Pushforward / Jvp rule for matrix-vector multiplication

$$f(\underline{\underline{A}}, \underline{x}) = \underline{\underline{A}}\,\underline{x} =: \underline{z}$$

$$\underline{\underline{A}} \in \mathbb{R}^{M \times N} \qquad \underline{x} \in \mathbb{R}^{N} \qquad \underline{z} \in \mathbb{R}^{M}$$

$$\dot{\underline{z}} = \frac{\partial f}{\partial \underline{\underline{A}}} \dot{\underline{\underline{A}}} + \frac{\partial f}{\partial \underline{x}} \dot{\underline{x}}$$

→ index notation

$$f(\underline{\underline{A}}, \underline{x}) = A_{ij} x_j = z_i$$

→ pushforward rule in index notation

$$\dot{z}_i = \frac{\partial z_i}{\partial A_{kl}} \dot{A}_{kl} + \frac{\partial z_i}{\partial x_k} \dot{x}_k$$

$$\rightarrow \frac{\partial z_i}{\partial A_{kl}} = \frac{\partial A_{ij}}{\partial A_{kl}} x_j = \delta_{ik} \delta_{jl} x_j$$

$$\rightarrow \frac{\partial z_i}{\partial x_k} = A_{ij} \frac{\partial x_j}{\partial x_k} = A_{ij} \delta_{jk}$$

$$\rightarrow \dot{z}_i = \delta_{ik} \delta_{jl} x_j \dot{A}_{kl} + A_{ij} \delta_{jk} \dot{x}_k$$

$$\dot{z}_i = x_j \dot{A}_{ij} + A_{ij} \dot{x}_j$$

→ back to symbolic notation

$$\dot{\underline{z}} = \dot{\underline{\underline{A}}}\,\underline{x} + \underline{\underline{A}}\,\dot{\underline{x}}$$

$$\boxed{\tilde{f}\left(f, (\underline{\underline{A}}, \underline{x}), (\dot{\underline{\underline{A}}}, \dot{\underline{x}})\right) = \left(\underbrace{(\underline{\underline{A}}\,\underline{x})}_{\underline{z}}, \underbrace{(\dot{\underline{\underline{A}}}\,\underline{x} + \underline{\underline{A}}\,\dot{\underline{x}})}_{\dot{\underline{z}}}\right)}$$