Sampling and Estimation
Part 3: Calibration Weights
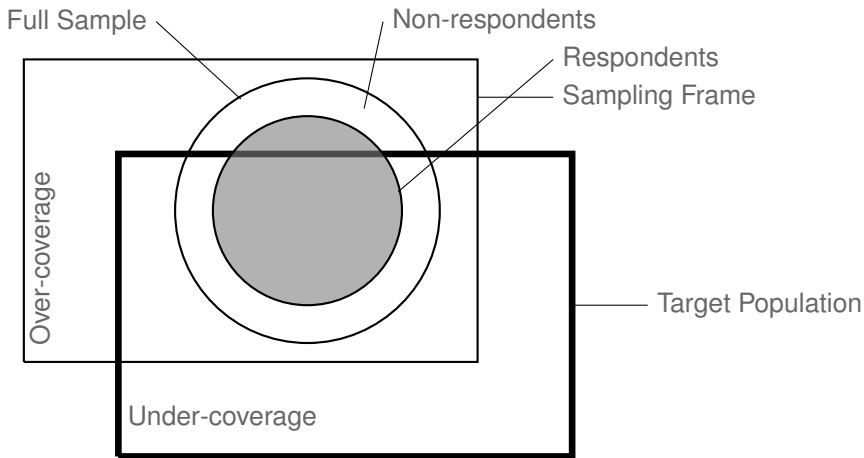
Stefan Zins[1] and Matthias Sand[2]

April 24, 2017

[1]Stefan.Zins@gesis.org
[2]Matthias.Sand@gesis.org

Survey Errors

Procedures based on the **available cases** only, i.e., only those cases that are completely recorded for the variables of interest.

Procedures based on the **available cases** only, i.e., only those cases that are completely recorded for the variables of interest.

Weighting procedures that adjust design weights to compensate the bias that a MAR non-response might inflict on HT-type estimators.

Procedures based on the **available cases** only, i.e., only those cases that are completely recorded for the variables of interest.

Weighting procedures that adjust design weights to compensate the bias that a MAR non-response might inflict on HT-type estimators.

Single imputation and correction of the variance estimates to account for imputation uncertainty.

Procedures based on the **available cases** only, i.e., only those cases that are completely recorded for the variables of interest.

Weighting procedures that adjust design weights to compensate the bias that a MAR non-response might inflict on HT-type estimators.

Single imputation and correction of the variance estimates to account for imputation uncertainty.

Multiple imputation (MI).

Procedures based on the **available cases** only, i.e., only those cases that are completely recorded for the variables of interest.

Weighting procedures that adjust design weights to compensate the bias that a MAR non-response might inflict on HT-type estimators.

Single imputation and correction of the variance estimates to account for imputation uncertainty.

Multiple imputation (MI).

$\longrightarrow$ Methods for handling coverage errors are not so widely spread, simply because there is often no reliable auxiliary information on just the target population. However if there is, it can receive a treatment similar to that of weighting by non-response.

Missing data is the norm, rather than the exception!
Missingness may be either

Missing data is the norm, rather than the exception!
Missingness may be either

MCAR missing completely at random,
every unit has same response propensity (RP);
respondents are a random sample of the initial sample

Missing Data

Missing data is the norm, rather than the exception!
Missingness may be either

MCAR missing completely at random,

every unit has same response propensity (RP);
respondents are a random sample of the initial sample

MAR missing at random, or

RP depends on auxiliary variables $\mathcal{X}$;
can be modeled, if $\mathcal{X}$ is known for both respondents &
non-respondents

MISSING DATA

Missing data is the norm, rather than the exception!
Missingness may be either

MCAR missing completely at random,

every unit has same response propensity (RP);
respondents are a random sample of the initial sample

MAR missing at random, or

RP depends on auxiliary variables $\mathcal{X}$;
can be modeled, if $\mathcal{X}$ is known for both respondents &
non-respondents

MNAR missing not at random

RP depends on variables of interest $\mathcal{Y}$;
cannot be modeled, because $\mathcal{Y}$ not known for
non-respondents

$\longrightarrow$ In multivariate analysis 30% to 40% of the data are often lost
with case deletion, assuming MCAR!

Weighting Methods

Calibration approach The design weighs are calibrated to the totals of some auxiliary variables $\mathcal{X}$.

Sample estimates using the calibrated weights will exactly replicated those totals.

If the used auxiliary variables help to explain the response process the calibrated weight can reduce the non-response error.

Two-phase approach The response process is modeled to obtain the response propensities ψ_k for all $k \in \mathfrak{s}$. The new weight of element k is $\frac{d_k}{\psi_k}$. (Two phases: 1. Sampling $\rightarrow$ 2. Responding).

In addition the new weights $\frac{d_k}{\psi_k}$ might then also be calibrated.

Often used models are:

> Response homogeneity classes, every element in a class has the same probability to respond.
> Generalized liner models (*probit*, *logit*, *log-log*), treating response as a latent variable.

The calibration approach is more direct, as the design weights are directly calibrated without considering the response propensities. Also, if the same models are used for both the modeling of the response propensities and the calibration, the two approaches can be equivalent.

Generic estimators for a total and a mean

$$\hat{\tau}_w = \sum_{k \in \delta} w_k y_k \quad \text{and} \quad \overline{y}_w = \frac{\sum_{k \in \delta} w_k y_k}{\sum_{k \in \delta} w_k} \, ,$$

where w_k is the survey weight of element k, with

Generic estimators for a total and a mean

$$\hat{\tau}_w = \sum_{k \in \delta} w_k y_k \quad \text{and} \quad \overline{y}_w = \frac{\sum_{k \in \delta} w_k y_k}{\sum_{k \in \delta} w_k} \,,$$

where w_k is the survey weight of element k, with

$$w_k = \begin{cases} d_k g_k & \text{for } k \in \delta \\ 0 & \text{else} \end{cases} \,.$$

$$w_k = \begin{cases} d_k g_k & \text{for } k \in \textit{s} \\ 0 & \text{else} \end{cases}.$$

Sometimes called base weights or design weights, the inverse of inclusion probabilities $d_k = \pi^{-1}$ is usually the first step in weighting. If we have $g_k = 1$, the $\hat{\tau}_w$ would be the HT estimator or π-estimator. The factor g_k adjusts the design weights to reduce

the sampling error (i.e. variance),

the non-response error, and

the coverage error

of estimator $\hat{\tau}_w$ or $\overline{y}_w$. Thereby the w_k's should not deviate to much from the d_k's as these weights ensure an unbiased estimation.

The general idea is to exploit the relationship between auxiliary variables and the variable of interest to improve the efficiency of estimators.

The following problem is solved with weight calibration:
For a given design $p(.)$ and a sample δ weights w_k for all $k \in \delta$ have to be found that minimize

$$\sum_{k \in \delta} G_k(w_k, d_k, c_k) \,,$$

subject to constraints

$$\sum_{k \in \delta} w_k \mathbf{x}_k = \sum_{k \in \mathcal{U}} \mathbf{x}_k = \boldsymbol{\tau}_x$$

where $\mathbf{x}_k = (x_{k1}, x_{k2}, \ldots, x_{kQ})^\top$ is a vector of q auxiliary variables for element k. G_k is a measure of distance between w_k and d_k and c_k is a factor that can be freely chosen for additional flexibility.

To calculate the weights, the $\mathbf{x}_k$'s are only needed for the elements in the net sample (i.e. typically only for the respondents), but τ_x, their population totals, need to be known.

The auxiliary variables can be metric (e.g. income or age) or categorical (e.g. gender or age groups).

Depending on the choice of G_k, different calibration estimators can be obtained, some of the most common are:

Post-stratification Estimator
Raking Estimator
Generalized Regression Estimator

Note that the w_k's typically depend on the sample δ, in contrast to the d_k, which are given by the sampling design.

Post-stratification is typically used if only categorical auxiliary variables are available. It is implemented by forming weighting cells by crossing *all* categories of the auxiliary variables. These weighting cells are the post-strata $\mathcal{U}_q$ with $q = 1, \ldots, Q$. The weight are then adjusted to replicate the counts in these cells. For $k \in \mathcal{U}_q$ we have

$$g_k = \frac{\tau_{x_q}}{\hat{\tau}_{x_q}} \,,$$

where $\tau_{x_q} = \sum_{k \in \mathcal{U}} x_{kq}$ and

$$x_{kq} = \begin{cases} 1 & \text{if } k \in \mathcal{U}_q \\ 0 & \text{else} \end{cases} .$$

$\hat{\tau}_{x_q \, \pi} = \sum_{k \in \delta} d_k x_{kq}$ its estimator for τ_{x_q} based on the design weights. The auxiliary variables are the post-stratum indicators, i.e. $\mathbf{x}_k = (x_{k1}, x_{k2}, \ldots, x_{kQ})^\top$. An adjustment to the totals of a metric variable within the post-strata would also be possible.

TABLE: Population Counts τ_{x_q} for Hair and Eye Colour

	Brown	Blue	Hazel	Green
Black	68	20	15	5
Brown	119	84	54	29
Red	26	17	14	14
Blond	7	94	10	16

TABLE: Population Counts τ_{x_q} for Hair and Eye Colour

	Brown	Blue	Hazel	Green
Black	68	20	15	5
Brown	119	84	54	29
Red	26	17	14	14
Blond	7	94	10	16

TABLE: Sample counts $\sum_{k \in \delta} x_{kq}$ in a SRS with $n = 150$

	Brown	Blue	Hazel	Green
Black	14	7	2	2
Brown	36	22	17	5
Red	7	3	1	4
Blond	1	23	1	5

TABLE: Population Counts τ_{x_q} for Hair and Eye Colour

	Brown	Blue	Hazel	Green
Black	68	20	15	5
Brown	119	84	54	29
Red	26	17	14	14
Blond	7	94	10	16

TABLE: Estimated totals $\hat{\tau}_{x_q \, \pi} = \sum_{k \in \Delta} x_{kq} d_k$

	Brown	Blue	Hazel	Green
Black	55.2533	27.6267	7.8933	7.8933
Brown	142.0800	86.8267	67.0933	19.7333
Red	27.6267	11.8400	3.9467	15.7867
Blond	3.9467	90.7733	3.9467	19.7333

TABLE: Population Counts τ_{x_q} for Hair and Eye Colour

	Brown	Blue	Hazel	Green
Black	68	20	15	5
Brown	119	84	54	29
Red	26	17	14	14
Blond	7	94	10	16

TABLE: Post-stratification $g_k = \dfrac{\tau_{x_q}}{\hat{\tau}_{x_q}}$

	Brown	Blue	Hazel	Green
Black	1.2307	0.7239	1.9003	0.6334
Brown	0.8376	0.9674	0.8048	1.4696
Red	0.9411	1.4358	3.5473	0.8868
Blond	1.7736	1.0355	2.5338	0.8108

Beware, there must be at least one element in the sample from each post-stratum, otherwise we divide by zero!

In raking, only the marginal totals are needed, *not* the totals for all the cross-categories. Raking can be implemented as iterative post-stratification to adjust the design weights to the margins of the different auxiliary variables.

The design weights of a SRSC cluster sample of school districts are raked to variables school type (`stype`) and the accomplishment of the growth target (`sch.wide`).

```
##                      dname           name stype sch.wide
## 1 Alameda City Unified     Alameda High     H      Yes
## 2 Alameda City Unified     Encinal High     H      Yes
## 3 Alameda City Unified  Chipman Middle     M      Yes
## 4 Alameda City Unified Lum (Donald D.)     E      Yes
## 5 Alameda City Unified Edison Elementa     E      Yes
## 6 Alameda City Unified Otis (Frank) El     E      Yes
```

TABLE: Population Counts τ_{x_q} for School Type (`stype`) and School Target (`sch.wide`)

	No	Yes	SUM
E	472	3949	4421
H	334	421	755
M	266	752	1018
SUM	1072	5122	6194

```
set.seed(-57844)
#selection the SRCS
apiclus <- apipop[apipop$dnum%in%sample(unique(apipop$dnum),10),]
apiclus$fpc <- length(unique(apipop$dnum))

dclus1<- svydesign(id=~dnum, data=apiclus, fpc=~fpc)
#initial weight
w1     <- weights(dclus1)
#convergence is declared if the maximum change in a
#table entry is less than 'eps' ...
eps    <- 1
#... otherwise the process stops after 'maxit' iterations
maxit <- 100

tau_stype     <- table(apipop$stype)
tau_sch.wide <- table(apipop$sch.wide)

#Raking (i.e. iterative post-stratification) for two variables
tab_x <- tab_y <- list()
```

```r
for (i in 1:maxit) {
    ## Post-stratification to the first variable
    w1 <- split(w1, apiclus$stype)
    adj1 <- tau_stype/sapply(w1, sum)
    # new weight
    w1. <- w1 <- mapply(function(x, y) x * y, w1, adj1)
    # return to original order
    w1 <- unlist(w1.)
    names(w1) <- unlist(sapply(w1., names))
    w1 <- w1[as.character(sort(as.numeric(names(w1))))]
    tab_x[[i]] <- tapply(w1, list(apiclus$stype, apiclus$sch.wide), sum)

    ## Post-stratification to the second variable
    w2 <- split(w1, apiclus$sch.wide)
    adj2 <- tau_sch.wide/sapply(w2, sum)
    # new weight
    w2. <- w2 <- mapply(function(x, y) x * y, w2, adj2)
    # return to original order
    w2 <- unlist(w2.)
    names(w2) <- unlist(sapply(w2., names))
    w2 <- w2[as.character(sort(as.numeric(names(w2))))]
    tab_y[[i]] <- tapply(w2, list(apiclus$stype, apiclus$sch.wide), sum)

    if (i > 1) {
        tab.diff <- abs(tab_y[[i - 1]] - tab_y[[i]])

        if (max(tab.diff) < eps)
            break
    }
    w1 <- w2

}
```

TABLE: Estimated Totals $\hat{\tau}_{x_q \pi} = \sum_{k \in s} x_{kq} d_k$ from a SRCS of Districts (`dname`) with $n_I = 10$

	No	Yes	SUM
E	984.1	4087.8	5071.9
H	378.5	302.8	681.3
M	378.5	832.7	1211.2
SUM	1741.1	5223.3	6964.4

TABLE: Estimated Totals after Adjustment to 'stype' in the 1 Interation

	No	Yes	SUM
E	857.8	3563.2	4421.0
H	419.4	335.6	755.0
M	318.1	699.9	1018.0
SUM	1595.4	4598.6	6194.0

TABLE: Estimated Totals after Adjustment to 'sch.wide' in the 1 Interation

	No	Yes	SUM
E	576.4	3968.7	4545.1
H	281.8	373.7	655.6
M	213.8	779.5	993.3
SUM	1072.0	5122.0	6194.0

TABLE: Estimated Totals after Adjustment to 'stype' in the 2 Interation

	No	Yes	SUM
E	560.7	3860.3	4421.0
H	324.6	430.4	755.0
M	219.1	798.9	1018.0
SUM	1104.3	5089.7	6194.0

TABLE: Estimated Totals after Adjustment to 'sch.wide' in the 2 Interation

	No	Yes	SUM
E	544.2	3884.9	4429.1
H	315.1	433.2	748.2
M	212.7	804.0	1016.7
SUM	1072.0	5122.0	6194.0

TABLE: Estimated Totals after Adjustment to 'stype' in the 3 Interation

	No	Yes	SUM
E	543.3	3877.7	4421.0
H	317.9	437.1	755.0
M	212.9	805.1	1018.0
SUM	1074.1	5119.9	6194.0

TABLE: Estimated Totals after Adjustment to 'sch.wide' in the 3 Interation

	No	Yes	SUM
E	542.2	3879.4	4421.5
H	317.3	437.3	754.6
M	212.5	805.4	1017.9
SUM	1072.0	5122.0	6194.0

TABLE: Estimated Totals after Adjustment to 'stype' in the 4 Interation

	No	Yes	SUM
E	542.1	3878.9	4421.0
H	317.5	437.5	755.0
M	212.5	805.5	1018.0
SUM	1072.1	5121.9	6194.0

TABLE: Estimated Totals after Adjustment to 'sch.wide' in the 4 Interation

	No	Yes	SUM
E	542.0	3879.0	4421.0
H	317.4	437.5	755.0
M	212.5	805.5	1018.0
SUM	1072.0	5122.0	6194.0

```
dclus1r <- rake( dclus1, list(~stype, ~sch.wide)
                ,list( table(stype=apipop$stype)
                      ,table(sch.wide=apipop$sch.wide)
                ))
svytable(~stype+sch.wide, dclus1r , round=TRUE)

##      sch.wide
## stype   No  Yes
##     E  542 3879
##     H  317  438
##     M  213  805

(w1/weights(dclus1r))[1:10]

##       863      1138      1139      1140      1141      1142      1143
## 0.9999724 1.0001319 0.9999724 0.9999724 0.9999724 0.9999724 0.9999724
##      1144      1145      1146
## 0.9999724 0.9999724 0.9999724

summary(w1/weights(dclus1r))

##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##       1       1       1       1       1       1
```

For the linear generalized regression estimator (GREG), the measure of distance G_k is

$$G_k(w, \pi, c) = G(w_k, d_k, c_k) = \frac{(w_k - d_k)^2}{2 d_k c_k} \,,$$

and we have

$$\hat{\tau}_{\mathsf{GREG}} = \hat{\tau}_\pi + (\boldsymbol{\tau}_x - \hat{\boldsymbol{\tau}}_{x\,\pi})^\top \widehat{\boldsymbol{\beta}},$$

where

$$\widehat{\boldsymbol{\beta}} = \left(\sum_{k \in \delta} d_k c_k \mathbf{x}_k (\mathbf{x}_k)^\top \right)^{-1} \sum_{k \in \delta} d_k c_k \mathbf{x}_k y_k \,,$$

and $\hat{\boldsymbol{\tau}}_{x\,\pi} = (\hat{\tau}_{x_1\,\pi}, \dots, \hat{\tau}_{x_Q\,\pi})^\top$.
The adjustment to the design weight g_k can be written as:

$$g_k = 1 + \left(\Big(\sum_{k \in \mathcal{U}} \mathbf{x}_k - \sum_{k \in \delta} d_k \mathbf{x}_k \Big)^\top \Big(\sum_{k \in \delta} d_k c_k \mathbf{x}_k (\mathbf{x}_k)^\top \Big)^{-1} \right) c_k \mathbf{x}_k$$

We want to estimate total expenditures of hospitals. To improve a possible estimate we use data from survey in 1998 to explore, if there are any useful predictors for our variable of interest.

```
library(PracTools)  #load the package
data(smho.N874)     #load the data set
head(smho.N874)

##   EXPTOTAL BEDS SEENCNT EOYCNT FINDIRCT hosp.type
## 1  9066430   81    1791    184        2         1
## 2  9853392   80    1870    244        2         1
## 3  3906074   26    1273      0        2         1
## 4  9853392   90    1781    154        2         1
## 5  9853392   71    1839    206        2         1
## 6  9853392   81    1823    196        2         1

#?smho.N874            #for a description of the variables

#only hospitals other than 'type 4' are considered
smho <- smho.N874[smho.N874$hosp.type != 4, ]
```

Generalized Regression Estimator
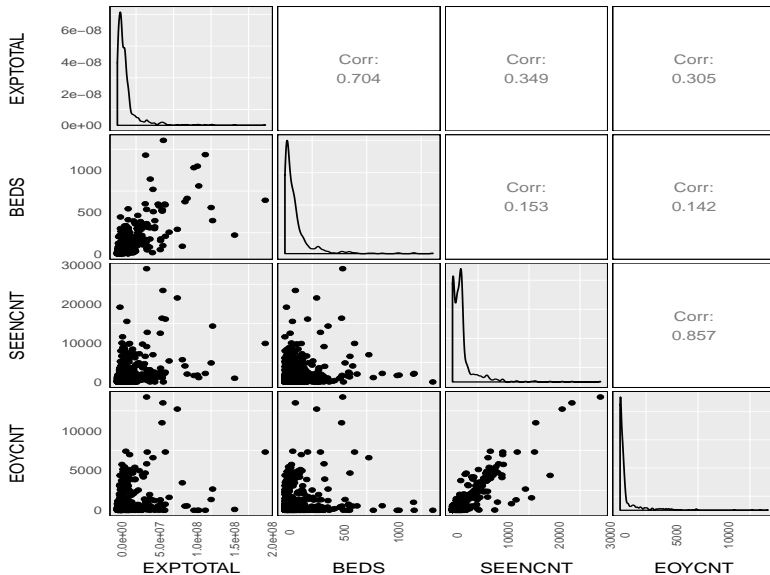
Fitting a linear model for `EXPTOTAL` with common slopes for `SEENCNT` and `EOYCNT` but a different slope for `BEDS` in each hospital type.

TABLE: Model Summary

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	1318589.11	912432.21	1.45	0.15
SEENCNT	1033.94	310.63	3.33	0.00
EOYCNT	2036.15	603.58	3.37	0.00
FINDIRCT2	78026.06	965237.62	0.08	0.94
hosp.type1:BEDS	98139.28	3318.84	29.57	0.00
hosp.type2:BEDS	39489.35	5644.51	7.00	0.00
hosp.type3:BEDS	77578.37	15082.20	5.14	0.00
hosp.type5:BEDS	36855.78	8650.48	4.26	0.00

We select a sample of hospitals with probability proportional to the square root of BEDS using a systematic sample.

```r
#######################################
### Select a pps to sqrt(BEDS) sample
#######################################
library(sampling)    #load the 'sample' package
                     #for the 'UPsystematic' function
smho. <-   # before sampling order the data set by hospital type
  smho.[order(smho.$hosp.type),]

x <- smho.[,"BEDS"]
x[x <= 5] <- 5       # recode small hospitals to have a minimum size
x <- sqrt(x)

n <- 80              #sample size
IP   <- n*x/sum(x)

set.seed(428274453)
sam <- UPsystematic(IP)

sam.dat <- smho.[sam==1, ]
sam.dat$IP <- IP[sam==1]     #the design weight
sam.dat$d <- 1/IP[sam==1]    #the design weight
```

Now we use the `survey` package to calibrate the weights.

```r
library(survey) #load the 'survey' package
#1. build a 'design' object
sam.dsgn <-
  svydesign( ids = ~1          # no clusters
            ,data = sam.dat    # the sample data
            ,fpc = ~IP         # incl. prob
            ,pps= "brewer")    # variance approx. method
 #the model we use for the GREG
lmod2 <- lm(EXPTOTAL ~ SEENCNT + EOYCNT + hosp.type:BEDS, data=smho.)
#2. compute pop totals of auxiliaries
pop.tots <- colSums(model.matrix(lmod2)) #Inefficient but convenient!

#3. use 'calibrate' to compute the new weights
sam.cal <-
  calibrate(design = sam.dsgn,
            formula = ~ SEENCNT + EOYCNT + hosp.type:BEDS,
            population = pop.tots,
            calfun='linear' )
```

Setting argument `calfun='linear'` in 'calibrate' results in the GREG weights, other calibration function are possible, already built-in are 'raking' and 'logit'.

Now we check if the calibration constrains are satisfied:

```
#BEDS by hospital type
svyby(~BEDS, by=~hosp.type, design=sam.cal, FUN=svytotal)

##   hosp.type  BEDS           se
## 1         1 37978 4.100694e-12
## 2         2 13066 1.122603e-12
## 3         3  9573 3.028628e-13
## 5         5 10077 7.135892e-13

#SEENCNT and EOYCNT
svytotal(~SEENCNT+EOYCNT, sam.cal)

##         total SE
## SEENCNT 1349241  0
## EOYCNT   505345  0

pop.tots

##     (Intercept)          SEENCNT          EOYCNT hosp.type1:BEDS
##             725          1349241          505345           37978
## hosp.type2:BEDS hosp.type3:BEDS hosp.type5:BEDS
##           13066            9573           10077
```

Nothing prevents the GREG weights from becoming negative, which is theoretically not a problem, as long as we infer to the population (or sub-populations) to which we calibrated.

Nothing prevents the GREG weights from becoming negative, which is theoretically not a problem, as long as we infer to the population (or sub-populations) to which we calibrated.

However the effects might be catastrophic of domain estimation, in case of estimation domains that where not considered in the calibration.

Nothing prevents the GREG weights from becoming negative, which is theoretically not a problem, as long as we infer to the population (or sub-populations) to which we calibrated.

However the effects might be catastrophic of domain estimation, in case of estimation domains that where not considered in the calibration.

In general, it is advisable to only use calibrated weights to infer to the whole population or sub-populations that are found in the marginal totals used for the calibration!

Nothing prevents the GREG weights from becoming negative, which is theoretically not a problem, as long as we infer to the population (or sub-populations) to which we calibrated.

However the effects might be catastrophic of domain estimation, in case of estimation domains that where not considered in the calibration.

In general, it is advisable to only use calibrated weights to infer to the whole population or sub-populations that are found in the marginal totals used for the calibration!

Design weights can always be used to do unbiased domain estimation, although the precision of these estimates can be very poor.

Variance Estimation for Complex Statistics

In practice design based variance estimation will *not* be an option for most surveys.

In practice design based variance estimation will *not* be an option for most surveys. Why?

In practice design based variance estimation will *not* be an option for most surveys. Why? We do not know the π_{kl} and in some cases not even the π_k and even if we would, the response process and frame imperfections remain still unknown.

In practice design based variance estimation will *not* be an option for most surveys. Why? We do not know the π_{kl} and in some cases not even the π_k and even if we would, the response process and frame imperfections remain still unknown. Thus we have to resort to second best solutions:

  Model based estimation (as shown for the estimation of the *deff*)

  Estimation of approximated variances

What exactly has variance estimation to fulfill?

    Deliver adequate quality measures

    Consider practical issues

What exactly has variance estimation to fulfill?

    Deliver adequate quality measures

    Consider practical issues

But

    Can we apply general methods like for point estimation?

In variance estimation we face often following obstacles:

Non-linear Statistics

In variance estimation we face often following obstacles:

- Non-linear Statistics
- Complex Designs (incl. non-response)

In variance estimation we face often following obstacles:

Non-linear Statistics

Complex Designs (incl. non-response)

Two of the commonly used methods to overcome these are

In variance estimation we face often following obstacles:

    Non-linear Statistics

    Complex Designs (incl. non-response)

Two of the commonly used methods to overcome these are

Linearization ; An approximation to the variance of the estimator is sought that is far easier to estimate.

In variance estimation we face often following obstacles:

Non-linear Statistics

Complex Designs (incl. non-response)

Two of the commonly used methods to overcome these are

Linearization ; An approximation to the variance of the estimator is sought that is far easier to estimate.

Resampling Methods ; The distribution of the estimator is simulated in order to estimate its variance.

In variance estimation we face often following obstacles:

Non-linear Statistics

Complex Designs (incl. non-response)

Two of the commonly used methods to overcome these are

Linearization ; An approximation to the variance of the estimator is sought that is far easier to estimate.

Resampling Methods ; The distribution of the estimator is simulated in order to estimate its variance.

The line between model and design based approaches is not so clear in some cases. There are resampling methods and linearization techniques that assume infinite population, i.e. they rely on models. We try to stay with the finite population *model*.

The estimator we use are often non-linear, like $\overline{y}_w$. Which is a problem, in particular for variance estimation. Mind that:

$$\mathsf{E}\left(\frac{\hat{\tau}_{y_1\,w}}{\hat{\tau}_{y_2\,w}}\right) \approx \frac{\mathsf{E}\left(\hat{\tau}_{y_1\,w}\right)}{\mathsf{E}\left(\hat{\tau}_{y_2\,w}\right)}$$

$$\mathsf{V}\left(\frac{\hat{\tau}_{y_1\,w}}{\hat{\tau}_{y_2\,w}}\right) \neq \frac{\mathsf{V}\left(\hat{\tau}_{y_1\,w}\right)}{\mathsf{V}\left(\hat{\tau}_{y_2\,w}\right)}$$

In case our statistic of interest θ can be displayed as a function f of Q totals

$$\theta = f(\boldsymbol{\tau}) \ ,$$

with $\boldsymbol{\tau} = (\tau_{x_1}, \ldots, \tau_{x_q}, \ldots, \tau_{x_Q})^\top$ and $\tau_{x_1} = \sum_{k \in \mathcal{U}} x_{kq}$. We estimate θ by

$$\hat{\theta} = f(\hat{\boldsymbol{\tau}}) \ ,$$

with $\hat{\boldsymbol{\tau}} = (\hat{\tau}_{x_1}, \ldots, \hat{\tau}_{x_q}, \ldots, \hat{\tau}_{x_Q})^\top$.

If *f* is continuously differentiable up to second order between τ and $\hat{\tau}$ we can use the Taylor series of estimator $\hat{\theta}$ to obtain a linearized version of it

$$\hat{\theta} = \theta + \sum_{q=1}^{Q} \left[\frac{\partial f(t_1, \ldots, t_Q)}{\partial t_q} \right]_{\mathbf{t}=\tau} \left(\hat{\tau}_{x_q} - \tau_{x_q} \right) + R(\hat{\tau}, \tau) \, ,$$

where

$$R(\hat{\tau}, \tau) = \frac{1}{2} \sum_{q=1}^{Q} \sum_{p=1}^{Q} \left[\frac{\partial^2 f(t_1, \ldots, t_Q)}{\partial t_q \partial t_p} \right]_{\mathbf{t}=\ddot{\tau}} (\hat{\tau}_{x_q} - \tau_{x_q})(\hat{\tau}_{x_p} - \tau_{x_p})$$

and $\ddot{\tau}$ is between $\hat{\tau}$ and τ. In most application the remainder term *R* is ignored for large enough sample sizes.

Thus we can approximate the variance of $\hat{\theta}$ by

$$V\left(\hat{\theta}\right) \approx V\left(\sum_{q=1}^{Q}\left[\frac{\partial f(t_1, \ldots, t_Q)}{\partial t_q}\right]_{\mathbf{t}=\boldsymbol{\tau}} \hat{\tau}_{x_q}\right)$$

$$= \sum_{q=1}^{Q} a_q^2 V\left(\hat{\tau}_{x_q}\right) + 2\sum_{q=1}^{Q}\sum_{\substack{p=1 \\ p<q}}^{Q} a_q a_p COV\left(\hat{\tau}_{x_q}, \hat{\tau}_{x_p}\right) ,$$

with $a_q = [\frac{\partial f(t_1, \ldots, t_Q)}{\partial t_q}]_{\mathbf{t}=\boldsymbol{\tau}}$.

For $\hat{\boldsymbol{\tau}} = (\hat{\tau}_{x_1 w}, \ldots, \hat{\tau}_{x_q w}, \ldots, \hat{\tau}_{x_Q w})^\top$ Woodruff (1971) proposes the transformation of x_{kq}

$$z_k = \sum_{q=1}^{Q} a_q x_{kq}$$

and use the following expression as an approximate variance of $\hat{\theta}$

$$V\left(\hat{\theta}\right) \approx V\left(\sum_{k \in \Delta} w_k z_k\right) .$$

This approximation is far more convenient to estimate then to estimate all the different variances and covariance in the above formula separately. z_k is also sometimes called the linearized variable.

Suppose we want to estimate $\theta = \frac{\tau_y}{N}$ the population mean of variable $\mathcal{Y}$. To do this we use estimator $\overline{y}_w = \overline{y}_\pi$, i.e. we set $w_k = d_k$ for all $k \in \Delta$. Now we have

$$\hat{\theta} = f(\boldsymbol{\tau}) = f((\hat{\tau}_{y,\pi}, \hat{\tau}_{x,\pi})) = \frac{\hat{\tau}_{y,\pi}}{\hat{\tau}_{x,\pi}} \ ,$$

were $\hat{\tau}_{y,\pi} = \sum_{k \in \Delta} d_k y_k$ and $\hat{\tau}_{x,\pi} = \sum_{k \in \Delta} d_k$, because $x_k = 1$ for all $k \in \mathcal{U}$. Our estimator is a function of Q=2 totals, and we have

$$a_1 = \frac{1}{\tau_x} \qquad\qquad a_2 = -\frac{\tau_y}{\tau_x^2} = -\frac{\theta}{\tau_x}$$

$$z_k = a_1 y_k + a_2 x_k = \frac{1}{\tau_x}(y_k - \theta)$$

To estimate the approximate variance $V\left(\sum_{k\in s} d_k z_k\right)$ we need estimates for the z_k's, because they involve unknown statistics τ_x and θ.

$$\hat{z}_k = \frac{1}{\hat{\tau}_{x,\pi}}\left(y_k - \overline{y}_\pi\right)$$

Our variance estimator would be $\widehat{V}\left(\sum_{k\in s} d_k \hat{z}_k\right)$ for which $\widehat{V}\left(.\right)_1$ or $\widehat{V}\left(.\right)_2$ could be used or an estimator for an approximate design variance.

Note that the variances of the $\hat{z}_k$'s (and the covariances between them) is often thought to be negligible and is therefore usually not considered.

THE VARIANCE OF THE GENERALIZED REGRESSION ESTIMATOR

Calibrated weights are **not** independent of the selected sample, i.e. they are random variables. Thus, the variance of the GREG estimator cannot be estimated as straightforwardly as for the π-estimator. We can write its approximate variance as

$$\text{AV}\left(\hat{\tau}_{\text{GREG}}\right) = \sum_{k\in\mathcal{U}}\sum_{l\in\mathcal{U}}\left(\pi_{kl} - \pi_k\pi_l\right)\frac{E_k}{\pi_k}\frac{E_l}{\pi_l} \, ,$$

where $E_k = y_k - y_k^0$, $y_k^0 = \mathbf{x}_x^\top\boldsymbol{\beta}$ and

$$\boldsymbol{\beta} = \left(\sum_{k\in\mathcal{U}}c_k\mathbf{x}_k(\mathbf{x}_k)^\top\right)^{-1}\sum_{k\in\mathcal{U}}c_k\mathbf{x}_ky_k \, .$$

Calibrated weights are **not** independent of the selected sample, i.e. they are random variables. Thus, the variance of the GREG estimator cannot be estimated as straightforwardly as for the π-estimator. A variance estimator for $\hat{\tau}_{\text{GREG}}$ is given by

$$\hat{V}\left(\hat{\tau}_{\text{GREG}}\right) = \sum_{k \in \delta}\sum_{l \in \delta} \frac{(\pi_{kl} - \pi_k \pi_l)}{\pi_{kl}} g_k \frac{e_k}{\pi_k} g_l \frac{e_l}{\pi_l} \ ,$$

where $e_k = y_k - \hat{y}_k$ and $\hat{y}_k = \mathbf{x}_x^\top \hat{\beta}$.

THE VARIANCE OF THE GENERALIZED REGRESSION ESTIMATOR

Calibrated weights are **not** independent of the selected sample, i.e. they are random variables. Thus, the variance of the GREG estimator cannot be estimated as straightforwardly as for the π-estimator. Compare the estimates for the total expenditure with calibrated and design weights.

```
svytotal(~EXPTOTAL, sam.dsgn)

##              total       SE
## EXPTOTAL 9.55e+09 7.72e+08


svytotal(~EXPTOTAL, sam.cal)

##              total       SE
## EXPTOTAL 9.03e+09 5.92e+08
```

We find that $\hat{\tau}_{\text{GREG}}/\hat{\tau}_{\pi} = 0.9451$, but $\widehat{V}(\hat{\tau}_{\text{GREG}})/\widehat{V}(\hat{\tau}_{\pi}) = 0.588$.

📕 R.J.A. Little, D.B. Rubin.
Statistical Analysis with Missing Data.
Wiley Interscience, 1999.

📕 C.-E. Särndal, S. Lundström.
Estimation in Surveys with Nonresponse.
Wiley, 2005.

📕 R.S. Woodruff.
A Simple Method for Approximating the Variance of a
Complicated Estimate.
Journal of the American Statistical Association, 1971.