Non-linear least squares method for Newton Gravity law using Gauss-Newton method

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1 Newton's Gravity law (ODE)

Newton's Second law

$$\vec{F} = m\vec{a} \Rightarrow \frac{d^2x}{dt^2} = \frac{\vec{F}}{m}$$

Newton's Gravity law

$$\vec{F} = \frac{GMm}{r^2}\frac{\vec{r}}{r} = \frac{GMm}{r^3}\vec{r} \Rightarrow \frac{d^2x}{dt^2} = \frac{GM}{r^3}\vec{r},$$

$$\text{where } x = \overrightarrow{r(t)} = (x(t), y(t), z(t))$$

$$\frac{d^2\vec{r}}{dt^2} = \frac{GM}{(x^2+y^2+z^2)^{\frac{3}{2}}}(x\ y\ z)$$

Split vector into scalar equations

$$\begin{cases} \frac{d^2x}{dt^2} = \frac{GM}{(x^2+y^2+z^2)^{\frac{3}{2}}}x \\ \frac{d^2y}{dt^2} = \frac{GM}{(x^2+y^2+z^2)^{\frac{3}{2}}}y \\ \frac{d^2z}{dt^2} = \frac{GM}{(x^2+y^2+z^2)^{\frac{3}{2}}}z \end{cases}$$

Make system first order ODE

$$\begin{cases} \frac{dv_x}{dt} = \frac{GM}{(x^2+y^2+z^2)^{\frac{3}{2}}}x \\ \frac{dv_y}{dt} = \frac{GM}{(x^2+y^2+z^2)^{\frac{3}{2}}}y \\ \frac{dv_z}{dt} = \frac{GM}{(x^2+y^2+z^2)^{\frac{3}{2}}}z \\ \frac{dx}{dt} = v_x \\ \frac{dy}{dt} = v_y \\ \frac{dz}{dt} = v_z \end{cases}$$

And for solving Cauchy problem add initial values:
$v(t_0) = (v_{x_0}, v_{y_0}, v_{z_0})$;
$r(t_0) = (x_0, y_0, z_0)$.

2 Gauss-Newton method

Regression model

$$y_i = g(x(t_i),\ q) + \varepsilon_i$$

Where:

y_i - row of observed values in t_i moment,
$\varepsilon_i \sim \mathcal{N}(a, \sigma^2)$ - row of errors in measurements,
g - nonlinear function that returns row of model values,
$x(t_i)$ - system state vector in moment t_i
q - reduction parameters (determines how we measure values)

Dynamical System equation

$$\begin{cases} \frac{dx}{dt} = F(x(t),\ p) \\ x(t_0) = x_0 \end{cases}$$

Where:

$x(t)$ - system state in time t
p - dynamical parameters (e.g. GM, other constants)

Residuals and regression parameters

$$r_i = y_i - g(x(t_i),\ q)$$

$$\beta = (x_0\ p\ q)$$

Parameters optimization using least squares

$$\sum_{i=1}^{N} r_i(\hat{\beta})^2 \to min$$

$$\hat{\beta}^{(s+1)} = \hat{\beta}^{(s)} - (J^T W J)^{-1} J^T W r(\hat{\beta}^{(s)})$$

Where:

J - Jacobian of residuals ($J_{ij} = \frac{\partial r_i}{\partial \beta_j}$)
W - weight matrix ($W = diag(\frac{1}{\sigma_1^2}, \ldots, \frac{1}{\sigma_N^2})$)

$$\frac{\partial r_i}{\partial \beta_j} = \begin{pmatrix} \frac{\partial r_i}{\partial x_{0_j}} & \frac{\partial r_i}{\partial p_j} & \frac{\partial r_i}{\partial q_j} \end{pmatrix}$$

$$\frac{\partial r_i}{\partial q_j} = -\frac{\partial g(x(t_i),\ q)}{\partial q_j}$$

$$\frac{\partial r_i}{\partial p_j} = -\frac{\partial g(x(t_i),\ q)}{\partial x}\frac{\partial x}{\partial p_j}$$

$$\frac{\partial r_i}{\partial x_{0_j}} = -\frac{\partial g(x(t_i),\ q)}{\partial x}\frac{\partial x}{\partial x_{0_j}}$$

Let's combine last two equations into one matrix $P \equiv (x_0\ p) \Rightarrow Q(t) = \frac{\partial x}{\partial P}$

$$\text{Additional Dynamical equation}$$
$$\frac{d}{dt}\frac{\partial x}{\partial P} = \frac{\partial F}{\partial P} + \frac{\partial F}{\partial x}\frac{\partial x}{\partial P} \;\Rightarrow\; \frac{d}{dt}Q(t) = \frac{\partial F}{\partial P} + \frac{\partial F}{\partial x}Q(t)$$

$$Q(t_0) = \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}$$
$$\underbrace{\qquad\qquad}_{x_0}\quad\underbrace{\qquad\qquad}_{p}$$

3 Newton law in Gauss-Newton method for Orbit correction

$$\text{Dynamical system (ODE)}$$
$$\begin{cases} F_1 = \frac{dv_x}{dt} = \frac{GM}{(x^2+y^2+z^2)^{\frac{3}{2}}}x \\ F_2 = \frac{dv_y}{dt} = \frac{GM}{(x^2+y^2+z^2)^{\frac{3}{2}}}y \\ F_3 = \frac{dv_z}{dt} = \frac{GM}{(x^2+y^2+z^2)^{\frac{3}{2}}}z \\ F_4 = \frac{dx}{dt} = v_x \\ F_5 = \frac{dy}{dt} = v_y \\ F_6 = \frac{dz}{dt} = v_z \\ v(t_0) = (v_{x_0}, v_{y_0}, v_{z_0}) \\ r(t_0) = (x_0, y_0, z_0) \end{cases}$$

In this case we have (x, y, z) as observed values therefore our nonlinear function $g(x(t)) = (x(t)\ y(t)\ z(t))$. Lets redefine state vector in function input as
$\vec{s}(t) = (x(t)\ y(t)\ z(t)\ v_x(t)\ v_y(t)\ v_z(t))$ from state $\Rightarrow g(s(t)) = (x(t)\ y(t)\ z(t)) = \vec{r}(t)$
We don't have any p or q parameters so differentiation much easier.

$$\frac{\partial r_i}{\partial s_{0_j}} = -\frac{\partial g(s(t_i))}{\partial s}\frac{\partial s}{\partial s_{0_j}} = \begin{bmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} & \frac{\partial x}{\partial z} & \frac{\partial x}{\partial v_x} & \frac{\partial x}{\partial v_y} & \frac{\partial x}{\partial v_z} \\ \frac{\partial y}{\partial y} & \frac{\partial y}{\partial y} & \frac{\partial y}{\partial z} & \frac{\partial y}{\partial v_x} & \frac{\partial y}{\partial v_y} & \frac{\partial y}{\partial v_z} \\ \frac{\partial z}{\partial z} & \frac{\partial z}{\partial z} & \frac{\partial z}{\partial z} & \frac{\partial z}{\partial v_x} & \frac{\partial z}{\partial v_y} & \frac{\partial z}{\partial v_z} \end{bmatrix} \frac{\partial s}{\partial s_{0_j}} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \end{bmatrix} \frac{\partial s}{\partial s_{0_j}}$$

And now to find last component $Q(t) = \frac{\partial s}{\partial s_{0_j}}$ we need to integrate one more equation

$$\frac{d}{dt}Q(t) = \frac{\partial F}{\partial s_0} + \frac{\partial F}{\partial s}Q(t)$$

$$Q(t_0) = \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \underbrace{0 \quad \cdots \quad 1}_{x_0} & & & \underbrace{0 \quad \cdots \quad 0}_{p} & & \end{bmatrix}$$

In that case first term equals 0 and we have simplified equation

$$\frac{d}{dt}Q(t) = \frac{\partial F}{\partial s}Q(t)$$

Lets find out what's $\frac{\partial F}{\partial s}$ equals to.

$$\frac{\partial F}{\partial s} = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} & \frac{\partial F_1}{\partial v_x} & \frac{\partial F_1}{\partial v_y} & \frac{\partial F_1}{\partial v_z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} & \frac{\partial F_2}{\partial v_x} & \frac{\partial F_2}{\partial v_y} & \frac{\partial F_2}{\partial v_z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} & \frac{\partial F_3}{\partial v_x} & \frac{\partial F_3}{\partial v_y} & \frac{\partial F_3}{\partial v_z} \\ \frac{\partial F_4}{\partial x} & \frac{\partial F_4}{\partial y} & \frac{\partial F_4}{\partial z} & \frac{\partial F_4}{\partial v_x} & \frac{\partial F_4}{\partial v_y} & \frac{\partial F_4}{\partial v_z} \\ \frac{\partial F_5}{\partial x} & \frac{\partial F_5}{\partial y} & \frac{\partial F_5}{\partial z} & \frac{\partial F_5}{\partial v_x} & \frac{\partial F_5}{\partial v_y} & \frac{\partial F_5}{\partial v_z} \\ \frac{\partial F_6}{\partial x} & \frac{\partial F_6}{\partial y} & \frac{\partial F_6}{\partial z} & \frac{\partial F_6}{\partial v_x} & \frac{\partial F_6}{\partial v_y} & \frac{\partial F_6}{\partial v_z} \end{bmatrix} = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} & 0 & 0 & 0 \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} & 0 & 0 & 0 \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} & 0 & 0 & 0 \\ \frac{a_x}{v_x} & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{a_y}{v_y} & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{a_z}{v_z} & 0 & 0 & 1 \end{bmatrix}$$

$a_x = \frac{\partial v_x}{\partial x} v_x \Rightarrow \frac{\partial v_x}{\partial x} = \frac{a_x}{v_x}$