Solving Generalized Multivariate Linear Rational Expectations Models*

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Abstract

We generalize the linear rational expectations solution method of Whiteman (1983) to the multivariate case. This facilitates the use of a generic exogenous driving process that must only satisfy covariance stationarity. Multivariate cross-equation restrictions linking the Wold representation of the exogenous process to the endogenous variables of the rational expectations model are obtained. We argue that this approach offers important insights into rational expectations models. We give two examples in the paper—an asset pricing model with incomplete information and a monetary model with observationally equivalent monetary-fiscal policy interactions. We relate our solution methodology to other popular approaches to solving multivariate linear rational expectations models, and provide user-friendly code that executes our approach.

Keywords: Solution Methods; Analytic Functions; Rational Expectations.

JEL Classification: C32, C62, C65, E63

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1 Introduction

Whiteman (1983) lays out a solution principle for solving stationary, linear rational expectations models. The four tenets of the solution principle are: [i.] Exogenous driving processes are taken to be zero-mean linearly regular covariance stationary stochastic processes with known Wold representation; [ii.] Expectations are formed rationally and are computed using Wiener-Kolmogorov formula; [iii.] Solutions are sought in the space spanned by time-independent square-summable linear combinations of the process fundamental for the driving process; [iv.] The rational expectations restrictions are required to hold for all realizations of the driving processes. The purpose of this paper is to extend Whiteman's solution principle to the multivariate setting.

The solution principle is general in the sense that the exogenous driving processes are assumed to only satisfy covariance stationarity. Solving for a rational expectations equilibrium is nontrivial under this assumption and Whiteman demonstrates how powerful z-transform techniques can be in deriving the appropriate fixed point conditions.

The techniques advocated in Whiteman (1983) are not well known. This could be because the literature contains several well-vetted solution procedures for linearized rational expectations models (e.g., Sims (2001b)) or because the solution procedure requires working knowledge of concepts unfamiliar to economists (e.g., z-transforms). We provide an introduction to these concepts and argue that there remain several advantages of Whiteman's approach on both theoretical and applied grounds. First, the approach only assumes that the exogenous driving processes possess a Wold representation, allowing for a relaxation of the standard assumption that exogenous driving processes follow an autoregressive process of order one, AR(1), specification. As recently emphasized in Curdia and Reis (2012), no justification is typically given for the AR(1) specification with little exploration into alternative stochastic processes despite obvious benefits to such deviations.[1] Second, models with incomplete information or heterogeneous beliefs are easier to solve using the z-transform approach advocated by Whiteman. Kasa (2000) and Walker (2007) show how these methods can be used to generate analytic solutions to problems that were approximated by Townsend (1983) and Singleton (1987).[2] Third, as shown in Kasa (2001) and Lewis and Whiteman (2008), the approach can easily be extended to allow for robustness as advocated by Hansen and Sargent (2011) or rational inattention as advocated by Sims (2001a). Finally, there are potential insights into the econometrics of rational expectations models. Qu and Tkachenko (2012) demonstrate how working in the frequency-domain can deliver simple identification conditions.

The contribution of the paper is to extend the approach of Whiteman (1983) to the multivariate setting and (re)introduce users of linear rational expectations models to the analytic function

[1]This is true despite the fact that Kydland and Prescott (1982), the paper that arguably started the real business cycle literature, contains an interesting deviation from the AR(1) specification.

[2]Taub (1989), Kasa, Walker, and Whiteman (2013), Rondina (2009), and Rondina and Walker (2013) also use the space of analytic functions to characterize equilibrium in models with informational frictions. Seiler and Taub (2008), Bernhardt and Taub (2008), and Bernhardt, Seiler, and Taub (2009) show how these methods can be used to accurately approximate asymmetric information equilibria in models with richer specifications of information.

solution technique. We provide sufficient (though not exhaustive) background by introducing a few key theorems in Section 2.1 and walking readers through the univariate example of Whiteman (1983) in Section 2.2. Section 3 establishes the main result of the paper. There is a chapter devoted to multivariate analysis in Whiteman (1983) that has known errors (see, Onatski (2006) and Sims (2007)). Section 3.3 provides an example of these errors and demonstrates why our approach does not suffer from the same setback. In effect, our approach is a straightforward way to maintain the methodology of Whiteman and provides robust existence and uniqueness criteria. Finally, Section 4 provides a few examples that demonstrate the usefulness of solving linear rational expectations models in the frequency-domain. An online Appendix B provides a user's guide to the MATLAB code that executes the solution procedure.

2 Preliminaries

Elementary results of the theory of stationary stochastic processes and the residue calculus are necessary for grasping the z-transform approach advocated here. This section introduces a few important theorems that are relatively well known but is by no means exhaustive. Interested readers are directed to Brown and Churchill (2013) and Whittle (1983) for good references on complex analysis and stochastic processes, and Kailath (1980) for results on matrix polynomials.[3]

2.1 A Few Useful Theorems

The first principle of Whiteman's solution procedure assumes that the exogenous driving processes are zero-mean linear covariance stationary stochastic processes with no other restrictions imposed. The Wold representation theorem allows for such a general specification.

Theorem 1. *[Wold Representation Theorem] Let $\{x_t\}$ be any $(n \times 1)$ covariance stationary stochastic process with $\mathbb{E}(x_t) = 0$. Then it can be uniquely represented in the form*

$$x_t = \eta_t + A(L)\varepsilon_t \tag{1}$$

where $A(L)$ is a matrix polynomial in the lag operator with $A(0) = I_n$ and $\sum_{s=1}^{\infty} A_s A_s'$ is convergent. The process ε_t is n-variate white noise with $\mathbb{E}(\varepsilon_t) = 0$, $\mathbb{E}(\varepsilon_t \varepsilon_t') = \Sigma$ and $\mathbb{E}(\varepsilon_t \varepsilon_{t-m}') = 0$ for $m \neq 0$. The process ε_t is the innovation in predicting x_t from its own past:

$$\varepsilon_t = x_t - \mathbb{P}[x_t | x_{t-1}, x_{t-2}, ...] \tag{2}$$

where $\mathbb{P}[\cdot]$ denotes linear projection. The process η_t is linearly deterministic; there exists an n vector c_0 and $n \times n$ matrices C_s such that without error $\eta_t = c_0 + \sum_{s=1}^{\infty} C_s \eta_{t-s}$ and $\mathbb{E}[\varepsilon_t \eta_{t-m}'] = 0$ for all m.

[3] Sargent (1987) provides a good introduction to these concepts and discusses economic applications.

The Wold representation theorem states that *any* covariance stationary process can be written as a linear combination of a (possibly infinite) moving average representation where the innovations are the linear forecast errors for x_t and a process that can be predicted arbitrarily well by a linear function of past values of x_t. The theorem is a *representation* determined by second moments of the stochastic process only and therefore may not fully capture the data generating process. For example, that the decomposition is linear suggests that a process could be deterministic in the strict sense and yet linearly non-deterministic; Whittle (1983) provides examples of such processes. The innovations in the Wold representation are generated by linear projections which need not be the same as the conditional expectation ($\mathbb{E}[x_t|x_{t-1}, x_{t-2}, ...]$). However, our focus here will be on linear Gaussian stochastic processes as is standard in the rational expectations literature. Under this assumption, the best conditional expectation coincides with linear projection.

The second principle advocated by Whiteman is that expectations are formed rationally and are computed using Wiener-Kolmogorov optimal prediction formula. Consider minimizing the forecast error associated with the k-step ahead prediction of $x_t = A(L)\varepsilon_t = \sum_{j=0}^{\infty} a_j \varepsilon_{t-j}$ by choosing $y_t = C(L)\varepsilon_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}$:

$$\begin{aligned}
\min_{y_t} \mathbb{E}(x_{t+k} - y_t)^2 &= \min_{\{c_j\}} \mathbb{E}\left(L^{-k}\sum_{j=0}^{\infty} a_j\varepsilon_{t-j} - \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}\right)^2 \\
&= \min_{\{c_j\}} \mathbb{E}\left(\sum_{j=0}^{k-1} a_j \varepsilon_{t+k-j} + \sum_{j=0}^{\infty}(a_{j+k} - c_j)\varepsilon_{t-j}\right)^2 \\
&= \sigma_\varepsilon^2 \sum_{j=0}^{k-1} a_j^2 + \sigma_\varepsilon^2 \sum_{j=0}^{\infty}(a_{j+k} - c_j)^2 \qquad (3)
\end{aligned}$$

Obviously, (3) is minimized by setting $c_j = a_{j+k}$, which yields the mean-square forecast error of $\sigma_\varepsilon^2 \sum_{j=0}^{k-1} a_j^2$. Due to the Riesz-Fischer Theorem, this sequential problem has an equivalent representation as a functional problem.

Theorem 2. *[Riesz-Fischer] Let $D(\sqrt{r})$ denote a disk in the complex plane of radius $\sqrt{r}$ centered at the origin. There is an equivalence (i.e. an isometric isomorphism) between the space of r-summable sequences $\sum_j r^j |f_j|^2 < \infty$ and the Hardy space of analytic functions $f(z)$ in $D(\sqrt{r})$ satisfying the restriction*

$$\frac{1}{2\pi i}\oint f(z) f(rz^{-1})\frac{dz}{z} < \infty$$

where $\oint$ denotes (counterclockwise) contour integration around $D(\sqrt{r})$. An analytic function satisfying the above condition is said to be r-integrable.[4]

The Riesz-Fischer theorem implies that the optimal forecasting rule can be derived by finding

[4]This theorem is usually proved for the case $r = 1$ and for functions defined on the boundary of a disk. For further exposition see Sargent (1987).

the the analytic function $C(z)$ on the unit disk $|z| \leq 1$ corresponding to the z-transform of the $\{c_j\}$ sequence, $C(z) = \sum_{j=0}^{\infty} c_j z^j$, that solves

$$\min_{C(z)\in H^2} \frac{1}{2\pi i} \oint |z^{-k}A(z) - C(z)|^2 \frac{dz}{z} \tag{4}$$

where H^2 denotes the Hardy space of square-integrable analytic functions on the unit disk, and $\oint$ denotes (counterclockwise) integration about the unit circle. The restriction $C(z) \in H^2$ ensures that the forecast is casual (i.e., that the forecast contains no future values of ε's).

The sequential forecasting rule, $c_j = a_{j+k}$, has the functional equivalent

$$C(z) = \sum_{j=0}^{\infty} c_j z^j = \left[\frac{A(z)}{z^k}\right]_+ \tag{5}$$

where $A(z) = \sum_{j=0}^{\infty} a_j z^j$ and the operator $[\cdot]_+$ is defined, for a sum that contains both positive and negative powers of z, as the sum containing only the nonnegative powers of z.[5] The beauty of the prediction formula (5) is its generality. It holds for any covariance stationary stochastic process. As an example, consider the AR(1) case, $x_t = \rho x_{t-1} + \varepsilon_t$ with $|\rho| < 1$. Here $A(z) = (1 - \rho z)^{-1}$ and (5) yields

$$\begin{aligned} C(z) &= \left[\frac{1}{(1-\rho z)z^k}\right]_+ = [z^{-k}(1 + \rho z + \rho^2 z^2 + \cdots)]_+ \\ &= \rho^k(1 + \rho z + \rho^2 z^2 + \cdots) = \frac{\rho^k}{1-\rho z} \end{aligned}$$

which delivers the well-known least-square predictor $\rho^k x_t$.[6]

The third principle assumes that solutions are sought in the space spanned by the time-independent square-summable linear combinations of the process fundamental for the driving process. Consider the moving average process $x_t = A(L)u_t$; the innovations u_t are said to be fundamental for the x_t process if $u_t \in \overline{\text{span}}\{x_{t-k}, k \geq 0\}$, i.e., if the innovations span the same space as the current and past observables. By construction, the innovations in the Wold representation theorem are fundamental. This implies that for any covariance stationary exogenous driving process, there will always exist a *unique* fundamental representation.[7]

Following Whiteman (1983), our solution procedure takes advantage of matrix polynomial factorization, in particular the Smith (or canonical) form decomposition. The following theorem and its proof and corollaries can be found in Kailath (1980).

[5]For a detailed derivation of (5) from (4), see Lewis and Whiteman (2008).

[6]It is often more convenient to express prediction formulas in terms of the x series as opposed to past forecast errors as in (5). If the process has an autoregressive representation, then one may write the prediction formula as $B(L)x_t$, where $B(z) = A(z)^{-1}[z^{-k}A(z)]_+$.

[7]The spanning conditions prove extremely convenient for backing out the information content of exogenous and endogenous variables in dynamic, incomplete information rational expectations equilibria.

Theorem 3. *[Smith Form] For any $m \times n$ polynomial matrix $P(z) = \sum_{j=0}^{s} P_j z^j$ there exists elementary row and column operations, or corresponding unimodular matrices $U(z)$ and $V(z)$ such that*

$$U(z)P(z)V(z) = \Lambda(z) \tag{6}$$

with

$$\Lambda(z) = \left(\begin{array}{cccc|c} \lambda_1(z) & 0 & \dots & & \\ 0 & \ddots & & & 0 \\ \vdots & & & \lambda_r(z) & \\ \hline & & 0 & & 0 \end{array}\right) \tag{7}$$

where r is the (normal) rank of $P(z)$ and the $\lambda_i(z)$'s are unique monic scalar polynomials such that $\lambda_i(z)$ is divisible by $\lambda_{i-1}(z)$; $U(z)$ and $V(z)$ are matrix polynomials of sizes $m \times m$ and $n \times n$, with constant nonzero determinants.

This decomposition is useful because it allows us to isolate the roots of the polynomial matrix $P(z)$ and identify roots inside (and outside) the unit circle as shown in the following corollary.

Corollary 4. *If $P(z)$ is a square polynomial matrix whose determinant is nonzero on the unit circle and $P(0)$ is nonsingular, then $P(z)$ can be written as $P(z) = S(z)T(z)$ where the roots of det $S(z)$ are inside the unit circle and those of det $T(z)$ are outside the unit circle.*

Given that $U(z)$ and $V(z)$ are unimodular, $U(z)^{-1}$ and $V(z)^{-1}$ exist. Factor each of the polynomials $\lambda_i(z)$ such that the roots of $\underline{\lambda}_i(z)$ are inside the unit circle and those of $\overline{\lambda}_i(z)$ are outside. Therefore we can write $P(z) = S(z)T(z)$ where $S(z) = U(z)^{-1}\text{diag}(\underline{\lambda}_1(z), ..., \underline{\lambda}_q(z))$ and $T(z) = \text{diag}(\overline{\lambda}_1(z), ..., \overline{\lambda}_q(z))V(z)^{-1}$.

2.2 Univariate Case It is instructive to work through a univariate example of Whiteman (1983). There is nothing new here but it will set the stage for the generalization in the next section. Consider the following generic rational expectations model

$$\mathbb{E}_t y_{t+1} - (\rho_1 + \rho_2)y_t + \rho_1\rho_2 y_{t-1} = x_t, \qquad x_t = A(L)\varepsilon_t, \qquad \varepsilon_t \overset{iid}{\sim} N(0,1) \tag{8}$$

where ε_t is assumed to be fundamental for x_t (i.e., $A(L)$ is assumed to have a one-sided inverse in non-negative powers of L). Following the solution principle, we will look for a solution that is square-summable in the Hilbert space generated by the fundamental shock ε, $y_t = C(L)\varepsilon_t$ (third tenet). If we invoke the optimal prediction formula (5), then $\mathbb{E}_t y_{t+1} = [C(L)/L]_+\varepsilon_t = L^{-1}[C(L) - C_0]\varepsilon_t$. Together with the fourth tenet of the solution principle (i.e., that the rational expectation restrictions hold for all realizations of ε), this implies that (8) can be written in z-transform as

$$z^{-1}[C(z) - C_0] - (\rho_1 + \rho_2)C(z) + \rho_1\rho_2 z C(z) = A(z)$$

Multiplying by z and rearranging delivers

$$C(z) = \frac{zA(z) + C_0}{(1 - \rho_1 z)(1 - \rho_2 z)} \tag{9}$$

Appealing to the Riesz-Fischer Thereom, square-summability (stationarity) in the time domain is tantamount to analyticity of $C(z)$ on the unit disk.

As shown in Whiteman (1983), there are three cases one must consider. First, assume that $|\rho_1| < 1$ and $|\rho_2| < 1$. Then (9) is an analytic function on $|z| < 1$ and the representation is given by

$$y_t = \left[\frac{LA(L) + C_0}{(1 - \rho_1 L)(1 - \rho_2 L)}\right]\varepsilon_t \tag{10}$$

For any finite value of C_0, this is a solution that lies in the Hilbert space generated by $\{x_t\}$ and satisfies the tenets of the solution principle. Thus, no unique solution exists because C_0 can be set arbitrarily.

The second case to consider is $|\rho_1| < 1 < |\rho_2|$. In this case, the function $C(z)$ has an isolated singularity at ρ_2^{-1}, implying that $C(z)$ is not analytic on the unit disk. In this case, the free parameter C_0 can be set to remove the singularity at ρ_2^{-1} by setting C_0 in such a way as to cause the residue of $C(\cdot)$ to be zero at ρ_2^{-1}

$$\lim_{z \to \rho_2^{-1}} (1 - \rho_2 z)C(z) = \frac{\rho_2^{-1}A(\rho_2^{-1}) + C_0}{1 - \rho_1\rho_2^{-1}} = 0$$

Solving for C_0 delivers $C_0 = -\rho_2^{-1}A(\rho_2^{-1})$. Substituting this into (10) yields the following rational expectations equilibrium

$$y_t = \left[\frac{LA(L) - \rho_2^{-1}A(\rho_2^{-1})}{(1 - \rho_2 L)(1 - \rho_1 L)}\right]\varepsilon_t \tag{11}$$

This is the unique solution that lies in the Hilbert space generated by $\{x_t\}$. The solution is the ubiquitous Hansen-Sargent prediction formula that clearly captures the cross-equation restrictions that are the "hallmark of rational expectations models" [Hansen and Sargent (1980)].

The final case to consider is $1 < |\rho_1|$ and $1 < |\rho_2|$. In this case, (9) has two isolated singularities at ρ_1^{-1} and ρ_2^{-1}, and C_0 cannot be set to remove both singularities.[8] Hence in this case, there is no

[8] As discussed by Whiteman (1983), the problem remains even if $\rho_1 = \rho_2$.

solution in the Hilbert space generated by $\{x_t\}$.

3 Generalization

This section extends the univariate solution method of Whiteman (1983) to the multivariate case. We also document how our approach is not susceptible to situations in which Whiteman's multivariate solution method falls apart.

3.1 Multivariate Case The multivariate linear rational expectations models can be cast in the form of

$$\mathbb{E}_t\left[\sum_{k=-n}^{m}\Gamma_k L^k y_t\right] = \mathbb{E}_t\left[\sum_{k=-n}^{l}\Psi_k L^k x_t\right] \tag{12}$$

where L is the lag operator: $L^k y_t = y_{t-k}$, y_t is a $(p\times 1)$ vector of endogenous variables, $\{\Gamma_k\}_{k=-n}^{m}$ and $\{\Psi_k\}_{k=-n}^{l}$ are $(p\times p)$ and $(p\times q)$ matrix coefficients, and $\mathbb{E}_t$ represents mathematical expectation given information available at time t including the model's structure and all past and present realizations of the exogenous and endogenous processes.[9] x_t is a $(q\times 1)$ vector covariance stationary exogenous driving process with known Wold representation

$$x_t = \sum_{k=0}^{\infty} A_k \varepsilon_{t-k} \equiv A(L)\varepsilon_t \tag{13}$$

where $\varepsilon_t = x_t - \mathbb{P}[x_t|x_{t-1}, x_{t-2}, \ldots]$ and $\mathbb{P}[x_t|x_{t-1}, x_{t-2}, \ldots]$ is the optimal linear predictor for x_t conditional on observing $\{x_{t-j}\}_{j=1}^{\infty}$. Also, each element of $\sum_{k=0}^{\infty} A_k A_k'$ is finite.

To illustrate how we get a model into the form of (12), consider the standard stochastic growth model with log preferences, inelastic labor supply, complete depreciation of capital, and Cobb-Douglas technology. The Euler equation and aggregate resource constraint, after log-linearizing, reduce to the following bivariate system in (c_t, k_t) which must hold for $t = 0, 1, 2, \ldots$, i.e.

$$E_t c_{t+1} = c_t + (\alpha-1)k_t + E_t a_{t+1}$$
$$\frac{1-\alpha\beta}{\alpha\beta}c_t + k_t = \frac{1}{\alpha\beta}a_t + \frac{1}{\beta}k_{t-1}$$

where (α, β) are parameters of preference and technology and a_t represents the technology shock. We can rewrite the above bivariate system into the form of (12)

[9] While not studied explicitly here, the approach taken in this paper can easily be adapted to study models with "sticky information" [Mankiw and Reis (2002)] or "withholding equations" [Whiteman (1983)] by replacing $\mathbb{E}_t$ with $\mathbb{E}_{t-j}$ for any finite j, or models with perfect foresight. Indeed, the inclusion of l periods of lags for exogenous driving processes already allows for the possibility that agents have foresight about some of the future endogenous variables.

$$\mathbb{E}_t\left[\left(\underbrace{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}}_{\Gamma_{-1}}L^{-1}+\underbrace{\begin{pmatrix}-1 & 1-\alpha\\ \frac{1-\alpha\beta}{\alpha\beta} & 1\end{pmatrix}}_{\Gamma_0}L^0+\underbrace{\begin{pmatrix}0 & 0\\ 0 & -\frac{1}{\beta}\end{pmatrix}}_{\Gamma_1}L\right)\underbrace{\begin{pmatrix}c_t\\ k_t\end{pmatrix}}_{y_t}\right]$$
$$=\mathbb{E}_t\left[\left(\underbrace{\begin{pmatrix}1\\ 0\end{pmatrix}}_{\Psi_{-1}}L^{-1}+\underbrace{\begin{pmatrix}0\\ \frac{1}{\alpha\beta}\end{pmatrix}}_{\Psi_0}L^0\right)\underbrace{a_t}_{x_t}\right]$$

where $n = m = 1$, $l = 0$, $p = 2$, and $q = 1$.

Analogous to the univariate solution procedure, we exploit the properties of polynomial matrices to establish conditions for the existence and uniqueness of solutions to multivariate linear rational expectations models driven by general exogenous driving processes. Following tenet [iii.], the solution will be sought in the space spanned by current and past ε. That is, we look for an equilibrium y_t to (12) that is of the form

$$y_t = \sum_{k=0}^{\infty} C_k \varepsilon_{t-k} \equiv C(L)\varepsilon_t \tag{14}$$

where $\{y_t\}$ is taken to be covariance stationary. Note that such moving average representation of the solution is convenient because it *is* the impulse response function. For example, the term $C_k(i,j)$ measures exactly the response of $y_{t+k}(i)$ to a shock $\varepsilon_t(j)$

$$(\mathbb{E}_t - \mathbb{E}_{t-1})y_{t+k}(i) = C_k(i,j)\varepsilon_t(j)$$

where $C_k(i,j)$ denotes the (i,j)-th element of C_k, $y_{t+k}(i)$ denotes the i-th component of y_{t+k}, and $\varepsilon_t(j)$ denotes the j-th component of ε_t.

3.2 Solution Procedure If we define η_t (resp. ν_t) as a $(p \times 1)$ vector of endogenous (resp. exogenous) expectational errors, satisfying $\eta_{t+k} = y_{t+k} - \mathbb{E}_t y_{t+k}$ (resp. $\nu_{t+k} = x_{t+k} - \mathbb{E}_t x_{t+k}$) for all $k > 0$ and hence $\mathbb{E}_t \eta_{t+k} = 0$ (resp. $\mathbb{E}_t \nu_{t+k} = 0$), then we may write (12) as

$$\sum_{k=-n}^{m} \Gamma_k L^k y_t = \sum_{k=-n}^{l} \Psi_k L^k x_t + \sum_{k=1}^{n} (\Gamma_{-k}\eta_{t+k} - \Psi_{-k}\nu_{t+k}) \tag{15}$$

Similar to Sims (2001b), it should be noted that the η terms are not given exogenously, but are instead determined as part of the model solution.

First, rewrite model (15) as

$$\Gamma(L)y_t = \Psi(L)x_t + \sum_{k=1}^{n} (\Gamma_{-k}\eta_{t+k} - \Psi_{-k}\nu_{t+k})$$

where $\Gamma(L) = \sum_{k=-n}^{m} \Gamma_k L^k$ and $\Psi(L) = \sum_{k=-n}^{l} \Psi_k L^k$. Applying (14) and the Wiener-Kolmogorov optimal prediction formula gives

$$\eta_{t+k} = y_{t+k} - E_t y_{t+k} = L^{-k}\left(\sum_{i=0}^{k-1} C_i L^i\right)\varepsilon_t$$
$$\nu_{t+k} = x_{t+k} - E_t x_{t+k} = L^{-k}\left(\sum_{i=0}^{k-1} A_i L^i\right)\varepsilon_t$$

Substituting the above expressions, (13), and (14) into (15) gives

$$\Gamma(L)C(L)\varepsilon_t = \left\{\Psi(L)A(L) + \sum_{k=1}^{n}\left[\Gamma_{-k}L^{-k}\left(\sum_{i=0}^{k-1} C_i L^i\right) - \Psi_{-k}L^{-k}\left(\sum_{i=0}^{k-1} A_i L^i\right)\right]\right\}\varepsilon_t$$

which must hold for all realizations of ε. Thus, the z-transform equivalent satisfies

$$z^n\Gamma(z)C(z) = z^n\Psi(z)A(z) + \sum_{t=1}^{n}\sum_{s=t}^{n}[\Gamma_{-s}C_{t-1} - \Psi_{-s}A_{t-1}]z^{n-s+t-1}$$

Next, just as in the univariate case, we need to determine the location of the zeros of the complex polynomial matrix $z^n\Gamma(z)$. This is achieved via the Smith canonical decomposition

$$U(z)z^n\Gamma(z)V(z) = \begin{pmatrix} f_1(z) & 0 & \cdots & \\ 0 & f_2(z) & & \\ \vdots & & \ddots & \\ & & & f_p(z) \end{pmatrix} \tag{16}$$

where $f_1, \ldots, f_p$ are monic polynomials in z, $f_{k+1}(z)$ is divisible by $f_k(z)$ for $1 \le k \le p-1$, $U(z)$ is a product of elementary row matrices, and $V(z)$ is a product of elementary column matrices. For $i = 1, \ldots, p$, let

$$f_i(z) = \underbrace{\prod_{j=1}^{\underline{r}_i}(z - \underline{z}_{ij})^{\underline{m}_{ij}}}_{\underline{f}_i} \cdot \underbrace{\prod_{j=1}^{\overline{r}_i}(z - \overline{z}_{ij})^{\overline{m}_{ij}}}_{\overline{f}_i}$$

where $\underline{z}_{ij}$'s are complex-valued roots inside the unit circle with multiplicity $\underline{m}_{ij}$ and $\overline{z}_{ij}$'s are complex-valued roots on or outside the unit circle with multiplicity $\overline{m}_{ij}$. Then

$$z^n\Gamma(z) = \underbrace{U(z)^{-1}\begin{pmatrix} \underline{f}_1 & & & \\ & \underline{f}_2 & & \\ & & \ddots & \\ & & & \underline{f}_p \end{pmatrix}}_{S(z)}\underbrace{\begin{pmatrix} \overline{f}_1 & & & \\ & \overline{f}_2 & & \\ & & \ddots & \\ & & & \overline{f}_p \end{pmatrix}V(z)^{-1}}_{T(z)}$$

where $S(z)$ is a polynomial matrix such that all roots of $\det[S(z)]$ lie inside the unit circle while $T(z)$ is a polynomial matrix with all roots of $\det[T(z)]$ outside the unit circle. Therefore, we have

$$S(z)^{-1} = \begin{pmatrix} \frac{U_{1\cdot}(z)}{\prod_{k=1}^{\underline{r}_1}(z-\underline{z}_{1k})^{\underline{m}_{1k}}} \\ \frac{U_{2\cdot}(z)}{\prod_{k=1}^{\underline{r}_2}(z-\underline{z}_{2k})^{\underline{m}_{2k}}} \\ \vdots \\ \frac{U_{p\cdot}(z)}{\prod_{k=1}^{\underline{r}_p}(z-\underline{z}_{pk})^{\underline{m}_{pk}}} \end{pmatrix}$$

where $U_{j\cdot}(z)$ is the jth row of $U(z)$. Substituting this into the equilibrium gives

$$T_{j\cdot}(z)C(z) = \frac{U_{j\cdot}(z)}{\prod_{k=1}^{\underline{r}_j}(z-\underline{z}_{jk})^{\underline{m}_{jk}}}\left\{z^n\Psi(z)A(z) + \sum_{t=1}^{n}\sum_{s=t}^{n}[\Gamma_{-s}C_{t-1} - \Psi_{-s}A_{t-1}]z^{n-s+t-1}\right\} \tag{17}$$

for $j = 1, \ldots, p$. These functions are not analytic on the unit disk due to the singularities at $z = \underline{z}_{jk}$ for $k = 1, \ldots, \underline{r}_j$.

We have np^2 free parameters $C_0, C_1, \ldots, C_{n-1}$ that can be set to remove these singularities. Existence and uniqueness will depend upon the extent to which these parameters can be used to remove the singularities. As in the univariate case, the parameters will be set such that the numerator of the right hand side of (17) vanishes at $z = \underline{z}_{jk}$ for $k = 1, \ldots, \underline{r}_j$. This places restrictions on the unknown coefficients:

$$A_{j\cdot} = -R_{j\cdot}C, \qquad j = 1, \ldots, p$$

where $C = [C_0', C_1', \ldots, C_{n-1}']'$ and the detailed expressions of the matrices $\{A_{j\cdot}, R_{j\cdot}\}_{j=1}^p$ can be found in Appendix A. Stacking the above restrictions over $j = 1, \ldots, p$ gives

$$\underset{[r\times q]}{A} = -\underset{[r\times np]}{R}\ \underset{[np\times q]}{C} \tag{18}$$

where $r = \sum_{j=1}^{p}\sum_{k=1}^{\underline{r}_j}\underline{m}_{jk}$.

Lastly, we establish the existence and uniqueness conditions of the model solution. The properties of (18) determine whether the rational expectations equilibrium exists. Existence cannot be established if at least one column of A is outside of the space spanned by the columns of R—there are not enough free parameters coming from the forecast errors to remove all of the singularities associated with (17). The precise existence condition is that columns of A are spanned by the columns of R, i.e.

$$\text{span}(A) \subset \text{span}(R) \tag{19}$$

To check whether (19) is satisfied, we follow Sims (2001b). Let the singular value decompositions of A and R be given by $A = U_A S_A V_A'$ and $R = U_R S_R V_R'$. Then R's column space spans A's if and only if $(I - U_R U_R')U_A = 0$, in which case one candidate of C can be computed as

$$C = -V_R S_R^{-1} U_R' A \tag{20}$$

When (19) is satisfied, we can obtain the analytical solution for y_t which is indexed by $C_0, C_1, \ldots, C_{n-1}$

$$C(L)\varepsilon_t = (L^n \Gamma(L))^{-1} \left\{ L^n \Psi(L) A(L) + \sum_{t=1}^{n} \sum_{s=t}^{n} [\Gamma_{-s} C_{t-1} - \Psi_{-s} A_{t-1}] L^{n-s+t-1} \right\} \varepsilon_t \tag{21}$$

The above solution captures the multivariate cross-equation restrictions linking the Wold representation of the exogenous process, $A(L)$, to the endogenous variables of the model, $C(L)$. This mapping is a multivariate version of the celebrated Hansen-Sargent formula, and serves as a key ingredient in the analysis and econometric evaluation of dynamic rational expectations models.

The uniqueness of solution (21) requires that we be able to determine $\{C_k\}_{k=0}^{\infty}$ from the parameter restrictions supplied by (18). Since $V(\cdot)$ has constant nonzero determinant, it has full rank and contains no zeros. Thus, it is equivalent to determining the coefficients $\{D_k\}_{k=0}^{\infty}$ of $D(z) = V(z)^{-1}C(z)$, or the quantities $Q_{j\cdot}C$ for $j = 1, \ldots, p$, where the detailed expressions of the matrices $\{Q_{j\cdot}\}_{j=1}^{p}$ are given in Appendix A. Stacking over $j = 1, \ldots, p$ yields QC. Note that the matrix Q has exactly the same structure as R but evaluated at the inverses of those roots outside the unit circle. In order for the solution to be unique, our knowledge of (18) must pin down all the error terms in the system that are influenced by the endogenous expectational error. That is, we use RC to determine QC for arbitrary choice of C and the solution is unique if and only if

$$\text{span}(Q') \subset \text{span}(R') \tag{22}$$

In other words, determinacy of the solution requires that the columns of R' span the space spanned by the columns of Q', in which case we will have $QC = \Phi RC$ for some matrix Φ.[10]

A more intuitive and perhaps insightful interpretation on the preceding mechanism in pinning

[10]Similar to the space spanning condition for existence, (22) can be verified using the singular value decompositions of Q' and R'.

down the free parameters C is as follows. A rational expectations equilibrium is a fixed point of the mapping between the perceived law of motion and the actual law of motion generated by those perceptions. Any candidate for C designates a perceived law of motion, or agents' belief about how endogenous variables evolve, and a solution obtained by plugging such C into the equilibrium conditions represents a new law of motion. The solution is consistent if and only if the two laws of motion be identical, implying that the z-transform approach can be thought of as solving a fixed point problem. By applying this argument, the existence and uniqueness conditions simply state that there exists one and only one fixed point.

The solution method derived in this section is intimately related to many other approaches proposed in the literature. In particular, we will focus our attention only on its connection to one of the most popular solution methodologies, i.e. that of Sims (2001b), and summarize the results in the following theorem.

Theorem 5. *Consider the multivariate linear rational expectations model*[11]

$$(\Gamma_{-1}L^{-1} + \Gamma_0)y_t = \Psi_{-1}L^{-1}x_t + \Gamma_{-1}\eta_{t+1} \tag{23}$$

Assume that Γ_{-1} *is of full rank, and both the eigenvalues of* $-\Gamma_{-1}^{-1}\Gamma_0$ *and the roots of* $\det[\Gamma_{-1}+z\Gamma_0] = 0$ *are nonzero and distinct. Then*

1. *Factorization equivalence: the eigenvalues of* $-\Gamma_{-1}^{-1}\Gamma_0$ *are exactly the inverses of the corresponding roots of* $\det[\Gamma_{-1} + z\Gamma_0] = 0$*;*

2. *Existence equivalence: the restrictions imposed by the unstable eigenvalues in Sims (2001b) are exactly those imposed by the roots inside the unit circle in this paper.*

3. *Uniqueness equivalence: the conditions under which the solution to (23) is unique are equivalent between Sims (2001b) and this paper.*

This theorem shows an equivalence relation between the time-domain approach of Sims (2001b) and the frequency-domain approach of this paper. We remark that one of the major differences lies in the fact that the solution is expressed in autoregressive moving average form in Sims (2001b) while this paper gives the solution in moving average form. See Appendix A for its proof.

3.3 Connection to Whiteman (1983) We demonstrate how our approach is different from that of Whiteman (1983) with a specific example. The example shows where Whiteman's multivariate solution methodology goes astray and how our approach does not suffer from the same logical inconsistency. The first theorem of Chapter IV of Whiteman (1983) states:

[11] Since all variables are taken to be zero-mean linearly regular covariance stationary stochastic processes in this paper, the vector of constants in Sims (2001b) drops off from (23).

Theorem 6. *[Whiteman (1983)] Suppose the model is*

$$\mathbb{E}_t\left[\sum_{j=0}^{n} F_j L^{-j} + \sum_{j=1}^{m} G_j L^j\right] y_t = x_t \tag{24}$$

where y_t and x_t are $(q\times 1)$, F_j and G_j are $(q\times q)$, and x_t has Wold representation given by (13). Suppose further that F_n is of full rank, that the roots of

$$det\left[z^n\left(\sum_{j=0}^{n} F_j z^{-j} + \sum_{j=1}^{m} G_j z^j\right)\right] = \sum_{j=0}^{p} f_j z^j$$

are distinct, and that rq of these roots are inside the unit circle while the other $p-rq \leq (n+m)q-rq$ roots lie outside the unit circle. Then

1. *if $r < n$, there are many solutions to (24).*
2. *if $r = n$, there is one solution to (24).*
3. *if $r > n$, there is no solution to (24).*

As noted in Onatski (2006) Section 3.3, there is a logical inconsistency between this multivariate theorem and the univariate counterpart described in Section 2.2. The following example clarifies this point.[12] Consider the following model consistent with (24),

$$F_1\mathbb{E}_t y_{t+1} + F_0 y_t + G_1 y_{t-1} = x_t$$

$$\text{where}\quad F_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix},\quad F_0 = \begin{pmatrix} -(\rho_1+\rho_2) & 0 \\ 0 & -(\varphi_1+\varphi_2)\end{pmatrix},\quad G_1 = \begin{pmatrix} \rho_1\rho_2 & 0 \\ 0 & \varphi_1\varphi_2\end{pmatrix}$$

and assume that $A(L)$ is diagonal. This simplifies to a system of two unrelated equations

$$\mathbb{E}_t y_{1t+1} - (\rho_1+\rho_2) y_{1t} + \rho_1\rho_2 y_{1t-1} = x_{1t}$$
$$\mathbb{E}_t y_{2t+1} - (\varphi_1+\varphi_2) y_{2t} + \varphi_1\varphi_2 y_{2t-1} = x_{2t}$$

each of which can be solved individually without reference to the other. These equations are identical to (8) described in the univariate section and the solution procedures outlined there will hold. Therefore we can write

$$y_{1t} = \frac{LA_{11}(L) + C_0(1,1)}{(1-\rho_1 L)(1-\rho_2 L)}\varepsilon_{1t}, \qquad y_{2t} = \frac{LA_{22}(L) + C_0(2,2)}{(1-\varphi_1 L)(1-\varphi_2 L)}\varepsilon_{2t} \tag{25}$$

Suppose $|\rho_1|, |\rho_2| < 1$ and $|\varphi_1|, |\varphi_2| > 1$ so that there are two roots inside the unit circle and two outside. We have $n = 1, m = 1, p = 4, q = 2, r = 1$, and according to Whiteman's theorem, we have

[12] We are indebted to an anonymous referee for this suggestion.

a unique rational expectations solution. However, it is clear from (25) and the results of Section 2.2 that y_{1t} has an infinite number of solutions and y_{2t} has no solution. Therefore, Whiteman's criterion is incorrect and inconsistent with the univariate case. The reason is that there is no way to set $C_0(1,1)$ to cancel the extra root inside the unit circle in y_{2t} due to the decoupled nature of the system. This criterion also shares the same setback as the "root-counting" criterion of Blanchard and Kahn (1980) that, as pointed out by Sims (2007), will break down in situations where the unstable eigenvalues (i.e., roots inside the unit circle by Theorem 5) occur in a part of the system that is decoupled from other expectational equations.[13]

Translating this example into our notation gives $\Gamma_{-1} = F_1, \Gamma_0 = F_0, \Gamma_1 = G_1$ and $\Psi_0 = I$, and the z-transform of (15) becomes $(\Gamma_{-1}+z\Gamma_0+z^2\Gamma_1)C(z) = zA(z)+\Gamma_{-1}C_0$. The Smith decomposition of $z\Gamma(z)$ gives

$$S(z) = \begin{pmatrix} 1 & 0 \\ 0 & (1-\varphi_1 z)(1-\varphi_2 z) \end{pmatrix}, \qquad T(z) = \begin{pmatrix} (1-\rho_1 z)(1-\rho_2 z) & 0 \\ 0 & 1 \end{pmatrix}$$

where the roots inside the unit circle in $S(z)$ place restrictions on the unknown coefficients C_0:

$$\begin{pmatrix} 0 & 1 \end{pmatrix} (zA(z) + \Gamma_{-1}C_0)|_{z=1/\varphi_1, 1/\varphi_2} = 0$$

Stacking the above restrictions yields

$$-\underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}}_{R} C_0 = \underbrace{\begin{pmatrix} 0 & \frac{1}{\varphi_1} A_{11}\left(\frac{1}{\varphi_1}\right) \\ 0 & \frac{1}{\varphi_2} A_{22}\left(\frac{1}{\varphi_2}\right) \end{pmatrix}}_{A}$$

Existence of solution requires that $\mathrm{span}(A) \subset \mathrm{span}(R)$, which is violated and hence the solution does not exist. This shows that, unlike Whiteman or root-counting criteria, our approach is immune to the troubles posed by decoupling issue.

4 Motivating Examples

We provide two examples that demonstrate the usefulness of solving linear rational expectations models in the frequency-domain. One is taken from the literature and therefore not rigorous, and the other is new in this paper.

4.1 Incomplete Information One of the more compelling reasons to solve models using the approach laid out above is the ease with which it handles incomplete information. The following example is a slightly modified version of Rondina and Walker (2013), which is based on Futia (1981).

[13] The root-counting criterion states that the solution exists and is unique when the number of unstable eigenvalues matches the number of forward-looking variables, which is clearly satisfied here.

Assume agents are risk neutral and discount the future at rate β. Agents trade an asset with price p_t and fundamentals given by s_t. Let there be a continuum of asymmetrically informed agents indexed by i. The model is given by

$$p_t = \beta \int_0^1 \mathbb{E}_t^i p_{t+1} di + s_t \tag{26}$$

where $\mathbb{E}_t^i$ is the conditional expectation of agent i taken with respect to a filtration Ω_t^i. The exogenous process (s_t) is driven by a Gaussian shock

$$s_t = A(L)\varepsilon_t, \qquad \varepsilon_t \overset{iid}{\sim} N(0, \sigma_\varepsilon^2) \tag{27}$$

where $A(L)$ is a square-summable polynomial in the lag operator L. Solving models of the form of (26) is nontrivial because a rational expectations equilibrium will consist of a fixed point in *endogenous* information and the coefficients of the price process.

Definition 7. *A Rational Expectations Equilibrium (REE) is a stochastic process for $\{p_t\}$ and a stochastic process for the information sets $\left\{\Omega_t^i, i \in [0,1]\right\}$ such that: (i) each agent i, given the price and the information set, optimally forms expectations; (ii) p_t satisfies the equilibrium condition (26).*

The solution procedure involves two steps: [i] guess a candidate solution that is minimal with respect to information and impose equilibrium conditions; [ii] check the invertibility of the endogenous variables to ensure the informational fixed point condition holds. Through market interactions, the information conveyed by the candidate solution may be larger than the initial information set of step [i]. If this is the case, the new enlarged information set is used to generate a new candidate solution, and the process is repeated until convergence. Since the expansion of the information set is bounded above by the full information benchmark, the iteration is sure to converge.

A critical component of the solution procedure is initializing the recursion in information. Here one can follow Radner (1979), who advocated forming an "exogenous information equilibrium" as an initial guess. The exogenous information equilibrium assumes agents are only able to condition on exogenous information (e.g., private exogenous signal), which places a lower-bound restriction on the initial condition for information. Radner argued that such an equilibrium would persist only if every agent remained unsophisticated and ignored the information coming from the endogenous variables. A dynamic interpretation of Radner is to say that a "sophisticated" agent acting rationally will not generate forecast errors that are serially correlated with respect to their own information sets.

Following Rondina and Walker (2013), suppose there are two types of agents, *informed* and *uninformed*. The proportion of the informed agents is denoted by $\mu \in [0,1]$ and they are assumed to observe the entire history of the structural shock ε up to time t. The remaining $1-\mu$ agents are uninformed in the sense that they observe only equilibrium outcomes (i.e., the price sequence).

The exogenous information is given by

$$\begin{aligned} U_t^i &= \mathbb{V}_t(\varepsilon) \quad \text{for} \quad i \in [0,\mu] \\ U_t^i &= \{0\} \quad \text{for} \quad i \in (\mu,1] \end{aligned}$$

The equilibrium is given by

$$p_t = \beta\left[\mu\mathbb{E}\big(p_{t+1}|V_t(\varepsilon) \vee \mathbb{M}_t(p)\big) + (1-\mu)\,\mathbb{E}\big(p_{t+1}|V_t(p) \vee \mathbb{M}_t(p)\big)\right] + s_t. \tag{28}$$

The following theorem is due to Rondina and Walker (2013).

Theorem 8. *Under the exogenous information assumption, a unique Information Equilibrium for* (28) *with* $|\beta| < 1$ *always exists and is determined as follows: If there exists a* $|\lambda| < 1$ *such that*

$$A(\lambda) - \frac{\mu\beta A(\beta)}{h(\beta)} = 0 \tag{29}$$

where

$$h(L) \equiv \mu\lambda - (1-\mu)\mathcal{B}_\lambda(L), \qquad \mathcal{B}_\lambda(L) \equiv \frac{L-\lambda}{1-\lambda L}$$

then the REE of (28) *is given by*

$$p_t = Q(L)(L-\lambda)\varepsilon_t = \frac{1}{L-\beta}\left\{LA(L) - \beta A(\beta)\frac{h(L)}{h(\beta)}\right\}\varepsilon_t \tag{30}$$

If restriction (29) *does not hold for* $|\lambda| < 1$*, the REE converges to a complete information equilibrium.*

The proof follows the solution procedure outlined above. The Radner "exogenous information equilibrium" is a price sequence given by

$$p_t = Q(L)(L-\lambda)\varepsilon_t \tag{31}$$

where $|\lambda| < 1$ is assumed, and $Q(L)$ is assumed to contain no zeros inside the unit circle. Viewed as an analytic function, the price process contains a zero inside the unit circle at $z = \lambda$. Thus, the right-hand side of (31) is not invertible. This implies that the price sequence p^t spans a smaller space than ε^t. For the uninformed agents, this space is characterized by a Blaschke factor [see Hansen and Sargent (1991), Lippi and Reichlin (1994)],

$$p_t = Q(L)(1-\lambda L)\tilde{\varepsilon}_t, \qquad \tilde{\varepsilon}_t = \left[\frac{L-\lambda}{1-\lambda L}\right]\varepsilon_t \tag{32}$$

Once we have an initial guess for our endogenous variables, we simply follow the solution procedure and rewrite the model as

$$(\beta L^{-1} - L^0)p_t = -L^0 s_t + \beta \eta_{t+1} \tag{33}$$

Take the conditional expectations for the informed and uninformed agents

$$\begin{aligned} E_t^I(p_{t+1}) &= L^{-1}[(L-\lambda)Q(L) + \lambda Q_0]\varepsilon_t \\ E_t^U(p_{t+1}) &= L^{-1}[(L-\lambda)Q(L) - Q_0 \mathcal{B}_\lambda(L)]\varepsilon_t \end{aligned}$$

Then the endogenous forecasting error can be evaluated as

$$\eta_{t+1} = p_{t+1} - \int_0^1 \mathbb{E}_t^i p_{t+1} di = -Q_0 h(L) L^{-1} \varepsilon_t$$

where $h(L) = \mu\lambda - (1-\mu)\mathcal{B}_\lambda(L)$. Substituting η_{t+1} into (33) gives the z-transform in ε_t space as

$$(z-\lambda)(z-\beta)Q(z) = zA(z) + \beta Q_0 h(z) \tag{34}$$

The $Q(\cdot)$ process will not be analytic unless the process vanishes at the poles $z \in \{\lambda, \beta\}$. This places restrictions on the unknown coefficient Q_0:

$$zA(z) + \beta Q_0 h(z)|_{z=\lambda,\beta} = 0 \tag{35}$$

Stacking the above restrictions yields

$$-\underbrace{\begin{pmatrix} \beta h(\beta) \\ \beta h(\lambda) \end{pmatrix}}_{R} Q_0 = \underbrace{\begin{pmatrix} \beta A(\beta) \\ \lambda A(\lambda) \end{pmatrix}}_{A}$$

Existence of solution requires that $\text{span}(A) \subseteq \text{span}(R)$. This is satisfied if and only if

$$A(\lambda) = \frac{\beta\mu A(\beta)}{h(\beta)}$$

which is (29). Substituting this into (34) delivers (30). Moreover, since there is no root outside the unit circle, the matrix Q is empty and the uniqueness condition $\text{span}(Q') \subseteq \text{span}(R')$ is trivially satisfied.

Therefore, the only additional step to solving models with incomplete information is forming an initial guess for the endogenous variables. We advocate following the recursion described by Radner (1979). We can then follow the stand solution procedure of Whiteman (1983). Moreover, working with analytic functions makes keeping track of the information content of endogenous and exogenous variables straightforward.

4.2 Observational Equivalence We apply our solution method to solve a cashless version of the model in Leeper (1991), and show that the two parameter regions of determinacy in this model can generate observationally equivalent equilibrium time series driven by carefully chosen exogenous driving processes. The model's essential elements include: an infinitely lived representative household endowed each period with a constant quantity of nondurable goods, y; government-issued nominal one-period bonds so that the price level P can be defined as the rate at which bonds exchange for goods; monetary authority follows nominal interest rate (R) rule whereas fiscal authority follows lump-sum taxation (τ) rule.

The household chooses a sequence of consumption and bonds, $\{c_t, B_t\}$, to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where $0 < \beta < 1$ is the discount factor, subject to the budget constraint

$$c_t + \frac{B_t}{P_t} + \tau_t = y + \frac{R_{t-1}B_{t-1}}{P_t}$$

taking prices and the initial principal and interest payments on debt, $R_{-1}B_{-1} > 0$, as given. Government spending is zero each period, so the government chooses a sequence of taxes and debt to satisfy its flow budget constraint

$$\frac{B_t}{P_t} + \tau_t = \frac{R_{t-1}B_{t-1}}{P_t}$$

given $R_{-1}B_{-1} > 0$. After imposing the goods market clearing condition, $c_t = y$ for $t \geq 0$, the household's consumption-Euler equation reduces to the simple Fisher relation

$$\frac{1}{R_t} = \beta E_t \frac{P_t}{P_{t+1}}$$

For analytical convenience, we close the model by specifying the following monetary and fiscal policy rules

$$\begin{aligned} R_t &= R^*(\pi_t/\pi^*)^{\alpha} e^{\theta_t}, && \theta_t \overset{iid}{\sim} N(0, \sigma_M^2) \\ \tau_t &= \tau^*(b_{t-1}/b^*)^{\gamma} e^{\psi_t}, && \psi_t \overset{iid}{\sim} N(0, \sigma_F^2) \end{aligned}$$

where $\pi_t \equiv P_t/P_{t-1}$, $b_t \equiv B_t/P_t$, and $*$ denotes the steady state value for the corresponding variable.

Log-linearizing the above equations around the steady states, the system can be reduced to a bivariate system in $(\hat{\pi}_t, \hat{b}_t)$ where $\hat{x}_t$ denotes the deviation of $\ln x_t$ from $\ln x^*$:

$$\begin{aligned} E_t\hat{\pi}_{t+1} &= \alpha\hat{\pi}_t + \theta_t \\ \hat{b}_t + \beta^{-1}\hat{\pi}_t &= [\beta^{-1} - \gamma(\beta^{-1} - 1)]\hat{b}_{t-1} + \alpha\beta^{-1}\hat{\pi}_{t-1} - (\beta^{-1} - 1)\psi_t + \beta^{-1}\theta_{t-1} \end{aligned}$$

for $t = 0, 1, 2, \ldots$. Putting these equations into the form of (15) gives

$$\Bigg[\underbrace{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}}_{\Gamma_{-1}} L^{-1} + \underbrace{\begin{pmatrix}-\alpha & 0\\ \frac{1}{\beta} & 1\end{pmatrix}}_{\Gamma_0} L^0 + \underbrace{\begin{pmatrix}0 & 0\\ -\frac{\alpha}{\beta} & -\left[\frac{1}{\beta} - \gamma(\frac{1}{\beta} - 1)\right]\end{pmatrix}}_{\Gamma_1} L\Bigg] \underbrace{\begin{pmatrix}\hat{\pi}_t\\ \hat{b}_t\end{pmatrix}}_{y_t}$$
$$= \Bigg[\underbrace{\begin{pmatrix}1 & 0\\ 0 & -(\frac{1}{\beta} - 1)\end{pmatrix}}_{\Psi_0} L^0 + \underbrace{\begin{pmatrix}0 & 0\\ \frac{1}{\beta} & 0\end{pmatrix}}_{\Psi_1} L\Bigg] \underbrace{\begin{pmatrix}\theta_t\\ \psi_t\end{pmatrix}}_{x_t} + \underbrace{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}}_{\Gamma_{-1}} \underbrace{\begin{pmatrix}\eta^{\pi}_{t+1}\\ \eta^{b}_{t+1}\end{pmatrix}}_{\eta_{t+1}}$$

where $n = m = l = 1$, $p = q = 2$, and $A(L)$ is taken to be a (2×2) identity matrix. Following the solution procedure outlined in Section 3.2, we compute the Smith decomposition of $z\Gamma(z)$ as

$$z\Gamma(z) = U(z)^{-1} \begin{pmatrix}1 & 0\\ 0 & z\left(z - \frac{1}{\alpha}\right)\left(z - \frac{1}{\frac{1}{\beta} - \gamma(\frac{1}{\beta} - 1)}\right)\end{pmatrix} V(z)^{-1}$$

Evidently, $\det[z\Gamma(z)]$ has three distinct roots, i.e. $z_1 = 0$, $z_2 = \frac{1}{\alpha}$, and $z_3 = \frac{1}{\frac{1}{\beta} - \gamma(\frac{1}{\beta} - 1)}$. A unique bounded equilibrium can exist if either $|\alpha| > 1$ and $|\gamma| > 1$, or $|\alpha| < 1$ and $|\gamma| < 1$. This implies that the policy parameter space is divided into four disjoint regions according to whether monetary and fiscal policies are, in Leeper (1991) terminology, "active" or "passive".

Case 1: $\alpha < 1$ and $\gamma > 1$. Then we have one root inside the unit circle, i.e. $z_1 = 0$, with the other two outside, i.e. $z_2 = \frac{1}{\alpha} > 1$ and $z_3 = \frac{1}{\frac{1}{\beta} - \gamma(\frac{1}{\beta} - 1)} > 1$. Therefore, $z\Gamma(z)$ can be decomposed as the product of

$$S(z) = U(z)^{-1} \begin{pmatrix}1 & 0\\ 0 & z\end{pmatrix}, \qquad T(z) = \begin{pmatrix}1 & 0\\ 0 & \left(z - \frac{1}{\alpha}\right)\left(z - \frac{1}{\frac{1}{\beta} - \gamma(\frac{1}{\beta} - 1)}\right)\end{pmatrix} V(z)^{-1}$$

where the root inside the unit circle in $S(z)$ places restrictions on the unknown coefficients C_0

$$U_{2\cdot}(z)[z\Psi(z) + \Gamma_{-1}C_0]|_{z=0} = 0$$

This gives the following system

$$-\underbrace{\begin{pmatrix}0 & 0\end{pmatrix}}_{R} C_0 = \underbrace{\begin{pmatrix}0 & 0\end{pmatrix}}_{A}$$

Since $\text{span}(A) \subset \text{span}(R)$ holds, the solution exists. Now we examine the uniqueness condition.

Notice that

$$R = U_{2\cdot}(z_1)\Gamma_{-1} = \begin{pmatrix} 0 & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} U_{2\cdot}(z_2^{-1})\Gamma_{-1} \\ U_{2\cdot}(z_3^{-1})\Gamma_{-1} \end{pmatrix} = \begin{pmatrix} \frac{\alpha(\alpha+1-\gamma+\beta\gamma)-(1+\beta)}{1-\gamma+\beta\gamma} & 0 \\ \frac{(\alpha+1-\gamma+\beta\gamma)(1-\gamma+\beta\gamma)-\beta(1+\beta)}{\alpha\beta^2} & 0 \end{pmatrix}$$

Since $\text{span}(Q') \not\subset \text{span}(R')$, any candidate of C_0 that satisfies the existence condition may lead to a different solution for y_t and hence there are infinite solutions.

CASE 2: $\alpha > 1$ and $\gamma > 1$. Then we have two roots inside the unit circle, i.e. $z_1 = 0$ and $z_2 = \frac{1}{\alpha} < 1$, with the other outside, $z_3 = \frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)} > 1$. Therefore, $z\Gamma(z)$ can be decomposed as the product of

$$S(z) = U(z)^{-1}\begin{pmatrix} 1 & 0 \\ 0 & z\left(z-\frac{1}{\alpha}\right) \end{pmatrix}, \qquad T(z) = \begin{pmatrix} 1 & 0 \\ 0 & z-\frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)} \end{pmatrix} V(z)^{-1}$$

where the roots inside the unit circle in $S(z)$ place restrictions on the unknown coefficients C_0[14]

$$U_{2\cdot}(z)[z\Psi(z) + \Gamma_{-1}C_0]|_{z=1/\alpha} = 0$$

This gives the following system

$$-\underbrace{\begin{pmatrix} \frac{1-\gamma+\beta\gamma-\alpha\beta}{\alpha^3(1-\gamma+\beta\gamma)} & 0 \end{pmatrix}}_{R} C_0 = \underbrace{\begin{pmatrix} \frac{1-\gamma+\beta\gamma-\alpha\beta}{\alpha^4(1-\gamma+\beta\gamma)} & 0 \end{pmatrix}}_{A}$$

Since $\text{span}(A) \subset \text{span}(R)$ holds, the solution exists with $C_0(1,1) = -\frac{1}{\alpha}$ and $C_0(1,2) = 0$. Now we examine the uniqueness condition. Notice that

$$R = \begin{pmatrix} U_{2\cdot}(z_1)\Gamma_{-1} \\ U_{2\cdot}(z_2)\Gamma_{-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{1-\gamma+\beta\gamma-\alpha\beta}{\alpha^3(1-\gamma+\beta\gamma)} & 0 \end{pmatrix} \quad \text{and} \quad Q = U_{2\cdot}(z_3^{-1})\Gamma_{-1} = \begin{pmatrix} \frac{(\alpha+1-\gamma+\beta\gamma)(1-\gamma+\beta\gamma)-\beta(1+\beta)}{\alpha\beta^2} & 0 \end{pmatrix}$$

Since $\text{span}(Q') \subset \text{span}(R')$ holds, any candidate of C_0 that satisfies the existence condition will lead to the same solution for y_t and hence the solution is unique. Finally, the z-transform of the coefficient matrices for y_t is given by

$$C(z) = (z\Gamma(z))^{-1}[z\Psi(z) + \Gamma_{-1}C_0] = \begin{pmatrix} -\frac{1}{\alpha} & 0 \\ \frac{-\frac{1}{\alpha}}{1-\gamma+\beta\gamma}\frac{1}{z-\frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)}} & \frac{1-\beta}{1-\gamma+\beta\gamma}\frac{1}{z-\frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)}} \end{pmatrix}$$

[14] Here we omit the restriction imposed by $z = 0$ because it is unrestrictive.

implying that

$$\begin{pmatrix}\hat{\pi}_t\\ \hat{b}_t\end{pmatrix} = C(L)\begin{pmatrix}\theta_t\\ \psi_t\end{pmatrix} = \underbrace{\begin{pmatrix}-\frac{1}{\alpha} & 0\\ \frac{1}{\alpha\beta} & 1-\frac{1}{\beta}\end{pmatrix}}_{C_0}\begin{pmatrix}\theta_t\\ \psi_t\end{pmatrix} + \sum_{k=1}^{\infty}\underbrace{\begin{pmatrix}0 & 0\\ \frac{\rho^k}{\alpha\beta} & (1-\frac{1}{\beta})\rho^k\end{pmatrix}}_{C_k}\begin{pmatrix}\theta_{t-k}\\ \psi_{t-k}\end{pmatrix}$$

where $\rho = \frac{1}{\beta} - \gamma(\frac{1}{\beta} - 1) < 1$ and C_0 not only satisfies the existence condition but is consistent as well. Also, observe that fiscal shock and its lags do not enter the solution for $\hat{\pi}_t$. This consequence is consistent with Sims (2001b) because we have one unstable eigenvalue ($\alpha > 1$) in the Fisher relation containing expectational terms, which allows it to evolve separately from the government budget constraint and hence $\hat{\pi}_t$ is not affected by the fiscal shocks.

Case 3: $\alpha < 1$ and $\gamma < 1$. Then we have two roots inside the unit circle, i.e. $z_1 = 0$ and $z_3 = \frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)} < 1$, with the other outside, $z_2 = \frac{1}{\alpha} > 1$. Therefore, $z\Gamma(z)$ can be decomposed as the product of

$$S(z) = U(z)^{-1}\begin{pmatrix}1 & 0\\ 0 & z\left(z - \frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)}\right)\end{pmatrix}, \qquad T(z) = \begin{pmatrix}1 & 0\\ 0 & z-\frac{1}{\alpha}\end{pmatrix}V(z)^{-1}$$

where the roots inside the unit circle in $S(z)$ place restrictions on the unknown coefficients C_0

$$U_{2\cdot}(z)[z\Psi(z) + \Gamma_{-1}C_0]\big|_{z=\frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)}} = 0$$

This gives the following system

$$-\underbrace{\begin{pmatrix}-\frac{\beta(1-\gamma+\beta\gamma-\alpha\beta)}{\alpha(1-\gamma+\beta\gamma)^3} & 0\end{pmatrix}}_{R}C_0 = \underbrace{\begin{pmatrix}0 & -\frac{\beta(1-\beta)(1-\gamma+\beta\gamma-\alpha\beta)}{\alpha(1-\gamma+\beta\gamma)^3}\end{pmatrix}}_{A}$$

Since $\text{span}(A) \subset \text{span}(R)$ holds, the solution exists with $C_0(1,1) = 0$ and $C_0(1,2) = \beta - 1$. Now we examine the uniqueness condition. Notice that

$$R = \begin{pmatrix}U_{2\cdot}(z_1)\Gamma_{-1}\\ U_{2\cdot}(z_3)\Gamma_{-1}\end{pmatrix} = \begin{pmatrix}0 & 0\\ -\frac{\beta(1-\gamma+\beta\gamma-\alpha\beta)}{\alpha(1-\gamma+\beta\gamma)^3} & 0\end{pmatrix} \quad \text{and} \quad Q = U_{2\cdot}(z_2^{-1})\Gamma_{-1} = \begin{pmatrix}\frac{\alpha(\alpha+1-\gamma+\beta\gamma)-(1+\beta)}{1-\gamma+\beta\gamma} & 0\end{pmatrix}$$

Since $\text{span}(Q') \subset \text{span}(R')$ holds, any candidate of C_0 that satisfies the existence condition will lead to the same solution for y_t and hence the solution is unique. Finally, the z-transform of the coefficient matrices for y_t is given by

$$C(z) = (z\Gamma(z))^{-1}[z\Psi(z) + \Gamma_{-1}C_0] = \begin{pmatrix}-\frac{1}{\alpha}\frac{z}{z-\frac{1}{\alpha}} & \frac{1-\beta}{\alpha}\frac{1}{z-\frac{1}{\alpha}}\\ 0 & 0\end{pmatrix}$$

implying that

$$\begin{pmatrix}\hat{\pi}_t\\ \hat{b}_t\end{pmatrix} = C(L)\begin{pmatrix}\theta_t\\ \psi_t\end{pmatrix} = \underbrace{\begin{pmatrix}0 & \beta-1\\ 0 & 0\end{pmatrix}}_{C_0}\begin{pmatrix}\theta_t\\ \psi_t\end{pmatrix} + \sum_{k=1}^{\infty}\underbrace{\begin{pmatrix}\alpha^{k-1} & (\beta-1)\alpha^k\\ 0 & 0\end{pmatrix}}_{C_k}\begin{pmatrix}\theta_{t-k}\\ \psi_{t-k}\end{pmatrix}$$

where C_0 not only satisfies the existence condition but is consistent as well. In contrast to the previous case, fiscal shock and its lags now enter the solution for $\hat{\pi}_t$. This consequence is also consistent with Sims (2001b) because the only unstable eigenvalue ($\frac{1}{\beta} - \gamma(\frac{1}{\beta} - 1) > 1$) stays in the government budget constraint containing no expectational term. Determinacy of solution thus requires that such unstable eigenvalue be imported into the Fisher relation which entails bringing the fiscal shocks in the solution for $\hat{\pi}_t$.

CASE 4: $\alpha > 1$ and $\gamma < 1$. Then all roots are inside the unit circle. Therefore, $z\Gamma(z)$ can be decomposed as the product of

$$S(z) = U(z)^{-1}\begin{pmatrix}1 & 0\\ 0 & z\left(z-\frac{1}{\alpha}\right)\left(z-\frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)}\right)\end{pmatrix}, \qquad T(z) = V(z)^{-1}$$

where the roots inside the unit circle in $S(z)$ place restrictions on the unknown coefficients C_0

$$U_{2\cdot}(z)[z\Psi(z) + \Gamma_{-1}C_0]|_{z=\frac{1}{\alpha},\frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)}} = 0$$

This gives the following system

$$-\underbrace{\begin{pmatrix}\frac{1-\gamma+\beta\gamma-\alpha\beta}{\alpha^3(1-\gamma+\beta\gamma)} & 0\\ -\frac{\beta(1-\gamma+\beta\gamma-\alpha\beta)}{\alpha(1-\gamma+\beta\gamma)^3} & 0\end{pmatrix}}_{R} C_0 = \underbrace{\begin{pmatrix}\frac{1-\gamma+\beta\gamma-\alpha\beta}{\alpha^4(1-\gamma+\beta\gamma)} & 0\\ 0 & -\frac{\beta(1-\beta)(1-\gamma+\beta\gamma-\alpha\beta)}{\alpha(1-\gamma+\beta\gamma)^3}\end{pmatrix}}_{A}$$

Since $\text{span}(A) \not\subset \text{span}(R)$, the solution does not exist.

Given the distinct equilibrium dynamics in the above example, it seems straightforward to distinguish an equilibrium time series generated by active monetary/passive fiscal policies (Case 2) from that generated by passive monetary/active fiscal policies (Case 3). Unfortunately, subtle observational equivalence results can make it difficult to identify whether a policy regime is active or passive. The solution methodology developed in this paper makes it possible to study such observational equivalence phenomenon and the implied identification challenge that potentially resides in many well-known DSGE models. In what follows, we highlight the point that simple monetary models show that two disjoint determinacy regions can generate observationally equivalent equilibrium time series driven by *generic* exogenous driving processes. This suggests that existing efforts to identify policy regimes may have been accomplished by imposing *ad hoc* identifying restrictions on the exogenous driving processes.

For simplicity, we assume that the Wold representations for the exogenous driving processes in Cases 2 and 3 are given by

$$\begin{pmatrix}\theta_t\\ \psi_t\end{pmatrix} = \underbrace{\begin{pmatrix}A_{11}(L) & 0\\ 0 & A_{22}(L)\end{pmatrix}}_{A(L)}\begin{pmatrix}\varepsilon_{1t}\\ \varepsilon_{2t}\end{pmatrix}, \qquad \begin{pmatrix}\theta_t\\ \psi_t\end{pmatrix} = \underbrace{\begin{pmatrix}B_{11}(L) & 0\\ 0 & B_{22}(L)\end{pmatrix}}_{B(L)}\begin{pmatrix}\varepsilon_{1t}\\ \varepsilon_{2t}\end{pmatrix}$$

where the functional forms for $\{A_{11}(\cdot), A_{22}(\cdot), B_{11}(\cdot), B_{22}(\cdot)\}$ are left unspecified.[15] We proceed by resolving the model for both cases. See Appendix A for derivation details.

Case 2: let $\alpha = \alpha_1 > 1$ and $\gamma = \gamma_1 > 1$. Then we have two roots inside the unit circle, i.e. 0 and $z_1^M = \frac{1}{\alpha_1} < 1$, with the other outside, $z_2^M = \frac{1}{\frac{1}{\beta} - \gamma_1(\frac{1}{\beta} - 1)} > 1$. The z-transform of the coefficient matrices for y_t is given by

$$C_1(z) = \begin{pmatrix} -z_1^M \frac{zA_{11}(z) - z_1^M A_{11}(z_1^M)}{z - z_1^M} & 0 \\ -\frac{1}{\beta}\frac{z_1^M z_2^M A_{11}(z_1^M)}{z - z_2^M} & (\frac{1}{\beta} - 1) z_2^M \frac{A_{22}(z)}{z - z_2^M} \end{pmatrix}$$

which gives the solution under active monetary/passive fiscal regime.

Case 3: let $\alpha = \alpha_2 < 1$ and $\gamma = \gamma_2 < 1$. Then we have two roots inside the unit circle, i.e. 0 and $z_2^F = \frac{1}{\frac{1}{\beta} - \gamma_2(\frac{1}{\beta} - 1)} < 1$, with the other outside, $z_1^F = \frac{1}{\alpha_2} > 1$. The z-transform of the coefficient matrices for y_t is given by

$$C_2(z) = \begin{pmatrix} -z_1^F \frac{zB_{11}(z)}{z - z_1^F} & (1-\beta)\frac{z_1^F B_{22}(z_2^F)}{z - z_1^F} \\ 0 & (\frac{1}{\beta} - 1) z_2^F \frac{B_{22}(z) - B_{22}(z_2^F)}{z - z_2^F} \end{pmatrix}$$

which gives the solution under passive monetary/active fiscal regime.

Equating the polynomial matrices $C_1(z)$ and $C_2(z)$ element by element delivers the following system of restrictions on the exogenous driving processes in both cases

$$\begin{aligned}
&\frac{zA_{11}(z) - z_1^M A_{11}(z_1^M)}{z - z_1^M} = \mu \frac{zB_{11}(z)}{z - z_1^F}\\
&A_{11}(z_1^M) = 0\\
&B_{22}(z_2^F) = 0\\
&\frac{A_{22}(z)}{z - z_2^M} = \nu \frac{B_{22}(z) - B_{22}(z_2^F)}{z - z_2^F}
\end{aligned}$$

where $\mu = \frac{z_1^F}{z_1^M}$ and $\nu = \frac{z_2^F}{z_2^M}$. This system seems overly restrictive but the fact that there are sequences of infinite undetermined coefficients in the polynomial functions $\{A_{11}(z), A_{22}(z), B_{11}(z), B_{22}(z)\}$ buys one enough freedom of matching the terms. The following theorem is due to Leeper, Tan, and Walker (2012).

[15] Obviously, this modified model is not readily solvable by conventional approaches.

Theorem 9. *Let* $\{A_{11}(z), A_{22}(z), B_{11}(z), B_{22}(z)\}$ *be given by*

$$A_{11}(z) = a_0 + a_1 z, \qquad A_{22}(z) = c_0 + c_1 z \tag{36}$$

$$B_{11}(z) = b_0 + b_1 z, \qquad B_{22}(z) = d_0 + d_1 z \tag{37}$$

Then there exist an infinite sequence of solutions satisfying the above system of restrictions, one of which is given by[16]

$$a_0 = 1, \qquad a_1 = -\frac{1}{z_1^M}, \qquad c_0 = 1, \qquad c_1 = -\frac{1}{z_2^M} \tag{38}$$

$$b_0 = 1, \qquad b_1 = -\frac{1}{z_1^F}, \qquad d_0 = 1, \qquad d_1 = -\frac{1}{z_2^F} \tag{39}$$

Its proof is trivial and thus omitted. This simple monetary model shows that two disjoint determinacy regions can generate observationally equivalent equilibrium time series driven by properly chosen exogenous driving processes. However, further study is needed to examine whether such conclusion extends to more complicated DSGE models that researchers and policy institutions employ to study monetary and fiscal policy interactions.

5 Concluding Comments

There are many other solution methodology papers in the literature that, like this one, expand the range of models beyond that of Blanchard and Kahn (1980) [Anderson and Moore (1985), Broze, Gouriroux, and Szafarz (1995), Klein (2000), Binder and Pesaran (1997), King and Watson (1998), McCallum (1998), Zadrozny (1998), Uhlig (1999), and Onatski (2006)]. There are compelling reasons for studying models with arbitrary number of lags of endogenous variables, or lagged expectations, or with expectations of more distant future values, and with generic exogenous driving processes that may be interesting to economists. From a purely methodological perspective, analyzing more general models gives new insights into methods developed under more restrictive assumptions and allows their deeper interpretation. Moreover, as we argue here, new (or old) techniques could prove useful for solving complicated linear rational expectation models.

We show that the advantage of this frequency-domain approach over other popular time-domain approaches derives from its provision of new insights into solving several well-known challenging problems, e.g. forecasting the forecasts of others in Townsend (1983) and observational equivalence of monetary and fiscal policy interactions in Leeper, Tan, and Walker (2012), etc. Therefore, our solution methodology proves to be an indispensable supplement to the existing approaches from both theoretical and applied perspectives.

One useful extension of our solution methodology would be to accommodate continuous-time processes as in Sims (2001b). On one hand, explicit extension to the continuous-time systems en-

[16] Under the specification given in Theorem 9, we have one free coefficient and hence there are infinite solutions.

ables one to tackle problems that can hardly be dealt with in the discrete-time systems and thus brings new insights to the table. On the other, a continuous-time extension makes it possible to study various non-stationary or near non-stationary features commonly present in almost all important macroeconomic time series data. These non-stationarities usually cannot be fully removed by simple detrending or transformations and very often, these detrending efforts may incur loss of important long-term information about the data that is potentially valuable to researchers. Therefore, an explicit extension of our solution methodology to the continuous-time setting proves to be both non-trivial and useful. We leave this for future research.

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Online Appendix

Appendix A: Derivations and Proofs First, we derive the expressions of the matrices $\{A_{j\cdot}, R_{j\cdot}, Q_{j\cdot}\}_{j=1}^{p}$ given in the text. Existence requires that the free parameters $C_0, C_1, \dots, C_{n-1}$ be set such that the numerator of the right hand side of (17) vanishes at $z = \underline{z}_{jk}$ for $k = 1, \dots, \underline{r}_j$:

$$\frac{d^i}{dz^i}\left[\prod_{k=1}^{\underline{r}_j}(z-\underline{z}_{jk})^{\underline{m}_{jk}}T_{j\cdot}(z)C(z)\right]\Bigg|_{z=\underline{z}_{jk}} = 0, \quad i = 0, \dots, \underline{m}_{jk}-1, \quad k = 1, \dots, \underline{r}_j$$

Stacking the above expressions yields

$$\underbrace{\begin{pmatrix} \left[U_{j\cdot}(\underline{z}_{j1})(\underline{z}_{j1}^{n}\Psi(\underline{z}_{j1})A(\underline{z}_{j1}) - \sum_{t=1}^{n}\sum_{s=t}^{n}\Psi_{-s}A_{t-1}\underline{z}_{j1}^{n-s+t-1})\right]^{(0)} \\ \vdots \\ \left[U_{j\cdot}(\underline{z}_{j1})(\underline{z}_{j1}^{n}\Psi(\underline{z}_{j1})A(\underline{z}_{j1}) - \sum_{t=1}^{n}\sum_{s=t}^{n}\Psi_{-s}A_{t-1}\underline{z}_{j1}^{n-s+t-1})\right]^{(\underline{m}_{j1}-1)} \\ \vdots \\ \left[U_{j\cdot}(\underline{z}_{j\underline{r}_j})(\underline{z}_{j\underline{r}_j}^{n}\Psi(\underline{z}_{j\underline{r}_j})A(\underline{z}_{j\underline{r}_j}) - \sum_{t=1}^{n}\sum_{s=t}^{n}\Psi_{-s}A_{t-1}\underline{z}_{j\underline{r}_j}^{n-s+t-1})\right]^{(0)} \\ \vdots \\ \left[U_{j\cdot}(\underline{z}_{j\underline{r}_j})(\underline{z}_{j\underline{r}_j}^{n}\Psi(\underline{z}_{j\underline{r}_j})A(\underline{z}_{j\underline{r}_j}) - \sum_{t=1}^{n}\sum_{s=t}^{n}\Psi_{-s}A_{t-1}\underline{z}_{j\underline{r}_j}^{n-s+t-1})\right]^{(\underline{m}_{j\underline{r}_j}-1)} \end{pmatrix}}_{A_{j\cdot}} =$$

$$-\underbrace{\begin{pmatrix} \left[U_{j\cdot}(\underline{z}_{j1})\sum_{s=1}^{n}\Gamma_{-s}\underline{z}_{j1}^{n-s}\right]^{(0)} & \cdots & \left[U_{j\cdot}(\underline{z}_{j1})\Gamma_{-n}\underline{z}_{j1}^{n-1}\right]^{(0)} \\ \vdots & \ddots & \vdots \\ \left[U_{j\cdot}(\underline{z}_{j1})\sum_{s=1}^{n}\Gamma_{-s}\underline{z}_{j1}^{n-s}\right]^{(\underline{m}_{j1}-1)} & \cdots & \left[U_{j\cdot}(\underline{z}_{j1})\Gamma_{-n}\underline{z}_{j1}^{n-1}\right]^{(\underline{m}_{j1}-1)} \\ \vdots & \ddots & \vdots \\ \left[U_{j\cdot}(\underline{z}_{j\underline{r}_j})\sum_{s=1}^{n}\Gamma_{-s}\underline{z}_{j\underline{r}_j}^{n-s}\right]^{(0)} & \cdots & \left[U_{j\cdot}(\underline{z}_{j\underline{r}_j})\Gamma_{-n}\underline{z}_{j\underline{r}_j}^{n-1}\right]^{(0)} \\ \vdots & \ddots & \vdots \\ \left[U_{j\cdot}(\underline{z}_{j\underline{r}_j})\sum_{s=1}^{n}\Gamma_{-s}\underline{z}_{j\underline{r}_j}^{n-s}\right]^{(\underline{m}_{j\underline{r}_j}-1)} & \cdots & \left[U_{j\cdot}(\underline{z}_{j\underline{r}_j})\Gamma_{-n}\underline{z}_{j\underline{r}_j}^{n-1}\right]^{(\underline{m}_{j\underline{r}_j}-1)} \end{pmatrix}}_{R_{j\cdot}} \underbrace{\begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_{n-1} \end{pmatrix}}_{C}$$

Further stacking over $j = 1, \dots, p$ gives

$$\underset{[r\times q]}{A} = -\underset{[r\times np]}{R}\ \underset{[np\times q]}{C}$$

where $r = \sum_{j=1}^{p}\sum_{k=1}^{\underline{r}_j}\underline{m}_{jk}$.

Uniqueness requires that we be able to determine $\{C_k\}_{k=0}^{\infty}$, or equivalently $\{D_k\}_{k=0}^{\infty}$, from the parameter restrictions supplied by $A = -RC$, which can be computed using the inversion formula

$$\begin{aligned} D_k &= \frac{1}{2\pi i}\oint D(z)z^{-k-1}dz \\ &= \text{sum of residues of } D(z^{-1})z^{k-1} \text{ at poles inside unit circle} \end{aligned}$$

Note that the jth row of $D(z^{-1})z^{k-1}$ is given by

$$\frac{U_{j\cdot}(z^{-1})z^{k-1}}{\prod_{k=1}^{r_j}(z^{-1}-\underline{z}_{jk})^{\underline{m}_{jk}}\prod_{k=1}^{\overline{r}_j}(z^{-1}-\overline{z}_{jk})^{\overline{m}_{jk}}}\left\{z^{-n}\Psi(z^{-1})A(z^{-1})+\sum_{t=1}^{n}\sum_{s=t}^{n}[\Gamma_{-s}C_{t-1}-\Psi_{-s}A_{t-1}]z^{-(n-s+t-1)}\right\}$$

which has poles inside unit circle at $\overline{z}_{jk}^{-1}$ with multiplicity $\overline{m}_{jk}$ for $k = 1, \ldots, \overline{r}_j$.[17] We can write the jth row of each D_k as a function of C that only shows up in the following common terms shared by all D_k's

$$\frac{d^i}{dz^i}\left[U_{j\cdot}(z^{-1})\sum_{t=1}^{n}\sum_{s=t}^{n}\Gamma_{-s}C_{t-1}z^{-(n-s+t-1)}\right]\Bigg|_{z=\overline{z}_{jk}^{-1}}, \quad i = 0, \ldots, \overline{m}_{jk}-1, \quad k = 1, \ldots, \overline{r}_j$$

Stacking the above expressions yields

$$\underbrace{\begin{pmatrix} \left[U_{j\cdot}(\overline{z}_{j1}^{-1})\sum_{s=1}^{n}\Gamma_{-s}\overline{z}_{j1}^{-(n-s)}\right]^{(0)} & \cdots & \left[U_{j\cdot}(\overline{z}_{j1}^{-1})\Gamma_{-n}\overline{z}_{j1}^{-(n-1)}\right]^{(0)} \\ \vdots & \ddots & \vdots \\ \left[U_{j\cdot}(\overline{z}_{j1}^{-1})\sum_{s=1}^{n}\Gamma_{-s}\overline{z}_{j1}^{-(n-s)}\right]^{(\overline{m}_{j1}-1)} & \cdots & \left[U_{j\cdot}(\overline{z}_{j1}^{-1})\Gamma_{-n}\overline{z}_{j1}^{-(n-1)}\right]^{(\overline{m}_{j1}-1)} \\ \vdots & \ddots & \vdots \\ \left[U_{j\cdot}(\overline{z}_{j\overline{r}_j}^{-1})\sum_{s=1}^{n}\Gamma_{-s}\overline{z}_{j\overline{r}_j}^{-(n-s)}\right]^{(0)} & \cdots & \left[U_{j\cdot}(\overline{z}_{j\overline{r}_j}^{-1})\Gamma_{-n}\overline{z}_{j\overline{r}_j}^{-(n-1)}\right]^{(0)} \\ \vdots & \ddots & \vdots \\ \left[U_{j\cdot}(\overline{z}_{j\overline{r}_j}^{-1})\sum_{s=1}^{n}\Gamma_{-s}\overline{z}_{j\overline{r}_j}^{-(n-s)}\right]^{(\overline{m}_{j\overline{r}_j}-1)} & \cdots & \left[U_{j\cdot}(\overline{z}_{j\overline{r}_j}^{-1})\Gamma_{-n}\overline{z}_{j\overline{r}_j}^{-(n-1)}\right]^{(\overline{m}_{j\overline{r}_j}-1)} \end{pmatrix}}_{Q_{j\cdot}}\underbrace{\begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_{n-1} \end{pmatrix}}_{C}$$

Further stacking over $j = 1, \ldots, p$ yields QC.

Next, we prove Theorem 5 regarding the equivalence relation of solution methodologies between Sims (2001b) and this paper.

Proof of (1): first, the eigenvalue λ of $-\Gamma_{-1}^{-1}\Gamma_0$ can be computed as $|\Gamma_{-1}^{-1}\Gamma_0 + \lambda I| = 0$. Also, since Γ_{-1} is assumed to be of full rank and $z \neq 0$, we have $|\Gamma_{-1} + z\Gamma_0| = |z\Gamma_{-1}||\Gamma_{-1}^{-1}\Gamma_0 + \frac{1}{z}I| = 0$, or $|\Gamma_{-1}^{-1}\Gamma_0 + \frac{1}{z}I| = 0$. This establishes $\lambda = \frac{1}{z}$.

Second, let $\Gamma_{-1} + z\Gamma_0 = U(z)^{-1}P(z)V(z)^{-1}$ where $U(z)$ and $V(z)$ are unimodular matrices and $P(z)$ is the Smith canonical form for $\Gamma_{-1} + z\Gamma_0$. Since $|U(z)|$ and $|V(z)|$ are nonzero constants, the roots of $|\Gamma_{-1} + z\Gamma_0| = 0$ are exactly those of $|P(z)| = 0$.

[17] For $k = 0$, there is an additional pole inside the unit circle at 0.

PROOF OF (2): first, we derive the restriction system in Sims (2001b). Since all eigenvalues of $-\Gamma_{-1}^{-1}\Gamma_0$ are distinct, we know that $-\Gamma_{-1}^{-1}\Gamma_0$ is diagonalizable and can be factorized as

$$-\Gamma_{-1}^{-1}\Gamma_0 = P\Lambda P^{-1}$$

where P is the matrix of right-eigenvectors, P^{-1} is the matrix of left-eigenvectors, and Λ is a diagonal matrix with all eigenvalues of $-\Gamma_{-1}^{-1}\Gamma_0$ on its main diagonal. Stability conditions then require that for all t

$$P^{U\cdot}(\Gamma_{-1}^{-1}\Psi_{-1}x_{t+1} + \eta_{t+1}) = 0 \tag{A.1}$$

where $P^{U\cdot}$ collects all the rows of P^{-1} corresponding to unstable eigenvalues.

Second, we derive the restriction system in this paper. Note that the polynomial matrix $\Gamma_{-1} + z\Gamma_0$ can be factorized as

$$\Gamma_{-1} + z\Gamma_0 = U(z)^{-1}P(z)V(z)^{-1} = \underbrace{U(z)^{-1}P_1(z)}_{S(z)}\underbrace{P_2(z)V(z)^{-1}}_{T(z)}$$

where $U(z)$ and $V(z)$ are unimodular matrices and $S(z)$ is the Smith canonical form for $\Gamma_{-1} + z\Gamma_0$. Also, $S(z)$ is a polynomial matrix such that all the roots of $\det[S(z)]$ lie inside the unit circle while $T(z)$ is a polynomial matrix with all the roots of $\det[T(z)]$ outside the unit circle. Since all the roots of $\det[\Gamma_{-1} + z\Gamma_0]$ are distinct, the property that the (i,i) entry of Smith canonical form is divisible by its $(i-1,i-1)$ entry for $i = 2,\ldots,p$ implies that $P_1(z)$ is of the form

$$P_1(z) = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \prod_{j=1}^{r}(z - \underline{z}_j) \end{pmatrix}$$

and hence

$$S(z)^{-1} = \begin{pmatrix} U_{1\cdot}(z) \\ \vdots \\ U_{p-1\cdot}(z) \\ \frac{1}{\prod_{j=1}^{r}(z-\underline{z}_j)}U_{p\cdot}(z) \end{pmatrix}$$

This implies the following restriction system

$$\begin{pmatrix} U_{p\cdot}(\underline{z}_1) \\ \vdots \\ U_{p\cdot}(\underline{z}_{\underline{r}}) \end{pmatrix} \Gamma_{-1}(\Gamma_{-1}^{-1}\Psi_{-1} + C_0) = 0 \tag{A.2}$$

Observe that for $\forall \underline{z}_j$ with $j = 1, 2, \ldots, \underline{r}$, we have the following equation

$$U(\underline{z}_j)\Gamma_{-1}\left(\Gamma_{-1}^{-1}\Gamma_0 + \frac{1}{\underline{z}_j}I\right) = \frac{1}{\underline{z}_j}P(\underline{z}_j)V(\underline{z}_j)^{-1}$$

where the last row is given by

$$U_{p\cdot}(\underline{z}_j)\Gamma_{-1}\left(\Gamma_{-1}^{-1}\Gamma_0 + \frac{1}{\underline{z}_j}I\right) = (0 \cdots 0)$$

This implies that $U_{p\cdot}(\underline{z}_j)\Gamma_{-1}$ is exactly the left eigenvector corresponding to the unstable eigenvalue $\frac{1}{\underline{z}_j}$ of $-\Gamma_{-1}^{-1}\Gamma_0$. Stacking $U_{p\cdot}(\underline{z}_j)\Gamma_{-1} = P^{j\cdot}$ for $j = 1, 2, \ldots, \underline{r}$ then gives

$$\begin{pmatrix} U_{p\cdot}(\underline{z}_1) \\ \vdots \\ U_{p\cdot}(\underline{z}_{\underline{r}}) \end{pmatrix} \Gamma_{-1} = P^{U\cdot}$$

This implies that (A.2) is equivalent to

$$P^{U\cdot}(\Gamma_{-1}^{-1}\Psi_{-1} + C_0) = 0 \tag{A.3}$$

The proof is completed by noticing that both (A.1) and (A.3) hold if and only if the columns of $P^{U\cdot}$ span the space spanned by the columns of $P^{U\cdot}\Gamma_{-1}^{-1}\Psi_{-1}$, i.e.

$$\operatorname{span}(P^{U\cdot}\Gamma_{-1}^{-1}\Psi_{-1}) \subset \operatorname{span}(P^{U\cdot})$$

Proof of (3): first, the uniqueness condition in Sims (2001b) requires that the knowledge of $P^{U\cdot}\eta$ can be used to determine $P^{S\cdot}\eta$, where $P^{S\cdot}$ is made up of all the rows of P^{-1} corresponding to stable eigenvalues.

Second, the uniqueness condition in this paper requires that the knowledge of

$$\begin{pmatrix} U_{p\cdot}(\underline{z}_1) \\ \vdots \\ U_{p\cdot}(\underline{z}_{\underline{r}}) \end{pmatrix} \Gamma_{-1}C_0$$

can be used to determine

$$\begin{pmatrix} U_{p\cdot}(\overline{z}_1^{-1}) \\ \vdots \\ U_{p\cdot}(\overline{z}_{\overline{r}}^{-1}) \end{pmatrix} \Gamma_{-1} C_0$$

where $\overline{z}_j$ for $j = 1, \ldots, \overline{r}$ are those roots outside unit circle for $\det[\Gamma_{-1} + z\Gamma_0] = 0$, and hence their inverses are exactly the stable eigenvalues of $-\Gamma_{-1}^{-1}\Gamma_0$ by part (1). Therefore, by part (2) the solution is unique when the knowledge of $P^{U\cdot}C_0$ can be used to determine $P^{S\cdot}C_0$.

The proof is completed by noticing that the uniqueness conditions in Sims (2001b) and this paper both hold if and only if the columns of $(P^{U\cdot})'$ span the space spanned by the columns of $(P^{S\cdot})'$, i.e.

$$\text{span}((P^{S\cdot})') \subset \text{span}((P^{U\cdot})')$$

Lastly, we resolve the simple monetary model in Section 4.2 but with generic exogenous driving processes in both regimes. First, let $\alpha = \alpha_1 > 1$ and $\gamma = \gamma_1 > 1$. Then we have two roots inside the unit circle, i.e. 0 and $z_1^M = \frac{1}{\alpha_1} < 1$, with the other outside, $z_2^M = \frac{1}{\frac{1}{\beta} - \gamma_1(\frac{1}{\beta} - 1)} > 1$. Therefore, $z\Gamma(z)$ can be decomposed as the product of

$$S(z) = U(z)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & z\left(z - z_1^M\right) \end{pmatrix}, \qquad T(z) = \begin{pmatrix} 1 & 0 \\ 0 & z - z_2^M \end{pmatrix} V(z)^{-1}$$

where the roots inside the unit circle in $S(z)$ place restrictions on the unknown coefficients C_0

$$U_{2\cdot}(z)[z\Psi(z)A(z) + \Gamma_{-1}C_0]|_{z=z_1^M} = 0$$

This gives the following system

$$-\begin{pmatrix} \frac{1-\gamma_1+\beta\gamma_1-\alpha_1\beta}{\alpha_1^3(1-\gamma_1+\beta\gamma_2)} & 0 \end{pmatrix} C_0 = \begin{pmatrix} \frac{1-\gamma_1+\beta\gamma_1-\alpha_1\beta}{\alpha_1^4(1-\gamma_1+\beta\gamma_1)} A_{11}(z_1^M) & 0 \end{pmatrix}$$

and hence $C_0(1,1) = -z_1^M A_{11}(z_1^M)$ and $C_0(1,2) = 0$. Finally, the z-transform of the coefficient matrices for y_t is given by

$$C_1(z) = \begin{pmatrix} -z_1^M \frac{zA_{11}(z) - z_1^M A_{11}(z_1^M)}{z - z_1^M} & 0 \\ -\frac{1}{\beta} \frac{z_1^M z_2^M A_{11}(z_1^M)}{z - z_2^M} & (\frac{1}{\beta} - 1) z_2^M \frac{A_{22}(z)}{z - z_2^M} \end{pmatrix}$$

which gives the solution under active monetary/passive fiscal regime.

Second, let $\alpha = \alpha_2 < 1$ and $\gamma = \gamma_2 < 1$. Then we have two roots inside the unit circle, i.e. 0 and $z_2^F = \frac{1}{\frac{1}{\beta} - \gamma_2(\frac{1}{\beta} - 1)} < 1$, with the other outside, $z_1^F = \frac{1}{\alpha_2} > 1$. Therefore, $z\Gamma(z)$ can be decomposed as the product of

$$S(z) = U(z)^{-1}\begin{pmatrix}1 & 0 \\ 0 & z\left(z - z_2^F\right)\end{pmatrix}, \qquad T(z) = \begin{pmatrix}1 & 0 \\ 0 & z - z_1^F\end{pmatrix} V(z)^{-1}$$

where the roots inside the unit circle in $S(z)$ place restrictions on the unknown coefficients C_0

$$U_{2\cdot}(z)[z\Psi(z)B(z) + \Gamma_{-1}C_0]|_{z=z_2^F} = 0$$

This gives the following system

$$-\begin{pmatrix}-\frac{\beta(1-\gamma+\beta\gamma-\alpha\beta)}{\alpha(1-\gamma+\beta\gamma)^3} & 0\end{pmatrix} C_0 = \begin{pmatrix}0 & -\frac{\beta(1-\beta)(1-\gamma+\beta\gamma-\alpha\beta)}{\alpha(1-\gamma+\beta\gamma)^3} B_{22}(z_2^F)\end{pmatrix}$$

and hence $C_0(1,1) = 0$ and $C_0(1,2) = (\beta - 1)B_{22}(z_2^F)$. Finally, the z-transform of the coefficient matrices for y_t is given by

$$C_2(z) = \begin{pmatrix}-z_1^F \frac{zB_{11}(z)}{z-z_1^F} & (1-\beta)\frac{z_1^F B_{22}(z_2^F)}{z-z_1^F} \\ 0 & (\frac{1}{\beta} - 1)z_2^F \frac{B_{22}(z) - B_{22}(z_2^F)}{z - z_2^F}\end{pmatrix}$$

which gives the solution under passive monetary/active fiscal regime.

Appendix B: User's Guide All of the routines required to implement this solution algorithm are written and compiled in MATLAB, which take the advantages of MATLAB *Symbolic Toolbox* and are executed with the following files:[18]

- **model.m** file serves as a template for inputting all of the matrix coefficients of a generalized multivariate linear rational expectations model of the form given by (12). It then calls the function **tranz(Gamma,Psi,A,n,T)** in **tranz.m**;

- **tranz.m** file serves as the main script that performs the z-transform algorithm for a given multivariate linear rational expectations model and computes its solution by invoking related functions in MATLAB *Symbolic Toolbox*. It also examines the model's existence and uniqueness conditions;

- **multroot.m** file finds all the distinct roots of a given polynomial with their corresponding multiplicities;

- **U.txt** file defines a MAPLE procedure that computes the (left) unimodular matrix $U(z)$ in the Smith canonical decomposition of a given polynomial matrix.

As an example, we use the model in Section 4.2 to outline how to implement the solution algorithm. There are a number of model-specific initializations that are specified by the user and

[18] This program is available upon request.

break down into several easily implementable steps:

- Step 1 – define the symbolic variable z and the numerical values of the model's parameters. MATLAB code:

```
syms z           % symbolic z
beta = 0.9804;   % discount factor
alpha = 1.5;     % active monetary
gamma = 1.2;     % passive fiscal
```

- Step 2 – specify the indices for both endogenous and exogenous variables. MATLAB code:

```
npi = 1;         % inflation
nb = 2;          % real debt
ntheta = 1;      % monetary shock
npsi = 2;        % fiscal shock
```

- Step 3 – define the matrix coefficients and relevant parameters. MATLAB code:

```
p  = 2;                      % system dimension
n = 1;                       % number of leads
m = 1;                       % number of endo lags
l = 1;                       % number of exo lags
Gamma = zeros(p,p,n+m+1);    % endo matrix polynomial
Psi = zeros(p,p,n+l+1);      % exo matrix polynomial
A = [1 0;0 1];               % driving matrix polynomial
```

- Step 4 – enter the equilibrium equations one by one. MATLAB code:

```
% (1) Fisher equation
Gamma(1,npi,1) = 1;
Gamma(1,npi,2) = -alpha;
Psi(1,ntheta,2) = 1;

% (2) Government budget constraint
Gamma(2,npi,2) = 1/beta;
Gamma(2,nb,2) = 1;
Gamma(2,npi,3) = -alpha/beta;
```

```
Gamma(2,nb,3) = -(1/beta-gamma*(1/beta-1));
Psi(2,npsi,2) = -(1/beta-1);
Psi(2,ntheta,3) = 1/beta;
```

- Step 5 – construct the matrix polynomials and solve the model by calling the function **tranz(Gamma,Psi,A,n,T)** in **tranz.m**. The program returns two elements, i.e. **eu** (existence and uniqueness) and **sol** (first T moving average matrix coefficients of the solution). MATLAB code:

```
% construct matrix polynomials
Gamma = Gamma(:,:,1)/z+Gamma(:,:,2)+Gamma(:,:,3)*z;
Psi = Psi(:,:,1)/z+Psi(:,:,2)+Psi(:,:,3)*z;
% solve model
[eu,sol] = tranz(Gamma,Psi,A,n,T);
```