## The Mathematics of the Five Card Trick

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arithmetic. Some self-working tricks are those based on counting, and require understanding and technique, but not necessarily the typical "magician" skills, like palming cards, false shuffling, or peeking. Many mathematical card tricks also require proficiency in those and other basic card handling skills.

#### Fitch Cheney's Five Card Trick

The trick I am going to present is credited to mathematician William Fitch Cheney, Jr. (1894-1974). Cheney was a mathematician and professor for most of his life, but enjoyed performing magic tricks for family and friends. He was a frequent contributor to the magazine M-U-M, the official publication of the Society of American Magicians. The magical effect of the Five Card trick is that the magician can name a card drawn randomly from a deck of cards. As originally published in Lee's (1950) book, *Math Miracles*, it was set up as a telephone trick. After a member of the audience selected five cards at random, one magician turned one card face down and laid the remaining four face up in a line on the table, and then phoned a second magician. Magician 1 would name the four cards faced up on the table and magician 2 would name the hidden card. In this way it was clear to all in the audience that there were no body language signs between the two magicians. In the version that will be examined here, the magician is simply out of the room. The assistant asks for a volunteer to draw five cards at random from a standard deck of 52 cards. The cards are drawn and given back to the assistant, who chooses one of the cards which will then be hidden. The assistant then lays the remaining four out on the table. The magician is called back, surveys the four cards on the table, and names the hidden card.

The probability of correctly naming 1 random card chosen from a standard deck of cards is 1 in 52. This means that we are looking for just one card, called the favorable outcome, in a

set of 52 cards, the total possible outcomes. The probability of naming the chosen card is 1 in 52 or about 2%. In this trick, since four additional cards are shown to the magician, the chance of naming the hidden card correctly is just slightly higher, because now the set of cards has been reduced from the total of 52 possible outcomes to 48. The probability of naming the hidden card is now 1 in 48, which is still approximately 2%. With that probability, there would be no magic trick. What do the magician and the assistant know in order to make this trick work? There must be more than what we can see, but there is no telepathy or secret body language involved, which is why it could originally be performed over the phone. The magic is in the mathematics behind the trick. It is essential that the assistant and magician work together in order to make the effect work.

Think about what the magician sees: there are four cards face up in a line on the table. Can these four cards give the magician a clue as to what the hidden card is? The cards are selected by a member of the audience, so the assistant does not get to choose them, but the assistant does have all cards before the magician sees them, and in fact chooses which of the cards will be hidden before laying out the other four. This is where the mathematics comes into play. There are four cards on the table and 48 possible cards that could be hidden. How can the assistant and magician use the four cards to identify the one that is hidden if every time the trick is performed different cards are chosen? The solution to the trick has three mathematical components: the pigeonhole principle, modular arithmetic, and permutations. Using these three principles and following some conventions previously practiced will make the most of the magical effect. We will look at how each one of these concepts works in the Five Card Trick.

### The Pigeonhole Principle

The pigeonhole principle is a simple, but important part of combinatorics. Let n = thenumber of pigeons and k = the number of pigeonholes. We define the ceiling, shown as  $\left| \frac{n}{k} \right|$ , to be the smallest integer  $\geq \frac{n}{k}$ . The floor of  $\frac{n}{k}$ , shown as  $\left| \frac{n}{k} \right|$ , is the greatest integer  $\leq \frac{n}{k}$ . When n pigeons are put into k pigeonholes, there exists at least 1 pigeonhole containing not less than  $\left| \frac{n}{k} \right|$  pigeons and at least 1 pigeonhole containing not more than  $\left| \frac{n}{k} \right|$  pigeons. For example, let k = 4, to represent the four suits in a deck of cards (the pigeonholes), and let n = 5 to represent the number of cards a volunteer draws from the deck for the trick (the pigeons). We know that  $\frac{5}{4}$  is  $1\frac{1}{4}$ , so the ceiling is 2. Therefore, at least one suit (a pigeonhole) has more than one representative among the 5 cards selected (the pigeons). Simply stated this means that if you draw any five cards, at least 2 of them will match suit.

To relate this to our trick we think of the 5 cards selected by a volunteer and the four possible suits. The pigeonhole principle guarantees that in any given draw of five cards, at least two of the cards are of the same suit. It is possible that less than all of the four suits are represented in the five cards selected, but we know that at least two of the cards must have the same suit. This is the first part of the solution.

One of the cards that share the same suit will become the hidden card and one will become the key card in the trick. The assistant scans the five cards to find a pair of cards of the same suit. The assistant must then quickly decide which of these two will be the hidden card and which will be the key card that the magician will see on the table with the other three cards. The assistant then shows the card that will be hidden to the audience and turns it face down on the

table. The key card is then placed in the first position face up on the table. Scanning left to right, it will be the first of the four cards the magician is allowed to see before naming the hidden card. The magician will know immediately the suit of the hidden card.

#### **Modular Arithmetic**

The assistant must also know how to count the numbers or faces of the chosen cards to find the difference between the value of the hidden card and the key card. This is done by using modular arithmetic. Modular arithmetic is a system that lets us think about operations of numbers using repetitive cycles outside the usual base ten system. If we want to add numbers using only those available to us in one suit of our standard deck of cards, we can use (mod 13). In each suit the numbered cards are given their corresponding value, and then we assign the value of one to the ace, 11 to the jack, 12 to the queen, and 13 to the king to make our system of thirteen complete. In base 10 we can easily add 4 and 6 to get ten, and it is the same (mod 13), but what happens when we want to add 10 and 6? We know 10 + 6 = 16, but in our deck of cards, we do not have a 16 card, we can only count as high as the number 13 since the suit has only 13 cards. Using (mod 13), numbers higher than 13 will be expressed by the congruence of their remainders in (mod 13). Let's imagine one of the suits from our deck arranged in a circle, like the face of a clock (see Figure 1).



Figure 1 Clock arrangement (mod 13)

Using (mod 13) we start counting clockwise with the ace (1), 2, 3, ..., 11 (jack), 12 (queen), 13 (king), and then we repeat the cycle. So, instead of saying 14, as we usually would, we start again with 1. We count 12, 13, 1, 2, ..., and so on. Therefore, if we add 10 and 6, we count 11 (jack), 12 (queen), 13 (king), 1, 2, and then 3. We can add from any number in the circle and when we reach 13, we go around again. The base ten equation 10 + 6 = 16 can become a new problem in (mod 13). When we divide 16 by 13 we get a quotient of 1 with a remainder

of 3. This remainder becomes our congruency answer in (mod 13). Using symbolic notation for (mod 13) we can show that  $10 + 6 \equiv 3 \pmod{13}$ . So, 10 add 6, counting clockwise, is 3.

Let's try another example. This time let's start with the queen (12) and add 2. We know in base ten 12 and 2 are 14. Looking at the clock arrangement of the suit, we can count clockwise from the queen 2 places and land on the 1 (ace). In (mod 13), we can say that  $14 \equiv 1 \pmod{13}$ . This means that when we divide 14 by 13 there is a remainder of 1.

This clock arrangement is a very quick and convenient way for the assistant in the trick to study the five cards the audience member has chosen and decide what to do next. Seen in this repetitive cycle any two cards are no more than six positions apart moving clockwise. These six spaces are important to the trick, and we will see why in the next section. How do the assistant and the magician use modular arithmetic to communicate the value of the hidden card? We have already established that the suit of the key card (the first upturned card) is the same as the suit of the hidden card. That narrows down the number of cards that could be the hidden card from 48 to 12; the remaining cards in the suit of the key card. When scanning the cards, the assistant chooses a pair of cards with the same suit. The assistant chooses which of the pair will be hidden and which will be the key card based on the difference in the values which cannot exceed six, for reasons we will see in the next section. For example, if the pair is the 2♥ and 8♥, the 8♥ will be hidden and the  $2 \heartsuit$  becomes the key card, since  $2 + 6 \equiv 8 \pmod{13}$ . The two cards cannot be switched, because to go clockwise from 8♥ to 2♥ the difference is 7 positions on the clock. Or we could say  $8 + 7 \equiv 2 \pmod{13}$ . The congruence is true in (mod 13), but this model will not work in our five card trick, where we must limit the differences to 6 positions.

In another example, let's say the pair is  $4 \heartsuit$  and  $Q \heartsuit$ . We need to find the arrangement of these two cards with a difference  $\leq 6$  positions. Counting clockwise from  $4 \heartsuit$  to  $Q \heartsuit$  is 8 positions,

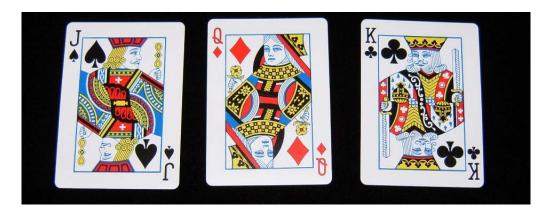
 $4 + 8 \equiv 12 \pmod{13}$ , but from  $Q \heartsuit$  to  $4 \heartsuit$  is only 5 positions,  $12 + 5 \equiv 4 \pmod{13}$ , so the  $4 \heartsuit$  is hidden and the  $Q \heartsuit$  becomes the key card. With the key card  $Q \heartsuit$  on the table as the first card faced up, how can the assistant use the remaining three cards to let the magician know which heart is hidden? And why must the differences in their positions be 6 or less? For the Cheney Five Card Trick, we use a code based on permutations.

### **Permutations**

Recall that the volunteer has drawn five cards at random and handed them to the magician's assistant, while the magician is out of the room. The assistant looks at the cards, decides which one will be hidden, shows it to the audience, and lays it face down on the table. The assistant has also identified the key card, which matches the suit of the hidden card, and has used (mod 13) to calculate the difference in positions on the clock of the two cards, keeping the difference of 6 or less positions moving clockwise between the key card and the hidden card. The three cards that the assistant still holds will all be distinct, but there is no way of knowing until the trick is being performed which specific cards from the deck they will be. The probability of simply guessing the hidden card now would be 1/12, an improvement over 1 in 48 as at the beginning, but there are still too many ways to go wrong to have a good magic trick. The third part of the solution involves using the possible permutations of the three cards to formulate a code which can transmit the value of the hidden card. The code is based on the arrangement of the three cards.

A permutation is an ordered arrangement of objects. If we have a collection of three objects, the permutations of three will show all the different ways those three things can be arranged. Let's say we have three cards J. O. K. as shown in Figure 2.

Figure 2 J♠, Q♦, K♣



How many different ways can we arrange them? These are the permutations of the three cards:

- 1) J♠, Q♠, K♣
- 2) J♠, K♣, Q♦
- 3) Q♦, J♠, K♣
- 4) Q♦, K♣, J♠
- 5) K♣, J♠, Q♦
- 6) K♣, Q♦, J♠

We can use factorial notation 3!, which means 3 x 2 x 1, to show that there are 6 ways to arrange 3 objects.

The magician and the assistant must have an order worked out ahead of time, so that the assistant can arrange the cards into one of the six possible permutations and the magician will decode the permutation to find the hidden card. Since there are only 6 possible arrangements of the three cards, the addend discussed in the previous section must be less than or equal to 6. This means that with any given pair, the hidden card can be a maximum of 6 positions from the key card when counting clockwise on the clock model. In this trick the suits are arranged in alphabetical order, such that clubs are the lowest, followed by diamonds, then hearts, and finally the highest suit is spades (see Figure 3). Within each suit, the cards are arranged with aces low. The aces are assigned the value of 1, number cards their corresponding value, jacks 11, queens 12, and kings 13. The order of the entire deck is determined first by suit and then by rank. Using

this order, the lowest card of the deck is the  $A \clubsuit$  and the highest is  $K \spadesuit$ . For example, look at the following five cards:  $2 \spadesuit$ ,  $4 \spadesuit$ ,  $8 \spadesuit$ , and  $K \spadesuit$ . Their arrangement from low to high following the order of suits for the trick would be:  $K \spadesuit$ ,  $4 \spadesuit$ ,  $8 \spadesuit$ ,  $4 \spadesuit$ ,  $8 \spadesuit$ ,  $4 \spadesuit$ ,  $2 \spadesuit$ .

Figure 3
Order of suits



No matter which three cards the magician's assistant has after choosing the hidden card and laying down the key card, the cards can be put in order as low, medium, or high. This order, or permutation, can represent a code for an addend. The code will tell the magician how many spaces on the clock to add to the key card, which is in the first position face up on the table, to get to the hidden card.

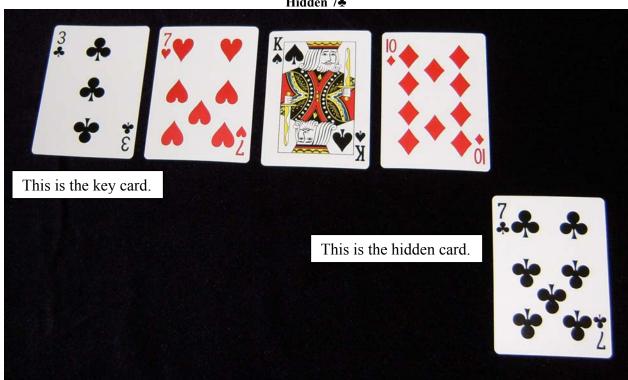
Using the permutations the 3 cards can be arranged in 3! = 6 distinct orders, as shown in the code below.

- 1 = low, middle, high
- 2 = low, high, middle
- 3 = middle, low, high
- 4 = middle, high, low
- 5 = high, low, middle
- 6 = high, middle, low.

Let's work out a few examples. The assistant gets the following cards from the volunteer: 3 - 7 - 10 - 7 = 10, and 6 - 10 - 10 = 10. The assistant chooses the 2 cards of the same

suit which are the 3 and 7 of clubs. The assistant decides that the 74 will be the hidden card since  $3 + 4 \equiv 7 \pmod{13}$ . Only this order will work since  $7 + 9 \equiv 3 \pmod{13}$  requires an addend of 9 and 9 > 6. The 3 $\clubsuit$ , the key card, is laid down in the first position on the table and the remaining three cards are arranged according to the order of suits corresponding to the permutation middle, high, low, the code to show that 4 should be added to the 34, to get to the hidden 7♣ using the clock model previously discussed.

Figure 4 Hidden 7♣



Now let's try an example going from a card with a higher value to a card with a lower one. This time let's say that the cards the volunteer selects are: 5♣, A♠, 8♣, 4♠ and J♠ (see Figure 5). The two cards of matching suit are the 4♠ and J♠. The assistant must hide the 4♠ and make the J♠ the key card. Using the clock arrangement (see Figure 1), and counting clockwise, we need to add six to the J $\triangleq$  to get to 4 $\triangleq$ . In (mod 13) we show this as 11 (jack) + 6  $\equiv$  4 (mod 13). The assistant needs to select the hidden card with care since moving from 4♠ to J♠ on the clock

would require adding seven positions,  $4 + 7 \equiv 11(\text{jack}) \pmod{13}$ . (With only the six permutations of 3 objects for the code, this arrangement does not work for the trick.) Therefore, the remaining three cards, 54, 84, A4, need to show "add 6", and so are arranged in high, middle, low  $(A \blacklozenge, 8 \clubsuit, 5 \clubsuit)$  order according to the permutation code.

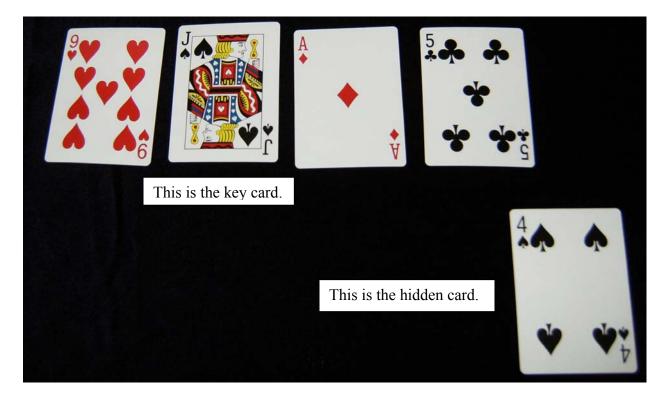
Figure 5 J♠ add 6 to 4♠ This is the key card. This is the hidden card.

#### **Variations**

After performing the trick once or twice, it might become obvious that the key card in the first position on the table is the clue to the suit of the hidden card. One way to vary the position of the key card is to use modular arithmetic in a new way. Look at the set of cards as shown in Figure 6:  $5 \clubsuit$ ,  $A \spadesuit$ ,  $9 \heartsuit$ ,  $4 \spadesuit$  and  $J \spadesuit$ . The  $4 \spadesuit$  is hidden, so we find the sum of the remaining four cards giving the ace the value of 1 and the face cards the values of jack = 11, queen = 12, and king = 13. In this case the sum of the four face up cards is 26. Convert the sum to its congruent value in (mod 4),  $26 \equiv 2 \pmod{4}$  and place the key card,  $J_{\bullet}$ , in the second position, which

corresponds to the congruency statement. The remaining 3 cards are put in the order of high, middle, low, around the key card for the permutation code indicating add 6. The magician comes in, quickly finds the sum of the four cards, determines the position of the key card, looks at the arrangement, and correctly announces the hidden 4. This variation would only require a bit more preparation and practice, but it would make the trick much more difficult for a spectator to figure out after seeing it just a couple of times.

Figure 6
Key card (mod 4)



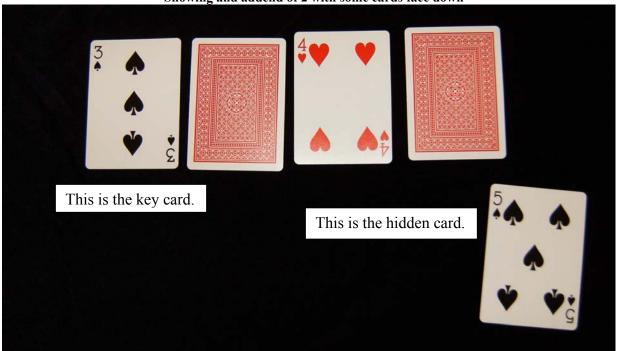
Now, let's imagine another variation to make the trick appear even more difficult for the magician. This time, let's change the code from the arrangement of the three cards from low, middle, and high, to face up or face down. A good bit of drama could be added here for effect. Can the magician guess the missing card with only three cards to see? What about two? Suppose the volunteer hands the assistant the following cards:  $K \clubsuit$ ,  $2 \spadesuit$ ,  $4 \heartsuit$ ,  $3 \spadesuit$ , and  $5 \spadesuit$ . The cards of matching suit are the  $3 \spadesuit$  and  $5 \spadesuit$ , so the  $5 \spadesuit$  is hidden. The  $3 \spadesuit$  is the key card and the remaining 3

cards need to show the magician to add 2 to the 3♠ to get to 5♠ using the clock model. Using a different code for the six permutations of the three remaining cards showing them face up (U) or face down (D), we can now present the six addends this way:

- 1 = U D D
- 2 = D U D
- 3 = DDU
- 4 = D U U
- 5 = UDU
- 6 = U U D

In this case with the key card,  $3 \spadesuit$ , and the hidden card,  $5 \spadesuit$ , we want to show add 2, so we arrange the cards D U D (see Figure 7) after the key card.

Figure 7 Showing and addend of 2 with some cards face down



There are even more variations of the trick. In one variation called Fitch Four Glory, four cards are selected and only three are displayed. To make this trick work, the suits need to be redefined from the usual four suits of 13 to just three suits of 17 each. These new suits are called alpha, beta, and gamma. Alpha is the entire suit of 13 clubs, A♣ through K♣, with the following

4 spades added after the K $\clubsuit$  to complete the suit of 17:  $2\spadesuit$  (14),  $3\spadesuit$  (15),  $4\spadesuit$  (16), and  $5\spadesuit$  (17). Beta is the entire suit of 13 diamonds with the next four spades added after the K♦ to complete the suit of 17: 64, 74, 84, and 94. The third suit, gamma, is the entire suit of 13 hearts with the next four spades added after the K♥ to complete the third suit of 17: 10♠, J♠, Q♠, and K♠. This leaves three "super" suits with 17 cards each and the A♠ left over. With these three new suits, we could use (mod 17) in the clock arrangement like we did with the original trick in (mod 13). In this variation there would be differences of up to eight positions between any two cards. Therefore, the code must be modified to find three new arrangements. The "super" suits are ordered low to high starting with alpha: A♣ through K♣, then 2♠, 3♠, 4♠, and 5♠; followed by beta: A♦ through K♦, 6♠, 7♠, 8♠, and 9♠; and finally gamma: A♥ through K♥, 10♠, J♠, Q♠, and K♠. Since A♠ is not part of the "super" suits, it does not follow this order. Combining the low, middle, high permutations and the face up and face down code, the addend 7 could be made by keeping the three cards face up and placing the key card in the first position, followed by the lower of the two remaining cards, and finally the higher of the two. The addend 8 could be made by arranging the three cards face up with the key card first, followed by the higher of the two remaining cards and finally the lower of the two. The complete code for the Fitch Four Glory variation would be:

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1 = U D D (key card in position 1)
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<sup>2 =</sup> D U D (key card in position 2)

<sup>3 =</sup> D D U (key card in position 3)

<sup>4 =</sup> D U U (key card in position 2)

<sup>5 =</sup> U D U (key card in position 1)

<sup>6 =</sup> U U D (key card in position 1)

<sup>7 =</sup> U key card, U lower, U higher

<sup>8 =</sup> U key card, U higher U lower

A = D D D

As seen in the code, each of the other arrangements 1-6 would include at least one card face up, so the first face up card would be the key card and would communicate the suit of the hidden card. If the hidden card is the  $A \spadesuit$ , which was excluded from our three super suits, the assistant would lay the remaining three cards face down and the magician would guess the  $A \spadesuit$ .

#### Conclusion

The Cheney Five Card Trick is a prime example of using mathematics for recreation in a magical application. The Five Card Trick, like the best mathematical magic, "combines the beauty of mathematical structure with the entertainment value of a trick" (Gardner, 1956, p. xi). We have seen that throughout history people have always enjoyed using math in games with dice and cards. People began by playing games of chance with just a few simple materials and concepts. Over time, our games have become more complex and elaborate, but the mathematical concepts, the thrill of chance, and the allure of the unknown has remained the same. When mathematics, magic, and mystery are combined like this we can truly experience the joy of mathematics.

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