(Multivariate) Completing the Square

$$ax^2+bx+c \longrightarrow a(x-d)^2+e$$

$$\underline{x}^T\underline{\underline{A}}\,\underline{x}+\underline{x}^T\underline{b}+c \longrightarrow (\underline{x}-\underline{d})^T\underline{\underline{A}}(\underline{x}-\underline{d})+e$$

Univariate Case

$$ax^2+bx+c \longrightarrow a(x-d)^2+e$$

$$a(x^2-2xd+d^2)+e$$

$$ax^2-2adx+ad^2+e$$

$$\boxed{a=a}$$

$$b=-2ad \iff \boxed{d=-\frac{b}{2a}}$$

$$c=ad^2+e$$

$$c=a\cdot\left(-\frac{b}{2a}\right)^2+e$$

$$=\cancel{a}\cdot\frac{b^2}{4a^{\cancel{2}}}+e$$

$$\iff \boxed{e=c-\frac{b^2}{4a}}$$

Example

$$5x^2-3x+7 \longrightarrow a=5,\ b=-3,\ c=7$$

$$a=5,\quad d=\frac{-(-3)}{2\cdot 5}=\frac{3}{10}$$

$$e=7-\frac{(-3)^2}{4\cdot 5}=$$

$$7-\frac{9}{20}=\frac{140-9}{20}=\frac{131}{20}$$

$$5\cdot\left(x-\frac{3}{10}\right)^2+\frac{131}{20}$$

$$5\cdot\left(x^2-\frac{3}{5}x+\frac{9}{100}\right)+\frac{131}{20}$$

$$5x^2-\frac{\cancel{15}^{\,3}}{\cancel{5}}x+\frac{45}{100}+\frac{131}{20}$$

$$5x^2-3x+\frac{(45+655)}{\cancel{100}}\ \cancel{700}\ 7$$

Multivariate Case

$$\underbrace{\underline{x}}_{\text{vector}}{}^T\underbrace{\underline{\underline{A}}}_{\text{matrix}}\underline{x}+\underbrace{\underline{b}}_{\text{vector}}{}^T\underline{x}+\underbrace{c}_{\text{scalar}} \longrightarrow (\underline{x}-\underline{d})^T\overset{\text{matrix}}{\underline{\underline{A}}}(\underline{x}-\overset{\text{vector}}{\underline{d}})+\overset{\text{scalar}}{e}$$

$$\underline{x}^T\underline{\underline{A}}\,\underline{x}-\underline{x}^T\underline{\underline{A}}\,\underline{d}-\underline{d}^T\underline{\underline{A}}\,\underline{x}$$
$$+\underline{d}^T\underline{\underline{A}}\,\underline{d}+e$$

Scalar ξ

$$\xi=\xi^T$$

$$\underline{d}^T\underline{\underline{A}}\,\underline{x}=(\underline{d}^T\underline{\underline{A}}\,\underline{x})^T$$

$$=\underline{x}^T\underline{\underline{A}}^T\underline{d}$$

let $\underline{\underline{A}}$ be symmetric

$$\underline{\underline{A}}=\underline{\underline{A}}^T$$

$$\underline{x}^T\underline{\underline{A}}\,\underline{x}+\underline{x}^T\underline{b}+c \qquad \underline{x}^T\underline{\underline{A}}\,\underline{x}-2\underline{x}^T\underline{\underline{A}}\,\underline{d}+\underline{d}^T\underline{\underline{A}}\,\underline{d}+e$$

$$\boxed{\underline{\underline{A}}=\underline{\underline{A}}}$$

$$\underline{b}=-2\underline{\underline{A}}\,\underline{d} \iff \boxed{\underline{d}=-\frac{1}{2}\underline{\underline{A}}^{-1}\underline{b}}$$

$$c=\underline{d}^T\underline{\underline{A}}\,\underline{d}+e$$

$$c=\left(-\frac{1}{2}\underline{\underline{A}}^{-1}\underline{b}\right)^T\underline{\underline{A}}\left(-\frac{1}{2}\underline{\underline{A}}^{-1}\underline{b}\right)+e$$

$$c=\frac{1}{4}\underline{b}^T\underline{\underline{A}}^{-T}\underbrace{\underline{\underline{A}}\,\underline{\underline{A}}^{-1}}_{\underline{\underline{I}}}\underline{b}+e$$

$$\underline{\underline{A}}=\underline{\underline{A}}^T \rightarrow \underline{\underline{A}}^{-1}=\underline{\underline{A}}^{-T}$$

$$c=\frac{1}{4}\underline{b}^T\underline{\underline{A}}^{-1}\underline{b}+e$$

$$\iff \boxed{e=c-\frac{1}{4}\underline{b}^T\underline{\underline{A}}^{-1}\underline{b}}$$

Example:

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}^T\underbrace{\begin{bmatrix}1&1\\1&2\end{bmatrix}}_{\underline{\underline{A}}}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}x_1\\x_2\end{bmatrix}^T\underbrace{\begin{bmatrix}2\\4\end{bmatrix}}_{\underline{b}}+\underbrace{3}_{c}$$

$$(\underline{\underline{A}}\ \ldots\ \text{sym}:\ \underline{\underline{A}}^T=\underline{\underline{A}})$$

$$\underline{\underline{A}}^{-1}=\frac{1}{2-1}\begin{bmatrix}2&-1\\-1&1\end{bmatrix}=\begin{bmatrix}2&-1\\-1&1\end{bmatrix}$$

(never invert matrices)

$$\underline{d}=-\frac{1}{2}\underline{\underline{A}}^{-1}\underline{b}=-\frac{1}{2}\begin{bmatrix}2&-1\\-1&1\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix}=-\frac{1}{2}\begin{bmatrix}0\\2\end{bmatrix}$$

$$=\begin{bmatrix}0\\-1\end{bmatrix}$$

$$e=c-\frac{1}{4}\underline{b}^T\underline{\underline{A}}^{-1}\underline{b}$$

$$=3-\frac{1}{4}\cdot\begin{bmatrix}2\\4\end{bmatrix}^T\underbrace{\begin{bmatrix}2&-1\\-1&1\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix}}_{\begin{bmatrix}0\\2\end{bmatrix}}$$

$$=3-\frac{1}{4}(2\cdot 0+4\cdot 2)=3-2=1$$

$$\Rightarrow (\underline{x}-\underline{d})^T\underline{\underline{A}}(\underline{x}-\underline{d})+e$$

$$\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\cancel{-}+\begin{bmatrix}0\\\cancel{-}1\end{bmatrix}\right)^T\begin{bmatrix}1&1\\1&2\end{bmatrix}\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\cancel{-}+\begin{bmatrix}0\\\cancel{-}1\end{bmatrix}\right)+1$$