A Frequency-Domain Approach to Dynamic Macroeconomic Models†

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Abstract

This article proposes a unified framework for solving and estimating linear rational expectations models with a variety of frequency-domain techniques, some established, some new. Unlike existing strategies, our starting point is to obtain the model solution entirely in the frequency domain. This solution method is applicable to a wide class of models and leads to straightforward construction of the spectral density for performing likelihood-based inference. To cope with potential model uncertainty, we also generalize the well-known spectral decomposition of the Gaussian likelihood function to a composite version implied by several competing models. Taken together, these techniques yield fresh insights into the model's theoretical and empirical implications beyond conventional time-domain approaches can offer. We illustrate the proposed framework using a prototypical new Keynesian model with fiscal details and two determinate monetary-fiscal policy regimes. The model is simple enough to deliver an analytical solution that makes the policy effects transparent under each regime, yet still able to shed light on the empirical interactions between U.S. monetary and fiscal policies along different frequencies.

Keywords: solution method; analytic function; Bayesian inference; spectral density; monetary and fiscal policy.

JEL Classification: C32, C51, C52, C65, E63, H63

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1 Introduction

In a collection of influential papers, Lucas and Sargent (1981) and Hansen and Sargent (1991) pioneered a research program on the so-called rational expectations econometrics, which aims to integrate dynamic economic models with econometric methods for the purpose of formulating and interpreting economic time series. At the core of this program lies Lucas' (1976) insight that sophisticated feedback relations exist between economic policy and the behavior of rational agents. Consequently, disentangling these relations is a prerequisite to conducting reliable econometric policy evaluation. Yet despite the tight link it promises between theory and estimation, rational expectations modelling at its early stage posed keen computational challenges to characterizing the concomitant cross-equation restrictions because they typically constrain the vector stochastic process of observables in a very complicated manner.

Subsequently, a variety of time-domain solution techniques had been proposed to solve linear rational expectations (LRE) models, allowing for a numerical characterization of the cross-equation restrictions even for high-dimensional systems [Blanchard and Kahn (1980), Uhlig (1999), Klein (2000), Sims (2002)]. Meanwhile, dynamic stochastic general equilibrium (DSGE) models had reached a level of sophistication that rendered it a useful tool for quantitative macroeconomic analysis in both academia and policymaking institutions. Lending credence to these developments and the continued improvement in model fit, it had become nearly the norm to estimate these models in the time domain using likelihood-based econometric procedures [Leeper and Sims (1994), Ireland (1997), Smets and Wouters (2007), An and Schorfheide (2007)].

While time-domain methods provide a popular framework for confronting theory with data, it necessarily precludes the additional insights into a model's cross-frequency implications that a spectral approach can complement. One compelling reason is that potential model misspecification along certain frequencies may produce spillover effects onto the whole spectrum and therefore contaminate statistical inference. As argued forcefully in Diebold et al. (1998), working in the frequency domain, on the other hand, is especially useful in communicating the strengths and weaknesses of a model over different frequency bands of interest.[1] Such flexibility of assessing model adequacy is difficult, if at all possible, to accomplish in the time domain. In light of the value added by spectral methods, this paper proposes a unified frequency-domain framework for conveniently solving and estimating dynamic linear models under the hypothesis of rational expectations. Indeed, most of the techniques described below are rooted in the spirit of Hansen and Sargent (1980) as well as many other early incarnations of rational expectations econometrics.

Unlike existing strategies that solve the model uniformly in the time domain, our starting point is to obtain the model solution entirely in the frequency domain. Whiteman (1983) outlined four

[1] Among others, see also Hansen and Sargent (1993), Watson (1993), Berkowitz (2001), Cogley (2001) and, more recently, Beaudry et al. (2016) who demonstrated the benefits of investigating dynamic economic models in the frequency domain.

tenets underlying this solution principle that distinguishes it from other work on solving linear expectational difference equations: [i] exogenous driving process is taken to be zero-mean linearly regular covariance stationary stochastic process with known Wold representation; [ii] expectations are formed rationally and computed using the Wiener-Kolmogorov optimal prediction formula; [iii] moving average solutions are sought in the space spanned by time-independent square-summable linear combinations of the process fundamental for the driving process; [iv] rational expectations restrictions are required to hold for all realizations of the driving process. The above principle is *generic* in that the exogenous driving process is assumed to only satisfy covariance stationarity, which lends itself well to solving a wide class of models, including dynamic economies with incomplete information, e.g., Kasa (2000), or heterogeneous beliefs, e.g., Walker (2007).

Without much loss of generality, we present a simplified but more accessible version of the solution method from Tan and Walker (2015), who extended Whiteman's (1983) principle to the multivariate setting, and accommodate their algorithm to allow for the possibility of equilibrium non-uniqueness, a phenomenon often referred to as indeterminacy.[2] While our analysis applies to any LRE model, we provide a step-by-step guideline for implementation with the aid of a generic univariate example. More broadly, our algorithm falls under the theory of linear systems. A related solution method can be found in Onatski (2006) and its generalization in Al-Sadoon (2018), who employ the Wiener–Hopf factorization to deliver simple conditions for existence and uniqueness of both particular and generic LRE models.

By virtue of the generic moving average solution, it is straightforward to construct the spectral density for performing likelihood-based inference. In particular, our econometric analysis is built upon a well-known property due to Hannan (1970) that the Gaussian log-likelihood function has an asymptotic linear decomposition in the frequency domain. In this vein, a number of authors have embraced such property to estimate and evaluate small to medium scale DSGE models based on the full spectrum or a set of preselected frequencies [Altug (1989), Christiano and Vigfusson (2003), Qu and Tkachenko (2012a,b), Qu (2014), Sala (2015)].

A more challenging situation, which oftentimes arises from the policymaking process, is that there can be several competing models available to the researcher. To cope with potential model uncertainty, we also generalize the spectral likelihood representation for a single model to a composite version formed by aggregating several component likelihoods, each of which corresponds to a candidate model. The idea of composite likelihood was originally introduced by Besag (1974) and Lindsay (1988) into the statistical literature and has recently found economic applications that differ in the composition of combined likelihoods.[3] For example, Canova and Matthes

[2] In the time domain, Lubik and Schorfheide (2003) and, more recently, Farmer et al. (2015) proposed modifications to the approach advocated by Sims (2002) that characterize the complete set of indeterminate equilibria. See Benhabib and Farmer (1999) for an overview of the related literature.

[3] Varin et al. (2011) surveyed the theory of composite likelihood and its wide range of application areas,

(2018) considered a mix of time-domain likelihoods of distinct structural or statistical models to address a number of estimation and inferential problems that are common in DSGE models. Qu (2018) developed a frequency-domain framework specifically for singular DSGE models by pooling a set of nonsingular submodel likelihoods corresponding to different observables. From a frequency-domain perspective, our aggregation scheme stands in contrast to these endeavors, which are equivalent to jointly fitting each model over the entire spectral density, in that component models are integrated according to their performance across different frequency bands. To the best of our knowledge, this extension is novel in the literature, enabling the relative importance of individual model to be assessed at each frequency. Together with the spectral solution method, these techniques yield fresh insights into the model's theoretical and empirical implications beyond conventional time-domain approaches can offer.

We illustrate the proposed framework using a prototypical new Keynesian model with fiscal details and two determinate policy regimes. Each regime embodies a completely different mechanism under which monetary and fiscal policy can jointly determine inflation and stabilize government debt. The model is kept simple enough to admit an analytical solution that is useful in characterizing the cross-equation restrictions and illustrating the complex interaction between policy behavior and price rigidity under each regime. Yet it is still able to shed light on the empirical interactions between U.S. monetary and fiscal policies along different frequencies. Our main findings are twofold. First, the combination of policy regimes, sample periods, and band spectra can generate markedly different posterior inferences for the model parameters. Second, in line with Kliem et al. (2016a,b), relatively low frequency relations in the data play an important role in discerning the underlying regime.

The rest of the paper is structured as follows. Section 2 describes the solution and econometric procedures within a unified framework. Section 3 illustrates the proposed framework using a simple monetary model for the study of price level determination. Section 4 concludes.

2 A Unified Framework

This section establishes the theoretical foundation of our frequency-domain approach and highlights its advantages vis-à-vis other popular time-domain approaches. While most of the apparatus described herein have been proposed in various strands of the literature, we present a unified framework for conveniently solving and estimating dynamic linear models under rational expectations. To keep the exposition self-contained, Section 2.1 briefly outlines the solution methodology and demonstrates its use via a simple univariate example. Section 2.2 derives the spectral likelihood function implied by the state space representation of the model, which is amenable to conducting classical or Bayesian inference based on selected band spectra of interest.

including geostatistics, spatial extremes, space-time models, etc.

2.1 Solution Method We consider a general class of multivariate LRE models that can be cast into the canonical form of Tan and Walker (2015)

$$\mathbb{E}_t\left[\sum_{k=-n}^{m} \Gamma_k L^k x_t\right] = \mathbb{E}_t\left[\sum_{k=-n}^{l} \Psi_k L^k d_t\right] \tag{2.1}$$

where L is the lag operator, i.e., $L^k x_t = x_{t-k}$, x_t is a $p \times 1$ vector of endogenous variables, $\{\Gamma_k\}_{k=-n}^m$ and $\{\Psi_k\}_{k=-n}^l$ are $p \times p$ and $p \times q$ coefficient matrices, and $\mathbb{E}_t$ represents mathematical expectation given information available at time t, including the model's structure and all past and current realizations of the endogenous and exogenous processes. Moreover, d_t is a $q \times 1$ vector of covariance stationary exogenous driving process with Wold decomposition

$$d_t = \sum_{k=0}^{\infty} A_k \epsilon_{t-k} \equiv A(L)\epsilon_t \tag{2.2}$$

where $\epsilon_t = d_t - \mathbb{P}[d_t | d_{t-1}, d_{t-2}, \ldots]$, $\mathbb{P}[d_t | d_{t-1}, d_{t-2}, \ldots]$ is the optimal linear predictor for d_t conditional on knowing $\{d_{t-k}\}_{k=1}^{\infty}$, and each element of $\sum_{k=0}^{\infty} A_k A_k'$ is finite.

We seek the solution to (2.1) in the Hilbert space generated by current and past shocks $\{\epsilon_{t-k}\}_{k=0}^{\infty}$

$$x_t = \sum_{k=0}^{\infty} C_k \epsilon_{t-k} \equiv C(L)\epsilon_t \tag{2.3}$$

where x_t is taken to be covariance stationary. Throughout this section, we use a generic univariate model below as an illustrative example to guide the reader through the key steps in deriving the content of $C(\cdot)$

$$\mathbb{E}_t x_{t+2} - (\rho_1 + \rho_2)\mathbb{E}_t x_{t+1} + \rho_1\rho_2 x_t = d_t \tag{2.4}$$

where $|\rho_1| > 1$ and $0 < |\rho_2| < 1$. The dimensions of this model are $p = q = 1$ with nonzero coefficient matrices $\Gamma_{-2} = 1$, $\Gamma_{-1} = -(\rho_1 + \rho_2)$, $\Gamma_0 = \rho_1\rho_2$, and $\Psi_0 = 1$.

Step 1: transform the time-domain system (2.1) into its equivalent frequency-domain representation. To this end, we define ν_t (η_t) as a vector of expectational errors satisfying $\nu_{t+k} \equiv d_{t+k} - \mathbb{E}_t d_{t+k}$ ($\eta_{t+k} \equiv x_{t+k} - \mathbb{E}_t x_{t+k}$) for all $k > 0$, which can be evaluated with (2.2)–(2.3) and the Wiener-Kolmogorov optimal prediction formula

$$\nu_{t+k} = L^{-k}\left(\sum_{i=0}^{k-1} A_i L^i\right)\epsilon_t, \quad \eta_{t+k} = L^{-k}\left(\sum_{i=0}^{k-1} C_i L^i\right)\epsilon_t$$

Substituting the above expressions and (2.2)–(2.3) into (2.1) gives

$$\Gamma(L)C(L)\epsilon_t = \left\{\Psi(L)A(L) + \sum_{k=1}^{n}\left[\Gamma_{-k}L^{-k}\left(\sum_{i=0}^{k-1} C_i L^i\right) - \Psi_{-k}L^{-k}\left(\sum_{i=0}^{k-1} A_i L^i\right)\right]\right\}\epsilon_t$$

where $\Gamma(L) \equiv \sum_{k=-n}^{m} \Gamma_k L^k$ and $\Psi(L) \equiv \sum_{k=-n}^{l} \Psi_k L^k$. Define the z-transform of $\{C_k\}_{k=0}^{\infty}$ (analogously to any sequence of coefficient matrices) as $C(z) \equiv \sum_{k=0}^{\infty} C_k z^k$, where z is a complex number. Since the above equation must hold for all realizations of ϵ_t, its coefficient matrices are related by the z-transform identities

$$z^n\Gamma(z)C(z) = z^n\Psi(z)A(z) + \sum_{t=1}^{n}\sum_{s=t}^{n}\left(\Gamma_{-s}C_{t-1} - \Psi_{-s}A_{t-1}\right) z^{n-s+t-1} \tag{2.5}$$

Specifically, the z-transform of the generic model (2.4) becomes

$$[1 - (\rho_1 + \rho_2)z + \rho_1\rho_2 z^2]C(z) = z^2 A(z) + [1 - (\rho_1 + \rho_2)z]C_0 + C_1 z$$

Appealing to the Riesz-Fischer Theorem [see Sargent (1987), p. 249–253], the square-summability (i.e., covariance stationarity) of $\{C_k\}_{k=0}^{\infty}$ implies that the infinite series in $C(z)$ converges in the mean square sense that $\lim_{j\to\infty} \oint |\sum_{k=0}^{j} C_k z^k - C(z)|^2 \frac{dz}{z} = 0$, where $\oint$ denotes counterclockwise integral about the unit circle, and $C(z)$ is analytic at least inside the unit circle. This requirement can be examined by a careful factorization of $z^n\Gamma(z)$ in the next step.

Step 2: apply the Smith canonical factorization to the polynomial matrix $z^n\Gamma(z)$

$$z^n\Gamma(z) = U(z)^{-1} \underbrace{\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \prod_{k=1}^{r^-}(z-\lambda_k^-) \end{pmatrix}}_{S(z)} \underbrace{\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \prod_{k=1}^{r^+}(z-\lambda_k^+) \end{pmatrix}}_{T(z)} V(z)^{-1}$$

where we factorize all roots inside the unit circle, λ_k^-'s, from those outside, λ_k^+'s, and collect them in the polynomial matrix $S(z)$. Moreover, both $U(z)$ and $V(z)$ are $p \times p$ polynomial matrices with nonzero constant determinants.[4] Regarding the generic model (2.4), we have $\lambda_1^- = 1/\rho_1$, $\lambda_1^+ = 1/\rho_2$, $U(z) = 1/\rho_1$, and $V(z) = 1/\rho_2$.

[4]The Smith factorization is available in MAPLE or MATLAB's Symbolic Toolbox. It decomposes any square polynomial matrix $P(z)$ as $U(z)P(z)V(z) = \Lambda(z)$ using elementary row and column operations, where $\Lambda(z) = \mathrm{diag}(\lambda_1(z), \ldots, \lambda_r(z))$ is diagonal and $\lambda_i(z)$'s are unique monic scalar polynomials such that $\lambda_i(z)$ is divisible by $\lambda_{i-1}(z)$. To simplify the exhibition, we assume that all roots are distinct. See Tan and Walker (2015) for the general case that allows for the possibility of repeated roots.

Step 3: examine the existence of solution. A covariance stationary solution exists if the free coefficient matrices $C_0, C_1 \ldots, C_{n-1}$ in (2.5) can be chosen to cancel those problematic roots in $S(z)$. To check that, multiply both sides of (2.5) by $S(z)^{-1}$ to obtain

$$T(z)C(z) = \begin{pmatrix} U_{1\cdot}(z) \\ \vdots \\ U_{(p-1)\cdot}(z) \\ \frac{1}{\prod_{k=1}^{r^-}(z-\lambda_k^-)} U_{p\cdot}(z) \end{pmatrix} \left[z^n \Psi(z) A(z) + \sum_{t=1}^{n} \sum_{s=t}^{n} \left(\Gamma_{-s} C_{t-1} - \Psi_{-s} A_{t-1}\right) z^{n-s+t-1} \right]$$

where $U_{j\cdot}$ is the jth row of $U(z)$. These identities are valid for all z on the open unit disk except at the singularities λ_k^-'s. But since $C(z)$ must be analytic for all $|z| < 1$, this condition places the following restrictions on $C_0, C_1 \ldots, C_{n-1}$

$$U_{p\cdot}(\lambda_k^-) \left[(\lambda_k^-)^n \Psi(\lambda_k^-) A(\lambda_k^-) + \sum_{t=1}^{n} \sum_{s=t}^{n} \left(\Gamma_{-s} C_{t-1} - \Psi_{-s} A_{t-1}\right) (\lambda_k^-)^{n-s+t-1} \right] = 0 \tag{2.6}$$

Stacking the restrictions in (2.6) over $k = 1, \ldots, r^-$ yields

$$\underbrace{\begin{pmatrix} U_{p\cdot}(\lambda_1^-)[(\lambda_1^-)^n \Psi(\lambda_1^-) A(\lambda_1^-) - \sum_{t=1}^{n} \sum_{s=t}^{n} \Psi_{-s} A_{t-1} (\lambda_1^-)^{n-s+t-1}] \\ \vdots \\ U_{p\cdot}(\lambda_{r^-}^-)[(\lambda_{r^-}^-)^n \Psi(\lambda_{r^-}^-) A(\lambda_{r^-}^-) - \sum_{t=1}^{n} \sum_{s=t}^{n} \Psi_{-s} A_{t-1} (\lambda_{r^-}^-)^{n-s+t-1}] \end{pmatrix}}_{A} = -$$

$$\underbrace{\begin{pmatrix} U_{p\cdot}(\lambda_1^-) \sum_{s=1}^{n} \Gamma_{-s} (\lambda_1^-)^{n-s} & \cdots & U_{p\cdot}(\lambda_1^-) \Gamma_{-n} (\lambda_1^-)^{n-1} \\ \vdots & \ddots & \vdots \\ U_{p\cdot}(\lambda_{r^-}^-) \sum_{s=1}^{n} \Gamma_{-s} (\lambda_{r^-}^-)^{n-s} & \cdots & U_{p\cdot}(\lambda_{r^-}^-) \Gamma_{-n} (\lambda_{r^-}^-)^{n-1} \end{pmatrix}}_{R} \underbrace{\begin{pmatrix} C_0 \\ \vdots \\ C_{n-1} \end{pmatrix}}_{C}$$

Apparently, the solution exists if and only if the column space of R spans the column space of A, i.e., $\text{span}(A) \subseteq \text{span}(R)$. This space spanning condition holds for the generic model (2.4) with $A = \rho_1^{-3} A(\rho_1^{-1})$ and $R = [-\rho_1^{-2}\rho_2, \rho_1^{-2}]$, though there are infinitely many choices of $C = [C_0', C_1']'$ satisfying $A = -RC$, which can be confirmed by checking the uniqueness condition below.

Step 4: examine the uniqueness of solution.[5] In order for the solution to be unique, we must be able to determine $\{C_k\}_{k=0}^{\infty}$ from the restrictions imposed by $A = -RC$. Since $V(z)$ is of full rank, this is equivalent to determining the coefficients $\{D_k\}_{k=0}^{\infty}$ of $D(z) \equiv V(z)^{-1} C(z)$. From the

[5]The criterion for model determinacy presented herein corrects an important error in the version originally derived in Tan and Walker (2015).

inversion formula we have

$$\begin{aligned} D_k &= \frac{1}{2\pi i}\oint D(z)z^{-k-1}dz \\ &= \text{sum of residues of } D(z^{-1})z^{k-1} \text{ at roots inside unit circle} \end{aligned}$$

where

$$D(z^{-1})z^{k-1} = \begin{pmatrix} U_{1\cdot}(z^{-1})z^{k-1} \\ \vdots \\ U_{(p-1)\cdot}(z^{-1})z^{k-1} \\ \frac{1}{\prod_{k=1}^{r^-}(z^{-1}-\lambda_k^-)\prod_{k=1}^{r^+}(z^{-1}-\lambda_k^+)}U_{p\cdot}(z^{-1})z^{k-1} \end{pmatrix}.$$

$$\left[z^{-n}\Psi(z^{-1})A(z^{-1}) + \sum_{t=1}^{n}\sum_{s=t}^{n}\left(\Gamma_{-s}C_{t-1} - \Psi_{-s}A_{t-1}\right)z^{-(n-s+t-1)} \right]$$

Note that only the last row of $D(z^{-1})z^{k-1}$ has roots inside unit circle at $1/\lambda_k^+$'s. It can be shown that $C_0, C_1 \dots, C_{n-1}$ affect D_k's only through the following common terms

$$U_{p\cdot}(\lambda_k^+)\sum_{t=1}^{n}\sum_{s=t}^{n}\Gamma_{-s}C_{t-1}(\lambda_k^+)^{n-s+t-1} \tag{2.7}$$

Stacking the expressions in (2.7) over $k = 1, \dots, r^+$ yields

$$\underbrace{\begin{pmatrix} U_{p\cdot}(\lambda_1^+)\sum_{s=1}^{n}\Gamma_{-s}(\lambda_1^+)^{n-s} & \cdots & U_{p\cdot}(\lambda_1^+)\Gamma_{-n}(\lambda_1^+)^{n-1} \\ \vdots & \ddots & \vdots \\ U_{p\cdot}(\lambda_{r^+}^+)\sum_{s=1}^{n}\Gamma_{-s}(\lambda_{r^+}^+)^{n-s} & \cdots & U_{p\cdot}(\lambda_{r^+}^+)\Gamma_{-n}(\lambda_{r^+}^+)^{n-1} \end{pmatrix}}_{Q}\underbrace{\begin{pmatrix} C_0 \\ \vdots \\ C_{n-1} \end{pmatrix}}_{C}$$

Therefore, the solution is unique if and only if the knowledge of RC can be used to pin down QC, which is tantamount to verifying whether the column space of R' spans the column space of Q', i.e., $\text{span}(Q') \subseteq \text{span}(R')$.[6] This space spanning condition fails for the generic model (2.4) with $Q = [-\rho_2^{-1}, \rho_1^{-1}\rho_2^{-1}]$ and $R = [-\rho_1^{-2}\rho_2, \rho_1^{-2}]$ due to $\rho_1 \neq \rho_2$.

Several remarks about the above solution methodology are in order. First, whenever the

[6] In practice, checking the space spanning criteria for existence and uniqueness and calculating the unknown coefficient matrix C can be achieved by applying the singular value decompositions of A, R, and Q. See Tan and Walker (2015) for computational details.

solution exists and is unique, its analytical form can be expressed as

$$C(L)\epsilon_t = (L^n\Gamma(L))^{-1}\left[L^n\Psi(L)A(L) + \sum_{t=1}^{n}\sum_{s=t}^{n}\left(\Gamma_{-s}C_{t-1} - \Psi_{-s}A_{t-1}\right)L^{n-s+t-1}\right]\epsilon_t \tag{2.8}$$

Such moving average representation leads directly to the impulse response function—the (i,j)th element of C_k, denoted $C_k(i,j)$, measures exactly the response of $x_{t+k}(i)$ to a shock $\epsilon_t(j)$. By linking the Wold representation of the exogenous process to the endogenous variables, (2.8) also captures all multivariate cross-equation restrictions imposed by the hypothesis of rational expectations, which Hansen and Sargent (1980) refer to as the "hallmark of rational expectations models".

Second, it is widely known that LRE models can have multiple equilibria in which both fundamental and sunspot shocks influence model dynamics. Loosely speaking, indeterminacy in our analysis stems from the lack of sufficient restrictions imposed by those roots inside the unit circle for uniquely determining the free coefficient matrices $C_0, C_1 \ldots, C_{n-1}$.[7] Based on the idea of Farmer et al. (2015), it is always possible to treat an indeterminate model as determinate and apply our solution method by redefining a subset of endogenous expectational errors as exogenous fundamental shocks. For example, consider again the generic model (2.4) reformulated as

$$\mathbb{E}_t y_{t+1} - (\rho_1 + \rho_2)y_t + \rho_1\rho_2 y_{t-1} = d_t - \rho_1\rho_2\eta_t$$

where $y_t = \mathbb{E}_t x_{t+1}$ and the forecast error $\eta_t = x_t - y_{t-1}$ is now taken as a fundamental shock. Suppose a covariance stationary solution is of the form $y_t = C(L)\epsilon_t + D(L)\eta_t$, where $\sum_{k=0}^{\infty} C_k^2 < \infty$ and $\sum_{k=0}^{\infty} D_k^2 < \infty$. Following the solution procedure outlined above, it is straightforward to verify that the solution is unique and given by

$$C(L) = \frac{LA(L) - \rho_1^{-1}A(\rho_1^{-1})}{(1-\rho_1 L)(1-\rho_2 L)}, \quad D(L) = \frac{\rho_2}{1-\rho_2 L}$$

from which we deduce that $x_t = LC(L)\epsilon_t + (1 + LD(L))\eta_t$.

Third, as advocated in Kasa (2000) and many others, models featuring dynamic signal extraction and infinite regress in expectations are more conveniently handled in the frequency domain. By circumventing the problem of matching an infinite sequence of coefficients in the time do-

[7] A corresponding time-domain notion of indeterminacy is that the endogenous forecast errors η are not uniquely determined by the exogenous fundamental shocks ϵ. Determinacy, however, does not necessarily require the mapping from ϵ to η be one-to-one; instead, it merely requires the knowledge from the unstable subsystem about equilibrium existence be able to pin down the error terms from the stable subsystem that are influenced by η [see Sims (2002), p. 7]. Thus, the degree of indeterminacy proposed by Lubik and Schorfheide (2003), which determines the dimension of sunspot shocks based solely on the existence condition, is only nominal.

main, our analytic function approach offers a tractable framework for the theoretical analysis of dynamic economies with incomplete information.

Finally, given the generic nature of exogenous driving processes, the moving average form of (2.8) allows for straightforward construction of the spectral density that provides the basis for performing likelihood-based inference over different frequency bands, which we elaborate in the next section.

2.2 Econometric Method

This section adopts the Bayesian perspective on taking dynamic macroeconomic models to the data. Our econometric analysis, including both parameter estimation and model evaluation, centers around a frequency-domain likelihood function implied by the LRE model (2.1). To that end, consider the following linear state space model parameterized by a vector of unknown parameters θ

$$y_t = Z_\theta(L)x_t + u_t, \quad u_t \sim \mathbb{N}(0, \Omega_\theta) \tag{2.9}$$

$$x_t = C_\theta(L)\epsilon_t, \quad \epsilon_t \sim \mathbb{N}(0, \Sigma_\theta) \tag{2.10}$$

where the measurement equation (2.9) links an $h \times 1$ vector of demeaned observable variables y_t to the model's (possibly latent) endogenous variables x_t subject to a vector of measurement errors u_t, and the transition equation (2.10) corresponds to the moving average solution to the model. Moreover, (u_t, ϵ_t) are mutually and serially uncorrelated at all leads and lags, and $\mathbb{N}(a, b)$ denotes the Gaussian distribution with mean vector a and covariance matrix b.

We will subsequently present the likelihood function associated with (2.9)–(2.10) and generalize it to a composite version when the underlying model space is taken to be incomplete—none of the models under consideration corresponds to the true data generating process. The latter approach has the flavor of linear prediction pools in the time domain that have been explored recently to assess the joint predictive performance of multiple macroeconomic models [Waggoner and Zha (2012), Negro et al. (2016), Amisano and Geweke (2017)].

2.2.1 Single Model

To begin with, suppose (2.10) is the only reduced form model available to the researcher. Then the model-implied spectral density matrix for the observables y_t can be conveniently formulated as

$$S_\theta(w) = \frac{1}{2\pi}\left[Z_\theta(e^{-iw})C_\theta(e^{-iw})\Sigma_\theta C_\theta(e^{-iw})^* Z_\theta(e^{-iw})^* + \Omega_\theta\right] \tag{2.11}$$

where $w \in [0, 2\pi]$ denotes the frequency, $i^2 = -1$, and the asterisk $(*)$ stands for the conjugate transpose.[8] Let $Y_{1:T}$ be a matrix that collects the sample for periods $t = 1, \ldots, T$ with row

[8] Without the inclusion of measurement errors, the spectral density matrix becomes singular for DSGE models with a small number of shocks and a larger number of observables, as is the case in Section 3.3. The conventional

observations y'_t. For any stationary Gaussian process y_t, it can be shown that the log-likelihood function has an asymptotic counterpart in the frequency domain [Hannan (1970), Harvey (1989)]:

$$\ln p(Y_{1:T}|\theta) = -\frac{1}{2}\sum_{k=0}^{T-1} s_k\{2\ln 2\pi + \ln[\det(S_\theta(w_k))] + \mathrm{tr}(S_\theta(w_k)^{-1}\mathrm{I}(w_k))\} \tag{2.12}$$

where $w_k = 2\pi k/T$ for $k = 0, 1, \ldots, T-1$, and $\det(\cdot)$ and $\mathrm{tr}(\cdot)$ denote the determinant and trace operators, respectively. In addition, the sample spectrum (or periodogram) $\mathrm{I}(w)$ is independent of θ and given by $\mathrm{I}(w) = y(w)y(-w)'/(2\pi T)$, where $y(w) = \sum_{t=1}^{T} y_t e^{-iwt}$ is the discrete Fourier transform of $Y_{1:T}$. In light of the excessive volatility of $\mathrm{I}(w)$, we follow Christiano and Vigfusson (2003) and compute its smoothed version $\tilde{\mathrm{I}}(w)$ by taking a centered, equally weighted average $\tilde{\mathrm{I}}(w_k) = \sum_{j=-5}^{5} \mathrm{I}(w_{k+j})/11$. For diagnostic purposes, we also incorporate pre-specified indicators s_k in (2.12) that takes value 1 if frequency w_k is included and value 0 otherwise.[9] This allows one to estimate and evaluate the model based on various frequency bands of interest.

From a computational perspective, since the summands in (2.12) are symmetric about π over the range $[0, 2\pi]$, there is no need to compute almost twice as many likelihood ordinates as are necessary. Also, the spectral density matrix (2.11) is the only part of the likelihood function that depends on θ and usually very easy to evaluate. The periodogram, on the other hand, is evaluated only once. These features lead to quite rapid calculations involved in an iterative estimation procedure even for high-dimensional systems.

2.2.2 Composite Model

In many situations, especially the policymaking process, there can be several competing models available to the researcher, giving rise to the natural question of model selection or composition. While Bayesian model averaging provides a useful way to account for model uncertainty, it operates under an implicit assumption that the underlying model space is complete—one of the models under consideration is correctly specified. An important consequence, as shown by Geweke and Amisano (2011), is that the full posterior weight will be assigned to whichever model that lies closest (in terms of the Kullback-Leibler divergence) to the true data generating process as $T \to \infty$. But more realistically, say, a prudent policymaker may view each model as misspecified along some aspects of the reality and therefore base her policy thinking beyond the implications from any single model. Recognizing the possibility of potential model misspecification over certain band spectrum, this section attempts to generalize the quasi-likelihood function (2.12) from the premise of an incomplete model space.

information matrix, though easily obtainable in the frequency domain, does not exist under singularity. This invalidates the well-known rank condition of Rothenberg (1971) for local identification of the unknown parameters. Qu and Tkachenko (2012b) derived simple frequency-domain identification conditions applicable to both singular and nonsingular DSGE models.

[9] This is justified by the fact that components of (2.12) formed over disjoint frequencies correspond to processes that are mutually orthogonal at all lags.

To make the idea concrete, suppose the expanded model space consists of two reduced form models, each of which is intended to fit a common set of observables y_t and can be represented in the linear state space form

$$\begin{aligned} y_t &= Z_{\theta_j}(L)x_{j,t} + u_{j,t}, \quad u_{j,t} \sim \mathbb{N}(0, \Omega_{\theta_j}) \\ x_{j,t} &= C_{\theta_j}(L)\epsilon_{j,t}, \quad \epsilon_{j,t} \sim \mathbb{N}(0, \Sigma_{\theta_j}) \end{aligned}$$

where $j \in \{1, 2\}$ denotes the model index and θ_j parameterizes model j. Let $S_{\theta_j}(w)$ be the spectral density matrix implied by model j and consider the following log-likelihood function

$$\begin{aligned} \ln p(Y_{1:T}|\theta_1, \theta_2, \{s_k\}_{k=0}^{T-1}) = &-\frac{1}{2}\sum_{k=0}^{T-1} s_k\{2\ln 2\pi + \ln[\det(S_{\theta_1}(w_k))] + \mathrm{tr}(S_{\theta_1}(w_k)^{-1}\mathrm{I}(w_k))\} \\ &-\frac{1}{2}\sum_{k=0}^{T-1}(1 - s_k)\{2\ln 2\pi + \ln[\det(S_{\theta_2}(w_k))] + \mathrm{tr}(S_{\theta_2}(w_k)^{-1}\mathrm{I}(w_k))\} \end{aligned} \tag{2.13}$$

which generalizes its single-model version (2.12) in two major aspects. First, rather than discarding the log-likelihood ordinates at some frequencies, we allow both candidate models to bear directly on mutually exclusive and collectively exhaustive frequencies. Formally, (2.13) can be understood as a frequency-domain instance of the composite likelihood, a term coined by Lindsay (1988)—a likelihood-type object formed by adding together individual component log-likelihoods, each of which corresponds to a marginal or conditional event. Second, the assignment of model-dependent log-likelihood ordinate to each frequency is now driven by a set of latent model-selection variables $\{s_k\}_{k=0}^{T-1}$ whose values are inferred from the data. By virtue of the symmetry of (2.13) about π, we require that $s_k = s_{T-k}$ for $k = 1, 2, \ldots, T/2 - 1$.[10]

The composite log-likelihood function (2.13) corresponds to its time-domain state space model (2.9)–(2.10) defined by $\theta = [\theta_1', \theta_2']'$ and

$$Z_\theta(L) = \begin{pmatrix} B(L)Z_{\theta_1}(L) & (\mathrm{I}_h - B(L))Z_{\theta_2}(L) \end{pmatrix}, \quad C_\theta(L) = \begin{pmatrix} C_{\theta_1}(L) & 0_p \\ 0_p & C_{\theta_2}(L) \end{pmatrix}$$

$$x_t = \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix}, \quad \epsilon_t = \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix}, \quad u_t = \begin{pmatrix} B(L) & \mathrm{I}_h - B(L) \end{pmatrix} \begin{pmatrix} u_{1,t} \\ u_{2,t} \end{pmatrix}$$

where I_h is an $h \times h$ identity matrix, 0_p a $p \times p$ zero matrix, and $B(L)$ a "random filter" that satisfies $B(e^{-iw_k}) = s_k\mathrm{I}_h$ for $k = 0, 1, \ldots, T-1$. Note that the set of coefficient matrices $\{b_j\}_{j=-\infty}^{\infty}$ for $B(\cdot)$, whose values depend on the realizations of $\{s_k\}_{k=0}^{T-1}$, can be determined via the inversion formula $b_j = \frac{1}{2\pi}\int_0^{2\pi} B(e^{-iw})e^{iwj}dw$ for all integers j and $b_j = b_{-j}$. In the special case of $B(L) = \mathrm{I}_h$

[10] We implicitly assume that T is even. The adjustment when T is odd is straightforward.

or $B(L) = 0_h$, (2.13) reduces to (2.12) so that only one model survives.

At a conceptual level, the unobserved indicators $\{s_k\}_{k=0}^{T-1}$ can be simply treated as additional unknown parameters from the Bayesian point of view. This motivates a full Bayesian procedure to estimate the model based on the idea of data augmentation [Tanner and Wong (1987)]. Specifically, we assume for convenience that $(\theta_1, \theta_2, \{s_k\}_{k=0}^{T-1})$ are a priori independent and sample from their joint posterior distribution $p(\theta_1, \theta_2, \{s_k\}_{k=0}^{T-1}|Y_{1:T})$ with the following Gibbs steps:

1. Simulate model 1's parameters θ_1 from

$$p(\theta_1|Y_{1:T}, \theta_2, \{s_k\}_{k=0}^{T-1}) \propto p(Y_{1:T}|\theta_1, \theta_2, \{s_k\}_{k=0}^{T-1})p(\theta_1)$$

   using the Metropolis-Hastings algorithm, where $p(Y_{1:T}|\theta_1, \theta_2, \{s_k\}_{k=0}^{T-1})$ is given by (2.13).

2. Like step 1, simulate model 2's parameters θ_2 from $p(\theta_2|Y_{1:T}, \theta_1, \{s_k\}_{k=0}^{T-1})$.

3. Simulate the indicator s_k from

$$p(s_k = j|Y_{1:T}, s_{-k}, \theta_1, \theta_2) \propto p(Y_{1:T}|\theta_1, \theta_2, \{s_k\}_{k=0}^{T-1})p(s_k = j)$$

   for $k = 0, \ldots, T/2$, where $s_{-k} = (s_0, \ldots, s_{k-1}, s_{k+1}, \ldots, s_{T-1})$. The normalizing constant of this kernel function is the sum of its values over $s_k = 0, 1$.

The above cycle is initialized at some starting values of $(\theta_1, \theta_2, \{s_k\}_{k=0}^{T-1})$ and then repeated a sufficiently large number of times until the posterior sampler has converged. Based on the draws from the joint posterior distribution, one can compute summary statistics such as posterior means and probability intervals.

3 Application to a New Keynesian Model

As an example, we illustrate the proposed framework using a prototypical new Keynesian model with fiscal details and two determinate monetary-fiscal policy regimes. This serves to keep the illustration simple and concrete, but it should be emphasized that these techniques are widely applicable for more richly structured models, which we leave for future research. Section 3.1 presents a linearized version of the model. Section 3.2 derives its analytical solution in the frequency domain that proves useful in characterizing the cross-equation restrictions and understanding the policy transmission mechanisms under each regime. Section 3.3 documents how the empirical performance of each regime varies across different frequency bands.

3.1 The Model We consider a textbook version of the new Keynesian model presented in Woodford (2003) and Galí (2008) but augmented with a fiscal policy rule. The model's essential elements include: a representative household and a continuum of firms, each producing a differentiated good; only a fraction of firms can reset their prices each period; a cashless economy with one-period nominal bonds B_t that sell at price $1/R_t$, where R_t is the monetary policy instrument; primary surplus s_t with lump-sum taxation and zero government spending so that consumption equals output, $c_t = y_t$; a monetary authority and a fiscal authority.

Let $\hat{x}_t \equiv \ln(x_t) - \ln(x)$ denote the log-deviation of a generic variable x_t from its steady state x. It is straightforward to show that a log-linear approximation to the model's equilibrium conditions around the steady state with zero net inflation leads to the following equations

$$\text{Dynamic IS equation:} \quad \hat{y}_t = \mathbb{E}_t\hat{y}_{t+1} - \sigma(\hat{R}_t - \mathbb{E}_t\hat{\pi}_{t+1}) \tag{3.1}$$

$$\text{New Keynesian Phillips curve:} \quad \hat{\pi}_t = \beta\mathbb{E}_t\hat{\pi}_{t+1} + \kappa\hat{y}_t \tag{3.2}$$

$$\text{Monetary policy:} \quad \hat{R}_t = \alpha\hat{\pi}_t + d_{M,t} \tag{3.3}$$

$$\text{Fiscal policy:} \quad \hat{s}_t = \gamma\hat{b}_{t-1} + d_{F,t} \tag{3.4}$$

$$\text{Government budget constraint:} \quad \hat{b}_t = \hat{R}_t + \beta^{-1}(\hat{b}_{t-1} - \hat{\pi}_t) - (\beta^{-1} - 1)\hat{s}_t \tag{3.5}$$

where $\sigma > 0$ is the elasticity of intertemporal substitution, $0 < \beta < 1$ is the discount factor, $\kappa > 0$ is the slope of the new Keynesian Phillips curve, $\pi_t = P_t/P_{t-1}$ is the inflation between periods $t-1$ and t, and $b_t = B_t/P_t$ is the real debt at the end of period t. $(d_{M,t}, d_{F,t})$ are exogenous policy shocks that evolve as autoregressive processes

$$d_{M,t} = \rho_M d_{M,t-1} + \epsilon_{M,t}, \quad d_{F,t} = \rho_F d_{F,t-1} + \epsilon_{F,t} \tag{3.6}$$

where $0 \leqslant \rho_M, \rho_F < 1$ and $(\epsilon_{M,t}, \epsilon_{F,t})$ are mutually and serially uncorrelated with bounded supports.

Equations (3.1)–(3.3) form the key building blocks of the standard new Keynesian model, (3.4) is the model analog to many surplus-debt regression studies that aim to test for fiscal sustainability, and (3.5) is the log-linearized version of the government's flow budget identity, $\frac{1}{R_t}\frac{B_t}{P_t} + s_t = \frac{B_{t-1}}{P_t}$. Taken together, (3.1)–(3.5) constitute a system of linear expectational difference equations in the variables $\{\hat{y}_t, \hat{\pi}_t, \hat{R}_t, \hat{s}_t, \hat{b}_t\}$, whose model dynamics lie at the core of most monetary DSGE models in the literature. For analytical clarity, we assume that the monetary authority does not respond to output deviations and shut down the serial correlation of policy shocks, i.e., $\rho_M = \rho_F = 0$.

3.2 Analytical Solution

An essential feature of this model is that all possible interactions between monetary and fiscal policies that are consistent with a uniquely determined price level must conform to the following relationship ubiquitous in any dynamic macroeconomic model with rational agents

$$\hat{b}_{t-1} - \hat{\pi}_t = -\beta \sum_{k=0}^{\infty} \beta^k \mathbb{E}_t \hat{r}_{t+k} + (1-\beta) \sum_{k=0}^{\infty} \beta^k \mathbb{E}_t \hat{s}_{t+k}, \quad \forall t \tag{3.7}$$

where $\hat{b}_{t-1}$ is predetermined in period t and $\hat{r}_{t+k} = \hat{R}_{t+k} - \mathbb{E}_{t+k}\hat{\pi}_{t+k+1}$ denotes the ex-ante real interest rate. The above *intertemporal equilibrium condition* can be obtained by substituting (3.1) into (3.5) and iterating forward. Reminiscent of any asset pricing relation, (3.7) simply states that the real value of government liabilities at the beginning of period t, $\hat{b}_{t-1} - \hat{\pi}_t$, stems from the present value of current and expected future primary surpluses. But importantly, it also makes clear two distinct financing schemes of government debt—surprise inflation and direct taxation, which are the key to understanding how policy shocks are transmitted to influence the endogenous variables in the subsequent analysis.

To simplify the exhibition, we substitute the policy rules (3.3)–(3.4) for $(\hat{R}_t, \hat{s}_t)$ in the model and solve the remaining trivariate LRE system using the frequency-domain approach in Section 2.1.[11] See the Online Appendix A for derivation details. Suppose a covariance stationary solution to the reduced model is of the form $x_t = \sum_{k=0}^{\infty} C_k \epsilon_{t-k}$, where $x_t = [\hat{y}_t, \hat{\pi}_t, \hat{b}_t]'$, $\epsilon_t = [\epsilon_{M,t}, \epsilon_{F,t}]'$, and each element of $\sum_{k=0}^{\infty} C_k C_k'$ is finite. In what follows, we fully characterize the model solution in two regions of the policy parameter space that imply unique bounded equilibria due to Leeper (1991).[12] To demonstrate the solution method in the presence of indeterminacy, we also explore a third region under which an indeterminate set of equilibria can arise. It is easy to verify that the Smith decomposition for this model gives rise to the following roots

$$\lambda_1 = \frac{\gamma_1 + \sqrt{\gamma_1^2 - 4\gamma_0}}{2\gamma_0}, \quad \lambda_2 = \frac{\gamma_1 - \sqrt{\gamma_1^2 - 4\gamma_0}}{2\gamma_0}, \quad \lambda_3 = \frac{\beta}{1 - \gamma(1-\beta)}$$

where $\gamma_0 = (1 + \alpha\sigma\kappa)/\beta$ and $\gamma_1 = (1 + \beta + \sigma\kappa)/\beta$. These roots also arise as the reciprocals of the eigenvalues from the reduced model viewed as a system of difference equations in $(\hat{y}_t, \hat{\pi}_t, \hat{b}_t)$.

3.2.1 Regime-M One region, $\alpha > 1$ and $\gamma > 1$, produces active monetary and passive fiscal policy or regime-M, yielding the conventional monetarist/Wicksellian perspective on inflation determination. Regime-M assigns monetary policy to target inflation and fiscal policy to stabilize debt—central banks can control inflation by systematically raising nominal interest rate more than one-for-one with inflation (i.e., the Taylor principle) and the government always adjusts taxes or spending to assure fiscal solvency. Given that $0 < \lambda_2 < \lambda_1 < 1 < \lambda_3$ under this regime, we can write output, inflation, and real debt as linear functions of all past and present policy

[11] An equivalent time-domain derivation can be found in Leeper and Leith (2015).

[12] These characterizations draw partly on Tan (2017), but see also Leeper and Li (2017) for a similar analysis based on a flexible-price endowment economy. Here we restrict $(\alpha, \gamma) \in [0, \infty) \times [0, \infty)$ because negative policy responses, though theoretically possible, make little economic sense.

shocks with unambiguously signed coefficients. In particular, output follows

$$\hat{y}_t = \underbrace{C_0(1,1)}_{<0} \epsilon_{M,t} \tag{3.8}$$

inflation follows

$$\hat{\pi}_t = \underbrace{C_0(2,1)}_{<0} \epsilon_{M,t} \tag{3.9}$$

and real debt follows

$$\hat{b}_t = \sum_{k=0}^{\infty} \underbrace{C_0(3,1)\left(\frac{1}{\lambda_3}\right)^k}_{>0} \epsilon_{M,t-k} + \sum_{k=0}^{\infty} \underbrace{C_0(3,2)\left(\frac{1}{\lambda_3}\right)^k}_{<0} \epsilon_{F,t-k} \tag{3.10}$$

where the contemporaneous responses are given by

$$C_0 = \begin{pmatrix} -\frac{\sigma}{1+\alpha\sigma\kappa} & 0 \\ -\frac{\sigma\kappa}{1+\alpha\sigma\kappa} & 0 \\ \frac{\beta+\sigma\kappa}{\beta(1+\alpha\sigma\kappa)} & \frac{\beta-1}{\beta} \end{pmatrix}$$

To the extent that fiscal shocks do not impinge on the equilibrium output and inflation, the analytical impulse response functions (3.8)–(3.10) immediately point to the familiar "Ricardian equivalence" result—a deficit-financed tax cut leaves aggregate demand unaffected because its positive wealth effect will be neutralized by the household's anticipation of higher future taxes whose present value matches exactly the initial debt expansion.

This anticipated backing of government debt also eliminates any fiscal consequence of monetary policy actions, freeing the central bank to control inflation. Take for instance a monetary contraction that aims to reduce inflation. Given sticky prices, a higher nominal interest rate translates into a higher real interest rate, which makes consumption today more costly relative to tomorrow. As a result, both output in (3.8) and inflation in (3.9) fall.[13] But the higher real rate also raises the household's real interest receipts and hence the real principal in (3.10). As the household feels wealthier and demands more goods, price levels are bid up, counteracting the monetary authority's original intention to lower inflation. This wealth effect, however, is unwarranted under the fiscal financing mechanism of regime-M because any increase in government debt now necessarily portends future fiscal contraction. If nothing else, it is such fiscal backing for monetary policy to achieve price stability that delivers Milton Friedman's (1970)

[13]Since $C_0(2,1) = \kappa C_0(1,1)$, the parameter κ determines the trade-off between output and inflation.

famous dictum that "inflation is always and everywhere a monetary phenomenon."

Another desirable outcome that appropriate fiscal backing affords the central bank to accomplish is greater macroeconomic stability. Because the initial impacts of monetary shock, $|C_0(1,1)|$, $|C_0(2,1)|$, and $|C_0(3,1)|$, are decreasing in α, and the decay factor of fiscal shock, $1/\lambda_3$, is decreasing in γ, a more aggressive monetary stance, in conjunction with a tighter fiscal discipline, can effectively reduce the volatilities of output, inflation, and government debt.

3.2.2 Regime-F

A second region, $0 \leqslant \alpha < 1$ and $0 \leqslant \gamma < 1$, consists of passive monetary and active fiscal policy or regime-F, producing the fiscal theory of the price level [Leeper (1991), Woodford (1995), Cochrane (1998), Davig and Leeper (2006), Sims (2013)]. In contrast to regime-M, policy roles are reversed under this alternative regime, with fiscal policy determining the price level and monetary policy acting to stabilize debt. Without much loss of generality, we consider the special case of an exogenous path for primary surpluses, i.e., $\gamma = 0$. This profligate fiscal policy requires that monetary authority raise the nominal rate only weakly with inflation to prevent debt service from growing too rapidly. It follows that $0 < \lambda_2 < \lambda_3 = \beta < 1 < \lambda_1$. Analogous to regime-M, we can write output, inflation, and real debt as linear functions of all past and present policy shocks with unambiguously signed coefficients. In particular, output follows

$$\begin{aligned}\hat{y}_t = \underbrace{C_0(1,1)}_{<0}\epsilon_{M,t} + \sum_{k=1}^{\infty}\underbrace{C_0(1,1)\left[\frac{1}{\lambda_1} - \frac{\beta-\lambda_2}{\beta\lambda_2(\beta-1+\sigma\kappa)}\right]\left(\frac{1}{\lambda_1}\right)^{k-1}}_{>0}\epsilon_{M,t-k} \\ + \sum_{k=0}^{\infty}\underbrace{C_0(1,2)\left(\frac{1}{\lambda_1}\right)^{k}}_{<0}\epsilon_{F,t-k}\end{aligned} \tag{3.11}$$

inflation follows

$$\begin{aligned}\hat{\pi}_t = \underbrace{C_0(2,1)}_{>0}\epsilon_{M,t} + \sum_{k=1}^{\infty}\underbrace{C_0(2,1)\left[\frac{1}{\lambda_1} - \frac{\lambda_2-\beta}{\beta\lambda_2}\right]\left(\frac{1}{\lambda_1}\right)^{k-1}}_{>0}\epsilon_{M,t-k} \\ + \sum_{k=0}^{\infty}\underbrace{C_0(2,2)\left(\frac{1}{\lambda_1}\right)^{k}}_{<0}\epsilon_{F,t-k}\end{aligned} \tag{3.12}$$

and real debt follows

$$\hat{b}_t = \sum_{k=0}^{\infty}\underbrace{C_0(3,1)\left(\frac{1}{\lambda_1}\right)^{k}}_{>0}\epsilon_{M,t-k} + \sum_{k=0}^{\infty}\underbrace{C_0(3,2)\left(\frac{1}{\lambda_1}\right)^{k}}_{<0}\epsilon_{F,t-k} \tag{3.13}$$

where the contemporaneous responses are given by

$$C_0 = \begin{pmatrix} \frac{\sigma\lambda_2^2(\beta-1+\sigma\kappa)}{\lambda_2-\beta} & -\frac{(1-\beta)\sigma[(\sigma\kappa+\beta)\lambda_2-\beta]}{\lambda_2-\beta} \\ -\frac{\sigma\kappa\lambda_2^2}{\lambda_2-\beta} & \frac{\sigma\kappa\lambda_2(1-\beta)}{\lambda_2-\beta} \\ \frac{\beta+\sigma\kappa}{(1+\alpha\sigma\kappa)\lambda_1} & \frac{\beta-1}{\lambda_1} \end{pmatrix}$$

The analytical impulse response functions (3.11)–(3.13), together with the intertemporal equilibrium condition (3.7), highlight a violation of "Ricardian equivalence"—unlike regime-M, expansions in government debt, due to either monetary contraction or fiscal expansion, will generate a positive wealth effect which in turn transmits into higher inflation and, in the presence of nominal rigidities, higher real activity.

Indeed, this non-Ricardian nature stems from a fundamentally different fiscal financing mechanism underlying the fiscal theory; while regime-M relies primarily on direct taxation, regime-F hinges crucially on the debt devaluation effect of surprise inflation. For example, consider the effects of a monetary contraction. With sticky prices, a higher nominal interest rate raises the real interest rate, inducing the household to save more in the current period. Thus, output falls initially in (3.11). The higher real rate also raises the real interest payments and hence the real principal in (3.13), making the household wealthier at the beginning of the next period. However, because future primary surpluses do not adjust to neutralize this wealth effect, aggregate demand increases in the next period, which pushes up both output in (3.11) and inflation in (3.12). More importantly, as evinced by (3.7), inflation must rise in the current as well as future periods to devalue the nominal government debt so as to guarantee its sustainability. This wealth effect channel triggers exactly the same macroeconomic impacts under a fiscal expansion. Given exogenous primary surpluses, (3.7) suggests that a deficit-financed tax cut shows up as a mix of higher current inflation and a lower path for real interest rates, which in turn leads to higher output. Through devaluation, the higher inflation again ensures that the government debt remains sustainable. The above policy implications should make it clear that inflation is fundamentally a fiscal phenomenon under regime-F.

Lastly, the role of inflation in stabilizing government debt under regime-F is also evident in that both the extent, $|C_0(2,1)|$ and $|C_0(2,2)|$, and the decay factor, $1/\lambda_1$, of the policy effects on inflation are increasing in α—a hawkish monetary stance not only amplifies the inflationary impacts of higher debt but makes these impacts more persistent as well, thereby reinforcing the fiscal theory mechanism.

3.2.3 Indeterminacy

The third region, $0 \leqslant \alpha < 1$ and $\gamma > 1$, combines passive monetary and passive fiscal policy, exhibiting an indeterminate set of equilibria [Clarida et al. (2000), Lubik and Schorfheide (2004), Bhattarai et al. (2012)]. Like regime-M, passive fiscal policy always selfensures government budget solvency, regardless of the monetary policy in place. Consequently, output and inflation can be determined without reference to the fiscal policy (3.4) and government budget constraint (3.5). Dropping these equations from the equilibrium system and introducing the inflation forecast error $\eta_{\pi,t} = \hat{\pi}_t - \mathbb{E}_{t-1}\hat{\pi}_t$ as a new fundamental shock, we can redefine $x_t = [\hat{y}_t, \hat{\pi}_t, \mathbb{E}_t\hat{\pi}_{t+1}]'$ and $\epsilon_t = [\epsilon_{M,t}, \eta_{\pi,t}]'$. Given that $0 < \lambda_2 < 1 < \lambda_1, \lambda_3$ under indeterminacy, output, inflation, and expected inflation can be expressed as linear functions of all past and present fundamental shocks with unambiguously signed coefficients. In particular, output follows

$$\hat{y}_t = \underbrace{C_0(1,1)}_{<0}\epsilon_{M,t} + \sum_{k=1}^{\infty} \underbrace{C_0(1,1)\left(\frac{1}{\lambda_1} - \frac{1}{\beta}\right)\left(\frac{1}{\lambda_1}\right)^{k-1}}_{>0} \epsilon_{M,t-k} + \sum_{k=0}^{\infty} \underbrace{C_0(1,2)\left(\frac{1}{\lambda_1}\right)^{k}}_{>0} \eta_{\pi,t-k} \tag{3.14}$$

inflation follows

$$\hat{\pi}_t = \sum_{k=1}^{\infty} \underbrace{C_0(3,1)\left(\frac{1}{\lambda_1}\right)^{k-1}}_{>0} \epsilon_{M,t-k} + \sum_{k=0}^{\infty} \underbrace{C_0(2,2)\left(\frac{1}{\lambda_1}\right)^{k}}_{>0} \eta_{\pi,t-k} \tag{3.15}$$

and expected inflation follows

$$\mathbb{E}_t\hat{\pi}_{t+1} = \sum_{k=0}^{\infty} \underbrace{C_0(3,1)\left(\frac{1}{\lambda_1}\right)^{k}}_{>0} \epsilon_{M,t-k} + \sum_{k=0}^{\infty} \underbrace{C_0(3,2)\left(\frac{1}{\lambda_1}\right)^{k}}_{>0} \eta_{\pi,t-k} \tag{3.16}$$

where the contemporaneous responses are given by

$$C_0 = \begin{pmatrix} -\sigma\lambda_2 & -\frac{(1+\alpha\sigma\kappa)\lambda_2 - 1}{\kappa} \\ 0 & 1 \\ \frac{\sigma\kappa}{(1+\alpha\sigma\kappa)\lambda_1} & \frac{1}{\lambda_1} \end{pmatrix}$$

Substituting (3.3)–(3.4) and (3.15) into (3.5) gives the solution for real debt.

The analytical impulse response functions (3.14)–(3.16) delineates the propagation of monetary policy shock and inflation forecast error under indeterminacy. In line with the previous findings of Lubik and Schorfheide (2003, 2004) and Bhattarai et al. (2012), both output in (3.14) and inflation in (3.15), though subject to one period lag, rise in response to a monetary contraction, thereby reminiscent of the prediction under regime-F. On the other hand, a higher forecast error (due to, e.g., an inflationary sunspot belief) induces agents to revise their expectations of future inflation upward in (3.16). The resulting lower real rate stimulates current consumption and thus aggregate demand, which pushes up output and inflation. Higher current inflation also validates the initial underestimate of inflation.

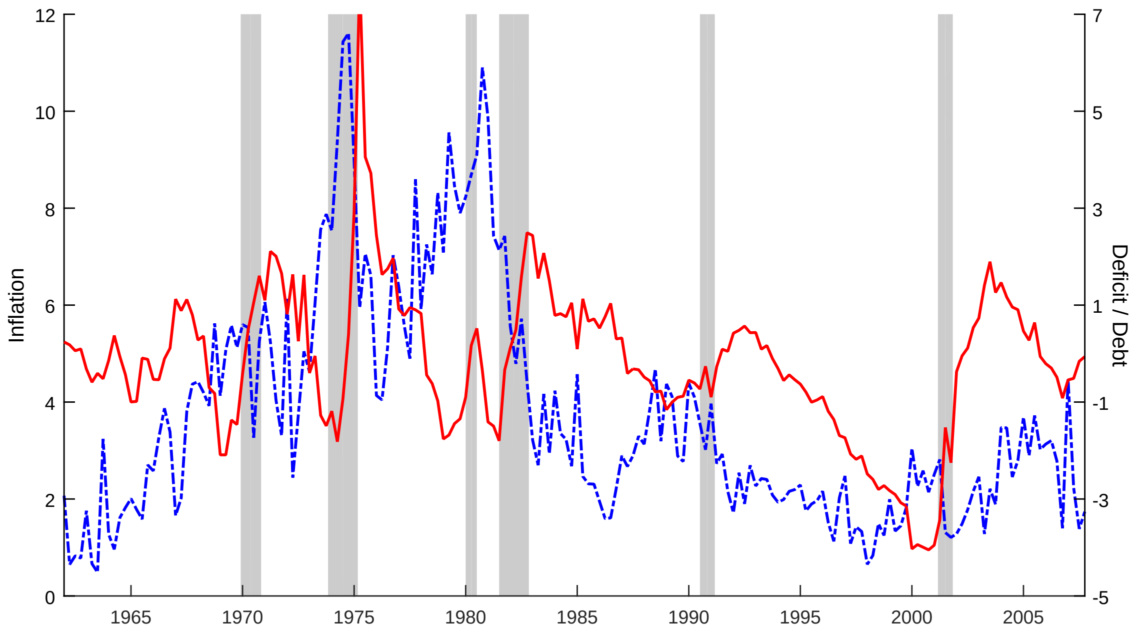

Figure 1: Inflation and fiscal stress. Notes: The left and right vertical axes measure the annualized inflation rate (blue dashed line, constructed as in Appendix B) and the deficit-to-debt ratio (red solid line, measured as in Sims (2011) by primary deficit as a proportion of lagged market value of privately held debt). Shaded bars indicate recessions as designated by the National Bureau of Economic Research.

3.3 Empirical Analysis As the previous section makes clear, regimes M and F imply starkly different mechanisms for inflation determination and debt stabilization. It is therefore a prerequisite to identify the prevailing regime in order to make appropriate policy choices. While the popular surplus-debt regressions are subject to potential simultaneity bias that may produce misleading inferences about fiscal sustainability, testing efforts based on general equilibrium models, on the other hand, find nearly uniform statistical support for regime-M in the pre-crisis U.S. data [Traum and Yang (2011), Leeper et al. (2017), Leeper and Li (2017)].[14] This consensus emerged even from periods of fiscal stress such as the 1970s, during which monetary policy appears to lose control over inflation (see Figure 1). Notice that fiscal variables, e.g., deficit-to-debt ratio as evinced by Figure 1, are persistent and primarily driven by low-frequency movements. As pointed out by Schorfheide (2013), however, DSGE models are typically misspecified with respect to certain low-frequency features of the data, and it was not until recently that academic attention has been paid to the empirical implications of each regime for the low-frequency relationship between measures of inflation and fiscal stress [Kliem et al. (2016a,b)].

In the frequency-domain context, formal regime comparison and selection along specific fre-

[14]Li et al. (2018) assessed the identification role of credit market imperfections in discerning the underlying regime. They found that adding financial frictions to a richly structured DSGE model improves the relative statistical fit of regime-F, to the extent that it can fundamentally alter the regime ranking found in the literature. See also Li and Tan (2018) for a more comprehensive (time-domain) exploration under both complete and incomplete model spaces.

quencies can be made possible by estimating marginal likelihoods and Bayes factors based on the corresponding spectral likelihood function. To that end, we first assume a complete model space and estimate each regime-dependent model over three frequency bands:

1. Full band: we set $s_k = 1$ for all frequencies $w_k = 2\pi k/T$, $k = 1, 2, \ldots, T-1$.[15] This is approximately tantamount to estimating the model in the time domain.

2. High-pass: we set $s_k = 1$ for frequencies $w_k \geqslant 2\pi/6$, corresponding to cycles with period 2 to 6 quarters. Similar to Sala (2015), this high-pass band contains conventional high frequencies (period 2 quarters to 1 year) but also partly overlaps the business cycle frequencies (period between 1 and 8 years) from its high end in order to keep enough data points in the estimation.

3. Low-pass: to separate and contrast the impacts of imposing different spectral bands, we set $s_k = 1$ for frequencies complementary to those on the high-pass band, i.e., $w_k \leqslant 2\pi/6$, corresponding to cycles with period 6 quarters to infinity. Again for the reason of retaining enough data points in the estimation, this low-pass band contains conventional low frequencies (period 8 years to infinity) but also partly overlaps the business cycle frequencies from its low end.

We consider two subsamples in the postwar U.S. data, separated by the appointment of Paul Volcker as Chairman of the Federal Reserve Board in August 1979: pre-Volcker era, 1954:Q3–1979:Q2; and post–Volcker era, 1984:Q1–2007:Q4.[16] The set of quarterly observables includes: per capita real output growth rate (YGR); annualized inflation rate (INF); annualized nominal interest rate (INT); and per capita real debt growth rate (BGR). The inclusion of BGR rather than deficit-to-debt ratio suggested by Sims (2011) and Kliem et al. (2016a,b) as a natural measure of fiscal stress is to avoid having the percentage change of a percentage in our simple model. See the Online Appendix B for details of the data construction. The demeaned observable variables are linked to the model variables through the following measurement equations

$$\begin{pmatrix} \mathrm{YGR}_t \\ \mathrm{INF}_t \\ \mathrm{INT}_t \\ \mathrm{BGR}_t \end{pmatrix} = \begin{pmatrix} \hat{y}_t - \hat{y}_{t-1} \\ 4\hat{\pi}_t \\ 4\hat{R}_t \\ \hat{b}_t - \hat{R}_t - \hat{b}_{t-1} + \hat{R}_{t-1} \end{pmatrix} + u_t, \quad u_t \sim \mathbb{N}(0, \Omega) \tag{3.17}$$

where $\bar{r} = 400(1/\beta - 1)$ is the annualized net real interest rate and Ω is a diagonal covariance

[15] We exclude $w_0 = 0$ because the model becomes stochastically singular at frequency zero.

[16] Our full sample begins when the federal funds rate data first became available and ends before the federal funds rate nearly hit its effective lower bound.

Table 1: Prior Distributions of Model Parameters

Parameter	Density	Para (1)	Para (2)
$1/\sigma$, relative risk aversion	$\mathbb{G}$	5.00	0.30
κ, slope of new Keynesian Phillips curve	$\mathbb{G}$	0.50	0.05
$\bar{r}$, s.s. annualized net real interest rate	$\mathbb{G}$	0.50	0.10
α, interest rate response to inflation, regime-M	$\mathbb{G}$	1.50	0.20
α, interest rate response to inflation, regime-F	$\mathbb{B}$	0.50	0.10
γ, surplus response to lagged debt, regime-M	$\mathbb{G}$	1.50	0.20
ρ_M, persistency of monetary shock	$\mathbb{B}$	0.50	0.10
ρ_F, persistency of fiscal shock	$\mathbb{B}$	0.50	0.10
$100\sigma_M$, scaled s.d. of monetary shock	$\mathbb{IG}$-1	0.40	12.00
$100\sigma_F$, scaled s.d. of fiscal shock	$\mathbb{IG}$-1	0.40	12.00

NOTES: Para (1) and Para (2) refer to the means and standard deviations for Gamma ($\mathbb{G}$) and Beta ($\mathbb{B}$) distributions; s and ν for the Inverse-Gamma Type-I ($\mathbb{IG}$-1) distribution with density $p(\sigma) \propto \sigma^{-\nu-1} \exp\left(-\frac{\nu s^2}{2\sigma^2}\right)$. The effective prior is truncated at the boundary of the determinacy region.

matrix.[17] In conjunction with the model solution under each regime, this leads to the state space form (2.9)–(2.10) whose likelihood function can be evaluated according to (2.12).

Table 1 summarizes the marginal prior distributions on the model parameters. For convenience, we place a prior on the coefficient of relative risk aversion, $1/\sigma$, that centers at a moderate value of 5. The prior mean of κ implies a somewhat smaller degree of price stickiness than the range of values typically found in the new Keynesian literature, and that of $\bar{r}$ translates into a β value of 0.998.[18] The relatively informed priors on $(1/\sigma, \kappa, \bar{r})$ are intended to help keep the posterior estimates in economically plausible regions of the parameter space. To reflect the two policy regimes, we specify two sets of priors on the policy parameters (α, γ), each of which places nearly all probability mass on regions of the parameter space that deliver unique model solution consistent with a regime. In particular, regime-M raises interest rate aggressively in response to inflation ($\alpha > 1$) and adjusts taxes or expenditures sufficiently to stabilize debt ($\gamma > 1$); regime-F makes interest rate respond only weakly to inflation ($0 \leqslant \alpha < 1$) and fiscal instrument unresponsive with regard to debt ($\gamma = 0$). The priors on the policy shock processes are harmonized: the autoregressive coefficients (ρ_M, ρ_F) are beta distributed with mean 0.5 and standard deviation 0.1; following standard practice, the standard deviation parameters (σ_M, σ_F),

[17] We set the square root of each diagonal element of Ω to 20% of the sample standard deviation of the corresponding observable variable.

[18] Two common ways to introduce sticky prices into new Keynesian models are through Rotemberg's (1982) price adjustment costs and Calvo's (1983) random price changes. It can be shown for both cases that κ depends inversely on the degree of price stickiness. As $\kappa \to \infty$, the model approaches to a flexible-price economy in which $\hat{y}_t = 0$ for all t.

Table 2: High-Pass Posterior Estimates

	Pre-Volcker Era				Post-Volcker Era			
	Regime-M		Regime-F		Regime-M		Regime-F	
Para	Mean	90% HPD	Mean	90% HPD	Mean	90% HPD	Mean	90% HPD
$1/\sigma$	3.60	[3.19,4.00]	4.85	[4.35,5.34]	3.91	[3.42,4.45]	4.61	[4.12,5.14]
κ	0.10	[0.07,0.12]	0.47	[0.39,0.54]	0.18	[0.10,0.27]	0.45	[0.38,0.53]
$\bar{r}$	0.51	[0.34,0.67]	0.49	[0.33,0.64]	0.51	[0.34,0.67]	0.49	[0.32,0.64]
α	1.41	[1.12,1.68]	0.59	[0.44,0.71]	1.57	[1.26,1.87]	0.52	[0.39,0.65]
γ	1.50	[1.15,1.81]	–	–	1.50	[1.18,1.83]	–	–
ρ_M	0.83	[0.81,0.86]	0.92	[0.88,0.96]	0.76	[0.72,0.80]	0.94	[0.91,0.97]
ρ_F	0.49	[0.32,0.65]	0.50	[0.34,0.66]	0.51	[0.33,0.67]	0.50	[0.34,0.67]
$100\sigma_M$	0.50	[0.39,0.61]	0.28	[0.23,0.33]	0.36	[0.29,0.42]	0.24	[0.20,0.28]
$100\sigma_F$	0.43	[0.28,0.58]	0.43	[0.28,0.57]	0.43	[0.28,0.57]	0.43	[0.28,0.56]
Ave Ineff		5.9		2.2		2.9		2.3

NOTES: The posterior means and 90% highest probability density (HPD) intervals [constructed as in Chen and Shao (1999)] are computed using 10,000 posterior draws after thinning. The last row reports the average of inefficiency factors defined as $1+2\sum_{j=1}^{K} w(j/K)\rho(j)$, where we set the truncation parameter $K = 200$ and weight the autocorrelation function $\rho(\cdot)$ using the Parzen kernel $w(\cdot)$.

all scaled by 100, follow inverse-gamma type-I distribution with mean 0.4 and standard deviation 0.1.

For each model, we sample a total of $210,000$ draws from the posterior distribution using the random-walk Metropolis-Hastings algorithm, discard the first $10,000$ draws as burn-in phase, and keep one every 20 draws afterwards.[19] The resulting $10,000$ draws form the basis for the posterior inference. Two aspects of the posterior estimates are worth highlighting.

First, the combination of regime-dependent priors, sample periods, and band spectra generates markedly different posterior inferences for some parameters reported in Tables 2–3. For example, regardless of the frequency bands, a cross-regime comparison reveals that the estimated relative risk aversion $1/\sigma$ tends to be somewhat lower in regime-M over both samples, whereas its estimated slope of the Phillips curve κ turns out to be much smaller than that of regime-F on the high-pass band, implying a significantly stronger degree of price stickiness. A flatter Phillips

[19]Diagnostics to check the convergence of Markov chains include graphical methods such as recursive means plot and the separated partial means test proposed by Geweke (1992). We also compute the inefficiency factors for the sequence of posterior draws for each parameter. In conjunction with a rejection rate of approximately 50% for each model, the low inefficiency factors suggest that the Markov chain mixes well. See Herbst and Schorfheide (2015) for a detailed textbook treatment of Bayesian estimation of DSGE models.

Table 3: Low-Pass Posterior Estimates

	Pre-Volcker Era				Post-Volcker Era			
	Regime-M		Regime-F		Regime-M		Regime-F	
Para	Mean	90% HPD	Mean	90% HPD	Mean	90% HPD	Mean	90% HPD
$1/\sigma$	4.90	[4.43,5.42]	5.35	[4.86,5.83]	4.91	[4.43,5.40]	5.51	[4.98,6.01]
κ	0.50	[0.42,0.58]	0.50	[0.42,0.58]	0.49	[0.41,0.57]	0.41	[0.34,0.49]
$\bar{r}$	0.50	[0.34,0.67]	0.50	[0.33,0.65]	0.50	[0.34,0.66]	0.50	[0.34,0.67]
α	1.78	[1.55,2.02]	0.54	[0.38,0.69]	2.30	[2.01,2.60]	0.45	[0.30,0.58]
γ	1.52	[1.16,1.88]	–	–	1.50	[1.16,1.81]	–	–
ρ_M	0.93	[0.92,0.95]	0.94	[0.91,0.97]	0.96	[0.94,0.97]	0.85	[0.80,0.90]
ρ_F	0.50	[0.33,0.66]	0.50	[0.34,0.66]	0.50	[0.33,0.66]	0.50	[0.34,0.66]
$100\sigma_M$	0.34	[0.26,0.42]	0.29	[0.23,0.34]	0.26	[0.21,0.31]	0.24	[0.20,0.28]
$100\sigma_F$	0.43	[0.28,0.57]	0.43	[0.29,0.57]	0.43	[0.29,0.57]	0.43	[0.28,0.57]
Ave Ineff		7.6		2.5		2.3		1.5

NOTES: See Table 2.

curve also emerges in regime-M when estimated over the high-pass band in comparison to the low-pass band because stronger-than-usual nominal rigidities are needed to account for the lack of higher frequency variations in the price data. Turning to the policy parameters, while the estimated monetary response to inflation α remains comparable in regime-F across sample periods and frequency bands, it increases to various extents in regime-M both from the pre-Volcker to post-Volcker sample and from the high-pass to low-pass band, reflecting a more aggressive policy stance during the Volcker-Greenspan period and towards long-run price stability. The estimated fiscal response to real debt γ, on the other hand, exhibits little updating on its prior other than sampling variation because it does not enter the equilibrium solutions for non-fiscal variables under regime-M. Similar patterns are shared by regime-M's estimated shock processes: a more persistent monetary shock (higher ρ_M) is required when all weights are given to lower frequencies, but the fiscal shock parameters are largely unidentified.

Second, the Bayes factors summarized in Table 4 suggest that changes to the frequency band to which the model is fit can lead to a complete reversal of the regime ranking.[20] For instance, while both samples overwhelmingly favor regime-M on the full spectrum and even more so on the high-pass band, removing frequencies from the high end of the spectrum substantially improves the *relative* statistical fit of regime-F, to the extent that it can fundamentally alter the regime

[20]Log marginal likelihoods are approximated using the modified harmonic mean estimator of Geweke (1999) with a truncation parameter of 0.5.

Table 4: Log Marginal Likelihood Estimates

	Pre-Volcker Era			Post-Volcker Era		
Frequency	Regime-M	Regime-F	ln BF	Regime-M	Regime-F	ln BF
Full band	−2102.18	−2144.14	41.96*	−2260.41	−2271.51	11.10*
	(0.01)	(0.01)		(0.02)	(0.02)	
High-pass	−959.91	−1022.26	62.35*	−859.18	−918.40	59.22*
	(0.02)	(0.01)		(0.02)	(0.01)	
Low-pass	−1113.63	−1145.01	31.38*	−1401.50	−1393.60	−7.90*
	(0.02)	(0.01)		(0.01)	(0.01)	

NOTES: Marginal likelihood estimates with numerical standard errors in parentheses and Bayes factors (BF) in favor of regime-M as opposed to regime-F are reported in logarithm scale. Asterisk ($*$) signifies decisive evidence, corresponding to a log Bayes factor whose absolute value exceeds 4.6 based on Jeffreys' (1961) criterion.

ranking when evaluated on the low-pass band—regime-F fares considerably better over the post-Volcker sample with a Bayes factor of approximately e^8.[21] This underscores the importance of relatively low frequency relations in the data for identifying the underlying regime, which largely corroborates the empirical findings of Kliem et al. (2016a,b).

Figures 2–3 compare the log spectra (diagonal panels) and coherence functions (off-diagonal panels) of the data with those implied by the model, which furnish additional information about the strengths and weaknesses of each regime in matching features of the data.[22] Both regimes can capture the smoothly declining spectra—the typical spectral shape of economic variables summarized by Granger (1966)—of inflation and interest rate fairly well, but fall short of fitting the spectra of variables in growth rates partly because the model does not feature a stochastic trend. In contrast, the coherence functions of the data appear more volatile over frequencies and a cross-regime divergence shows up in a number of cases. Focusing on the comovements between nominal and fiscal variables (i.e., INF vs. BGR and INT vs. BGR), the hump-shaped pattern produced by regime-F, which is absent in regime-M, helps accommodate the coherence spikes in the low range of the business cycle frequencies relatively well.

It is also instructive to examine the cross-correlograms estimated over different frequency

[21] The decisive evidence in favor of regime-F on the low-pass band is partly due to the inclusion of fiscal data (i.e., BGR) in the estimation, which features more prominent lower frequency variations than other aggregate variables.

[22] In considering the strength of comovement between two variables, it is more convenient to work with their coherence function rather than cross-spectrum because the latter is in general a complex-valued function. For a given spectral density matrix $S(w)$ of two variables, the coherence at frequency w, being analogous to the R^2 statistic, is defined as $R^2(w) = |S_{12}(w)|^2/(S_{11}(w)S_{22}(w))$.

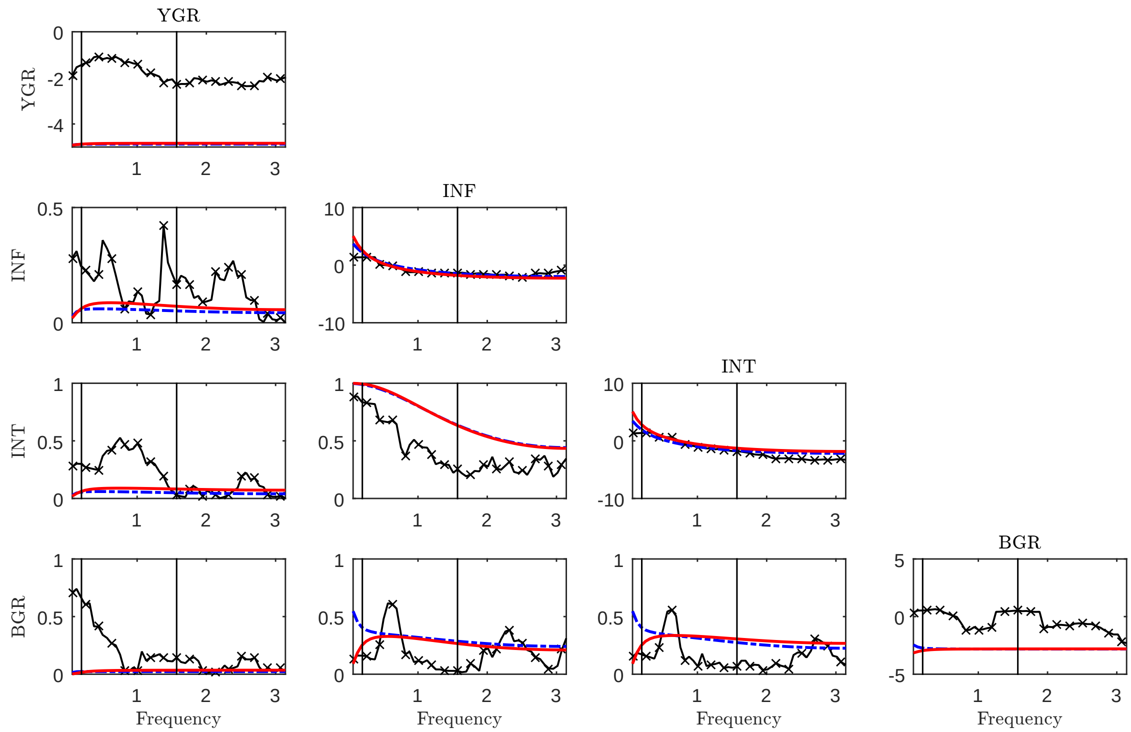

Figure 2: Pre-Volcker log spectrum and coherence. Notes: The diagonal (off-diagonal) panels compare the log spectra (coherence functions) of the data (black solid line with cross) with those of regime-M (blue dashed line) and regime-F (red solid line) evaluated with the posterior mean over the full spectrum. Vertical bars separate the frequency domain into three regions: low, business cycle, and high (see Figure 4 notes).

bands (see Figures 5–8 of Appendix C).[23] Not surprisingly, the data exhibit little persistence on the high-pass band but damped oscillations in the auto and cross-correlation functions, for which both regimes can replicate reasonably well. This success carries more or less over to the slowly decaying autocorrelation functions on the low-pass band, although regime-M generates less persistence than regime-F does due to its Ricardian equivalence nature. The cross-correlations on the same band, however, pose some challenges for the model to match with. Among those exceptions, focus again on the comovements between nominal and fiscal variables that may run counter to the conventional belief. These low frequency correlations in the data agree with those under pre-Volcker regime-M and post-Volcker regime-F.

Another look at how the empirical performance of each regime varies along different frequencies can be achieved through the lens of an incomplete model space. In lieu of estimating individual regime over pre-specified frequency bands, we next perform a joint estimation of both regimes as well as all regime-selection variables $\{s_k\}_{k=0}^{T-1}$ using the composite likelihood function (2.13)

[23]The cross-correlograms can be computed as $\rho_\Omega(k) = \int_\Omega S_{12}(w)e^{iwk}dw/(\sqrt{\int_\Omega S_{11}(w)dw}\sqrt{\int_\Omega S_{22}(w)dw})$ via numerical integration, where Ω denotes the relevant frequency band and k the number of lags.

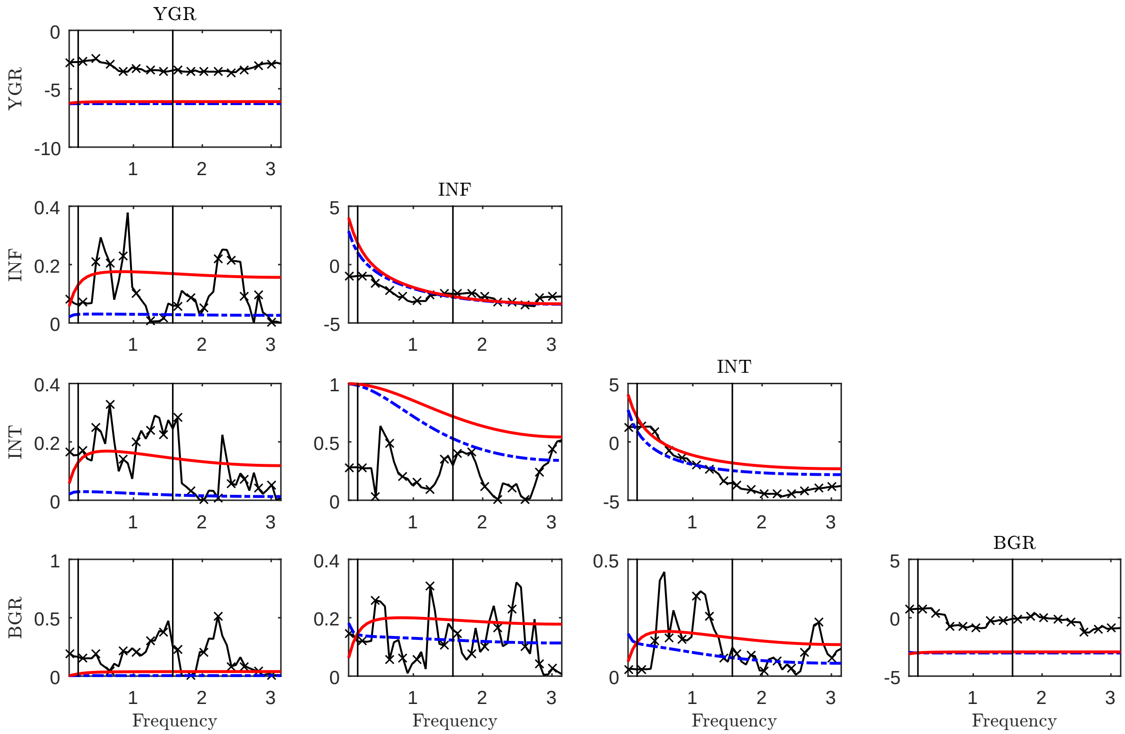

Figure 3: Post-Volcker log spectrum and coherence. Notes: See Figure 2.

and the Metropolis-Hastings-within-Gibbs algorithm outlined in Section 2.2.2. Our approach thus affords a stronger voice to the data when assessing the relative importance of regimes M and F at each frequency. Specifically, let s_k take value one (zero) if regime-M (F) is selected at frequency w_k so that its expected value can be readily interpreted as regime-M's importance weight. In addition to the prior distributions in Table 1 for the composite model, we adopt an agnostic prior view on s_k, i.e., $p(s_k = 0) = p(s_k = 1) = \frac{1}{2}$.

Figure 4 delineates the estimated regime-selection variables (solid line) based on the posterior draws over the full spectrum.[24] It displays prima facie evidence of cross-frequency variations in the relative importance of each regime. Overall, both samples predominantly prefer regime-F at frequencies near the low end of the spectrum but assign more weights to regime-M throughout most of the business cycle and high frequencies. Moreover, the estimated weights exhibit pronounced dips that hover around, e.g., $w = 1.4$ (period 4.5 quarters) in the pre-Volcker sample and $w = 1.8$ (period 3.5 quarters) in the post-Volcker sample. These patterns are by and large in line with a cross-regime comparison of the likelihoods evaluated with the posterior mean over the full spectrum, whose log differentials (dashed line) at each frequency are depicted in Figure

[24]By symmetry Figure 4 only plots the range $[0, \pi]$. To conserve space, we do not display the spectrum associated with the composite model because it simply equals the weighted average of all componenent spectra. Neither do we compute its marginal likelihood, which can be a daunting task due to the presence of regime-selection variables and the resulting high-dimensional integration problem.

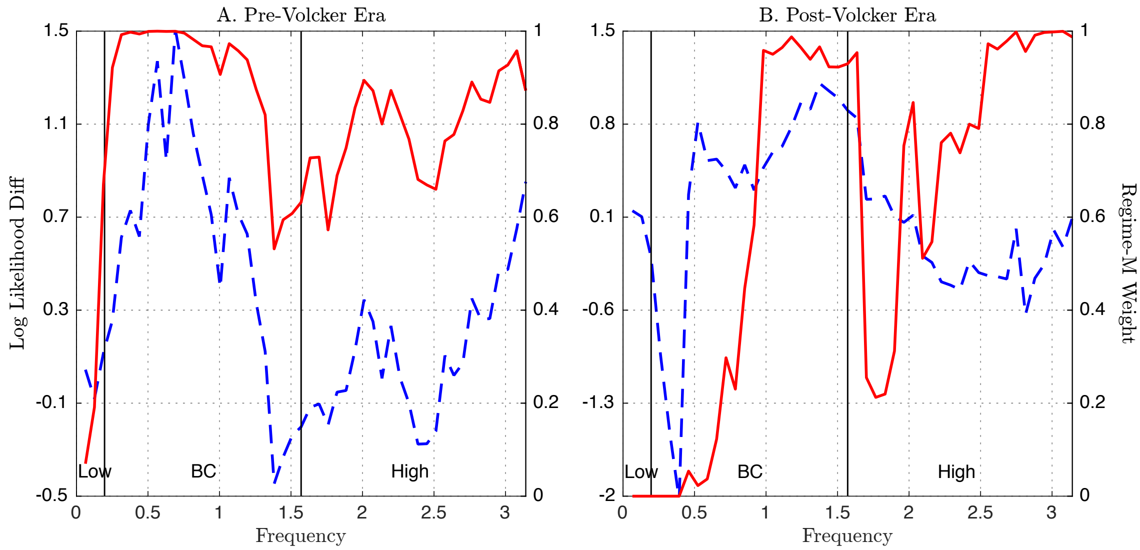

Figure 4: Log likelihood differential and regime-M weight. Notes: The left vertical axis measures the log likelihood of regime-M less that of regime-F (blue dashed line) at each frequency evaluated with the posterior mean over the full spectrum. The right vertical axis measures the posterior mean of regime-selection variable (red solid line). Vertical bars demarcate three frequency bands: low (period 32 quarters to infinity), business cycle (period 4 to 32 quarters, labeled BC), and high (period 2 to 4 quarters).

Table 5: Posterior Estimates of Composite Model

	Pre-Volcker Era				Post-Volcker Era			
	Regime-M		Regime-F		Regime-M		Regime-F	
Para	Mean	90% HPD	Mean	90% HPD	Mean	90% HPD	Mean	90% HPD
$1/\sigma$	4.91	[4.40,5.37]	4.94	[4.45,5.44]	3.88	[3.32,4.45]	5.37	[4.86,5.90]
κ	0.50	[0.42,0.58]	0.51	[0.43,0.59]	0.17	[0.07,0.28]	0.41	[0.34,0.49]
$\bar{r}$	0.50	[0.35,0.67]	0.50	[0.33,0.65]	0.50	[0.34,0.67]	0.49	[0.33,0.65]
α	1.77	[1.54,1.98]	0.53	[0.37,0.69]	1.59	[1.28,1.90]	0.49	[0.35,0.64]
γ	1.50	[1.17,1.81]	–	–	1.51	[1.19,1.84]	–	–
ρ_M	0.92	[0.90,0.94]	0.66	[0.48,0.83]	0.77	[0.71,0.82]	0.89	[0.85,0.94]
ρ_F	0.50	[0.34,0.66]	0.50	[0.33,0.66]	0.50	[0.33,0.66]	0.50	[0.33,0.66]
$100\sigma_M$	0.35	[0.27,0.43]	0.34	[0.26,0.42]	0.35	[0.28,0.42]	0.25	[0.20,0.29]
$100\sigma_F$	0.43	[0.28,0.57]	0.43	[0.28,0.57]	0.43	[0.28,0.57]	0.43	[0.28,0.57]
Ave Ineff		2.3		10.6		6.7		2.3

NOTES: See Table 2.

4. Consistent with the posterior weights of each regime are the parameter estimates of the composite model reported in Table 5. Since the post-Volcker sample brings about a more balanced assignment of frequencies between regimes, its regime-M (F) estimates closely resemble those obtained on the high-pass (low-pass) band (see Tables 2–3). The post-Volcker sample, on the other hand, discounts regime-F's weight more heavily, to the extent that it pushes regime-F (M) estimates towards the prior (full-band, see Table 6 of Appendix C) values. Given the apparent evidence of cross-frequency regime uncertainty, these findings point to a broader message that policymakers should routinely examine alternative monetary-fiscal policy specifications in their policymaking process.

4 Concluding Remarks

This article contributes to the research program on rational expectations econometrics by proposing a unified framework for conveniently solving and estimating dynamic macroeconomic models in the frequency domain. In spite of the popularity and continued dominance of time-domain analysis, we argue that there remain several advantages of our approach on both theoretical and empirical grounds. First, the z-transform solution method is applicable for solving a wide class of models, including dynamic economies featuring signal extraction and infinite regress in expectations. Second, given generic exogenous driving processes, the moving average representation of the solution permits straightforward construction of the spectral density for performing likelihood-based inference. Third, the spectral decomposition of the Gaussian likelihood function is useful in assessing model adequacy over different frequency bands of interest as well as identifying promising avenues for further model development. Finally, in the presence of potential model uncertainty, the composite spectral likelihood function implied by all candidate models allows the relative importance of individual model to be evaluated at each frequency.

The proposed framework is applied to solve and estimate a simple new Keynesian model with fiscal details and two determinate monetary-fiscal policy regimes. The closed-form solution derived herein is useful in characterizing the cross-equation restrictions and illustrating the complex interaction between policy behavior and price rigidity under each regime. Based on the postwar U.S. data, we find strong evidence of cross-frequency variations in the importance weight of each regime and that relatively low frequency relations in the data play an important role in discerning the underlying regime. Nevertheless, it is still worthwhile to examine whether these empirical findings carry over to a more richly structured DSGE model with low-frequency features, e.g., persistent long-run component in fiscal policy rule as in Sims (2012) and long-term nominal debt as in Cochrane (1998), Sims (2013), and Leeper and Leith (2015). Another interesting application is to take models with incomplete information or heterogenous beliefs to the data. We defer these extensions to a sequel to this paper.

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Online Appendix

Appendix A: Model Solution To simplify the exhibition, we substitute the policy rules (3.3)–(3.4) for $(\hat{R}_t, \hat{s}_t)$ in the model and rewrite the remaining trivariate LRE system in the canonical form (2.1)

$$\mathbb{E}_t \left[\underbrace{\begin{pmatrix} 1 & \sigma & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\Gamma_{-1}} L^{-1} + \underbrace{\begin{pmatrix} -1 & -\alpha\sigma & 0 \\ \kappa & -1 & 0 \\ 0 & \beta^{-1} - \alpha & 1 \end{pmatrix}}_{\Gamma_0} L^0 + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma(\beta^{-1} - 1) - \beta^{-1} \end{pmatrix}}_{\Gamma_1} L \right] \underbrace{\begin{pmatrix} \hat{y}_t \\ \hat{\pi}_t \\ \hat{b}_t \end{pmatrix}}_{x_t}$$

$$= \underbrace{\begin{pmatrix} \sigma & 0 \\ 0 & 0 \\ 1 & 1 - \beta^{-1} \end{pmatrix}}_{\Psi_0} L^0 \underbrace{\begin{pmatrix} d_{M,t} \\ d_{F,t} \end{pmatrix}}_{d_t} \tag{A.1}$$

where

$$\underbrace{\begin{pmatrix} d_{M,t} \\ d_{F,t} \end{pmatrix}}_{d_t} = \underbrace{\begin{pmatrix} \frac{1}{1-\rho_M L} & 0 \\ 0 & \frac{1}{1-\rho_F L} \end{pmatrix}}_{A(L)} \underbrace{\begin{pmatrix} \epsilon_{M,t} \\ \epsilon_{F,t} \end{pmatrix}}_{\epsilon_t} \tag{A.2}$$

and the solution $x_t = C(L)\epsilon_t$ to (A.1) is taken to be covariance stationary. Below we closely follow the solution procedure laid out in Section 2.1 and the notations established therein to derive the content of $C(\cdot)$.

Determinacy First, transform the time-domain system (A.1) into its equivalent frequency-domain representation. Appealing to the Wiener-Kolmogorov optimal prediction formula, we can evaluate the vector of expectational errors as $\eta_{t+1} = C_0 L^{-1} \epsilon_t$. Define $\Gamma(L) \equiv \Gamma_{-1} L^{-1} + \Gamma_0 + \Gamma_1 L$ and substitute x_t, η_{t+1}, and (A.2) into (A.1)

$$\Gamma(L)C(L)\varepsilon_t = (\Psi_0 A(L) + \Gamma_{-1} C_0 L^{-1})\epsilon_t$$

which must hold for all realizations of ϵ_t. Therefore, the coefficient matrices are related by the z-transform identities

$$z\Gamma(z)C(z) = z\Psi_0 A(z) + \Gamma_{-1} C_0$$

where $C(z)$ needs to have only non-negative powers of z and be analytic inside the unit circle so that its coefficients are square-summable by covariance stationarity.

Second, apply the Smith canonical factorization to the polynomial matrix $z\Gamma(z)$

$$z\Gamma(z) = U(z)^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z(z-\lambda_1)(z-\lambda_2)(z-\lambda_3) \end{pmatrix} V(z)^{-1}$$

with

$$\lambda_1 = \frac{\gamma_1 + \sqrt{\gamma_1^2 - 4\gamma_0}}{2\gamma_0}, \quad \lambda_2 = \frac{\gamma_1 - \sqrt{\gamma_1^2 - 4\gamma_0}}{2\gamma_0}, \quad \lambda_3 = \frac{\beta}{1-\gamma(1-\beta)}$$

where $\gamma_0 = (1+\alpha\sigma\kappa)/\beta$ and $\gamma_1 = (1+\beta+\sigma\kappa)/\beta$. The zero root arises whenever the model is forward-looking, i.e., $\Gamma_{-1} \neq 0$.[25] The root λ_3 emerges as the reciprocal of the eigenvalue from the government budget constraint (3.5) viewed as a difference equation in $\hat{b}$. To see where the pair of roots (λ_1, λ_2) comes from, combine the dynamic IS equation (3.1) and the new Keynesian Phillips curve (3.2) and substitute out $\hat{y}$ to obtain a second order expectational difference equation for inflation

$$\mathbb{E}_t\hat{\pi}_{t+2} - \frac{1+\beta+\sigma\kappa}{\beta}\mathbb{E}_t\hat{\pi}_{t+1} + \frac{1+\alpha\sigma\kappa}{\beta}\hat{\pi}_t = -\frac{\sigma\kappa}{\beta}\epsilon_{M,t}$$

The eigenvalues governing the dynamics of this equation are exactly $(1/\lambda_1, 1/\lambda_2)$.

Lastly, examine the existence and uniqueness of solution. Under regime-M with $\alpha > 1$ and $\gamma > 1$, it follows that $0 < \lambda_2 < \lambda_1 < 1 < \lambda_3$. Collect the roots inside the unit circle in $S(z)$ and multiply both sides of the z-transform identities by $S(z)^{-1}$

$$T(z)C(z) = \begin{pmatrix} U_{1\cdot}(z) \\ U_{2\cdot}(z) \\ \frac{1}{z(z-\lambda_1)(z-\lambda_2)}U_{3\cdot}(z) \end{pmatrix}(z\Psi_0 A(z) + \Gamma_{-1}C_0)$$

These identities are valid for all z on the open unit disk except for $z = 0, \lambda_1, \lambda_2$. But since $C(z)$ must be well-defined for all $|z| < 1$, this condition places the following restrictions on the unknown coefficient matrix C_0

$$U_{3\cdot}(z)(z\Psi_0 A(z) + \Gamma_{-1}C_0)|_{z=0,\lambda_1,\lambda_2} = 0$$

[25] Below we omit the restriction imposed by $z = 0$ because it is unrestrictive.

Stacking the above restrictions yields

$$-\underbrace{\begin{pmatrix} \frac{\lambda_1^2\lambda_3\kappa(\alpha\beta-1)}{(1+\alpha\sigma\kappa)\beta} & \frac{\lambda_1^2\lambda_3(\alpha\beta-1)(\sigma\kappa+\beta)}{(1+\alpha\sigma\kappa)\beta}-\frac{\lambda_1\lambda_3(\alpha\beta-1)}{1+\alpha\sigma\kappa} & 0 \\ \frac{\lambda_2^2\lambda_3\kappa(\alpha\beta-1)}{(1+\alpha\sigma\kappa)\beta} & \frac{\lambda_2^2\lambda_3(\alpha\beta-1)(\sigma\kappa+\beta)}{(1+\alpha\sigma\kappa)\beta}-\frac{\lambda_2\lambda_3(\alpha\beta-1)}{1+\alpha\sigma\kappa} & 0 \end{pmatrix}}_{R} C_0 = \underbrace{\begin{pmatrix} \frac{\lambda_1^3\lambda_3\sigma\kappa(\alpha\beta-1)}{(1+\alpha\sigma\kappa)\beta(1-\rho_M\lambda_1)} & 0 \\ \frac{\lambda_2^3\lambda_3\sigma\kappa(\alpha\beta-1)}{(1+\alpha\sigma\kappa)\beta(1-\rho_M\lambda_2)} & 0 \end{pmatrix}}_{A}$$

Apparently, the solution exists because $\text{span}(A) \subseteq \text{span}(R)$ is satisfied here. In order for the solution to be unique, we must be able to pin down the terms

$$QC_0 = U_{3\cdot}(\lambda_3)\Gamma_{-1}C_0 = \begin{pmatrix} \frac{\lambda_3^3\kappa(\alpha\beta-1)}{(1+\alpha\sigma\kappa)\beta} & \frac{\lambda_3^3(\alpha\beta-1)(\sigma\kappa+\beta)}{(1+\alpha\sigma\kappa)\beta}-\frac{\lambda_3^2(\alpha\beta-1)}{1+\alpha\sigma\kappa} & 0 \end{pmatrix} C_0$$

from the knowledge of RC_0. This is tantamount to verifying $\text{span}(Q') \subseteq \text{span}(R')$, which is also satisfied here.

Now the unique solution can be computed as

$$\begin{aligned} \begin{pmatrix} \hat{y}_t \\ \hat{\pi}_t \\ \hat{b}_t \end{pmatrix} &= (L\Gamma(L))^{-1}(L\Psi_0 A(L) + \Gamma_{-1}C_0) \begin{pmatrix} \epsilon_{M,t} \\ \epsilon_{F,t} \end{pmatrix} \\ &= \begin{pmatrix} C_0(1,1)\frac{1}{1-\rho_M L} & 0 \\ C_0(2,1)\frac{1}{1-\rho_M L} & 0 \\ C_0(3,1)\frac{1}{\left(1-\frac{1}{\lambda_3}L\right)(1-\rho_M L)} & C_0(3,2)\frac{1}{\left(1-\frac{1}{\lambda_3}L\right)(1-\rho_F L)} \end{pmatrix} \begin{pmatrix} \epsilon_{M,t} \\ \epsilon_{F,t} \end{pmatrix} \end{aligned}$$

where the contemporaneous responses are given by

$$C_0 = \begin{pmatrix} -\frac{\sigma(1-\beta\rho_M)\lambda_1\lambda_2}{(1-\rho_M\lambda_1)(1-\rho_M\lambda_2)\beta} & 0 \\ -\frac{\sigma\kappa\lambda_1\lambda_2}{(1-\rho_M\lambda_1)(1-\rho_M\lambda_2)\beta} & 0 \\ 1-\frac{(\alpha\beta-1)[\sigma\kappa-\rho_M\kappa C_0(1,1)-\rho_M(\sigma\kappa+\beta)C_0(2,1)]}{\beta(1+\alpha\sigma\kappa)} & \frac{\beta-1}{\beta} \end{pmatrix}$$

Setting $\rho_M = \rho_F = 0$ gives the expression for C_0 in Section 3.2.1.

Under regime-F with $0 \leqslant \alpha < 1$ and $\gamma = 0$, it follows that $0 < \lambda_2 < \lambda_3 = \beta < 1 < \lambda_1$. Collect the roots inside the unit circle in $S(z)$ and multiply both sides of the z-transform identities by

$S(z)^{-1}$

$$T(z)C(z) = \begin{pmatrix} U_{1\cdot}(z) \\ U_{2\cdot}(z) \\ \frac{1}{z(z-\lambda_2)(z-\lambda_3)} U_{3\cdot}(z) \end{pmatrix} (z\Psi_0 A(z) + \Gamma_{-1} C_0)$$

These identities are valid for all z on the open unit disk except for $z = 0, \lambda_2, \lambda_3$. But since $C(z)$ must be well-defined for all $|z| < 1$, this condition places the following restrictions on the unknown coefficient matrix C_0

$$U_{3\cdot}(z)(z\Psi_0 A(z) + \Gamma_{-1} C_0)|_{z=0,\lambda_2,\lambda_3} = 0$$

Stacking the above restrictions yields

$$\underbrace{-\begin{pmatrix} \frac{\lambda_2^2 \kappa(\alpha\beta-1)}{1+\alpha\sigma\kappa} & \frac{\lambda_2^2(\alpha\beta-1)(\sigma\kappa+\beta)}{1+\alpha\sigma\kappa} - \frac{\lambda_2 \beta(\alpha\beta-1)}{1+\alpha\sigma\kappa} & 0 \\ \frac{\beta^2 \kappa(\alpha\beta-1)}{1+\alpha\sigma\kappa} & \frac{\beta^2(\alpha\beta-1)(\beta-1+\sigma\kappa)}{1+\alpha\sigma\kappa} & 0 \end{pmatrix}}_{R} C_0 = \underbrace{\begin{pmatrix} \frac{\sigma\kappa\lambda_2^3(\alpha\beta-1)}{(1+\alpha\sigma\kappa)(1-\rho_M\lambda_2)} & 0 \\ 0 & \frac{\beta^2\sigma\kappa(1-\beta)(\alpha\beta-1)}{(1+\alpha\sigma\kappa)(1-\rho_F\beta)} \end{pmatrix}}_{A}$$

Apparently, the solution exists because $\text{span}(A) \subseteq \text{span}(R)$ is satisfied here. In order for the solution to be unique, we must be able to pin down the terms

$$QC_0 = U_{3\cdot}(\lambda_1)\Gamma_{-1}C_0 = \begin{pmatrix} \frac{\lambda_1^2 \kappa(\alpha\beta-1)}{1+\alpha\sigma\kappa} & \frac{\lambda_1^2(\alpha\beta-1)(\sigma\kappa+\beta)}{1+\alpha\sigma\kappa} - \frac{\lambda_1 \beta(\alpha\beta-1)}{1+\alpha\sigma\kappa} & 0 \end{pmatrix} C_0$$

from the knowledge of RC_0. This is tantamount to verifying $\text{span}(Q') \subseteq \text{span}(R')$, which is also satisfied here.

Now the unique solution can be computed as

$$\begin{pmatrix} \hat{y}_t \\ \hat{\pi}_t \\ \hat{b}_t \end{pmatrix} = (L\Gamma(L))^{-1}(L\Psi_0 A(L) + \Gamma_{-1}C_0) \begin{pmatrix} \epsilon_{M,t} \\ \epsilon_{F,t} \end{pmatrix}$$
$$= \begin{pmatrix} C_0(1,1)\frac{1-\frac{\beta-(1+\rho_M\beta)\lambda_2+\rho_M\beta(1+\alpha\sigma\kappa)\lambda_2^2}{\beta\lambda_2(\beta-1+\sigma\kappa)}L}{\left(1-\frac{1}{\lambda_1}L\right)(1-\rho_M L)} & C_0(1,2)\frac{1}{1-\frac{1}{\lambda_1}L} \\ C_0(2,1)\frac{1-\frac{(1+\rho_M\beta)\lambda_2-\beta}{\beta\lambda_2}L}{\left(1-\frac{1}{\lambda_1}L\right)(1-\rho_M L)} & C_0(2,2)\frac{1}{1-\frac{1}{\lambda_1}L} \\ C_0(3,1)\frac{1}{\left(1-\frac{1}{\lambda_1}L\right)(1-\rho_M L)} & C_0(3,2)\frac{1}{\left(1-\frac{1}{\lambda_1}L\right)(1-\rho_F L)} \end{pmatrix} \begin{pmatrix} \epsilon_{M,t} \\ \epsilon_{F,t} \end{pmatrix}$$

where the contemporaneous responses are given by

$$C_0 = \begin{pmatrix} \frac{\sigma\lambda_2^2(\beta-1+\sigma\kappa)}{(\lambda_2-\beta)(1-\rho_M\lambda_2)} & -\frac{(1-\beta)\sigma[(\sigma\kappa+\beta)\lambda_2-\beta]}{(\lambda_2-\beta)(1-\rho_F\beta)} \\ -\frac{\sigma\kappa\lambda_2^2}{(\lambda_2-\beta)(1-\rho_M\lambda_2)} & \frac{\sigma\kappa\lambda_2(1-\beta)}{(\lambda_2-\beta)(1-\rho_F\beta)} \\ \frac{\beta}{\lambda_1} + \frac{(1-\alpha\beta)[\sigma\kappa-\rho_M\kappa C_0(1,1)-\rho_M(\sigma\kappa+\beta)C_0(2,1)]}{\lambda_1(1+\alpha\sigma\kappa)} & \frac{\beta-1}{\lambda_1} + \frac{\rho_F(\alpha\beta-1)[\kappa C_0(1,2)+(\sigma\kappa+\beta)C_0(2,2)]}{\lambda_1(1+\alpha\sigma\kappa)} \end{pmatrix}$$

Setting $\rho_M = \rho_F = 0$ gives the expression for C_0 in Section 3.2.2.

Indeterminacy Dropping the fiscal policy (3.4) and government budget constraint (3.5) and introducing the inflation forecast error $\eta_{\pi,t} = \hat{\pi}_t - \mathbb{E}_{t-1}\hat{\pi}_t$ into the equilirium system, (A.1)–(A.2) can be modified as

$$\mathbb{E}_t\left[\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\Gamma_{-1}} L^{-1} + \underbrace{\begin{pmatrix} -1 & -\alpha\sigma & \sigma \\ \kappa & -1 & \beta \\ 0 & 1 & 0 \end{pmatrix}}_{\Gamma_0} L^0 + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{\Gamma_1} L\right] \underbrace{\begin{pmatrix} \hat{y}_t \\ \hat{\pi}_t \\ \mathbb{E}_t\hat{\pi}_{t+1} \end{pmatrix}}_{x_t} = \underbrace{\begin{pmatrix} \sigma & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\Psi_0} L^0 \underbrace{\begin{pmatrix} d_{M,t} \\ d_{\pi,t} \end{pmatrix}}_{d_t} \tag{A.3}$$

where

$$\underbrace{\begin{pmatrix} d_{M,t} \\ d_{\pi,t} \end{pmatrix}}_{d_t} = \underbrace{\begin{pmatrix} \frac{1}{1-\rho_M L} & 0 \\ 0 & 1 \end{pmatrix}}_{A(L)} \underbrace{\begin{pmatrix} \epsilon_{M,t} \\ \eta_{\pi,t} \end{pmatrix}}_{\epsilon_t} \tag{A.4}$$

and we treat $\eta_{\pi,t}$ as a new fundamental shock.

Next, apply the Smith canonical factorization to the polynomial matrix $z\Gamma(z)$

$$z\Gamma(z) = U(z)^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z(z-\lambda_1)(z-\lambda_2) \end{pmatrix} V(z)^{-1}$$

where the roots (λ_1, λ_2) are identical to those under determinacy.

Finally, examine the existence and uniqueness of solution. Under indeterminacy with $0 \leqslant \alpha < 1$, it follows that $0 < \lambda_2 < 1 < \lambda_1$. Collect the roots inside the unit circle in $S(z)$ and multiply both sides of the z-transform identities by $S(z)^{-1}$

$$T(z)C(z) = \begin{pmatrix} U_{1\cdot}(z) \\ \frac{1}{z}U_{2\cdot}(z) \\ \frac{1}{z(z-\lambda_2)}U_{3\cdot}(z) \end{pmatrix} (z\Psi_0 A(z) + \Gamma_{-1}C_0)$$

These identities are valid for all z on the open unit disk except for $z = 0, \lambda_2$. But since $C(z)$ must be well-defined for all $|z| < 1$, this condition places the following restrictions on the unknown coefficient matrix C_0

$$\begin{aligned} U_{2\cdot}(z)(z\Psi_0 A(z) + \Gamma_{-1}C_0)|_{z=0} &= 0 \\ U_{3\cdot}(z)(z\Psi_0 A(z) + \Gamma_{-1}C_0)|_{z=0,\lambda_2} &= 0 \end{aligned}$$

Stacking the effective restrictions yields

$$-\underbrace{\begin{pmatrix} -\frac{\lambda_2\kappa}{1+\alpha\sigma\kappa} & 0 & 0 \end{pmatrix}}_{R} C_0 = \underbrace{\begin{pmatrix} -\frac{\lambda_2^2\sigma\kappa}{(1+\alpha\sigma\kappa)(1-\rho_M\lambda_2)} & -\frac{\lambda_2[\lambda_2(1+\alpha\sigma\kappa)-1]}{1+\alpha\sigma\kappa} \end{pmatrix}}_{A}$$

Apparently, the solution exists because $\text{span}(A) \subseteq \text{span}(R)$ is satisfied here. In order for the solution to be unique, we must be able to pin down the terms

$$QC_0 = U_{3\cdot}(\lambda_1)\Gamma_{-1}C_0 = \begin{pmatrix} -\frac{\lambda_1\kappa}{1+\alpha\sigma\kappa} & 0 & 0 \end{pmatrix} C_0$$

from the knowledge of RC_0. This is tantamount to verifying $\text{span}(Q') \subseteq \text{span}(R')$, which is also satisfied here.

Now the unique solution can be computed as

$$\begin{aligned} \begin{pmatrix} \hat{y}_t \\ \hat{\pi}_t \\ \mathbb{E}_t\hat{\pi}_{t+1} \end{pmatrix} &= (L\Gamma(L))^{-1}(L\Psi_0 A(L) + \Gamma_{-1}C_0) \begin{pmatrix} \epsilon_{M,t} \\ \eta_{\pi,t} \end{pmatrix} \\ &= \begin{pmatrix} C_0(1,1)\frac{1-\frac{1}{\beta}L}{(1-\rho_M L)\left(1-\frac{1}{\lambda_1}L\right)} & C_0(1,2)\frac{1}{1-\frac{1}{\lambda_1}L} \\ C_0(3,1)\frac{L}{(1-\rho_M L)\left(1-\frac{1}{\lambda_1}L\right)} & C_0(2,2)\frac{1}{1-\frac{1}{\lambda_1}L} \\ C_0(3,1)\frac{1}{(1-\rho_M L)\left(1-\frac{1}{\lambda_1}L\right)} & C_0(3,2)\frac{1}{1-\frac{1}{\lambda_1}L} \end{pmatrix} \begin{pmatrix} \epsilon_{M,t} \\ \eta_{\pi,t} \end{pmatrix} \end{aligned}$$

where the contemporaneous responses are given by

$$C_0 = \begin{pmatrix} -\frac{\sigma\lambda_2}{1-\rho_M\lambda_2} & -\frac{(1+\alpha\sigma\kappa)\lambda_2-1}{\kappa} \\ 0 & 1 \\ \frac{\sigma\kappa}{(1-\rho_M\lambda_2)(1+\alpha\sigma\kappa)\lambda_1} & \frac{1}{\lambda_1} \end{pmatrix}$$

Setting $\rho_M = 0$ gives the expression for C_0 in Section 3.2.3.

Appendix B: Data Set Unless otherwise stated, the following data are drawn from the National Income and Product Accounts (NIPA) released by the Bureau of Economic Analysis. All data in levels from NIPA are nominal values and divided by 4. The quarterly observable sequences in the text are constructed as follows.

1. Per capita real output growth rate, YGR. Per capita real output is obtained by dividing the gross domestic product (Table 1.1.5, line 1) by the civilian noninstitutional population (series "CNP16OV", Federal Reserve Economic Data, St. Louis Fed) and deflating using the implicit price deflator for gross domestic product (Table 1.1.9, line 1). Growth rates are computed using quarter-to-quarter log difference and converted into percentage by multiplying by 100.

2. Annualized inflation rate, INF, is defined as the quarter-to-quarter log difference of the implicit price deflator for gross domestic product and converted into percentage by multiplying by 400.

3. Annualized nominal interest rate, INT, corresponds to the effective federal funds rate (Board of Governors of the Federal Reserve System) and is in percentage.

4. Per capita real debt growth rate, BGR. Per capita real debt is obtained by dividing the market value of privately held gross federal debt (Federal Reserve Bank of Dallas) by the civilian noninstitutional population and deflating using the implicit price deflator for gross domestic product. Growth rates are computed using quarter-to-quarter log difference and converted into percentage by multiplying by 100.

Appendix C: Supplementary Tables and Figures Table 6 reports the posterior estimates of model parameters based on the full band. Figures 5–8 compare the cross-correlograms of the data (black solid line with cross) with those of regime-M (blue dashed line) and regime-F (red solid line) evaluated with the posterior mean over partial bands.

Table 6: Full Band Posterior Estimates

	Pre-Volcker Era				Post-Volcker Era			
	Regime-M		Regime-F		Regime-M		Regime-F	
Para	Mean	90% HPD	Mean	90% HPD	Mean	90% HPD	Mean	90% HPD
$1/\sigma$	4.92	[4.42,5.41]	5.26	[4.77,5.77]	4.93	[4.44,5.42]	5.14	[4.62,5.66]
κ	0.51	[0.42,0.59]	0.45	[0.38,0.53]	0.51	[0.42,0.59]	0.39	[0.33,0.46]
$\bar{r}$	0.50	[0.33,0.66]	0.50	[0.33,0.65]	0.50	[0.34,0.66]	0.51	[0.36,0.67]
α	1.80	[1.57,2.01]	0.56	[0.44,0.69]	2.24	[1.98,2.48]	0.47	[0.35,0.59]
γ	1.51	[1.18,1.82]	–	–	1.50	[1.16,1.82]	–	–
ρ_M	0.93	[0.91,0.95]	0.97	[0.95,0.98]	0.95	[0.94,0.97]	0.95	[0.93,0.97]
ρ_F	0.49	[0.34,0.67]	0.50	[0.34,0.67]	0.51	[0.35,0.67]	0.50	[0.34,0.67]
$100\sigma_M$	0.34	[0.27,0.42]	0.25	[0.21,0.29]	0.27	[0.22,0.32]	0.21	[0.18,0.24]
$100\sigma_F$	0.43	[0.28,0.57]	0.42	[0.28,0.57]	0.43	[0.28,0.57]	0.43	[0.29,0.57]
Ave Ineff		2.3		2.3		2.2		10.1

NOTES: See Table 2.

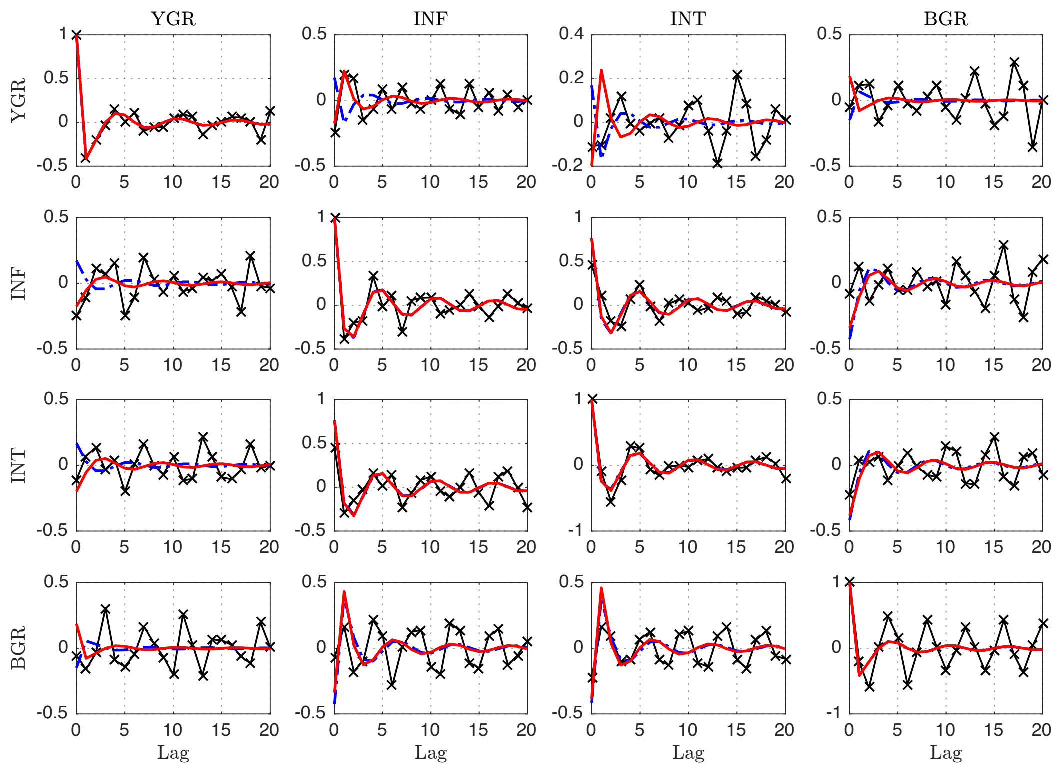

Figure 5: Pre-Volcker cross-correlogram estimated on high-pass band.

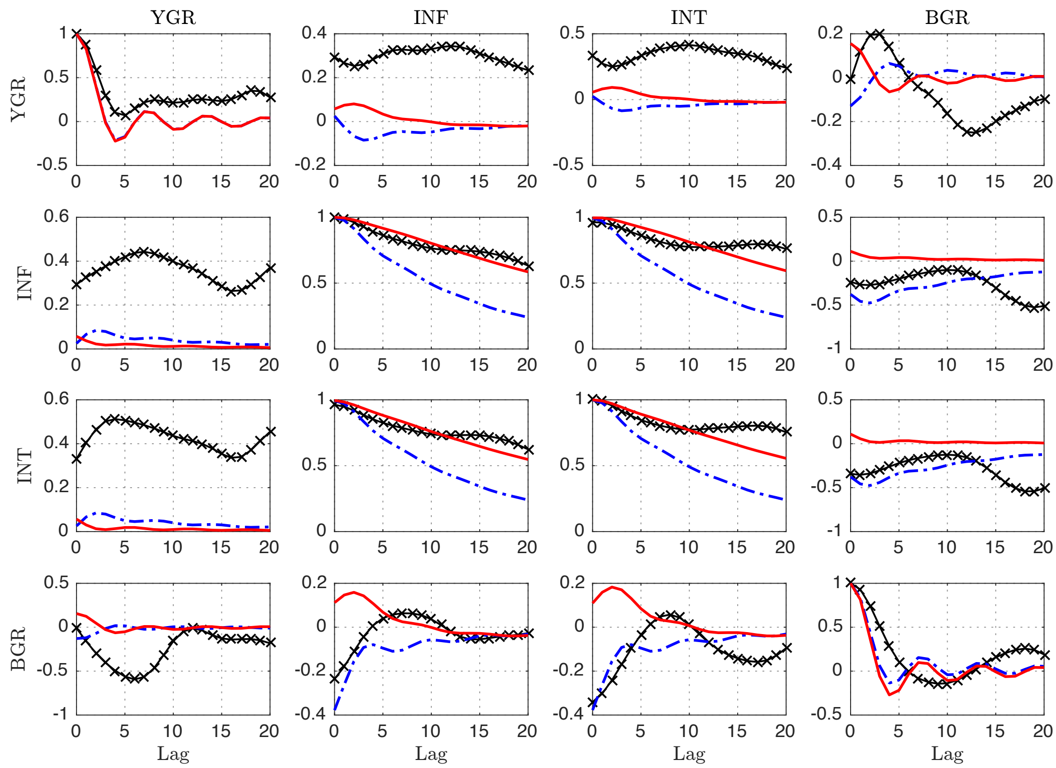

Figure 6: Pre-Volcker cross-correlogram estimated on low-pass band.

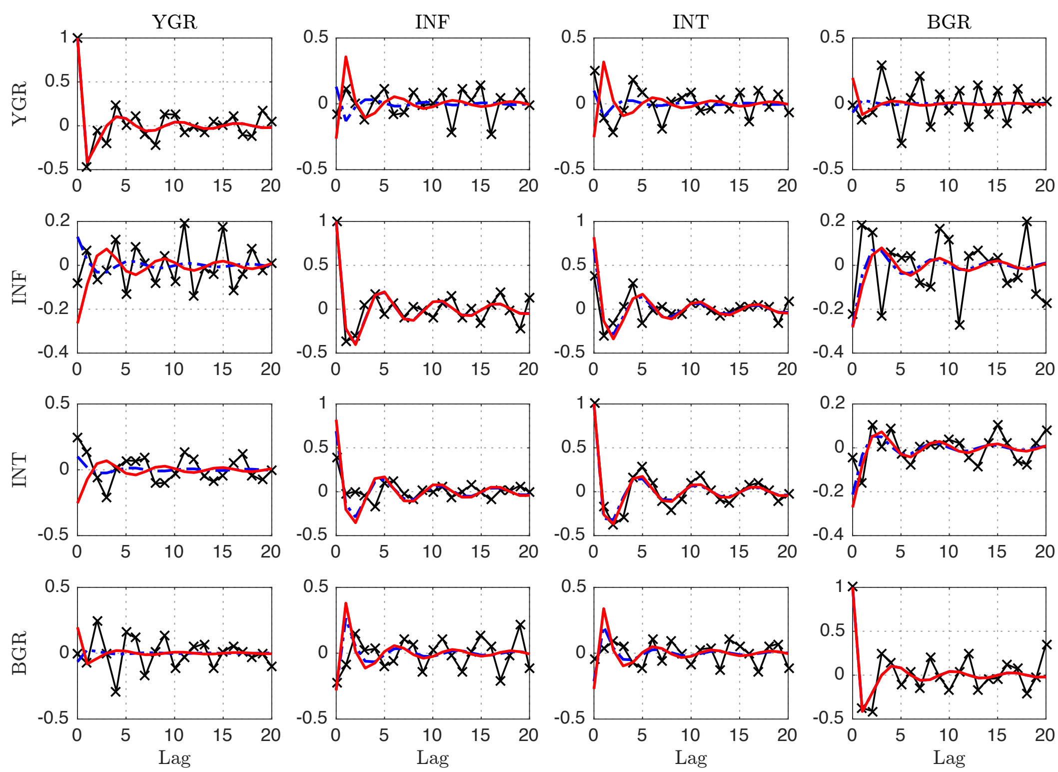

Figure 7: Post-Volcker cross-correlogram estimated on high-pass band.

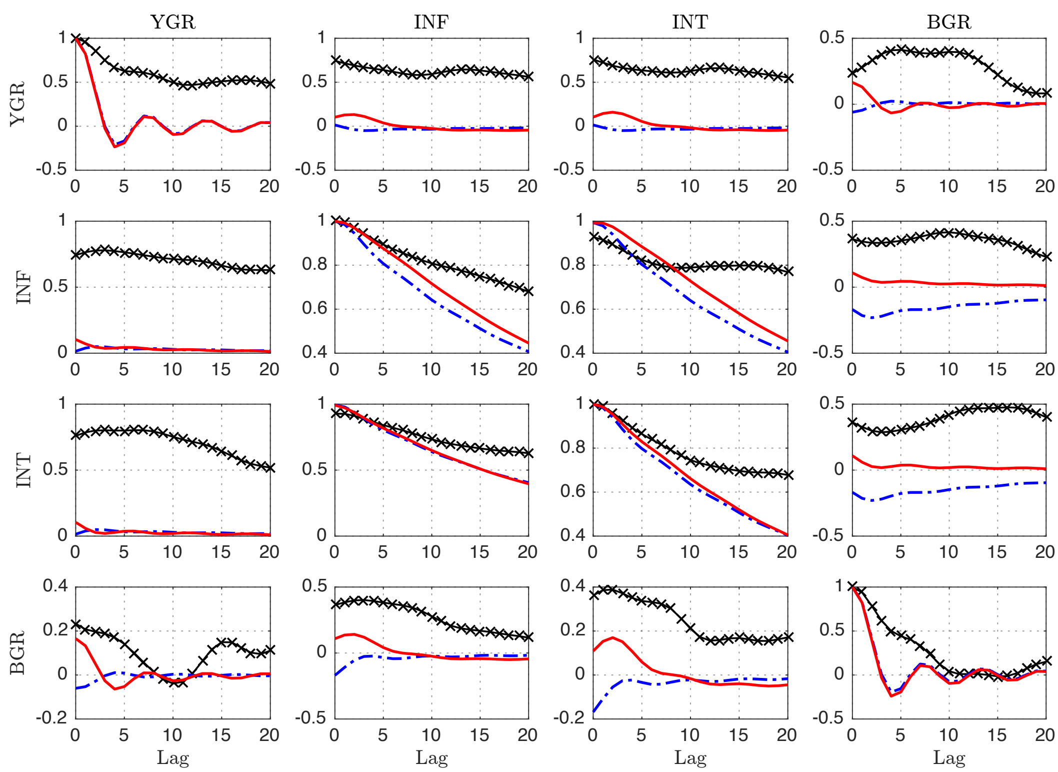

Figure 8: Post-Volcker cross-correlogram estimated on low-pass band.