Pullback/vJp rule for matrix-vector multiplication

$$f(\underline{\underline{A}}, \underline{x}) = \underline{\underline{A}}\,\underline{x} =: \underline{z}$$

$$\underline{\underline{A}} \in \mathbb{R}^{M \times N} \qquad \underline{x} \in \mathbb{R}^{N} \qquad \underline{z} \in \mathbb{R}^{M}$$

task: backpropagate $\bar{\underline{z}} \in \mathbb{R}^{M}$ to $\bar{\underline{\underline{A}}} \in \mathbb{R}^{M \times N}$

& $\bar{\underline{x}} \in \mathbb{R}^{N}$

$$\bar{\underline{\underline{A}}} = \bar{\underline{z}}^T \frac{\partial f}{\partial \underline{\underline{A}}} \qquad \bar{\underline{x}} = \bar{\underline{z}}^T \frac{\partial f}{\partial \underline{x}}$$

index notation

primal: $z_i = f(\underline{\underline{A}}, \underline{x}) = A_{ij} x_j$

(2) $$\bar{A}_{kl} = \bar{z}_i \frac{\partial z_i}{\partial A_{kl}}$$

(1) $$\bar{x}_k = \bar{z}_i \frac{\partial z_i}{\partial x_k}$$

(1) $$\Rightarrow \frac{\partial z_i}{\partial x_k} = A_{ij} \frac{\partial x_j}{\partial x_k} = A_{ij} \delta_{jk}$$

$$\bar{x}_k = \bar{z}_i A_{ij} \delta_{jk}$$

$$\bar{x}_k = \bar{z}_i A_{ik}$$

back to symbolic notation

$$\bar{\underline{x}} = \underline{\underline{A}}^T \bar{\underline{z}}$$

(2) $$\Rightarrow \frac{\partial z_i}{\partial A_{kl}} = \frac{\partial A_{ij}}{\partial A_{kl}} x_j = \delta_{ik} \delta_{jl} x_j$$

$$\bar{A}_{kl} = \bar{z}_i \delta_{ik} \delta_{jl} x_j$$

$$\Rightarrow \bar{A}_{kl} = \bar{z}_k x_l$$

in symbolic notation

$$\bar{\underline{\underline{A}}} = \bar{\underline{z}}\, \underline{x}^T$$

$$\boxed{\mathcal{B}\left(f, (\underline{\underline{A}}, \underline{x}), (\bar{\underline{z}},)\right) = \left((\underbrace{\underline{\underline{A}}\,\underline{x}}_{\underline{z}}), (\underbrace{\bar{\underline{z}}\,\underline{x}^T}_{\bar{\underline{\underline{A}}}}, \underbrace{\underline{\underline{A}}^T \bar{\underline{z}}}_{\bar{\underline{x}}}) \right)}$$