

# Lecture 2   Posterior Distributions and Inference

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# Choice of Likelihood Function

- ▶ Bayesian approaches: transparent finite-sample inference but must specify likelihood function
- ▶ Such specification is part of prior information and requires justification, e.g.

$$y_i = \mu + u_i, \quad u_i \sim_{i.i.d.} t_\nu(0, \sigma^2), \quad i = 1, \dots, n$$

- ▶ distributional assumption: normal ( $\nu = \infty$ ) vs. Student- $t$
  - ▶ posterior odds comparison w.r.t.  $\nu$
- ▶ Frequentist approaches: MLE also requires distribution; GMM is free of distribution but relies on large sample

# The Road Ahead...

- ▶ Multiparameter case
- ▶ Bayesian updating
- ▶ Large-sample posterior
- ▶ Identification
- ▶ Posterior estimates: point & interval
- ▶ Model comparison
- ▶ Prediction

## Vector Case: $\theta = (\theta_1, \dots, \theta_d)$

### Marginal vs. conditional posterior

$$\underbrace{\pi(\theta_1|y)}_{\text{marginal}} = \int \underbrace{\pi(\theta_1|\theta_2, \dots, \theta_d, y)}_{\text{conditional}} \underbrace{\pi(\theta_2, \dots, \theta_d|y)}_{\text{weight}} d\theta_2 \cdots d\theta_d$$

where  $\pi(\theta_1|\theta_2, \dots, \theta_d, y) = \underbrace{\pi(\theta_1, \theta_2, \dots, \theta_d|y)}_{\text{joint}} / \pi(\theta_2, \dots, \theta_d|y)$

- ▶ From joint to marginal posteriors
  - ▶  $\pi(\theta_1|y)$  accounts for uncertainty over  $(\theta_2, \dots, \theta_d)$  by averaging  $\pi(\theta_1|\theta_2, \dots, \theta_d, y)$  weighted by  $\pi(\theta_2, \dots, \theta_d|y)$
  - ▶ analytical vs. numerical integral
- ▶ Focus on one/two-dim marginals (readily graphed)

# Die-Tossing Example

- ▶ Multinomial joint likelihood:  $(y_1, \dots, y_d) \sim \mathbb{M}_d(\theta, n)$

$$f(y_1, \dots, y_d | \theta_1, \dots, \theta_d) = \frac{n!}{\prod_{i=1}^d y_i!} \prod_{i=1}^d \theta_i^{y_i}$$

- ▶  $d$  outcomes: probabilities  $\sum \theta_i = 1$ , counts  $\sum y_i = n$
- ▶ Bernoulli ( $d = 2, n = 1$ ), binomial ( $d = 2, n > 1$ )
- ▶ Dirichlet joint prior:  $\theta \sim \mathbb{D}(\alpha_1, \dots, \alpha_d)$

$$\pi(\theta) = \frac{\Gamma(\sum_{i=1}^d \alpha_i)}{\prod_{i=1}^d \Gamma(\alpha_i)} \prod_{i=1}^d \theta_i^{\alpha_i - 1}, \quad \sum_{i=1}^d \theta_i = 1, \quad \alpha_i > 0$$

- ▶ Dirichlet joint posterior:  $\theta | y \sim \mathbb{D}(y_1 + \alpha_1, \dots, y_d + \alpha_d)$
- ▶ Beta marginal posterior:  $\theta_j | y \sim \mathbb{B}(y_j + \alpha_j, \sum_{i \neq j} y_i + \alpha_i)$

# Bayesian Updating

## Bayes theorem

$$\text{Old data } y_1: \underbrace{\pi(\theta|y_1)}_{\text{posterior}} \propto \underbrace{f(y_1|\theta)}_{\text{likelihood}} \underbrace{\pi(\theta)}_{\text{prior}}$$

$$\text{New data } y_2: \underbrace{\pi(\theta|y_1, y_2)}_{\text{posterior}} \propto \underbrace{f(y_2|y_1, \theta)}_{\text{likelihood}} \underbrace{\pi(\theta|y_1)}_{\text{prior}}$$

- ▶ Consider coin-tossing example
  - ▶ prior:  $\theta \sim \mathbb{B}(\alpha_0, \beta_0)$
  - ▶ posterior after initial  $n_1$  tosses:  $\theta|y_1 \sim \mathbb{B}(\alpha_1, \beta_1)$ , where  $\alpha_1 = \alpha_0 + \sum y_{1,i}$ ,  $\beta_1 = \beta_0 + n_1 - \sum y_{1,i}$
  - ▶ posterior after another  $n_2$  tosses:  $\theta|y_1, y_2 \sim \mathbb{B}(\alpha_2, \beta_2)$ , where  $\alpha_2 = \alpha_1 + \sum y_{2,i}$ ,  $\beta_2 = \beta_1 + n_2 - \sum y_{2,i}$
- ▶ For sequential data, Bayesian updates ‘prior’ with new information to obtain ‘posterior’

# Large Samples

## Posterior with independent data

$$\pi(\theta|y) \propto \pi(\theta) \exp[n\bar{l}(\theta|y)], \quad \bar{l}(\theta|y) = \frac{1}{n} \sum_{i=1}^n \log[f(y_i|\theta)]$$

- ▶ Effects of large  $n$  on posterior
  - ▶ data/likelihood dominates prior
  - ▶ ‘consistency’:  $\bar{l}(\theta|y) \rightarrow_{n \rightarrow \infty} \bar{l}(\theta_0|y)$  so posterior degenerates to point mass at true value of  $\theta$
  - ▶ ‘asymptotic normality’: take 2nd-order Taylor expansion around  $\hat{\theta}_{\text{MLE}}$

$$\pi(\theta|y) \propto \pi(\theta) \underbrace{\exp \left[ -\frac{n}{2v} (\theta - \hat{\theta}_{\text{MLE}})^2 \right]}_{\text{Gaussian kernel}}, \quad v = -\bar{l}''(\hat{\theta}|y)^{-1} > 0$$

provided  $\pi(\hat{\theta}_{\text{MLE}}) \neq 0$  (exercise: multiparameter case)

# Identification

- ▶ Identification through data/likelihood
  - ▶ model A & model B are observationally equivalent if  $f(y|\theta_A) = f(y|\theta_B)$  for *all*  $y \Rightarrow \theta$  not identified
  - ▶ no observational equivalence  $\Rightarrow \theta$  identified
- ▶ Important special case:  $f(y|\theta_1, \theta_2) = f(y|\theta_1)$ 
  - ▶  $\theta_2$  not identified, e.g. linear regression with both constant and complete set of dummies
  - ▶ Identification through prior: if  $\pi(\theta_2|\theta_1) \neq \pi(\theta_2)$

$$\pi(\theta_2|y) = \int \pi(\theta_1|y)\pi(\theta_2|\theta_1)d\theta_1 \neq \pi(\theta_2)$$

- ▶ Be cautious when interpreting difference between prior-posterior for unidentified parameters



# Posterior Estimates

- ▶ Bayes estimator minimizes expected loss

$$\hat{\theta} = \arg \min_{\tilde{\theta}} \mathbb{E}[L(\tilde{\theta}, \theta)] = \arg \min_{\tilde{\theta}} \int L(\tilde{\theta}, \theta) \pi(\theta|y) d\theta$$

- ▶ quadratic loss  $L(\tilde{\theta}, \theta) = (\tilde{\theta} - \theta)^2 \Rightarrow \hat{\theta} = \mathbb{E}(\theta|y)$
- ▶ frequentist criteria: unbiasedness, consistency, efficiency
- ▶  $\mathbb{E}$  operator: Bayesian  $f(\theta)|y$  vs. frequentist  $f(y)|\theta$
- ▶ Credible interval: e.g.  $\mathbb{P}(\theta_l \leq \theta \leq \theta_u) = 0.9$ 
  - ▶  $\min(\theta_u - \theta_l) \Rightarrow$  highest probability density (HPD) interval
  - ▶ frequentist confidence intervals entail all possible  $y$

# Model Comparison

## Posterior odd & marginal likelihood

$$\underbrace{\frac{\pi(M_1|y)}{\pi(M_2|y)}}_{\text{posterior odds}} = \underbrace{\frac{\pi(M_1)}{\pi(M_2)}}_{\text{prior odds}} \underbrace{\frac{m(y|M_1)}{m(y|M_2)}}_{\text{Bayes factor}}$$

where  $\underbrace{m(y|M_i)}_{\text{marginal likelihood}} = \int f(y|\theta_i, M_i) \pi(\theta_i|M_i) d\theta_i$

- Effects of large  $n$  on log Bayes factor

$$\log(B_{12}) \approx \underbrace{\log\left(\frac{f(\hat{\theta}_{1,\text{MLE}}|y)}{f(\hat{\theta}_{2,\text{MLE}}|y)}\right)}_{\text{log likelihood ratio}} - \underbrace{\frac{d_1 - d_2}{2} \log(n)}_{\text{penalty on dim}(\theta)} + \underbrace{C}_{\text{free of } n}$$

- Jeffreys guideline vs. frequentist hypothesis test
- nested vs. non-nested model comparison

# Prediction

## Predicting new data

$$f(y_f|y) = \int f(y_f|\theta, y) \pi(\theta|y) d\theta$$

- ▶ Recall coin-tossing example
  - ▶ posterior:  $\theta|y \sim \mathbb{B}(\alpha_1, \beta_1)$ , where  $\alpha_1 = \alpha_0 + \sum y_i$ ,  
 $\beta_1 = \beta_0 + n - \sum y_i$
  - ▶ exercise: verify  $f(y_{n+1} = 1|y) = \frac{\alpha_0 + \sum y_i}{\alpha_0 + \beta_0 + n} = \mathbb{E}(\theta|y)$
- ▶ Prediction with multiple models via model averaging

$$f(y_f|y) = \sum_{i=1}^m \pi(M_i|y) f(y_f|y, M_i)$$

# Readings

- ▶ Jeffreys (1961), "Theory of Probability," Clarendon Press