{GM}{(x^2+y^2+z^2)^{\frac{3}{2}}}z \\ \frac{dx}{dt} = v_x \\ \frac{dy}{dt} = v_y \\ \frac{dz}{dt} = v_z \end{cases}$$

And for solving Cauchy problem add initial values:
$v(t_0) = (v_{x_0}, v_{y_0}, v_{z_0})$;
$r(t_0) = (x_0, y_0, z_0)$.

2 Gauss-Newton method

Regression model

$$y_i = g(x(t_i),\ q) + \varepsilon_i$$

Where:

y_i - row of observed values in t_i moment,
$\varepsilon_i \sim \mathcal{N}(0, \sigma^2)$ - row of errors in measurements,
g - nonlinear function that returns row of model values,
$x(t_i)$ - system state vector in moment t_i
q - reduction parameters (determines how we measure values)

Dynamical System equation

$$\begin{cases} \frac{dx}{dt} = F(x(t),\ p) \\ x(t_0) = x_0 \end{cases}$$

Where:

$x(t)$ - system state in time t
p - dynamical parameters (e.g. GM, other constants)

Residuals and regression parameters

$$r_i = y_i - g(x(t_i),\ q)$$
$$\beta = (x_0\ p\ q)$$

$$\boxed{\begin{array}{r} \text{Parameters optimization using least squares} \\ \sum_{i=1}^{N} r_i(\hat{\beta})^2 \rightarrow min \\ \hat{\beta}^{(s+1)} = \hat{\beta}^{(s)} - (J^T W J)^{-1} J^T W r(\hat{\beta}^{(s)}) \end{array}}$$

Where:

J - Jacobian of residuals ($J_{ij} = \frac{\partial r_i}{\partial \beta_j}$)
W - weight matrix ($W = diag(\frac{1}{\sigma_1^2}, \dots, \frac{1}{\sigma_N^2})$)

$$\frac{\partial r_i}{\partial \beta_j} = \begin{pmatrix} \frac{\partial r_i}{\partial x_{0_j}} & \frac{\partial r_i}{\partial p_j} & \frac{\partial r_i}{\partial q_j} \end{pmatrix}$$

$$\frac{\partial r_i}{\partial q_j} = -\frac{\partial g(x(t_i),\ q)}{\partial q_j}$$
$$\frac{\partial r_i}{\partial p_j} = -\frac{\partial g(x(t_i),\ q)}{\partial x}\frac{\partial x}{\partial p_j}$$
$$\frac{\partial r_i}{\partial x_{0_j}} = -\frac{\partial g(x(t_i),\ q)}{\partial x}\frac{\partial x}{\partial x_{0_j}}$$

Let's combine last two equations into one matrix $P \equiv (x_0\ p) \Rightarrow Q(t) = \frac{\partial x}{\partial P}$

$$\text{Additional Dynamical equation}$$
$$\frac{d}{dt}\frac{\partial x}{\partial P} = \frac{\partial F}{\partial P} + \frac{\partial F}{\partial x}\frac{\partial x}{\partial P} \;\Rightarrow\; \frac{d}{dt}Q(t) = \frac{\partial F}{\partial P} + \frac{\partial F}{\partial x}Q(t)$$
$$Q(t_0) = \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}$$
$$\qquad\qquad\quad \underbrace{\qquad\quad}_{x_0}\ \underbrace{\qquad\quad}_{p}$$

3 Newton law in Gauss-Newton method for Orbit correction

$$\text{Dynamical system (ODE)}$$
$$\begin{cases} F_1 = \frac{dv_x}{dt} = \frac{GM}{(x^2+y^2+z^2)^{\frac{3}{2}}}x \\ F_2 = \frac{dv_y}{dt} = \frac{GM}{(x^2+y^2+z^2)^{\frac{3}{2}}}y \\ F_3 = \frac{dv_z}{dt} = \frac{GM}{(x^2+y^2+z^2)^{\frac{3}{2}}}z \\ F_4 = \frac{dx}{dt} = v_x \\ F_5 = \frac{dy}{dt} = v_y \\ F_6 = \frac{dz}{dt} = v_z \\ v(t_0) = (v_{x_0}, v_{y_0}, v_{z_0}) \\ r(t_0) = (x_0, y_0, z_0) \end{cases}$$

In this case we have (x, y, z) as observed values therefore our nonlinear function $g(x(t)) = (x(t)\ y(t)\ z(t))$. Lets redefine state vector in function input as $\vec{s}(t) = (x(t)\ y(t)\ z(t)\ v_x(t)\ v_y(t)\ v_z(t))$ from state $\Rightarrow g(s(t)) = (x(t)\ y(t)\ z(t)) = \vec{r}(t)$
We don't have any p or q parameters so differentiation much easier.

$$\frac{\partial r_i}{\partial s_{0_j}} = -\frac{\partial g(s(t_i))}{\partial s}\frac{\partial s}{\partial s_{0_j}} = \begin{bmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} & \frac{\partial x}{\partial z} & \frac{\partial x}{\partial v_x} & \frac{\partial x}{\partial v_y} & \frac{\partial x}{\partial v_z} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} & \frac{\partial y}{\partial z} & \frac{\partial y}{\partial v_x} & \frac{\partial y}{\partial v_y} & \frac{\partial y}{\partial v_z} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & \frac{\partial z}{\partial z} & \frac{\partial z}{\partial v_x} & \frac{\partial z}{\partial v_y} & \frac{\partial z}{\partial v_z} \end{bmatrix} \frac{\partial s}{\partial s_{0_j}} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \end{bmatrix} \frac{\partial s}{\partial s_{0_j}}$$

$$\frac{\partial s}{\partial s_{0_j}} = \begin{bmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} & \frac{\partial x}{\partial z_0} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} & \frac{\partial y}{\partial z_0} \\ \frac{\partial z}{\partial x_0} & \frac{\partial z}{\partial y_0} & \frac{\partial z}{\partial z_0} \end{bmatrix}$$

3.1 Cartesian to Right ascension & Declination

In our case we have $g(s(t)) = (ra(t)\ dec(t))$ which maps with Cartesian this way:

$$ra = atan2(y, x)$$
$$dec = atan2(z, \sqrt{(x^2 + y^2)})$$

Therefore

$$\frac{\partial g}{\partial s} = \begin{bmatrix} \frac{\partial ra}{\partial x} & \frac{\partial ra}{\partial y} & \frac{\partial ra}{\partial z} & \frac{\partial ra}{\partial v_x} & \frac{\partial ra}{\partial v_y} & \frac{\partial ra}{\partial v_z} \\ \frac{\partial dec}{\partial x} & \frac{\partial dec}{\partial y} & \frac{\partial dec}{\partial z} & \frac{\partial dec}{\partial v_x} & \frac{\partial dec}{\partial v_y} & \frac{\partial dec}{\partial v_z} \end{bmatrix} = \begin{bmatrix} \frac{\partial ra}{\partial x} & \frac{\partial ra}{\partial y} & \frac{\partial ra}{\partial z} & 0 & 0 & 0 \\ \frac{\partial dec}{\partial x} & \frac{\partial dec}{\partial y} & \frac{\partial dec}{\partial z} & 0 & 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \\ -\frac{xz}{\sqrt{(x^2+y^2)}(x^2+y^2+z^2)} & -\frac{yz}{\sqrt{(x^2+y^2)}(x^2+y^2+z^2)} & \frac{\sqrt{(x^2+y^2)}}{x^2+y^2+z^2} \end{bmatrix}$$

And now to find last component $Q(t) = \frac{\partial s}{\partial s_{0_j}}$ we need to integrate one more equation

$$\frac{d}{dt}Q(t) = \frac{\partial F}{\partial s_0} + \frac{\partial F}{\partial s}Q(t)$$

$$Q(t_0) = \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underbrace{\qquad\qquad}_{s_0} \quad \underbrace{\qquad\qquad}_{p}$$

In that case first term equals 0 and we have simplified equation

$$\frac{d}{dt}Q(t) = \frac{\partial F}{\partial s}Q(t)$$

Lets find out what's $\frac{\partial F}{\partial s}$ equals to.

$$\frac{\partial F}{\partial s} = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} & \frac{\partial F_1}{\partial v_x} & \frac{\partial F_1}{\partial v_y} & \frac{\partial F_1}{\partial v_z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} & \frac{\partial F_2}{\partial v_x} & \frac{\partial F_2}{\partial v_y} & \frac{\partial F_2}{\partial v_z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} & \frac{\partial F_3}{\partial v_x} & \frac{\partial F_3}{\partial v_y} & \frac{\partial F_3}{\partial v_z} \\ \frac{\partial F_4}{\partial x} & \frac{\partial F_4}{\partial y} & \frac{\partial F_4}{\partial z} & \frac{\partial F_4}{\partial v_x} & \frac{\partial F_4}{\partial v_y} & \frac{\partial F_4}{\partial v_z} \\ \frac{\partial F_5}{\partial x} & \frac{\partial F_5}{\partial y} & \frac{\partial F_5}{\partial z} & \frac{\partial F_5}{\partial v_x} & \frac{\partial F_5}{\partial v_y} & \frac{\partial F_5}{\partial v_z} \\ \frac{\partial F_6}{\partial x} & \frac{\partial F_6}{\partial y} & \frac{\partial F_6}{\partial z} & \frac{\partial F_6}{\partial v_x} & \frac{\partial F_6}{\partial v_y} & \frac{\partial F_6}{\partial v_z} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{GM}{(x^2+y^2+z^2)^{\frac{5}{2}}}(-2x^2+y^2+z^2) & \frac{GM}{(x^2+y^2+z^2)^{\frac{5}{2}}}(-3xy) & \frac{GM}{(x^2+y^2+z^2)^{\frac{5}{2}}}(-3xz) & 0 & 0 & 0 \\ \frac{GM}{(x^2+y^2+z^2)^{\frac{5}{2}}}(-3yx) & \frac{GM}{(x^2+y^2+z^2)^{\frac{5}{2}}}(x^2-2y^2+z^2) & \frac{GM}{(x^2+y^2+z^2)^{\frac{5}{2}}}(-3yz) & 0 & 0 & 0 \\ \frac{GM}{(x^2+y^2+z^2)^{\frac{5}{2}}}(-3zx) & \frac{GM}{(x^2+y^2+z^2)^{\frac{5}{2}}}(-3zy) & \frac{GM}{(x^2+y^2+z^2)^{\frac{5}{2}}}(x^2+y^2-2z^2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\frac{\partial v_x}{\partial x} = 0$ same for y, z