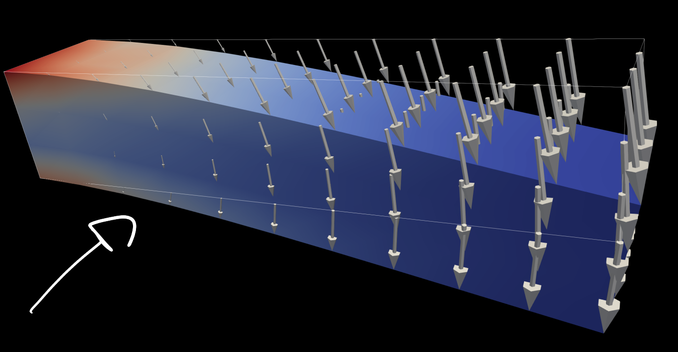
Weak Form for Linear Elasticity

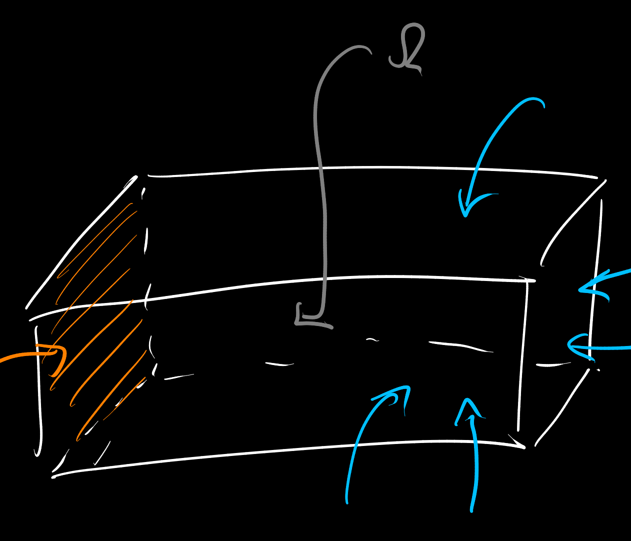


strong

$$-\nabla \cdot \underline{\underline{\sigma}} = \underline{f}$$

weak:

$$\int_{\Omega} \underline{\underline{\sigma}}(\underline{u}) : \underline{\underline{\varepsilon}}(\underline{v})\, dV = \int_{\Omega} \underline{f}^T \underline{v}\, dV + \int_{\partial\Omega} \underline{t}^T \underline{v}\, dS$$



free faces

$\Rightarrow$ no traction

$\underline{t} = \underline{0}$

face clamped

$\underline{u} = \underline{0}$

$\rightarrow$ under own weight with gravity $\underline{g}$

Boundary Value Problem

$$\begin{cases} -\nabla \cdot \underline{\underline{\sigma}} = \underline{f} & , \underline{x} \in \Omega \\ \underline{\underline{\sigma}} = \lambda\, tr(\underline{\underline{\varepsilon}})\, \underline{\underline{I}}_3 + 2\mu \underline{\underline{\varepsilon}} & \\ \underline{\underline{\varepsilon}} = \frac{1}{2}\left(\nabla \underline{u} + (\nabla \underline{u})^T\right) & \\ \underline{u} = \underline{0} & \text{on left face} \\ \underline{\underline{\sigma}}\, \underline{n} = \underline{0} & \text{on all other faces} \end{cases}$$

Derive weak form

(1) Multiply with a test function $\underline{v}$ ← a 3D vector field like displacement $\underline{u}$

$$\underbrace{(\underbrace{-\nabla \cdot \underline{\underline{\sigma}}}_{\in \mathbb{R}^3})^T \underline{v}}_{\in \mathbb{R}} = \underline{f}^T \underline{v}$$

(2) Integrate over the domain Ω

$$\int_{\Omega} (-\nabla \cdot \underline{\underline{\sigma}})^T \underline{v}\, dV = \int_{\Omega} \underline{f}^T \underline{v}\, dV$$

(3) Integration by parts

turn $(\nabla \cdot \underline{\underline{\sigma}})^T \underline{v}$ into sth. like "$\underline{\underline{\sigma}} : \nabla v$"

reverse product rule

$$\underbrace{\nabla \cdot (\underbrace{\underbrace{\underline{\underline{\sigma}}}_{\mathbb{R}^{3\times3}} \underbrace{\underline{v}}_{\mathbb{R}^3}}_{\mathbb{R}^3})}_{\mathbb{R}} = \underbrace{\underbrace{(\nabla \cdot \underbrace{\underline{\underline{\sigma}}}_{\mathbb{R}^{3\times3}})^T}_{\mathbb{R}^3} \underline{v}}_{\mathbb{R}} + \underbrace{\underbrace{\underline{\underline{\sigma}}}_{\mathbb{R}^{3\times3}} : \underbrace{\nabla \underbrace{\underline{v}}_{\mathbb{R}^3}}_{\mathbb{R}^{3\times3}}}_{\mathbb{R}}$$

$$\frac{\partial(\sigma_{ij} v_j)}{\partial x_i} = \underbrace{\frac{\partial \sigma_{ij}}{\partial x_i}}_{\text{divergence}} v_j + \sigma_{ij} \underbrace{\frac{\partial v_j}{\partial x_i}}_{\text{gradient}}$$

$$\Rightarrow -(\nabla \cdot \underline{\underline{\sigma}})^T \underline{v} = -\nabla \cdot (\underline{\underline{\sigma}}\, \underline{v}) + \underline{\underline{\sigma}} : (\nabla \underline{v})$$

$\rightarrow$ back into integral

$$\underbrace{\int_{\Omega} -\nabla \cdot (\underline{\underline{\sigma}}\, \underline{v})\, dV}_{\text{Gauss / Divergence theorem}} + \int_{\Omega} \underline{\underline{\sigma}} : (\nabla \underline{v})\, dV = \int_{\Omega} \underline{f}^T \underline{v}\, dV$$

$$\hat{=} \int_{\partial\Omega} -(\underline{\underline{\sigma}}\, \underline{v})^T \underline{n}\, dS$$

$\rightarrow$ Preliminary weak form

$$\boxed{\int_{\Omega} \underbrace{\underline{\underline{\sigma}} : (\nabla \underline{v})}_{\mathrm{II}}\, dV = \int_{\Omega} \underline{f}^T \underline{v}\, dV + \int_{\partial\Omega} \underbrace{(\underline{\underline{\sigma}}\, \underline{v})^T \underline{n}}_{\mathrm{I}}\, dS}$$

(I) $(\underline{\underline{\sigma}}\, \underline{v})^T \underline{n} = \underline{v}^T \underline{\underline{\sigma}}^T \underline{n} = \underline{v}^T \underbrace{\underline{\underline{\sigma}}\, \underline{n}}_{=:\underline{t}} = \underline{v}^T \underline{t} = \underline{t}^T \underline{v}$ ← here $\underline{t} = \underline{0}$

(II) $\nabla \underline{v} \in \mathbb{R}^{3\times3}$

$$\nabla \underline{v} = sym(\nabla \underline{v}) + antisym(\nabla \underline{v})$$

← $= \frac{1}{2}(\underline{\underline{A}} + \underline{\underline{A}}^T)$

e.g. $\underline{\underline{A}} = \begin{bmatrix} 8 & 2 & 1 \\ 3 & 7 & 2 \\ 2 & 2 & 3 \end{bmatrix} \rightarrow sym(\underline{\underline{A}}) = \begin{bmatrix} 8 & 2.5 & 1.5 \\ 2.5 & 7 & 2 \\ 1.5 & 2 & 3 \end{bmatrix}$

← $\underline{\underline{A}} - sym(\underline{\underline{A}})$

$\rightarrow antisym(\underline{\underline{A}}) = \begin{bmatrix} 0 & -0.5 & -0.5 \\ 0.5 & 0 & 0 \\ 0.5 & 0 & 0 \end{bmatrix}$ $\rightarrow$ skew-symmetric

$$\underline{\underline{\sigma}} : \nabla \underline{v} = \underline{\underline{\sigma}} : sym(\nabla \underline{v}) + \underbrace{\underline{\underline{\sigma}} : antisym(\nabla \underline{v})}_{=0}$$

$\underline{\underline{\sigma}} = \begin{bmatrix} 5 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

$\underline{\underline{\sigma}} : antisym(\underline{\underline{A}}) = -0.5 - 1 + 0.5 + 1 = 0$

$\Rightarrow$ product of a skew-symmetric and a symmetric matrix will always be 0

$$\Rightarrow \underline{\underline{\sigma}} : \nabla \underline{v} = \underline{\underline{\sigma}} : sym(\nabla \underline{v}) = \underline{\underline{\sigma}}(\underline{u}) : \underline{\underline{\varepsilon}}(\underline{v})$$

Final weak form

$$\boxed{\int_{\Omega} \underline{\underline{\sigma}}(\underline{u}) : \underline{\underline{\varepsilon}}(\underline{v})\, dV = \int_{\Omega} \underline{f}^T \underline{v}\, dV + \int_{\partial\Omega} \underline{t}^T \underline{v}\, dS}$$