

# SAMPLING AND ESTIMATION

## DAY 3: ESTIMATION IN COMPLEX SURVEY DESIGNS

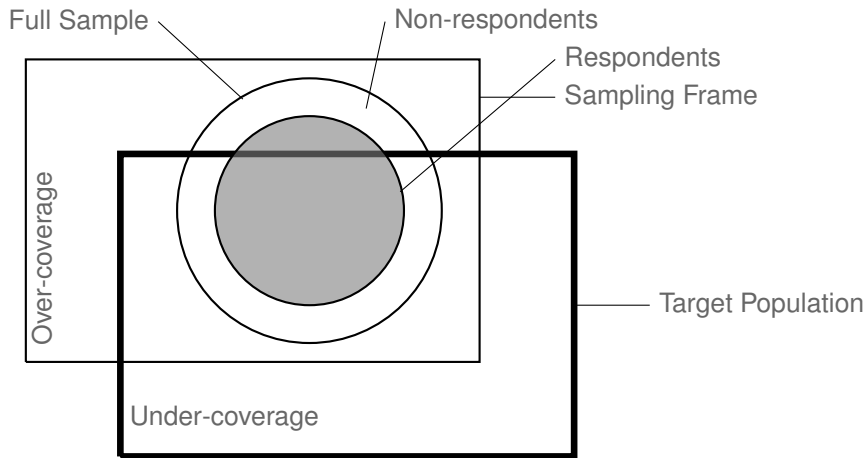
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—> Methods for handling coverage errors are not so widely spread, simply because there is often no reliable auxiliary information on just the target population. However if there is, it can receive a treatment similar to that of weighting by non-response.

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**MNAR** missing not at random  
RP depends on variables of interest  $\mathcal{Y}$   
cannot be modeled, because  $\mathcal{Y}$  not known for  
non-respondents

[Rubin and Little 2002]

→ In multivariate analysis often 30% to 40% of the data are lost  
with case deletion assuming MCAR!

**Calibration approach** The design weights are calibrated to the totals of some auxiliary variables  $\mathcal{X}$ .

Sample estimates using the calibrated weights will exactly replicated those totals.

If the used auxiliary variables help to explain the response process the calibrated weight can reduce the non-response error.

**Two-phase approach** The response process is modeled to obtain the response propensities  $\psi_k$  for all  $k \in \mathcal{A}$ . The new weight of element  $k$  is  $\frac{d_k}{\psi_k}$ . (Two phases: 1. Sampling  $\rightarrow$  2. Responding).

In addition the new weights  $\frac{d_k}{\psi_k}$  might then also be calibrated.

Often used models are:

- Response homogeneity classes, every element in a class has the same probability to respond.

- Generalized liner models (*probit*, *logit*, *log-log*), treating response as a latent variable.

The calibration approach is more direct as the design weights are directly calibrated without considering the response propensities. Also, if the same models are used for both the modeling of the response propensities and the calibration the two approaches can be equivalent.

Generic estimators for a total and a mean

$$\hat{\tau}_w = \sum_{k \in \delta} w_k y_k \quad \text{and} \quad \bar{y}_w = \frac{\sum_{k \in \delta} w_k y_k}{\sum_{k \in \delta} w_k},$$

where  $w_k$  is the survey weight of element  $k$ , with

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where  $w_k$  is the survey weight of element  $k$ , with

$$w_k = \begin{cases} d_k g_k & \text{for } k \in \mathcal{A} \\ 0 & \text{else} \end{cases}.$$



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Sometimes called base weights or design weights, the inverse of inclusion probabilities  $d_k = \pi^{-1}$  is usually the first step in weighting. If we have  $g_k = 1$  the  $\hat{\tau}_w$  would be the HT estimator or  $\pi$ -estimator. The factor  $g_k$  adjusts the design weights to reduce

- the sampling error (i.e. variance),

- the non-response error, and

- the coverage error

of estimator  $\hat{\tau}_w$  or  $\bar{y}_w$ . Thereby the  $w_k$ 's should not deviate to much from the  $d_k$ 's as these weights ensure an unbiased estimation.

The general idea is to exploit the relationship between auxiliary variables and the variable of interest to improve the efficiency of estimators.

The following problem is solved with weight calibration:

For a give design  $p(\cdot)$  and a sample  $\Delta$  weights  $w_k$  for all  $k \in \Delta$  have to be found that minimize

$$\sum_{k \in \Delta} G_k(w_k, d_k, c_k) ,$$

subject to constraints

$$\sum_{k \in \Delta} w_k \mathbf{x}_k = \sum_{k \in \mathcal{U}} \mathbf{x}_k = \boldsymbol{\tau}_x$$

where  $\mathbf{x}_k = (x_{k1}, x_{k2}, \dots, x_{kQ})^\top$  is a vector of  $q$  auxiliary variables for element  $k$ .  $G_k$  is a measure of distance between  $w_k$  and  $d_k$  and  $c_k$  is a factor that can be freely chosen for additional flexibility.

To calculate the weights the  $\mathbf{x}_k$ 's are only needed for the elements in the net sample (i.e. typically only for the respondents), but  $\tau_x$ , their population totals need to be known.

The auxiliary variables can be metric (e.g. income or age) or categorical (e.g. gender or age groups).

Depending on the choice of  $G_k$  different calibration estimators can be obtained, some of the most common are:

- Post-stratification Estimator

- Raking Estimator

- Generalized Regression Estimator

Note that the  $w_k$ 's typically depend on the sample  $\mathcal{A}$ , in contrast to the  $d_k$ , which are given by the sampling design.

Post-stratification is typically used if only categorical auxiliary variables are available. It is implemented by forming weighting cells by crossing *all* categories of the auxiliary variables. These weighting cells are the post-strata  $\mathcal{U}_q$  with  $q = 1, \dots, Q$ . The weight are then adjusted to replicate the counts in these cells. For  $k \in \mathcal{U}_q$  we have

$$g_k = \frac{\tau_{x_q}}{\hat{\tau}_{x_q}},$$

where  $\tau_{x_q} = \sum_{k \in \mathcal{U}} x_{kq}$  and

$$x_{kq} = \begin{cases} 1 & \text{if } k \in \mathcal{U}_q \\ 0 & \text{else} \end{cases}.$$

$\hat{\tau}_{x_q \pi} = \sum_{k \in \mathcal{U}} d_k x_{kq}$  its estimator for  $\tau_{x_q}$  based on the design weights. The auxiliary variables are the post-stratum indicators, i.e.  $\mathbf{x}_k = (x_{k1}, x_{k2}, \dots, x_{kQ})^\top$ . An adjustment to the totals of a metric variable within the post-strata would also be possible.

TABLE: Population Counts  $\tau_{xq}$  for Hair and Eye Colour

	Brown	Blue	Hazel	Green
Black	68	20	15	5
Brown	119	84	54	29
Red	26	17	14	14
Blond	7	94	10	16

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TABLE: Sample counts  $\sum_{k \in \Delta} x_{kq}$  in a SRS with  $n = 150$

	Brown	Blue	Hazel	Green
Black	14	7	2	2
Brown	36	22	17	5
Red	7	3	1	4
Blond	1	23	1	5

TABLE: Population Counts  $\tau_{xq}$  for Hair and Eye Colour

	Brown	Blue	Hazel	Green
Black	68	20	15	5
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Blond	7	94	10	16

TABLE: Estimated totals  $\hat{\tau}_{xq} \pi = \sum_{k \in \Delta} x_{kq} d_k$

	Brown	Blue	Hazel	Green
Black	55.2533	27.6267	7.8933	7.8933
Brown	142.0800	86.8267	67.0933	19.7333
Red	27.6267	11.8400	3.9467	15.7867
Blond	3.9467	90.7733	3.9467	19.7333



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	Brown	Blue	Hazel	Green
Black	68	20	15	5
Brown	119	84	54	29
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Blond	7	94	10	16

TABLE: Post-stratification  $g_k = \frac{\tau_{xq}}{\hat{\tau}_{xq}}$

	Brown	Blue	Hazel	Green
Black	1.2307	0.7239	1.9003	0.6334
Brown	0.8376	0.9674	0.8048	1.4696
Red	0.9411	1.4358	3.5473	0.8868
Blond	1.7736	1.0355	2.5338	0.8108

Beware, there must be at least one element in the sample from each post-stratum, otherwise we divide by null!

In raking only the marginal totals are need, *not* the totals for all the cross-categories. Raking can be implemented as iterative post-stratification to adjust the design weights to the margins of the different auxiliary variables.

The design weights of a SRSC cluster sample of school districts are raked to variables school type (stype) and the accomplishment of the growth target (sch.wide).

```
##          dname          name stype sch.wide
## 1 Alameda City Unified   Alameda High      H      Yes
## 2 Alameda City Unified   Encinal High      H      Yes
## 3 Alameda City Unified  Chipman Middle      M      Yes
## 4 Alameda City Unified Lum (Donald D.)      E      Yes
## 5 Alameda City Unified Edison Elementa      E      Yes
## 6 Alameda City Unified Otis (Frank) El      E      Yes
```

TABLE: Population Counts  $\tau_{xq}$  for School Type (stype) and School Target (sch.wide)

	No	Yes	SUM
E	472	3949	4421
H	334	421	755
M	266	752	1018
SUM	1072	5122	6194

```
data(api)
set.seed(-57844)
#selection the SRCS
apiclus <- apipop[apipop$dnum%in%sample(unique(apipop$dnum),10),]
apiclus$fpc <- length(unique(apipop$dnum))

dclus1<- svydesign(id=~dnum, data=apiclus, fpc=~fpc)
#initial weight
w1      <- weights(dclus1)
#convergence is declared if the maximum change in a
#table entry is less than 'eps' ...
eps     <- 1
#... otherwise the process stops after 'maxit' iterations
maxit   <- 100

tau_type    <- table(apipop$type)
tau_sch.wide <- table(apipop$sch.wide)

#Raking (i.e. iterative post-stratification) for two variables
tab_x <- tab_y <- list()
```

```

for (i in 1:maxit) {
  ## Post-stratification to the first variable
  w1 <- split(w1, apiclus$type)
  adj1 <- tau_type/sapply(w1, sum)
  # new weight
  w1. <- w1 <- mapply(function(x, y) x * y, w1, adj1)
  # return to original order
  w1 <- unlist(w1.)
  names(w1) <- unlist(sapply(w1., names))
  w1 <- w1[as.character(sort(as.numeric(names(w1))))]
  tab_x[[i]] <- tapply(w1, list(apiclus$type, apiclus$sch.wide), sum)

  ## Post-stratification to the second variable
  w2 <- split(w1, apiclus$sch.wide)
  adj2 <- tau_sch.wide/sapply(w2, sum)
  # new weight
  w2. <- w2 <- mapply(function(x, y) x * y, w2, adj2)
  # return to original order
  w2 <- unlist(w2.)
  names(w2) <- unlist(sapply(w2., names))
  w2 <- w2[as.character(sort(as.numeric(names(w2))))]
  tab_y[[i]] <- tapply(w2, list(apiclus$type, apiclus$sch.wide), sum)

  if (i > 1) {
    tab.diff <- abs(tab_y[[i - 1]] - tab_y[[i]])

    if (max(tab.diff) < eps)
      break
  }
  w1 <- w2
}

```

TABLE: Estimated Totals  $\hat{\tau}_{x_q \pi} = \sum_{k \in \Delta} x_{kq} d_k$  from a SRCS of Districts (dname) with  $n_l = 10$

	No	Yes	SUM
E	984.1	4087.8	5071.9
H	378.5	302.8	681.3
M	378.5	832.7	1211.2
SUM	1741.1	5223.3	6964.4

TABLE: Estimated Totals after Adjustment to 'stypc' in the 1 Iteration

	No	Yes	SUM
E	857.8	3563.2	4421.0
H	419.4	335.6	755.0
M	318.1	699.9	1018.0
SUM	1595.4	4598.6	6194.0

TABLE: Estimated Totals after Adjustment to 'sch.wide' in the 1 Iteration

	No	Yes	SUM
E	576.4	3968.7	4545.1
H	281.8	373.7	655.6
M	213.8	779.5	993.3
SUM	1072.0	5122.0	6194.0



TABLE: Estimated Totals after Adjustment to 'stypc' in the 2 Iteration

	No	Yes	SUM
E	560.7	3860.3	4421.0
H	324.6	430.4	755.0
M	219.1	798.9	1018.0
SUM	1104.3	5089.7	6194.0

TABLE: Estimated Totals after Adjustment to 'sch.wide' in the 2 Iteration

	No	Yes	SUM
E	544.2	3884.9	4429.1
H	315.1	433.2	748.2
M	212.7	804.0	1016.7
SUM	1072.0	5122.0	6194.0

TABLE: Estimated Totals after Adjustment to 'stypc' in the 3 Iteration

	No	Yes	SUM
E	543.3	3877.7	4421.0
H	317.9	437.1	755.0
M	212.9	805.1	1018.0
SUM	1074.1	5119.9	6194.0

TABLE: Estimated Totals after Adjustment to 'sch.wide' in the 3 Iteration

	No	Yes	SUM
E	542.2	3879.4	4421.5
H	317.3	437.3	754.6
M	212.5	805.4	1017.9
SUM	1072.0	5122.0	6194.0

TABLE: Estimated Totals after Adjustment to 'stype' in the 4 Iteration

	No	Yes	SUM
E	542.1	3878.9	4421.0
H	317.5	437.5	755.0
M	212.5	805.5	1018.0
SUM	1072.1	5121.9	6194.0

TABLE: Estimated Totals after Adjustment to 'sch.wide' in the 4 Iteration

	No	Yes	SUM
E	542.0	3879.0	4421.0
H	317.4	437.5	755.0
M	212.5	805.5	1018.0
SUM	1072.0	5122.0	6194.0

```
dclus1r <- rake( dclus1, list(~stype, ~sch.wide)
                ,list( table(stype=apipop$stype)
                      ,table(sch.wide=apipop$sch.wide)
                ))
```

```
svytable(~stype+sch.wide, dclus1r , round=TRUE)
```

```
##      sch.wide
## stype   No  Yes
##      E  542 3879
##      H  317  438
##      M  213  805
```

```
(w1/weights(dclus1r))[1:10]
```

```
##      863      1138      1139      1140      1141      1142      1143
## 0.9999724 1.0001319 0.9999724 0.9999724 0.9999724 0.9999724 0.9999724
##      1144      1145      1146
## 0.9999724 0.9999724 0.9999724
```

```
summary(w1/weights(dclus1r))
```

```
##      Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##      1         1         1         1         1         1
```

For the linear generalized regression estimator (GREG) the measure of distance  $G_k$  is

$$G_k(w, \pi, c) = G(w_k, d_k, c_k) = \frac{(w_k - d_k)^2}{2d_k c_k},$$

and we have

$$\hat{\tau}_{\text{GREG}} = \hat{\tau}_{\pi} + (\tau_x - \hat{\tau}_{x\pi})^{\top} \hat{\beta},$$

where

$$\hat{\beta} = \left( \sum_{k \in \delta} d_k c_k \mathbf{x}_k (\mathbf{x}_k)^{\top} \right)^{-1} \sum_{k \in \delta} d_k c_k \mathbf{x}_k y_k,$$

and  $\hat{\tau}_{x\pi} = (\hat{\tau}_{x_1\pi}, \dots, \hat{\tau}_{x_Q\pi})^{\top}$ .

The adjustment to the design weight  $g_k$  can be written as:

$$g_k = 1 + \left( \left( \sum_{k \in \mathcal{U}} \mathbf{x}_k - \sum_{k \in \delta} d_k \mathbf{x}_k \right)^{\top} \left( \sum_{k \in \delta} d_k c_k \mathbf{x}_k (\mathbf{x}_k)^{\top} \right)^{-1} \right)^{\top} c_k \mathbf{x}_k$$



# GRAPHICAL PRESENTATION OF $\pi$ - AND GREG ESTIMATOR

We want to estimate total expenditures of hospitals. To improve a possible estimate we use data from survey in 1998 to explore if there are any useful predictors for our variable of interest.

```
library(PracTools) #load the package
data(smho.N874)    #load the data set
head(smho.N874)
```

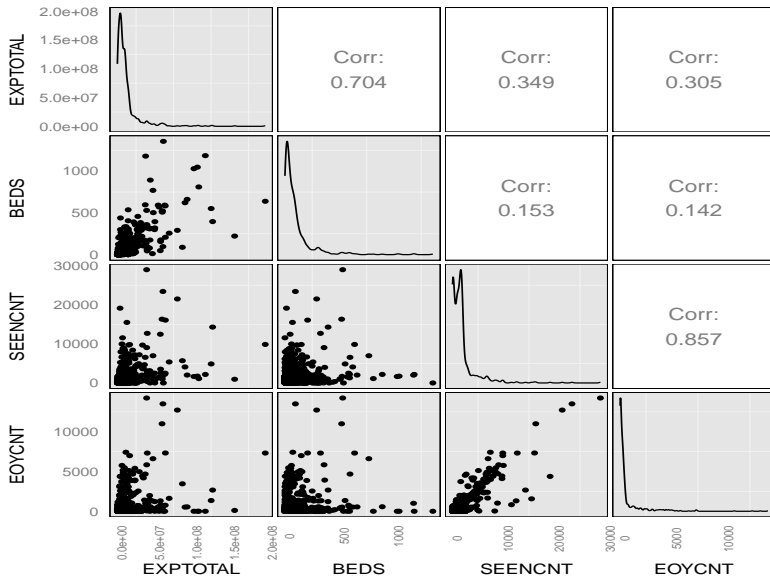
##	EXPTOTAL	BEDS	SEENCNT	EOYCNT	FINDIRCT	hosp.type
## 1	9066430	81	1791	184	2	1
## 2	9853392	80	1870	244	2	1
## 3	3906074	26	1273	0	2	1
## 4	9853392	90	1781	154	2	1
## 5	9853392	71	1839	206	2	1
## 6	9853392	81	1823	196	2	1

```
##?smho.N874           #for a description of the variables
```

*#only hospitals other than 'type 4' are considered*

```
smho <- smho.N874[smho.N874$hosp.type != 4, ]
```

# GENERALIZED REGRESSION ESTIMATOR



Fitting a linear model for EXPTOTAL with common slopes for SEENCNT and EOYCNT but a different slope for BEDS in each hospital type.

TABLE: Model Summary

	Estimate	Std. Error	t value	Pr(> t )
(Intercept)	1318589.11	912432.21	1.45	0.15
SEENCNT	1033.94	310.63	3.33	0.00
EOYCNT	2036.15	603.58	3.37	0.00
FINDIRECT2	78026.06	965237.62	0.08	0.94
hosp.type1:BEDS	98139.28	3318.84	29.57	0.00
hosp.type2:BEDS	39489.35	5644.51	7.00	0.00
hosp.type3:BEDS	77578.37	15082.20	5.14	0.00
hosp.type5:BEDS	36855.78	8650.48	4.26	0.00



We select a sample of hospitals with probability proportional to the square root of BEDS using a systematic sample.

```
#####  
## Select a pps to sqrt(BEDS) sample  
#####  
library(sampling)      #load the 'sample' package  
                        #for the 'UPsystematic' function  
smho. <-               # before sampling order the data set by hospital type  
  smho.[order(smho.$hosp.type),]  
  
x <- smho.[,"BEDS"]  
x[x <= 5] <- 5          # recode small hospitals to have a minimum size  
x <- sqrt(x)  
  
n <- 80                  #sample size  
IP  <- n*x/sum(x)  
  
set.seed(428274453)  
sam <- UPsystematic(IP)  
  
sam.dat <- smho.[sam==1, ]  
sam.dat$d <- 1/IP[sam==1] #the design weight
```

Now we use the survey package to calibrate the weights.

```
library(survey) #load the 'survey' package
#1. build a 'design' object
sam.dsgn <-
  svydesign(ids = ~1,           # no clusters
            strata = NULL,      # no strata
            data = sam.dat,     # the sample data
            weights = ~d)       # the design weight
  #the model we use for the GREG
lmod2 <- lm(EXPTOTAL ~ SEENCNT + EOYCNT + hosp.type:BEDS, data=samho.)
#2. compute pop totals of auxiliaries
pop.tots <- colSums(model.matrix(lmod2)) #Inefficient but convenient!

#3. use 'calibrate' to compute the new weights
sam.cal <-
  calibrate(design = sam.dsgn,
            formula = ~ SEENCNT + EOYCNT + hosp.type:BEDS,
            population = pop.tots,
            calfun='linear' )
```

Setting argument `calfun='linear'` in 'calibrate' results in the GREG weights, other calibration function are possible, already built-in are 'raking' and 'logit'.

Now we check if the calibration constraints are satisfied:

*#BEDS by hospital type*

```
svyby(~BEDS, by=~hosp.type, design=sam.cal, FUN=svytotal)
```

```
##      hosp.type  BEDS              se
## 1           1 37978 3.951866e-12
## 2           2 13066 1.421532e-12
## 3           3  9573 5.260079e-13
## 5           5 10077 5.811345e-13
```

*#SEENCNT and EOYCNT*

```
svytotal(~SEENCNT+EOYCNT, sam.cal)
```

```
##              total SE
## SEENCNT 1349241  0
## EOYCNT   505345  0
```

pop.tots

```
##      (Intercept)          SEENCNT          EOYCNT hosp.type1:BEDS
##              725          1349241          505345          37978
## hosp.type2:BEDS hosp.type3:BEDS hosp.type5:BEDS
##              13066              9573              10077
```



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However the effects might be catastrophic of domain estimation, in case of estimation domains that were not considered in the calibration.

In general it is advisable to only use calibrated weights to infer to the whole population or sub-populations that are found in the marginal totals used for the calibration!

Design weights can always be used to do unbiased domain estimation, although the precision of these estimates can be very poor.

# THE VARIANCE OF THE GENERALIZED REGRESSION ESTIMATOR

Calibrated weights are **not** independent of the selected sample, i.e. they are random variables. Thus the variance of the GREG estimator cannot be estimated as straightforwardly as for the  $\pi$ -estimator. We can write its approximate variance as

$$AV(\hat{t}_{\text{GREG}}) = \sum_{k \in \mathcal{U}} \sum_{l \in \mathcal{U}} (\pi_{kl} - \pi_k \pi_l) \frac{E_k}{\pi_k} \frac{E_l}{\pi_l},$$

where  $E_k = y_k - y_k^0$ ,  $y_k^0 = \mathbf{x}_k^\top \boldsymbol{\beta}$  and

$$\boldsymbol{\beta} = \left( \sum_{k \in \mathcal{U}} c_k \mathbf{x}_k (\mathbf{x}_k)^\top \right)^{-1} \sum_{k \in \mathcal{U}} c_k \mathbf{x}_k y_k.$$

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Calibrated weights are **not** independent of the selected sample, i.e. they are random variables. Thus the variance of the GREG estimator cannot be estimated as straightforwardly as for the  $\pi$ -estimator. A variance estimator for  $\hat{\tau}_{\text{GREG}}$  is given by

$$\hat{V}(\hat{\tau}_{\text{GREG}}) = \sum_{k \in \mathcal{A}} \sum_{l \in \mathcal{A}} \frac{(\pi_{kl} - \pi_k \pi_l)}{\pi_{kl}} g_k \frac{e_k}{\pi_k} g_l \frac{e_l}{\pi_l},$$

where  $E_k = y_k - \hat{y}_k$  and  $\hat{y}_k = \mathbf{x}_k^\top \hat{\boldsymbol{\beta}}$ .



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