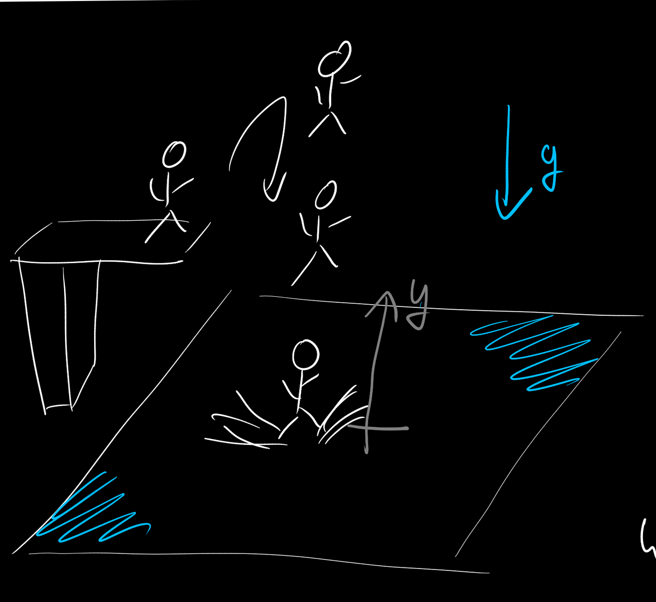
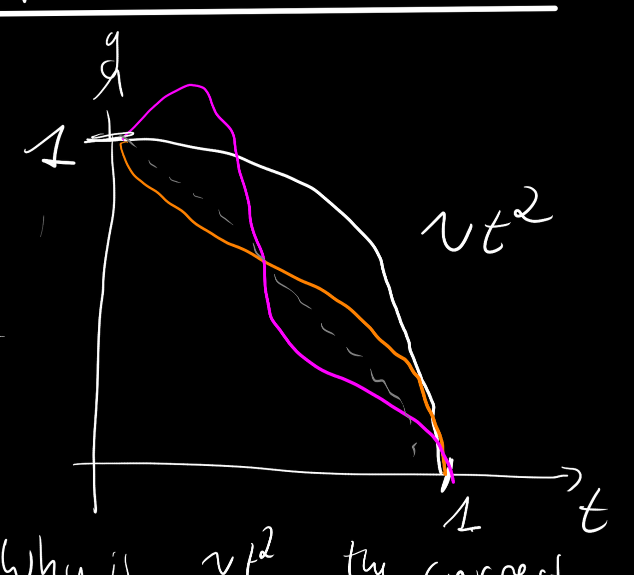
Intro Functionals & Functional Derivatives





Why is vt^2 the correct solution?

(also solvable by Newtonian Mechanics: $F = m a$)
↳ not here

$F = m \cdot \ddot{y}$

$-\cancel{m} \cdot g = \cancel{m} \cdot \ddot{y}$

$E_{pot} = m \cdot g \cdot h = m \cdot g \cdot y$ m … mass of the diver

g … acc. due to gravity

$E_{kin} = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{y}^2$ $h = y$ $v = \dot{y} = \frac{dy}{dt}$

if $\underbrace{m = 2\,kg \quad g = 2\,\frac{m}{s^2}}_{\text{see later why}}$ then if $H = 1m$ then $T = 1s$

↳ $E_{kin} = \dot{y}^2$ $E_{pot} = 4y$

$$\int_0^1 E_{kin} - E_{pot} \quad dt$$

(→ Lagrangian Mechanics)

$$I(y) = \int_0^1 \dot{y}_{(t)}^2 - 4 y_{(t)} \, dt$$

loose: "average difference in energy" → it's not energy
↳ ensures feasibility

I … takes in a function y and outputs a scalar

⇒ Functional

"function of function"

$$I(y = 1-t) = \int_0^1 \left(\dot{\overline{1-t}}\right)^2 - 4 \cdot (1-t) \, dt$$

$$= \int_0^1 (-1)^2 - 4 + 4t \quad dt$$

$$= \int_0^1 -3 + 4t \, dt = \Big[-3t + 2t^2\Big]_0^1 = -3 + 2 = \underline{\underline{-1}}$$

$$I(y = 1-t^2) = \int_0^1 \left(\dot{\overline{1-t^2}}\right)^2 - 4 \cdot (1-t^2) \, dt$$

$$= \int_0^1 (-2t)^2 - 4 + 4t^2 \, dt$$

$$= \int_0^1 8t^2 - 4 \, dt$$

$$= \Big[\tfrac{8}{3} t^3 - 4t\Big]_0^1 = \tfrac{8}{3} - 4 = -\tfrac{4}{3} \approx \underline{\underline{-1.333}}$$

$$I(y = 1-t^3) = \int_0^1 \left(\dot{\overline{1-t^3}}\right)^2 - 4(1-t^3) \, dt$$

$$= \int_0^1 (-3t^2)^2 - 4 + 4t^3 \, dt$$

$$= \int_0^1 9t^4 - 4 + 4t^3 \, dt$$

$$= \Big[\tfrac{9}{5} t^5 - 4t + t^4\Big]_0^1$$

$$= \frac{9}{5} - 4 + 1 = \frac{9 - 20 + 5}{5} = -\frac{6}{5}$$

$$= \underline{\underline{-1.2}}$$

$I(y = 1-t) = -1$

$I(y = 1-t^2) = -1.33$ ← Minimum (Minimum energy principle)

$I(y = 1-t^3) = -1.2$

What is a Functional?

$y: \overset{\text{scalar}}{\mathbb{R}} \to \overset{\text{scalar}}{\mathbb{R}}$

$I: (\;\ldots\;) \to \overset{\text{scalar}}{\mathbb{R}}$

function space

↳ e.g.: • infinitly often continuously differentiable functions $C^\infty[\mathbb{R}, \mathbb{R}]$

• Square-integrable functions $L_2[\mathbb{R}, \mathbb{R}]$

• Space of polynomials

The true solution is attained at Minimum Energy

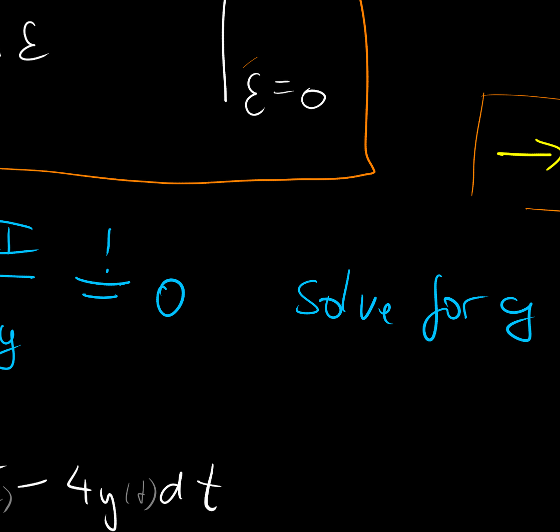
↳ How to minimize a Functional?

(build derivative & set to 0)

↳ How to do a derivative?

Functional Derivative

$$\frac{\delta I}{\delta y}$$



← function space

infinite-dimensional (vector space)

$$\boxed{\frac{\delta I}{\delta y} = \frac{d\, I(y + \varepsilon\varphi)}{d\varepsilon}\bigg|_{\varepsilon = 0}}$$

→ directional derivative in function space

→ Gateaux Derivative

↳ just $\frac{\delta I}{\delta y} \overset{!}{=} 0$ solve for y

$$I(y) = \int_0^1 \dot{y}_{(t)}^2 - 4 y_{(t)} \, dt$$

$$I(y + \varepsilon\varphi) = \int_0^1 \left(\dot{\overline{y + \varepsilon\varphi}}\right)^2 - 4 \cdot (y + \varepsilon\varphi) \, dt$$

(ε does not depend on t)

$$= \int_0^1 (\dot{y} + \varepsilon\dot{\varphi})^2 - 4y - 4\varepsilon\varphi \, dt$$

$$= \int_0^1 \cancel{\dot{y}^2} + 2\dot{y}\varepsilon\dot{\varphi} + \varepsilon^2\dot{\varphi}^2 - \cancel{4y} - 4\varepsilon\varphi \, dt$$

$$\frac{d\, I(y + \varepsilon\varphi)}{d\varepsilon} = \int_0^1 0 + 2\dot{y}\dot{\varphi} + 2\varepsilon\dot{\varphi}^2 - 0 - 4\varphi \, dt$$

$$= \int_0^1 2\dot{y}\dot{\varphi} + 2\varepsilon\dot{\varphi}^2 - 4\varphi \, dt$$

$$\frac{d\, I(y + \varepsilon\varphi)}{d\varepsilon}\bigg|_{\varepsilon = 0} = \int_0^1 2\dot{y}\dot{\varphi} \quad - 4\varphi \, dt = \frac{\delta I}{\delta y}$$

How to rearrange for y

$$\int_0^1 2\dot{y}\dot{\varphi} - 4\varphi \, dt = \int_0^1 2\dot{y}\dot{\varphi} \, dt - \int_0^1 4\varphi \, dt$$

$$\overset{\text{Int by parts}}{=} \underbrace{\Big[2\dot{y}\varphi\Big]_0^1}_{=0,\ \varphi \text{ is a test function}} - \int_0^1 2\ddot{y}\varphi \, dt - \int_0^1 4\varphi \, dt$$

$\forall \varphi \in V$

$$= -\int_0^1 2\ddot{y}\varphi + 4\varphi \, dt$$

$$= -\int_0^1 (2\ddot{y} + 4) \cdot \varphi \, dt = \frac{\delta I}{\delta y} \overset{!}{=} 0$$

↳ Fundamental Lemma of Calculus of Variations

↳ $-(2\ddot{y} + 4) = 0$

$$2\ddot{y} + 4 = 0$$

$$\boxed{\ddot{y} = -2}$$

Condition for the optimal Solution

because g = 2

we rediscovered Newtonian Mechanics

ODE

$$\dot{y} = -2 \cdot t + \underbrace{\dot{y}_0}_{=0}$$

$y = -t^2 + y_0$ $y_0 = 1$

$$\boxed{y(t) = 1 - t^2}$$

$y(t) = 1 - t^2$ is indeed the correct solution as it minimizes the energy function

↳ Minimum energy principle