Bibliograpy report

Strain-Based Parameterization Approach For a Cosserat Rod

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Chapter 1

Cosserat Rod Theory

The Cosserat rod theory was presented by the Cosserat brothers in 1909 [1]. More recent studies applied this theory to define the behavior of strings and rod [2]. Nowadays, this theory is exstensively studied and implemented in order to model the behavior of deformable bodies [3] [4].

In this chapter, we present the outline of the theory needed for modeling the deformable links of a CPR. We will start with the static modeling of one rod and then we proceed to present its dynamic modeling.

1.1 Statics

We choose to start considering the static modeling in order to give a more gentle and light introduction to the theory. In fact, in the static case, the rod shape and kinematics are defined only with respect of the arc-length, being thus time invariant.

1.1.1 Geometrical Description

The rod, assumed as a slender body, is virtually sectioned infinite times, creating a stack on cross-sections stacked on top of each other. For every section, we can define the center of mass.

$$\mathbf{c} = \frac{\int_0^m \mathbf{r} dx}{\int\int \rho dA} \tag{1.1}$$

Where ρ is the specific weight of the material composing the cross-section, A is the area of the cross section, m are all the mass points of the cross section and $\mathbf{r}$ is the position of every mass point.

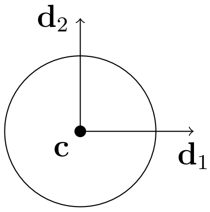

Figure 1.1: Figure representing the center of mass $\mathbf{c}$ and the directors $\mathbf{d}_1$ and $\mathbf{d}_2$ for a circle.

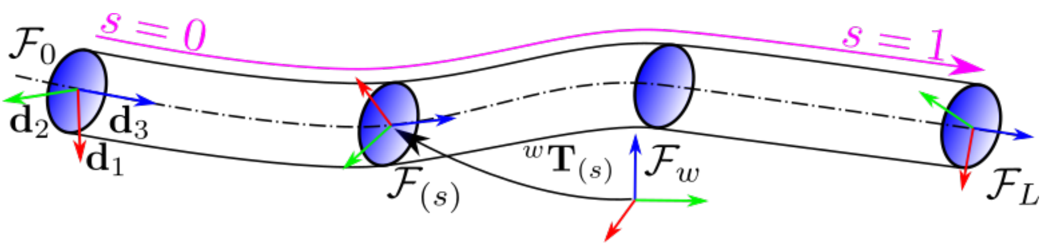

Figure 1.2: Parameterisation of the rod as a one dimensional body, in the dimension of its arc length s.

The various centers of mass define a line along the slender body: the centerline, which evolves in the arc-length of the body, namely $s \in [0, \ell]$ with ℓ the length of the rod. This line is used to describe the body, making a Cosserat rod a one dimensional body in the dimension of its arc-length.

As the domain of definition of the reference configuration is a bounded interval, then, without loss of generality, we scale the length variable s to lie in the unit interval $[0, 1]$, defining the label $X = \frac{s}{\ell}$. Note that we do not parametrize the curve with its arc length s. The parameter X, which identifies material points, is far more convenient on mathematical and physical grounds. In this document, while presenting different theories and application, we will make use of both s and X.

Each cross-section has its center of mass point and we can also define two directors $\mathbf{d}_{1(s)}$ and $\mathbf{d}_{2(s)}$ with $\mathbf{d}_{1(s)}, \mathbf{d}_{2(s)} \in \mathbb{R}^3 \mid \left\|\mathbf{d}_{1(s)}\right\| = \left\|\mathbf{d}_{2(s)}\right\| = 1$. The two directors are reciprocally orthogonal and lies on the plane defined by the cross-section, as shown in Figure 1.1. Typically, they are placed along the principal moment of inertia of the cross section. For a circular cross-section, they can be placed arbitrarly.

The two othogonal directors $\mathbf{d}_{1(s)}$ and $\mathbf{d}_{2(s)}$ define a third unique director $\mathbf{d}_{3(s)} = \mathbf{d}_{1(s)} \times \mathbf{d}_{2(s)}$. In the non-deformed configuration, *i.e.* when the rod is load free, $\mathbf{d}_{3(s)}$ is tangent to the rod centerline.

The directors are used in turn to define a frame attached to the rod cross section, namely the body frame $\mathcal{F}_s = \langle \mathbf{p}_{(s)}, \mathbf{d}_{1(s)}, \mathbf{d}_{2(s)}, \mathbf{d}_{3(s)} \rangle$.This basis will remain orthonormal to the cross section, but could lose orthogonality with respect to the centerline due to shear stresses. This frame is defined with respect to an inertial frame as commonly described in mathematics [5]. A graphic representation of these three axes is given in Figure 1.2.

In this work, we assume that:

- The cross-sections remain plane
- The cross-sections do not change in shape or size

Therefore, the cross-sections can only translate or rotate as a rigid bodies. Moreover, we decide to consider only the case of initially straight rod even if it is possible to model non-initially straight rods.

Similarly to what stated before, we can define the frame attached to the rod cross section at $s = 0$ as $\mathcal{F}_0 = \langle \mathbf{r}_0, \mathbf{d}_{10}, \mathbf{d}_{20}, \mathbf{d}_{30} \rangle$. The frame $\mathcal{F}_0$ is defined with respect to a reference frame, or world frame, $\mathcal{F}_w$. In this case, we will have the $SE(3)$ pose of the first rod cross section in absolute coordinates as ${}^w\mathbf{T}_0 = \langle {}^w\mathbf{r}_0, {}^w\mathbf{R}_0 \rangle$, where ${}^w\mathbf{R}_0$ is the rotation matrix defined in Appendix A.

With these assumptions, for each section of the rod, we define position and orientation w.r.t. a reference frame as ${}^w\mathbf{r}_{(s)} = \begin{bmatrix} {}^w r_{x(s)} & {}^w r_{y(s)} & {}^w r_{z(s)} \end{bmatrix}^T$ and ${}^w\mathbf{R}_{(s)} \in SO(3)$. From this each section pose can be defined with respect to the reference frame with the affine transformation:

$$
{}^w\mathbf{T}_{(s)} = \begin{bmatrix} {}^w\mathbf{R}_{(s)} & {}^w\mathbf{r}_{(s)} \\ \mathbf{0} & 1 \end{bmatrix} \tag{1.2}
$$

In Figure 1.2 we show a graphical interpretation of this parameterisation.

We follow a notation coming from geometrical mechanics: the uppercase letters are reserved for quantities in the local coordinate frame while the lowercase letters are for quantities expressed with respect of a fixed frame. We can thus drop the suffix ${}^w\cdot$.

However, we make an exception for the expression of the pose of the cross sections. Typically, in geometrical mechanics, a Lie group notation is privileged, using $\mathbf{g}$ to indicate an homogeneous transformation. Here we use $\mathbf{T}$, which is widely used in robotics for the homogeneous transformation of a frame. Morevoer, we have the notation for $\mathbf{r}_{(s)}$ and $\mathbf{R}_{(s)}$. For these two elements, expressed in fixed frame coordinates, we follow again the conventional robotics notation, using $\mathbf{R}$ for the rotation matrix and $\mathbf{r}$ for the position vector.

Note that the position vector $\mathbf{r}_{(s)}$ is the only one that can be expressed only with respect of the fixed frame and not in a local frame. In fact, any other vector, forces, velocities, orientations, are tangent to the group. However, the position vector identifies a point on the group. It cannot be expressed in a local coordinates frame without changing its meaning.

We can also define the derivative of the position vector $\mathbf{r}_{(s)}$and the rotation matrix $\mathbf{R}_{(s)}$with respect to the arc-length, as follows.

$$\begin{aligned} \mathbf{r}'_{(s)} &= \frac{d}{ds}\mathbf{r}_{(s)} \\ \mathbf{R}'_{(s)} &= \frac{d}{ds}\mathbf{R}_{(s)} \\ \mathbf{Q}'_{(s)} &= \frac{d}{ds}\mathbf{Q}_{(s)} \end{aligned} \tag{1.3}$$

Note that in (1.3) the quaternion $\mathbf{Q}_{(s)}$is expressed in the coordinate frame relative to the cross sections, as is it represented by a bold uppercase letter.

These derivatives are expressed with respect to the reference frame. We could express their counterpart in the coordinates of the corresponding local cross sectional frame with $\mathbf{K}_{(s)} = (K_x, K_y, K_z)$ as the rate of change of the orientation matrix $\mathbf{R}_{(s)}$ and $\mathbf{\Gamma}_{(s)} = (\Gamma_x, \Gamma_y, \Gamma_z)$ the rate of change of the position vector $\mathbf{r}_{(s)}$, both with respect to the arc-length.

$$\mathbf{K}_{(s)} = \left[\mathbf{R}^T_{(s)}\mathbf{R}'_{(s)}\right]^\vee \tag{1.4}$$

$$\mathbf{\Gamma}_{(s)} = \mathbf{R}^T_{(s)}\mathbf{r}'_{(s)} \tag{1.5}$$

These two quantities express the curvature of the rod. Definition of the operator $\cdot^\vee$ is given in Appendix A.

Here the two derivatives in the local coordinate frame deserve some attention. In fact, these two defines how the directors changes with respect to the arc-length(following the Poisson theorem).

$$\mathbf{d}_{k}{}'_{(s)} = \mathbf{K}_{(s)} \times \mathbf{d}_{k(s)}, \quad \mathbf{r}' = \Gamma_k \mathbf{d}_{k(s)}, \quad \forall k \in [1, 2, 3] \tag{1.6}$$

We can now introduce the condition that $\mathbf{r}'_{(s)} \cdot \mathbf{d}_{1(s,t)} > 0$. This condition implies $\|\mathbf{r}'\| > 0$ and this ensures that every cross-section cannot undergo a total shear. With $\mathbf{K}_{(s)}$and $\mathbf{\Gamma}_{(s)}$we can define the strain of the cross section

$$\boldsymbol{\xi}_{(s)} = \begin{bmatrix} \mathbf{K}_{(s)} \\ \mathbf{\Gamma}_{(s)} \end{bmatrix} \tag{1.7}$$

As a map from $s \in [0, L]$ to $se(3) \cong \mathbb{R}^6$. This operator is expressed in local coordinates and can be used to compute the derivative of Equation (1.2).

$$\mathbf{T}'_{(s)} = \mathbf{T}_{(s)}\hat{\boldsymbol{\xi}} \tag{1.8}$$

The definition of the matrix $\hat{\boldsymbol{\xi}}$ is given in the Appendinx A.

Equations (1.4) and (1.5) can be inverted in order to obtain the relation between the strain and the the arc-length rate of change of $\mathbf{r}_{(s)}$and $\mathbf{R}_{(s)}$.

$$\begin{aligned} \mathbf{r}'_{(s)} &= \mathbf{R}_{(s)}\boldsymbol{\Gamma}_{(s)} \\ \mathbf{R}'_{(s)} &= \mathbf{R}_{(s)}\hat{\mathbf{K}}_{(s)} \end{aligned} \tag{1.9}$$

Another formulation, that accounts for the quaternions is

$$\begin{aligned} \mathbf{Q}'_{(s)} &= \frac{1}{2}\mathbf{A}_{(\mathbf{K}_{(s)})}\mathbf{Q}_{(s)} \\ \mathbf{r}'_{(s)} &= \mathbf{R}_{(\mathbf{Q}_{(s)})}\boldsymbol{\Gamma}_{(s)} \end{aligned} \tag{1.10}$$

Where the elements $\mathbf{A}_{(\mathbf{K}_{(s)})}$ and $\mathbf{R}_{(\mathbf{Q}_{(s)})}$ are discussed in Appendix A.

Here we see that, comparing to what is discussed in Appendinx A, the rate of change of the pose in $SE\,(3)$of the cross section is an element in $se(3)$. This means that very cross section behaves like a rigid body connected with the previus one with a virtual joint allowing every kind of rotation and motion.

From the definition of the terms $\mathbf{r}_{(s)}$and $\mathbf{R}_{(s)}$, we have that a Cosserat rod configutation space is defined as follows.

$$\mathcal{C} = \mathbf{T} : X \in [0,1] \mapsto \mathbf{T}_{(X)} \in SE\,(3) = SE\,(3) \times \mathbb{S} \tag{1.11}$$

The right hand side of (1.11) explicit that the configuration space of a Cosserat rod is defined by the $SE\,(3)$ pose of its first cross-section and the shape space of the Cosserar rod, namely $\mathbb{S}$. As the rod is a continous system, $\mathbb{S}$ is a functional space: an infinite dimensional space of X-parametrized curves in $\mathbb{R}^6$.

$$\mathbb{S} = \boldsymbol{\xi} : X \in [0,1] \mapsto \boldsymbol{\xi}_{(X)} \in \mathbb{R}^6 \tag{1.12}$$

1.1.2 Equilibrium Equation

Considering the static equilibrium of the rod, we can define a set of rules to link internal forces and moments with external distributed loads. We can develop our formulations analyzing an infinitesimal section of the rod modeled as a free body in space. This infinitesimal cross section is defined between the two centerline coordinates s_1 and s_2 having $s_2 = s_1 + ds$ where ds is an infinitesimal displacement on the arc length coordinate s.

The figure 1.3 shows a section of the rod, subjected to the internal and distributed forces and moment. In each section, we have the wrench coming from the internal forces and moment, express in the coordinates of the

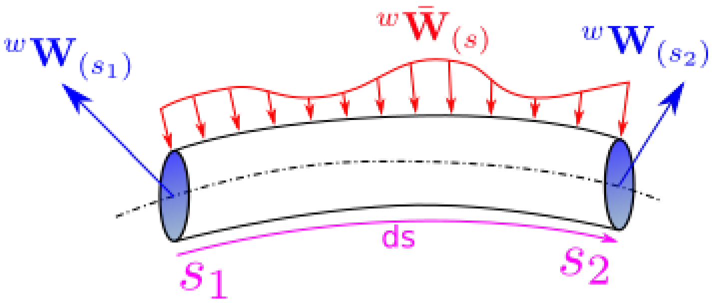

Figure 1.3: Section of a Cosserat rod with the two internal wrenches at the two ends.

reference frame as follows.

$$\mathbf{W}_{(s)} = \begin{bmatrix} \mathbf{n}_{(s)} \\ \mathbf{m}_{(s)} \end{bmatrix} \tag{1.13}$$

This wrench is defined as the wrench that from a cross section past s_2 applies on the section s_2.

Moreover, the rod could be subjected to some distributed forces $\bar{\mathbf{f}}_{(s)}$ and moment $\bar{\mathbf{l}}_{(s)}$, which are represented in the distributed wrench.

$$\bar{\mathbf{W}}_{(s)} = \begin{bmatrix} \bar{\mathbf{f}}_{(s)} \\ \bar{\mathbf{l}}_{(s)} \end{bmatrix} \tag{1.14}$$

We now need to define the effects of the external distributed forces and moments on the internal ones. In order to develop the equilibrium equations, let us consider the balance of forces and moments of the infinitesimal rod section between s_1 and s_2.

$$\mathbf{n}_{s_2} - \mathbf{n}_{s_1} + \int_{s_1}^{s_2} \bar{\mathbf{f}}_{(\sigma)} \, d\sigma = \mathbf{0} \tag{1.15}$$

$$\mathbf{m}_{s_2} + \mathbf{r}_{s_2} \times \mathbf{n}_{s_2} - \mathbf{m}_{s_1} + \mathbf{r}_{s_1} \times \mathbf{n}_{s_1} + \int_{s_1}^{s_2} \left(\mathbf{r}_\sigma \times \bar{\mathbf{f}}_{(\sigma)} + \bar{\mathbf{l}}_{(\sigma)} \right) d\sigma = \mathbf{0} \tag{1.16}$$

From the derivative of the equilibrium equations (1.15) we obtain the following relations

$$\mathbf{n}'_{(s)} = -\bar{\mathbf{f}}_{(s)} \tag{1.17}$$

$$\mathbf{m}'_{(s)} = -\left(\mathbf{r}'_{(s)} \times \mathbf{n}_{(s)} + \bar{\mathbf{l}}_{(s)} \right) \tag{1.18}$$

1.1.3 Constituves Laws

We need to define how the stresses, acting on the rod, deformate the body. Thus, we need an operator which acts as a map from the space of the internal stresses to the space of the internal deformations.

We introduce the notion of initial curvature as the shape of the rod when free from any internal or external wrench. Typically the rod is straight when it is in a load free configuration. However, in some case the rod has a pre-curvature, which could be desired, such as in the work of Rucker et al. [3] [6] [7]. The initial curvature is described with the quantities $\mathbf{\Gamma}^0_{(s)}$ and $\mathbf{K}^0_{(s)}$, acting both from $s \in [0, \ell]$to $\mathbb{R}^3$. These quantities defines the initial strain $\boldsymbol{\xi}^0_{(s)} = \left(\mathbf{\Gamma}^0_{(s)}, \mathbf{K}^0_{(s)}\right)$ As an example, if the rod is initially straight, then the two initial curvatures are $\mathbf{\Gamma}^0_{(s)} = (0, 0, 1)$ and $\mathbf{K}^0_{(s)} = \mathbf{0} \in \mathbb{R}^3$.

Thus, we define the deformation of the rod as the changes from the non-deformed case.

$$\Delta\boldsymbol{\xi}_{(s)} = \boldsymbol{\xi}_{(s)} - \boldsymbol{\xi}^0_{(s)} = \begin{bmatrix} \Delta\mathbf{K}_{(s)} \\ \Delta\mathbf{\Gamma}_{(s)} \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{(s)} - \mathbf{K}^0_{(s)} \\ \mathbf{\Gamma}_{(s)} - \mathbf{\Gamma}^0_{(s)} \end{bmatrix} \tag{1.19}$$

In the Equation (1.19) we have that $\Delta\mathbf{\Gamma}_{(s)} = \begin{bmatrix} \Delta\Gamma_x & \Delta\Gamma_y & \Delta\Gamma_z \end{bmatrix}^T$ and $\Delta\mathbf{K}_{(s)} = \begin{bmatrix} \Delta K_x & \Delta K_y & \Delta K_z \end{bmatrix}^T$. We can give a physical significance to these terms. The terms $\Delta\Gamma_x$ and $\Delta\Gamma_y$ defines the deformation due to the shear, thus the motion of two sections sliding with respect to each other, while $\Delta\Gamma_z$ define the deformation due to a force that is normal to the section, moving the two sections closer together or further apart. The Figure 1.4 is a graphical representation of what discussed above.

We define the vector $\mathbf{E}_\ell$ as $\mathbf{E}_\ell = \mathbf{\Gamma} - \mathbf{N}$. Having $\mathbf{N} = (1, 0, 0)$ as the normal to the cross section. Then, the vector $\mathbf{E}_\ell$ expresses how the cross section slides with respect of the material centerline.

On the other and, the components ΔK_x and ΔK_y are the two angular deformations due to moments along the two principal axis of inertia while ΔK_z is the deformation due to the torsion, a moment acting on the axis perpendicular to the cross section.

Note that the geometrical meaning of the strain is: how the material evolvers with respect of itself. It is an absolute quantity as it does not depend on the past state nor on a reference state. On the other hand, the deformation is a difference: it expresses the changes in the shape with respect of a reference configuration.

Depending on the number of DoFs that are admitted between two subsequent cross sections, different theories have been developed. A Reissner rod allows every motion, thus all the six deformations: ΔK_x, ΔK_y, ΔK_z, $\Delta\Gamma_x$, $\Delta\Gamma_y$, $\Delta\Gamma_z$. Such formulation are threated in [8].

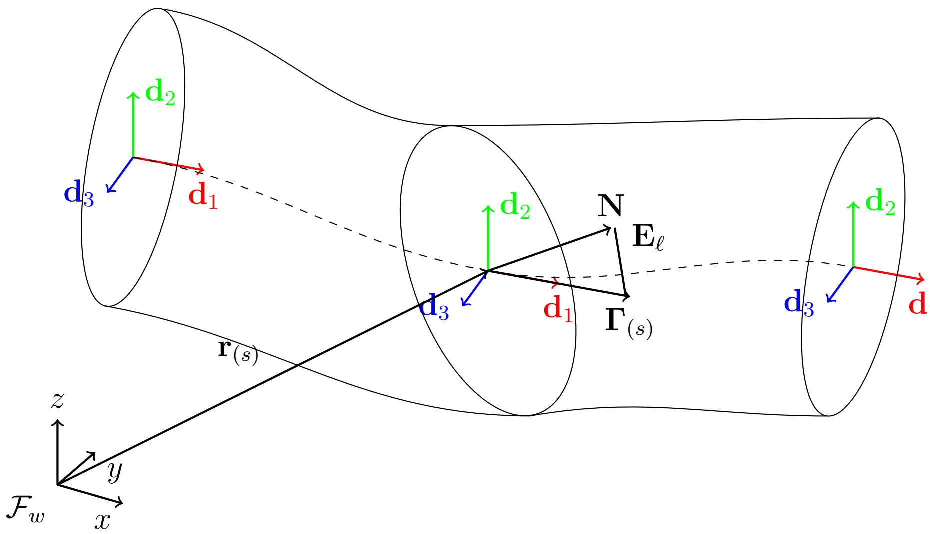

Figure 1.4: In this figure, a graphical representation of $\mathbf{\Gamma}_{(s)}$having the vector $\mathbf{N}$ as the normal to the cross section.

On the other hand, some deformation can be neglected; it is common for slender bodies to neglect the shear and normal deformation. The Kirchhoff rod theory, accounts only for the deformation in the rod curvature. In this case, the strain $\mathbf{\Gamma}$ is considered constant and equal to $\mathbf{\Gamma} = (1, 0, 0)$.

Assumptions on the body If we assume that:

- The material has an elastic behaviour.
- The stress-strain characteristic of the material is linear.
- The material is isotropic.
- The body frame into the reference state is aligned to the principal inertia axis of the cross-section.

Then we can define an element $\boldsymbol{\mathcal{H}} \in \mathbb{R}^{6\times 6}$ which acts as a map from the space of the deformation to the space of the internal forces and moments, expressed in the local coordinates frame of the rod cross section. The linear map $\boldsymbol{\mathcal{H}} : \mathbb{R}^6 \to \mathbb{R}^6$ is the matrix of the Hooke tensors along the beam, expressed in the local coordinates frame of the rod cross section, defined as follows.

$$\boldsymbol{\mathcal{H}} = \begin{bmatrix} \boldsymbol{\mathcal{H}}_{ang} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\mathcal{H}}_{lin} \end{bmatrix} \tag{1.20}$$

$$\boldsymbol{\mathcal{H}}_{ang} = \begin{bmatrix} EI_{xx} & 0 & 0 \\ 0 & EI_{yy} & 0 \\ 0 & 0 & GJ \end{bmatrix} \quad \boldsymbol{\mathcal{H}}_{lin} = \begin{bmatrix} GA & 0 & 0 \\ 0 & GA & 0 \\ 0 & 0 & EA \end{bmatrix} \tag{1.21}$$

In Equation (1.21) the matrices $\boldsymbol{\mathcal{H}}_{ang}$ and $\boldsymbol{\mathcal{H}}_{lin}$ express the link between deformations and internal forces and moments respectively. In (1.21), A is the area of the cross section, I_{xx}, I_{yy} and J the moments of inertia along the principal axes, with $J = I_{xx} + I_{yy}$. On the other hand, E is the Young modulus and G is the shear modulus.

Using the matrix $\boldsymbol{\mathcal{H}}$, we map the deformations into the space of the internal stresses.

$$\boldsymbol{\Lambda}_{(s)} = \boldsymbol{\mathcal{H}}\Delta\boldsymbol{\xi} \tag{1.22}$$

Note that, in Equation (1.22), the wrench $\boldsymbol{\Lambda}_{(s)}$is the expression in the coordinate frame of $\mathcal{F}_{(s)}$ of the internal wrench ${}^{w}\mathbf{W}_{(s)}$, which was previously defined in Equation (1.13). Reminding the definition of $\mathbf{W}_{(s)}$ we obtain the following relation.

$$\mathbf{W}_{(s)} = \boldsymbol{\mathcal{R}}_{(s)}\boldsymbol{\Lambda}_{(s)} = \boldsymbol{\mathcal{R}}_{(s)}\boldsymbol{\mathcal{H}}\Delta\boldsymbol{\xi}_{(s)} \tag{1.23}$$

Where $\boldsymbol{\mathcal{R}}$ is a block diagonal matrix containing the rotation matris $\mathbf{R}_{(s)}$.

$$\boldsymbol{\mathcal{R}}_{(s)} = \begin{bmatrix} \mathbf{R}_{(s)} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{(s)} \end{bmatrix} \tag{1.24}$$

We can rewrite Equation (1.22) exploiting the internal forces and moments in the global or reference frame: ${}^{w}\mathbf{n}_{(s)}$ and ${}^{w}\mathbf{m}_{(s)}$ as well as the matrices $\boldsymbol{\mathcal{H}}_{a}$ and $\boldsymbol{\mathcal{H}}_{\ell}$ mapping the deformations into the stress space.

$$\begin{aligned} \mathbf{m}_{(s)} &= \mathbf{R}_{(s)}\boldsymbol{\mathcal{H}}_{ang}\left[\mathbf{K}_{(s)} - \mathbf{K}^{0}_{(s)}\right] \\ \mathbf{n}_{(s)} &= \mathbf{R}_{(s)}\boldsymbol{\mathcal{H}}_{lin}\left[\boldsymbol{\Gamma}_{(s)} - \boldsymbol{\Gamma}^{0}_{(s)}\right] \end{aligned} \tag{1.25}$$

By inverting the equation (1.25) we can obtain the value of the curvature at any $s \in [0, \ell]$, having ${}^{w}\mathbf{n}_{(s)}$, ${}^{w}\mathbf{m}_{(s)}$ and the initial curvatures $\mathbf{K}^{0}_{(s)}$ and $\boldsymbol{\Gamma}^{0}_{(s)}$

$$\mathbf{K}_{(s)} = \boldsymbol{\mathcal{H}}^{-1}_{ang}\mathbf{R}_{(s)}\mathbf{n}_{(s)} + \mathbf{K}^{0}_{(s)} \tag{1.26}$$

$$\boldsymbol{\Gamma}_{(s)} = \boldsymbol{\mathcal{H}}^{-1}_{lin}\mathbf{R}^{T}_{(s)}\,\mathbf{m}_{(s)} + \boldsymbol{\Gamma}^{0}_{(s)} \tag{1.27}$$

Equations (1.26) and (1.27) can be conbined, reminding the definition of $\boldsymbol{\xi}_{(s)}$ and $\boldsymbol{\xi}_{0(s)}$, and introducing the definition of $\boldsymbol{\Lambda}_{(s)}$ given in Equation (1.22) giving us the following equation, which directly express the relation between the stresses and the strain fields.

$$\begin{aligned} \boldsymbol{\xi}_{(s)} &= \boldsymbol{\mathcal{H}}^{-1}\mathbf{R}^T_{(s)}\mathbf{W}_{(s)} + \boldsymbol{\xi}^0_{(s)} \\ &= \boldsymbol{\mathcal{H}}^{-1}\boldsymbol{\Lambda}_{(s)} + \boldsymbol{\xi}^0_{(s)} \\ &= \boldsymbol{\xi}_{a(s)} + \boldsymbol{\xi}^0_{(s)} \end{aligned} \tag{1.28}$$

Where we have $\boldsymbol{\xi}_{a(s)}$ as the strain field coming from the internal stress of the rod.

1.2 Internal Stresses

Considering an external distributed wrench $\bar{\mathbf{W}}$ we express its counterpart in the local coordinates frame as $\mathbf{F}_{ext}$. We then need to realte its influence on the internal stresses. To this aim, we follow the results of [9] where they apply the Hamilton principle on the dynamics of the cosserat rod. First, we impose the boundary conditions o : $\left(\frac{\delta\mathcal{L}}{\delta\boldsymbol{\xi}}\right)_{(0)} = \mathbf{F}^-_{ext}$ and $\left(\frac{\delta\mathcal{L}}{\delta\boldsymbol{\xi}}\right)_{(\ell)} = -\mathbf{F}^+_{ext}$.

Then, neglecting the terms relative to the dynamics of the rod, the PDE obtained in [9] can be simplified into the following ODE.

$$-\boldsymbol{\Lambda}' + ad^T_{\boldsymbol{\xi}}\boldsymbol{\Lambda} = \bar{\mathbf{F}}_{ext} \quad \begin{cases} \boldsymbol{\Lambda}_0 = \mathbf{F}^-_{ext} \\ \boldsymbol{\Lambda}_\ell = -\mathbf{F}^+_{ext} \end{cases} \tag{1.29}$$

This equation gives the progression of the internal stresses considering the external action of wrenches.

1.3 Conclusions

In this chapter we presented how a deformable body can be represented using the Cosserat rod theory. We defined the kinematic behavior and a relation between internal stresses and external loads. In the next chapter, we will describe how we can use these quantities to model the deformable body.

Chapter 2

Strain Parameterization Based Approach

In this chapter we detail the Strain Parameterization Based Approach. In a deformable bodies, the strains define how the material evolves. The strains field is considered variable and reduced on a truncated basis of polynomials, or strain functions.

As a consequence the kinematics quantities of the rod are not analytically integrable, but a numerical integration is required in order to reconstruct the rod kinematics.

In the next section we will analyze the kinematics of a Cosserat rod beam applying these assumptions to what introduced in Chapter 1.

2.1 Introduction of the Assumptions

We start recalling the presentation of the Cosserat rod theory of Chapter 1. The configuration space of a Cosserat rod, represented in Equation (1.11), is an infinite dimension space. Here we want to reduce the space to a finite dimension, in order to find numerical solution for the modeling. As detailed in Section 1.1.3, different assumptions on the admitted deformations defined different types of rod. Moreover, from Equation (1.28) we can deduce that the strains acting on the rod can be divided into a set of active strains $\boldsymbol{\xi}_a$ and a set of strains $\boldsymbol{\xi}_0$ describing the configuration of the rod when load free.

Here, we combine these two ideas in order to reduce the configuration space of the Cosserat rod. First, we make an assumption on the admitted motions between the cross-section of the rod. For example if all the motions are allowed then we have a Reissner beam. On the other hand, when only the three rotations are allowed, while the translation part of the strain is constant and equal to $\mathbf{\Gamma} = (1, 0, 0)$, we have a Kirchhoff rod. If we define the number of admitted motions as n_a with $0 < n_a \leq 6$, we can identify an admitted field of strains $\boldsymbol{\xi}_a \in \mathbb{R}^{n_a} \mid \boldsymbol{\xi}_a \subseteq \boldsymbol{\xi}$ and a constrained field of strains $\boldsymbol{\xi}_c \in \mathbb{R}^{n_c} \mid \boldsymbol{\xi}_c \subset \boldsymbol{\xi}$ with $n_c = 6 - n_a$. We then define two complentary matrices $\mathbf{B} \in \mathbb{R}^{6\times n_a}$ and $\bar{\mathbf{B}} \in \mathbb{R}^{6\times n_c}$ that perform the maps from the subset of admitted and constrained strains into the full strain field.

$$\boldsymbol{\xi} = \mathbf{B}\boldsymbol{\xi}_a + \bar{\mathbf{B}}\boldsymbol{\xi}_c \tag{2.1}$$

Equation (2.1) shows how the strain field can be decomposed in the two subfield of admitted and constrained strains. The two complementary matrices $\mathbf{B}$ and $\bar{\mathbf{B}}$ also have the fallowing properties: $\mathbf{B}^T\mathbf{B} = \mathbb{1}_{n_a}$, $\mathbf{B}^T\bar{\mathbf{B}} = \mathbf{0}_{n_a}$ and $\bar{\mathbf{B}}^T\bar{\mathbf{B}} = \mathbb{1}_{n_a}$. Where $\mathbb{1}_{n_a}$ is the identity matrix and $\mathbf{0}_{n_a}$ a matrix of zeros both having n_a rows and n_a columns.

With the constant strain field $\boldsymbol{\xi}_c$ the kinematics of the Cosserat rod is defined by the field $\boldsymbol{\xi}_a$, and the shape space descibed in Equation (1.12) is now reduced.

$$\mathbb{S} = \boldsymbol{\xi}_a : X \in [0, 1] \mapsto \boldsymbol{\xi}_{a(X)} \in \mathbb{R}^{n_a} \tag{2.2}$$

As we did in Equation (1.28), the admitted strain field can be decomposed in a reference component $\boldsymbol{\xi}_{a_0(X)}$ and a varing component $\boldsymbol{\xi}_{a_q(X)}$.

$$\boldsymbol{\xi}_{a(X)} = \boldsymbol{\xi}_{a_0(X)} + \boldsymbol{\xi}_{a_q(X)} \tag{2.3}$$

The component $\boldsymbol{\xi}_{a_0(X)}$ represents the set of strains when the rod is load free. On the other hand, the time varying component $\boldsymbol{\xi}_{a_q(X)}$ can be reduced on a truncated basis of strain functions, as anticipated in the introduction. To this aim, we discretize every component of $\boldsymbol{\xi}_{a_q(X)}$ with a polynomial, or a shape function, of order $n_e - 1$ which depends on the normalized arc-length $X \in [0, 1]$. This polynomial is paired with a vector of coefficients, or strain-generalized coordinates, namely $\mathbf{q}_e \in \mathbb{R}^{n_e}$. As a result, every component of the admitted strain field $\boldsymbol{\xi}_{a_i(X)}$, with $i = 1, \ldots, n_a$ can be defined as follows.

$$\begin{aligned}
\boldsymbol{\xi}_{a_i(X)} &= \boldsymbol{\xi}_{a_{0\,i(X)}} + \phi_{i_1} q_{e_{i_1}} + \phi_{i_2(X)} q_{e_{i_2}} + \ldots + \phi_{i_{n_e}(X)} q_{e_{i_{n_e}}} \\
&= \boldsymbol{\xi}_{a_{0\,i(X)}} + \begin{bmatrix} \phi_{i_1} & \phi_{i_2(X)} & \cdots & \phi_{i_{n_e}(X)} \end{bmatrix} \begin{bmatrix} q_{e_{i_1}} \\ q_{e_{i_2}} \\ \vdots \\ q_{e_{i_{n_e}}} \end{bmatrix} \\
&= \boldsymbol{\xi}_{a_{0\,i(X)}} + \mathbf{\Phi}_{i(X)} \cdot \mathbf{q}_{e_i}
\end{aligned} \tag{2.4}$$

In Equation (2.4), the coefficients ϕ_{i_j}, with $j \in [1, n_e]$, can be computed using the different polynomials. Common choices are orthogonal or orthonormal polynomial like the Chebyshev and Legendre polynomial respectively. In Appendix B, we detail the Chebyshev polynomials. Adopting these polynomials, the coeffients are defined as $\phi_{i_{j(X)}} = \mathbf{T}_{j\left(x_{(X)}\right)}$. Note that the Chebyshev point $x_{(X)}$ is a function of X. In fact, we need to report the corresponding point in the rod material coordinates $X \in [0,1]$ into the Chebyshev domain $x \in [-1,1]$, as detailed in Appendix B. To this aim, we define the ratio $x_{(X)} = \frac{2X-(a+b)}{a-b}$, with $a = 0$ and $b = 1$. On the other hand, the vector $\mathbf{q}_{e_i} \in \mathbb{R}^{n_e}$ stack all the generalized coordinates for the i^{th} component of $\boldsymbol{\xi}_a$. In order to discretize the whole field $\boldsymbol{\xi}_a$, we define a vector of generalized coordinates $\mathbf{q}_e = \left(\mathbf{q}_{e_1}, \mathbf{q}_{e_2}, \ldots, \mathbf{q}_{e_{n_a}}\right)$. This vector contains the n_e generalized coordinates for the n_a components of $\boldsymbol{\xi}_a$. Naming $n = n_a \cdot n_e$ we have that $\mathbf{q}_e \in \mathbb{R}^n$.

As a result, the whole field $\boldsymbol{\xi}_a$ can be discretized as follows.

$$\boldsymbol{\xi}_{a(X,t)} = \boldsymbol{\xi}_{a_0(X)} + \boldsymbol{\Phi}_{(X)}\mathbf{q}_{e(t)} \tag{2.5}$$

Where $\boldsymbol{\Phi}_{(X)} \in \mathbb{R}^{n_a \times n}$ is the matrix of the shape functions, which has the following definition.

$$\boldsymbol{\Phi}_{(X)} = \begin{bmatrix} \boldsymbol{\Phi}_1 & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Phi}_2 & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \boldsymbol{\Phi}_{n_a} \end{bmatrix} \tag{2.6}$$

With this last assumption, the shape space of the Cosserat rod reduced to a finite dimension $\mathbb{R}^n$. We thus have that the configuration space of a floating Cosserat rod is

$$\mathcal{C} = SE(3) \times \mathbb{R}^n \tag{2.7}$$

Using the matrix $\mathbf{B}$ we can also select from the linear mapping $\boldsymbol{\mathcal{H}}$ the components relative to $\boldsymbol{\xi}_a$.

$$\boldsymbol{\mathcal{H}}_a = \mathbf{B}^T\boldsymbol{\mathcal{H}}\mathbf{B} \tag{2.8}$$

Moreover, the decomposition on the field of strain has its counterpart in the field of stress, as follows.

$$\boldsymbol{\Lambda} = \mathbf{B}\boldsymbol{\Lambda}_a + \bar{\mathbf{B}}\boldsymbol{\Lambda}_c \tag{2.9}$$

Where $\boldsymbol{\Lambda}_a = \mathbf{B}^T\boldsymbol{\Lambda} = \boldsymbol{\mathcal{H}}_a\left(\boldsymbol{\xi}_a - \boldsymbol{\xi}_{a0}\right)$ and $\boldsymbol{\Lambda}_c$ imposes the internal constrains $\boldsymbol{\xi}_c = \bar{\mathbf{B}}^T\boldsymbol{\xi}$.

2.1.1 Expression of Material Properties

In this subsection we want to apply the principle of virtual power in order to retrive the relation between the set of generalized strain coordinates and the internal elasticity and dumping. We start by applying the principle of virtual power to the internal stresses and strains of the Cosserat rod.

$$\mathcal{P}_{int} = \int_0^1 \boldsymbol{\Lambda}_{a(X,t)}^T \delta\boldsymbol{\xi}_{a(X,t)}\, dX \tag{2.10}$$

In Equation (2.10) the term $\delta\boldsymbol{\xi}_a$ expresses a virtual derivative on the strain field. We then introduce the definition of the admitted internal stresses $\boldsymbol{\Lambda}_{a(X,t)}$ in Equation (2.10) in order to obtain the following formulation.

$$\mathcal{P}_{int} = \int_0^1 \left(\boldsymbol{\xi}_{a(X,t)} - \boldsymbol{\xi}_{a_0(X)}\right)^T \boldsymbol{\mathcal{H}}_a^T \delta\boldsymbol{\xi}_{a(X,t)}\, dX \tag{2.11}$$

Introducing (2.5) into Equation (2.11) allows to simply with respect of $\boldsymbol{\xi}_{a_0}$.Moreover, as this terms is constant, its derivatives cancels out having $\delta\boldsymbol{\xi}_a = \delta\boldsymbol{\xi}_{a_0} + \boldsymbol{\Phi}\delta\mathbf{q}_e = \boldsymbol{\Phi}\delta\mathbf{q}_e$. Equation (2.11) now becames:

$$\begin{aligned}
\mathcal{P}_{int} &= \int_0^1 \boldsymbol{\xi}_{a(X,t)}^T \boldsymbol{\mathcal{H}}_a^T \delta\boldsymbol{\xi}_{a(X,t)}\, dX \\
&= \int_0^1 \delta\boldsymbol{\xi}_{a(X,t)}^T \boldsymbol{\mathcal{H}}_a \boldsymbol{\xi}_{a(X,t)}\, dX \\
&= \int_0^1 \delta\mathbf{q}_{e(t)}^T \boldsymbol{\Phi}_{(X)}^T \boldsymbol{\mathcal{H}}_a \boldsymbol{\Phi}_{(X)} \mathbf{q}_{e(t)}\, dX \\
&= \delta\mathbf{q}_{e(t)}^T \int_0^1 \boldsymbol{\Phi}_{(X)}^T \boldsymbol{\mathcal{H}}_a \boldsymbol{\Phi}_{(X)}\, dX\ \mathbf{q}_{e(t)}
\end{aligned} \tag{2.12}$$

Being valid for all $\delta\mathbf{q}_{e(t)}$ Equation (2.12) shows that the internal energy due to the elasticity of the material is given by the followig realation.

$$\mathcal{E}_{int} = \int_0^1 \boldsymbol{\Phi}_{(X)}^T \boldsymbol{\mathcal{H}}_a \boldsymbol{\Phi}_{(X)}\, dX\ \mathbf{q}_{e(t)} \tag{2.13}$$

From Equation (2.13), we see that the integral only contains terms which are defined; in fact, the Hook matrix $\boldsymbol{\mathcal{H}}_a$ is known along the rod arc-length as well as the basis function $\boldsymbol{\Phi}_{(X)}$. We can thus compute the integral and introduce the internal stiffness matrix as follows.

$$\mathbf{K}_{\epsilon\epsilon} = \int_0^1 \mathbf{\Phi}^T_{(X)} \boldsymbol{\mathcal{H}}_a \mathbf{\Phi}_{(X)} \, dX \tag{2.14}$$

We have seen that the internal stresses are related to the deformation by the Hook matrix in Equation (1.22). This equation is valid for all passive rod.

On the other hand, when we have an actuated rod, this formaulation changes. An actuated rod is a Cosserat rod provided with some sort of actuation. In the simplest case, we have an actuator attached to the rod base. The internal actuation is defined by the term $\mathbf{\Lambda}_{ad(X)}$.

$$\mathbf{\Lambda}_{a(X,t)} = \mathbf{\Lambda}_{ad(X,t)} + \boldsymbol{\mathcal{H}}_a \left(\boldsymbol{\xi}_{a(X,t)} - \boldsymbol{\xi}_{a(X)} \right) \tag{2.15}$$

The internal actuation can be mapped into the generalized coordinate space with the matrix $\mathbf{\Phi}_{(X)}$. We thus obtain the vector of fittitius internal generalized forces as the contribution on the internal actuation stresses along the rod arc-length.

$$\mathbf{Q}_{ad(t)} = -\int_0^1 \mathbf{\Phi}^T_{(X)} \mathbf{\Lambda}_{ad(X,t)} \, dX \tag{2.16}$$

2.1.2 Inverse Geometric Model

From the results obtained in the last section, we can obtain the Inverse Geometric Model (IGM) for a Cosserat rod. This model serves as a input-output map: given the wrench at the rod tip and the one distributed on its arc-length, it returns the set of generalized forces acting on the Cosserat rod. The name IGM comes from the fact that the wrench from the rod tip are reported to the rod base, thus goint from $X = 1$ to $X = 0$ instead of from $X = 0$ to $X = 1$.

First, in order to recover the effect of the wrenches on the Cosserat rod body, we need to define the kinematics, *i.e.*, configuration, velocities and accelerations of the rod. To this aim, we combine Equation (1.8) into a system of ODEs. Thanks to the definition of $\boldsymbol{\xi}$, we can express the term in Equation (1.8) with the contribution of quaternion and position, separately. Note that the label X is bounded in the its domain $X \in [0, 1]$. Thus, in order to consider the real length of the rod, defined as ℓ, we can multiply the ODEs by the rod length to scale the numerical integration.

$$\begin{cases} \mathbf{Q}'_{(X)} &= \ell \left(\frac{1}{2} \mathbf{A}_{(\mathbf{K}_{(X)})} \mathbf{Q}_{(X)} \right) \\ \mathbf{r}'_{(X)} &= \ell \left(\mathbf{R}_{(\mathbf{Q}_{(X)})} \mathbf{\Gamma}_{(X)} \right) \end{cases} \tag{2.17}$$

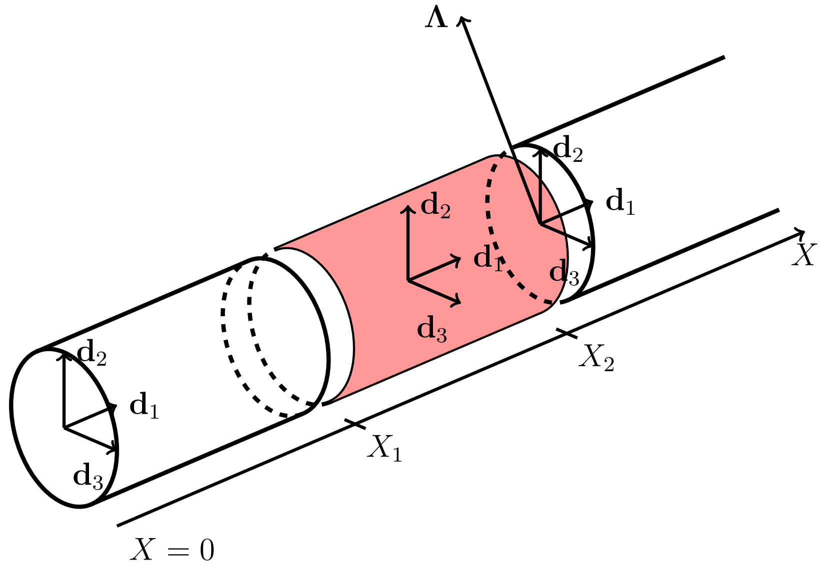

Figure 2.1: In this figure, a graphical representation of the action of $\mathbf{\Lambda}$ for the section at X_2, it acts from the section $X_2 + dX$.

Integrating the ODEs system in Equation (2.17) from $X = 0$ to $X = 1$ gives the kinematic configuration of the Cosserat rod. We can then compute configuration dependent quantities such as the inertia and gravity components. In order to map the influence of the wrenches into the rod body, we backward integrate (from $X = 1$ to $X = 0$) the Equation (1.29), rearranged in oder to isolate $\mathbf{\Lambda}'_{(s,t)}$ and (2.16), which gives us the set of generalized forces $\mathbf{Q}_{ad(X,t)}$ that enforce $\mathbf{\Lambda}_{(s,t)}$ into the Cosserat rod. Note that, as represented in Figure 2.1, $\mathbf{\Lambda}_{(s,t)}$ acts from the tip to the base.

$$\begin{cases} \mathbf{Q}'_{(X)} & = \ell \left(\frac{1}{2} \mathbf{A}_{(\mathbf{K}_{(X)})} \mathbf{Q}_{(X)} \right) \\ \mathbf{r}'_{(X)} & = \ell \left(\mathbf{R}_{(\mathbf{Q}_{(X)})} \mathbf{\Gamma}_{(X)} \right) \\ \mathbf{\Lambda}'_{(X)} & = \ell \left(ad^T_{\boldsymbol{\xi}} \mathbf{\Lambda}_{(X)} - \bar{\mathbf{F}}_{ext} \right) \\ \mathbf{Q}_{ad}{}'_{(X)} & = -\ell \left(\mathbf{\Phi}^T_{(X)} \mathbf{B}^T \mathbf{\Lambda}_{(X)} \right) \end{cases} \tag{2.18}$$

Note that in Equation (2.18) we need to backward integrate the ODEs system (2.17) previously forward integrated as the right hand side of (1.29) requires knowledge on the rod kinematics, as it needs to access values of twists $\boldsymbol{\eta}$ and acceleration $\dot{\boldsymbol{\eta}}$ in the backward integration range $[1, 0]$.

The output of the backward integration of (2.18) gives us the set of generalized actuation forces $\mathbf{Q}_{ad}$ and the wrench at the rod base $\mathbf{\Lambda}_{(0)}$. We can thus encapsulate the forward and backward integration in an unique operator called IGM.

$$IGM : \mathbf{q}_e, \mathbf{T}_0 \mapsto \mathbf{F}_0, \mathbf{Q}_a \tag{2.19}$$

The IGM operator can be seen as a function whose domain is in the configuration space of the rod positon and strains while its image is in the space of generalized forces and wrench at the rod base.

2.2 Boundary Value Problem

Having the IGM algorithm we solve the problem of finding the configuration for a given load on the rod. The problem can be stated as follows: given a known wrench at the rod tip, find the corresponding deformed configuration of the rod. This is essentialy a Boundary Value Problem (BVP) where given the condition at the boundary, in this case $\mathbf{F}^{+}_{ext}$, we find the corresponding set of variables that solve the problem, in this case the set of strains $\mathbf{q}$. Note that in the BVP we could impose other conditions such as wrench at the base $\mathbf{F}^{-}_{ext}$ and the distributed one $\bar{\mathbf{F}}_{ext}$.

This problem is of course non liear due to the non linearities of the deformable body. Thus, the solution is found numerically unsing a non lineat solver such as the Levenberg-Marquardt algorithm.

In order to find a relation between the internal generalized strain corrdinates $\mathbf{q}$ and the external wrenches, we make use again of the results obtained by Boyer et al. [9]. They obtained the dynamic model for the Cosserat rod in the lagrangian form. Starting from the dynamic model, we simplify it neglegting all the dynamics terms. The model simplifies in the following equation.

$$\mathbf{K}_{ee}\mathbf{q}_e = -\mathbf{Q}_{ad(X)} \tag{2.20}$$

2.3 Conclusions

In this chapter, we have seen how the equation of a Cosserat rod can be used to define a model based on a strain parameterization of the rod. This approach is general and geometrically exact in terms of the rod kinematics, givin us a reliable model.

However, this model requires two numerical integrations (forward and backward) in order to obtain the IGM of the deformable body. On the other hand, looking at the system of ODEs, one may notice that the equations are decoupled. In other words, thanks to the strain parameterization, the ODEs can be solved indipendently in descendent order. That is, the evolution of the quaternions can be integrated indipendently, then we can integrate the positions and so on.

This represents an huge advantage as, in the next chapter, we will see how a differential equation can be converted in a linear algebra problem.

Chapter 3

Applied Spectral Numerical Integration

In Appendix B we have seen how an ODE can be converted into a problem of linear algebra.

In order to extend that formulation for the case of a cosserat rod, we need to account for an ODE of the form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b} \tag{3.1}$$

In Equation (3.1), the terms $\mathbf{A}$ and $\mathbf{b}$ are known. Respectively, we have that $\mathbf{A}$ is a matrix of coefficient multiplying the state vector $\mathbf{x}$ while $\mathbf{b}$ is a vector of known parameters not dependent on $\mathbf{x}$.

We can consider our system a two dymensional and we define the quantities as follows.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{3.2}$$

We define the domain of this system as $X \in [0, 1]$. We can choose to observe this system at the Chebyshev points. For the sake of concision, we limit the number of Chebyshev points Nc to 3, thus, $Nc = 3$. Under this assumption, we observe the system at the Chebyshev points t_0, t_1 and t_2, as represented in Figure 3.1.

Note that the numbering of the Chebyshev point goes from left to right while the domain is going from the right to the left. In fact, the first Chebyshev point is a $X = 1$ as can be deduced from Equation (B.1). The layout of the system makes necessary to stack the vector in a matrix form column wise and not row-wise, as we did in the Appendix B. In fact, the linear operator

$X = 1 \quad t_0 \qquad\qquad t_1 \qquad\qquad t_2 \quad X = 0$

$$\mathbf{x}_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} \qquad \mathbf{x}_1 = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} \qquad \mathbf{x}_2 = \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix}$$

$$\mathbf{A}_0 = \begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix} \qquad \mathbf{A}_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \qquad \mathbf{A}_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

$$\mathbf{b}_0 = \begin{bmatrix} b_{01} \\ b_{02} \end{bmatrix} \qquad \mathbf{b}_1 = \begin{bmatrix} b_{11} \\ b_{12} \end{bmatrix} \qquad \mathbf{b}_2 = \begin{bmatrix} b_{21} \\ b_{22} \end{bmatrix}$$

Figure 3.1: Representation the observation of the system at the Chebyshev points.

$\mathbf{A}$ needs to multiply the rows of the state vector $\mathbf{x}$, and for this reason they must be stacked in a proper matrix.

We can stack all the observed vector $\mathbf{x}$ in a matrix that we call $\boldsymbol{\mathcal{X}}$.

$$\boldsymbol{\mathcal{X}} = \begin{bmatrix} \mathbf{x}_0 & \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} x_{01} & x_{11} & x_{21} \\ x_{02} & x_{12} & x_{22} \end{bmatrix} \tag{3.3}$$

We stack this matrix row-wise into a vector $\boldsymbol{\chi}$. Similarly, we stack the vector $\mathbf{b}$ observed at the Chebyshev points into the vector $\boldsymbol{\beta}$

$$\boldsymbol{\chi} = \begin{bmatrix} x_{01} \\ x_{11} \\ x_{21} \\ x_{02} \\ x_{12} \\ x_{22} \end{bmatrix} \tag{3.4}$$

$$\boldsymbol{\beta} = \begin{bmatrix} b_{01} \\ b_{11} \\ b_{21} \\ b_{02} \\ b_{12} \\ b_{22} \end{bmatrix} \tag{3.5}$$

On the other hand, we define a matrix $\boldsymbol{\mathcal{A}}$which preserves the form of the system $\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{b}$ for the new stack-vectors $\boldsymbol{\chi}$ and $\boldsymbol{\beta}$.

$$\boldsymbol{\mathcal{A}} = \begin{bmatrix} a_0 & 0 & 0 & b_0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & b_1 & 0 \\ 0 & 0 & a_2 & 0 & 0 & b_2 \\ c_0 & 0 & 0 & d_0 & 0 & 0 \\ 0 & c_1 & 0 & 0 & d_1 & 0 \\ 0 & 0 & c_2 & 0 & 0 & d_2 \end{bmatrix} \tag{3.6}$$

Note that in Equations (3.3), (3.4), (3.5) and (3.6) we highlight in red some terms. In the case of Equation (3.3) and (3.4) the terms is red correspond to the initial condions, or $\mathbf{x}_2$ and are the only known terms of the vector. They are the terms relative to the last Chebyshev points, or the first point following the arc-length at $X = 0$. In fact, in the context of the numerical integration, only the first point is known and the others must be determined. On the other hand, in Equations (3.5) and (3.6), we have that both $\boldsymbol{\beta}$ and $\boldsymbol{\mathcal{A}}$ are known and we highlighted in red the terms that are relative to $\mathbf{x}_2$.

Starting from this assumption, we need to rearrange our equations. In fact, the system $\dot{\boldsymbol{\chi}} = \boldsymbol{\mathcal{A}}\boldsymbol{\chi} + \boldsymbol{\beta}$ has the form.

$$\dot{\begin{bmatrix} x_{01} \\ x_{11} \\ x_{21} \\ x_{02} \\ x_{12} \\ x_{22} \end{bmatrix}} = \begin{bmatrix} a_0 & 0 & 0 & b_0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & b_1 & 0 \\ 0 & 0 & a_2 & 0 & 0 & b_2 \\ c_0 & 0 & 0 & d_0 & 0 & 0 \\ 0 & c_1 & 0 & 0 & d_1 & 0 \\ 0 & 0 & c_2 & 0 & 0 & d_2 \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{11} \\ x_{21} \\ x_{02} \\ x_{12} \\ x_{22} \end{bmatrix} + \begin{bmatrix} b_{01} \\ b_{11} \\ b_{21} \\ b_{02} \\ b_{12} \\ b_{22} \end{bmatrix} \tag{3.7}$$

The Equation (3.7) it is not practical in terms of implementation as the known terms are mixed with the unknows. We thus define a new vector $\tilde{\boldsymbol{\chi}}$ which is defined as follows.

$$\tilde{\boldsymbol{\chi}} = \begin{bmatrix} \mathbf{x}_2 \\ \boldsymbol{\chi}_{NN} \end{bmatrix} \tag{3.8}$$

Where $\boldsymbol{\chi}_{NN}$ is the stack of the unknown vectors. Simarly, we define a new vector $\tilde{\boldsymbol{\beta}} = \begin{bmatrix} \mathbf{b}_2^T & \boldsymbol{\beta}_{NN}^T \end{bmatrix}^T$ To this aim, we define $\mathbf{P}$ as the permutation matrix mapping from $\boldsymbol{\chi}$ to $\tilde{\boldsymbol{\chi}}$, $\mathbf{P} : \boldsymbol{\chi} \mapsto \tilde{\boldsymbol{\chi}}$. In our case this matrix is defined as follows.

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.9}$$

We can apply this operator to the vectors $\boldsymbol{\chi}$, $\boldsymbol{\beta}$ and the matrix $\boldsymbol{\mathcal{A}}$, in order to shift the position of the known components.

$$\boldsymbol{\chi}_p = \mathbf{P}\boldsymbol{\chi} \Rightarrow \begin{bmatrix} x_{21} \\ x_{22} \\ x_{01} \\ x_{02} \\ x_{12} \\ x_{11} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{11} \\ x_{21} \\ x_{02} \\ x_{12} \\ x_{22} \end{bmatrix} \tag{3.10}$$

$$\boldsymbol{\beta}_p = \mathbf{P}\boldsymbol{\beta} \Rightarrow \begin{bmatrix} b_{21} \\ b_{22} \\ b_{01} \\ b_{02} \\ b_{12} \\ b_{11} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_{01} \\ b_{11} \\ b_{21} \\ b_{02} \\ b_{12} \\ b_{22} \end{bmatrix} \tag{3.11}$$

Similarly, we define a permuted version of the matrix $\boldsymbol{\mathcal{A}}$, namely $\boldsymbol{\mathcal{A}}_p$

$$\begin{gathered} \boldsymbol{\mathcal{A}}_p = \mathbf{P}^T \boldsymbol{\mathcal{A}} \mathbf{P} \\ \begin{bmatrix} a_2 & b_2 & 0 & 0 & 0 & 0 \\ c_2 & d_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_0 & b_0 & 0 & 0 \\ 0 & 0 & c_0 & d_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_1 & c_1 \\ 0 & 0 & 0 & 0 & b_1 & a_1 \end{bmatrix} = \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 & 0 & 0 & b_0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & b_1 & 0 \\ 0 & 0 & a_2 & 0 & 0 & b_2 \\ c_0 & 0 & 0 & d_0 & 0 & 0 \\ 0 & c_1 & 0 & 0 & d_1 & 0 \\ 0 & 0 & c_2 & 0 & 0 & d_2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{gathered} \tag{3.12}$$

We thus have obtained the system with the reorded terms

$$
\begin{gathered}
\dot{\boldsymbol{\chi}}_p = \boldsymbol{\mathcal{A}}_p \boldsymbol{\chi}_p + \boldsymbol{\beta}_p \\
\begin{bmatrix} \dot{\mathbf{x}}_2 \\ \dot{\boldsymbol{\chi}}_{NN} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{NN} \end{bmatrix} \begin{bmatrix} \mathbf{b}_2 \\ \boldsymbol{\beta}_{NN} \end{bmatrix} \\
\begin{bmatrix} \dot{x}_{21} \\ x_{22} \\ x_{01} \\ x_{02} \\ x_{12} \\ x_{11} \end{bmatrix} = \begin{bmatrix} a_2 & b_2 & 0 & 0 & 0 & 0 \\ c_2 & d_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_0 & b_0 & 0 & 0 \\ 0 & 0 & c_0 & d_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_1 & c_1 \\ 0 & 0 & 0 & 0 & b_1 & a_1 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \\ x_{01} \\ x_{02} \\ x_{12} \\ x_{11} \end{bmatrix} + \begin{bmatrix} b_{21} \\ b_{22} \\ b_{01} \\ b_{02} \\ b_{12} \\ b_{11} \end{bmatrix}
\end{gathered}
\tag{3.13}
$$

Similar cosiderations can be applied to the differentiation matrix $\mathbf{D}_n$. In Appendix B we introduced this linear operator for a set of vector stacked row-wise. In our case, the vectors are stacked in one column vector. As a result, the differentiation matrix $\mathbf{D}$ is defined as follows.

$$
\begin{gathered}
\dot{\boldsymbol{\chi}} = \mathbf{D}\boldsymbol{\chi} = \begin{bmatrix} \mathbf{D}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_n \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{11} \\ x_{21} \\ x_{02} \\ x_{12} \\ x_{22} \end{bmatrix} \\
\begin{bmatrix} \dot{x}_{01} \\ x_{11} \\ x_{21} \\ x_{02} \\ x_{12} \\ x_{22} \end{bmatrix} \begin{bmatrix} d_{00} & d_{01} & d_{02} & 0 & 0 & 0 \\ d_{10} & d_{11} & d_{12} & 0 & 0 & 0 \\ d_{20} & d_{21} & d_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{00} & d_{01} & d_{02} \\ 0 & 0 & 0 & d_{10} & d_{11} & d_{12} \\ 0 & 0 & 0 & d_{20} & d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{11} \\ x_{21} \\ x_{02} \\ x_{12} \\ x_{22} \end{bmatrix}
\end{gathered}
\tag{3.14}
$$

However, similarly to what we did to matrix of coefficients $\boldsymbol{\mathcal{A}}$, we need to permute the elements of $\mathbf{D}$. We thus define the matrix $\mathbf{D}_p$.

$$\mathbf{D}_p = \mathbf{P}^T \mathbf{D} \mathbf{P}$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} d_{00} & d_{01} & d_{02} & 0 & 0 & 0 \\ d_{10} & d_{11} & d_{12} & 0 & 0 & 0 \\ d_{20} & d_{21} & d_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{00} & d_{01} & d_{02} \\ 0 & 0 & 0 & d_{10} & d_{11} & d_{12} \\ 0 & 0 & 0 & d_{20} & d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} d_{22} & 0 & d_{20} & 0 & 0 & d_{21} \\ 0 & d_{22} & 0 & d_{20} & d_{21} & 0 \\ d_{02} & 0 & d_{00} & 0 & 0 & d_{01} \\ 0 & d_{02} & 0 & d_{00} & d_{01} & 0 \\ 0 & d_{12} & 0 & d_{10} & d_{11} & 0 \\ d_{12} & 0 & d_{10} & 0 & 0 & d_{11} \end{bmatrix} \tag{3.15}$$

Note that the permuted differentiation matrix $\mathbf{D}_p$ has the following structure.

$$\mathbf{D}_p = \begin{bmatrix} d_{22} & 0 & d_{20} & 0 & 0 & d_{21} \\ 0 & d_{22} & 0 & d_{20} & d_{21} & 0 \\ d_{02} & 0 & d_{00} & 0 & 0 & d_{01} \\ 0 & d_{02} & 0 & d_{00} & d_{01} & 0 \\ 0 & d_{12} & 0 & d_{10} & d_{11} & 0 \\ d_{12} & 0 & d_{10} & 0 & 0 & d_{11} \end{bmatrix} = \begin{bmatrix} \mathbf{D}_{II} & \mathbf{D}_{NI} \\ \mathbf{D}_{IN} & \mathbf{D}_{NN} \end{bmatrix} \tag{3.16}$$

Where $\mathbf{D}_{II}$ represents the influence of the initial condition onto themself, $\mathbf{D}_{NI}$ represents the influence of the the vector $\boldsymbol{\chi}_{NN}$ onto the initial conditions $\mathbf{x}_2$, $\mathbf{D}_{IN}$ represents the influence of the initial conditions to the other components in $\boldsymbol{\chi}_{NN}$. Finally, $\mathbf{D}_{NN}$ represents the relations among the values in $\boldsymbol{\chi}_{NN}$.

We can now apply this formulation to the system.

$$\begin{bmatrix} \mathbf{D}_{II} & \mathbf{D}_{NI} \\ \mathbf{D}_{IN} & \mathbf{D}_{NN} \end{bmatrix} \begin{bmatrix} \mathbf{x}_2 \\ \boldsymbol{\chi}_{NN} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{NN} \end{bmatrix} \begin{bmatrix} \mathbf{x}_2 \\ \boldsymbol{\chi}_{NN} \end{bmatrix} + \begin{bmatrix} \mathbf{b}_2 \\ \boldsymbol{\beta}_{NN} \end{bmatrix} \tag{3.17}$$

In Equation (3.17), the system is now a linear algebra problem and can be simply solved numerically. We now define two components $\mathbf{D}_{IT}$ and $\mathbf{A}_{IT}$ as follows.

$$\mathbf{D}_{IT} = \begin{bmatrix} \mathbf{D}_{II} \\ \mathbf{D}_{IN} \end{bmatrix} \qquad \mathbf{A}_{IT} = \begin{bmatrix} \mathbf{A}_2 \\ \mathbf{0} \end{bmatrix} \tag{3.18}$$

Moreover, we define the element $\mathbf{b}_{IT} = \begin{bmatrix} \mathbf{b}_2 \\ \mathbf{ivp} \end{bmatrix}$ as the vector composed by the initial and known term $\mathbf{b}_2$ and **ivp** which is the vector describing the Initial Value Problem (IVP). The IVP is the problem of finding the influence of the condition at the beginning of the sequence on the other terms.In other words, **ivp** tells us how the initial conditions propagates from $X = 0$ to $X = 1$.

$$\mathbf{D}_{IT}\mathbf{x}_2 = \mathbf{A}_{IT}\mathbf{x}_2 + \mathbf{b}_{IT} \tag{3.19}$$

From this last equation, we can explicit $\mathbf{b}_{IT}$ in order to compute the influence of the initial point on the others.

$$\begin{gathered} \mathbf{b}_{IT} = \left(\mathbf{D}_{IT} - \mathbf{A}_{IT}\right)\mathbf{x}_2 \\ \begin{bmatrix} \mathbf{b}_2 \\ \mathbf{ivp} \end{bmatrix} = \left(\begin{bmatrix} \mathbf{D}_{II} \\ \mathbf{D}_{IN} \end{bmatrix} - \begin{bmatrix} \mathbf{A}_2 \\ \mathbf{0} \end{bmatrix} \right) \mathbf{x}_2 \end{gathered} \tag{3.20}$$

Note that, the first line of Equation (3.20) gives the solution of the linear system for the known initial point $\mathbf{x}_2$. On the other hand, the bottom row can be sintatically rewritten as: $\mathbf{ivp} = \mathbf{D}_{IN}\mathbf{x}_2$. This relation gives the influence of the initial condition on the set of other points $\boldsymbol{\chi}_{NN}$.

We now perform the matrix multiplication of Equation (3.17) and we obtain the following result.

$$\begin{bmatrix} \mathbf{D}_{II}\mathbf{x}_2 + \mathbf{D}_{NI}\boldsymbol{\chi}_{NN} \\ \mathbf{D}_{IN}\mathbf{x}_2 + \mathbf{D}_{NN}\boldsymbol{\chi}_{NN} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_2\mathbf{x}_2 \\ \mathbf{A}_{NN}\boldsymbol{\chi}_{NN} \end{bmatrix} + \begin{bmatrix} \mathbf{b}_2 \\ \boldsymbol{\beta}_{NN} \end{bmatrix} \tag{3.21}$$

At this point, we have all the terms in order to compute the values of the other vectors $\mathbf{x}_0$ and $\mathbf{x}_1$. In fact, we can impose the IVP, computed in Equation (3.20), into the explicit version of the permuted system in Equation (3.21).

$$\mathbf{ivp} + \mathbf{D}_{NN}\boldsymbol{\chi}_{NN} = \mathbf{A}_{NN}\boldsymbol{\chi}_{NN} + \boldsymbol{\beta}_{NN} \tag{3.22}$$

The solution, accounting for the intial conditions, is given solving Equation (3.22) for $\boldsymbol{\chi}_{NN}$.This relation is given as follows.

$$\left(\mathbf{D}_{NN} - \mathbf{A}_{NN}\right)\boldsymbol{\chi}_{NN} = \boldsymbol{\beta}_{NN} - \mathbf{ivp} \tag{3.23}$$

$$\boldsymbol{\chi}_{NN} = \left(\mathbf{D}_{NN} - \mathbf{A}_{NN}\right)^{-1}\left(\boldsymbol{\beta}_{NN} - \mathbf{ivp}\right) \tag{3.24}$$

Equation (3.24) is the solution of the linear system. It gives us the values of out state vector at all the chebyshev points, accounting for the initial conditions $\mathbf{x}_2$.

Appendix A

Lie Algebra Notations

In this chapter we discuss some Lie algebra notation we use.

We define a reference frame $\mathcal{F}_w$which is an time invariant fixed frame. This frame is defined as

$$\mathcal{F}_w = \langle \mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle \tag{A.1}$$

Where $\mathbf{0} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$ and $\mathbf{e}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, $\mathbf{e}_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ and $\mathbf{e}_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ three unit vector, or principal axes, defining an orthonormal base.

Note that being the base orthonormal, it is sufficient to define two element to retrieve the last one with the cross product law. The cross product between two vector, namely $\mathbf{v}$ and $\mathbf{u}$ is defined as follows.

$$\mathbf{w} = \mathbf{v} \times \mathbf{u} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \times \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} v_y u_z - u_y v_z \\ v_z u_x - u_z v_x \\ v_x u_y - u_x v_y \end{bmatrix} \tag{A.2}$$

We thus have that: $\mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_3$, $\mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_1$ and $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$. Moreover, we introduce the notation $\hat{\cdot}$ that define a screw symmetric matrix for which

$$\mathbf{w} = \hat{\mathbf{v}}\mathbf{u} \tag{A.3}$$

$$\hat{\mathbf{v}} = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix} \tag{A.4}$$

Thus we have that $\hat{\cdot}$ is a map $\hat{\cdot} : \mathbb{R}^3 \mapsto SO(3)$. On the other hand, we define the inverse map of $\hat{\cdot}$ with $\cdot^V$ the operator that maps $\cdot^V : SO(3) \mapsto \mathbb{R}^3$.

A.1 Affine Transformation

Any other frame can be defined with respect to $\mathcal{F}_w$. Specifically, having a generic frame $\mathcal{F}_a = \langle {}^w\mathbf{p}_a, {}^w\mathbf{x}_a, {}^w\mathbf{y}_a, {}^w\mathbf{z}_a \rangle$ it can be uniquely defined with the element ${}^{w}\mathbf{T}_a \in SE(3)$.

$$ {}^{w}\mathbf{T}_a = \begin{bmatrix} {}^{w}\mathbf{R}_a & {}^{w}\mathbf{p}_a \\ \mathbf{0} & 1 \end{bmatrix} \tag{A.5} $$

Where ${}^{w}\mathbf{p}_a \in \mathbb{R}^3$ express the position of the origin of $\mathcal{F}_a$ with respect to the origin of $\mathcal{F}_w$. On the other hand, the matrix ${}^{w}\mathbf{R}_a \in SO(3)$ expresses the orientation of the frame $\mathcal{F}_a$ with respect to $\mathcal{F}_w$. This matrix is defined as follows.

$$ {}^{w}\mathbf{R}_a = \begin{bmatrix} {}^{w}\mathbf{x}_a & {}^{w}\mathbf{y}_a & {}^{w}\mathbf{z}_a \end{bmatrix} \tag{A.6} $$

Where ${}^{w}\mathbf{x}_a$, ${}^{w}\mathbf{y}_a$ and ${}^{w}\mathbf{z}_a$ are the principal axes of the frame $\mathcal{F}_a$ expressed in the coordinates of frame $\mathcal{F}_w$.

Among the many properties of the rotation matrix, we cite the followings.

$$ \mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbb{1} \tag{A.7} $$

$$ det\mathbf{R} = \pm 1 \tag{A.8} $$

A more complete discussion can be found in the work of Seling et al.[10] and Murray et al. [5]

A.2 Homogeneous Coordinates

The operator expressed in equation A.5 in an affine transformation matrix. This matrix operates on vectors that are expressed in homogeneous coordinates. In homogeneous coordinates, a point $\mathbf{p}$ is expressed as $\mathbf{p} = \begin{bmatrix} p_x & p_y & p_z & 1 \end{bmatrix}$ and a vector $\mathbf{v}$ is defined as $\mathbf{v} = \begin{bmatrix} v_x & v_y & v_z & 0 \end{bmatrix}$. We thus append a 1 or a 0 depending if we deal with vectors of points. The reason behind this difference is that vectors are difference between points. As a result, if we exploit the definition of a vector $\mathbf{v}$ defined as $\mathbf{v}_{ab} = \mathbf{p}_b - \mathbf{p}_a$ we have that:

$$ \mathbf{v}_{ab} = \begin{bmatrix} p_{b_x} \\ p_{b_y} \\ p_{b_z} \\ 1 \end{bmatrix} - \begin{bmatrix} p_{a_x} \\ p_{a_y} \\ p_{a_z} \\ 1 \end{bmatrix} = \begin{bmatrix} v_{ab_x} \\ v_{ab_y} \\ v_{ab_z} \\ 0 \end{bmatrix} \tag{A.9} $$

From Equation A.9 can determine the following properties of homogeneous representation, as detailed by Murray [5]

1. Sums and differences of vectors are vectors.

2. The sum of a vector and a point is a point.

3. The difference between two points is a vector.

4. The sum of two points is meaningless.

We have that the affine transformation in Equation A.5 performs a rigid displacement of an element in $\mathbb{R}^4$. However, the last row of the matrix can be changed in order to add scaling of an object, by changing the 1 on the bottom right corner, and performing some perspective transformations by changing the last bottom zeros row of the matrix. This results in the affine transformation not representing a rigid body transformation anymore.

In our work, we only deal with rigid body transformations. These transformations satisfy the composition rule. In fact, describing with ${}^{w}\mathbf{T}_a$ and ${}^{w}\mathbf{T}_b$ the affine transformations from the the reference frame $\mathcal{F}_w$ to the frames $\mathcal{F}_a$ and $\mathcal{F}_b$ respectively, if we also know the rigid body transformation from frame $\mathcal{F}_a$ to frame $\mathcal{F}_b$, namely ${}^{a}\mathbf{T}_b$, then we have that:

$$\begin{aligned} {}^{w}\mathbf{T}_b &= {}^{w}\mathbf{T}_a\, {}^{a}\mathbf{T}_b \\ &= \begin{bmatrix} {}^{w}\mathbf{R}_a {}^{a}\mathbf{R}_b & {}^{w}\mathbf{R}_a {}^{a}\mathbf{v}_{ab} + {}^{w}\mathbf{p}_a \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} {}^{w}\mathbf{R}_b & {}^{w}\mathbf{v}_{ab} + {}^{w}\mathbf{p}_a \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} {}^{w}\mathbf{R}_b & {}^{w}\mathbf{p}_b \\ \mathbf{0} & 1 \end{bmatrix} \end{aligned} \tag{A.10}$$

The relation in A.10 can be inverted: we can obtain the affine transformation ${}^{w}\mathbf{T}_a$, knowing ${}^{w}\mathbf{T}_b$ and ${}^{a}\mathbf{T}_b$. In fact, we define the inverse of a rigid body transformation between two frames, in this case: $\mathcal{F}_a$ and $\mathcal{F}_b$, as

$${}^{a}\mathbf{T}_b{}^{-1} = \begin{bmatrix} {}^{a}\mathbf{R}_b{}^{T} & -{}^{a}\mathbf{R}_b{}^{T}\,{}^{a}\mathbf{p}_b \\ \mathbf{0} & 1 \end{bmatrix} \tag{A.11}$$

The, we obtain the rigid body transformation ${}^{w}\mathbf{T}_a$.

$${}^{w}\mathbf{T}_a = {}^{a}\mathbf{T}_b{}^{-1}\,{}^{w}\mathbf{T}_b \tag{A.12}$$

A.3 Quaternions

Quaternions are used in pure mathematics, but also have practical uses in applied mathematics, particularly for calculations involving three-dimensional rotations. They are another way of representing the $SO(3)$ group. Compared to rotation matrices, quaternions are more compact, efficient, and numerically stable. Compared to Euler angles, they are simpler to compose. However, they are not as intuitive and easy to understand. A quaternion can be seen as an extension of complex number as it has an expression of the form

$$\mathbf{q} = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{A.13}$$

Where w, x, y, z are real numbers and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are symbols that can be interpreted as unit-vectors pointing along the three spatial axes.

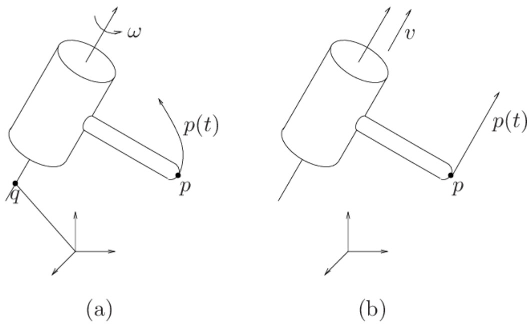

Figure A.1: Representation of the rigid body motions due to the action of a) revolute joint and b) a prismatic joint [5].

Thus, the null rotation is the real element $\mathbf{q} = 1 + 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$. A quaternion is said unit quaternion is its norm $\|\mathbf{q}\| = 1$, where the norm of a quaternion is defined as follows.

$$\|\mathbf{q}\| = \sqrt{\mathbf{q}\mathbf{q}} = \sqrt{w^2 + x^2 + y^2 + z^2} \tag{A.14}$$

A quaternion can be converted into a rotation matrix with the following relation.

$$\mathbf{R}_{\mathbf{q}} = \begin{bmatrix} 1 - 2y^2 - 2z^2 & 2xy - 2wz & 2xz + 2wy \\ 2xy + 2wz & 1 - 2x^2 - 2z^2 & 2yz - 2wx \\ 2xz - 2wy & 2yz + 2wx & 1 - 2x^2 - 2y^2 \end{bmatrix} \tag{A.15}$$

Moreover, we can apply a skew simmetric matrix, namely $\hat{\mathbf{v}}$ on a quaternion with the operator $\mathbf{A}_{(\hat{\mathbf{v}})}$. Having $\mathbf{v} = \begin{bmatrix} v_x & v_y & v_z \end{bmatrix}$ we have that $\mathbf{A}_{(\hat{\mathbf{v}})}$ is defined as follows.

$$\mathbf{A}_{(\hat{\mathbf{v}})} = \begin{bmatrix} 0 & -v_x & -v_y & -v_z \\ v_x & 0 & v_z & -v_y \\ v_y & -v_z & 0 & v_x \\ v_z & v_y & -v_x & 0 \end{bmatrix} \tag{A.16}$$

A.4 Rigid Motion in Space

In order to define the motion of a rigid body in $SE(3)$ we can start with the example in Figure A.1

Considering the revolute joint, we can define: the axis of rotation ω, the velocity of rotation λ, a point on the axis of rotation $\mathbf{q} \in \mathbb{R}^3$ and a point on the joint body $\mathbf{p}_{(t)}$ which changes with respect to time. The velocity of this point is defined in Equation A.17

$$\dot{\mathbf{p}}_{(t)} = \lambda\omega \times \left[\mathbf{p}_{(t)} - \mathbf{q}\right] = \lambda\omega \times \mathbf{p}_{(t)} - \lambda\omega \times \mathbf{q}$$
$$\dot{\mathbf{p}}_{(t)} = \lambda\hat{\omega}\mathbf{p}_{(t)} + \lambda\mathbf{v} \tag{A.17}$$

If we consider a unit angular velocity λ, we can express Equation A.17 in homogeneous coordinates as follows.

$$\begin{bmatrix} \dot{\mathbf{p}}_{(t)} \\ 0 \end{bmatrix} = \hat{\xi} \begin{bmatrix} \mathbf{p}_{(t)} \\ 1 \end{bmatrix} \tag{A.18}$$

Where $\hat{\xi}$ is the 4×4 matrix defined from the twist ξ of the joint.

$$\xi = \begin{bmatrix} \omega \\ \mathbf{v} \end{bmatrix}$$
$$\hat{\xi} = \begin{bmatrix} \hat{\omega} & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix} \tag{A.19}$$

In the case of a non unitary angular velocity λ, Equation A.18 becomes:

$$\begin{bmatrix} \dot{\mathbf{p}}_{(t)} \\ 0 \end{bmatrix} = \lambda\, \hat{\xi} \begin{bmatrix} \mathbf{p}_{(t)} \\ 1 \end{bmatrix} \tag{A.20}$$

Similarly, we can define, for the prismatic joint, an axis of translation $\mathbf{v}$ and a linear velocity λ. In this case, the velocity of the point $\mathbf{p}_{(t)}$ is given by

$$\dot{\mathbf{p}}_{(t)} = \lambda\mathbf{v} \tag{A.21}$$

We thus have that the resulting twist of the joint is $\xi = \begin{bmatrix} \mathbf{0} & \mathbf{v} \end{bmatrix}$

Using the element ξ we can compute the velocity of a frame $\mathcal{F}_a$, with respect to the reference frame $\mathcal{F}_w$, as follows.

$${}^{w}\dot{\mathbf{T}}_a = {}^{w}\,\mathbf{T}_a\, \hat{\xi} \tag{A.22}$$

On the other hand, if we want to project into the frame $\mathcal{F}_a$ a twist that expressed in the coordinates of another frame $\mathcal{F}_b$, namely ${}^{b}\xi$, we define the Adjoint transformation: a 6×6 matrix which transforms twists from one coordinate frame to another. This matrix is defined from the affine transformation between the two frames ${}^{a}\mathbf{T}_b = \langle {}^{a}\mathbf{R}_b, {}^{a}\mathbf{p}_b \rangle$.

$$Ad_{\mathbf{T}_{ab}} = \begin{bmatrix} {}^{a}\mathbf{R}_b & {}^{a}\hat{\mathbf{p}}_b{}^{a}\mathbf{R}_b \\ \mathbf{0} & {}^{a}\mathbf{R}_b \end{bmatrix} \tag{A.23}$$

And thus we have that

$${}^{a}\xi = Ad_{\mathbf{T}_{ab}}\,{}^{b}\xi = \begin{bmatrix} {}^{a}\mathbf{R}_b & {}^{a}\hat{\mathbf{p}}_b{}^{a}\mathbf{R}_b \\ 0 & {}^{a}\mathbf{R}_b \end{bmatrix} \begin{bmatrix} {}^{b}\omega \\ {}^{b}\mathbf{v} \end{bmatrix} \tag{A.24}$$

A.5 Wrenches

Similarly to the definition of the velocity in $SE(3)$ we can define the action of a force $\mathbf{f} \in \mathbb{R}^3$ and a torque $\mathbf{m} \in \mathbb{R}^3$ on a rigid body with the definition of a wrench $\mathbf{W} \in \mathbb{R}^6$

$$ {}^{w}\mathbf{W} = \begin{bmatrix} {}^{w}\mathbf{f} \\ {}^{w}\mathbf{m} \end{bmatrix} \tag{A.25} $$

Where we indicate with ${}^{w}\mathbf{f}$ and ${}^{w}\mathbf{f}$ some force and torques expressed with respect to reference frame $\mathcal{F}_w$.

We give the definition of equivalent wrenches as wrenches that generate the same work for every possible rigid body motion.

Having a wrench ${}^{b}\mathbf{W}_b$ applied at the origin of a frame $\mathcal{F}_b$, and expressed in its local coordinate, we want to define an equivalent wrench ${}^{a}\mathbf{W}_a$ applied at the origin of a frame $\mathcal{F}_a$, and expressed in its local coordinate.

We can obtain this relation having ${}^{b}\mathbf{T}_a$ the affine transformation from frame $\mathcal{F}_b$ to frame $\mathcal{F}_a$. We thus have that

$$ {}^{a}\mathbf{W}_a = \mathbf{Ad}_{\mathbf{T}_{ba}}\, {}^{b}\mathbf{W}_b \tag{A.26} $$

The Equation A.26 can be exploited in order to define the adjoint transformation $\mathbf{Ad}_{\mathbf{T}_{ba}}$.

$$ \begin{bmatrix} {}^{a}\mathbf{f}_a \\ {}^{a}\mathbf{m}_a \end{bmatrix} = \begin{bmatrix} {}^{b}\mathbf{R}_a{}^{T} & \mathbf{0} \\ -{}^{b}\mathbf{R}_a{}^{T}\, {}^{b}\hat{\mathbf{p}}_a & {}^{b}\mathbf{R}_a{}^{T} \end{bmatrix} \begin{bmatrix} {}^{b}\mathbf{f}_b \\ {}^{b}\mathbf{m}_b \end{bmatrix} \tag{A.27} $$

A.6 Mappings and transformations

We define the operator $ad \mid ad : \mathbb{R}^6 \mapsto \mathbb{R}^6$ as an operator defining the Lie commutator. Given two generic vector $\boldsymbol{\chi}_1, \boldsymbol{\chi}_2 \in se(3)$

$$ ad_{\boldsymbol{\chi}_1}\boldsymbol{\chi}_2 = (\hat{\boldsymbol{\chi}}_1\hat{\boldsymbol{\chi}}_2 - \hat{\boldsymbol{\chi}}_2\hat{\boldsymbol{\chi}}_1)^{\vee} = -ad_{\boldsymbol{\chi}_2}\boldsymbol{\chi}_1 \tag{A.28} $$

Having $\boldsymbol{\chi} = \begin{bmatrix} \boldsymbol{\phi} \\ \boldsymbol{\psi} \end{bmatrix}$ we can define the components of $ad_{\boldsymbol{\chi}}$ as follows.

$$ ad_{\boldsymbol{\chi}} = \begin{bmatrix} \hat{\boldsymbol{\phi}} & \mathbf{0} \\ \hat{\boldsymbol{\psi}} & \hat{\boldsymbol{\phi}} \end{bmatrix} \tag{A.29} $$

Appendix B

Chebyshev Polynomials

The Chebyshev polynomials are two sequences of polynomials related to the cosine and sine functions, notated as $\mathbf{T}_{n(x)}$ and $\mathbf{U}_{n(x)}$ respectively. In this document, we are interested only in $\mathbf{T}_{n(x)}$, which is related to the cosine function, these polynomials are also called the Chebyshev polynomials of the first kind. In this document, the pedix $\cdot_n$ indicates the order of the polynomial.

The Chebyshev polinomials are classified as othogonal. In mathematics, an orthogonal polynomial sequence is a family of polynomials such that any two different polynomials in the sequence are orthogonal to each other under some inner product.

These polynomials live in a defined domain $x \in [-1, 1]$, and are given by the following relation, valid for N_c Chebyshev points

$$x_j = -cos\left(\frac{\pi j}{N_c}\right) \qquad 0 \leq j \leq N_c - 1 \tag{B.1}$$

Figure B.1 shows a graphical representations of these points. The figure also shows that these points are not equally spaced in x coordinates, but equally spaced on the unit circle. These points are used to calculate the values of the Chebyshev polynomials.

The Chebyshev polynomials of the first kind are obtained from the recurrence relation

$$\begin{aligned} \mathbf{T}_{0(x)} &= 1 \\ \mathbf{T}_{1(x)} &= x \\ \mathbf{T}_{n+1(x)} &= 2x\mathbf{T}_{n(x)} - \mathbf{T}_{n-1(x)} \end{aligned} \tag{B.2}$$

The standard Chebyshev polynomials are reported in Equation B.3, until

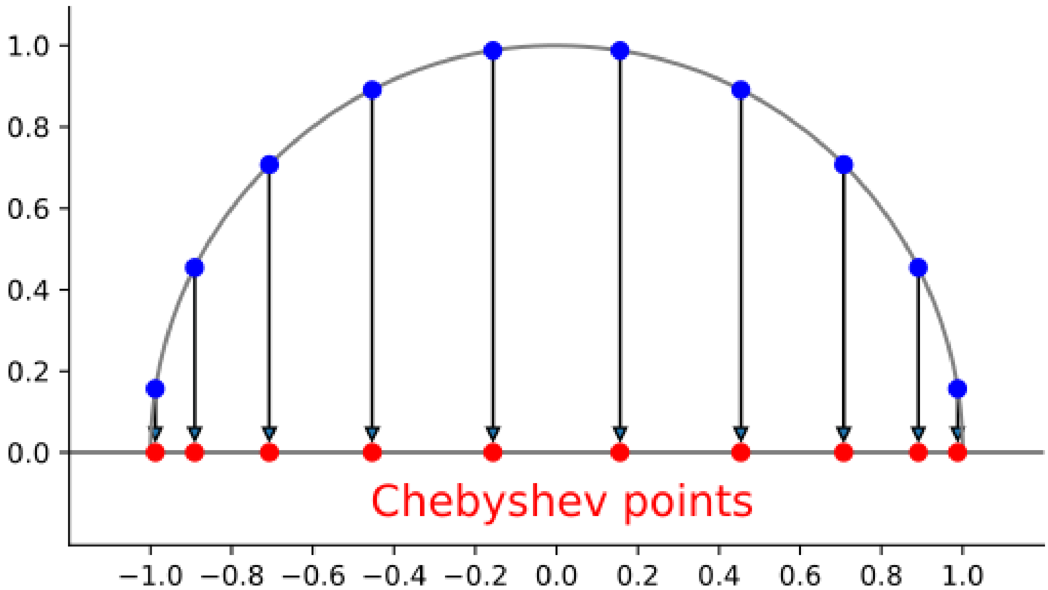

Figure B.1: Representation of N Chebyshev points on the unit circle.

the order nine.

$$
\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x \\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1 \\
T_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x \\
T_8(x) &= 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1 \\
T_9(x) &= 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x
\end{aligned} \tag{B.3}
$$

The Figure B.2 reports the values of the first four polynomials in the interval $[-1, 1]$.

Following the procedure of [11], we now define how these polynomials can be used to define a differentiation matrix.

If we now define a polynomial p of degree $< N$ with $p_{(x_j)} = v_j$ with $0 \leq j \leq N$, we can define the derivative of the polynomial with respect to the Chebyshev points as $\frac{dp_{(x_j)}}{dx_j} = p'_j = w_j$. As the polynomial $p_{(x_j)}$ is linear, its derivative can de defined in a matrix form. This defines the linear

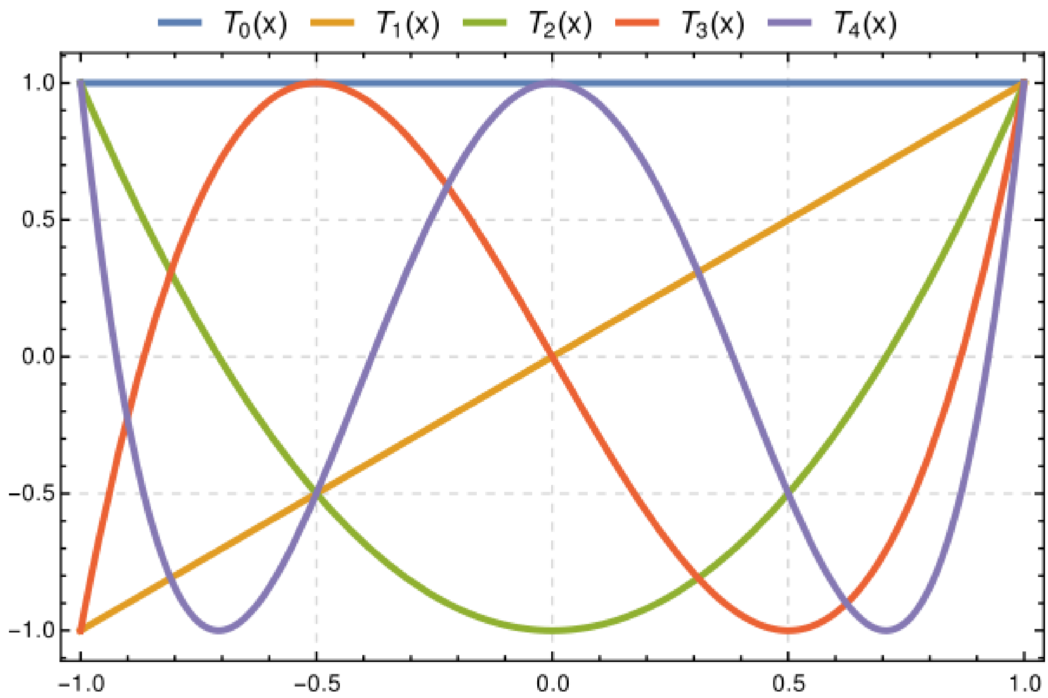

Figure B.2: Values for the first 4 Chebyshev polynomials in the interval $[-1, 1]$.

operator D_N as follows.

$$w_j = D_N v_j \tag{B.4}$$

We can give a definition of this matrix starting from the example of first order polynomial, having $N = 1$. This polynomial in the Lagrangian form has the following form, for the two chebyshev points $x_0 = 1$ and $x_1 = -1$.

$$p_{(x_j)} = \frac{1}{2}(1 + x)v_0 + (1 - x)v_1 \tag{B.5}$$

In fact, we have that for x_0 we have $p_{x_0} = \frac{1}{2}(1 + 1)v_0 = v_0$; similarly, for x_1 we have $p_{(x_j)} = v_1$. Taking the derivative in Equation (B.5), we obtain the following.

$$p'_{(x_j)} = \frac{1}{2}v_0 - \frac{1}{2}v_1 \tag{B.6}$$

For this trivial case, the derivative is constant. However, we can define the differentiation matrix starting by the vector form of the polynomial $\mathbf{p}_{(x_j)} = \begin{bmatrix} v_0 \\ v_1 \end{bmatrix}$. We wish to obtain the derivative vector $\mathbf{w}_{(x_j)} = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}$. To this aim, we stack the value for the derivative in a matrix obtaining the Chebyshev differentiation matrix for a first order polynomial.

$$D_1 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \tag{B.7}$$

Note that the matrix is constant as the derived polynomial has a constant form.

We can now address the more elaborate case of a second order polynomial, having $N = 2$. In this case the Chebyshev points are defined as $x_0 = 1$, $x_1 = 0$ and $x_2 = -1$. The polynomial, in its Lagrangian form has the following definition.

$$p_{(x_j)} = \frac{1}{2}x(1+x)v_0 + (1+x)(1-x)v_1 + \frac{1}{2}x(x-1)v_2 \tag{B.8}$$

Derivating the polynomial with respect to x we obtain the following.

$$p'_{(x_j)} = (\frac{1}{2}+x)v_0 - 2xv_1 + (x-\frac{1}{2})v_2 \tag{B.9}$$

In this case, the vector form of the polynomial is the following.

$$\mathbf{w}_{(x_j)} = \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = \mathbf{D}_2 \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} \tag{B.10}$$

In order to define $\mathbf{D}_2$ we need to compute the value for the derivative $p'_{(x_j)}$ for all the Chebyshev points, *i.e.*, $x_0 = 1$, $x_1 = 0$ and $x_2 = -1$.

$$\begin{aligned} p'_{(x_0)} &= (\frac{1}{2}+1)v_0 - 2v_1 + (1-\frac{1}{2})v_2 = \frac{3}{2}v_0 - 2v_1 + \frac{1}{2}v_2 \\ p'_{(x_1)} &= (\frac{1}{2}+0)v_0 - 0v_1 + (0-\frac{1}{2})v_2 = \frac{1}{2}v_0 - \frac{1}{2}v_2 \\ p'_{(x_2)} &= (\frac{1}{2}-1)v_0 + 2v_1 + (-1-\frac{1}{2})v_2 = -\frac{1}{2}v_0 + 2v_1 - \frac{3}{2}v_2 \end{aligned} \tag{B.11}$$

We can stack the coefficients in the right-most hand side of Equation (B.11) in a matrix form obtaining the CHebyshev differentiation matrix D_2.

$$\mathbf{D}_2 = \begin{bmatrix} \frac{3}{2} & -2 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 2 & -\frac{3}{2} \end{bmatrix} \tag{B.12}$$

In order to define this matrix for an albitrary number of Chebyshev points, we give a definition for its terms. Assuming that the matrix has the following form.

$$\mathbf{D}_N = \begin{bmatrix} d_{00} & \dots & d_{0j} & \dots & d_{0(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ d_{i0} & \dots & d_{ij} & \dots & d_{i(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ d_{(N-1)0} & \dots & d_{(N-1)j} & \dots & d_{(N-1)(N-1)} \end{bmatrix} \tag{B.13}$$

We can give a formulation for the components of the Chebyshev matrix as follows.

$$d_{00} = \frac{2N^2+1}{6} \qquad d_{(N-1)(N-1)} = -\frac{2N^2+1}{6} \tag{B.14}$$

$$d_{ij} = \frac{-x_j}{2\left(1-x_j^2\right)} \qquad i = j, \quad i,j = 1,\dots,N-2 \tag{B.15}$$

$$d_{ij} = \frac{c_i}{c_j}\frac{(-1)^{i+j}}{(x_i - x_j)} \qquad i \neq j, \quad i,j = 1,\dots,N-2 \tag{B.16}$$

In the case where we observe some quantities on the Chebyshev domain, we can use the differentiation matrix to compute in an analytical way the derivatives.

In fact, assuming that we observe the values of a parameter v in the Chebyshev domain using N points, we can define a vector of observations : $\mathbf{v} = \begin{bmatrix} v_0 & v_1 & \dots & v_N \end{bmatrix}^T$. The values of these derivatives can then be computed directly.

$$\mathbf{w} = \mathbf{D}_N \mathbf{v} \tag{B.17}$$

Where $\mathbf{w} = \begin{bmatrix} w_0 & w_1 & \dots & w_N \end{bmatrix}^T$ is the vector of derivatives for the parameters vector $\mathbf{v}$.

We can extend this formulation in the case when the observed quantities are vectors. In this case we have that at each point we observe the set of m indipendent parameter $\mathbf{v}_j = \begin{bmatrix} v_{j_1} & v_{j_2} & \dots & v_{j_m} \end{bmatrix}^T$. In this case, the set of observed parameters will be the stack of the parameters vectors.

$$\mathbf{V} = \begin{bmatrix} v_{01} & v_{02} & \dots & v_{0m} \\ \vdots & \vdots & \ddots & \vdots \\ v_{j_1} & v_{j_2} & \dots & v_{j_m} \\ \vdots & \vdots & \ddots & \vdots \\ v_{N1} & v_{N2} & \dots & v_{Nm} \end{bmatrix} \tag{B.18}$$

We thus have that the stack of the derivatives is given by the the following formulation.

$$\begin{aligned}\mathbf{W} &= \begin{bmatrix} w_{01} & w_{02} & \dots & w_{0m} \\ \vdots & \vdots & \ddots & \vdots \\ w_{j1} & w_{j2} & \dots & w_{jm} \\ \vdots & \vdots & \ddots & \vdots \\ w_{N1} & w_{N2} & \dots & w_{Nm} \end{bmatrix} \\ &= \begin{bmatrix} d_{00} & d_{01} & \dots & d_{0(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ d_{i0} & d_{i1} & \dots & d_{i(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ d_{(N-1)0} & d_{(N-1)j} & \dots & d_{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} v_{01} & v_{02} & \dots & v_{0m} \\ \vdots & \vdots & \ddots & \vdots \\ v_{j1} & v_{j2} & \dots & v_{jm} \\ \vdots & \vdots & \ddots & \vdots \\ v_{N1} & v_{N2} & \dots & v_{Nm} \end{bmatrix}\end{aligned} \tag{B.19}$$

We can give an explicit formulation using the definition of $\mathbf{D}_N$ for $N = 2$ given in Equation (B.12), namely $\mathbf{D}_2$. In this case, we have $\mathbf{v}_j = \begin{bmatrix} v_{j1} & v_{j2} & v_{j3} \end{bmatrix}^T$. In this case we have the following relation.

$$\begin{gathered}\mathbf{W} = \mathbf{D}_2\mathbf{V} \\ \begin{bmatrix} w_{01} & w_{02} & w_{03} \\ w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -2 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 2 & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} v_{01} & v_{02} & v_{03} \\ v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \end{bmatrix}\end{gathered} \tag{B.20}$$

We can state that the Chebshev differentiation matrix can be used to compute the derivative of an observed quantity with a linear operation. This result is practical as it allows to a fast computation of a derivative. Morevoer, once the number of points are defined, this matrix remains constant.

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