Part II: Likelihood Evaluation of Linear DSGE Models

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Mini-Course Prepared for School of Finance
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June 23, 2018

Introduction

- State space representation of DSGE models amenable to likelihood evaluation
  - linearity + Gaussian shocks
  - analytical evaluation via Kalman filter
- Main references
  - Kalman (1960), "*A New Approach to Linear Filtering and Prediction Problems*", Journal of Basic Engineering
  - DeJong & Dave (2011), "*Structural Macroeconometrics*", Princeton University Press
  - Herbst & Schorfheide (2015), "*Bayesian Estimation of DSGE Models*", Princeton University Press

The Road Ahead...

- Linear state space models
  - small new Keynesian DSGE
  - state space representation
- Likelihood evaluation
  - Generic filtering algorithm
  - Kalman filter & derivation (see notes)

Small New Keynesian DSGE

- Dynamic IS relation

$$\hat{y}_t = \mathbb{E}_t \hat{y}_{t+1} + \hat{g}_t - \mathbb{E}_t \hat{g}_{t+1} - \frac{1}{\tau}(\hat{R}_t - \mathbb{E}_t \hat{\pi}_{t+1} - \mathbb{E}_t \hat{z}_{t+1})$$

- New Keynesian Phillips curve

$$\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \kappa(\hat{y}_t - \hat{g}_t)$$

- Monetary policy rule

$$\hat{R}_t = \rho_R \hat{R}_{t-1} + (1-\rho_R)\psi_1 \hat{\pi}_t + (1-\rho_R)\psi_2(\hat{y}_t - \hat{g}_t) + \epsilon_{R,t}$$

- Exogenous driving processes

$$\hat{g}_t = \rho_G \hat{g}_{t-1} + \epsilon_{G,t}, \quad \hat{z}_t = \rho_Z \hat{z}_{t-1} + \epsilon_{Z,t}$$

State Space Representation

- Transition equations
  - time domain (our focus), e.g. Sims (2001)

$$s_t = C(\theta) + G(\theta)s_{t-1} + M(\theta)\epsilon_t$$

  - frequency domain, e.g. Tan (2018)

$$s_t = \sum_{k=0}^{\infty} C_k(\theta)\epsilon_{t-k} \equiv C_\theta(L)\epsilon_t$$

- Measurement equations

$$\underbrace{\begin{pmatrix} \mathsf{YGR}_t \\ \mathsf{INF}_t \\ \mathsf{INT}_t \end{pmatrix}}_{y_t} = \underbrace{\begin{pmatrix} \gamma^{(Q)} \\ \pi^{(A)} \\ \pi^{(A)} + r^{(A)} + 4\gamma^{(Q)} \end{pmatrix}}_{D(\theta)} + \underbrace{\begin{pmatrix} \hat{y}_t - \hat{y}_{t-1} + \hat{z}_t \\ 4\hat{\pi}_t \\ 4\hat{R}_t \end{pmatrix}}_{Z(\theta)s_t} + u_t$$

- Distributional assumption: $\epsilon_t \sim \mathbb{N}(0, \Sigma_\epsilon(\theta))$, $u_t \sim \mathbb{N}(0, \Sigma_u(\theta))$

Generic Filtering Algorithm

- Initialization: set $p(s_0|y_0,\theta) = p(s_0|\theta)$
- recursion: for $t = 1, \ldots, T$

  step 1: forecasting s_t via model solution

  $$p(s_t|y_{1:t-1},\theta) = \int p(s_t|s_{t-1},\cancel{y_{1:t-1}},\theta)p(s_{t-1}|y_{1:t-1},\theta)ds_{t-1}$$

  step 2: forecasting y_t via measurement equations

  $$p(y_t|y_{1:t-1},\theta) = \int p(y_t|s_t,\cancel{y_{1:t-1}},\theta)p(s_t|y_{1:t-1},\theta)ds_t$$

  step 3: filtering s_t via Bayes' Theorem

  $$p(s_t|y_{1:t},\theta) = \frac{p(y_t|s_t,\cancel{y_{1:t-1}},\theta)p(s_t|y_{1:t-1},\theta)}{p(y_t|y_{1:t-1},\theta)}$$

- Likelihood evaluation: $p(y_{1:T}|\theta) = \prod_{t=1}^{T} p(y_t|y_{1:t-1},\theta)$

Kalman Filter

- Initialization: set $(s_{0|0}, P_{0|0})$
- recursion: for $t = 1, \ldots, T$

  step 1: forecasting s_t via model solution

$$
\begin{aligned}
s_{t|t-1} &= C(\theta) + G(\theta) s_{t-1|t-1} \\
P_{t|t-1} &= G(\theta) P_{t-1|t-1} G(\theta)' + M(\theta) \Sigma_\epsilon(\theta) M(\theta)'
\end{aligned}
$$

  step 2: forecasting y_t via measurement equations

$$
\begin{aligned}
y_{t|t-1} &= D(\theta) + Z(\theta) s_{t|t-1} \\
F_{t|t-1} &= Z(\theta) P_{t|t-1} Z(\theta)' + \Sigma_u(\theta)
\end{aligned}
$$

  step 3: filtering s_t via Bayes' Theorem

$$
\begin{aligned}
s_{t|t} &= s_{t|t-1} + P_{t|t-1} Z(\theta)' F_{t|t-1}^{-1} (y_t - y_{t|t-1}) \\
P_{t|t} &= P_{t|t-1} - P_{t|t-1} Z(\theta)' F_{t|t-1}^{-1} Z(\theta) P_{t|t-1}
\end{aligned}
$$

- Likelihood evaluation: $p(y_{1:T}|\theta) = \prod_{t=1}^{T} p_{\mathbb{N}}(y_t | y_{t|t-1}, F_{t|t-1})$

MATLAB Pseudo-Code

```
function [m_fs,V_fs,loglik] = KalmanFilter(Y,SSR)
[...]
for t = 1:T
    % Period-(t-1) predictive density
    m_ps = C+G*m_fs(:,t-1);
    V_ps = G*V_fs(:,:,t-1)*G'+M*V_e*M';

    % Period-t log likelihood
    m_py = D+Z*m_ps;
    V_py = Z*V_ps*Z'+V_u;
    loglik(t) = mvt_pdf(Y(t,:),m_py',V_py,inf);

    % Period-t filtering density
    gain = (V_ps*Z')/V_py;
    m_fs(:,t) = m_ps+gain*(Y(t,:)'-m_py);
    V_fs(:,:,t) = V_ps-gain*Z*V_ps;
end
```