Sampling and Estimation

Part 1: Introduction to Design Based Inference

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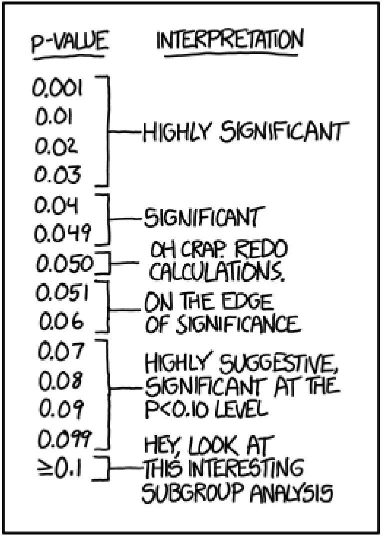
Welcome and Introduction

- What do you want to do?

- 

Source: http://xkcd.com/1478/

Motivation

- What do you want to do?

- How do you plan on doing it?

Motivation

- What do you want to do?
- How do you plan on doing it?
- What problems do you foresee?

What is a representative sample?

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The popular concept of a representative sample is that the sample is a *miniature* of the population.

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We want to estimate a statistic of interest with a certain level of precision and if the level of precision is high enough, we say our estimation *strategy* is representative.

Framework and Notation

Finite population, Sample, and Sampling Design

$\mathcal{Y} = \{y_1, y_2, \ldots, y_k, \ldots, y_N\}$	finite population of size N
$\mathcal{U} = \{1, 2, \ldots, k, \ldots, N\}$	sampling frame
$s \subset \mathcal{U}$	sample of size n
$\mathcal{P}(\mathcal{U})$	all possible subsets of $\mathcal{U}$

The discrete probability distribution $p(.)$ over $\mathcal{P}(\mathcal{U})$ is called a *sampling design* and $\mathcal{G} = \{s | s \in \mathcal{P}(\mathcal{U}),\ p(s) > 0\}$ is called the support of $p(.)$ with

$$\sum_{s \in \mathcal{G}} p(s) = 1$$

Hence, $p: \ \mathcal{G} \mapsto (0, 1]$.

$$\theta = f(\mathcal{Y})$$ statistic of interest

$$\hat{\theta} = f(\mathcal{Y}, s)$$ estimator for θ

$$\mathrm{E}\left(\hat{\theta}\right) = \sum_{s \in \mathcal{G}} p(s) f(\mathcal{Y}, s)$$ expected value of $\hat{\theta}$

$$\mathrm{V}\left(\hat{\theta}\right) = \mathrm{E}\left(\hat{\theta}^2\right) - \mathrm{E}\left(\hat{\theta}\right)^2$$ variance of $\hat{\theta}$

$$\mathrm{MSE}\left(\hat{\theta}\right) = \mathrm{E}\left((\hat{\theta} - \theta)^2\right) = \left(\mathrm{E}\left(\hat{\theta}\right) - \theta\right)^2 + \mathrm{V}\left(\hat{\theta}\right)$$ mean square error of $\hat{\theta}$

$\mathrm{E}(.)$, $\mathrm{V}(.)$, and $\mathrm{MSE}(.)$ are always with respect to the sampling design $p()$ and an estimator is said to be unbiased if

$$\mathrm{E}\left(\hat{\theta}\right) = \theta\ .$$

Expectation and Variance of a Random Sample

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S_k	number of times element k is selected
$I_k = \begin{cases} 1 & \text{if } k \in s \\ 0 & \text{else} \end{cases}$	sampling indicator element k
$\mathrm{E}(S_k) = \nu_k$	expected selection frequency of element k
$\mathrm{E}(S_k S_l) = \nu_{kl}$	joint expectation of S_k and S_l
$\mathrm{E}(I_k) = \pi_k$	inclusion probability of element k
$\mathrm{E}(I_k I_l) = \pi_{kl}$	joint expectation of I_k and I_l
$\sum_{k \in \mathcal{U}} \nu_k = \mathrm{E}(n)$	expected sample size

Simple Random Sampling

Simple random sampling without replacement (SRS): Drawing n elements out of a urn without putting them back (i.e. $S_k \geqslant I_k$) and without remembering the order of the selected element.

$$\mathcal{G} = \binom{N}{n} \tag{1}$$

$$p(\mathcal{s}) = \binom{N}{n}^{-1} \tag{2}$$

$$\pi_k = \nu_k = \frac{n}{N} \tag{3}$$

$$\pi_{kl} = \nu_{kl} = \frac{n(n-1)}{N(N-1)} \text{ for } k \neq l \tag{4}$$

$$\theta = \mu = \frac{1}{N}\sum_{k\in\mathcal{U}} y_k, \quad \hat{\theta} = \bar{y} = \sum_{k\in\mathcal{s}} \frac{y_k}{n}, \quad \sigma^2 = \frac{1}{N}\sum_{k\in\mathcal{U}} (y_k - \mu)^2, \quad V^2 = \sigma^2 \frac{N}{N-1}$$

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Expected value

$$\begin{aligned} \mathrm{E}\left(\overline{y}\right) &= \mathrm{E}\left(\sum_{k\in\mathcal{U}} S_k \frac{y_k}{n}\right) \\ &= \frac{1}{n}\sum_{k\in\mathcal{U}} \mathrm{E}\left(S_k\right) y_k \\ &= \frac{1}{n}\sum_{k\in\mathcal{U}} \pi_k y_k \\ &= \frac{1}{N}\sum_{k\in\mathcal{U}} y_k \end{aligned}$$

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&= \frac{1}{n}\sum_{k\in\mathcal{U}} \mathrm{E}\left(S_k\right) y_k \\
&= \frac{1}{n}\sum_{k\in\mathcal{U}} \pi_k y_k \\
&= \frac{1}{N}\sum_{k\in\mathcal{U}} y_k
\end{aligned}$$

Variance

$$\begin{aligned}
\mathrm{V}\left(\overline{y}\right) &= \mathrm{V}\left(\sum_{k\in\mathcal{U}} S_k \frac{y_k}{n}\right) \\
&= \frac{1}{n^2}\sum_{k\in\mathcal{U}}\sum_{l\in\mathcal{U}} \mathrm{COV}\left(S_k,\, S_l\right) y_k y_l \\
&= -\frac{1}{2}\frac{1}{n^2}\sum_{k\in\mathcal{U}}\sum_{l\in\mathcal{U}} \left(\pi_{kl} - \pi_k \pi_l\right)\left(y_k - y_l\right)^2 \\
&= \frac{N-n}{N-1}\frac{\sigma^2}{n} = \left(1 - \frac{n}{N}\right)\frac{V^2}{n}
\end{aligned}$$

Complex Sampling Designs - Stratification

- Stratification
- Cluster Sampling: Not elementary units are selected but *clusters* containing multiple elements.
- Multistage Sampling: The population is structured by hierarchically ordered clusters that are nested within each other. The sampling procedure has multiple selecting stages.

The universe $\mathcal{U}$ is decomposed into H non-overlapping groups, $\mathcal{U}_1, \ldots, \mathcal{U}_H$,called strata.

- $\mathcal{U} = \bigcup_{h=1}^{H} \mathcal{U}_h$, where set $\mathcal{U}_h$ is the h-th strata.
- A sample s_h is selected from $\mathcal{U}_h$ according to a design $p_h(.)$, for all $h = 1, \ldots, H$.
- The number of elements in $\mathcal{U}_h$ is called stratum size and denote with N_h
- The number of elements in s_h is denoted with n_h.

In stratified random sampling the sub-populations are called strata. For the *h*-ht stratum we get:

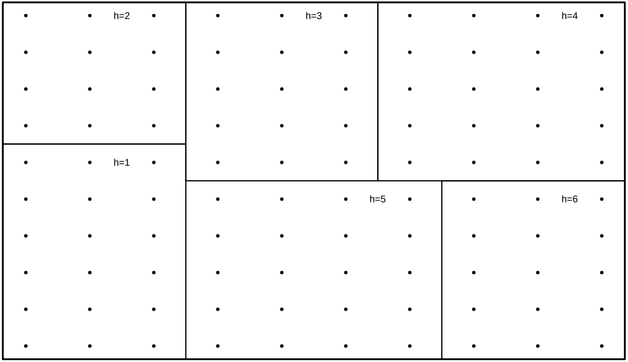
$$\mu_h = \frac{1}{N_h} \sum_{k=1}^{N_h} y_{kh}$$ mean of stratum h

$$\sigma_h^2 = \frac{1}{N_h} \sum_{k=1}^{N_h} (y_{kh} - \mu_h)^2$$ variance of stratum *h*

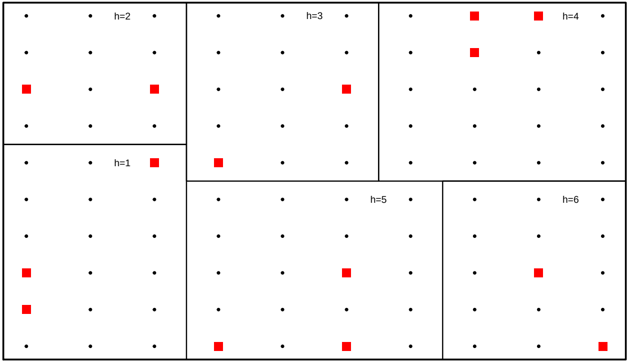
$$V_h^2 = \sigma_h^2 \frac{N_h}{N_h - 1}$$

Where y_{kh} as the *k*-th element in the *h*-th stratum. Sampling from stratified populations is called stratified random sampling (StrRS).

A Population of 100 elements is stratified into $H = 6$ strata.



A Population of 100 elements is stratified into $H = 6$ strata.
14 elements are selected from the population and their allocation is given by $n_1 = 2$ $n_2 = 3$ $n_3 = 2$ $n_4 = 3$ $n_5 = 3$ $n_6 = 2$



Estimator for the mean:

$$\overline{y}_{\text{str}} = \sum_{h=1}^{H} \gamma_h \overline{y}_h$$

where $\gamma_h = \dfrac{N_h}{N}$ and $\mathrm{E}\left(\overline{y}_{\text{str}}\right) = \mu$ for SRS and SRSWR within each stratum.

Variance and variance estimator:

$$\mathrm{V}\left(\overline{y}_{\text{str}}\right)_{\text{SRS}} = \sum_{h=1}^{H} \frac{N_h - n_h}{N_h} \gamma_h^2 \frac{V_h^2}{n_h}$$

$$\hat{\mathrm{V}}\left(\overline{y}_{\text{str}}\right)_{\text{SRS}} = \sum_{h=1}^{H} \frac{N_h - n_h}{N_h} \gamma_h^2 \frac{s_h^2}{n_h}$$

$$s_h^2 = \frac{1}{n_h - 1} \sum_{k \in s_h} \left(y_k - \overline{y}_h\right)^2$$

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Issues with Stratification

- Why should stratification be used?
  - To reduce the sampling variance of estimators.
  - Sometimes it is necessary because of organizational reasons (e.g. no joint sampling frame).
- How should the population be stratified?
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- How should the overall sample size be allocated to the strata?
  - Achieve proportionality between sample and population (i.e. the frame)
  - Fulfill precision constraints for certain estimation domains

Table: Population ANOVA

Source	df	Sum of Squares
Between strata	$H - 1$	$\text{SSB} = \sum_{h=1}^{H} N_h(\mu_h - \mu)^2$
Within strata	$N - H$	$\text{SSW} = \sum_{h=1}^{H}(N_h - 1)V_h^2$
Total, about μ_y	$N - 1$	$\text{SSTO} = (N - 1)V^2$

The more homogeneous the strata are the higher is the gain in efficiency from using stratified simple random sample sampling (StrSRS) instead of SRS. Because then SSW (variance within) is considerably small in contrast to SSB (variance between). This is called the effect of stratification.

For all $h = 1, \ldots, H$

$$n_h = \begin{cases} \dfrac{n}{H} & \text{equal allocation} \\ \dfrac{N_h}{N} n & \text{proportional allocation} \\ \dfrac{N_h V_h}{\sum_{h=1}^{H} N_h V_h} n & \text{optimal allocation} \\ \dfrac{c}{\overline{c}_h} \dfrac{N_h V_h \sqrt{\overline{c}_h}}{\sum_{h=1}^{H} N_h V_h \sqrt{\overline{c}_h}} & \text{cost-optimal allocation} \end{cases},$$

where $\overline{c}_h$ are average cost of selecting a element from stratum h and $c = \sum_{h=1}^{H} n_h \overline{c}_h$ are the total costs of the survey. For the cost-optimal allocation c is given, not n.

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If $n_h = \frac{N_h}{N} n$

$$\mathrm{V}\left(\bar{y}_{\text{str}}\right)_{\text{StrSRS}} = \left(\frac{N-n}{N}\right) \frac{1}{n} \sum_{h=1}^{H} N_h V_h^2 \qquad \text{and}$$

$$\begin{aligned} \mathrm{V}\left(\bar{y}\right)_{\text{SRS}} &= \left(\frac{N-n}{N}\right) \frac{1}{n(N-1)} (\text{SSW} + \text{SSB}) \\ &= \mathrm{V}\left(\bar{y}_{\text{str}}\right)_{\text{StrSRS}} + \left(\frac{N-n}{N}\right) \frac{1}{n(N-1)} \left[\text{SSB} - \sum_{h=1}^{H} \frac{N-N_h}{N} V_h^2\right] . \end{aligned}$$

Thus, StrSRS with prop. allocation will always result in an equal or smaller variance than SRS if

$$SSB > \sum_{h=1}^{H} \frac{N-N_h}{N} V_h^2 .$$

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