Euler-Lagrange Equation

$$I(y) = \int_0^T \dot{y}^2_{(t)} - 4y_{(t)}\, dt$$

Optimize?

$$\frac{\partial I}{\partial y} \overset{!}{=} 0$$

$$= \frac{d I(y+\varepsilon\varphi)}{d\varepsilon}\Big|_{\varepsilon=0} \overset{!}{=} 0$$

Is there a simpler way?

Tedious

$$I(y) = \int_a^b \underbrace{\mathcal{L}(t, y_{(t)}, \dot{y})}_{\text{Lagrangian Mechanics}}\, dt$$

e.g. $\mathcal{L} = \dot{y}^2 - 4y$

$$\frac{\partial I}{\partial y} \overset{!}{=} 0$$

$$\frac{d I(y+\varepsilon\varphi)}{d\varepsilon}\Big|_{\varepsilon=0} \overset{!}{=} 0$$

does not depend on time

$$I(y+\varepsilon\varphi) = \int_a^b \mathcal{L}(t, y+\varepsilon\varphi, \dot{\overline{y+\varepsilon\varphi}})\, dt$$

$$= \int_a^b \mathcal{L}(t, \overset{y}{y+\varepsilon\varphi}, \overset{\dot{y}}{\dot{y}+\varepsilon\dot{\varphi}})\, dt$$

$$\frac{d I(y+\varepsilon\varphi)}{d\varepsilon} = \int_a^b \frac{\partial\mathcal{L}}{\partial y}\cdot\frac{\partial(y+\varepsilon\varphi)}{\partial\varepsilon} + \frac{\partial\mathcal{L}}{\partial\dot{y}}\frac{\partial(\dot{y}+\varepsilon\dot{\varphi})}{\partial\varepsilon}\, dt$$

Substantial

$$= \int_a^b \frac{\partial\mathcal{L}}{\partial y}\varphi + \frac{\partial\mathcal{L}}{\partial\dot{y}}\dot{\varphi}\, dt$$

$$\frac{d I(y+\varepsilon\varphi)}{d\varepsilon}\Big|_{\varepsilon=0} = \int_a^b \frac{\partial\mathcal{L}}{\partial y}\varphi + \frac{\partial\mathcal{L}}{\partial\dot{y}}\dot{\varphi}\, dt$$

P.I.

$$= \int_a^b \frac{\partial\mathcal{L}}{\partial y}\varphi\, dt + \int_a^b \frac{\partial\mathcal{L}}{\partial\dot{y}}\dot{\varphi}\, dt$$

$$= \int_a^b \frac{\partial\mathcal{L}}{\partial y}\varphi\, dt + \left[\frac{\partial\mathcal{L}}{\partial\dot{y}}\varphi\right]_a^b - \int_a^b \left(\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{y}}\right)\right)\varphi\, dt$$

$=0$, φ vanishes at boundary

$$= \int_a^b \underbrace{\left(\frac{\partial\mathcal{L}}{\partial y} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{y}}\right)\right)}_{\text{Fundamental Lemma of Calculus of Variations}}\varphi\, dt \overset{!}{=} 0$$

Euler Lagrange Equation

$$\boxed{\frac{\partial\mathcal{L}}{\partial y} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{y}}\right) = 0}$$

Example:

$$\mathcal{L} = \dot{y}^2 - 4y$$

$$-4 - \frac{d}{dt}(2\dot{y}) = 0$$

$$-4 - 2\ddot{y} = 0$$

$$\Leftrightarrow \boxed{\ddot{y} = -2}$$

Newton Mechanics

$F = m a$

One more thing

consider $\mathcal{L}(t, y)$ instead of $\mathcal{L}(t, y, \dot{y})$

then

$$\frac{\partial I}{\partial y} \overset{!}{=} 0 \quad \rightarrow \quad \frac{\partial\mathcal{L}}{\partial y} - \underbrace{\frac{d}{dt}\left(\cancel{\frac{\partial\mathcal{L}}{\partial\dot{y}}}\right)}_{=0} = 0$$

instead of Functional Derivative we can do a classical partial derivative