Lecture 6 Simulation by MCMC Methods

Fei Tan

Department of Economics
Chaifetz School of Business
Saint Louis University

E6930 Applied Bayesian Statistics

December 31, 2022

Big Picture of MCMC

Central equation

$$\int_A \int_{\mathbb{R}^d} p(x,y) f(x) dx dy = \int_A f(y) dy, \quad \forall A \in \mathcal{B}(\mathbb{R}^d)$$

- What is Markov chain theory doing? Know transition kernel $p(\cdot,\cdot)$, find invariant distribution $f(\cdot)$

$$\int_A \int_{\mathbb{R}^d} p(x,y) f^{(n-1)}(x) dx dy = \int_A f^{(n)}(y) dy \to \int_A f(y) dy$$

- Markov chain Monte Carlo (MCMC) is doing opposite: know $f(\cdot)$, find corresponding $p(\cdot,\cdot)$ such that

$$f(x)p(x,y) = f(y)p(y,x) \quad \text{(reversibility)}$$

- MCMC methods greatly broaden Bayesian scope though at cost of simulating *dependent* samples

The Road Ahead...

- Gibbs algorithm
- Metropolis-Hastings (MH) algorithm
- Calculation of marginal likelihood
- Measures of convergence

Gibbs Algorithm

Algorithm 1

1. Choose $x^{(0)} = (x_1^{(0)}, \ldots, x_d^{(0)})$ and set $g = 0$
2. Sample $x_i^{(g+1)} \sim f(x_i|x_{-i}^{(g)})$ for $i = 1, \ldots, d$
3. Set $g = g + 1$ and go to step 2

- Represent joint $f(x)$ by sampling conditional $f(x_i|x_{-i})$
  - discard burn-in phase, $\{x^{(g)}\}_{g=1}^{G}$ approximate $f(x)$
  - Rao-Blackwellization: $\hat{f}(x_i) = \frac{1}{G}\sum_{g=1}^{G} f(x_i|x_{-i}^{(g)})$
  - rule of thumb: highly correlated x_i's in one block
  - what if some $f(x_i|x_{-i})$ cannot be sampled directly?
- Exercise: prove for $d = 2$ blocks, Gibbs kernel

$$p(x, y) = f(y_1|x_2) f(y_2|y_1), \quad x = (x_1, x_2), \quad y = (y_1, y_2)$$

has invariant distribution $f(\cdot)$

Gibbs Algorithm (Cont'd)

- Consider Gaussian model
  - likelihood: $y_i \sim_{i.i.d.} \mathbb{N}(\mu, h^{-1})$, $i = 1, \ldots, n$
  - conditionally conjugate prior: $\mu \sim \mathbb{N}(\mu_0, h_0^{-1})$, $h \sim \mathbb{G}(\frac{\alpha_0}{2}, \frac{\delta_0}{2})$
  - conditional posteriors are of same family
- Gibbs algorithm
  - step 1: choose $\mu = \mu^{(0)}$, $h = h^{(0)}$, set $g = 0$
  - step 2: sample recursively

$$
\begin{aligned}
\mu^{(g+1)} \quad & \sim \quad \mathbb{N}\left(\frac{h_0 + \mu_0 + h^{(g)} n\bar{y}}{h_0 + h^{(g)} n}, (h_0 + h^{(g)} n)^{-1}\right) \\
h^{(g+1)} \quad & \sim \quad \mathbb{G}\left(\frac{\alpha_0 + n}{2}, \frac{\delta_0 + \sum_{i=1}^{n} (y_i - \mu^{(g+1)})^2}{2}\right)
\end{aligned}
$$

  - step 3: set $g = g + 1$ and go to step 2

Marginal Likelihood

Marginal likelihood identity

$$m(y) = \frac{f(y|\theta^*)\pi(\theta^*)}{\pi(\theta^*|y)}, \qquad \forall \theta^* \in \Theta$$

- Chib (1995) computes $\pi(\theta^*|y)$ at high-density point θ^* from Gibbs output, e.g.

$$\pi(\theta_1^*, \theta_2^*, \theta_3^*|y) = \pi(\theta_1^*|y)\pi(\theta_2^*|\theta_1^*, y)\pi(\theta_3^*|\theta_1^*, \theta_2^*, y)$$

  - full run: $\hat{\pi}(\theta_1^*|y) = \frac{1}{G}\sum_{g=1}^{G} \pi(\theta_1^*|\theta_2^{(g)}, \theta_3^{(g)}, y)$, where $\theta^{(g)} \sim \pi(\theta|y) \Rightarrow (\theta_2^{(g)}, \theta_3^{(g)}) \sim \pi(\theta_2, \theta_3|y)$
  - reduced run: $\hat{\pi}(\theta_2^*|\theta_1^*, y) = \frac{1}{G}\sum_{g=1}^{G} \pi(\theta_2^*|\theta_1^*, \theta_3^{(g)}, y)$, where $\theta_{-1}^{(g)} \sim \pi(\theta_{-1}|\theta_1^*, y) \Rightarrow \theta_2^{(g)} \sim \pi(\theta_2|\theta_1^*, y)$, $\theta_3^{(g)} \sim \pi(\theta_3|\theta_1^*, y)$
  - $\pi(\theta_3^*|\theta_1^*, \theta_2^*, y)$ can be evaluated directly

Metropolis-Hastings Algorithm

Algorithm 2

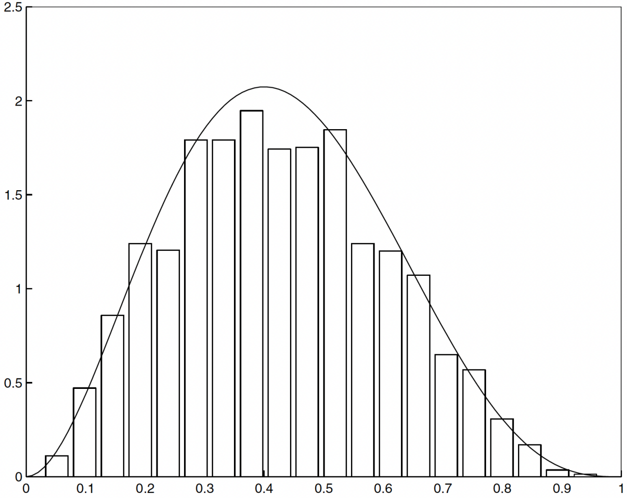
1. Choose $x^{(0)}$ and set $g = 0$
2. Sample proposal $y \sim q(x^{(g)}, y)$, $u \sim \mathbb{U}(0,1)$. If

$$u \leq \alpha(x^{(g)}, y) = \begin{cases} \min\left\{\frac{f(y)q(y,x^{(g)})}{f(x^{(g)})q(x^{(g)},y)}, 1\right\}, & \text{if } f(x^{(g)})q(x^{(g)}, y) > 0 \\ 0, & \text{otherwise} \end{cases}$$

   set $x^{(g+1)} = y$; otherwise, set $x^{(g+1)} = x^{(g)}$
3. Set $g = g + 1$ and go to step 2

- Chib & Greenberg (1995): MH kernel $p(x, y) = \alpha(x, y)q(x, y)$ is reversible and has invariant distribution $f(\cdot)$
  - choice of proposal: random-walk/independence, but good mixing requires 'tailoring' proposal to target
  - more generally, MH-within-Gibbs algorithm

MH Algorithm (Cont'd)



- Target: $\mathbb{B}(3, 4)$; proposal: $\mathbb{U}(0, 1)$; $G = 5,000$ draws

Marginal Likelihood Revisited

Marginal likelihood identity

$$m(y) = \frac{f(y|\theta^*)\pi(\theta^*)}{\pi(\theta^*|y)}, \qquad \forall \theta^* \in \Theta$$

- Chib and Jeliazkov (2001) computes $\pi(\theta^*|y)$ at high-density point θ^* from MH output, e.g. for one-block case

$$\alpha(\theta, \theta^*|y) q(\theta, \theta^*|y) \pi(\theta|y) = \alpha(\theta^*, \theta|y) q(\theta^*, \theta|y) \pi(\theta^*|y)$$

from which

$$\pi(\theta^*|y) = \frac{\int \alpha(\theta, \theta^*|y) q(\theta, \theta^*|y) \pi(\theta|y) d\theta}{\int \alpha(\theta^*, \theta|y) q(\theta^*, \theta|y) d\theta}$$

  - numerator: $\frac{1}{G}\sum_{g=1}^{G} \alpha(\theta^{(g)}, \theta^*|y) q(\theta^{(g)}, \theta^*|y)$, $\theta^{(g)} \sim \pi(\theta|y)$
  - denominator: $\frac{1}{G}\sum_{g=1}^{G} \alpha(\theta^*, \theta^{(g)}|y)$, $\theta^{(g)} \sim q(\theta^*, \theta|y)$

Convergence

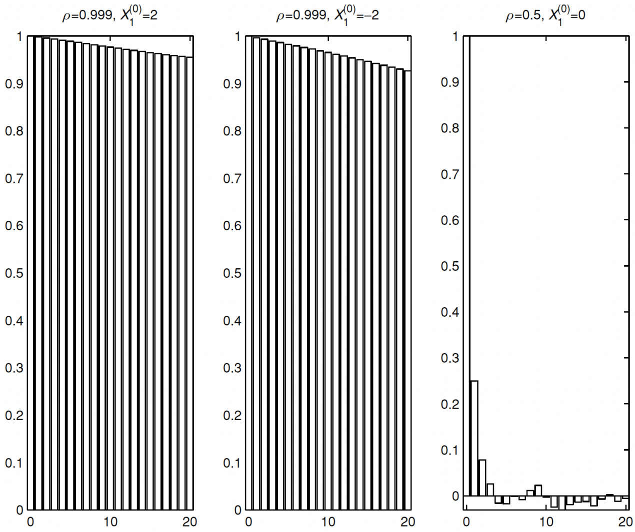
- Measures of convergence
  - autocorrelation function $\rho(\cdot)$
  - inefficiency factor

$$\frac{\text{numerical variance of MCMC draws}}{\text{numerical variance of i.i.d. draws}} \approx 1 + 2\sum_{j=1}^{K} w(j/K)\rho(j)$$

  $\rho(\cdot)$ is truncated by K and weighted by Parzen kernel $w(\cdot)$

- Judging convergence is as much art as science: ‘low’ serial correlation and inefficiency factor

Convergence (Cont'd)



- Gibbs sampler for $\mathbb{N}(0, \Sigma)$, $\Sigma = [1, \rho; \rho, 1]$

Readings

- Chib (1995), "Marginal Likelihood from the Gibbs Output," *Journal of the American Statistical Association*
- Chib & Greenberg (1995), "Understanding the Metropolis-Hastings Algorithm," *The American Statistician*
- Chib & Jeliazkov (2001), "Marginal Likelihood from the Metropolis-Hastings Output," *Journal of the American Statistical Association*