COMP6247 - **Reinforcement & Online Learning (MSc)**

Online Learning

Week 6: Sequential Monte Carlo

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University of Southampton

Spring Semester 2020/2021

Recap: Monte Carlo Sampling

Basic Monte Carlo Sampling:

Approximate a target distribution by random sampling

$$p(\mathbf{y}) \approx \frac{1}{L} \sum_{\ell=1}^{L} \delta_{\tilde{\mathbf{y}}^{(\ell)}}(\mathbf{y})$$

where $\tilde{\mathbf{y}}^{(\ell)} \sim p(\mathbf{y}) \, \forall \ell \in \{1, \ldots, L\}$ and $\delta_{\tilde{\mathbf{y}}}(\mathbf{y}) = \begin{cases} 1, & \text{if } \mathbf{y} = \tilde{\mathbf{y}}^{(\ell)} \\ 0, & \text{otherwise} \end{cases}$

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Problems with Rejection Sampling:

1) Exponential decrease in acceptance rate with increasing dimensionality of $\mathbf{y}$
2) Multimodal distributions: Difficult to find a good proposal distribution

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2) Multimodal distributions: Difficult to find a good proposal distribution

Problems with Markov Chain Monte Carlo:

1) Successive samples are highly correlated
2) Iterative approach: Assume data is readily available in a single batch.

Lecture Overview

Week 4: Bayesian Inference

Week 5: Markov Chain Monte Carlo (MCMC)

Week 6: Importance Sampling & Sequential Monte Carlo

Part 1: Importance Sampling
Part 2: Sequential Importance Sampling
Part 3: Sequential Monte Carlo (aka. Sequential Importance Resampling)

Learning Outcomes

Following this week's lecture on Sequential Monte Carlo methods, you should be able to:

1) **Explain the difference between Importance Sampling, Sequential Importance Sampling, and Sequential Monte Carlo**
2) **Analyse and mitigate degeneracy of sequential importance sampling methods;**
3) **Apply sequential Monte Carlo for online learning.**

Further Reading

Textbook:

K. Murphy, “Chapter 23: Monte Carlo Inference,” *in Machine Learning: A Probabilistic Perspective*, MIT Press, 2012.

Tutorial Papers:

M. S. Arulampalam, S. Maskell, N. Gordon and T. Clapp, "A tutorial on particle filters for online nonlinear/non-Gaussian Bayesian tracking," in IEEE Transactions on Signal Processing, vol. 50, no. 2, 2002. doi: 10.1109/78.978374

A. Doucet, S. Godsill, & C. Andrieu, “On sequential Monte Carlo sampling methods for Bayesian filtering,” *in Statistics and Computing* **10,** 2000. doi: 10.1023/A:1008935410038

A. Doucet & A.M. Johansen, “A Tutorial on Particle Filtering and Smoothing: 15 Years Later," *in Oxford Handbook of Nonlinear Filtering*, Oxford University Press, 2011.

C. A. Naesseth, F. Lindsten and T. B. Schön, "Elements of Sequential Monte Carlo", Foundations and Trends in Machine Learning: Vol. 12: No. 3, 2019. doi: 10.1561/2200000074 and arXiv: 1903.04797

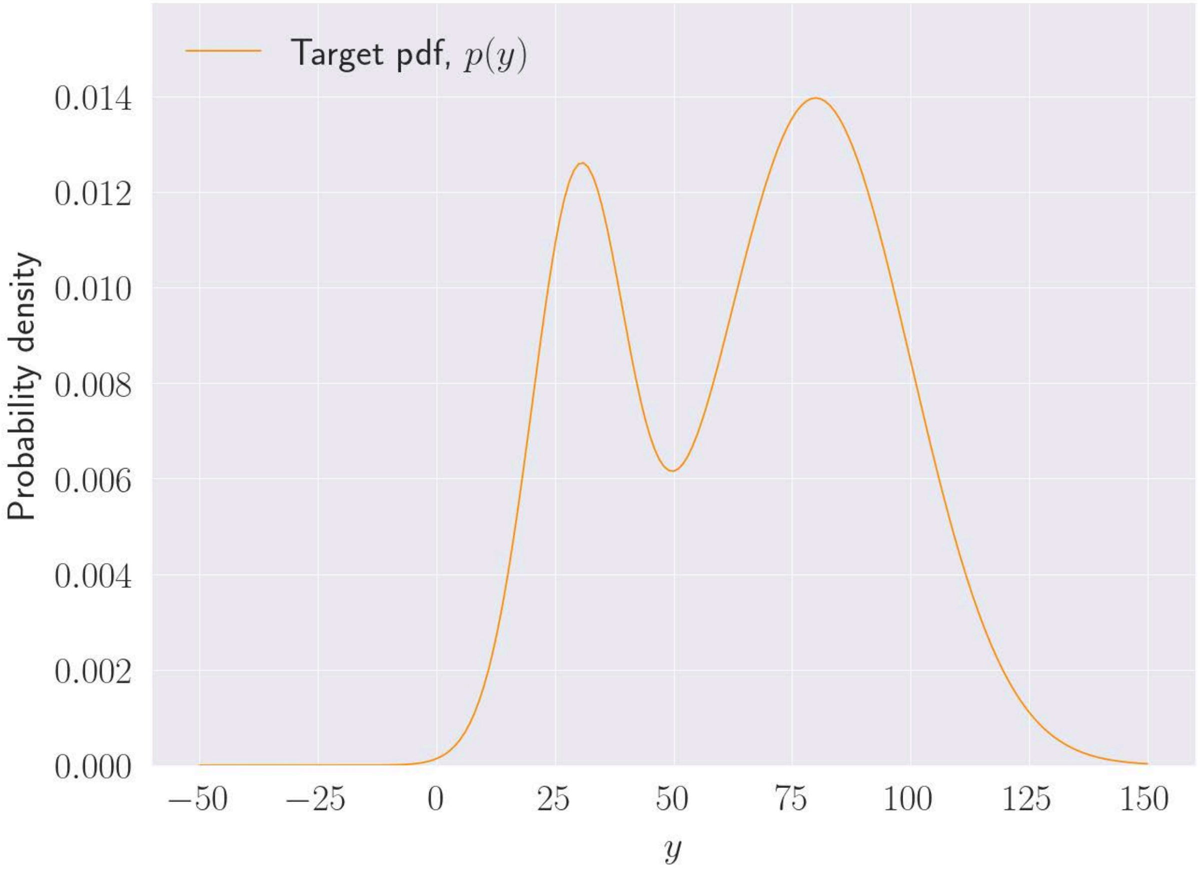
Importance Sampling

Importance Sampling Illustrated

Aim:
Penalise, rather than rejecting, samples

$$p(\mathbf{y}) = \frac{\tilde{p}(\mathbf{y})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y})\, d\mathbf{y}}$$

$p(\mathbf{y})$: Target pdf, $\tilde{p}(\mathbf{y})$: Unnormalised target pdf
$\tilde{w}^{(\ell)}$: Normalised weight, $w^{(\ell)}$: Unnormalised weight



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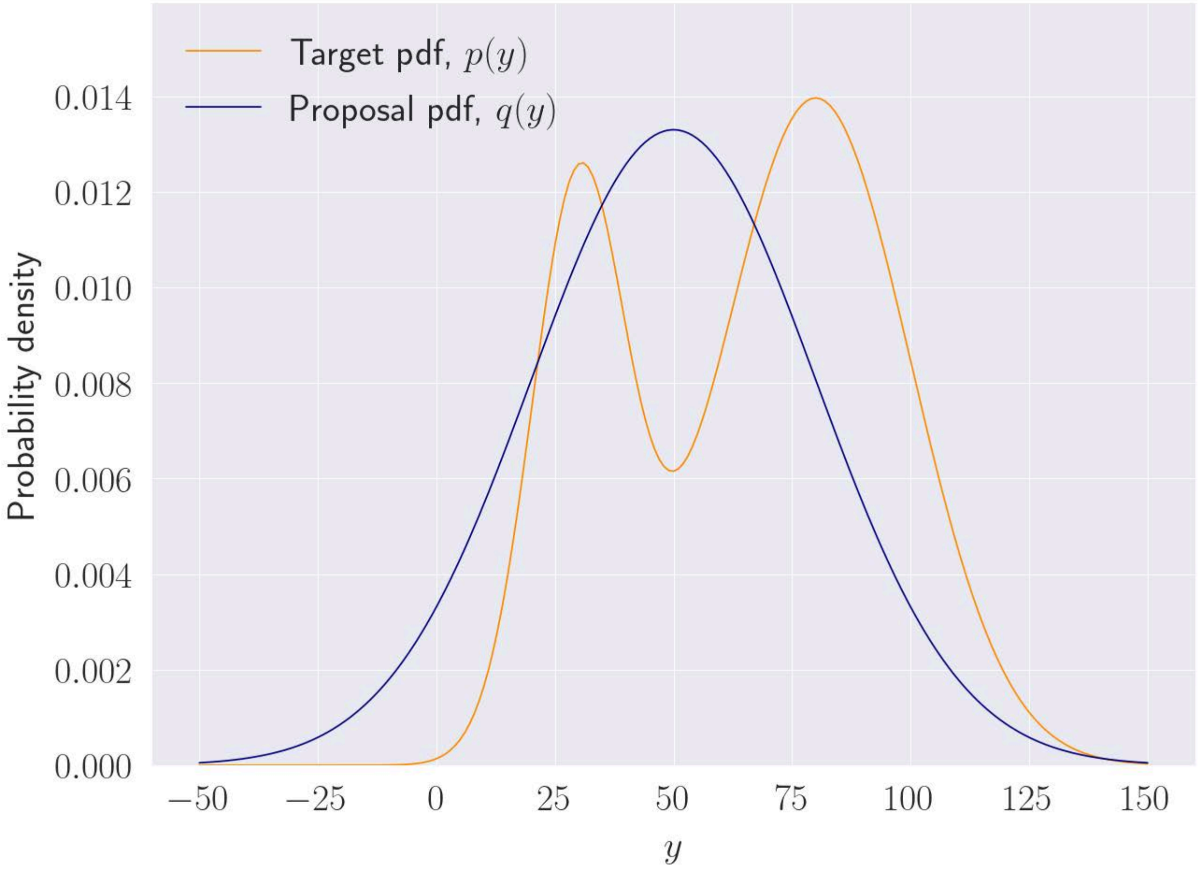

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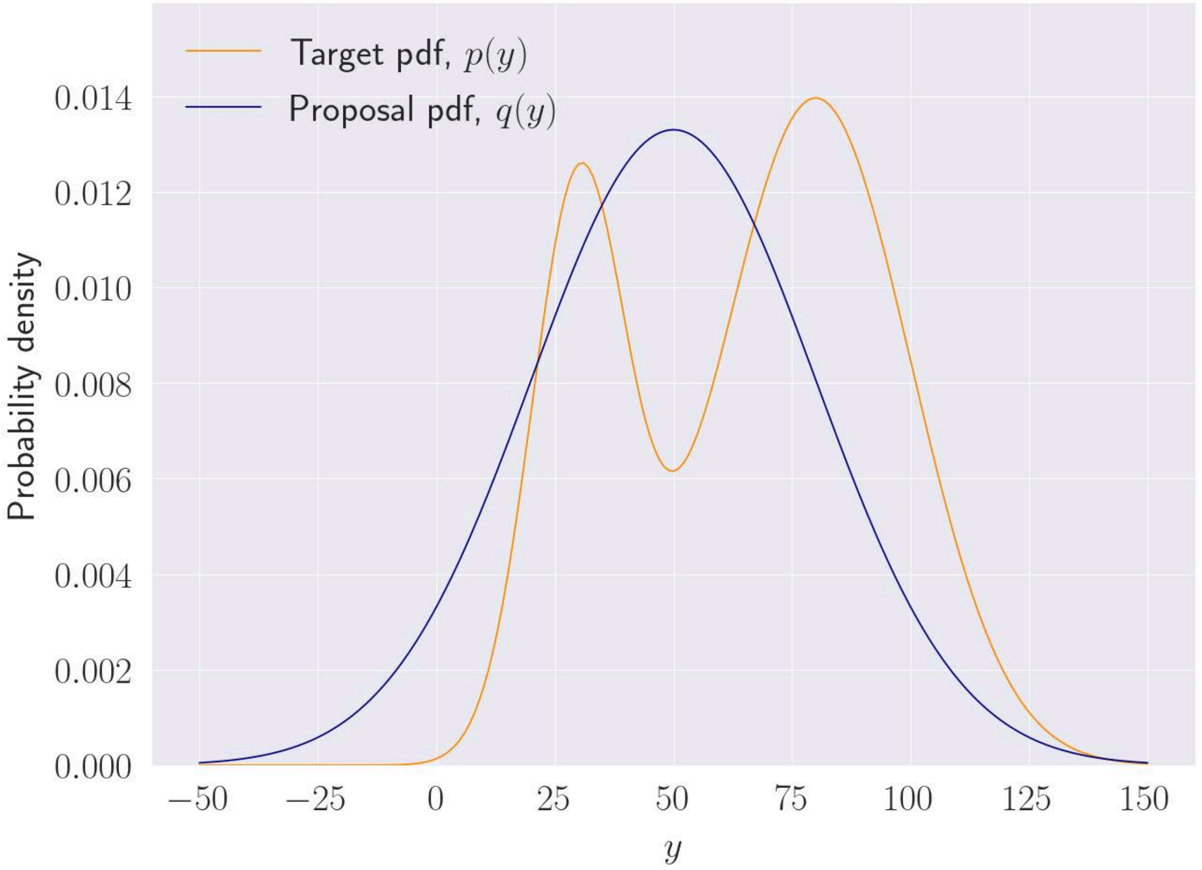

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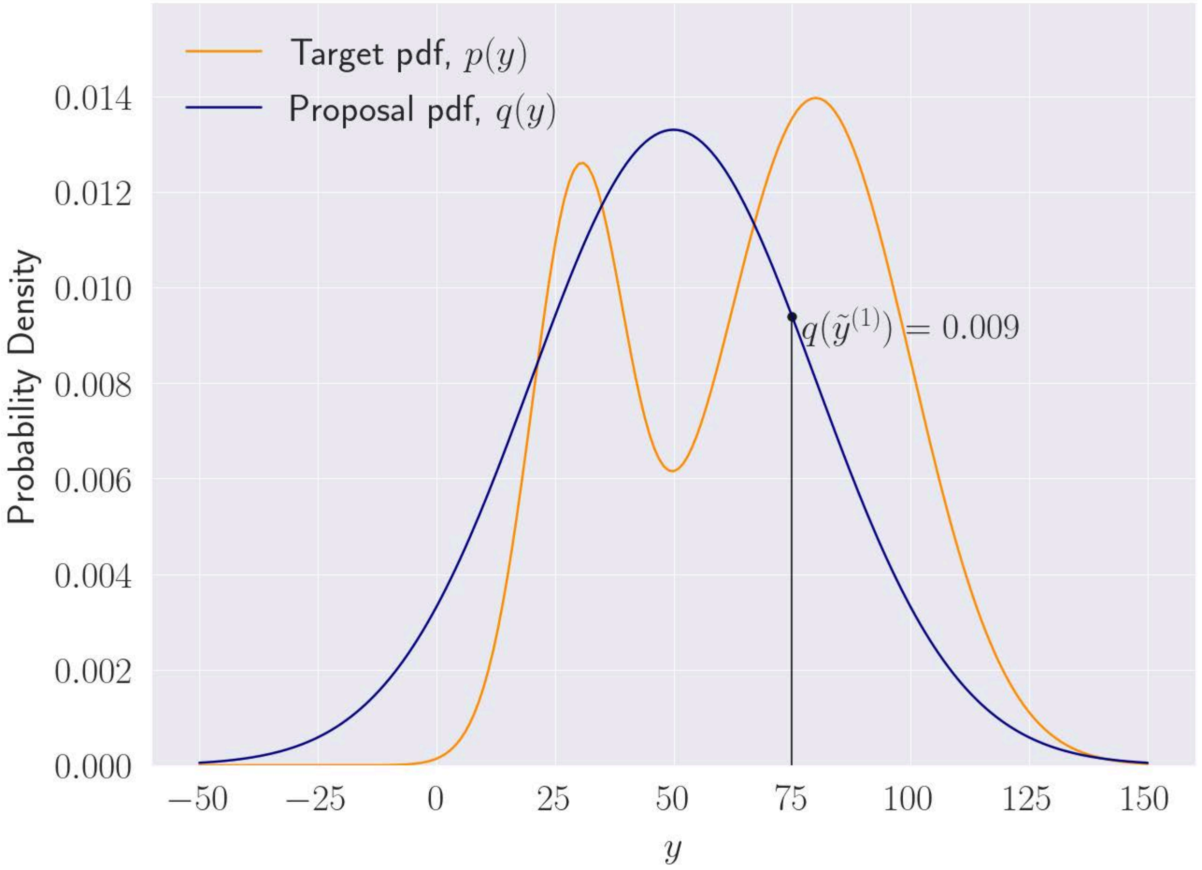

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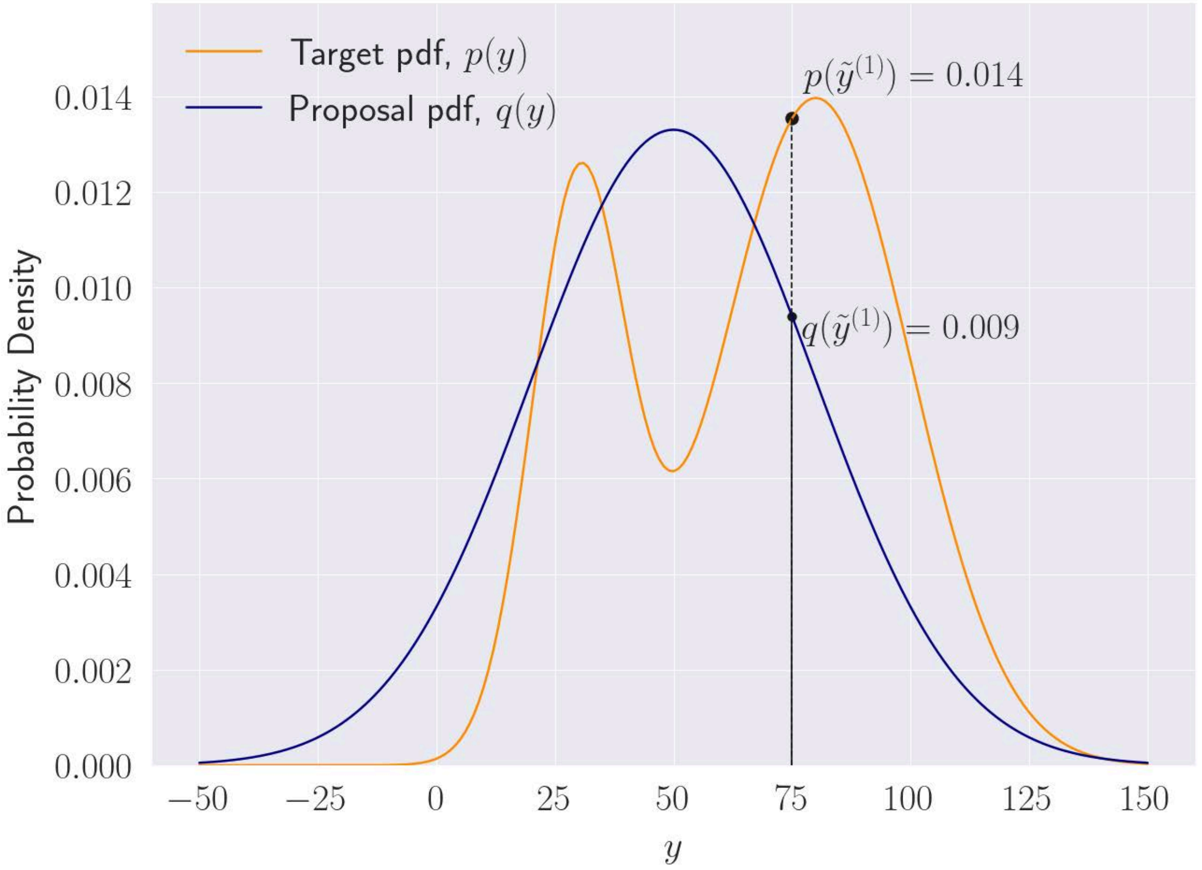

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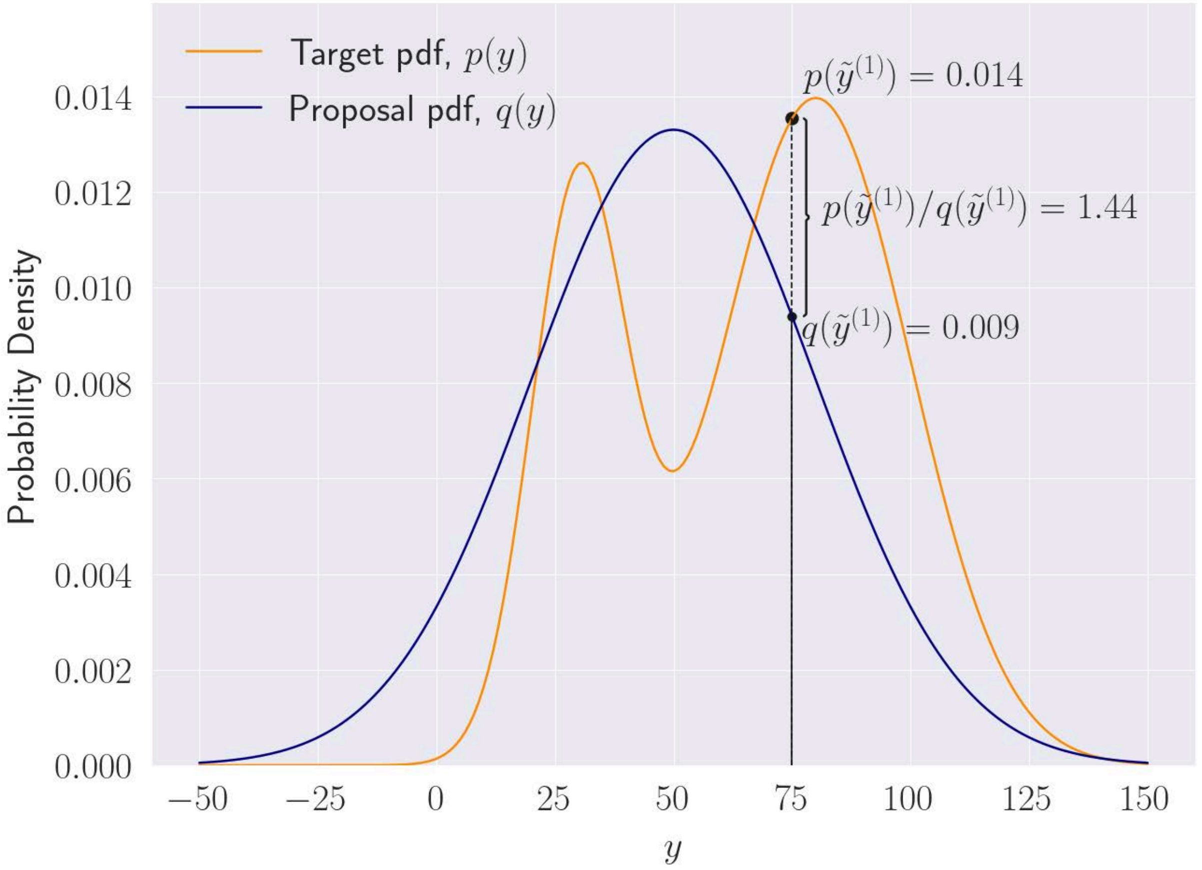

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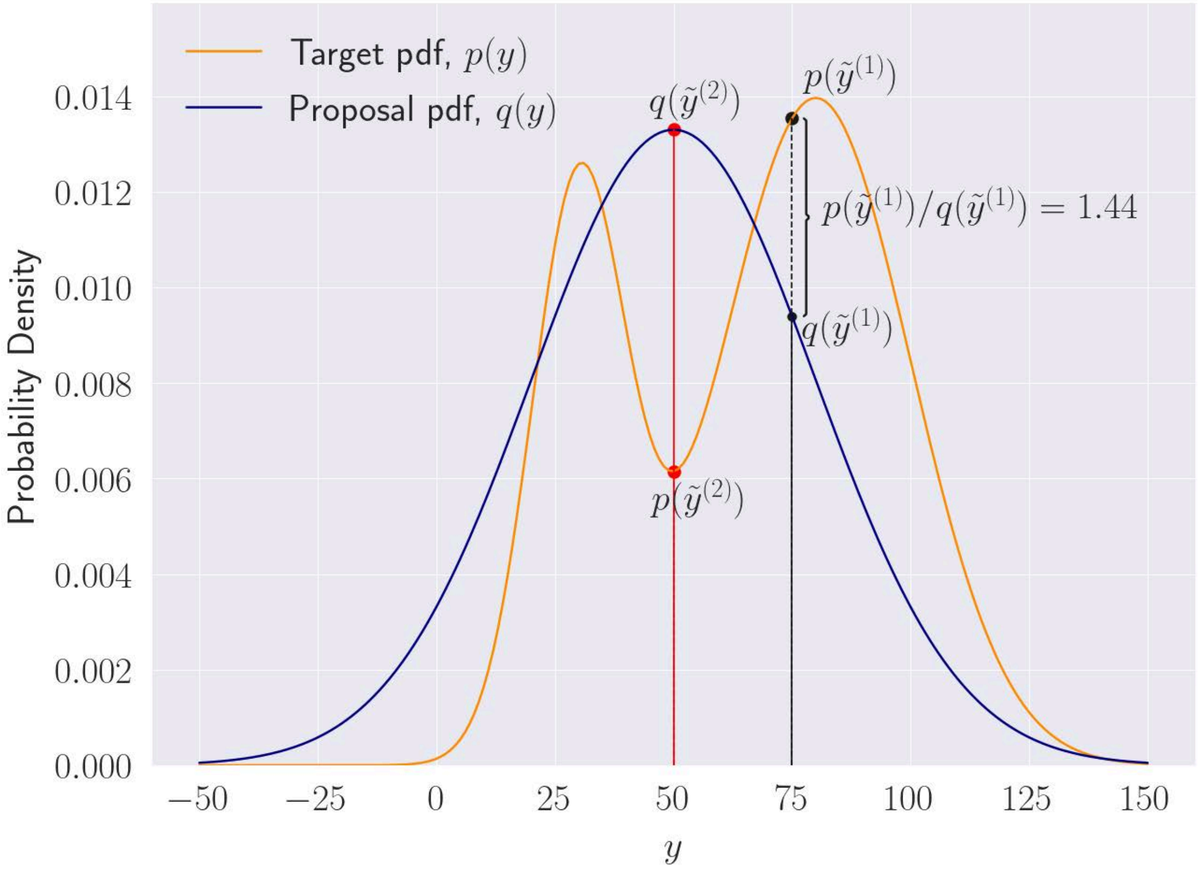

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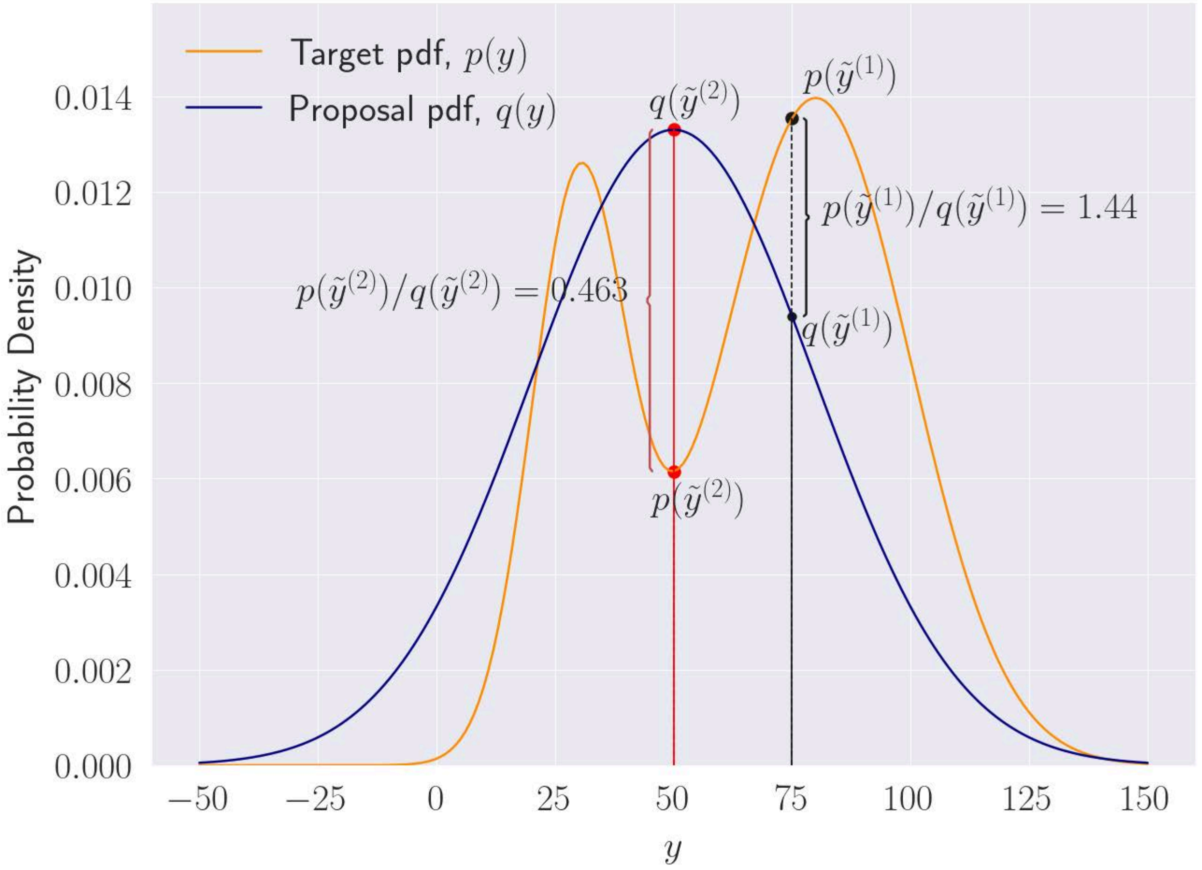

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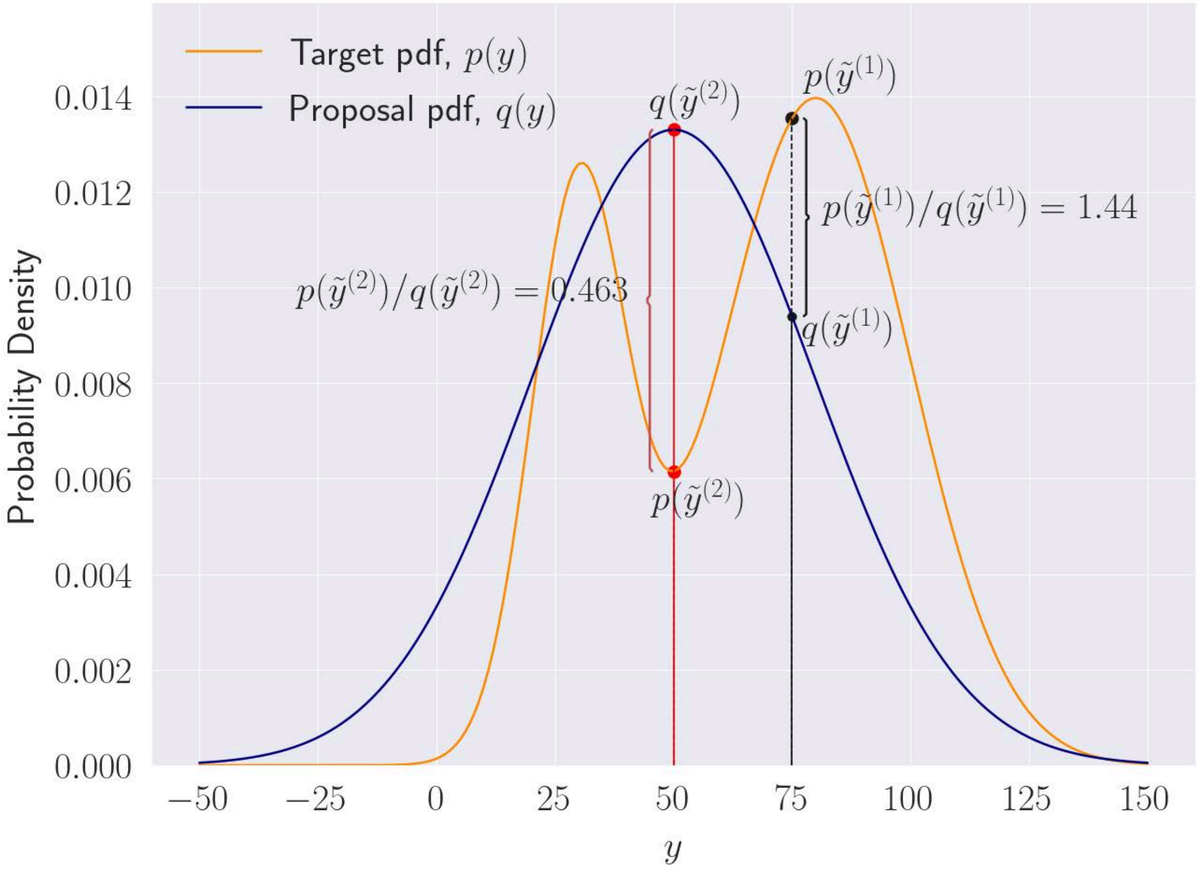

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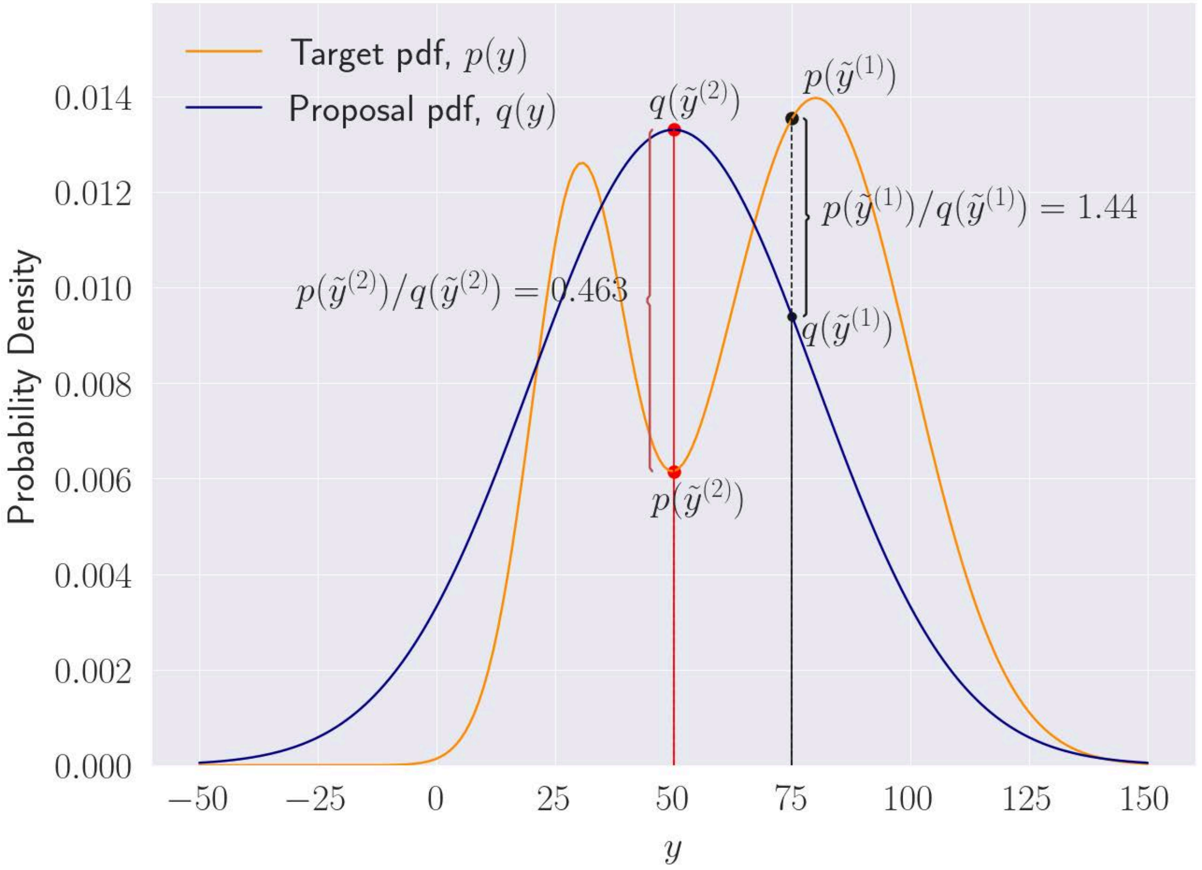

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Proof:

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$$p(\mathbf{y}) = \frac{\tilde{p}(\mathbf{y})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y})\, d\mathbf{y}} = \frac{\frac{\tilde{p}(\mathbf{y})}{q(\mathbf{y})} q(\mathbf{y})}{\int_{\mathcal{Y}} \frac{\tilde{p}(\mathbf{y})}{q(\mathbf{y})} q(\mathbf{y})\, d\mathbf{y}}$$

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Proof:

Unnormalised weight function, $w(\mathbf{y})$

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Importance Sampling

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Proof:

Unnormalised weight function, $w(\mathbf{y})$

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Example: Gaussian Mixture Model

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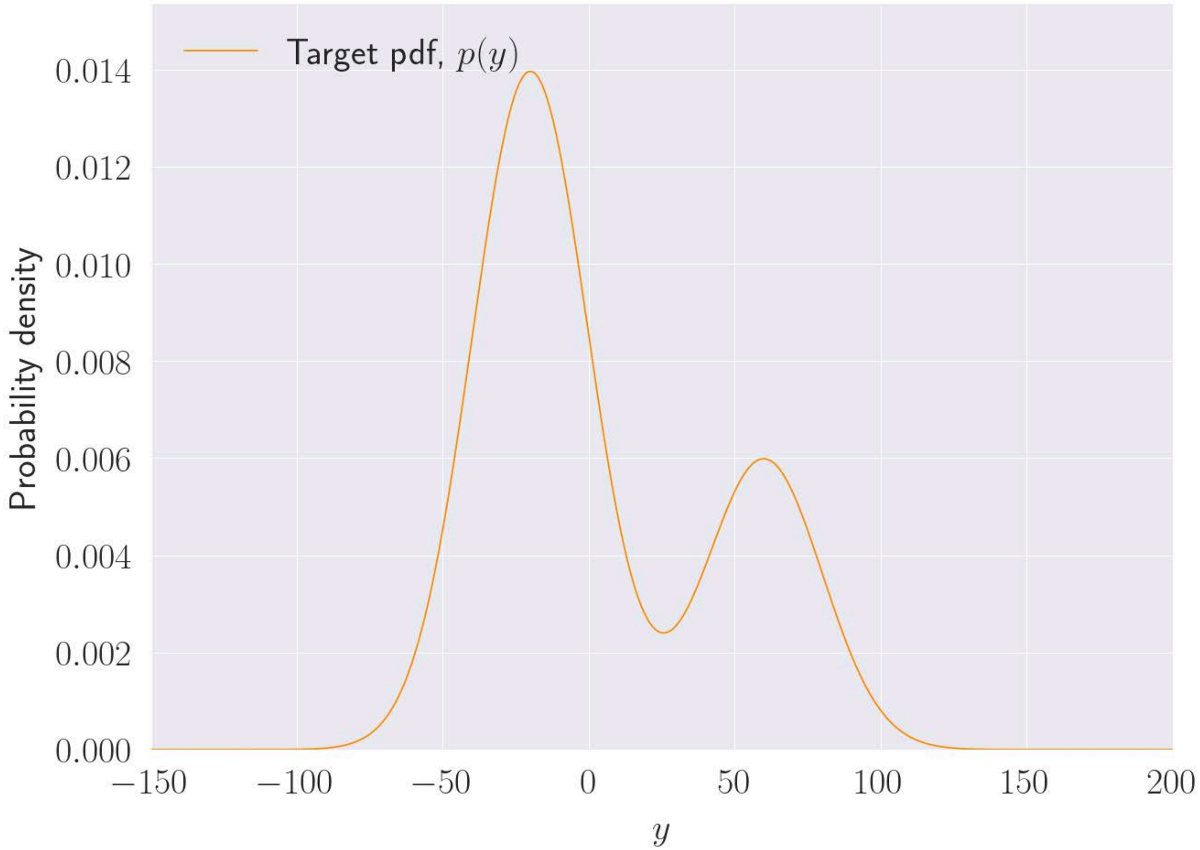

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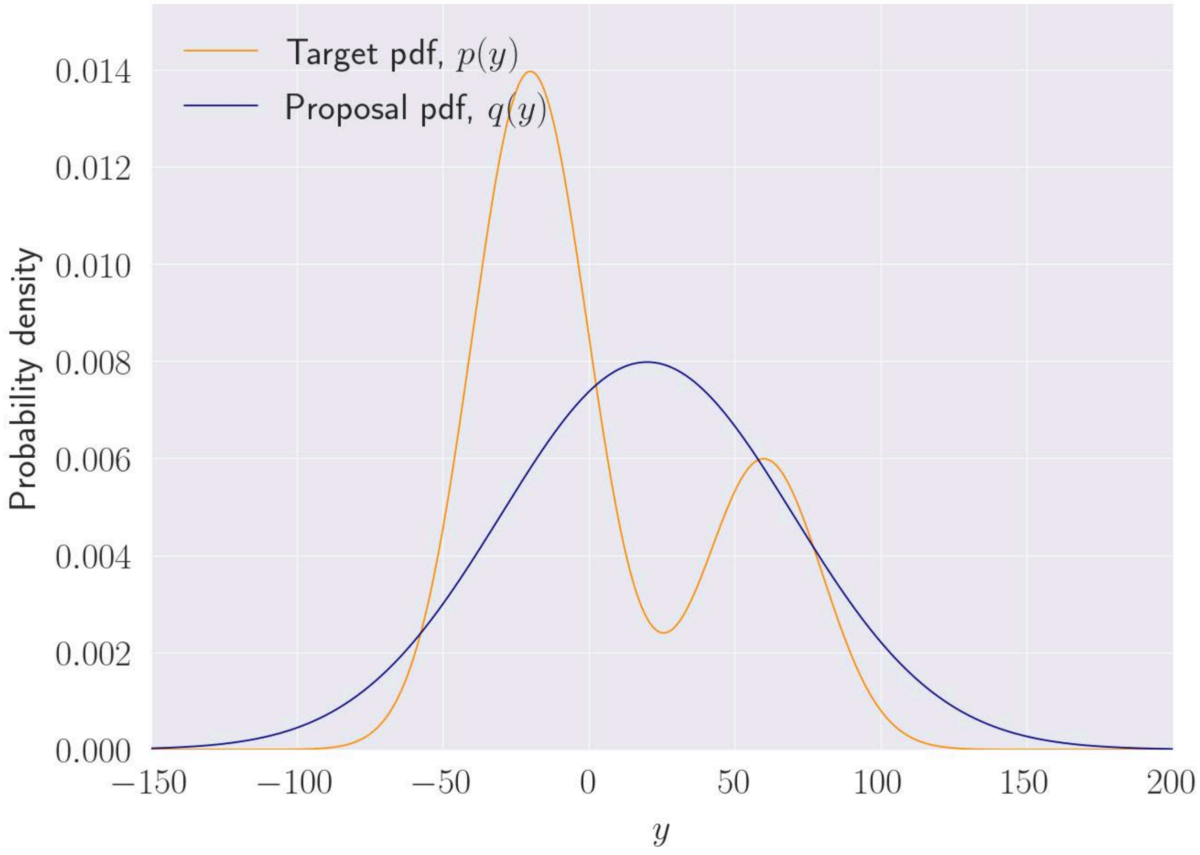

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Importance Sampling:

For $\ell = 1, \ldots, L$:
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$\tilde{p}(\cdot)$: Unnormalised target pdf
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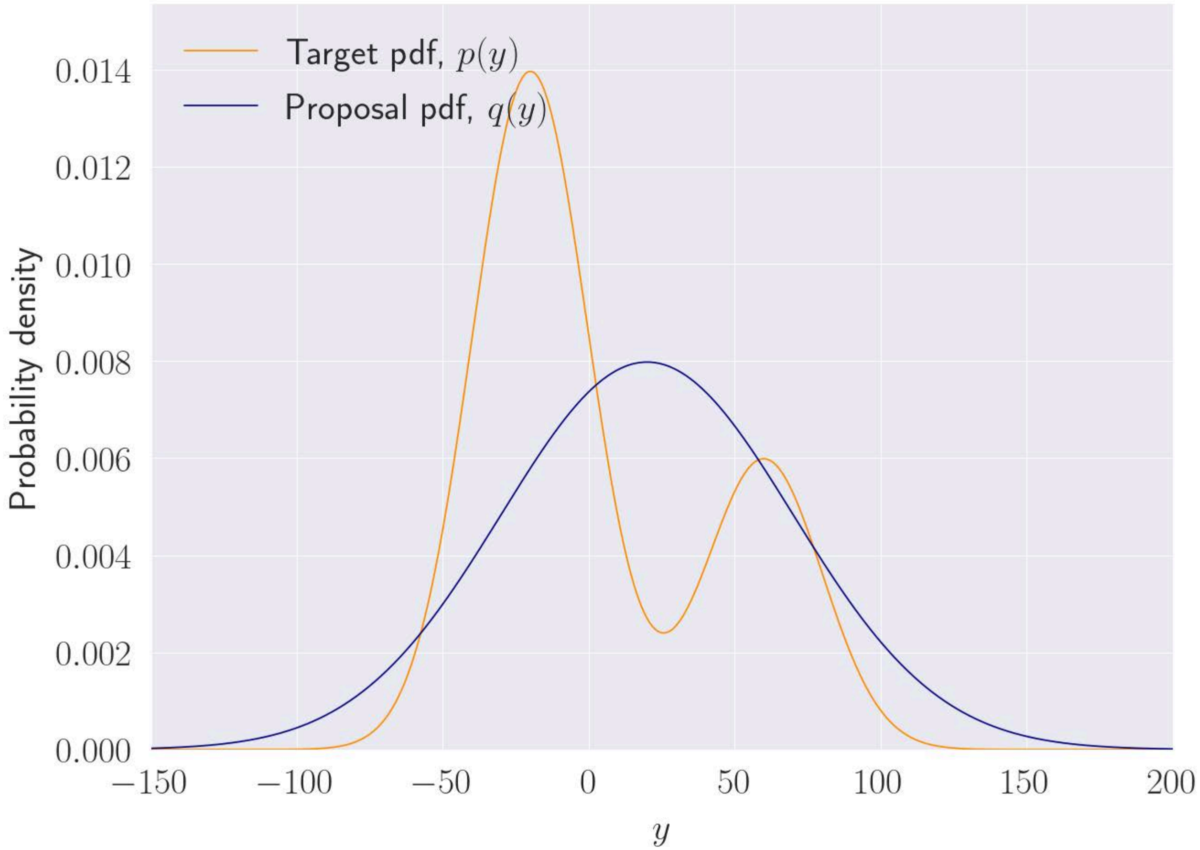

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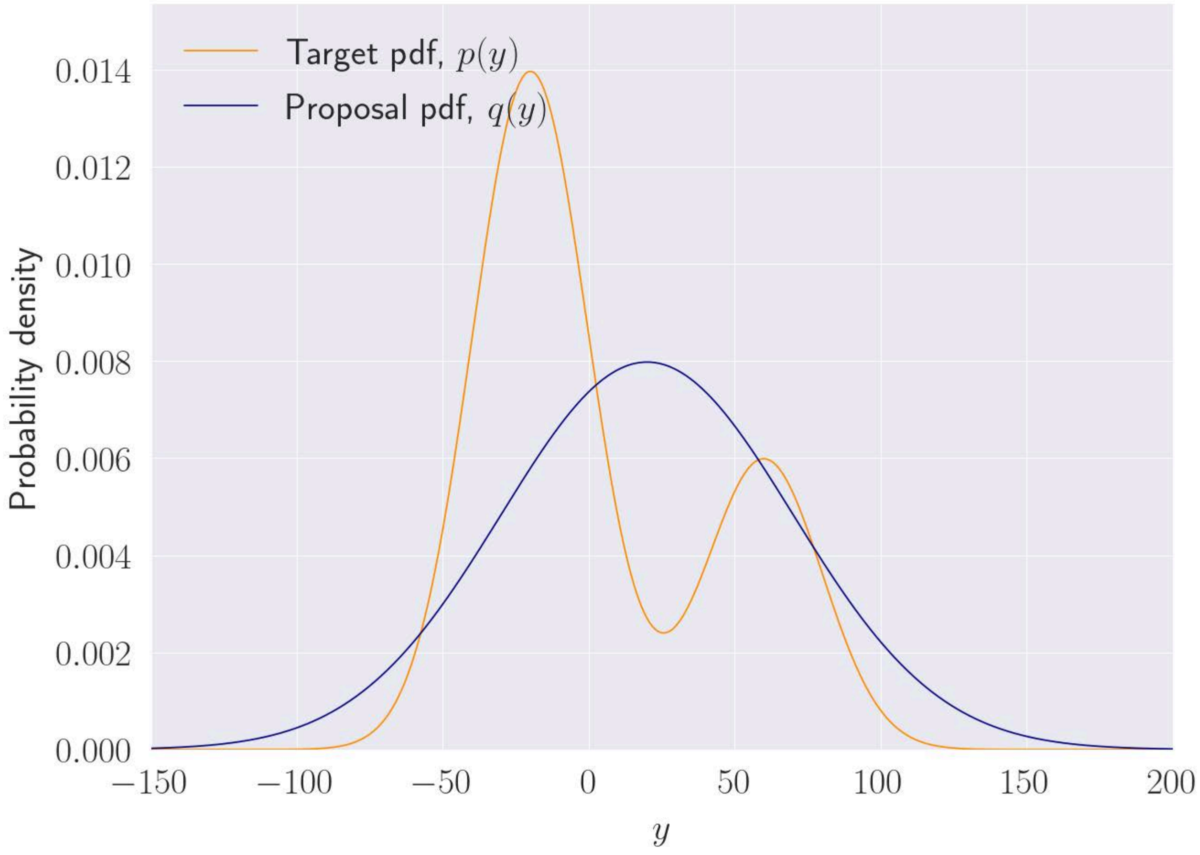

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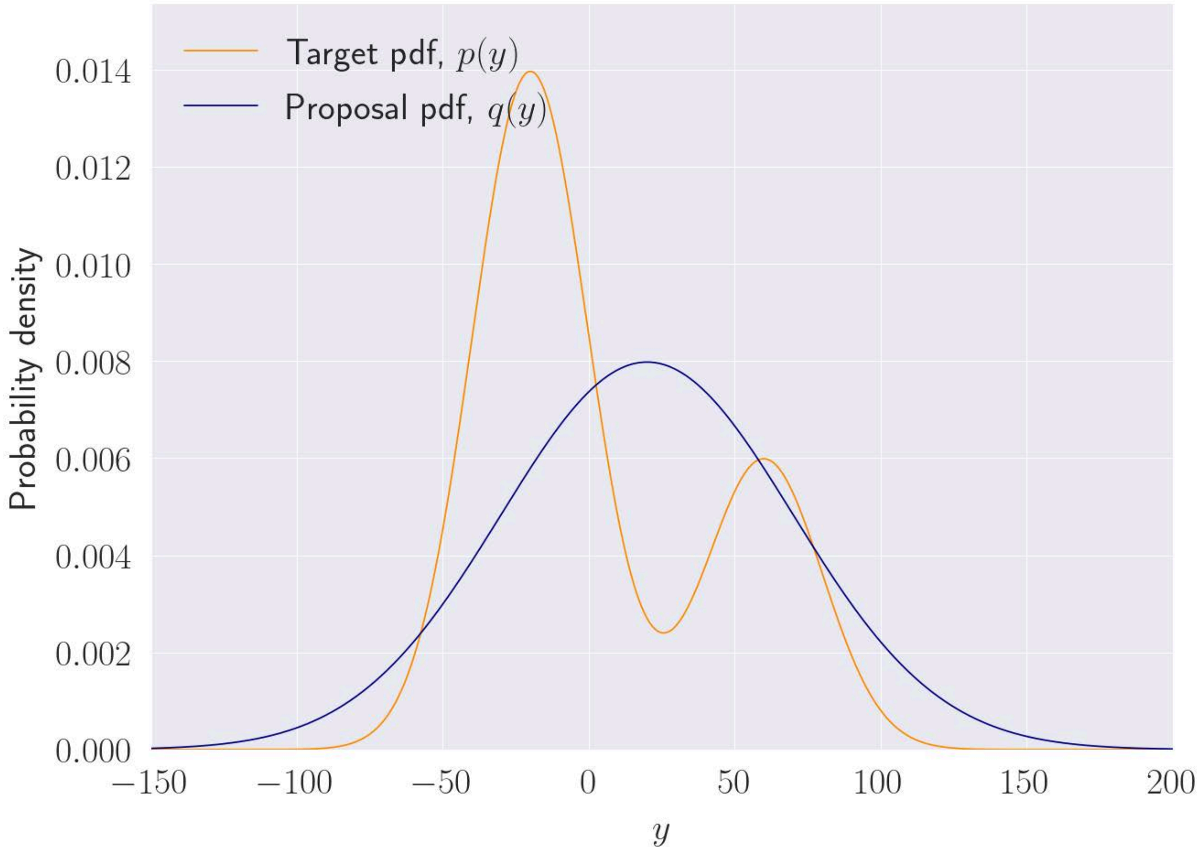

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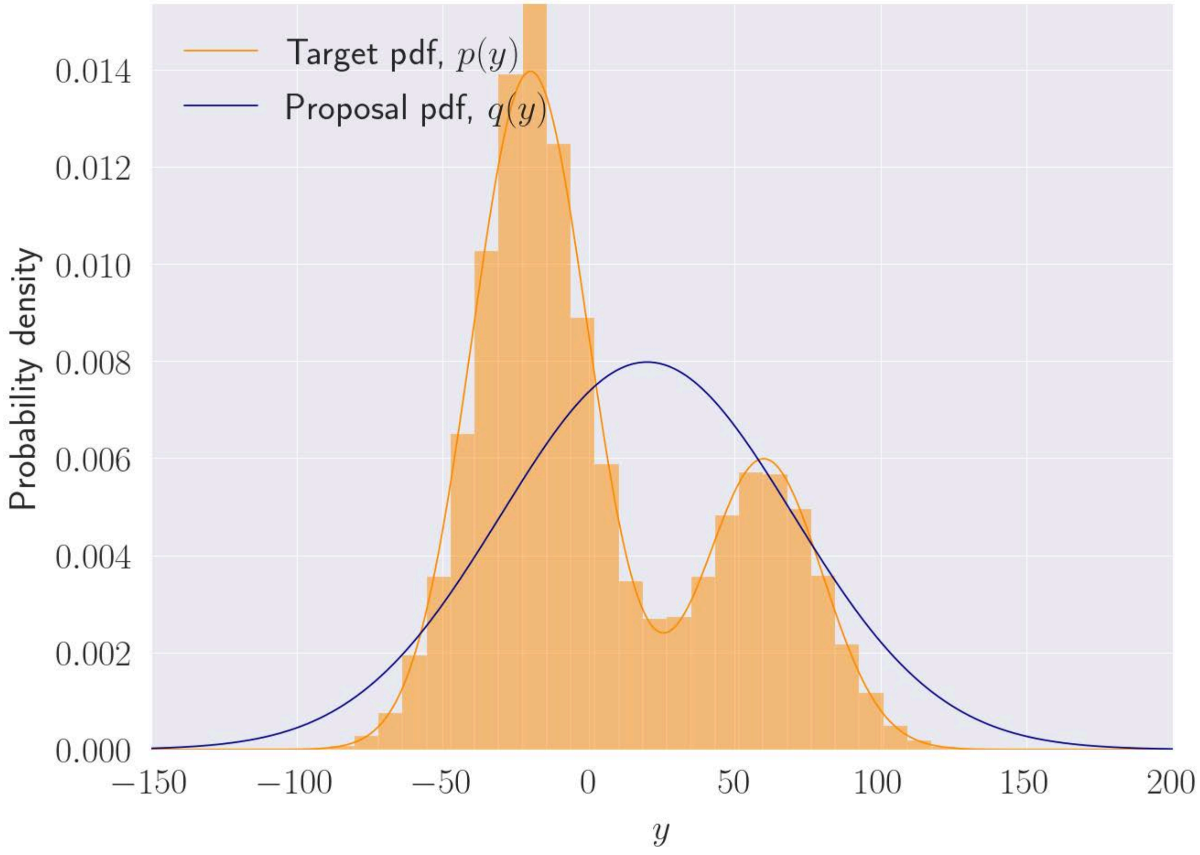

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Sequential Importance Sampling (SIS)

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Sample sequentially from a sequence of target densities

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} \approx \sum_{\ell=1}^{L} \tilde{w}^{(\ell)} \delta_{\tilde{\mathbf{y}}_{1:t}^{(\ell)}}\left(\mathbf{y}_{1:t}\right)$$

$\mathbf{y}_{1:t} \triangleq \left[\mathbf{y}_1^T, \ldots, \mathbf{y}_t^T\right]^T$, $p(\,\cdot\,)$: Target pdf, $\tilde{p}(\,\cdot\,)$: Unnormalised target pdf, $q(\,\cdot\,)$: Proposal distribution

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Target pdf: $$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}}$$

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$$\tilde{\mathbf{y}}_t^{(\ell)} \sim q\left(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)}\right) \Rightarrow \tilde{\mathbf{y}}_{1:t}^{(\ell)} = \left(\tilde{\mathbf{y}}_{1:t-1}^{(\ell)}, \tilde{\mathbf{y}}_t^{(\ell)}\right)$$

$$w_t^{(\ell)} = w_{t-1}^{(\ell)} \frac{\tilde{p}(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}{q(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}, \quad \text{and} \quad \tilde{w}_t^{(\ell)} = w_t^{(\ell)} \Big/ \sum_{\ell=1}^{L} w_t^{(\ell)}$$

$\mathbf{y}_{1:t} \triangleq \left[\mathbf{y}_1^T, \dots, \mathbf{y}_t^T\right]^T$, $p(\cdot)$: Target pdf, $\tilde{p}(\cdot)$: Unnormalised target pdf, $q(\cdot)$: Proposal distribution

Target pdf:

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} = \frac{\frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} q(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} q(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}}$$

Sequential Importance Sampling (SIS)

Sample sequentially from a sequence of target densities

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} \approx \sum_{\ell=1}^{L} \tilde{w}^{(\ell)} \delta_{\tilde{\mathbf{y}}_{1:t}^{(\ell)}}(\mathbf{y}_{1:t})$$

where

$$\tilde{\mathbf{y}}_t^{(\ell)} \sim q\left(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)}\right) \Rightarrow \tilde{\mathbf{y}}_{1:t}^{(\ell)} = \left(\tilde{\mathbf{y}}_{1:t-1}^{(\ell)}, \tilde{\mathbf{y}}_t^{(\ell)}\right)$$

$$w_t^{(\ell)} = w_{t-1}^{(\ell)} \frac{\tilde{p}(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}{q(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}, \quad \text{and} \quad \tilde{w}_t^{(\ell)} = w_t^{(\ell)} \Big/ \sum_{\ell=1}^{L} w_t^{(\ell)}$$

$\mathbf{y}_{1:t} \triangleq \left[\mathbf{y}_1^T, \ldots, \mathbf{y}_t^T\right]^T$, $p(\cdot)$: Target pdf, $\tilde{p}(\cdot)$: Unnormalised target pdf, $q(\cdot)$: Proposal distribution

Target pdf:

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} = \frac{\frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} q(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} q(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} = \frac{w(\mathbf{y}_{1:t})\, q(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} w(\mathbf{y}_{1:t})\, q(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}}$$

Sequential Importance Sampling (SIS)

Sample sequentially from a sequence of target densities

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} \approx \sum_{\ell=1}^{L} \tilde{w}^{(\ell)} \delta_{\tilde{\mathbf{y}}_{1:t}^{(\ell)}}(\mathbf{y}_{1:t})$$

where

$$\tilde{\mathbf{y}}_t^{(\ell)} \sim q\left(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)}\right) \Rightarrow \tilde{\mathbf{y}}_{1:t}^{(\ell)} = \left(\tilde{\mathbf{y}}_{1:t-1}^{(\ell)}, \tilde{\mathbf{y}}_t^{(\ell)}\right)$$

$$w_t^{(\ell)} = w_{t-1}^{(\ell)} \frac{\tilde{p}(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}{q(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}, \quad \text{and} \quad \tilde{w}_t^{(\ell)} = w_t^{(\ell)} \Big/ \sum_{\ell=1}^{L} w_t^{(\ell)}$$

$\mathbf{y}_{1:t} \triangleq \left[\mathbf{y}_1^T, \dots, \mathbf{y}_t^T\right]^T$, $p(\cdot)$: Target pdf, $\tilde{p}(\cdot)$: Unnormalised target pdf, $q(\cdot)$: Proposal distribution

Target pdf:

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} = \frac{\frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} q(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} q(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} = \frac{w(\mathbf{y}_{1:t})\, q(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} w(\mathbf{y}_{1:t})\, q(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}}$$

Weight function:

$$w(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})}$$

Sequential Importance Sampling (SIS)

Sample sequentially from a sequence of target densities

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} \approx \sum_{\ell=1}^{L} \tilde{w}^{(\ell)} \delta_{\tilde{\mathbf{y}}_{1:t}^{(\ell)}}(\mathbf{y}_{1:t})$$

where

$$\tilde{\mathbf{y}}_t^{(\ell)} \sim q\left(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)}\right) \Rightarrow \tilde{\mathbf{y}}_{1:t}^{(\ell)} = \left(\tilde{\mathbf{y}}_{1:t-1}^{(\ell)}, \tilde{\mathbf{y}}_t^{(\ell)}\right)$$

$$w_t^{(\ell)} = w_{t-1}^{(\ell)} \frac{\tilde{p}(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}{q(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}, \quad \text{and} \quad \tilde{w}_t^{(\ell)} = w_t^{(\ell)} \Big/ \sum_{\ell=1}^{L} w_t^{(\ell)}$$

$\mathbf{y}_{1:t} \triangleq \left[\mathbf{y}_1^T, \ldots, \mathbf{y}_t^T\right]^T$, $p(\cdot)$: Target pdf, $\tilde{p}(\cdot)$: Unnormalised target pdf, $q(\cdot)$: Proposal distribution

Target pdf:

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} = \frac{\frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} q(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} q(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} = \frac{w(\mathbf{y}_{1:t})\, q(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} w(\mathbf{y}_{1:t})\, q(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}}$$

Weight function:

$$w(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})}$$

$$q(\mathbf{y}_{1:t}) = q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1})\, q(\mathbf{y}_{1:t-1})$$

Sequential Importance Sampling (SIS)

Sample sequentially from a sequence of target densities

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} \approx \sum_{\ell=1}^{L} \tilde{w}^{(\ell)} \delta_{\tilde{\mathbf{y}}_{1:t}^{(\ell)}}(\mathbf{y}_{1:t})$$

where

$$\tilde{\mathbf{y}}_t^{(\ell)} \sim q\left(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)}\right) \Rightarrow \tilde{\mathbf{y}}_{1:t}^{(\ell)} = \left(\tilde{\mathbf{y}}_{1:t-1}^{(\ell)}, \tilde{\mathbf{y}}_t^{(\ell)}\right)$$

$$w_t^{(\ell)} = w_{t-1}^{(\ell)} \frac{\tilde{p}(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}{q(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}, \quad \text{and} \quad \tilde{w}_t^{(\ell)} = w_t^{(\ell)} \Big/ \sum_{\ell=1}^{L} w_t^{(\ell)}$$

$\mathbf{y}_{1:t} \triangleq \left[\mathbf{y}_1^T, \ldots, \mathbf{y}_t^T\right]^T$, $p(\cdot)$: Target pdf, $\tilde{p}(\cdot)$: Unnormalised target pdf, $q(\cdot)$: Proposal distribution

Target pdf: $$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} = \frac{\frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} q(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} q(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} = \frac{w(\mathbf{y}_{1:t})\, q(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} w(\mathbf{y}_{1:t})\, q(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}}$$

Weight function: $$w(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} = \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1}) q(\mathbf{y}_{1:t-1})}$$

Sequential Importance Sampling (SIS)

Sample sequentially from a sequence of target densities

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} \approx \sum_{\ell=1}^{L} \tilde{w}^{(\ell)}\, \delta_{\tilde{\mathbf{y}}_{1:t}^{(\ell)}}(\mathbf{y}_{1:t})$$

where

$$\tilde{\mathbf{y}}_t^{(\ell)} \sim q\left(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)}\right) \Rightarrow \tilde{\mathbf{y}}_{1:t}^{(\ell)} = \left(\tilde{\mathbf{y}}_{1:t-1}^{(\ell)}, \tilde{\mathbf{y}}_t^{(\ell)}\right)$$

$$w_t^{(\ell)} = w_{t-1}^{(\ell)} \frac{\tilde{p}(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}{q(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}, \quad \text{and} \quad \tilde{w}_t^{(\ell)} = w_t^{(\ell)} \Big/ \sum_{\ell=1}^{L} w_t^{(\ell)}$$

$\mathbf{y}_{1:t} \triangleq \left[\mathbf{y}_1^T, \ldots, \mathbf{y}_t^T\right]^T$, $p(\,\cdot\,)$: Target pdf, $\tilde{p}(\,\cdot\,)$: Unnormalised target pdf, $q(\,\cdot\,)$: Proposal distribution

Target pdf:

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} = \frac{\frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} q(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} q(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} = \frac{w(\mathbf{y}_{1:t})\, q(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} w(\mathbf{y}_{1:t})\, q(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}}$$

Weight function:

$$w(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} = \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1}) q(\mathbf{y}_{1:t-1})} = \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1}) q(\mathbf{y}_{1:t-1})} \frac{\tilde{p}(\mathbf{y}_{1:t-1})}{\tilde{p}(\mathbf{y}_{1:t-1})}$$

Sequential Importance Sampling (SIS)

Sample sequentially from a sequence of target densities

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} \approx \sum_{\ell=1}^{L} \tilde{w}^{(\ell)} \delta_{\tilde{\mathbf{y}}_{1:t}^{(\ell)}}(\mathbf{y}_{1:t})$$

where

$$\tilde{\mathbf{y}}_t^{(\ell)} \sim q\left(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)}\right) \Rightarrow \tilde{\mathbf{y}}_{1:t}^{(\ell)} = \left(\tilde{\mathbf{y}}_{1:t-1}^{(\ell)}, \tilde{\mathbf{y}}_t^{(\ell)}\right)$$

$$w_t^{(\ell)} = w_{t-1}^{(\ell)} \frac{\tilde{p}(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}{q(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}, \quad \text{and} \quad \tilde{w}_t^{(\ell)} = w_t^{(\ell)} \Big/ \sum_{\ell=1}^{L} w_t^{(\ell)}$$

$\mathbf{y}_{1:t} \triangleq \left[\mathbf{y}_1^T, \ldots, \mathbf{y}_t^T\right]^T$, $p(\cdot)$: Target pdf, $\tilde{p}(\cdot)$: Unnormalised target pdf, $q(\cdot)$: Proposal distribution

Target pdf:

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} = \frac{\frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} q(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} q(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} = \frac{w(\mathbf{y}_{1:t})\, q(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} w(\mathbf{y}_{1:t})\, q(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}}$$

Weight function:

$$w(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} = \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1}) q(\mathbf{y}_{1:t-1})} = \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1}) q(\mathbf{y}_{1:t-1})} \frac{\tilde{p}(\mathbf{y}_{1:t-1})}{\tilde{p}(\mathbf{y}_{1:t-1})} = \frac{\tilde{p}(\mathbf{y}_{1:t-1})}{q(\mathbf{y}_{1:t-1})} \frac{\tilde{p}(\mathbf{y}_{1:t})}{\tilde{p}(\mathbf{y}_{1:t-1})\, q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1})}$$

Sequential Importance Sampling (SIS)

University of Southampton

Sample sequentially from a sequence of target densities

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} \approx \sum_{\ell=1}^{L} \tilde{w}^{(\ell)} \delta_{\tilde{\mathbf{y}}_{1:t}^{(\ell)}}(\mathbf{y}_{1:t})$$

where

$$\tilde{\mathbf{y}}_t^{(\ell)} \sim q\left(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)}\right) \Rightarrow \tilde{\mathbf{y}}_{1:t}^{(\ell)} = \left(\tilde{\mathbf{y}}_{1:t-1}^{(\ell)}, \tilde{\mathbf{y}}_t^{(\ell)}\right)$$

$$w_t^{(\ell)} = w_{t-1}^{(\ell)} \frac{\tilde{p}(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}{q(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}, \quad \text{and} \quad \tilde{w}_t^{(\ell)} = w_t^{(\ell)} \Big/ \sum_{\ell=1}^{L} w_t^{(\ell)}$$

$\mathbf{y}_{1:t} \triangleq \left[\mathbf{y}_1^T, \dots, \mathbf{y}_t^T\right]^T$, $p(\cdot)$: Target pdf, $\tilde{p}(\cdot)$: Unnormalised target pdf, $q(\cdot)$: Proposal distribution

Target pdf:

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} = \frac{\frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} q(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} q(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} = \frac{w(\mathbf{y}_{1:t})\, q(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} w(\mathbf{y}_{1:t})\, q(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}}$$

Weight function:

$$w(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} = \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1}) q(\mathbf{y}_{1:t-1})} = \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1}) q(\mathbf{y}_{1:t-1})} \frac{\tilde{p}(\mathbf{y}_{1:t-1})}{\tilde{p}(\mathbf{y}_{1:t-1})} = \underbrace{\frac{\tilde{p}(\mathbf{y}_{1:t-1})}{q(\mathbf{y}_{1:t-1})}}_{w(\mathbf{y}_{1:t-1})} \frac{\tilde{p}(\mathbf{y}_{1:t})}{\tilde{p}(\mathbf{y}_{1:t-1})\, q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1})}$$

Sequential Importance Sampling (SIS)

Sample sequentially from a sequence of target densities

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} \approx \sum_{\ell=1}^{L} \tilde{w}^{(\ell)} \delta_{\tilde{\mathbf{y}}_{1:t}^{(\ell)}}(\mathbf{y}_{1:t})$$

where

$$\tilde{\mathbf{y}}_t^{(\ell)} \sim q\left(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)}\right) \Rightarrow \tilde{\mathbf{y}}_{1:t}^{(\ell)} = \left(\tilde{\mathbf{y}}_{1:t-1}^{(\ell)}, \tilde{\mathbf{y}}_t^{(\ell)}\right)$$

$$w_t^{(\ell)} = w_{t-1}^{(\ell)} \frac{\tilde{p}(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}{q(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}, \text{ and } \tilde{w}_t^{(\ell)} = w_t^{(\ell)} \Big/ \sum_{\ell=1}^{L} w_t^{(\ell)}$$

$\mathbf{y}_{1:t} \triangleq \left[\mathbf{y}_1^T, \ldots, \mathbf{y}_t^T\right]^T$, $p(\cdot)$: Target pdf, $\tilde{p}(\cdot)$: Unnormalised target pdf, $q(\cdot)$: Proposal distribution

Target pdf:

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} = \frac{\frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} q(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} q(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} = \frac{w(\mathbf{y}_{1:t})\, q(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} w(\mathbf{y}_{1:t})\, q(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}}$$

Weight function:

$$w(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} = \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1}) q(\mathbf{y}_{1:t-1})} = \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1}) q(\mathbf{y}_{1:t-1})} \frac{\tilde{p}(\mathbf{y}_{1:t-1})}{\tilde{p}(\mathbf{y}_{1:t-1})} = \frac{\tilde{p}(\mathbf{y}_{1:t-1})}{q(\mathbf{y}_{1:t-1})} \frac{\tilde{p}(\mathbf{y}_{1:t})}{\tilde{p}(\mathbf{y}_{1:t-1})\, q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1})}$$

$$= w(\mathbf{y}_{1:t-1}) \frac{\tilde{p}(\mathbf{y}_{1:t})}{\tilde{p}(\mathbf{y}_{1:t-1})\, q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1})}$$

Sequential Importance Sampling (SIS)

Sample sequentially from a sequence of target densities

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} \approx \sum_{\ell=1}^{L} \tilde{w}^{(\ell)}\, \delta_{\tilde{\mathbf{y}}_{1:t}^{(\ell)}}(\mathbf{y}_{1:t})$$

where

$$\tilde{\mathbf{y}}_t^{(\ell)} \sim q\left(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)}\right) \Rightarrow \tilde{\mathbf{y}}_{1:t}^{(\ell)} = \left(\tilde{\mathbf{y}}_{1:t-1}^{(\ell)}, \tilde{\mathbf{y}}_t^{(\ell)}\right)$$

$$w_t^{(\ell)} = w_{t-1}^{(\ell)} \frac{\tilde{p}(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}{q(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}, \quad \text{and} \quad \tilde{w}_t^{(\ell)} = w_t^{(\ell)} \Big/ \sum_{\ell=1}^{L} w_t^{(\ell)}$$

$\mathbf{y}_{1:t} \triangleq \left[\mathbf{y}_1^T, \ldots, \mathbf{y}_t^T\right]^T$, $p(\cdot)$: Target pdf, $\tilde{p}(\cdot)$: Unnormalised target pdf, $q(\cdot)$: Proposal distribution

Target pdf:

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} = \frac{\frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} q(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} q(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} = \frac{w(\mathbf{y}_{1:t})\, q(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} w(\mathbf{y}_{1:t})\, q(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}}$$

Weight function:

$$w(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} = \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1}) q(\mathbf{y}_{1:t-1})} = \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1}) q(\mathbf{y}_{1:t-1})} \frac{\tilde{p}(\mathbf{y}_{1:t-1})}{\tilde{p}(\mathbf{y}_{1:t-1})} = \frac{\tilde{p}(\mathbf{y}_{1:t-1})}{q(\mathbf{y}_{1:t-1})} \frac{\tilde{p}(\mathbf{y}_{1:t})}{\tilde{p}(\mathbf{y}_{1:t-1})\, q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1})}$$

$$= w(\mathbf{y}_{1:t-1}) \frac{\tilde{p}(\mathbf{y}_{1:t})}{\tilde{p}(\mathbf{y}_{1:t-1})\, q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1})}$$

Sequential Importance Sampling (SIS)

Sample sequentially from a sequence of target densities

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} \approx \sum_{\ell=1}^{L} \tilde{w}^{(\ell)} \delta_{\tilde{\mathbf{y}}_{1:t}^{(\ell)}}(\mathbf{y}_{1:t})$$

where

$$\tilde{\mathbf{y}}_t^{(\ell)} \sim q\left(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)}\right) \Rightarrow \tilde{\mathbf{y}}_{1:t}^{(\ell)} = \left(\tilde{\mathbf{y}}_{1:t-1}^{(\ell)}, \tilde{\mathbf{y}}_t^{(\ell)}\right)$$

$$w_t^{(\ell)} = w_{t-1}^{(\ell)} \frac{\tilde{p}(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}{q(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}, \quad \text{and} \quad \tilde{w}_t^{(\ell)} = w_t^{(\ell)} \Big/ \sum_{\ell=1}^{L} w_t^{(\ell)}$$

$\mathbf{y}_{1:t} \triangleq \left[\mathbf{y}_1^T, \ldots, \mathbf{y}_t^T\right]^T$, $p(\cdot)$: Target pdf, $\tilde{p}(\cdot)$: Unnormalised target pdf, $q(\cdot)$: Proposal distribution

Target pdf:

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} = \frac{\frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} q(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} q(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} = \frac{w(\mathbf{y}_{1:t})\, q(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} w(\mathbf{y}_{1:t})\, q(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}}$$

Weight function:

$$w(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} = \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1}) q(\mathbf{y}_{1:t-1})} = \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1}) q(\mathbf{y}_{1:t-1})} \frac{\tilde{p}(\mathbf{y}_{1:t-1})}{\tilde{p}(\mathbf{y}_{1:t-1})} = \frac{\tilde{p}(\mathbf{y}_{1:t-1})}{q(\mathbf{y}_{1:t-1})} \frac{\tilde{p}(\mathbf{y}_{1:t})}{\tilde{p}(\mathbf{y}_{1:t-1})\, q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1})}$$

$$= w(\mathbf{y}_{1:t-1}) \frac{\tilde{p}(\mathbf{y}_{1:t})}{\tilde{p}(\mathbf{y}_{1:t-1})\, q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1})} = w(\mathbf{y}_{1:t-1}) \frac{\tilde{p}(\mathbf{y}_0) \prod_{\tau=1}^{t} \tilde{p}(\mathbf{y}_\tau \,|\, \mathbf{y}_{1:\tau-1})}{\tilde{p}(\mathbf{y}_0) \prod_{\tau=1}^{t-1} \tilde{p}(\mathbf{y}_\tau \,|\, \mathbf{y}_{1:\tau-1})\, q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1})}$$

Sequential Importance Sampling (SIS)

Sample sequentially from a sequence of target densities

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} \approx \sum_{\ell=1}^{L} \tilde{w}^{(\ell)}\, \delta_{\tilde{\mathbf{y}}_{1:t}^{(\ell)}}(\mathbf{y}_{1:t})$$

where

$$\tilde{\mathbf{y}}_t^{(\ell)} \sim q\left(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)}\right) \Rightarrow \tilde{\mathbf{y}}_{1:t}^{(\ell)} = \left(\tilde{\mathbf{y}}_{1:t-1}^{(\ell)}, \tilde{\mathbf{y}}_t^{(\ell)}\right)$$

$$w_t^{(\ell)} = w_{t-1}^{(\ell)} \frac{\tilde{p}(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}{q(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}, \quad \text{and} \quad \tilde{w}_t^{(\ell)} = w_t^{(\ell)} \Big/ \sum_{\ell=1}^{L} w_t^{(\ell)}$$

$\mathbf{y}_{1:t} \triangleq \left[\mathbf{y}_1^T, \ldots, \mathbf{y}_t^T\right]^T$, $p(\cdot)$: Target pdf, $\tilde{p}(\cdot)$: Unnormalised target pdf, $q(\cdot)$: Proposal distribution

Target pdf:

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} = \frac{\frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} q(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} q(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} = \frac{w(\mathbf{y}_{1:t})\, q(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} w(\mathbf{y}_{1:t})\, q(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}}$$

Weight function:

$$w(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} = \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1}) q(\mathbf{y}_{1:t-1})} = \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1}) q(\mathbf{y}_{1:t-1})} \frac{\tilde{p}(\mathbf{y}_{1:t-1})}{\tilde{p}(\mathbf{y}_{1:t-1})} = \frac{\tilde{p}(\mathbf{y}_{1:t-1})}{q(\mathbf{y}_{1:t-1})} \frac{\tilde{p}(\mathbf{y}_{1:t})}{\tilde{p}(\mathbf{y}_{1:t-1})\, q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1})}$$

$$= w(\mathbf{y}_{1:t-1}) \frac{\tilde{p}(\mathbf{y}_{1:t})}{\tilde{p}(\mathbf{y}_{1:t-1})\, q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1})} = w(\mathbf{y}_{1:t-1}) \frac{\tilde{p}(\mathbf{y}_0) \prod_{\tau=1}^{t} \tilde{p}(\mathbf{y}_\tau \,|\, \mathbf{y}_{1:\tau-1})}{\tilde{p}(\mathbf{y}_0) \prod_{\tau=1}^{t-1} \tilde{p}(\mathbf{y}_\tau \,|\, \mathbf{y}_{1:\tau-1})\, q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1})}$$

Sequential Importance Sampling (SIS)

Sample sequentially from a sequence of target densities

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} \approx \sum_{\ell=1}^{L} \tilde{w}^{(\ell)} \delta_{\tilde{\mathbf{y}}_{1:t}^{(\ell)}}(\mathbf{y}_{1:t})$$

where

$$\tilde{\mathbf{y}}_t^{(\ell)} \sim q\left(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)}\right) \Rightarrow \tilde{\mathbf{y}}_{1:t}^{(\ell)} = \left(\tilde{\mathbf{y}}_{1:t-1}^{(\ell)}, \tilde{\mathbf{y}}_t^{(\ell)}\right)$$

$$w_t^{(\ell)} = w_{t-1}^{(\ell)} \frac{\tilde{p}(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}{q(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}, \quad \text{and} \quad \tilde{w}_t^{(\ell)} = w_t^{(\ell)} \Big/ \sum_{\ell=1}^{L} w_t^{(\ell)}$$

$\mathbf{y}_{1:t} \triangleq \left[\mathbf{y}_1^T, \dots, \mathbf{y}_t^T\right]^T$, $p(\cdot)$: Target pdf, $\tilde{p}(\cdot)$: Unnormalised target pdf, $q(\cdot)$: Proposal distribution

Target pdf:

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} = \frac{\frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} q(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} q(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} = \frac{w(\mathbf{y}_{1:t})\, q(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} w(\mathbf{y}_{1:t})\, q(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}}$$

Weight function:

$$w(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_{1:t})} = \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1}) q(\mathbf{y}_{1:t-1})} = \frac{\tilde{p}(\mathbf{y}_{1:t})}{q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1}) q(\mathbf{y}_{1:t-1})} \frac{\tilde{p}(\mathbf{y}_{1:t-1})}{\tilde{p}(\mathbf{y}_{1:t-1})} = \frac{\tilde{p}(\mathbf{y}_{1:t-1})}{q(\mathbf{y}_{1:t-1})} \frac{\tilde{p}(\mathbf{y}_{1:t})}{\tilde{p}(\mathbf{y}_{1:t-1})\, q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1})}$$

$$= w(\mathbf{y}_{1:t-1}) \frac{\tilde{p}(\mathbf{y}_{1:t})}{\tilde{p}(\mathbf{y}_{1:t-1})\, q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1})} = w(\mathbf{y}_{1:t-1}) \frac{\tilde{p}(\mathbf{y}_0) \prod_{\tau=1}^{t} \tilde{p}(\mathbf{y}_\tau \,|\, \mathbf{y}_{1:\tau-1})}{\tilde{p}(\mathbf{y}_0) \prod_{\tau=1}^{t-1} \tilde{p}(\mathbf{y}_\tau \,|\, \mathbf{y}_{1:\tau-1})\, q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1})} = w(\mathbf{y}_{1:t-1}) \frac{\tilde{p}(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1})}{q(\mathbf{y}_t \,|\, \mathbf{y}_{1:t-1})}$$

SIS for Filtering

Sample sequentially from a sequence of target densities

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} \approx \sum_{\ell=1}^{L} \tilde{w}^{(\ell)}\, \delta_{\tilde{\mathbf{y}}_{1:t}^{(\ell)}}(\mathbf{y}_{1:t})$$

where

$$\tilde{\mathbf{y}}_t^{(\ell)} \sim q\left(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)}\right) \Rightarrow \tilde{\mathbf{y}}_{1:t}^{(\ell)} = \left(\tilde{\mathbf{y}}_{1:t-1}^{(\ell)}, \tilde{\mathbf{y}}_t^{(\ell)}\right)$$

$$w_t^{(\ell)} = w_{t-1}^{(\ell)} \frac{\tilde{p}(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}{q(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}, \quad \text{and} \quad \tilde{w}_t^{(\ell)} = w_t^{(\ell)} \Big/ \sum_{\ell=1}^{L} w_t^{(\ell)}$$

$\mathbf{y}_{1:t} \triangleq \left[\mathbf{y}_1^T, \ldots, \mathbf{y}_t^T\right]^T$, $p(\cdot)$: Target pdf, $\tilde{p}(\cdot)$: Unnormalised target pdf, $q(\cdot)$: Proposal distribution

Example: Filtering

Posterior pdf: $$p(\mathbf{z}_{1:t} \,|\, \mathbf{x}_{1:t}) = \frac{p(\mathbf{x}_{1:t} \,|\, \mathbf{z}_{1:t})\, p(\mathbf{z}_{1:t})}{\int_{\mathcal{Z}} p(\mathbf{x}_{1:t} \,|\, \mathbf{z}_{1:t})\, p(\mathbf{z}_{1:t})\, d\mathbf{z}_{1:t}} \approx \sum_{\ell=1}^{L} \tilde{w}_t^{(\ell)}\, \delta_{\tilde{\mathbf{z}}_{1:t}^{(\ell)}}(\mathbf{z}_{1:t})$$

SIS for Filtering

Sample sequentially from a sequence of target densities

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} \approx \sum_{\ell=1}^{L} \tilde{w}^{(\ell)} \delta_{\tilde{\mathbf{y}}_{1:t}^{(\ell)}}(\mathbf{y}_{1:t})$$

where

$$\tilde{\mathbf{y}}_t^{(\ell)} \sim q\left(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)}\right) \Rightarrow \tilde{\mathbf{y}}_{1:t}^{(\ell)} = \left(\tilde{\mathbf{y}}_{1:t-1}^{(\ell)}, \tilde{\mathbf{y}}_t^{(\ell)}\right)$$

$$w_t^{(\ell)} = w_{t-1}^{(\ell)} \frac{\tilde{p}(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}{q(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}, \quad \text{and} \quad \tilde{w}_t^{(\ell)} = w_t^{(\ell)} \Big/ \sum_{\ell=1}^{L} w_t^{(\ell)}$$

$\mathbf{y}_{1:t} \triangleq \left[\mathbf{y}_1^T, \ldots, \mathbf{y}_t^T\right]^T$, $p(\cdot)$: Target pdf, $\tilde{p}(\cdot)$: Unnormalised target pdf, $q(\cdot)$: Proposal distribution

Example: Filtering

Posterior pdf:

$$p(\mathbf{z}_{1:t} \,|\, \mathbf{x}_{1:t}) = \frac{p(\mathbf{x}_{1:t} \,|\, \mathbf{z}_{1:t})\, p(\mathbf{z}_{1:t})}{\int_{\mathcal{Z}} p(\mathbf{x}_{1:t} \,|\, \mathbf{z}_{1:t})\, p(\mathbf{z}_{1:t})\, d\mathbf{z}_{1:t}} \approx \sum_{\ell=1}^{L} \tilde{w}_t^{(\ell)} \delta_{\tilde{\mathbf{z}}_{1:t}^{(\ell)}}(\mathbf{z}_{1:t})$$

Importance samples:

$$\tilde{\mathbf{z}}_t^{(\ell)} \sim q\left(\mathbf{z}_t \,|\, \tilde{\mathbf{z}}_{1:t-1}^{(\ell)}, \mathbf{x}_{1:t}\right)$$

SIS for Filtering

Sample sequentially from a sequence of target densities

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} \approx \sum_{\ell=1}^{L} \tilde{w}^{(\ell)}\, \delta_{\tilde{\mathbf{y}}_{1:t}^{(\ell)}}(\mathbf{y}_{1:t})$$

where

$$\tilde{\mathbf{y}}_t^{(\ell)} \sim q\left(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)}\right) \Rightarrow \tilde{\mathbf{y}}_{1:t}^{(\ell)} = \left(\tilde{\mathbf{y}}_{1:t-1}^{(\ell)}, \tilde{\mathbf{y}}_t^{(\ell)}\right)$$

$$w_t^{(\ell)} = w_{t-1}^{(\ell)} \frac{\tilde{p}(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}{q(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}, \quad \text{and} \quad \tilde{w}_t^{(\ell)} = w_t^{(\ell)} \Big/ \sum_{\ell=1}^{L} w_t^{(\ell)}$$

$\mathbf{y}_{1:t} \triangleq \left[\mathbf{y}_1^T, \ldots, \mathbf{y}_t^T\right]^T$, $p(\,\cdot\,)$: Target pdf, $\tilde{p}(\,\cdot\,)$: Unnormalised target pdf, $q(\,\cdot\,)$: Proposal distribution

Example: Filtering

Posterior pdf: $$p(\mathbf{z}_{1:t} \,|\, \mathbf{x}_{1:t}) = \frac{p(\mathbf{x}_{1:t} \,|\, \mathbf{z}_{1:t})\, p(\mathbf{z}_{1:t})}{\int_{\mathcal{Z}} p(\mathbf{x}_{1:t} \,|\, \mathbf{z}_{1:t})\, p(\mathbf{z}_{1:t})\, d\mathbf{z}_{1:t}} \approx \sum_{\ell=1}^{L} \tilde{w}_t^{(\ell)}\, \delta_{\tilde{\mathbf{z}}_{1:t}^{(\ell)}}(\mathbf{z}_{1:t})$$

Importance samples: $$\tilde{\mathbf{z}}_t^{(\ell)} \sim q\left(\mathbf{z}_t \,|\, \tilde{\mathbf{z}}_{1:t-1}^{(\ell)}, \mathbf{x}_{1:t}\right) \text{ and } \tilde{\mathbf{z}}_{1:t}^{(\ell)} = \left(\tilde{\mathbf{z}}_{1:t-1}^{(\ell)}, \tilde{\mathbf{z}}_t^{(\ell)}\right)$$

SIS for Filtering

Sample sequentially from a sequence of target densities

$$p(\mathbf{y}_{1:t}) = \frac{\tilde{p}(\mathbf{y}_{1:t})}{\int_{\mathcal{Y}} \tilde{p}(\mathbf{y}_{1:t})\, d\mathbf{y}_{1:t}} \approx \sum_{\ell=1}^{L} \tilde{w}^{(\ell)} \delta_{\tilde{\mathbf{y}}_{1:t}^{(\ell)}}(\mathbf{y}_{1:t})$$

where

$$\tilde{\mathbf{y}}_t^{(\ell)} \sim q\left(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)}\right) \Rightarrow \tilde{\mathbf{y}}_{1:t}^{(\ell)} = \left(\tilde{\mathbf{y}}_{1:t-1}^{(\ell)}, \tilde{\mathbf{y}}_t^{(\ell)}\right)$$

$$w_t^{(\ell)} = w_{t-1}^{(\ell)} \frac{\tilde{p}(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}{q(\mathbf{y}_t \,|\, \tilde{\mathbf{y}}_{1:t-1}^{(\ell)})}, \quad \text{and} \quad \tilde{w}_t^{(\ell)} = w_t^{(\ell)} \Big/ \sum_{\ell=1}^{L} w_t^{(\ell)}$$

$\mathbf{y}_{1:t} \triangleq \left[\mathbf{y}_1^T, \ldots, \mathbf{y}_t^T\right]^T$, $p(\,\cdot\,)$: Target pdf, $\tilde{p}(\,\cdot\,)$: Unnormalised target pdf, $q(\,\cdot\,)$: Proposal distribution

Example: Filtering

Posterior pdf:

$$p(\mathbf{z}_{1:t} \,|\, \mathbf{x}_{1:t}) = \frac{p(\mathbf{x}_{1:t} \,|\, \mathbf{z}_{1:t})\, p(\mathbf{z}_{1:t})}{\int_{\mathcal{Z}} p(\mathbf{x}_{1:t} \,|\, \mathbf{z}_{1:t})\, p(\mathbf{z}_{1:t})\, d\mathbf{z}_{1:t}} \approx \sum_{\ell=1}^{L} \tilde{w}_t^{(\ell)} \delta_{\tilde{\mathbf{z}}_{1:t}^{(\ell)}}(\mathbf{z}_{1:t})$$

Importance samples:

$$\tilde{\mathbf{z}}_t^{(\ell)} \sim q\left(\mathbf{z}_t \,|\, \tilde{\mathbf{z}}_{1:t-1}^{(\ell)}, \mathbf{x}_{1:t}\right) \text{ and } \tilde{\mathbf{z}}_{1:t}^{(\ell)} = \left(\tilde{\mathbf{z}}_{1:t-1}^{(\ell)}, \tilde{\mathbf{z}}_t^{(\ell)}\right)$$

Importance weights:

$$w_t^{(\ell)} = w_{t-1}^{(\ell)} \frac{p(\mathbf{x}_t \,|\, \tilde{\mathbf{z}}_{1:t}^{(\ell)})\, p(\tilde{\mathbf{z}}_t^{(\ell)} \,|\, \tilde{\mathbf{z}}_{1:t-1}^{(\ell)})}{q(\tilde{\mathbf{z}}_t^{(\ell)} \,|\, \tilde{\mathbf{z}}_{1:t-1}^{(\ell)}, \mathbf{x}_{1:t})},$$

Example: SIS for Filtering

Sequential Importance Sampling for Filtering:

$$p(\mathbf{z}_{1:t} \mid \mathbf{x}_{1:t}) = \frac{p(\mathbf{x}_{1:t} \mid \mathbf{z}_{1:t})\, p(\mathbf{z}_{1:t})}{\int_{\mathcal{Z}} p(\mathbf{x}_{1:t} \mid \mathbf{z}_{1:t})\, p(\mathbf{z}_{1:t})\, d\mathbf{z}_{1:t}} \approx \sum_{\ell=1}^{L} \tilde{w}_t^{(\ell)}\, \delta_{\tilde{\mathbf{z}}_{1:t}^{(\ell)}}\left(\mathbf{z}_{1:t}\right)$$

$$\tilde{\mathbf{z}}_t^{(\ell)} \sim q\left(\mathbf{z} \mid \tilde{\mathbf{z}}_{1:t-1}^{(\ell)}\right) \qquad w_t^{(\ell)} = w_{t-1}^{(\ell)} \frac{p(\mathbf{x}_t \mid \tilde{\mathbf{z}}_{1:t}^{(\ell)})\, p(\tilde{\mathbf{z}}_t^{(\ell)} \mid \tilde{\mathbf{z}}_{1:t-1}^{(\ell)})}{q(\tilde{\mathbf{z}}_t^{(\ell)} \mid \tilde{\mathbf{z}}_{1:t-1}^{(\ell)}, \mathbf{x}_{1:t})},$$

C. A. Naesseth *et al.*, "Elements of Sequential Monte Carlo", *in Foundations and Trends in Machine Learning*, 2019.

Example: SIS for Filtering

Sequential Importance Sampling for Filtering:

$$p(\mathbf{z}_{1:t} \mid \mathbf{x}_{1:t}) = \frac{p(\mathbf{x}_{1:t} \mid \mathbf{z}_{1:t})\, p(\mathbf{z}_{1:t})}{\int_{\mathcal{Z}} p(\mathbf{x}_{1:t} \mid \mathbf{z}_{1:t})\, p(\mathbf{z}_{1:t})\, d\mathbf{z}_{1:t}} \approx \sum_{\ell=1}^{L} \tilde{w}_t^{(\ell)}\, \delta_{\tilde{\mathbf{z}}_{1:t}^{(\ell)}}\left(\mathbf{z}_{1:t}\right)$$

$$\tilde{\mathbf{z}}_t^{(\ell)} \sim q\left(\mathbf{z} \mid \tilde{\mathbf{z}}_{1:t-1}^{(\ell)}\right) \qquad w_t^{(\ell)} = w_{t-1}^{(\ell)} \frac{p(\mathbf{x}_t \mid \tilde{\mathbf{z}}_{1:t}^{(\ell)})\, p(\tilde{\mathbf{z}}_t^{(\ell)} \mid \tilde{\mathbf{z}}_{1:t-1}^{(\ell)})}{q(\tilde{\mathbf{z}}_t^{(\ell)} \mid \tilde{\mathbf{z}}_{1:t-1}^{(\ell)}, \mathbf{x}_{1:t})},$$

Prior pdf: $z_t = \phi\, z_{t-1} + v_t, \quad v_t \sim \mathcal{N}\left(0, \sigma_z^2\right)$

C. A. Naesseth *et al.*, "Elements of Sequential Monte Carlo", *in Foundations and Trends in Machine Learning*, 2019.

Example: SIS for Filtering

Sequential Importance Sampling for Filtering:

$$p(\mathbf{z}_{1:t} \,|\, \mathbf{x}_{1:t}) = \frac{p(\mathbf{x}_{1:t} \,|\, \mathbf{z}_{1:t})\, p(\mathbf{z}_{1:t})}{\int_{\mathcal{Z}} p(\mathbf{x}_{1:t} \,|\, \mathbf{z}_{1:t})\, p(\mathbf{z}_{1:t})\, d\mathbf{z}_{1:t}} \approx \sum_{\ell=1}^{L} \tilde{w}_t^{(\ell)}\, \delta_{\tilde{\mathbf{z}}_{1:t}^{(\ell)}}\left(\mathbf{z}_{1:t}\right)$$

$$\tilde{\mathbf{z}}_t^{(\ell)} \sim q\left(\mathbf{z} \,|\, \tilde{\mathbf{z}}_{1:t-1}^{(\ell)}\right) \qquad w_t^{(\ell)} = w_{t-1}^{(\ell)} \frac{p(\mathbf{x}_t \,|\, \tilde{\mathbf{z}}_{1:t}^{(\ell)})\, p(\tilde{\mathbf{z}}_t^{(\ell)} \,|\, \tilde{\mathbf{z}}_{1:t-1}^{(\ell)})}{q(\tilde{\mathbf{z}}_t^{(\ell)} \,|\, \tilde{\mathbf{z}}_{1:t-1}^{(\ell)}, \mathbf{x}_{1:t})},$$

Prior pdf: $z_t = \phi\, z_{t-1} + v_t, \quad v_t \sim \mathcal{N}\left(0, \sigma_z^2\right)$ $\Rightarrow$ $p(z_t \,|\, z_{t-1}) = \mathcal{N}\left(z_t \,|\, \phi\, z_{t-1}, \sigma_z^2\right)$

C. A. Naesseth *et al.*, "Elements of Sequential Monte Carlo", *in Foundations and Trends in Machine Learning*, 2019.

Example: SIS for Filtering

Sequential Importance Sampling for Filtering:

$$p(\mathbf{z}_{1:t} \,|\, \mathbf{x}_{1:t}) = \frac{p(\mathbf{x}_{1:t} \,|\, \mathbf{z}_{1:t})\, p(\mathbf{z}_{1:t})}{\int_{\mathcal{Z}} p(\mathbf{x}_{1:t} \,|\, \mathbf{z}_{1:t})\, p(\mathbf{z}_{1:t})\, d\mathbf{z}_{1:t}} \approx \sum_{\ell=1}^{L} \tilde{w}_t^{(\ell)}\, \delta_{\tilde{\mathbf{z}}_{1:t}^{(\ell)}}\left(\mathbf{z}_{1:t}\right)$$

$$\tilde{\mathbf{z}}_t^{(\ell)} \sim q\left(\mathbf{z} \,|\, \tilde{\mathbf{z}}_{1:t-1}^{(\ell)}\right) \qquad w_t^{(\ell)} = w_{t-1}^{(\ell)} \frac{p(\mathbf{x}_t \,|\, \tilde{\mathbf{z}}_{1:t}^{(\ell)})\, p(\tilde{\mathbf{z}}_t^{(\ell)} \,|\, \tilde{\mathbf{z}}_{1:t-1}^{(\ell)})}{q(\tilde{\mathbf{z}}_t^{(\ell)} \,|\, \tilde{\mathbf{z}}_{1:t-1}^{(\ell)}, \mathbf{x}_{1:t})},$$

Prior pdf: $z_t = \phi\, z_{t-1} + v_t, \quad v_t \sim \mathcal{N}\left(0, \sigma_z^2\right)$ $\Rightarrow$ $p(z_t \,|\, z_{t-1}) = \mathcal{N}\left(z_t \,|\, \phi\, z_{t-1}, \sigma_z^2\right)$

Likelihood function: $x_t = \sum_{\tau=1}^{t} \beta^{t-\tau}\, z_\tau + w_t, \quad w_t \sim \mathcal{N}\left(0, \sigma_x^2\right)$

C. A. Naesseth *et al.*, "Elements of Sequential Monte Carlo", *in Foundations and Trends in Machine Learning*, 2019.

Example: SIS for Filtering

Sequential Importance Sampling for Filtering:

$$p(\mathbf{z}_{1:t} \,|\, \mathbf{x}_{1:t}) = \frac{p(\mathbf{x}_{1:t} \,|\, \mathbf{z}_{1:t})\, p(\mathbf{z}_{1:t})}{\int_{\mathcal{Z}} p(\mathbf{x}_{1:t} \,|\, \mathbf{z}_{1:t})\, p(\mathbf{z}_{1:t})\, d\mathbf{z}_{1:t}} \approx \sum_{\ell=1}^{L} \tilde{w}_t^{(\ell)}\, \delta_{\tilde{\mathbf{z}}_{1:t}^{(\ell)}}\left(\mathbf{z}_{1:t}\right)$$

$$\tilde{\mathbf{z}}_t^{(\ell)} \sim q\left(\mathbf{z} \,|\, \tilde{\mathbf{z}}_{1:t-1}^{(\ell)}\right) \qquad w_t^{(\ell)} = w_{t-1}^{(\ell)} \frac{p(\mathbf{x}_t \,|\, \tilde{\mathbf{z}}_{1:t}^{(\ell)})\, p(\tilde{\mathbf{z}}_t^{(\ell)} \,|\, \tilde{\mathbf{z}}_{1:t-1}^{(\ell)})}{q(\tilde{\mathbf{z}}_t^{(\ell)} \,|\, \tilde{\mathbf{z}}_{1:t-1}^{(\ell)}, \mathbf{x}_{1:t})},$$

Prior pdf: $z_t = \phi\, z_{t-1} + v_t, \quad v_t \sim \mathcal{N}\left(0, \sigma_z^2\right) \Rightarrow p(z_t \,|\, z_{t-1}) = \mathcal{N}\left(z_t \,|\, \phi\, z_{t-1}, \sigma_z^2\right)$

Likelihood function: $x_t = \sum_{\tau=1}^{t} \beta^{t-\tau} z_\tau + w_t, \quad w_t \sim \mathcal{N}\left(0, \sigma_x^2\right) \Rightarrow p(x_t \,|\, \mathbf{z}_{1:t}) = \mathcal{N}\left(x_t \,|\, \sum_{\tau=1}^{t} \beta^{t-\tau} z_\tau, \sigma_x^2\right)$

C. A. Naesseth *et al.*, "Elements of Sequential Monte Carlo", *in Foundations and Trends in Machine Learning*, 2019.

Example: SIS for Filtering

Sequential Importance Sampling for Filtering:

$$p(\mathbf{z}_{1:t} \,|\, \mathbf{x}_{1:t}) = \frac{p(\mathbf{x}_{1:t} \,|\, \mathbf{z}_{1:t})\, p(\mathbf{z}_{1:t})}{\int_{\mathcal{Z}} p(\mathbf{x}_{1:t} \,|\, \mathbf{z}_{1:t})\, p(\mathbf{z}_{1:t})\, d\mathbf{z}_{1:t}} \approx \sum_{\ell=1}^{L} \tilde{w}_t^{(\ell)}\, \delta_{\tilde{\mathbf{z}}_{1:t}^{(\ell)}}(\mathbf{z}_{1:t})$$

$$\tilde{\mathbf{z}}_t^{(\ell)} \sim q\left(\mathbf{z} \,|\, \tilde{\mathbf{z}}_{1:t-1}^{(\ell)}\right) \qquad w_t^{(\ell)} = w_{t-1}^{(\ell)} \frac{p(\mathbf{x}_t \,|\, \tilde{\mathbf{z}}_{1:t}^{(\ell)})\, p(\tilde{\mathbf{z}}_t^{(\ell)} \,|\, \tilde{\mathbf{z}}_{1:t-1}^{(\ell)})}{q(\tilde{\mathbf{z}}_t^{(\ell)} \,|\, \tilde{\mathbf{z}}_{1:t-1}^{(\ell)}, \mathbf{x}_{1:t})},$$

Prior pdf: $z_t = \phi\, z_{t-1} + v_t, \quad v_t \sim \mathcal{N}\left(0, \sigma_z^2\right) \Rightarrow p(z_t \,|\, z_{t-1}) = \mathcal{N}\left(z_t \,|\, \phi\, z_{t-1}, \sigma_z^2\right)$

Likelihood function: $x_t = \sum_{\tau=1}^{t} \beta^{t-\tau} z_\tau + w_t, \quad w_t \sim \mathcal{N}\left(0, \sigma_x^2\right) \Rightarrow p(x_t \,|\, \mathbf{z}_{1:t}) = \mathcal{N}\left(x_t \,\middle|\, \sum_{\tau=1}^{t} \beta^{t-\tau} z_\tau, \sigma_x^2\right)$

Proposal distribution: $\tilde{z}_t^{(\ell)} \sim q(z_t \,|\, \mathbf{z}_{1:t-1}) = p(z_t \,|\, z_{t-1}) = \mathcal{N}\left(z_t \,|\, \phi\, z_{t-1}, \sigma_z^2\right)$

C. A. Naesseth *et al.*, "Elements of Sequential Monte Carlo", *in Foundations and Trends in Machine Learning*, 2019.

Example: SIS for Filtering

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Sequential Importance Sampling for Filtering:

$$p(\mathbf{z}_{1:t} \mid \mathbf{x}_{1:t}) = \frac{p(\mathbf{x}_{1:t} \mid \mathbf{z}_{1:t})\, p(\mathbf{z}_{1:t})}{\int_{\mathcal{Z}} p(\mathbf{x}_{1:t} \mid \mathbf{z}_{1:t})\, p(\mathbf{z}_{1:t})\, d\mathbf{z}_{1:t}} \approx \sum_{\ell=1}^{L} \tilde{w}_t^{(\ell)}\, \delta_{\tilde{\mathbf{z}}_{1:t}^{(\ell)}}(\mathbf{z}_{1:t})$$

$$\tilde{\mathbf{z}}_t^{(\ell)} \sim q\left(\mathbf{z} \mid \tilde{\mathbf{z}}_{1:t-1}^{(\ell)}\right) \qquad w_t^{(\ell)} = w_{t-1}^{(\ell)} \frac{p(\mathbf{x}_t \mid \tilde{\mathbf{z}}_{1:t}^{(\ell)})\, p(\tilde{\mathbf{z}}_t^{(\ell)} \mid \tilde{\mathbf{z}}_{1:t-1}^{(\ell)})}{q(\tilde{\mathbf{z}}_t^{(\ell)} \mid \tilde{\mathbf{z}}_{1:t-1}^{(\ell)}, \mathbf{x}_{1:t})},$$

Prior pdf: $z_t = \phi\, z_{t-1} + v_t, \quad v_t \sim \mathcal{N}\left(0, \sigma_z^2\right) \Rightarrow p(z_t \mid z_{t-1}) = \mathcal{N}\left(z_t \mid \phi\, z_{t-1}, \sigma_z^2\right)$

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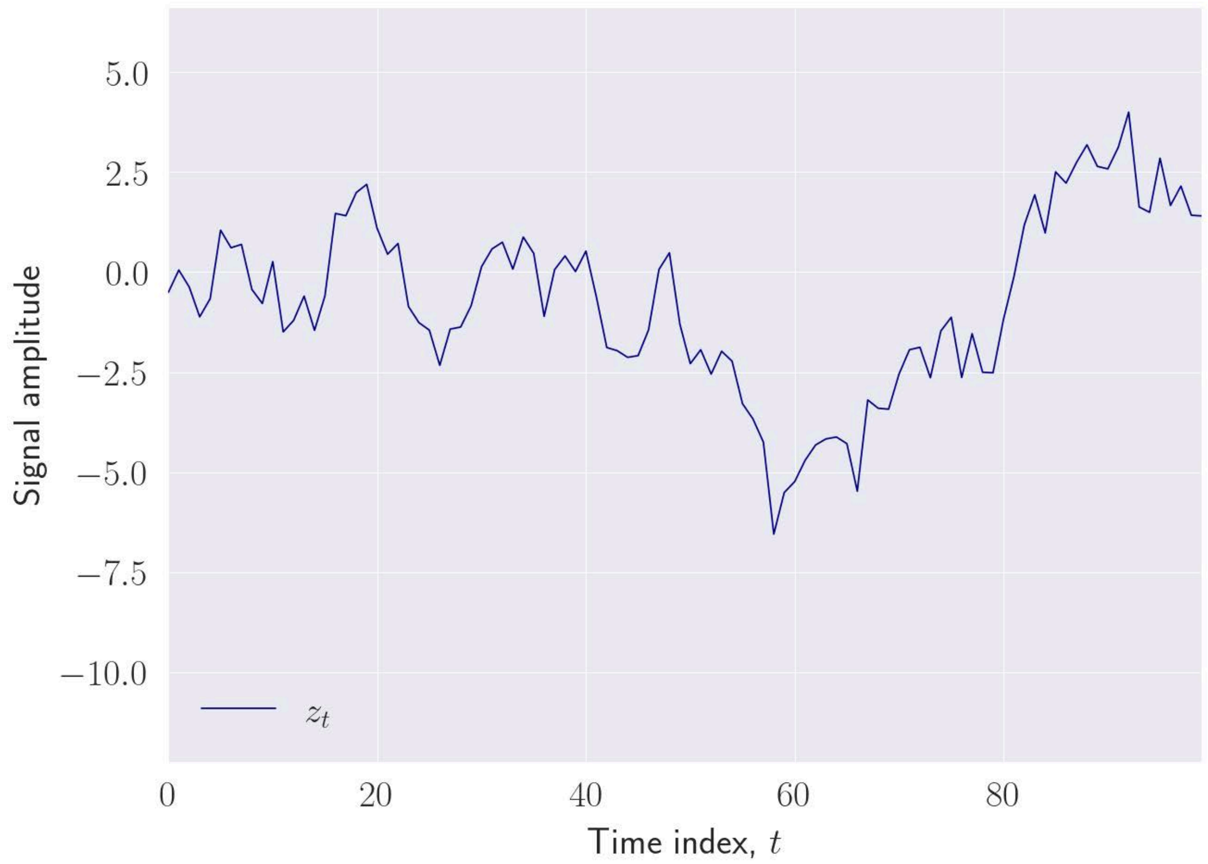
Importance weights: $w_t^{(\ell)} = w_{t-1}^{(\ell)} \frac{p(x_t \mid \tilde{\mathbf{z}}_{1:t}^{(\ell)})\, p(\tilde{\mathbf{z}}_t^{(\ell)} \mid \tilde{\mathbf{z}}_{1:t-1}^{(\ell)})}{q(\tilde{\mathbf{z}}_t^{(\ell)} \mid \tilde{\mathbf{z}}_{1:t-1}^{(\ell)}, \mathbf{x}_{1:t})} = w_{t-1}^{(\ell)}\, p(x_t \mid \tilde{\mathbf{z}}_{1:t}^{(\ell)})$

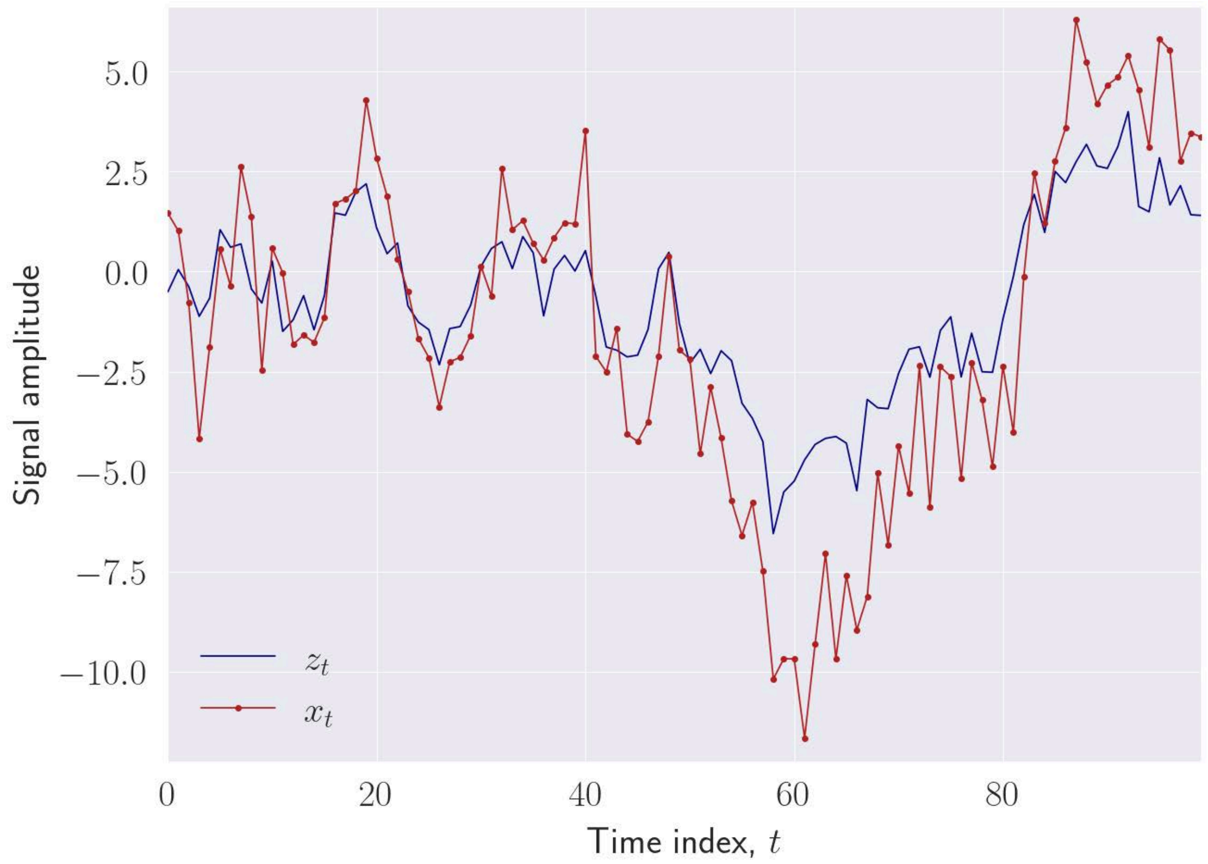
C. A. Naesseth *et al.*, "Elements of Sequential Monte Carlo", *in Foundations and Trends in Machine Learning*, 2019.

Example: SIS for Filtering



Process model: $p(z_t \mid z_{t-1}) = \mathcal{N}\left(z_t \mid \phi\, z_{t-1}, \sigma_z^2\right)$

Example: SIS for Filtering



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Example: SIS for Filtering

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SIS

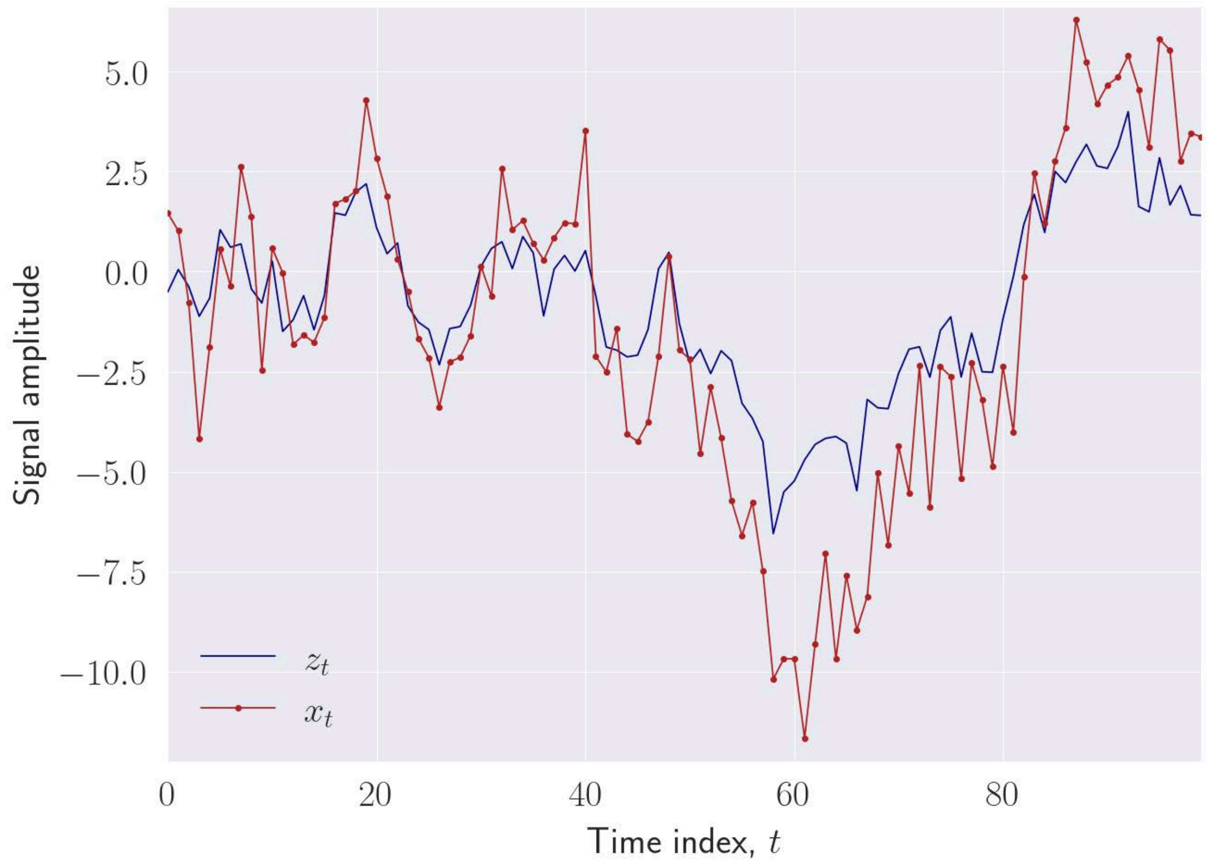
Input: $\{\phi, \beta, \sigma_x^2, \sigma_z^2, \mathbf{x}_{1:t}\}$

Initialise: $\left\{(w_0^{(1)}, \tilde{z}_0^{(1)}), \ldots, (w_0^{(L)}, \tilde{z}_0^{(L)})\right\}$

For $t = 1, \ldots, T$:

  For $\ell = 1, \ldots, L$:

   1. Sample $\tilde{z}_t^{(\ell)} \sim \mathcal{N}\left(z_t \,|\, \phi\, z_{t-1}^{(\ell)}, \sigma_z^2\right)$



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Example: SIS for Filtering

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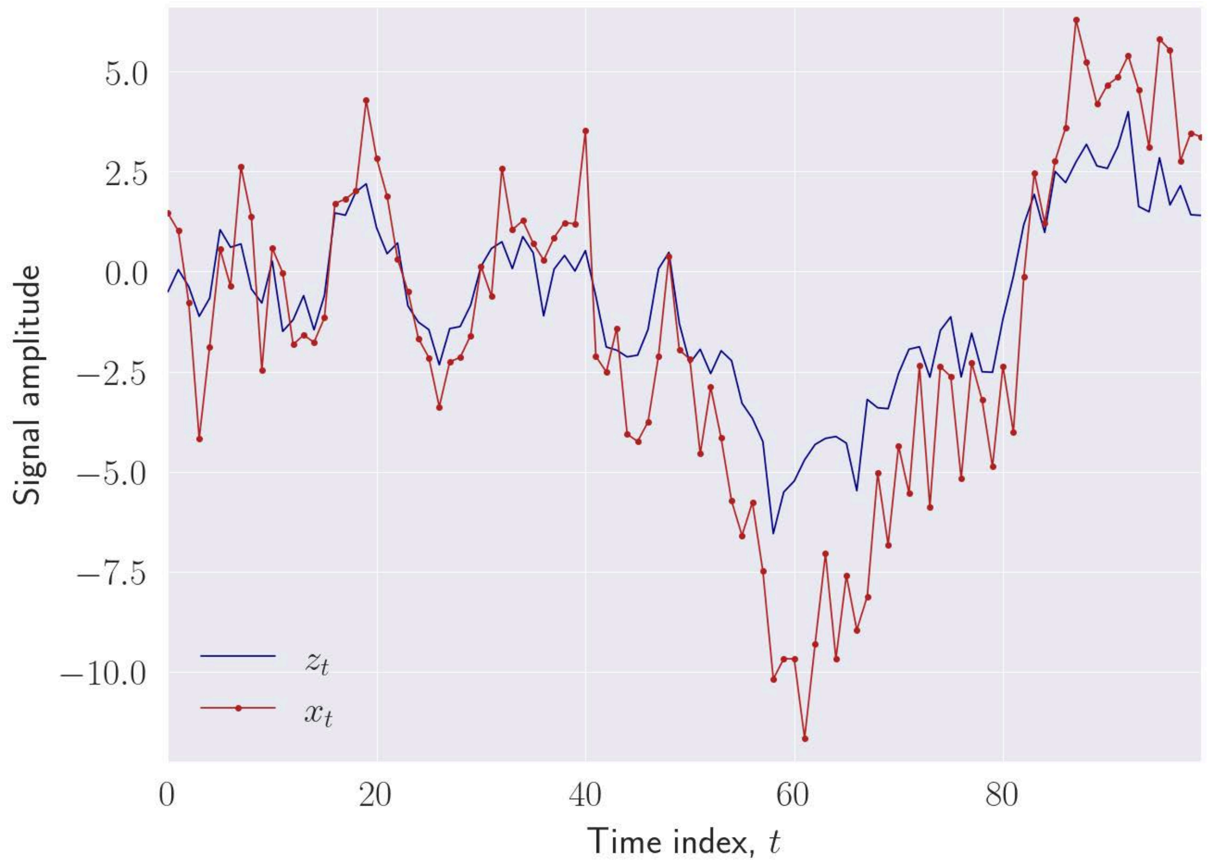

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Example: SIS for Filtering

SIS

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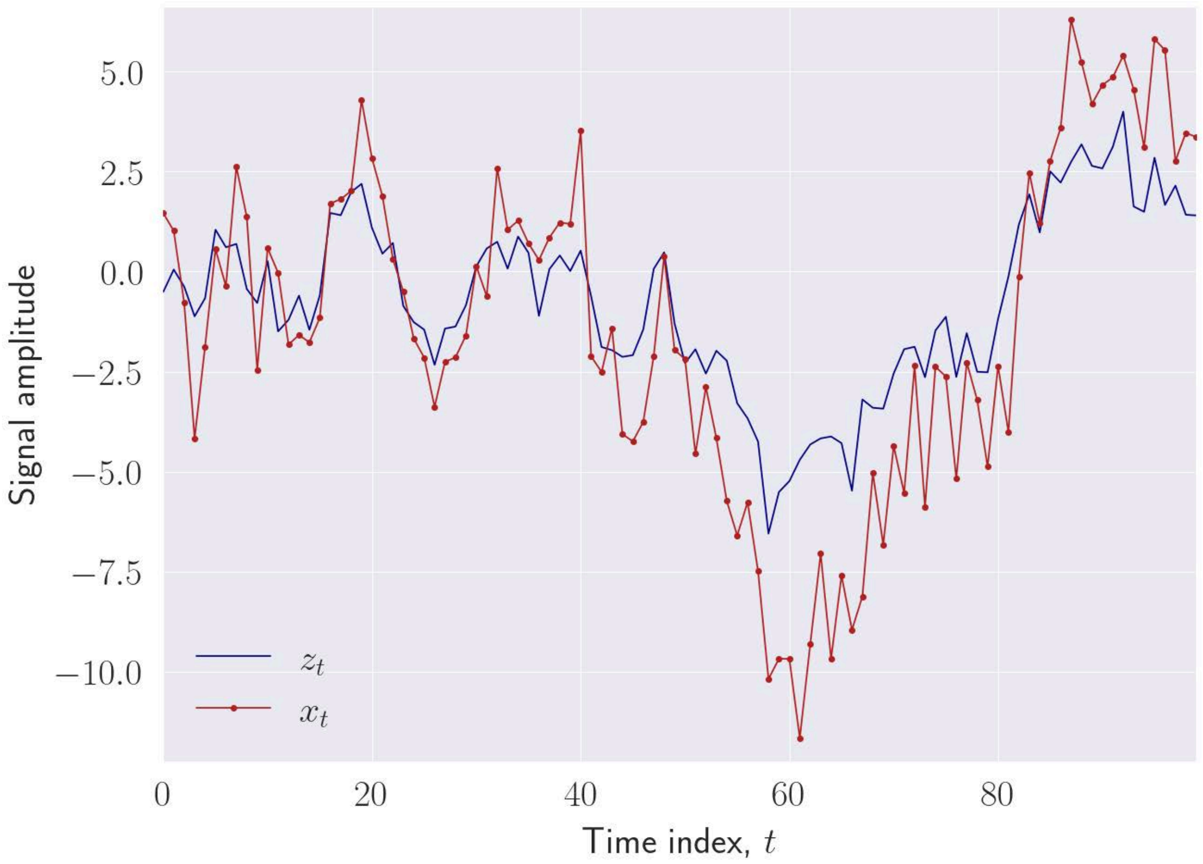

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Example: SIS for Filtering

SIS

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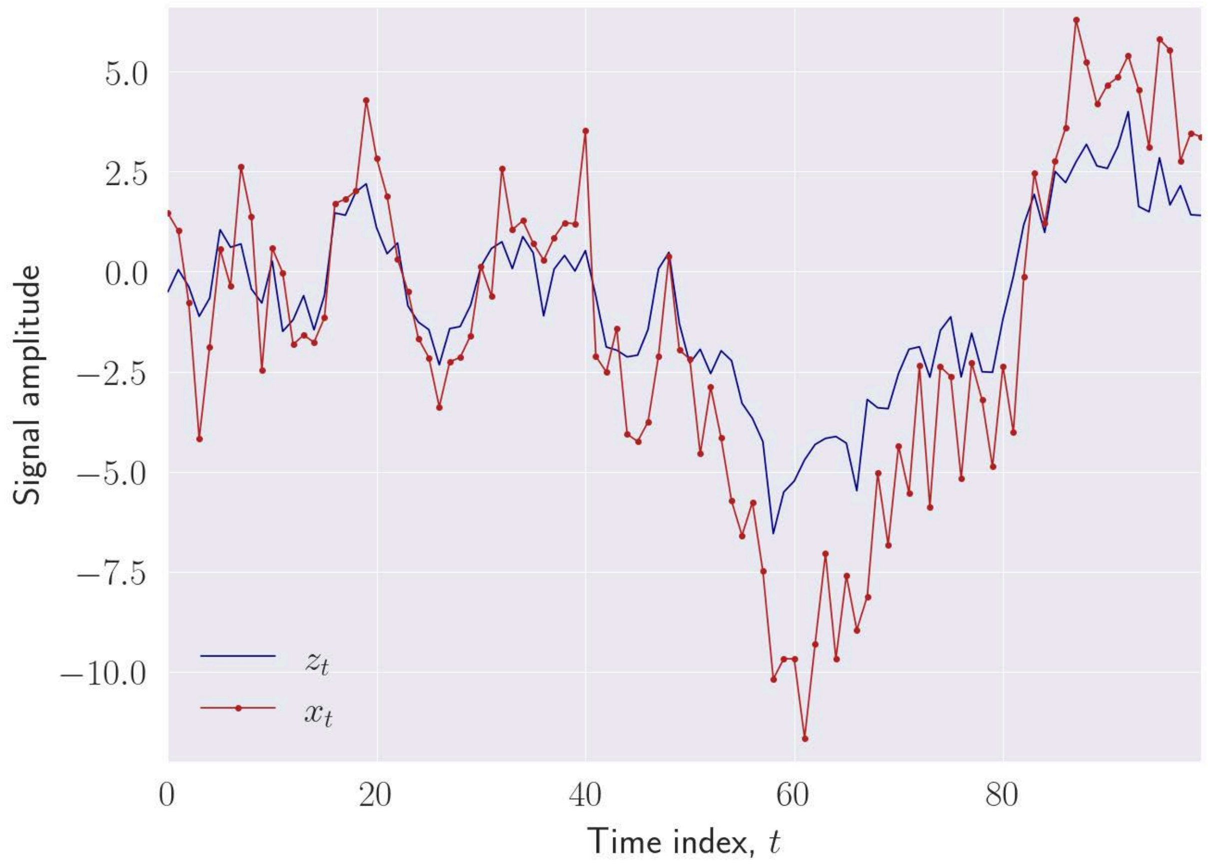
For $\ell = 1, \ldots, L$:

3. Normalise $\tilde{w}_t^{(\ell)} = w_t^{(\ell)} \Big/ \sum_{\ell'=1}^{L} w_t^{(\ell')}$



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Example: SIS for Filtering

SIS

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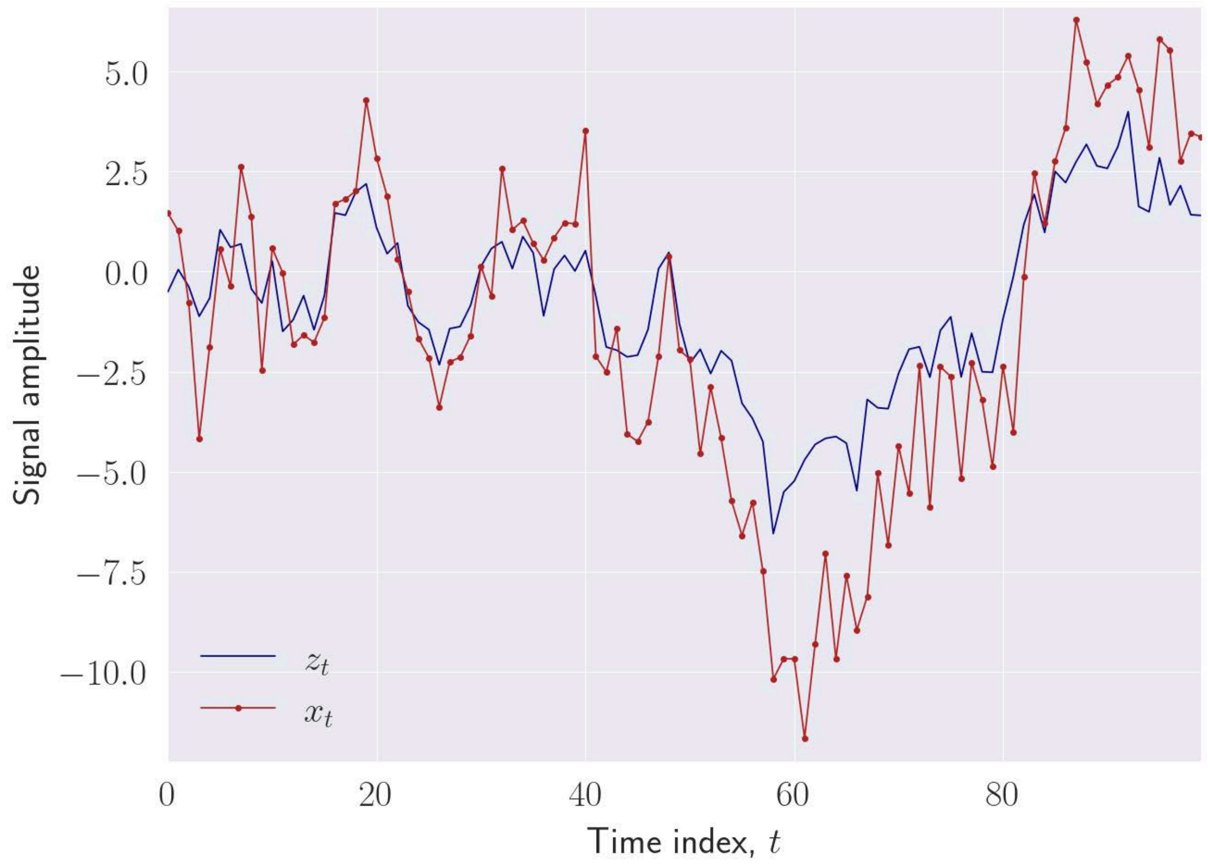

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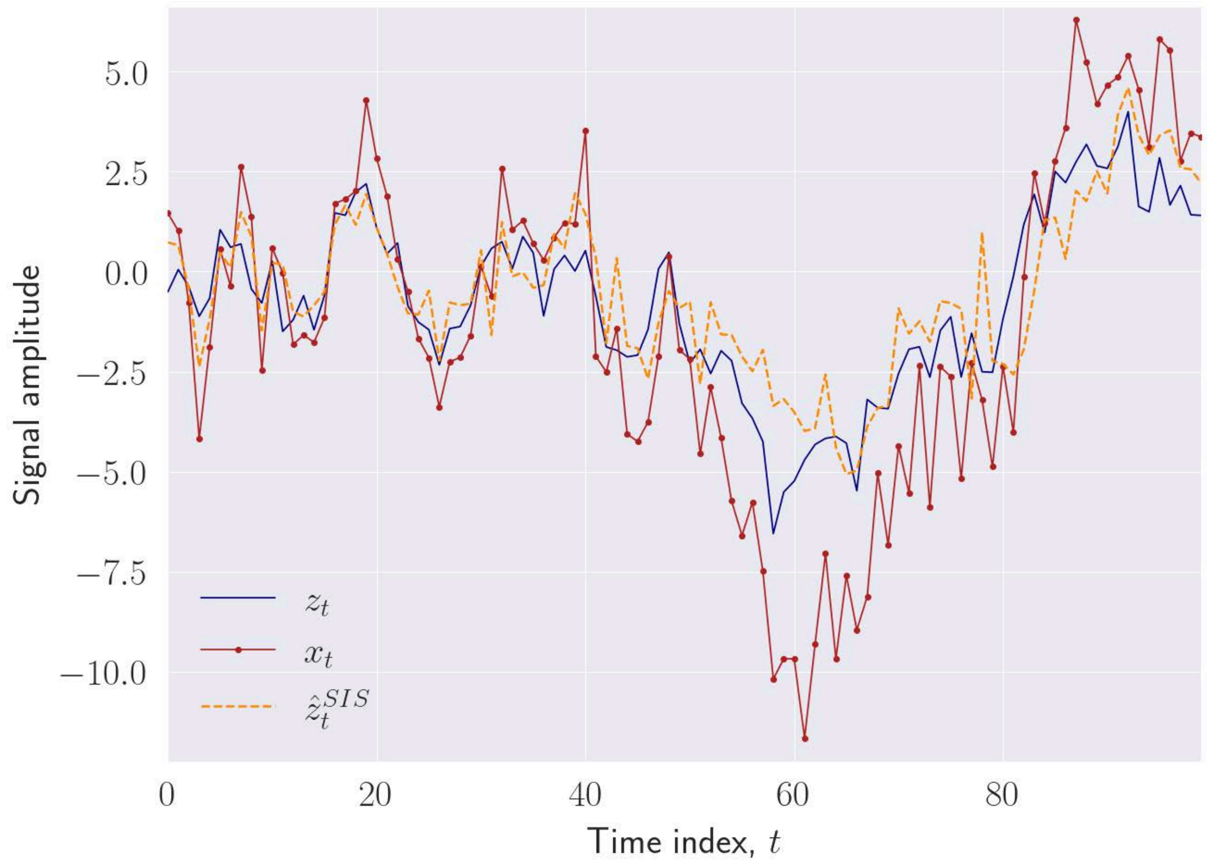

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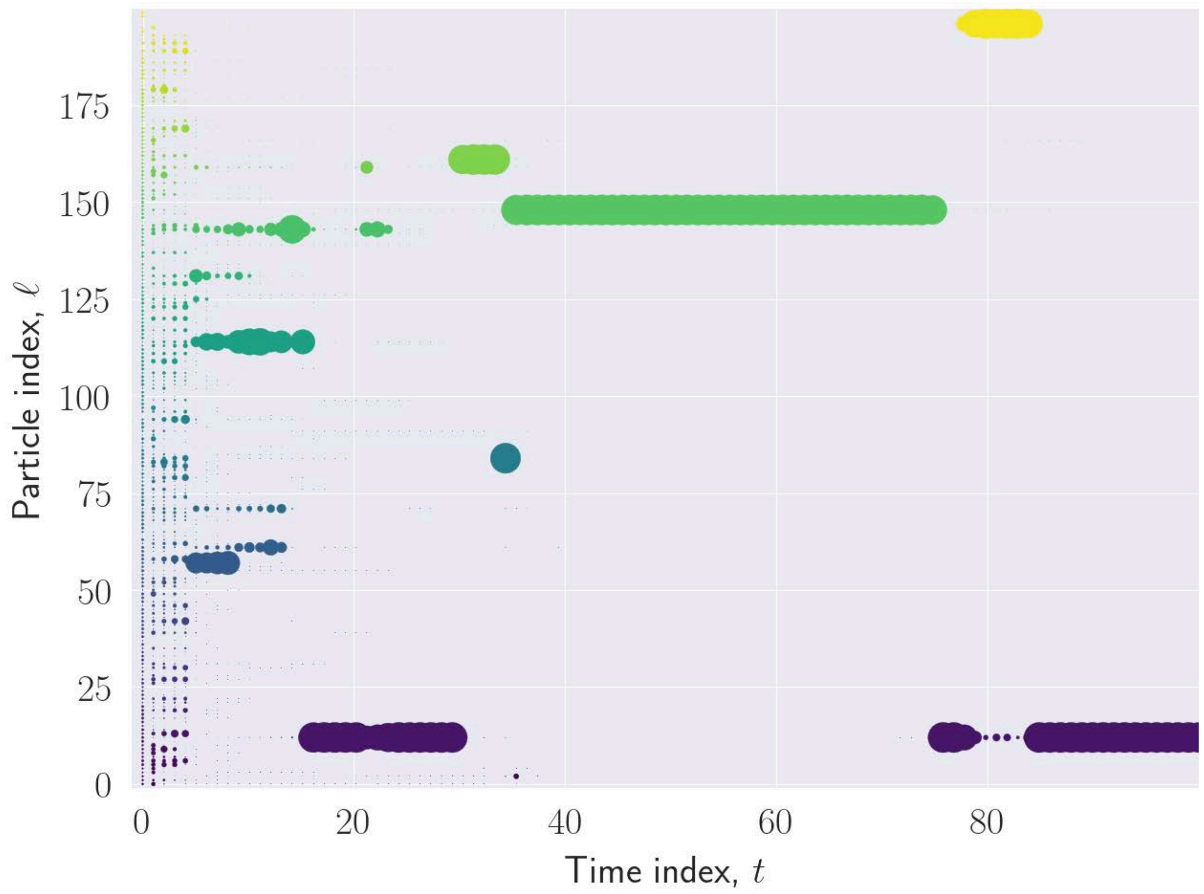
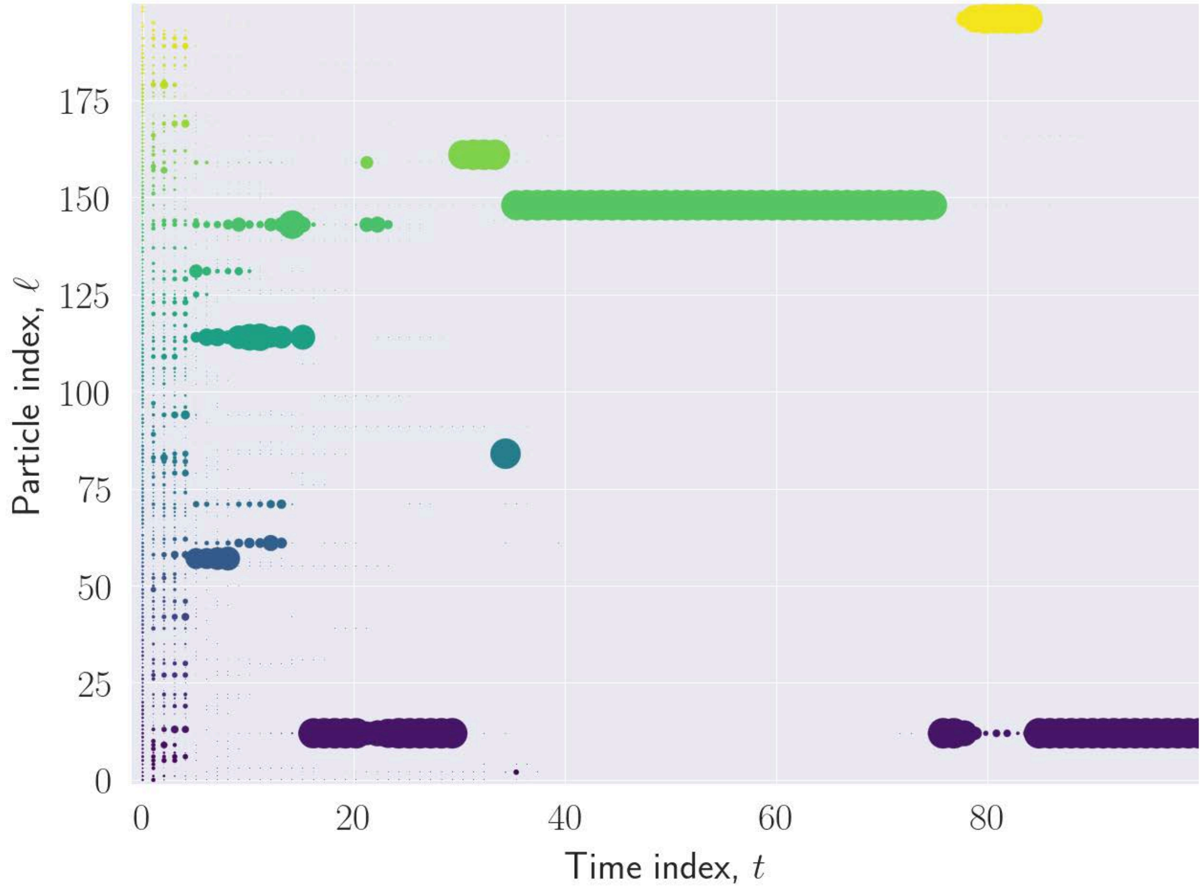
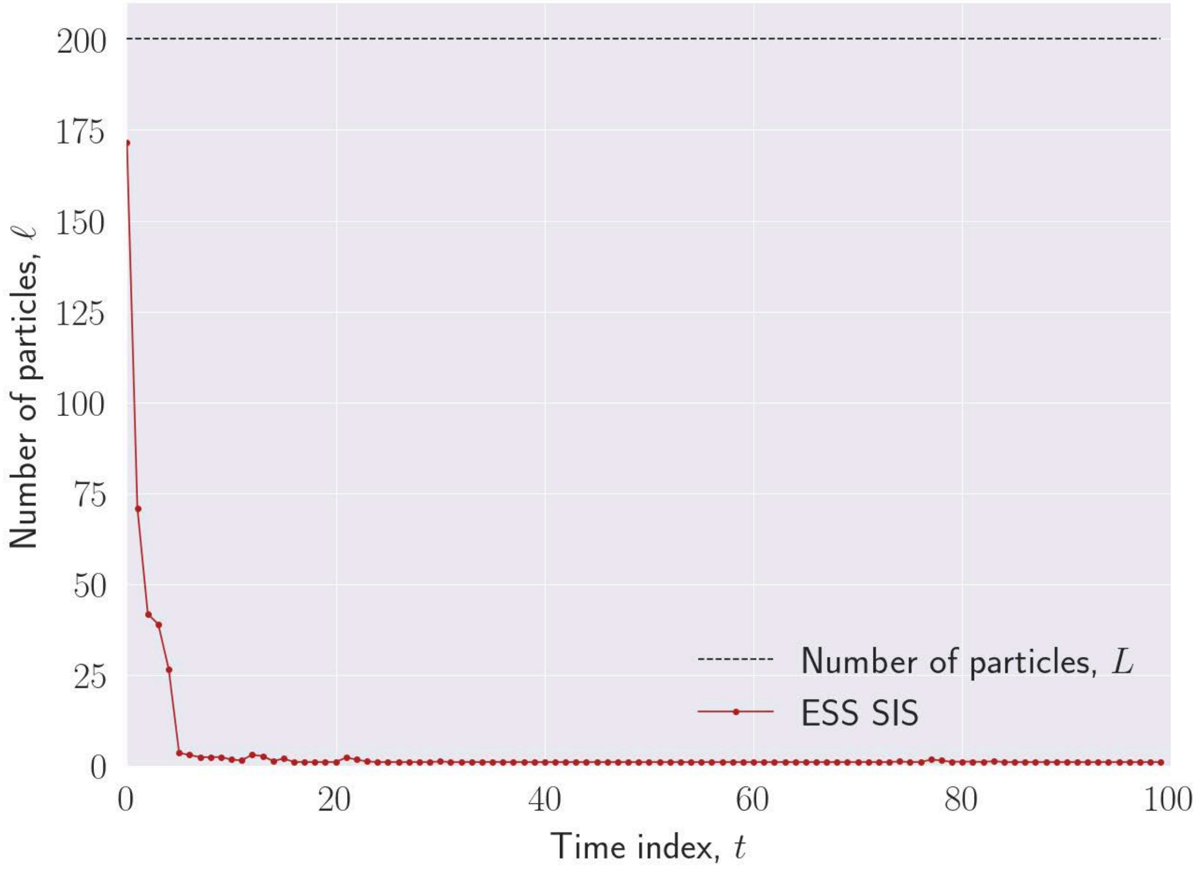
Weight Degeneracy



Weight Degeneracy



Effective Sample Size: $\text{ESS} = \left(\sum_{\ell=1}^{L} \left[w_t^{(\ell)} \right]^2 \right)^{-1}$



Conclusion: SIS

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Assumptions:

- Sample sequentially from a proposal (or importance) function
- Rather than throwing away samples, penalise samples

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Conclusion: SIS

University of Southampton

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- Sample sequentially from a proposal (or importance) function
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Advantages:

- Avoids issues due to high rejection rates
- Suitable for high-dimensional and multi-modal distributions

Limitations:

- Variance of the estimates increases exponentially with the number of time steps
- Sample impoverishment $\rightarrow$ Weight degeneracy:
    - Approximate target distribution using a single sample

Sequential Monte Carlo (SMC)

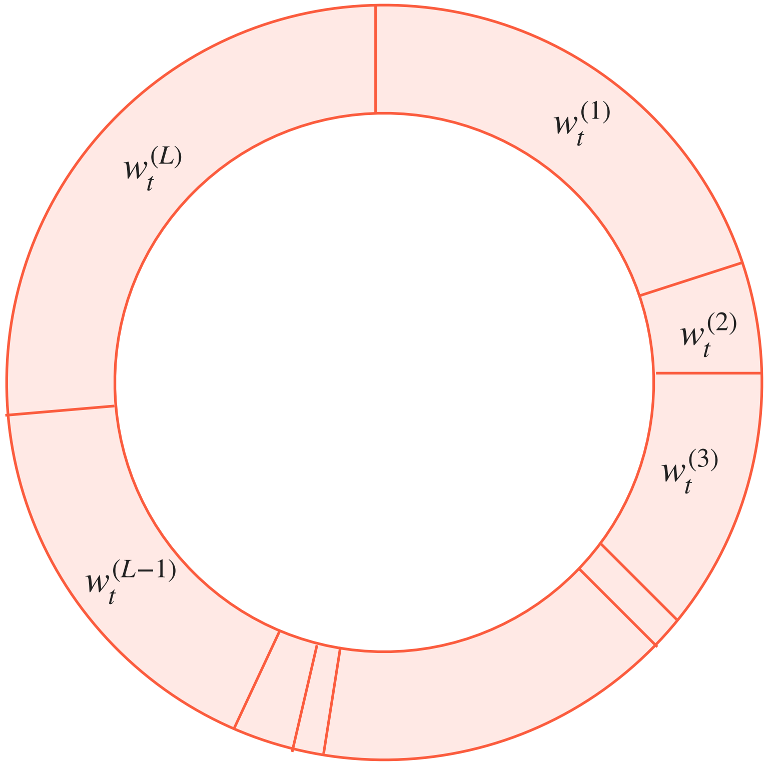
aka. Sequential Importance Resampling (SIR)
or Sequential Importance Sampling and Resampling (SIS/R)

Resampling

'Survival of the Fittest': Only retain stochastically relevant samples

Resampling

‘Survival of the Fittest’: Only retain stochastically relevant samples



Resampling

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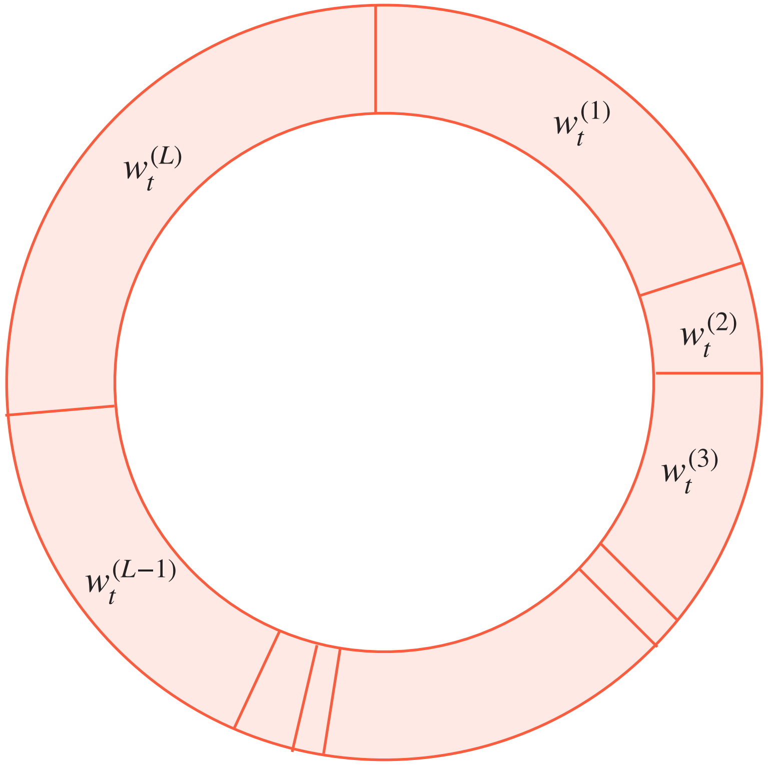


Systematic Resampling:

Input: $\left\{\left(w_t^{(1)}, \tilde{\mathbf{y}}_t^{(1)}\right) \dots, \left(w_t^{(L)}, \tilde{\mathbf{y}}_t^{(L)}\right)\right\}$

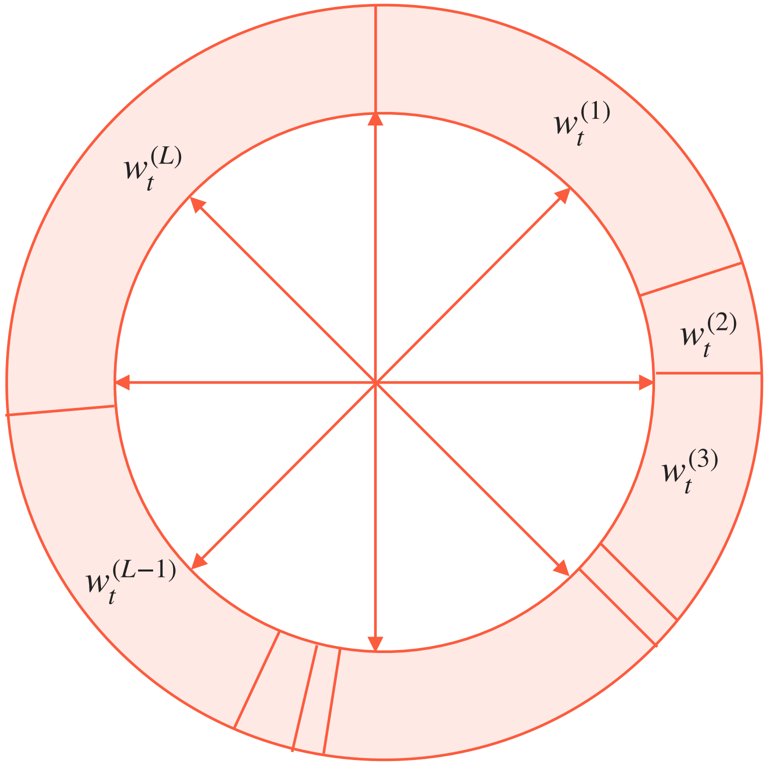
Initialise: $\mathbf{s}_t = \emptyset, c_1 = 0, i = 1$

For $\ell = 1, \dots, L$:

1. Sample $c_\ell = c_{\ell-1} + w^{(\ell)}$ (CDF)

Resampling

'Survival of the Fittest': Only retain stochastically relevant samples



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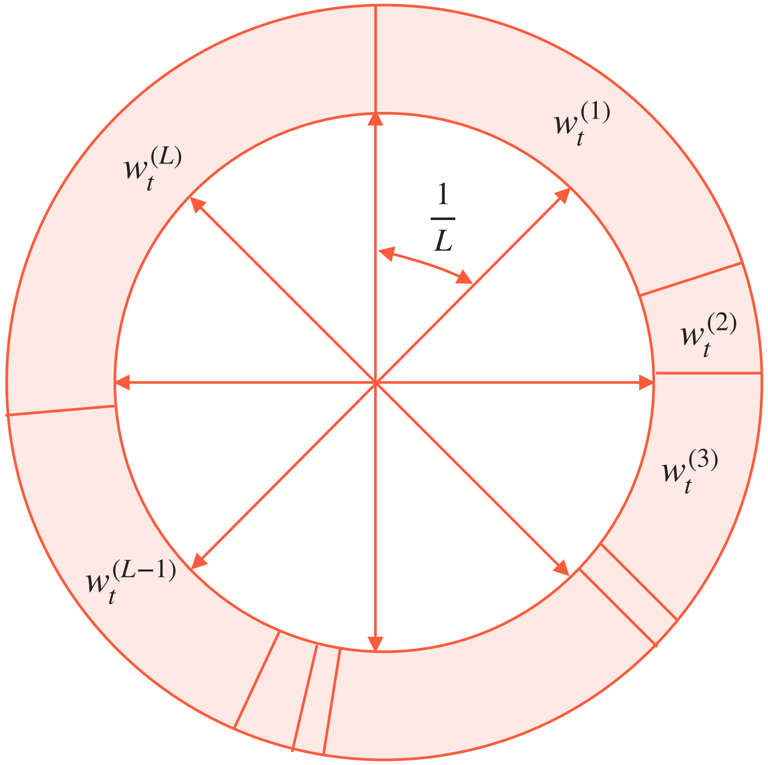

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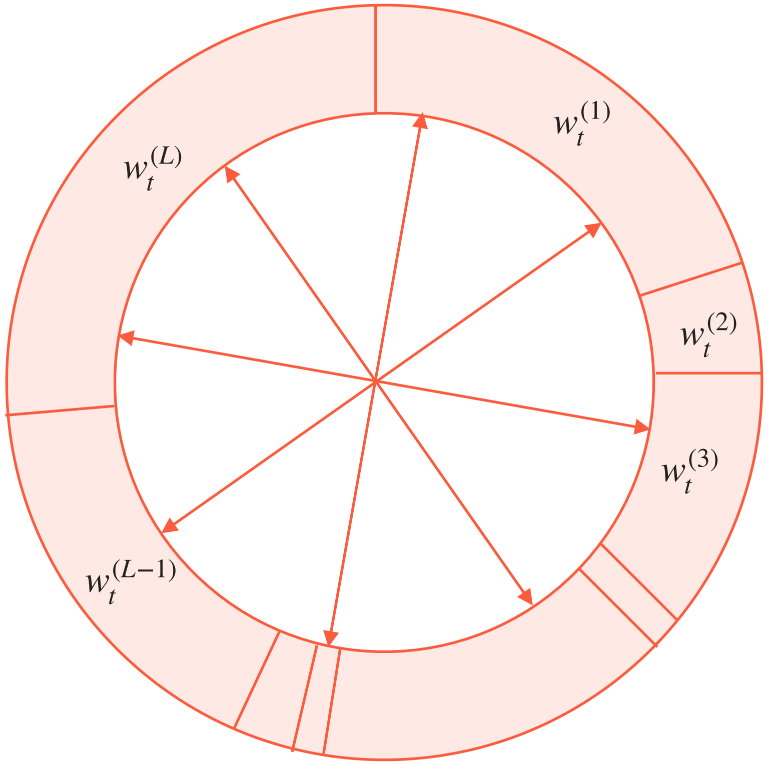

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Resampling

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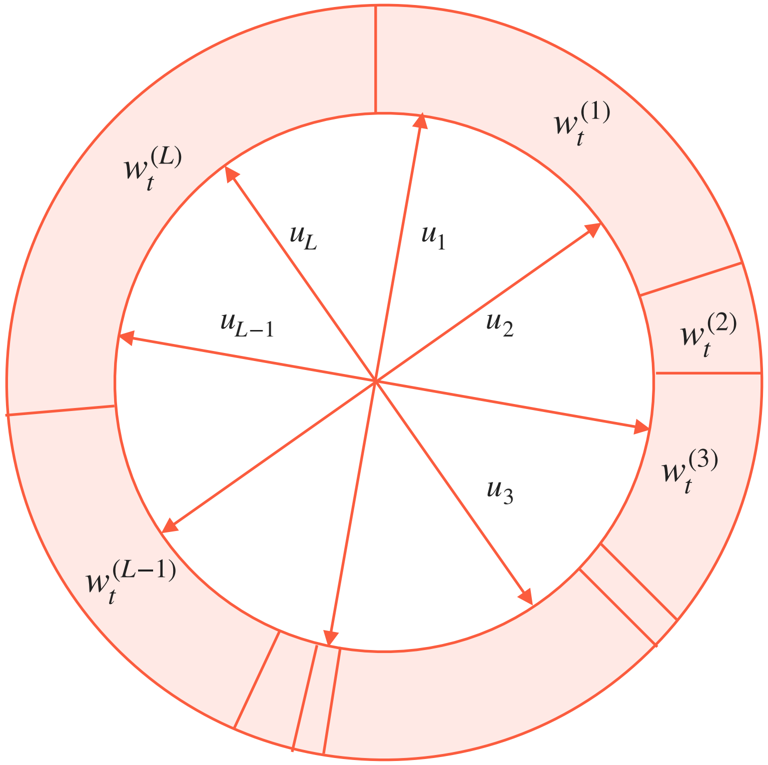

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For $\ell = 1, \ldots, L$:

3. Evaluate thresholds: $u_{\ell+1} = u_\ell + \frac{1}{L}$

Resampling

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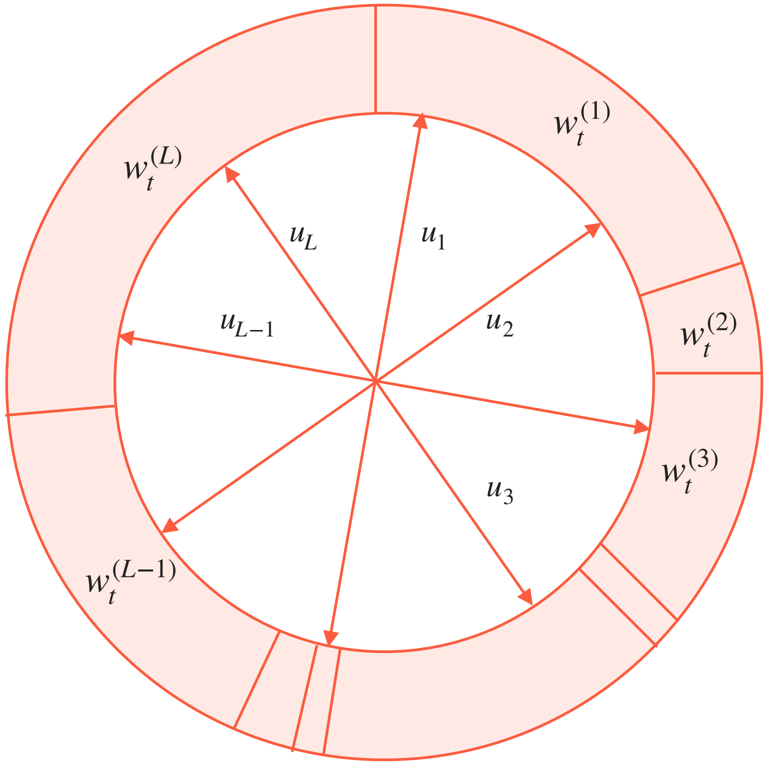

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For $\ell = 1, \ldots, L$:

While $u_\ell > c_i$ **:**

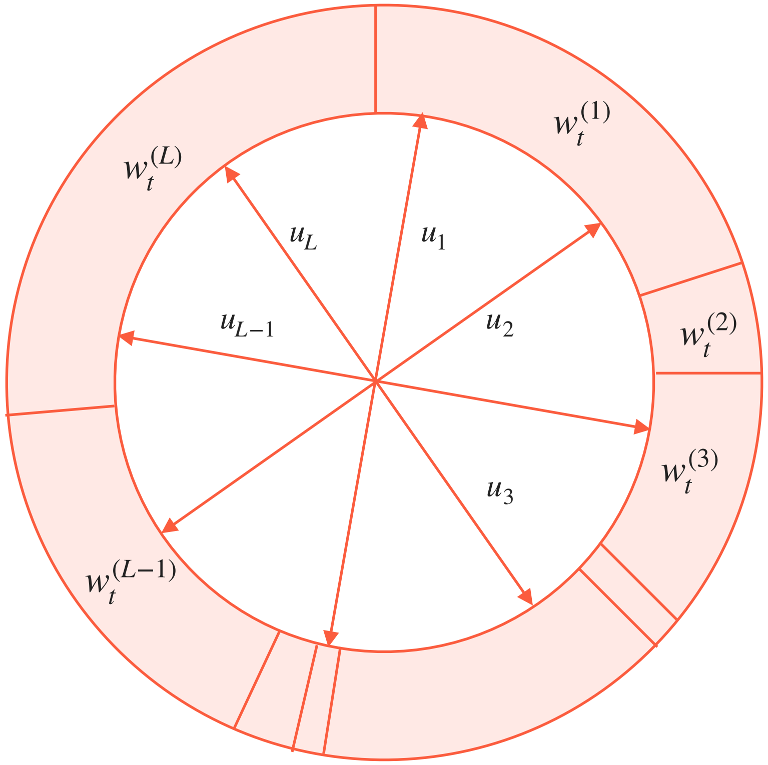
3. Set $i = i + 1$

4. Append sample: $\mathbf{s}_t = \left(\mathbf{s}_t, \tilde{\mathbf{y}}_t^{(i)}\right)$

Out: $\left\{\left(\frac{1}{L}, \mathbf{s}_t^{(1)}\right) \ldots, \left(\frac{1}{L}, \mathbf{s}_t^{(L)}\right)\right\}$

Resampling

'Survival of the Fittest': Only retain stochastically relevant samples



$\mathbf{s} = (1, 1, 3, \ldots, L-1, L-1, L)$

Systematic Resampling:

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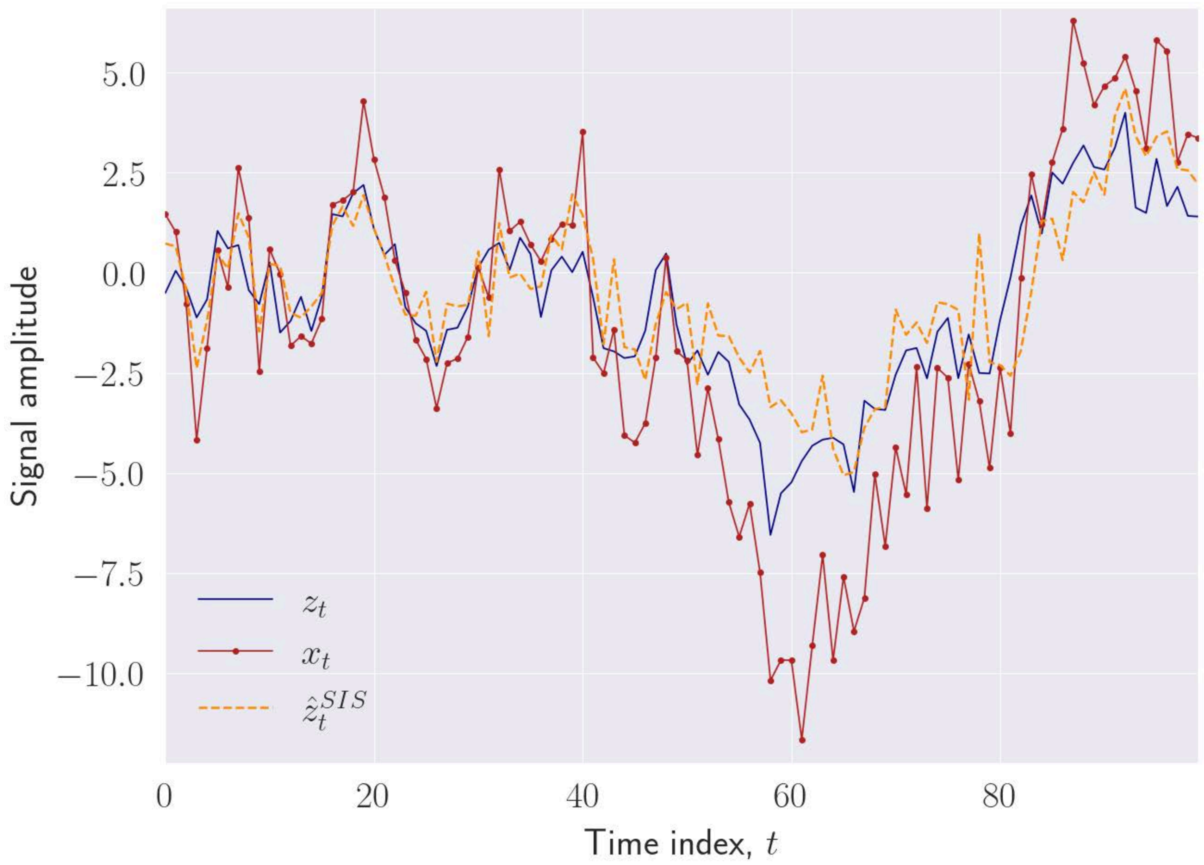
Sequential Monte Carlo



Sequential Monte Carlo

SMC:

Input: $\{\phi, \beta, \sigma_x^2, \sigma_z^2, \mathbf{x}_{1:t}\}$

Initialise: $\left\{(w_0^{(1)}, \tilde{z}_0^{(1)}), \ldots, (w_0^{(L)}, \tilde{z}_0^{(L)})\right\}$

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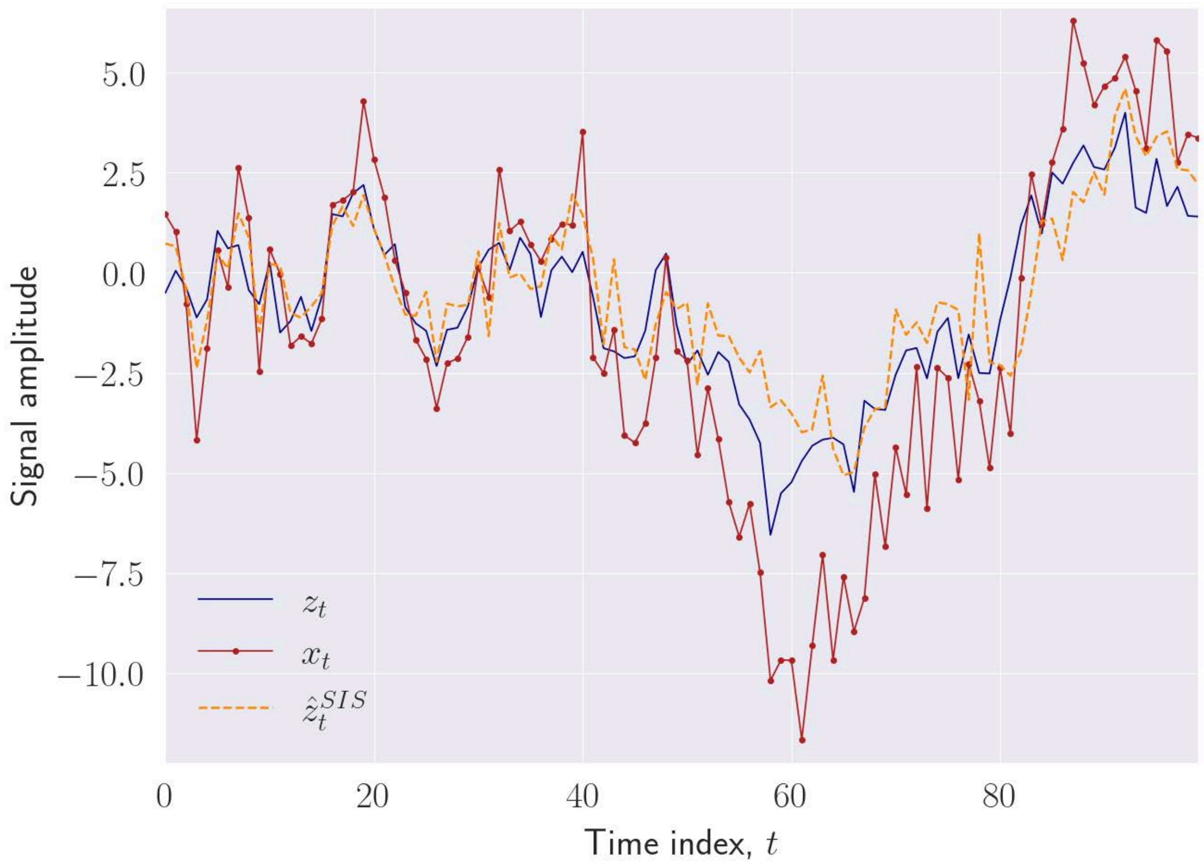

Sequential Monte Carlo

UNIVERSITY OF Southampton

SMC:

Input: $\{\phi, \beta, \sigma_x^2, \sigma_z^2, \mathbf{x}_{1:t}\}$

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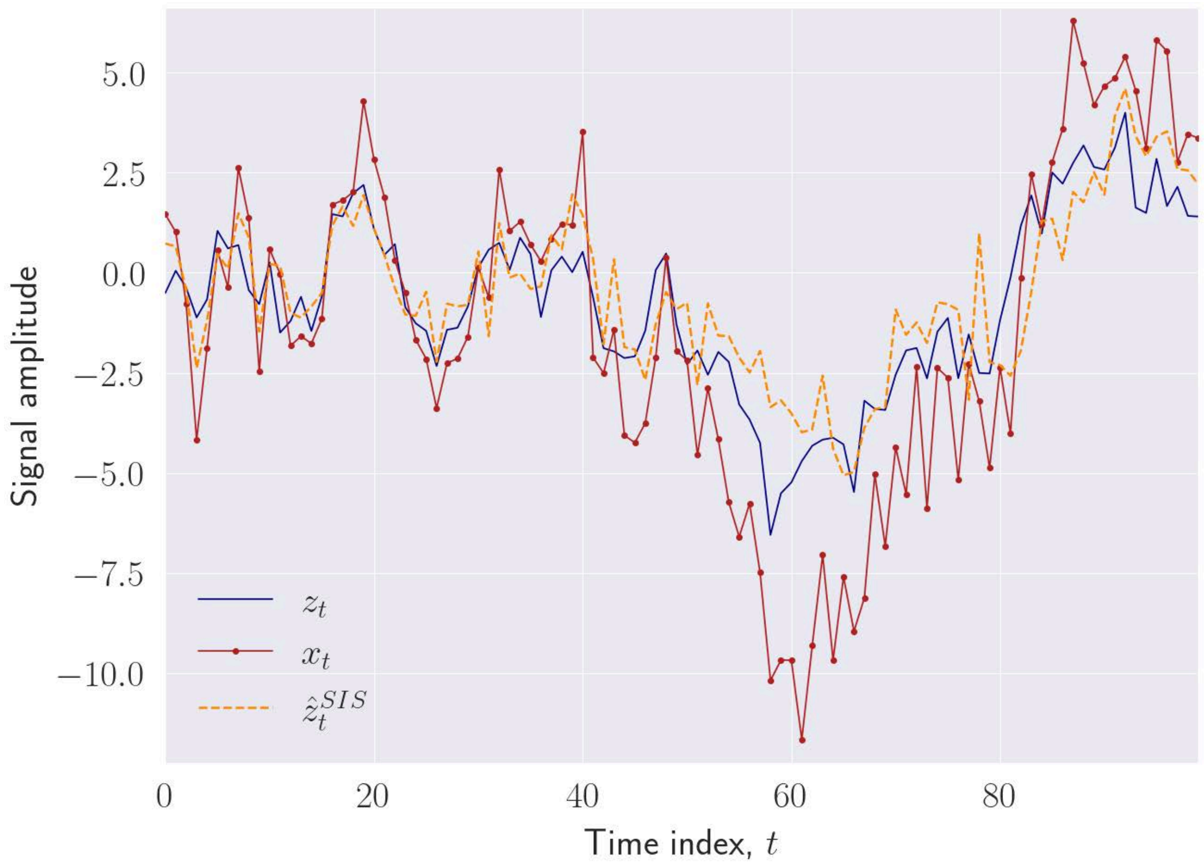

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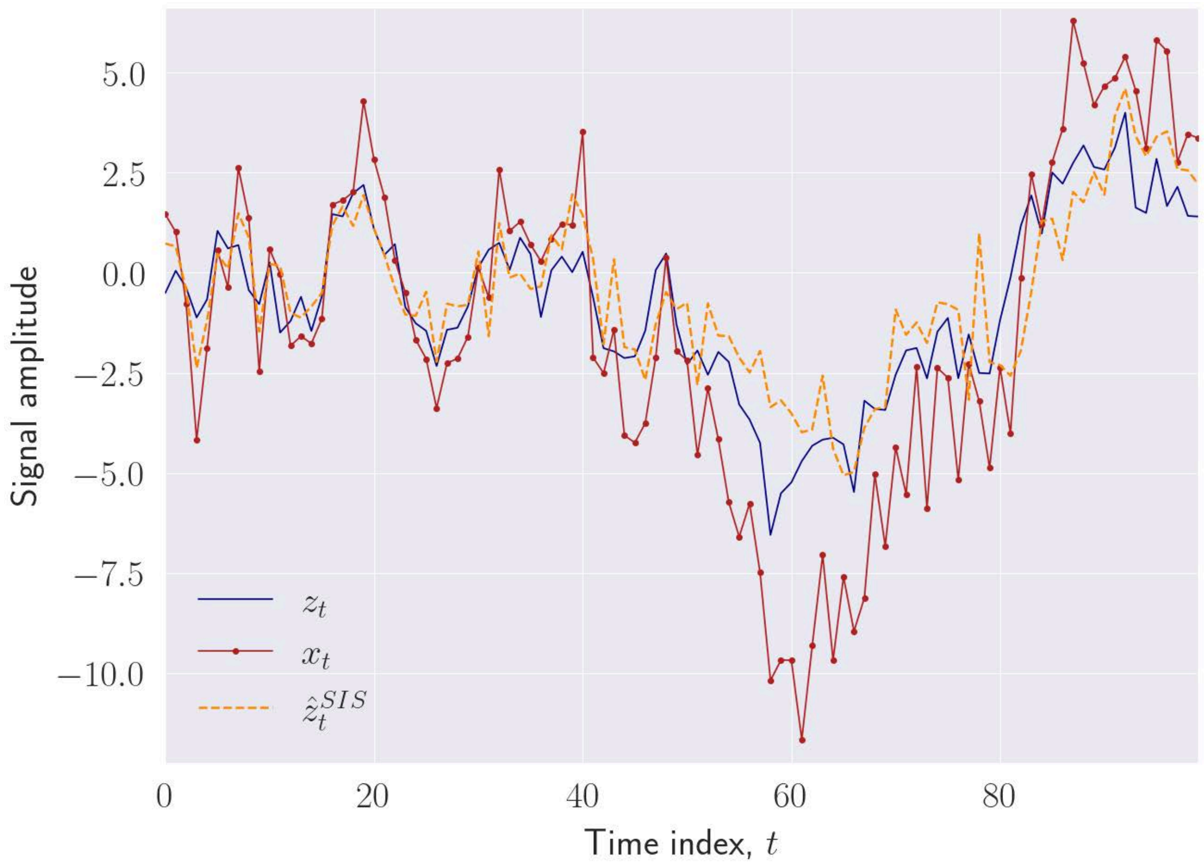

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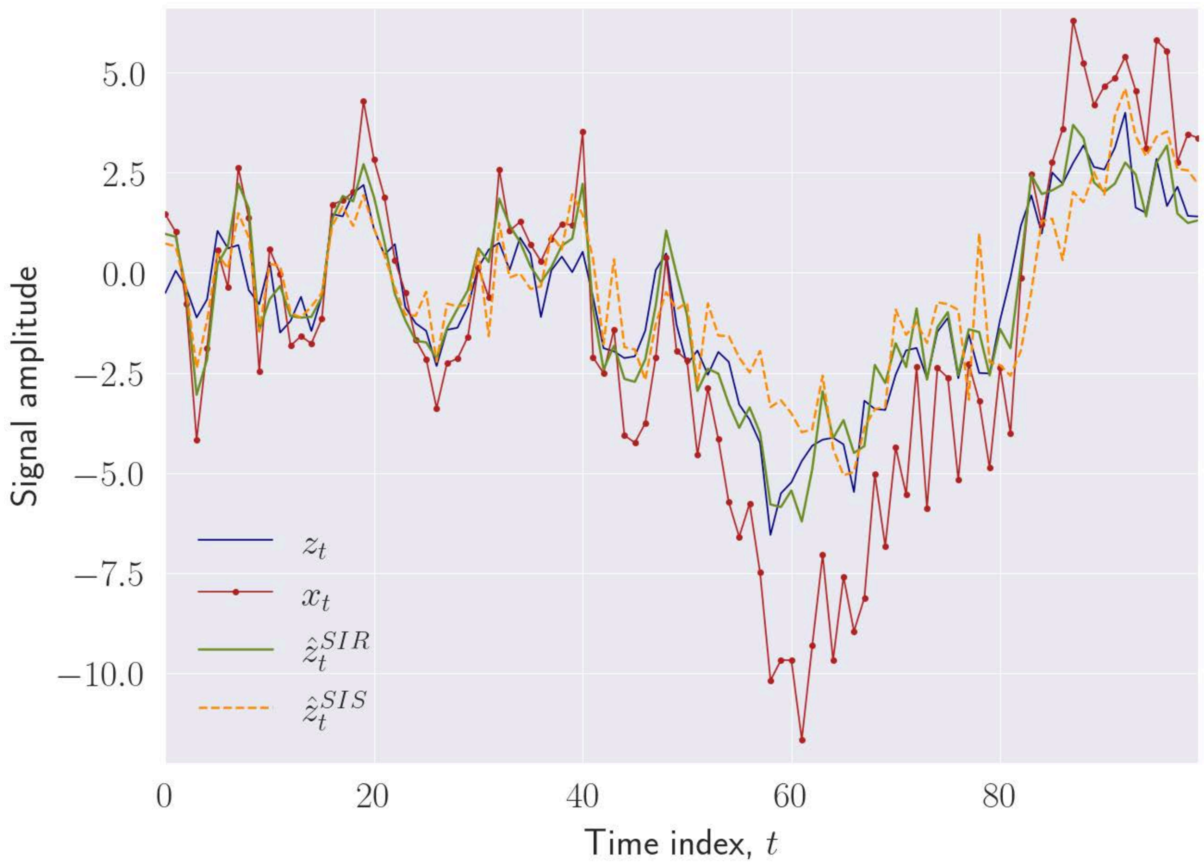
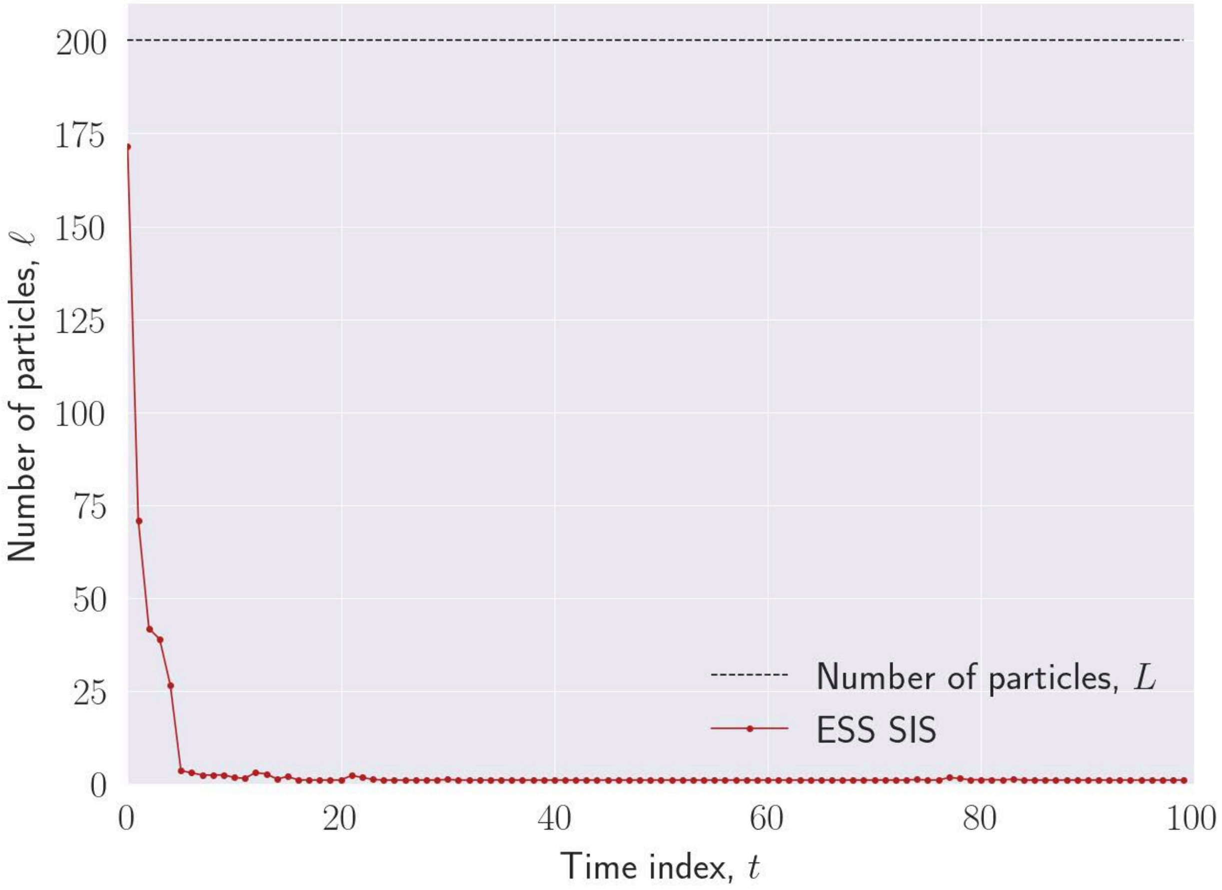
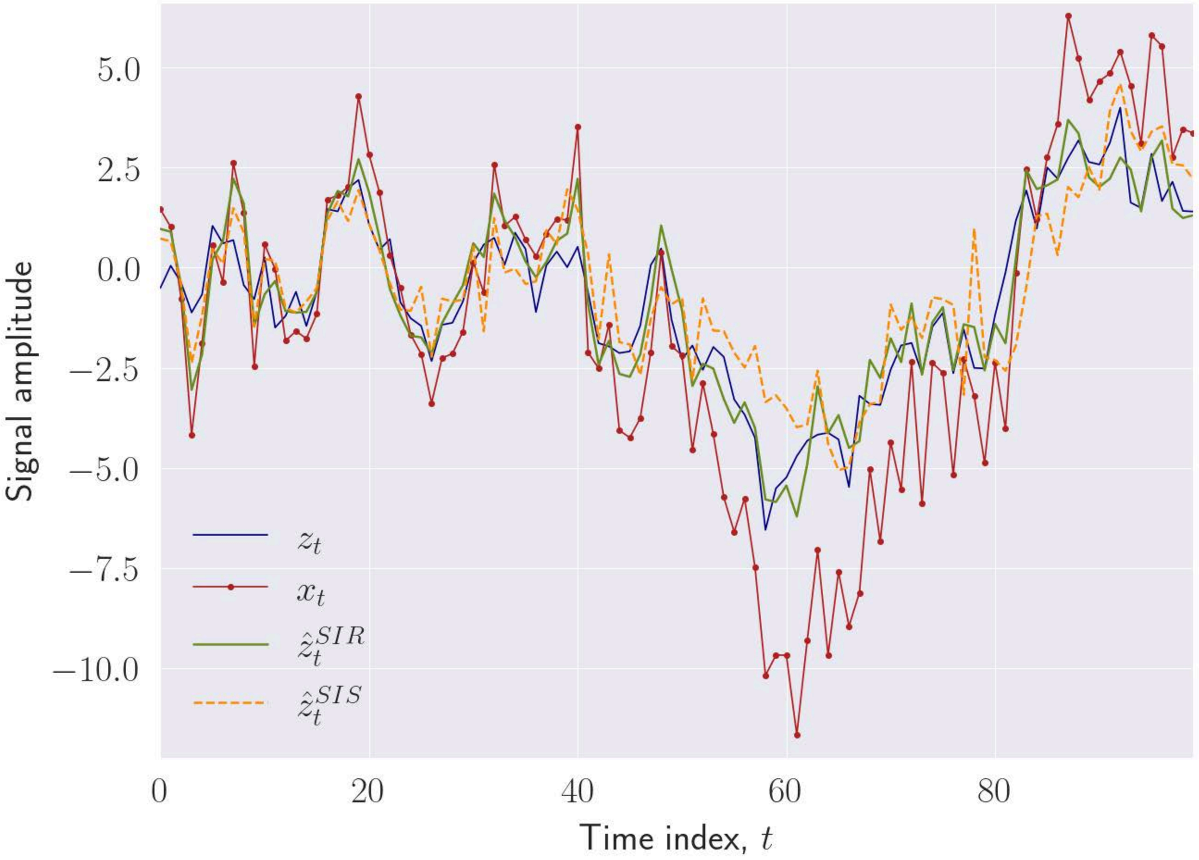


Sequential Monte Carlo





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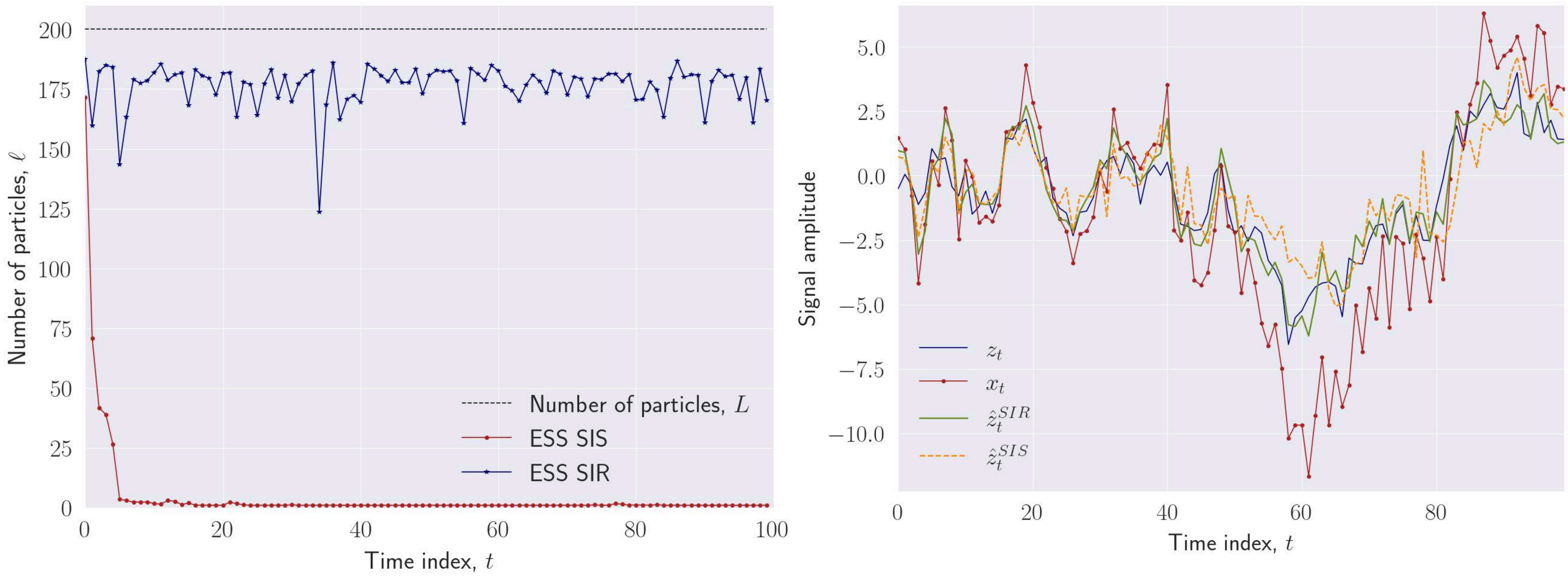
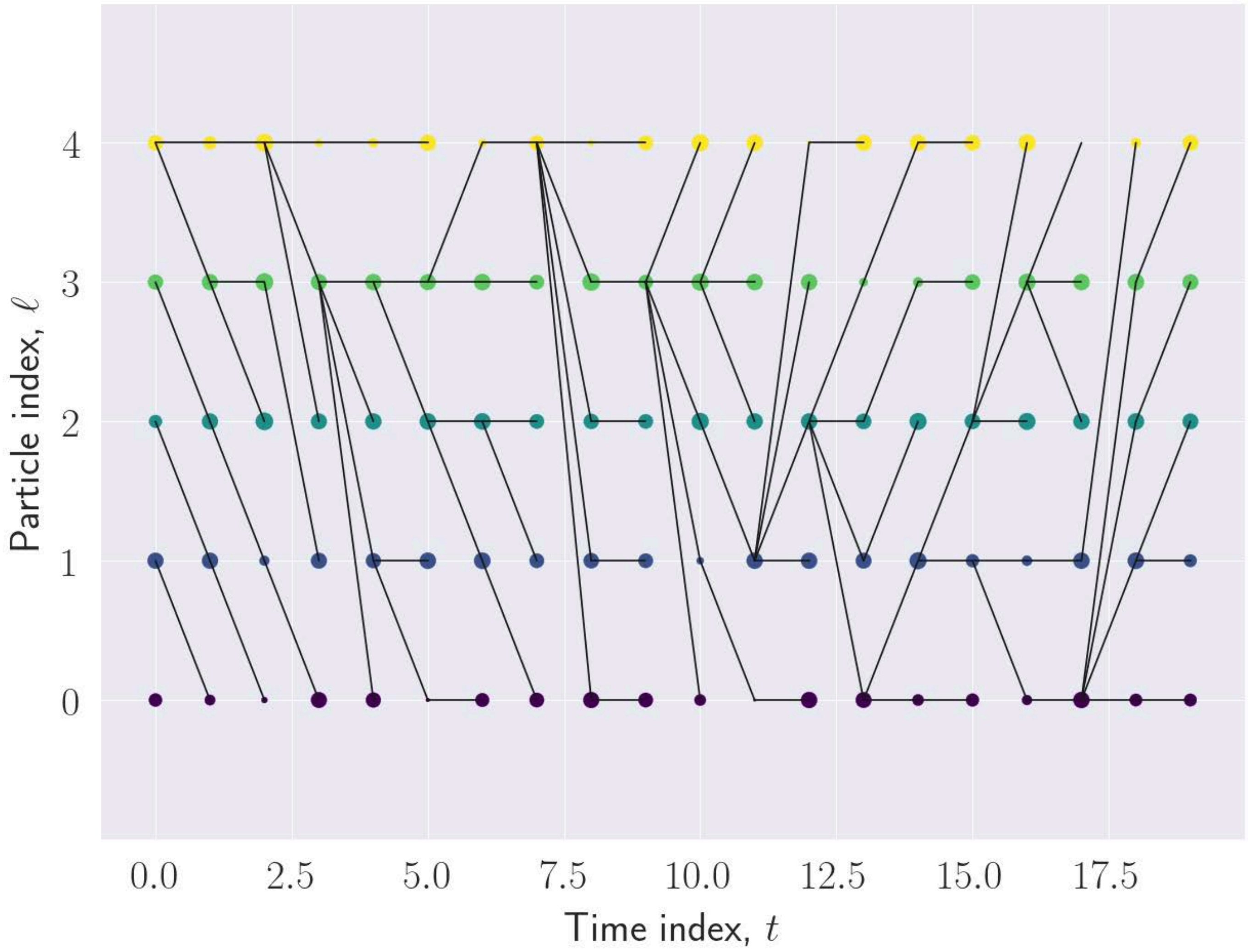


Path Degeneracy



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Limitations:

- System of particles that interact (through resampling)
- Path degeneracy: Particles share same ‘ancestor’

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2) **Analyse and mitigate degeneracy of sequential importance sampling methods;**
3) **Apply sequential Monte Carlo for online learning.**