Karuhn-Kush-Tucker (KKT) conditions

Equality-Constrained Optimization

$$\min_{\underline{x}} f(\underline{x}) \quad \text{s.t.} \quad \underline{h}(\underline{x}) = \underline{0}$$

↳ Lagrangian

$$\mathcal{L}(\underline{x}, \underline{\lambda}) = f(\underline{x}) + \underline{\lambda}^T \underline{h}(\underline{x})$$

$$\nabla \mathcal{L} \overset{!}{=} \underline{0}$$

Inequality-Constrained Optimization

$$\min_{\underline{x}} f(\underline{x}) \quad \text{s.to} \quad \underline{g}(\underline{x}) \leq \underline{0}$$

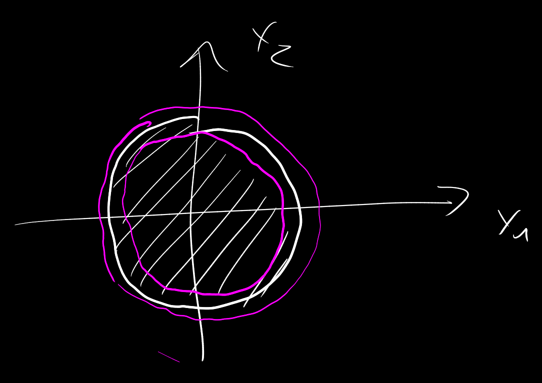
What to do here?

Also just build the Lagrangian

$$\mathcal{L}(\underline{x}, \underline{\mu}) = f(\underline{x}) + \underline{\mu}^T \underline{g}(\underline{x}) = f(\underline{x}) + \sum_{i=0}^{M-1} \mu_i g_i(\underline{x})$$

and set gradient to 0

does not work (:



↳ only finds solutions where $\underline{g}(\underline{x}) = \underline{0}$

We need sth. different

→ Recover target from Lagrangian

$$f(\underline{x}) = \max_{\underline{\mu} \geq \underline{0}} \left(\mathcal{L}(\underline{x}, \underline{\mu}) \right) = \max_{\underline{\mu} \geq 0} \left(f(\underline{x}) + \sum_{i=0}^{M-1} \mu_i g_i(\underline{x}) \right)$$

all $\mu_i \geq 0 \ \forall i$

- assume $M=3$ (3 ineq constraints)
  violate $g_1(\underline{x})$
  $g_1(\underline{x}) > 0$
  then $\max_{\underline{\mu} \geq 0}(\ldots) = \infty$
- no $g_i(\underline{x})$ is violated
  $g_i(\underline{x}) \leq 0$
  $g_i(\underline{x}) < 0 \ \rightarrow \mu_i = 0$ to have the max
  $g_i(\underline{x}) = 0$ (constraint is active) → μ_i is free but doesn't affect maximum

$$f(\underline{x}) = \max_{\underline{\mu} \geq 0} \left(f(\underline{x}) + \sum_{i=0}^{M-1} \mu_i g_i(\underline{x}) \right)$$

Original problem

$$\min_{\underline{x}} f(\underline{x}) \quad \text{s.t.} \quad \underline{g}(\underline{x}) \leq \underline{0} \quad \xrightarrow{\text{transformation}} \quad \min_{\underline{x}} \max_{\underline{\mu} \geq 0} \left(f(\underline{x}) + \sum_{i=0}^{M-1} \mu_i g_i(\underline{x}) \right)$$

↳ unconstrained optimization in $\underline{x}$

Disclaimer:

We would inf instead of ~~min~~ and sup instead of max

We need the original problem in standard

- minimization (change sign of target)
- inequalities are "≤" smaller equal (multiply by (-1))

→ Max-Min Inequality (holds in general)

$$\underbrace{\min_{\underline{x}} \max_{\underline{\mu} \geq \underline{0}} \left(\overbrace{f(\underline{x}) + \sum_{i=0}^{M-1} \mu_i g_i(\underline{x})}^{\mathcal{L}(\underline{x}, \underline{\mu})} \right)}_{f(\underline{x}) = \max_{\underline{\mu} \geq 0} \mathcal{L}(\underline{x}, \underline{\mu})} \geq \max_{\underline{\mu} \geq \underline{0}} \underbrace{\min_{\underline{x}} \left(\overbrace{f(\underline{x}) + \sum_{i=0}^{M-1} \mu_i g_i(\underline{x})}^{\mathcal{L}(\underline{x}, \underline{\mu})} \right)}_{\varphi(\underline{\mu}) = \min_{\underline{x}} \mathcal{L}(\underline{x}, \underline{\mu})}$$

$$\underbrace{\begin{pmatrix} \min_{\underline{x}} f(\underline{x}) \\ \text{s.t. } \underline{g}(\underline{x}) \leq 0 \end{pmatrix}}_{\text{primal version}} = p^* \geq q^* = \underbrace{\begin{pmatrix} \max_{\underline{\mu}} \varphi(\underline{\mu}) \\ \text{s.to } \underline{\mu} \geq 0 \end{pmatrix}}_{\text{dual version}}$$

p^* optimal value of the primal

q^* ... optimal problem of the dual

$p^* - q^*$ ... duality gap

Can we say sth. about the duality gap

↳ it is "0" for "regular" problems (*)

→ strong duality

now assume we have a regular

$$\begin{pmatrix} \min_{\underline{x}} f(\underline{x}) \\ \text{s.t. } \underline{g}(\underline{x}) \leq \underline{0} \end{pmatrix} = p^* = q^* = \begin{pmatrix} \max_{\underline{\mu}} \varphi(\underline{\mu}) \\ \text{s.t. } \underline{\mu} \geq 0 \end{pmatrix}$$

p^* is attained at x^* $\qquad$ q^* is attained at $\underline{\mu}^*$

Hence:

$$f(\underline{x}^*) = p^* = q^* = \varphi(\underline{\mu}^*)$$
$$= \min_{\underline{x}} \left(f(\underline{x}) + \sum_{i=0}^{M-1} \mu_i^* g_i(\underline{x}) \right)$$
$$= f(\underline{x}^*) + \underbrace{\sum_{i=0}^{M-1} \mu_i^* g_i(\underline{x}^*)}_{=0}$$
$$= f(\underline{x}^*)$$

KKT for $(\underline{x}^*, \underline{\mu}^*)$ point in order to have $\underline{x}^*$ as a constrained minimizer

(I) Primal feasibility:

$$\underline{g}(\underline{x}^*) \leq \underline{0}$$

(II) Stationarity:

$$\nabla f(\underline{x})\Big|_{\underline{x}=\underline{x}^*} + \sum_{i=0}^{M-1} \mu_i^* \nabla g(\underline{x})\Big|_{\underline{x}=x^*} \overset{!}{=} 0$$

(III) Dual feasibility:

$$\underline{\mu}^* \geq \underline{0}$$

(IV) Complimentary Slackness

$$\underbrace{\sum_{i=0}^{M-1} \underbrace{\mu_i^*}_{\text{will always be pos}} \underbrace{g_i(\underline{x}^*)}_{\text{will always be neg}}}_{\text{will always neg}} = 0$$

→ all components in summation have to be 0

$$\mu_i^* g_i(\underline{x}^*) = 0 \quad \forall i$$

"either μ_i^* is zero or $g_i(\underline{x}^*)$ is zero"

*** Regularity?**

When is a problem regular (there are plenty of conditions)

↳ Slater's condition (→ convex problem)

$\exists \, \tilde{\underline{x}}$

- satisfies all non-affine but convex constraints strictly:
  $g_{[\ldots]}(\tilde{\underline{x}}) < 0$
- satisfies all affine constraints:
  $g_{[\ldots]}(\tilde{\underline{x}}) \leq 0$