SAMPLING AND ESTIMATION
DAY 1: SAMPLING DESIGNS

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- What do you want to do?

Source: http://xkcd.com/1478/

- What do you want to do?

- How do you plan on doing it?

- What do you want to do?

- How do you plan on doing it?

- What problems do you foresee?

What is a representative sample?

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The popular concept of a representative sample it that the sample is
a *miniature* of the population.

However, what do we really want?

However, what do we really want?
We want to estimate a statistic of interest with a certain level of
precision and if the level of precision is high enough we say our
estimation *strategy* is representative.

$$\mathcal{Y} = \{y_1, y_2, \ldots, y_k, \ldots, y_N\} \qquad \text{finite population of size } N$$
$$\mathcal{U} = \{1, 2, \ldots, k, \ldots, N\} \qquad \text{sampling frame}$$
$$\delta \subset \mathcal{U} \qquad \text{sample of size } n$$
$$\mathcal{P}(\mathcal{U}) \qquad \text{all possible subsets of } \mathcal{U}$$

The discrete probability distribution $p(.)$ over $\mathcal{P}(\mathcal{U})$ is called a *sampling design* and $\mathcal{G} = \{\delta | \delta \in \mathcal{P}(\mathcal{U}), p(\delta) > 0\}$ is called the support of $p(.)$ with

$$\sum_{\delta \in \mathcal{G}} p(\delta) = 1$$

Hence, $p : \mathcal{G} \mapsto (0, 1]$.

$$\theta = f(\mathcal{Y}) \qquad \text{statistic of interest}$$

$$\hat{\theta} = f(\mathcal{Y}, \delta) \qquad \text{estimator for } \theta$$

$$\mathsf{E}\left(\hat{\theta}\right) = \sum_{\delta \in \mathcal{G}} p(\delta) f(\mathcal{Y}, \delta) \qquad \text{expected value of } \hat{\theta}$$

$$\mathsf{V}\left(\hat{\theta}\right) = \mathsf{E}\left(\hat{\theta}^2\right) - \mathsf{E}\left(\hat{\theta}\right)^2 \qquad \text{variance of } \hat{\theta}$$

$$\mathsf{MSE}\left(\hat{\theta}\right) = \mathsf{E}\left((\hat{\theta} - \theta)^2\right)$$

$$= \left(\mathsf{E}\left(\hat{\theta}\right) - \theta\right)^2 + \mathsf{V}\left(\hat{\theta}\right) \qquad \text{mean square error of } \hat{\theta}$$

$\mathsf{E}\,(.)$, $\mathsf{V}\,(.)$, and $\mathsf{MSE}\,(.)$ are always with respect to the sampling design $p()$ and an estimator is said to be unbiased if

$$\mathsf{E}\left(\hat{\theta}\right) = \theta \ .$$

EXPECTATION AND VARIANCE OF A RANDOM SAMPLE

$$S_k \qquad \text{number of times element } k \text{ is selected}$$

$$I_k = \begin{cases} 1 & \text{if } k \in \mathcal{s} \\ 0 & \text{else} \end{cases} \qquad \text{sampling indicator element } k$$

$$\mathsf{E}\left(S_k\right) = \nu_k \qquad \text{expected selection frequency of element } k$$

$$\mathsf{E}\left(S_k S_l\right) = \nu_{kl} \qquad \text{joint expectation of } S_k \text{ and } S_l$$

$$\mathsf{E}\left(I_k\right) = \pi_k \qquad \text{inclusion probability of element } k$$

$$\mathsf{E}\left(I_k I_l\right) = \pi_{kl} \qquad \text{joint expectation of } I_k \text{ and } I_l$$

$$\sum_{k \in \mathcal{U}} \nu_k = \mathsf{E}\left(n\right) \qquad \text{expected sample size}$$

Simple random sampling without replacement (SRS): Drawing *n* elements out of a urn without putting them back (i.e. $S_k = I_k$) and without remembering the order of the selected element.

$$\mathcal{G} = \binom{N}{n} \tag{1}$$

$$p(\delta) = \binom{N}{n}^{-1} \tag{2}$$

$$\pi_k = \nu_k = \frac{n}{N} \tag{3}$$

$$\pi_{kl} = \nu_{kl} = \frac{n(n-1)}{N(N-1)} \text{ for } k \neq l \tag{4}$$

$$\theta = \mu = \frac{1}{N} \sum_{k \in \mathcal{U}} y_k, \quad \hat{\theta} = \overline{y} = \sum_{k \in \mathscr{s}} \frac{y_k}{n}, \quad \sigma^2 = \frac{1}{N} \sum_{k \in \mathcal{U}} (y_k - \mu)^2, \quad V^2 = \sigma^2 \frac{N}{N-1}$$

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Expected value

$$
\begin{aligned}
\mathsf{E}\left(\overline{y}\right) &= \mathsf{E}\left(\sum_{k \in \mathcal{U}} S_k \frac{y_k}{n}\right) \\
&= \frac{1}{n} \sum_{k \in \mathcal{U}} \mathsf{E}\left(S_k\right) y_k \\
&= \frac{1}{n} \sum_{k \in \mathcal{U}} \pi_k y_k \\
&= \frac{1}{N} \sum_{k \in \mathcal{U}} y_k
\end{aligned}
$$

$$\theta = \mu = \frac{1}{N}\sum_{k\in\mathcal{U}} y_k, \quad \hat{\theta} = \overline{y} = \sum_{k\in s}\frac{y_k}{n}, \quad \sigma^2 = \frac{1}{N}\sum_{k\in\mathcal{U}}(y_k - \mu)^2, \quad V^2 = \sigma^2\frac{N}{N-1}$$

Expected value

$$\begin{aligned}
\mathsf{E}(\overline{y}) &= \mathsf{E}\left(\sum_{k\in\mathcal{U}} S_k \frac{y_k}{n}\right) \\
&= \frac{1}{n}\sum_{k\in\mathcal{U}} \mathsf{E}(S_k)\, y_k \\
&= \frac{1}{n}\sum_{k\in\mathcal{U}} \pi_k y_k \\
&= \frac{1}{N}\sum_{k\in\mathcal{U}} y_k
\end{aligned}$$

Variance

$$\begin{aligned}
\mathsf{V}(\overline{y}) &= \mathsf{V}\left(\sum_{k\in\mathcal{U}} S_k \frac{y_k}{n}\right) \\
&= \frac{1}{n^2}\sum_{k\in\mathcal{U}}\sum_{l\in\mathcal{U}} \mathsf{COV}(S_k, S_l)\, y_k y_l \\
&= -\frac{1}{2}\frac{1}{n^2}\sum_{k\in\mathcal{U}}\sum_{l\in\mathcal{U}}(\pi_{kl} - \pi_k\pi_l)(y_k - y_l)^2 \\
&= \frac{N-n}{N-1}\frac{\sigma^2}{n} = \left(1 - \frac{n}{N}\right)\frac{V^2}{n}
\end{aligned}$$

SIMPLE RANDOM SAMPLING
WITH REPLACEMENT

Simple random sampling with replacement (SRSWR): Drawing *n* elements out of a urn by making *n* successive draws and putting after each draw the element back (i.e. $S_k \neq I_k$). The order of the selected elements is also not remembered.

$$\mathcal{G} = \binom{N+n-1}{n} \quad p(\delta) = \binom{N+n-1}{n}^{-1}$$

$$\nu_k = \frac{n}{N} \qquad \pi_k = 1 - \left(\frac{N-1}{N}\right)^n$$

$$\nu_{kl} = \frac{n(n-1)}{N^2} \qquad \pi_{kl} = 1 - 2\left(\frac{N-1}{N}\right)^n + \left(\frac{N-2}{N}\right)^n \qquad \text{for } k \neq l$$

Expected value

$$
\begin{aligned}
\mathsf{E}\left(\overline{y}\right) &= \mathsf{E}\left(\sum_{k \in \mathcal{U}} S_k \frac{y_k}{n}\right) \\
&= \frac{1}{n} \sum_{k \in \mathcal{U}} \mathsf{E}\left(S_k\right) y_k \\
&= \frac{1}{n} \sum_{k \in \mathcal{U}} \nu_k y_k \\
&= \frac{1}{N} \sum_{k \in \mathcal{U}} y_k
\end{aligned}
$$

Expected value

$$E\left(\overline{y}\right) = E\left(\sum_{k\in\mathcal{U}} S_k \frac{y_k}{n}\right)$$

$$= \frac{1}{n}\sum_{k\in\mathcal{U}} E\left(S_k\right) y_k$$

$$= \frac{1}{n}\sum_{k\in\mathcal{U}} \nu_k y_k$$

$$= \frac{1}{N}\sum_{k\in\mathcal{U}} y_k$$

Variance

$$V\left(\overline{y}\right) = V\left(\sum_{k\in\mathcal{U}} S_k \frac{y_k}{n}\right)$$

$$= \frac{1}{n^2}\sum_{k\in\mathcal{U}}\sum_{l\in\mathcal{U}} \text{COV}\left(S_k, S_l\right) y_k y_l$$

$$= -\frac{1}{2}\frac{1}{n^2}\sum_{k\in\mathcal{U}}\sum_{l\in\mathcal{U}} \left(\nu_{kl} - \nu_k \nu_l\right) \left(y_k - y_l\right)^2$$

$$= \frac{\sigma^2}{n}$$

Expected value

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$$= \frac{\sigma^2}{n}$$

Note: $\frac{\sigma^2}{n} > \left(1 - \frac{n}{N}\right) \frac{V^2}{n}$, if $n > 1$.

$$\widehat{V}\left(\overline{y}\right)_{\text{SRS}} = \frac{N-n}{N}\frac{s^2}{n}$$

$$\widehat{V}\left(\overline{y}\right)_{\text{SRSWR}} = \frac{s^2}{n}$$

Sample variance

$$s^2 = \frac{1}{n-1}\sum_{k\in s}(y_k-\overline{y})^2 = \frac{1}{n(n-1)}\sum_{k\in\mathcal{U}}\sum_{l\in\mathcal{U}}(y_k-y_l)^2 S_k S_l$$

$$E\left(s^2\right)_{\text{SRS}} = \frac{N}{N-1}\sigma^2 = V^2$$

$$E\left(s^2\right)_{\text{SRSWR}} = \sigma^2$$

- Stratification
- Cluster Sampling: Not elementary units are selected but *clusters* containing multiple elements.
- Multistage Sampling: The population is structured by hierarchically ordered clusters that are nested within each other. The sampling procedure has multiple selecting stages.

The universe $\mathcal{U}$ is decomposed into H non-overlapping groups, $\mathcal{U}_1, \ldots, \mathcal{U}_H$,called strata.

- $\mathcal{U} = \bigcup\limits_{h=1}^{H} \mathcal{U}_h$, where set $\mathcal{U}_h$ is the h-th strata.
- A sample $\mathfrak{s}_h$ is selected from $\mathcal{U}_h$ according to a design $p_h(.)$, for all $h = 1, \ldots, H$.
- The number of elements in $\mathcal{U}_h$ is called stratum size and denote with N_h
- The number of elements in $\mathfrak{s}_h$ is denoted with n_h.

In stratified random sampling the sub-populations are called strata. For the *h*-ht stratum we get:

$$\mu_h = \frac{1}{N_h} \sum_{k=1}^{N_h} y_{kh} \qquad \text{mean of stratum h}$$
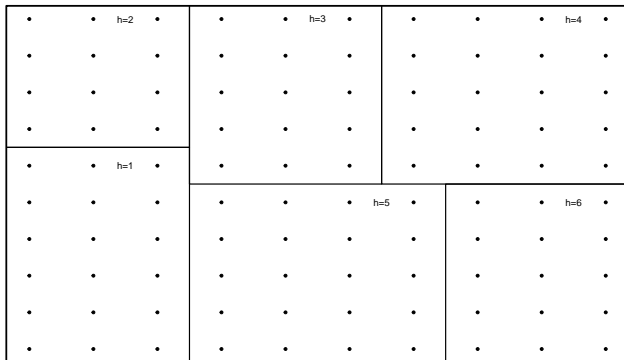
$$\sigma_h^2 = \frac{1}{N_h} \sum_{k=1}^{N_h} (y_{kh} - \mu_h)^2 \qquad \text{variance of stratum } h$$

$$V_h^2 = \sigma_h^2 \frac{N_h}{N_h - 1}$$

Where y_{kh} as the k-th element in the h-th stratum. Sampling from stratified populations is called stratified random sampling (StrRS).

A Population of 100 elements is stratified into $H = 6$ strata.

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14 elements are selected from the population and their allocation is
given by $n_1 = 2$ $n_2 = 3$ $n_3 = 2$ $n_4 = 3$ $n_5 = 3$ $n_6 = 2$

Estimator for the mean:

$$\overline{y}_{\text{str}} = \sum_{h=1}^{H} \gamma_h \overline{y}_h$$

where $\gamma_h = \dfrac{N_h}{N}$ and $E(\overline{y}_{\text{str}}) = \mu$ for SRS and SRSWR within each stratum.

Variance and variance estimator:

$$V(\overline{y}_{\text{str}})_{\text{SRS}} = \sum_{h=1}^{H} \frac{N_h - n_h}{N_h} \gamma_h^2 \frac{V_h^2}{n_h}$$

$$\widehat{V}(\overline{y}_{\text{str}})_{\text{SRS}} = \sum_{h=1}^{H} \frac{N_h - n_h}{N_h} \gamma_h^2 \frac{s_h^2}{n_h}$$

$$s_h^2 = \frac{1}{n_h - 1} \sum_{k \in s_h} (y_k - \overline{y}_h)^2$$

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  - (You might check out the `SamplingStrata` package, it implements a method to determine of the best stratification of a sampling frame, of minimal cost under the condition to satisfy precision constraints in a multivariate and multi-domain case.)
- How should the overall sample size be allocated to the strata?
  - Achieve proportionality between sample and population (i.e. the frame)
  - Fulfill precision constraints for certain estimation domains

TABLE: Population ANOVA

Source	df	Sum of Squares
Between strata	$H - 1$	$\text{SSB} = \sum_{h=1}^{H} N_h(\mu_h - \mu)^2$
Within strata	$N - H$	$\text{SSW} = \sum_{h=1}^{H}(N_h - 1)V_h^2$
Total, about μ_y	$N - 1$	$\text{SSTO} = (N - 1)V^2$

The more homogeneous the strata are the higher is the gain in efficiency from using stratified simple random sample sampling (StrSRS) instead of SRS. Because then SSW (variance within) is considerably small in contrast to SSB (variance between). This is called the effect of stratification.

For all $h = 1, \ldots, H$

$$
n_h =
\begin{cases}
\dfrac{n}{H} & \text{equal allocation} \\[2ex]
\dfrac{N_h}{N} n & \text{proportional allocation} \\[2ex]
\dfrac{N_h V_h}{\sum_{h=1}^{H} N_h V_h} n & \text{optimal allocation} \\[2ex]
\dfrac{c}{\overline{c}_h} \dfrac{N_h V_h \sqrt{\overline{c}_h}}{\sum_{h=1}^{H} N_h V_h \sqrt{\overline{c}_h}} & \text{cost-optimal allocation}
\end{cases}
,
$$

where $\overline{c}_h$ are average cost of selecting a element from stratum h and $c = \sum_{h=1}^{H} n_h \overline{c}_h$ are the total costs of the survey. For the cost-optimal allocation c is given, not n.

If $n_h = \dfrac{N_h}{N} n$

$$V\left(\overline{y}_{\text{str}}\right)_{\text{StrSRS}} = \left(\frac{N-n}{N}\right) \frac{1}{n} \sum_{h=1}^{H} N_h V_h^2 \qquad \text{and}$$

$$\begin{aligned}
V\left(\overline{y}\right)_{\text{SRS}} &= \left(\frac{N-n}{N}\right) \frac{1}{n(N-1)} \left(\text{SSW} + \text{SSB}\right) \\
&= V\left(\overline{y}_{\text{str}}\right)_{\text{StrSRS}} + \left(\frac{N-n}{N}\right) \frac{1}{n(N-1)} \left[\text{SSB} - \sum_{h=1}^{H} \frac{N-N_h}{N} V_h^2\right] .
\end{aligned}$$

Thus, StrSRS with prop. allocation will always result in an equal or smaller variance than SRS if

$$SSB > \sum_{h=1}^{H} \frac{N-N_h}{N} V_h^2 .$$

- It is not assured that $\gamma_h n$ is an integer. If $n_h^* = [n_h]$ is used instead, the allocation is no longer strictly proportional. Furthermore $\sum_{h=1}^{H} n_h^* = n$ is also not assured.
- However stochastic techniques can be used that ensure that $E\left(n_h^*\right) = n_h$ and thus $E\left(\sum_{h=1}^{H} n_h^*\right) = n$.

$$n_h^* = \begin{cases} \lfloor n_h \rfloor & \text{with prob. } 1 - (n_h \mod 1) \\ \lceil n_h \rceil & \text{with prob. } (n_h \mod 1) \end{cases}$$

There are stochastic rounding procedures that are controlled and unbiased, i.e. $\sum_{h=1}^{H} n_h = n$ and $E\left(n_h^*\right) = \frac{N_h}{N} n$ and $|n_h^* - n_h| \leqslant 1$ [Cox, 1987].

SIMPLE SYSTEMATIC SAMPLING

The elements of a population of size $N = nH$ are ordered in a specific way (every unit having a unique rank). Starting form a random number k, with $1 \leqslant k \leqslant H$ and $k \in \{1, 2, \ldots, H\}$ the sample is defined as the elements with ranks

$$k, k + H, k + 2H, k + 3H, \ldots, k + (n-1)H.$$

$$\overline{y}_k = \frac{1}{n} \sum_{i=0}^{n-1} y_{(k+iH)} \qquad \text{sample mean}$$

$$\mathsf{E}\left(\overline{y}_k\right)_{\text{SyS}} = \mu \qquad \text{expected value of } \overline{y}_k$$

$$\mathsf{V}\left(\overline{y}_k\right)_{\text{SyS}} = \frac{1}{H} \sum_{h=1}^{H} \left(\overline{y}_k - \mu\right)^2 \qquad \text{variance of } \overline{y}_k$$

There is no unbiased variance estimator for systematic sampling. Ordering the population with respect to certain variables has a similar effects as stratification by the same variable with proportional allocation.

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