

Karush-Kuhn-Tucker (KKT) conditions

Equality-constrained optimization

$$\begin{aligned} \min_x \quad & f(x) \\ \text{s.t.} \quad & h(x) = \underline{0} \end{aligned}$$

Lagrangian

$$\mathcal{L}(x, \lambda) = f(x) + \lambda^T h(x)$$

$$p_d \stackrel{!}{=} \underline{0}$$

Inequality-constrained optimization

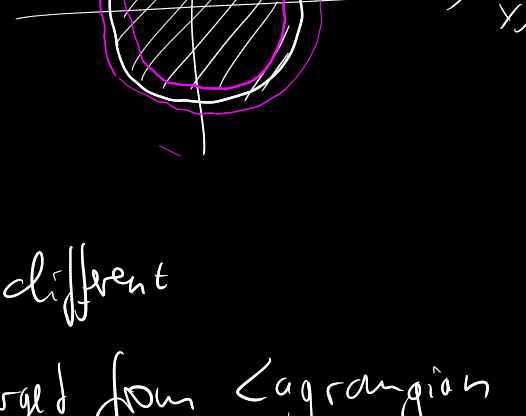
$$\begin{aligned} \min_x \quad & f(x) \\ \text{s.t.} \quad & g(x) \leq \underline{0} \end{aligned}$$

What to do here?

$$\begin{aligned} \mathcal{L}(x, \mu) &= f(x) + \mu^T \underline{g}(x) \\ &= f(x) + \sum_{i=0}^{n-1} \mu_i g_i(x) \end{aligned}$$

and set gradient to 0

does not work $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



only finds solutions where $\underline{g}(x) = \underline{0}$

We need sth. different

→ Recover target from Lagrangian

$$f(x) = \max_{\mu \geq \underline{0}} \left(\mathcal{L}(x, \mu) \right) = \max_{\mu \geq \underline{0}} \left(f(x) + \sum_{i=0}^{n-1} \mu_i g_i(x) \right)$$

all $\mu_i \geq 0 \forall i$

• assume $M=3$ (3 ineq constraints)

violate $g_1(x)$

$$g_1(x) > 0$$

$$\text{then } \max_{\mu \geq 0} (\dots) = \varnothing$$

• no $g_i(x)$ is violated

$$g_i(x) \leq 0$$

$$g_i(x) < 0 \rightarrow \mu_i = 0 \text{ to have the max}$$

$$g_i(x) = 0 \rightarrow \mu_i \text{ is free but doesn't affect maximum}$$

$$f(x) = \max_{\mu \geq \underline{0}} \left(f(x) + \sum_{i=0}^{n-1} \mu_i g_i(x) \right)$$

Original problem

$$\begin{aligned} \min_x \quad & f(x) \\ \text{s.t.} \quad & \underline{g}(x) \leq \underline{0} \end{aligned}$$

transformation

$$\min_x \max_{\mu \geq \underline{0}} \left(f(x) + \sum_{i=0}^{n-1} \mu_i g_i(x) \right)$$

→ unconstrained optimization

Disclaimer:

We would inf instead of min and sup instead of max

We need the original problem in standard

• minimization (change sign of target)

• inequalities are " \leq " smaller equal (multiply by (-1))

$$\min_x \max_{\mu \geq \underline{0}} \left(\underbrace{f(x) + \sum_{i=0}^{n-1} \mu_i g_i(x)}_{f(x) = \max_{\mu \geq \underline{0}} \mathcal{L}(x, \mu)} \right) \geq \max_{\mu \geq \underline{0}} \min_x \left(\underbrace{f(x) + \sum_{i=0}^{n-1} \mu_i g_i(x)}_{\varphi(\mu) = \min_x \mathcal{L}(x, \mu)} \right)$$

→ Max-Min Inequality (holds in general)

$$\left(\begin{array}{l} \min_x f(x) \\ \text{s.t. } \underline{g}(x) \leq \underline{0} \end{array} \right) = p^* \geq q^* = \left(\begin{array}{l} \max_{\mu} \varphi(\mu) \\ \text{s.t. } \mu \geq \underline{0} \end{array} \right)$$

primal version

dual version

p^*
optimal value of the primal

q^* ... optimal problem of the dual

$p^* - q^*$... duality gap

Can we say sth. about the duality gap

↳ it is " 0 " for "regular" problems

→ strong duality

\star

Now assume we have a regular

$$\left(\begin{array}{l} \min_x f(x) \\ \text{s.t. } \underline{g}(x) \leq \underline{0} \end{array} \right) = p^* = q^* = \left(\begin{array}{l} \max_{\mu} \varphi(\mu) \\ \text{s.t. } \mu \geq \underline{0} \end{array} \right)$$

p^* is attained at x^*

q^* is attained at μ^*

$$\begin{aligned} \text{Hence: } f(x^*) &= p^* = q^* = \varphi(\mu^*) \\ &= \min_x \left(f(x) + \sum_{i=0}^{n-1} \mu_i^* g_i(x) \right) \\ &= f(x^*) + \underbrace{\sum_{i=0}^{n-1} \mu_i^* g_i(x^*)}_{=0} \\ &= f(x^*) \end{aligned}$$

KKT for (x^*, μ^*) point in order to have x^* as a constrained min

I Primal feasibility:

$$\underline{g}(x^*) \leq \underline{0}$$

II Stationarity:

$$\nabla f(x) \Big|_{x=x^*} + \sum_{i=0}^{n-1} \mu_i^* \nabla g_i(x) \Big|_{x=x^*} \stackrel{!}{=} \underline{0}$$

III Dual feasibility:

$$\mu^* \geq \underline{0}$$

IV Complementary Slackness

$$\sum_{i=0}^{n-1} \mu_i^* g_i(x^*) = 0$$

with always be pos with always be neg with always neg

→ all components in summation have to be 0

$$\mu_i^* g_i(x^*) = 0 \quad \forall i$$

"either μ_i^* is zero or $g_i(x^*)$ is zero"

* Regularity?

When is a problem regular

(there are plenty of conditions)

↳ Slater's condition (→ convex problem)

$$\exists \tilde{x}$$

• satisfies all non-affine but convex constraints strictly:

$$g_{[1 \dots j]}(\tilde{x}) < 0$$

• satisfies all affine constraints:

$$g_{[j+1 \dots J]}(\tilde{x}) \leq 0$$