(Multivariate) Sum of 2 Quadratic Forms

$$f(x) = a(x-y)^2 + b(x-z)^2 \quad \rightarrow f(x) = c(x-d)^2 + e$$

$$f(\mathbf{x}) = (\mathbf{x}-\mathbf{y})^T \mathbf{A} (\mathbf{x}-\mathbf{y}) + (\mathbf{x}-\mathbf{z})^T \mathbf{B} (\mathbf{x}-\mathbf{z})$$

$$\rightarrow f(\mathbf{x}) = (\mathbf{x}-\mathbf{d})^T \mathbf{C} (\mathbf{x}-\mathbf{d}) + e$$

Univariate

$$f(x) = a(x-y)^2 + b(x-z)^2$$
$$= a(x^2 - 2xy + y^2) + b(x^2 - 2xz + z^2)$$
$$= ax^2 - 2axy + ay^2 + bx^2 - 2bxz + bz^2$$
$$= \underbrace{(a+b)}_{a'}x^2 \underbrace{-2(ay+bz)}_{b'}x + \underbrace{ay^2 + bz^2}_{c'}$$

Completing the Square

$$= a'x^2 + b'x + c'$$
$$= a'(x-d')^2 + e'$$

$a' = a'$ $\qquad d' = -\frac{b'}{2a'}$ $\qquad e' = c' - \frac{b'^2}{4a'}$

$$d' = \cancel{-}\frac{\cancel{-2}(ay+bz)}{\cancel{2}(a+b)} = \frac{ay+bz}{a+b}$$

$$e' = ay^2 + bz^2 - \frac{(-\cancel{2}(ay+bz))^2}{\cancel{4}(a+b)}$$

$$e' = ay^2 + bz^2 - \frac{(ay+bz)^2}{a+b}$$
$$= \frac{ay^2(a+b) + bz^2(a+b) - (a^2y^2 + 2aybz + b^2z^2)}{a+b}$$
$$= \frac{\cancel{a^2y^2} + aby^2 + bz^2a + \cancel{b^2z^2} - \cancel{a^2y^2} - 2aybz - \cancel{b^2z^2}}{a+b}$$
$$= \frac{aby^2 - 2aybz + bz^2a}{a+b}$$
$$= ab \cdot \frac{y^2 - 2yz + z^2}{a+b}$$
$$= ab \cdot \frac{(y-z)^2}{a+b} = e'$$

$$= a'(x-d')^2 + e'$$
$$= (a+b)\left(x - \frac{ay+bz}{a+b}\right)^2 + ab \cdot \frac{(y-z)^2}{a+b}$$
$$= a(x-y)^2 + b(x-z)^2$$
$$= c(x-d)^2 + e$$

$$\boxed{c = a+b}$$

$$\boxed{d = \frac{ay+bz}{a+b}}$$

$$\boxed{e = ab \cdot \frac{(y-z)^2}{a+b}}$$

example:

$$f(x) = \underbrace{5}_{a}(x-\underbrace{2}_{y})^2 + \underbrace{2}_{b}(x-\underbrace{2}_{z})^2$$

$a = 5$
$y = 2$
$b = 2$
$z = 2$

$$c = 7$$
$$d = \frac{5\cdot 2 + 2\cdot 2}{5+2} = \frac{10+4}{7} = \frac{14}{7} = 2$$
$$e = (5\cdot 2)\cdot \frac{(2-2)^2}{5+2} = 0$$

$$f(x) = 7\cdot (x-2)^2 + 0$$

Multivariate Case

Sym Matrices $\quad \mathbf{A} = \mathbf{A}^T$, $\mathbf{B} = \mathbf{B}^T$

$$f(\mathbf{x}) = \underbrace{(\mathbf{x}-\mathbf{y})^T \mathbf{A} (\mathbf{x}-\mathbf{y})}_{\text{Scalar}} + \underbrace{(\mathbf{x}-\mathbf{z})^T \mathbf{B} (\mathbf{x}-\mathbf{z})}_{\text{Scalar}}$$

$$= \mathbf{x}^T\mathbf{A}\mathbf{x} - 2\mathbf{y}^T\mathbf{A}\mathbf{x} + \mathbf{y}^T\mathbf{A}\mathbf{y} + \mathbf{x}^T\mathbf{B}\mathbf{x} - 2\mathbf{z}^T\mathbf{B}\mathbf{x} + \mathbf{z}^T\mathbf{B}\mathbf{z}$$

$$= \mathbf{x}^T\underbrace{(\mathbf{A}+\mathbf{B})}_{\mathbf{A}'}\mathbf{x} \underbrace{-2(\mathbf{y}^T\mathbf{A} + \mathbf{z}^T\mathbf{B})}_{\mathbf{b}'}\mathbf{x} + \underbrace{\mathbf{y}^T\mathbf{A}\mathbf{y} + \mathbf{z}^T\mathbf{B}\mathbf{z}}_{c}$$

$$(\mathbf{A}\mathbf{y} + \mathbf{B}\mathbf{z})^T$$

not transpose

$$= \mathbf{x}^T\mathbf{A}'\mathbf{x} + \mathbf{b}'^T\mathbf{x} + c$$

Completing the square

$$= (\mathbf{x}-\mathbf{d}')^T\mathbf{A}'(\mathbf{x}-\mathbf{d}') + e'$$

$$\mathbf{d}' = -\frac{1}{2}\mathbf{A}'^{-1}\mathbf{b}'$$

$$e' = c' - \frac{1}{4}\mathbf{b}'^T\mathbf{A}'\mathbf{b}'$$

$$\mathbf{d}' = \cancel{-}\frac{1}{\cancel{2}}\cdot(\mathbf{A}+\mathbf{B})^{-1}(\cancel{-2}\cdot(\mathbf{A}\mathbf{y} + \mathbf{B}\mathbf{z}))$$
$$= (\mathbf{A}+\mathbf{B})^{-1}(\mathbf{A}\mathbf{y} + \mathbf{B}\mathbf{z})$$

$$e' = \mathbf{y}^T\mathbf{A}\mathbf{y} + \mathbf{z}^T\mathbf{B}\mathbf{z} - \frac{1}{\cancel{4}}\cdot(\cancel{-2}(\mathbf{A}\mathbf{y}+\mathbf{B}\mathbf{z}))^T(\mathbf{A}+\mathbf{B})^{-1}(\cancel{-2}(\mathbf{A}\mathbf{y}+\mathbf{B}\mathbf{z}))$$

$$= \mathbf{y}^T\mathbf{A}\mathbf{y} + \mathbf{z}^T\mathbf{B}\mathbf{z} - \mathbf{y}^T\mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{A}\mathbf{y}$$
$$- 2\mathbf{y}^T\mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{B}\mathbf{z}$$
$$- \mathbf{z}^T\mathbf{B}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{B}\mathbf{z}$$

$\mathbf{I} = (\mathbf{A}+\mathbf{B})^{-1}(\mathbf{A}+\mathbf{B})$

$$\mathbf{y}^T(\mathbf{A} - \mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{A})\mathbf{y}$$
$$+ \mathbf{z}^T(\mathbf{B} - \mathbf{B}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{B})\mathbf{z}$$
$$- 2\mathbf{y}^T\mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{B}\mathbf{z}$$

$$= \mathbf{y}^T(\mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}(\cancel{\mathbf{A}}+\mathbf{B}) - \cancel{\mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{A}})\mathbf{y}$$
$$+ \mathbf{z}^T(\mathbf{B}(\mathbf{A}+\mathbf{B})^{-1}(\mathbf{A}+\cancel{\mathbf{B}}) - \cancel{\mathbf{B}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{B}})\mathbf{z}$$
$$- 2\mathbf{y}^T\mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{B}\mathbf{z}$$

$$= \mathbf{y}^T(\mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{B})\mathbf{y} + \mathbf{z}^T(\cancel{\mathbf{B}}\,\mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}\cancel{\mathbf{A}}\,\mathbf{B})\mathbf{z}$$
$$- 2(\mathbf{y}^T\mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{B})\mathbf{z}$$

$$= (\mathbf{y}-\mathbf{z})^T(\mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{B})\cdot(\mathbf{y}-\mathbf{z}) = e'$$

Woodbury Matrix Identity

$$(\mathbf{A}+\mathbf{B})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}(\mathbf{A}^{-1}+\mathbf{B}^{-1})^{-1}\mathbf{A}^{-1}$$

$$= (\mathbf{y}-\mathbf{z})^T(\mathbf{A}(\mathbf{A}^{-1} - \mathbf{A}^{-1}(\mathbf{A}^{-1}+\mathbf{B}^{-1})^{-1}\mathbf{A}^{-1})\mathbf{B})(\mathbf{y}-\mathbf{z})$$

$$= (\mathbf{y}-\mathbf{z})^T(\underbrace{\cancel{\mathbf{A}\mathbf{A}^{-1}}}_{\mathbf{I}}\mathbf{B} - \underbrace{\cancel{\mathbf{A}\mathbf{A}^{-1}}}_{\mathbf{I}}(\mathbf{A}^{-1}+\mathbf{B}^{-1})^{-1}\mathbf{A}^{-1}\mathbf{B})(\mathbf{y}-\mathbf{z})$$

$\mathbf{I} = (\mathbf{A}^{-1}+\mathbf{B}^{-1})^{-1}(\mathbf{A}^{-1}+\mathbf{B}^{-1})$

$$= (\mathbf{y}-\mathbf{z})^T((\mathbf{A}^{-1}+\mathbf{B}^{-1})^{-1}(\mathbf{A}^{-1}+\mathbf{B}^{-1})\mathbf{B} - (\mathbf{A}^{-1}+\mathbf{B}^{-1})^{-1}\mathbf{A}^{-1}\mathbf{B})(\mathbf{y}-\mathbf{z})$$

$$= (\mathbf{y}-\mathbf{z})^T((\mathbf{A}^{-1}+\mathbf{B}^{-1})^{-1}((\cancel{\mathbf{A}^{-1}}+\underbrace{\mathbf{B}^{-1})\mathbf{B}}_{\mathbf{I}} - \cancel{\mathbf{A}^{-1}\mathbf{B}}))(\mathbf{y}-\mathbf{z})$$

$$= (\mathbf{y}-\mathbf{z})^T(\mathbf{A}^{-1}+\mathbf{B}^{-1})^{-1}(\mathbf{y}-\mathbf{z}) = e'$$

$$f(\mathbf{x}) = (\mathbf{x}-\mathbf{d}')^T\mathbf{A}'(\mathbf{x}-\mathbf{d}') + e'$$
$$= (\mathbf{x} - (\mathbf{A}+\mathbf{B})^{-1}(\mathbf{A}\mathbf{y}+\mathbf{B}\mathbf{z}))(\mathbf{A}+\mathbf{B})(\mathbf{x} - (\mathbf{A}+\mathbf{B})^{-1}(\mathbf{A}\mathbf{y}+\mathbf{B}\mathbf{z}))$$
$$+ (\mathbf{y}-\mathbf{z})^T(\mathbf{A}^{-1}+\mathbf{B}^{-1})^{-1}(\mathbf{y}-\mathbf{z})$$

Original $\quad (\mathbf{x}-\mathbf{y})^T\mathbf{A}(\mathbf{x}-\mathbf{y}) + (\mathbf{x}-\mathbf{z})^T\mathbf{B}(\mathbf{x}-\mathbf{z})$

$$= (\mathbf{x}-\mathbf{d})^T\mathbf{C}(\mathbf{x}-\mathbf{d}) + e$$

$$\boxed{\mathbf{C} = \mathbf{A}+\mathbf{B}}$$

$$\boxed{\mathbf{d} = (\mathbf{A}+\mathbf{B})^{-1}(\mathbf{A}\mathbf{y}+\mathbf{B}\mathbf{z})}$$

$$\boxed{e = (\mathbf{x}-\mathbf{y})^T(\mathbf{A}^{-1}+\mathbf{B}^{-1})^{-1}(\mathbf{x}-\mathbf{y})}$$

equivalently:

$$\boxed{e = (\mathbf{x}-\mathbf{y})^T\mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{B}(\mathbf{x}-\mathbf{y})}$$