Lecture 9 Time Series

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State Space Model

Gaussian linear case

$$
\begin{aligned}
\text{Measurement:} \quad y_t &= D + Z\epsilon_t + v_t, \quad v_t \sim \mathbb{N}(0, \Sigma_v) \\
\text{Transition:} \quad \epsilon_t &= C + G\epsilon_{t-1} + u_t, \quad u_t \sim \mathbb{N}(0, \Sigma_u)
\end{aligned}
$$

- Kalman filter: set $(\epsilon_{0|0}, P_{0|0})$, for $t = 1, \ldots, T$
  1. forecast ϵ_t via transition: $\epsilon_t | y_{1:t-1} \sim \mathbb{N}(\epsilon_{t|t-1}, P_{t|t-1})$

$$
\begin{aligned}
\epsilon_{t|t-1} &= C + G\epsilon_{t-1|t-1} \\
P_{t|t-1} &= GP_{t-1|t-1}G' + \Sigma_u
\end{aligned}
$$

  2. forecast y_t via measurement: $y_t | y_{1:t-1} \sim \mathbb{N}(y_{t|t-1}, F_{t|t-1})$

$$
\begin{aligned}
y_{t|t-1} &= D + Z\epsilon_{t|t-1} \\
F_{t|t-1} &= ZP_{t|t-1}Z' + \Sigma_v
\end{aligned}
$$

  3. filter ϵ_t via Bayes theorem: $\epsilon_t | y_{1:t} \sim \mathbb{N}(\epsilon_{t|t}, P_{t|t})$

$$
\begin{aligned}
\epsilon_{t|t} &= \epsilon_{t|t-1} + P_{t|t-1}Z'F_{t|t-1}^{-1}(y_t - y_{t|t-1}) \\
P_{t|t} &= P_{t|t-1} - P_{t|t-1}Z'F_{t|t-1}^{-1}ZP_{t|t-1}
\end{aligned}
$$

Python Pseudo Code

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To be added...
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The Road Ahead...

- Autoregressive model
- Regime-switching model

Autoregressive (AR) Model

Chib's (1993) regression with AR(p) error

$$
\begin{aligned}
y_t &= x_t'\beta + \epsilon_t, \quad t = 1, \ldots, T \\
\phi(L)\epsilon_t &= u_t, \quad \phi(L) = 1 - \phi_1 L - \cdots - \phi_p L^p, \quad L\epsilon_t = \epsilon_{t-1}
\end{aligned}
$$

- Likelihood function under $u_t \sim_{i.i.d.} \mathbb{N}(0, \sigma^2)$

$$
f(y_{p+1:T}|y_{1:p}, \beta, \sigma^2, \phi) \propto \sigma^{-(n-p)} \exp\left[-\frac{1}{2\sigma^2}\sum_{t=p+1}^{T}(y_t - y_{t|t-1})^2\right]
$$

where $y_{t|t-1} = (1 - \phi(L))y_t + \phi(L)x_t'\beta$

- Joint prior (stationarity: $\Phi_s = \{\phi : \phi(z) \neq 0, \ \forall |z| < 1\}$)

$$
\pi(\beta, \sigma^2, \phi) = \underbrace{\mathbb{N}(\beta_0, \sigma^2 B_0)}_{\pi(\beta|\sigma^2)} \underbrace{\mathbb{IG}\text{-}2(\nu_0/2, \delta_0/2)}_{\pi(\sigma^2)} \underbrace{\mathbb{N}(\phi_0, \Phi_0)1\{\phi \in \Phi_s\}}_{\pi(\phi)}
$$

Gibbs Algorithm

- Gibbs sampler for $\pi(\beta, \sigma^2, \phi|y)$

$$\begin{aligned}
\beta|y, \sigma^2, \phi &\sim \mathbb{N}(\beta_1, \sigma^2 B_1) \\
\sigma^2|y, \beta, \phi &\sim \mathbb{IG}\text{-}2(\nu_1/2, \delta_1/2) \\
\phi|y, \beta, \sigma^2 &\sim \mathbb{N}(\phi_1, \Phi_1)1\{\phi \in \Phi_s\}
\end{aligned}$$

where

$$\begin{aligned}
B_1 &= (X^{*\prime}X^* + B_0^{-1})^{-1}, \quad x_t^* = \phi(L)x_t \\
\beta_1 &= B_1(X^{*\prime}y^* + B_0^{-1}\beta_0), \quad y_t^* = \phi(L)y_t \\
\nu_1 &= \nu_0 + k, \quad k = \text{dim}(x_t) \\
\delta_1 &= \delta_0 + (\beta - \beta_0)' B_0^{-1}(\beta - \beta_0) + (y^* - X^{*\prime}\beta)'(y^* - X^{*\prime}\beta) \\
\Phi_1 &= (\sigma^{-2}E'E + \Phi_0^{-1})^{-1}, \quad \epsilon = E\phi + u \text{ (known given } y, \beta, \sigma^2) \\
\phi_1 &= \Phi_1(\sigma^{-2}E'\epsilon + \Phi_0^{-1}\phi_0)
\end{aligned}$$

- Extension to ARMA(p, q) error [Chib & Greenberg (1994)]

Python Pseudo Code

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To be added...
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Regime-Switching Model

Chib's (1996) Markov mixture model

$$\begin{aligned} f(y_t|y_{1:t-1}, s_{t-1}, \theta) &= \sum_{k=1}^{m} f(y_t|y_{1:t-1}, \theta_k) p(s_t = k|s_{t-1}) \\ p(s_t = j|s_{t-1} = i) &= p_{ij}, \quad s_t \in \{1, \ldots, m\}, \quad t = 1, \ldots, T \end{aligned}$$

- Chib's (1998) change-point model

$$P = \begin{bmatrix} p_{11} & 1 - p_{11} & 0 & \cdots & 0 & 0 \\ 0 & p_{22} & 1 - p_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{mm} & 1 - p_{mm} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

- Which specification is more general?

Gibbs Algorithm

- Sample $s_{1:T}$ in one block

$$\begin{aligned} p(s_{1:T}|y_{1:T}) &= p(s_T|y_{1:T}) \cdots {\color{red} p(s_t|y_{1:T}, s_{t+1:T})} \cdots p(s_1|y_{1:T}, s_{2:T}) \\ {\color{red} p(s_t|y_{1:T}, s_{t+1:T})} &\propto {\color{blue} p(s_t|y_{1:t})} p(s_{t+1}|s_t), \quad \text{where for } t = 1, \ldots, T \end{aligned}$$

  - prediction step: for $j = 1, \ldots, m$

$$p(s_t = j|y_{1:t-1}) = \sum_{k=1}^{m} p(s_t = j|s_{t-1} = k) p(s_{t-1} = k|y_{1:t-1})$$

  - update step: for $j = 1, \ldots, m$

$${\color{blue} p(s_t|y_{1:t})} \propto p(s_t = j|y_{1:t-1}) f(y_t|y_{1:t-1}, \theta_j)$$

- Sample each row of P from Dirichlet distribution

$$p_i \sim \mathbb{D}(\alpha_{i1}, \ldots, \alpha_{im}) \quad \Rightarrow \quad p_i|s_{1:T} \sim \mathbb{D}(\alpha_{i1} + n_{i1}, \ldots, \alpha_{im} + n_{im})$$

where n_{im} = # of one-step transitions $i \to j$ in $s_{1:T}$

Python Pseudo Code

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To be added...
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Readings

- Chib (1993), "Bayes Regression with Autoregressive Errors: A Gibbs Sampling Approach," *Journal of Econometrics*
- Chib & Greenberg (1994), "Bayes Inference in Regression Models with ARMA(p, q) Erros," *Journal of Econometrics*
- Chib (1996), "Calculating Posterior Distributions and Modal Estimates in Markov Mixture Models," *Journal of Econometrics*
- Chib (1998), "Estimation and Comparison of Multiple Change-Point Models," *Journal of Econometrics*