Pushforward / Jvp rule of scalar root finding

$$f(\theta) = \{\text{solve } g(x,\theta)=0 \text{ for } x\} =: x$$

e.g. by:
- bisection method
- Newton-Raphson
- ⋮

$x \in \mathbb{R}$ $\quad \theta \in \mathbb{R}$

$g: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$f: \mathbb{R} \to \mathbb{R}$

(assume f always converges)

task: propagate $\dot{\theta} \in \mathbb{R}$ to $\dot{x} \in \mathbb{R}$

without AD through the root solver

$$\dot{x} = \frac{\partial f}{\partial \theta} \dot{\theta} \qquad \frac{\partial f}{\partial \theta} \in \mathbb{R} \qquad \left(\frac{\partial f}{\partial \theta} \hat{=} \frac{\partial x}{\partial \theta} \right)$$

total derivative of optimality condition

$$\frac{dg}{d\theta} = \frac{\partial g}{\partial x} \underline{\underline{\frac{\partial x}{\partial \theta}}} + \frac{\partial g}{\partial \theta} \overset{!}{=} 0$$

$$\Leftrightarrow \quad \frac{\partial x}{\partial \theta} = - \frac{\overset{\in \mathbb{R}}{\frac{\partial g}{\partial \theta}}}{\underset{\in \mathbb{R}}{\frac{\partial g}{\partial x}}}$$

plug back into pushforward

$$\dot{x} = - \left(\frac{\partial g}{\partial x} \right)^{-1} \frac{\partial g}{\partial \theta} \dot{\theta}$$

obtain by forward-mode AD

$$\boxed{\mathcal{J}(f, (\theta,), (\dot{\theta},)) = \Big((\underbrace{f(\theta)}_{x},), (\underbrace{-\Big(\underset{\in\mathbb{R}}{\frac{\partial g}{\partial x}}\Big)^{-1} \underset{\in\mathbb{R}}{\frac{\partial g}{\partial \theta}} \dot{\theta}}_{\dot{x}}) \Big)}$$