Pullback/vJp rule for L_2-Loss

$\hat{=}$ squared L_2 norm

$$\ell(\underline{y}, \underline{y}^r) = \frac{1}{2} \| \underline{y} - \underline{y}^r \|_2^2 =: L$$

$\underline{y} \in \mathbb{R}^N$, $\underline{y}^r \in \mathbb{R}^N$, $L \in \mathbb{R}$, $\ell : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$

task: backpropagate cotangent information on the output $\bar{L} \in \mathbb{R}$ to the inputs $\underline{\bar{y}} \in \mathbb{R}^N$ & $\underline{\bar{y}}^r \in \mathbb{R}^N$

often just 1.0

often not interesting

in general

$$\underline{\bar{y}}^T = \bar{L} \frac{\partial \ell}{\partial \underline{y}} \qquad \underline{\bar{y}}^r = \bar{L} \frac{\partial \ell}{\partial \underline{y}^r}$$

$$\frac{\partial \ell}{\partial \underline{y}} = \frac{\partial}{\partial \underline{y}} \left(\frac{1}{2} \| \underline{y} - \underline{y}^r \|_2^2 \right) = \frac{\partial}{\partial \underline{y}} \left(\frac{1}{2} (\underline{y} - \underline{y}^r)^T (\underline{y} - \underline{y}^r) \right)$$

in index notation

$$\frac{\partial \ell}{\partial y_i} = \frac{\partial}{\partial y_i} \left(\frac{1}{2} \sum_{j=1}^{N} (y_j - y_j^r)^2 \right)$$

$$= \frac{1}{2} \sum_{j=1}^{N} \frac{\partial}{\partial y_i} (y_j - y_j^r)^2$$

$$= \frac{1}{2} \sum_{j=1}^{N} 2 (y_j - y_j^r) \frac{\partial y_j}{\partial y_i}$$

$= \delta_{ji}$

$$\frac{\partial -y_j^r}{\partial y_i^r} = -\delta_{ji}$$

$$= \frac{1}{2} \sum_{j=1}^{N} 2 (y_j - y_j^r) \delta_{ji}$$

$$= \frac{1}{\cancel{2}} \cancel{2} (y_i - y_i^r)$$

$$= y_i - y_i^r$$

back to symbolic notation

$$\frac{\partial \ell}{\partial \underline{y}} = (\underline{y} - \underline{y}^r)^T \qquad \frac{\partial \ell}{\partial \underline{y}^r} = (\underline{y}^r - \underline{y})^T = -(\underline{y} - \underline{y}^r)^T$$

backprop

$$\underline{\bar{y}}^T = \bar{L} (\underline{y} - \underline{y}^r)^T \quad | \, (\cdot)^T$$

$$\underline{\bar{y}} = \bar{L} (\underline{y} - \underline{y}^r)$$

$$\underline{\bar{y}}^r = \bar{L} (\underline{y}^r - \underline{y})$$

Full pullback rule

$$\boxed{\mathcal{B}(\ell, (\underline{y}, \underline{y}^r), (\bar{L},)) = \Big(\big(\underbrace{\tfrac{1}{2} \| \underline{y} - \underline{y}^r \|_2^2}_{L} \big), \big(\underbrace{\bar{L}(\underline{y} - \underline{y}^r)}_{\underline{\bar{y}}}, \underbrace{\bar{L}(\underline{y}^r - \underline{y})}_{\underline{\bar{y}}^r} \big) \Big)}$$

$$\underline{y} - \underline{y}^r = \Delta \underline{y}$$