# Lagrange Multipliers

$f(\underline{x}) = f(x_1, x_2, \dots, x_N)$

$\min f(\underline{x})$

s.t. $g(\underline{x}) = 0$

(Equality Constraint)

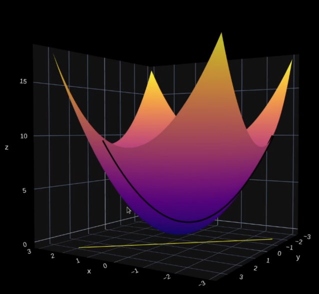
$\downarrow$

?

$\min_{\underline{x}} f(\underline{x})$

$\downarrow$

$$\nabla f(\underline{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_N} \end{bmatrix} \stackrel{!}{=} \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

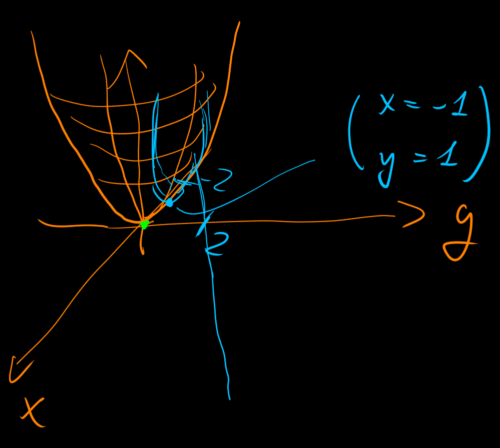


$\min_{\underline{x}} f(x,y) = x^2 + y^2$

s.t. $y - x = 2$

$y = 2 + x$

## Lagrange Multiplies for Equality-Constrained Minimization Problems




$\min_{\underline{x}} f(\underline{x})$

s.t. $\underline{g}(\underline{x}) = \underline{0}$ ($\underline{g}(\underline{x})$ vector of constraint)

($M$ equality constraints)

① Build a Lagrangian $\mathcal{L}(\underline{x}, \underline{\lambda}) = f(\underline{x}) + \underline{\lambda}^T \underline{g}(\underline{x})$

Transformed constrained problem to unconstrained problem

② Take gradient and set to zero

$$\nabla \mathcal{L}(\underline{x}, \underline{\lambda}) = \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial x_1} \\ \vdots \\ \frac{\partial \mathcal{L}}{\partial x_N} \\ \frac{\partial \mathcal{L}}{\partial \lambda_1} \\ \vdots \\ \frac{\partial \mathcal{L}}{\partial \lambda_M} \end{bmatrix} = \underline{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

③ Solve for $\underline{x}$ to be the minimum of the constrained problem

## Example

$\min_{\underline{x}} x^2 + y^2$

s.t. $y - x = 2$ → Reformulate to only have "0" on side of the eq.

$$\underbrace{y - x - 2}_{g(\underline{x}) = g(x,y)} = 0$$

① $\mathcal{L}(x, y, \lambda) = x^2 + y^2 + \lambda \cdot (y - x - 2)$

②

$$\frac{\partial \mathcal{L}}{\partial x} = 2x - \lambda \stackrel{!}{=} 0 \quad \text{I}$$

$$\frac{\partial \mathcal{L}}{\partial y} = 2y + \lambda \stackrel{!}{=} 0 \quad \text{II}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = y - x - 2 \stackrel{!}{=} 0 \quad \text{III}$$

by III: $y = x + 2$

plug into II: $2 \cdot (x+2) + \lambda = 0$

$\lambda = -2x - 4$

plug into I: $2x - (-2x - 4) = 0$

$4x + 4 = 0$

$x^* = -1$

$\lambda^* = -2$

$y^* = 1$

$$f(x^*, y^*) = x^{*2} + y^{*2} = (-1)^2 + (1)^2 = 2$$