1D Diffusion using the Finite Element Method in Fenics with Python

Initial- Boundary- Value Problem

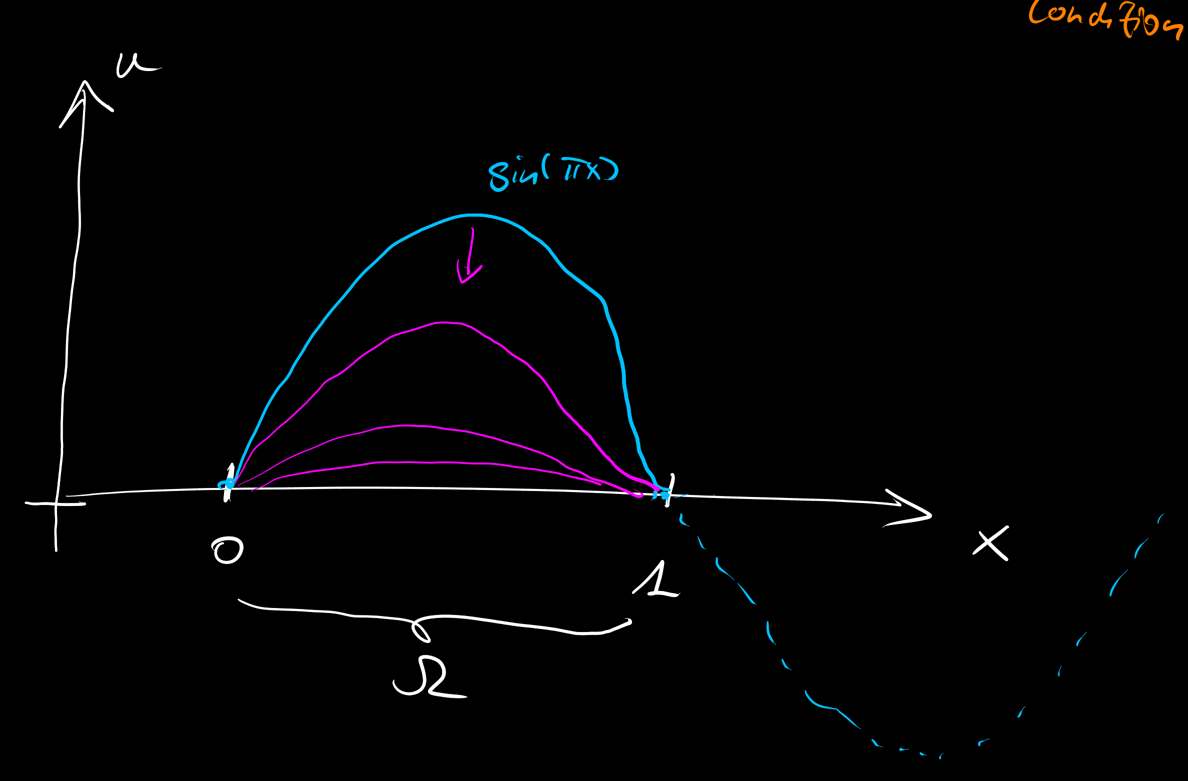
$$\begin{cases} \dfrac{\partial u}{\partial t} = \alpha \dfrac{\partial^2 u}{\partial x^2}, & x \in \Omega = (0,1),\ t > 0 \\ u(t=0, x) = \sin(\pi x), & x \in \Omega = (0,1),\ t = 0 \\ u(t, x=0) = 0 = u(t, x=1), & t > 0 \end{cases}$$

← PDE

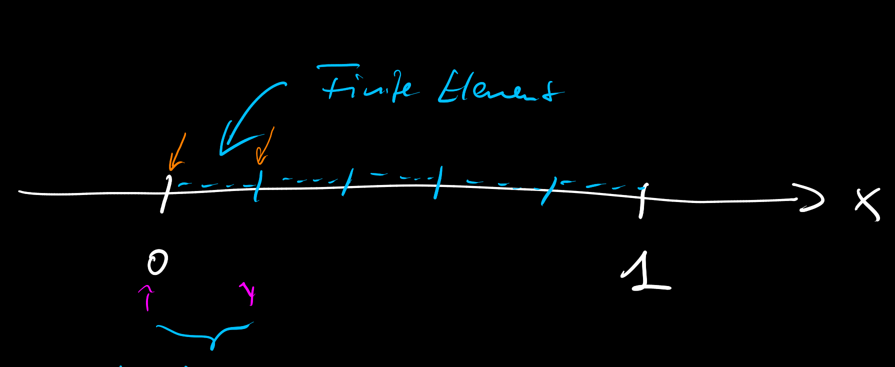
← Initial Condition

← Homogeneous Dirichlet Boundary Condition

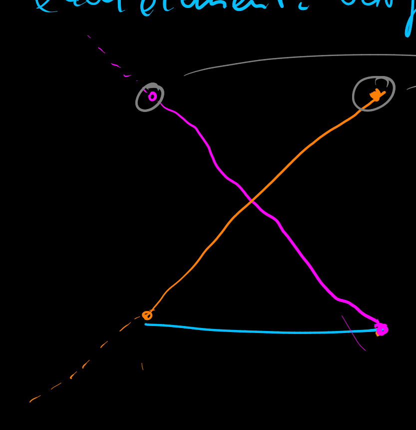
IC:



FEM subdivides the domain Ω into Finite Elements

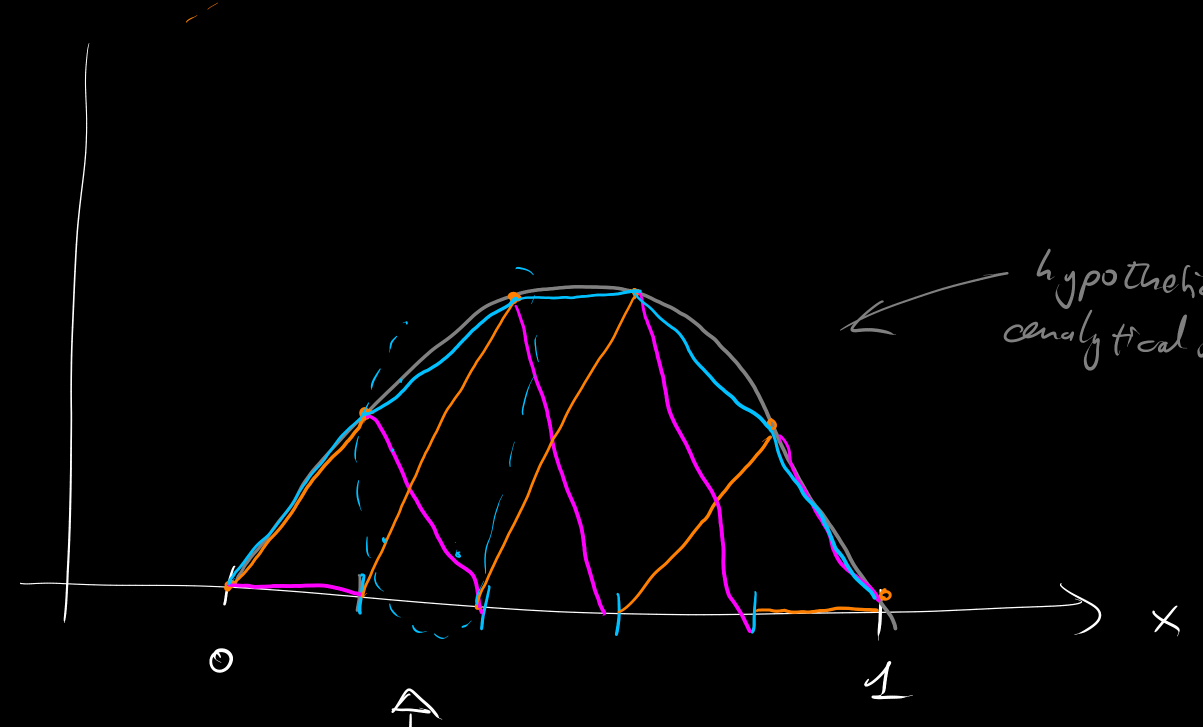


each element: outfitted with a shape / ansatz function



scaled by nodal values

↳ Superposition is the Solution in the domain



Lagrange Elements of order 1

↳ Define the function space

for Fenics: weak form

strong → weak

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

① Multiply with test function $v \in V$

$$\frac{\partial u}{\partial t} v = \alpha \frac{\partial^2 u}{\partial x^2} v$$

② Integrate over domain

$$\int_0^1 \frac{\partial u}{\partial t} v \, dx = \int_0^1 \alpha \frac{\partial^2 u}{\partial x^2} v \, dx$$

③ Apply integration by parts

$$\int_0^1 \frac{\partial u}{\partial t} v \, dx = \underbrace{\left[\alpha \frac{\partial u}{\partial x} v \right]_{x=0}^{x=1}}_{\Rightarrow 0} - \int_0^1 \alpha \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} dx$$

the test function has to comply with the BC of u, i.e. it is zero on the boundary

$$\int_0^1 \frac{\partial u}{\partial t} v \, dx = - \int_0^1 \alpha \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} dx$$

⟺ residuum form

$$\int_0^1 \frac{\partial u}{\partial t} v + \alpha \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} dx = 0$$

introduce $u^{[t+1]}$ & $u^{[t]}$

$$\int_0^1 \frac{u^{[t+1]} - u^{[t]}}{\Delta t} \cdot v + \alpha \frac{\partial u^{[t+1]}}{\partial x} \frac{\partial v}{\partial x} dx = 0 \quad | \cdot \Delta t$$

primary unknown

$$\int_0^1 u^{[t+1]} \cdot v - u^{[t]} v + \alpha \Delta t \frac{\partial u^{[t+1]}}{\partial x} \frac{\partial v}{\partial x} dx = 0$$

Derive weak form in higher dim case $\underline{x} \in \mathbb{R}^d,\ d \geq 2$

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u$$

$(= \Delta)$ Laplace operator

[in 2D: $\dfrac{\partial u}{\partial t} = \alpha \cdot \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} \right)$

① Multiply with a test function

$$\frac{\partial u}{\partial t} v = \alpha \nabla^2 u \, v$$

② Integrate over the domain

$$\int_\Omega \frac{\partial u}{\partial t} v \, dV = \int_\Omega \alpha (\nabla^2 u) v \, dV$$

③ Multidimensional Integration by parts (Gauss / Divergence Theorem)

$$\int_\Omega \alpha (\nabla^2 u) v \, dV = \ \ldots ?$$

$$\nabla^2 = \Delta = \nabla \cdot \nabla = \mathrm{div}(\mathrm{grad}(\ldots))$$

$$\Rightarrow (\nabla \cdot \nabla u) v$$

in 1D: $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x} v \right) = \dfrac{\partial^2 u}{\partial x^2} v + \dfrac{\partial u}{\partial v} \dfrac{\partial v}{\partial x}$

reverse product rule: imagine we had: $\nabla \cdot ((\nabla u) v)$

$$= (\nabla \cdot \nabla u) v + (\nabla u) \cdot (\nabla v)$$

$$\Rightarrow (\nabla^2 u) v = \nabla \cdot ((\nabla u) \cdot v) - (\nabla u) \cdot (\nabla v)$$

$$\rightarrow \int_\Omega \frac{\partial u}{\partial t} v \, dV = \underbrace{\int_\Omega \alpha \nabla \cdot ((\nabla u) v) \, dV}_{\text{calls for the divergence theorem}} - \int_\Omega \alpha (\nabla u) \cdot (\nabla v) \, dV$$

$$= \underbrace{\int_{\partial\Omega} \alpha \, \underline{n} \cdot ((\nabla u) v) \, dS}_{\Rightarrow 0} - \int_\Omega \alpha (\nabla u) \cdot (\nabla v) \, dV$$

the test function is zero on the domain boundary

$$\Rightarrow \int_\Omega \frac{\partial u}{\partial t} v \, dV = - \int_\Omega \alpha (\nabla u) \cdot (\nabla v) \, dV$$

$$\int_\Omega \frac{\partial u}{\partial t} v + \alpha (\nabla u) \cdot (\nabla v) \, dV = 0$$

$$\int_\Omega \frac{u^{[t+1]} - u^{[t]}}{\Delta t} v + \alpha (\nabla u^{[t+1]}) \cdot (\nabla v) \, dV = 0$$

$$\boxed{\int_\Omega u^{[t+1]} v \, dV - \int_\Omega u^{[t]} v \, dV + \int_\Omega \Delta t \, \alpha (\nabla u^{[t+1]}) \cdot (\nabla v) \, dV}$$