Notes to Matlab code that solves model in "Information, heterogeneity and market incompleteness"

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1 Description of files

1.1 Model definition

The file model.m contains the definition of the model along with parameters, settings etc.

1.2 Calculation

HierarchySolver.m contains the main code for the solver. The solution is a fixed point of equations for M, N, η, β as described in section A.1 of the supplementary material. The code attempts to find the fixed point simply by iterating on the equations. There are many ways to carry out the iterations, and we've tried most of them. For simplicity, this version of the code is set to work with the method that we've found to be most reliable:

- taking η as given and iterating M, N and β , until they are mutually consistent
- taking M, N and β as given and iterating η until it converges
- repeating until M, N, η, β are mutually consistent

1.3 Output

The file Simul.m calculates a time series given the models solution, and PlotIrf.m plots impulse responses, both controlled by settings in the model.m file.

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1.4 Convergence

For some calibrations and starting values, the iteration will not converge. The code gives a simple way ot dealing with this by allowing a starting value to be chosen. To implement this:

1. Run the code for a case that solves
2. In the section commented "Smart starting values" (around line 132 of "Model.m")
   - entered the desired values for It_start (whcih should be the value that you know works), It_end (the value you want to sovle for) and It_step (the smaller the steps, the more likely is convergence)
   - uncomment the "for" statement on line 136
   - uncomment the expression on line 137 and replace the LHS with the variable you want to change
   - set "LOAD_VARS" to 1
   - uncomment the "end" statement on line 314
3. The code will now solve the model for values of the given variable starting at It_start and ending at It_endwith steps of It_step. It will use the values of M, N, η, β from the previous solution as starting values for the next.
4. When returning to the ordinary method of solving, undo steps 1 - 3

2 Truncating the hierarchy

If the state variables relevant to agent s's decision are $X_t^{s,h}$ conjecture a consumption function

$$c_t^s = \underset{1xhr}{\eta'} \underset{hr\times 1}{E_t^s X_t^s} \tag{1}$$

Then aggregate consumption is

$$c_t = \eta' X_t^{h+1} \tag{2}$$

where the sum adds an extra order to the hierarchy of expectations, so X^{h+1} has the same dimension as X^s

$$X_t^{h+1} = \left[\begin{array}{cc} X_t^h & \left[\begin{array}{cc} \xi_{t|t}^{(h+1)} & \chi_{t|t}^{(h+1)} \end{array} \right] \end{array} \right]' \tag{3}$$

But then the states relevant to agent s's decision are actually $X_t^{s,h+1}$.

So to truncate the hierarchy consistently define an approximate consumption function

$$c_t^s = \eta' E_t^s X_t^{s,h-1} \tag{4}$$

and a truncation matrix T_3 is which cuts off the bottom order of the hierarchy so

$$T_3 X_t^{s,h} = X_t^{s,h-1} \tag{5}$$

Define the consumption - relevant expected states by $\widehat{X}_t^s$

$$\widehat{X}_t^s = E_t^s X_t^{s,h-1} = T_3 E_t^s X_t^{s,h} \tag{6}$$

where

$$T_3 = \begin{bmatrix} I_{hr} & 0_{hr\times r} \end{bmatrix} \tag{7}$$

and

$$c_t^s = \eta' T_3 E_t^s X_t^{s,h} = \eta' \widehat{X}_t^s \tag{8}$$

Then aggregate consumption is given by

$$c_t = \eta' X_t^{n^e} = T_2 X_t^s \tag{9}$$

and define another truncation matrix T_2 such that

$$T_2 = \begin{bmatrix} 0_{hr\times r} & I_{hr} \end{bmatrix} \tag{10}$$

From now on we write the system for some h in terms of X_t^s and $\widehat{X}_t^s$

2.1 Solution with truncation

Conjecture that the actual law of motion of the states has an AR representation

$$X_{t+1}^s = Lc_t^s + MX_t^s + Nv_{t+1}^s \tag{11}$$

where

$$L = \begin{bmatrix} F_s \\ 0_{(h-1)rx1} \end{bmatrix} \tag{12}$$

Taking expectations of (16) gives

$$E_t^s X_{t+1}^s = LE_t^s c_t^s + ME_t^s X_t^s \tag{13}$$

since $E_t^s v_{t+1}^s = 0$. Substituting from (8)

$$E_t^s X_{t+1}^s = (LT_3\eta' + M)\, E_t^s X_t^s \tag{14}$$

The updating rule is

$$E_{t+1}^s X_{t+1}^s = E_t^s X_t^s + \beta \left(y_{t+1}^s - E_t^s y_{t+1}^s \right) \tag{15}$$

where

$$y_t^s = \begin{bmatrix} H_\xi' & H_\chi & 0 & .... \end{bmatrix} X_t^s = H_X' X_t^s + H_c c_t + H_s c_t^s \tag{16}$$

so

$$E_{t+1}^s X_{t+1}^s = E_t^s X_{t+1}^s + \beta \left[H_X' \left(X_{t+1}^s - E_t^s X_{t+1}^s \right) + H_c \left(c_{t+1} - E_t^s c_{t+1} \right) + H_s \left(c_{t+1}^s - E_t^s c_{t+1}^s \right) \right] \tag{17}$$

substituting for aggregate and idiosyncractic consumption from (8) and (9)

$$c_t^s = \eta' T_3 E_t^s X_t^{s,n^e} = \eta' \widehat{X}_t^s \tag{18}$$

Then aggregate consumption is given by

$$c_t = \eta' X_t^{n^e} = T_2 X_t^s \tag{19}$$

$$E_{t+1}^s X_{t+1}^s = E_t^s X_{t+1}^s + \beta \left[\begin{array}{c} H_X' \left(X_{t+1}^s - E_t^s X_{t+1}^s \right) + H_c \eta' T_2 \left(X_{t+1}^s - E_t^s X_{t+1}^s \right) \\ + H_s \eta' T_3 \left(E_{t+1}^s X_{t+1}^{s,n^e} - E_t^s E_{t+1}^s X_{t+1}^{s,n^e} \right) \end{array} \right]$$

by the law of iterated expectations the last term cancels out so

$$E_{t+1}^s X_{t+1}^s = E_t^s X_{t+1}^s + \beta \left[\left(H_X' + H_c \eta' T_2 \right) \left(X_{t+1}^s - E_t^s X_{t+1}^s \right) \right] \tag{20}$$

Define

$$H' = H_X' + H_c \eta' T_2 \tag{21}$$

Then substituting for X_{t+1}^s from (11) and from (14)

$$\begin{aligned} E_{t+1}^s X_{t+1}^s &= \left(L T_3 \eta' + M \right) E_t^s X_t^s + N E_t^s v_{t+1}^s + \\ & \quad \beta H' \left(L c_t^s + M X_t^s + N v_{t+1}^s - L E_t^s c_t^s - M E_t^s X_t^s - N E_t^s v_{t+1}^s \right) \\ &= \left(L T_3 \eta' + \left(I - \beta H' \right) M \right) E_t^s X_t^s + \beta H' \left(M X_t^s + N v_{t+1}^s \right) + \end{aligned}$$

$$X_t^s \approx T_1 T_3 X_t^s$$

and, noting that $T_3 T_1 T_3 = T_3$, this means we can write

$$\begin{aligned} T_3 E_{t+1}^s X_{t+1}^s &= T_3 \left[L \eta' + \left(I - \beta H' \right) M T_1 \right] T_3 E_t^s X_t^s + T_3 \beta H' \left(M X_t^s + N v_{t+1}^s \right) \\ \widehat{X}_{t+1}^s &= T_3 \left[L \eta' + \left(I - \beta H' \right) M T_1 \right] \widehat{X}_t^s + T_3 \beta H' M X_t^s + T_3 \beta H_W' v_{t+1}^s \end{aligned} \tag{22}$$

Averaging across agents

$$X_{t+1}^h = T_3 \left[L \eta' + \left(I - \beta H' \right) M T_1 \right] X_t^h + T_3 \beta H' M T_7 X_t^s + T_3 \beta H_W' T_8 v_{t+1}^s \tag{23}$$

where T_7 is an averaging matrix which zeros the idiosyncratic elements of X_t^s i.e.

$$T_7 = \left[\begin{array}{ccc} I_{r-r^s x r-r^s} & 0 & 0 \\ 0 & 0_{r^s x r^s} & 0 \\ 0 & 0 & I_{hrxhr} \end{array} \right]$$

and T_8 does the same for v_{t+1}^s

$$T_8 = \left[\begin{array}{cc} I_{r-r^s x r-r^s} & 0 \\ 0 & 0_{r^s x r^s} \end{array} \right]$$

since averaging over agents averages away the idiosyncratic elements of the observable variables, and these do not depend on the hierarchy of expectations.

So now we have a law of motion for the hierarchy of expectations X^h_{t+1} and the consumption relevant states $\widehat{X}^s_{t+1}$

The parallel filtering problem is

$$\begin{aligned} W^s_{t+1} &= F_W W^s_t + F_c c_t + v^s_{t+1} && (24)\\ &= F_W W^s_t + F_c \eta' X^h_t + v^s_{t+1} && (25)\\ &= \begin{bmatrix} F_W & F_c\eta' \end{bmatrix} \begin{bmatrix} W^s_t \\ X^h_t \end{bmatrix} + v^s_{t+1} && (26)\end{aligned}$$

stacking this on top of (23) gives

$$\underset{(n^e+1)r\text{x}1}{\begin{bmatrix} W^s_{t+1} \\ X^h_{t+1} \end{bmatrix}} = \left\{ \underset{(n^e+1)r\text{x}(n^e+1)r}{\begin{bmatrix} F_W & F_c\eta' \\ \underset{n^e r\text{x}r}{0} & \underset{n^e r\text{x}n^e r}{T_3\left[L\eta' + (I-\beta H')\, M T_1\right]} \end{bmatrix}} + \begin{bmatrix} \underset{r\text{x}r}{0} & \underset{r\text{x}n^e r}{0} \\ \underset{n^e r\text{x}n^e r}{T_3\beta H' M T_7} & \underset{r\text{x}r}{0} \end{bmatrix} \right\} \cdot \begin{bmatrix} W^s_t \\ X^h_t \end{bmatrix} +$$

$$+ \underset{(n^e r+1)\text{x}r}{\begin{bmatrix} \underset{r\text{x}r}{I} \\ T_3\beta H'_W T_8 \end{bmatrix}} \underset{r\text{x}1}{v^s_{t+1}} \tag{27}$$

The full system is then

$$Z_{t+1} = \begin{bmatrix} X^s_{t+1} \\ \widehat{X}^s_{t+1} \end{bmatrix} = A_Z Z_t + A_v v^s_{t+1} \tag{28}$$

$$Z_{t+1} = \begin{bmatrix} W^s_{t+1} \\ X^h_{t+1} \\ \widehat{X}^s_{t+1} \end{bmatrix}$$

we can write

$$A_Z = \begin{bmatrix} F_W & F_c\eta' & F_s\eta' \\ 0 & T_3\left[L\eta' + (I-\beta H')\, M T_1\right] & 0 \\ 0 & 0 & T_3\left[L\eta' + (I-\beta H')\, M T_1\right] \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ T_3\beta H' M T_7 & 0 \\ T_3\beta H' M & 0 \end{bmatrix}$$

where the second matrix partitions the columns by X^s_t and $\widehat{X}^s_t$, and

$$A_v = \begin{bmatrix} I \\ T_3\beta H'_W T_8 \\ T_3\beta H'_W \end{bmatrix}$$

2.2 Euler equation

The Euler equation is

$$E^s_t c^s_{t+1} - \sigma E^s_t r_{t+1} = E^s_t c^s_t \tag{29}$$

The return can be written

$$
\begin{aligned}
r_t &= \mu_w W_t^s + \mu_c c_t && (30)\\
&= R X_t^s && (31)
\end{aligned}
$$

where

$$R = \left(\begin{bmatrix} \mu_w & 0 \end{bmatrix} + \mu_c \eta' T_2 \right) \tag{32}$$

Substituting this and the conjectured consumption (8) function into the Euler equation (29) gives

$$\left(\eta' T_3 - \sigma R \right) E_t^s X_{t+1}^s = \eta' T_3 E_t^s X_t^s \tag{33}$$

then using he conjectured law of motion for the states (taking expectations of Appendix D.4 and substituting for c_t^s from (8))

$$E_t^s X_{t+1}^s = \left(L \eta' T_3 + M \right) E_t^s X_t^s \tag{34}$$

gives

$$
\begin{aligned}
\left(\eta' T_3 - \sigma R \right) \left(L \eta' T_3 + M \right) E_t^s X_t^s &= \eta' T_3 E_t^s X_t^s && (35)\\
\left(\eta' T_3 - \sigma R \right) \left(L \eta' T_3 + M \right) &= \eta' T_3 && (36)\\
\left(\eta' T_3 - \sigma R \right) \left(L \eta' T_3' + M \right) T_1 &= \eta' && (37)
\end{aligned}
$$

using $T_3 T_1 = I$