Weak Form for Navier-Stokes with Chorin's Projection

$$\begin{cases} \dfrac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \underline{\nabla})\underline{u} = -\dfrac{1}{\rho}\nabla p + \nu \nabla^2 \underline{u} \quad (+\underline{f}) \\ \underline{\nabla} \cdot \underline{u} = 0 \end{cases}$$

$\rho = 1$ here

BC: $\underline{u} = \underline{0}$ everywhere on domain boundary, except on top $\underline{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

IC: $\underline{u}(t=0, x) = \underline{0}$

Chorin's Projection $\Rightarrow$ weak form $\rightarrow$ FEniCS

Chorin's Projection Method

transient + convection = pressure grad + diffusion

$$\frac{\partial u}{\partial t} = -\text{conv} + \text{pres.grd} + \text{diffusion}$$

$$\frac{\underline{u}^{\cancel{[t+1]}*} - \underline{u}^{[t]}}{\Delta t} = -\text{conv} + \underbrace{\cancel{\text{pres.grad}}}_{=0} + \text{diff}$$

$$\frac{\underline{u}^{*} - \underline{u}^{[t]}}{\Delta t} = -\text{conv} + \text{diff}$$

$\Rightarrow$ operator splitting

$$\frac{\underline{u}^{[t+1]} - \underline{u}^{*}}{\Delta t} = \text{pres.grad.}$$

1. $$\frac{\underline{u}^{*} - \underline{u}^{[t]}}{\Delta t} = \underbrace{-(\underline{u}^{[t]} \cdot \underline{\nabla})\underline{u}^{[t]}}_{\text{fully explicit convection}} + \underbrace{\nu \nabla^2 \underline{u}^{*}}_{\text{fully implicit diffusion}}$$ for $\underline{u}^{*}$

2. obtain the pressure ? How?

3. $$\frac{\underline{u}^{[t+1]} - \underline{u}^{*}}{\Delta t} = -\nabla p^{\cancel{[t+1]}}$$ $\rightarrow$ for $\underline{u}^{[t+1]}$

implicit/explicit does not matter here

Find pressure Poisson problem by taking the divergence of the update equation (3)

$$\nabla \cdot \left(\frac{\underline{u}^{[t+1]}}{\Delta t} - \frac{\underline{u}^{*}}{\Delta t} \right) = \nabla \cdot (-\nabla p)$$

$$\frac{\nabla \cdot \underline{u}^{[t+1]}}{\Delta t} - \frac{\nabla \cdot \underline{u}^{*}}{\Delta t} = -\nabla^2 p$$

$=0$, because velocity at next time step is incompressible

$$-\frac{\nabla \cdot \underline{u}^{*}}{\Delta t} = -\nabla^2 p$$

$$\nabla^2 p = \frac{1}{\Delta t} \nabla \cdot \underline{u}^{*}$$

In summary (Chorin's projection in strong form)

1. $$\frac{\underline{u}^{*} - \underline{u}^{[t]}}{\Delta t} = -(\underline{u}^{[t]} \cdot \underline{\nabla})\underline{u}^{[t]} + \nu \nabla^2 \underline{u}^{*}$$ "Tentative Momentum step"
2. $$\nabla^2 p = \frac{1}{\Delta t} \underline{\nabla} \cdot \underline{u}^{*}$$ "Pressure Poisson"
3. $$\frac{\underline{u}^{[t+1]} - \underline{u}^{*}}{\Delta t} = -\nabla p$$ "Velocity Correction/Projection"

$\Rightarrow$ Now: Find weak forms to all 3 equations

1. Tentative Momentum step

I. Multiply with a test function $\underline{v}$

$$\left(\frac{\underline{u}^{*} - \underline{u}^{[t]}}{\Delta t}\right)^T \underline{v} = -\left((\underline{u}^{[t]} \cdot \underline{\nabla})\underline{u}^{[t]}\right)^T \underline{v} + (\nu \nabla^2 \underline{u}^{*})^T \underline{v}$$

II. Integration over the domain Ω

$$\int_\Omega \left(\frac{\underline{u}^{*} - \underline{u}^{[t]}}{\Delta t}\right)^T \underline{v} \, dV = -\int_\Omega \left((\underline{u}^{[t]} \cdot \underline{\nabla})\underline{u}^{[t]}\right)^T \underline{v} \, dV + \int_\Omega (\nu \nabla^2 \underline{u}^{*})^T \underline{v} \, dV$$

III. Integration by parts

only here

$$\int_\Omega (\nabla^2 \underline{u})^T \underline{v} \, dV$$

(Typical problem, weak form of Laplace operator

in index notation (Einstein summation convention)

$$\frac{\partial^2 u_i}{\partial x_j \partial x_j} v_i$$

reverse product rule

$$\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} v_i\right) = \frac{\partial^2 u_i}{\partial x_j \partial x_j} v_i + \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}$$

symbolic notation

$$\underline{\nabla} \cdot ((\underline{\nabla}\underline{u})\underline{v}) = (\nabla^2 \underline{u})^T \underline{v} + (\nabla \underline{u}) : (\nabla \underline{v})$$

re-arrange for this

$$(\nabla^2 \underline{u})^T \underline{v} = \underline{\nabla} \cdot ((\nabla \underline{u})\underline{v}) - (\nabla \underline{u}) : (\nabla \underline{v})$$

back into the integral

$$\nu \int_\Omega (\nabla^2 \underline{u}^*)^T \underline{v} \, dV = \nu \underbrace{\int_\Omega \underline{\nabla} \cdot ((\underline{\nabla}\underline{u}^*)\underline{v}) \, dV}_{\text{Gauss / Divergence theorem}} - \nu \int_\Omega (\nabla \underline{u}^*) : (\nabla \underline{v}) \, dV$$

$$\stackrel{1}{=} \underbrace{\nu \int_{\partial\Omega} \underline{n}^T ((\underline{\nabla}\underline{u}^*)\underline{v}) \, dS}_{=0}$$

$\Rightarrow 0$ on the boundary

$\rightarrow$ weak form of the tentative momentum step

$$\int_\Omega \left(\frac{\underline{u}^{*} - \underline{u}^{[t]}}{\Delta t}\right)^T \underline{v} \, dV = -\int_\Omega \left((\underline{\nabla}\underline{u}^{[t]})\underline{u}^{[t]}\right)^T \underline{v} \, dV - \nu \int_\Omega (\underline{\nabla}\underline{u}^*) : (\underline{\nabla}\underline{v}) \, dV$$

2. Pressure Poisson Problem

I. Multiply with test function q

$$(\nabla^2 p) q = \frac{1}{\Delta t} (\nabla \cdot \underline{u}^*) q$$

II. Integrate over the domain

$$\int_\Omega q \nabla^2 p \, dV = \int_\Omega \frac{1}{\Delta t} (\underline{\nabla} \cdot \underline{u}^*) q \, dV$$

III. Integration by parts

$$-\int_\Omega (\underline{\nabla} p)^T (\underline{\nabla} q) \, dV = \int_\Omega \frac{1}{\Delta t} (\underline{\nabla} \cdot \underline{u}^*) q \, dV$$

Final weak Form

3. Velocity update step

I. Multiply with test function $\underline{v}$

$$\left(\frac{\underline{u}^{[t+1]} - \underline{u}^*}{\Delta t}\right)^T \underline{v} = -(\underline{\nabla} p)^T \underline{v}$$

II. Integration over domain

$$\int_\Omega \left(\frac{\underline{u}^{[t+1]} - \underline{u}^*}{\Delta t}\right)^T \underline{v} \, dV = \int_\Omega -(\underline{\nabla} p)^T \underline{v} \, dV$$

final weak form

III. Integration by parts $\rightarrow$ not necessary

Summary:

1. Tentative Momentum Step

$$0 = \frac{1}{\Delta t} \langle \underline{u}^* - \underline{u}^{[t]}, \underline{v} \rangle + \langle (\underline{\nabla}\underline{u}^{[t]})\underline{u}^{[t]}, \underline{v} \rangle + \nu \langle \underline{\nabla}\underline{u}^*, \underline{\nabla}\underline{v} \rangle$$

$\hookrightarrow$ for $\underline{u}^*$

2. Pressure Poisson Problem

$$\langle \underline{\nabla} p, \underline{\nabla} q \rangle = -\frac{1}{\Delta t} \langle \underline{\nabla} \cdot \underline{u}^*, q \rangle$$

$\hookrightarrow$ for p

3. Velocity Correction / Projection

$$\langle \underline{u}^{[t+1]}, \underline{v} \rangle = \langle \underline{u}^*, \underline{v} \rangle - \Delta t \langle \underline{\nabla} p, \underline{v} \rangle$$

$\hookrightarrow$ for $\underline{u}^{[t+1]}$