Lecture 9 Time Series

Fei Tan

Department of Economics
Chaifetz School of Business
Saint Louis University

E6930 Applied Bayesian Statistics

April 10, 2023

State Space Representation

Gaussian linear model + Kalman filter

$$\begin{array}{rrcll} \text{Measurement:} & y_t & = & D + Z\epsilon_t + v_t, & v_t \sim \mathcal{N}(0, \Sigma_v) \\ \text{Transition:} & \epsilon_t & = & C + G\epsilon_{t-1} + u_t, & u_t \sim \mathcal{N}(0, \Sigma_u) \end{array}$$

1. forecast ϵ_t via transition: $\epsilon_t | y_{1:t-1} \sim \mathcal{N}(\epsilon_{t|t-1}, P_{t|t-1})$

$$\begin{aligned} \epsilon_{t|t-1} &= C + G\epsilon_{t-1|t-1} \\ P_{t|t-1} &= GP_{t-1|t-1}G' + \Sigma_u \end{aligned}$$

2. forecast y_t via measurement: $y_t | y_{1:t-1} \sim \mathcal{N}(y_{t|t-1}, F_{t|t-1})$

$$\begin{aligned} y_{t|t-1} &= D + Z\epsilon_{t|t-1} \\ F_{t|t-1} &= ZP_{t|t-1}Z' + \Sigma_v \end{aligned}$$

3. filter ϵ_t via Bayes theorem: $\epsilon_t | y_{1:t} \sim \mathcal{N}(\epsilon_{t|t}, P_{t|t})$

$$\begin{aligned} \epsilon_{t|t} &= \epsilon_{t|t-1} + P_{t|t-1}Z'F_{t|t-1}^{-1}(y_t - y_{t|t-1}) \\ P_{t|t} &= P_{t|t-1} - P_{t|t-1}Z'F_{t|t-1}^{-1}ZP_{t|t-1} \end{aligned}$$

The Road Ahead...

- Autoregressive model
- Regime-switching model

Autoregressive (AR) Model

Chib's (1993) regression with AR(p) error

$$
\begin{aligned}
y_t &= x_t'\beta + \epsilon_t, \quad t = 1, \ldots, T \\
\phi(L)\epsilon_t &= u_t, \quad \phi(L) = 1 - \phi_1 L - \cdots - \phi_p L^p, \quad L\epsilon_t = \epsilon_{t-1}
\end{aligned}
$$

- Likelihood function under $u_t \sim_{i.i.d.} \mathcal{N}(0, \sigma^2)$

$$
f(y_{p+1:T}|y_{1:p}, \beta, \sigma^2, \phi) \propto \sigma^{-(n-p)} \exp\left[-\frac{1}{2\sigma^2} \sum_{t=p+1}^{T} (y_t - y_{t|t-1})^2\right]
$$

where $y_{t|t-1} = (1 - \phi(L))y_t + \phi(L)x_t'\beta$

- Joint prior (stationarity: $\Phi_s = \{\phi : \phi(z) \neq 0, \ \forall |z| < 1\}$)

$$
\pi(\beta, \sigma^2, \phi) = \underbrace{\mathcal{N}(\beta_0, \sigma^2 B_0)}_{\pi(\beta|\sigma^2)} \underbrace{\mathcal{IG}\text{-}2(\nu_0/2, \delta_0/2)}_{\pi(\sigma^2)} \underbrace{\mathcal{N}(\phi_0, \Phi_0) 1\{\phi \in \Phi_s\}}_{\pi(\phi)}
$$

Gibbs Algorithm

- Gibbs sampler for $\pi(\beta, \sigma^2, \phi | y)$

$$
\begin{aligned}
\beta | y, \sigma^2, \phi &\sim \mathcal{N}(\beta_1, \sigma^2 B_1) \\
\sigma^2 | y, \beta, \phi &\sim \mathcal{IG}\text{-}2(\nu_1/2, \delta_1/2) \\
\phi | y, \beta, \sigma^2 &\sim \mathcal{N}(\phi_1, \Phi_1) 1\{\phi \in \Phi_s\}
\end{aligned}
$$

  where

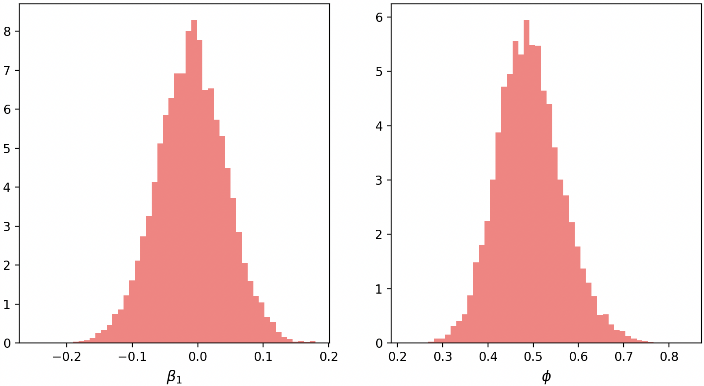
$$
\begin{aligned}
B_1 &= (X^{*\prime} X^* + B_0^{-1})^{-1}, \quad x_t^* = \phi(L) x_t \\
\beta_1 &= B_1 (X^{*\prime} y^* + B_0^{-1} \beta_0), \quad y_t^* = \phi(L) y_t \\
\nu_1 &= \nu_0 + k + T - p, \quad k = \text{dim}(x_t) \\
\delta_1 &= \delta_0 + (\beta - \beta_0)' B_0^{-1} (\beta - \beta_0) + (y^* - X^* \beta)' (y^* - X^* \beta) \\
\Phi_1 &= (\sigma^{-2} E' E + \Phi_0^{-1})^{-1}, \quad \epsilon = E\phi + u \text{ (known given } y, \beta, \sigma^2) \\
\phi_1 &= \Phi_1 (\sigma^{-2} E' \epsilon + \Phi_0^{-1} \phi_0)
\end{aligned}
$$

- Extension to ARMA(p, q) error [Chib & Greenberg (1994)]

Python Code

```
def ar_err(y, x, s, b0, B0, nu0, d0, p0, P0):
    for i in range(1, s):
        # Sample beta
        y2 = filter(y, s['phi'][i - 1, :])
        x2 = filter(x, s['phi'][i - 1, :])
        ...
        s['beta'][i, :] = multivariate_normal.rvs(
            size=1, mean=b1, cov=B1)
        # Sample sig2
        ...
        s['sig2'][i] = invgamma.rvs(nu1 / 2, size
            =1, scale=d1 / 2)
        # Sample phi
        for t in range(T - p):
            for j in range(p):
                E[t, j] = err[t + p - j - 1]
        ...
        s['phi'][i, :] = multivariate_normal.rvs(
            size=1, mean=p1, cov=P1)
    return s
```

Application: Phillips Curve



- $\pi_t = \beta_0 + \beta_1 u_t + \epsilon_t$, $\epsilon_t = \phi \epsilon_{t-1} + u_t$, 2000–2022 monthly
- $\beta_0, \beta_1 \sim \mathcal{N}(0, \sigma^2)$, $\sigma^2 \sim \mathcal{IG}-2(\frac{5}{2}, \frac{5}{2})$, $\phi \sim \mathcal{N}(0.5, 0.15)$
- Phillips curve has flattened considerably since 2000

Regime-Switching Model

Chib's (1996) Markov mixture model

$$
\begin{aligned}
f(y_t | y_{1:t-1}, s_{t-1}, \theta) &= \sum_{k=1}^{m} f(y_t | y_{1:t-1}, \theta_k) p(s_t = k | s_{t-1}) \\
p(s_t = j | s_{t-1} = i) &= p_{ij}, \quad s_t \in \{1, \ldots, m\}, \quad t = 1, \ldots, T
\end{aligned}
$$

- Chib's (1998) change-point model

$$
P = \begin{bmatrix}
p_{11} & 1 - p_{11} & 0 & \cdots & 0 & 0 \\
0 & p_{22} & 1 - p_{22} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & p_{mm} & 1 - p_{mm} \\
0 & 0 & 0 & \cdots & 0 & 1
\end{bmatrix}
$$

- Which specification is more general?

Gibbs Algorithm

- Sample $s_{1:T}$ in one block

$$p(s_{1:T}|y_{1:T}) = p(s_T|y_{1:T}) \cdots p(s_t|y_{1:T}, s_{t+1:T}) \cdots p(s_1|y_{1:T}, s_{2:T})$$
$$p(s_t|y_{1:T}, s_{t+1:T}) \propto p(s_t|y_{1:t})p(s_{t+1}|s_t), \quad \text{where for } t = 1, \ldots, T$$

  - prediction step: for $j = 1, \ldots, m$

$$p(s_t = j|y_{1:t-1}) = \sum_{k=1}^{m} p(s_t = j|s_{t-1} = k)p(s_{t-1} = k|y_{1:t-1})$$

  - update step: for $j = 1, \ldots, m$

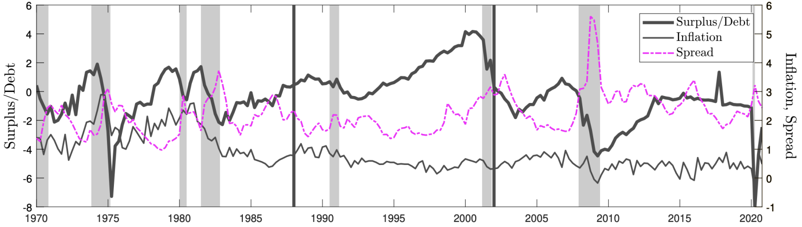
$$p(s_t|y_{1:t}) \propto p(s_t = j|y_{1:t-1})f(y_t|y_{1:t-1}, \theta_j)$$

- Sample each row of P from Dirichlet distribution

$$p_i \sim \mathcal{D}(\alpha_{i1}, \ldots, \alpha_{im}) \quad \Rightarrow \quad p_i|s_{1:T} \sim \mathcal{D}(\alpha_{i1} + n_{i1}, \ldots, \alpha_{im} + n_{im})$$

where n_{ij} = # of one-step transitions $i \to j$ in $s_{1:T}$

Application: Debt Cycles



- $y_t = \Phi_{0,s_t} + \Phi_{1,s_t} y_{t-1} + \Phi_{2,s_t} y_{t-2} + u_t,\ u_t \sim \mathcal{N}(0, \Sigma_{s_t}),$

$$P = \begin{bmatrix} p_{11} & 1 - p_{11} & 0 \\ 0 & p_{22} & 1 - p_{22} \\ 0 & 0 & 1 \end{bmatrix}$$

- 1970–2020 quarterly data: GDP growth rate, inflation rate, nominal interest rate, surplus-debt ratio, credit spread
- Tan (2022), “Appetite for Treasuries, Debt Cycles, and Fiscal Inflation,” manuscript

Readings

- Chib (1993), "Bayes Regression with Autoregressive Errors: A Gibbs Sampling Approach," *Journal of Econometrics*
- Chib & Greenberg (1994), "Bayes Inference in Regression Models with ARMA(p, q) Erros," *Journal of Econometrics*
- Chib (1996), "Calculating Posterior Distributions and Modal Estimates in Markov Mixture Models," *Journal of Econometrics*
- Chib (1998), "Estimation and Comparison of Multiple Change-Point Models," *Journal of Econometrics*